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VISCOSITY SOLUTIONS TO PARABOLIC MASTER EQUATIONS AND MCKEAN–VLASOV SDES WITH CLOSED-LOOP CONTROLS

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The master equation is a type of PDE whose state variable involves the distribution of certain underlying state process. It is a powerful tool for studying the limit behavior of large interacting systems, including mean field games and systemic risk. It also appears naturally in stochastic control problems with partial information and in time inconsistent problems. In this paper we propose a novel notion of viscosity solution for parabolic master equations, arising mainly from control problems, and establish its wellposedness. Our main innovation is to restrict the involved measures to a certain set of semimartingale measures which satisfy the desired compactness. As an important example, we study the HJB master equation associated with the control problems for McKean–Vlasov SDEs. Due to practical considerations, we consider closed-loop controls. It turns out that the regularity of the value function becomes much more involved in this framework than the counterpart in the standard control problems. Finally, we build the whole theory in the path dependent setting, which is often seen in applications. The main result in this part is an extension of Dupire's (2009) functional Itô formula. This Itô formula requires a special structure of the derivatives with respect to the measures, which was originally due to Lions in the state dependent case. We provided an elementary proof for this well known result in the short note (2017), and the same arguments work in the path dependent setting here.

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1. Introduction. Initiated independently by Caines, Huang and Malhame [24] and Lasry and Lions [26], mean field games and the closely related mean field control problems have received very strong attention in the past decade. Such problems consider the limit behavior of large systems where the agents interact with each other in certain symmetric way, with the systemic risk as a notable application. There have been numerous publications on the subject; see, for example, Cardaliaguet [8], Bensoussan, Frehse and Yam [4], Carmona and Delarue [11, 12], and the references therein. The master equation is a powerful and inevitable tool in this framework, which plays the role of the PDE in the standard literature of controls and games. The main feature of the master equation is that its state variable contains probability measures, typically the distribution of certain underlying state process, so it can be viewed as a PDE on the Wasserstein space. By nature this is an infinite dimensional problem. The master equation is also a convenient tool for (standard) control problems with partial information (see, e.g., Bandini, Cosso, Fuhrman and Pham [1, 2] and Saporito and Zhang [39]), and for some time inconsistent problems as we will see in this paper.

Our main goal of this paper is to propose an intrinsic notion of viscosity solutions for parabolic master equations which mainly arise from control problems or zero-sum game problems in the McKean–Vlasov setting. There have been serious efforts on classical solutions for master equations in various settings; see, for example, Buckdahn, Li, Peng and Rainer [7], Cardaliaguet, Delarue, Lasry and Lions [9], Chassagneux, Crisan and Delarue [13], Saporito and Zhang [39], and Bensoussan, Graber and Yam [5]. However, due to its infinite dimensionality, all these works require very strong technical conditions. So there is a cry for an appropriate notion of weak solutions. We remark that a classical solution requires the candidate solution (typically the value function of certain control/game problem) to be in $C^{1,2}$ (in appropriate sense), while a viscosity solution theory will allow us to reduce the regularity requirement to C^0. It is in general very challenging to establish the differentiability of the value function (especially that with respect to the measures), so such a relaxation of regularity requirement is desirable in many applications.

There have already been some works on viscosity solutions. A natural approach is to use smooth test functions on the Wasserstein space; see, for example, Carmona and Delarue [10]. However, the involved space lacks the local compactness, which is crucial for the viscosity theory, and thus the comparison principle does not seem possible in this approach. In an alternative approach Pham and Wei [34] lift the functions on the Wasserstein space to those on the Hilbert space of random variables and then apply the existing viscosity theory on Hilbert spaces; see, for example, Lions [27–29] and Fabbri, Gozzi and Swiech [20]. Along this approach one could obtain both existence and uniqueness. However, this notion is not intrinsic, in particular, it is not clear to us that a classical solution (with smoothness in the Wasserstein space of probability measures instead of the Hilbert space of random variables) would be a viscosity solution in their sense. Moreover, the viscosity theory on Hilbert spaces is not available in the path dependent case (see Ren and Rosestolato [37] for some recent progress along this direction though), and thus it will be difficult to extend their results to the path dependent case which is important in applications and is another major goal of this paper. We remark that we are in the stochastic setting and thus the master equation is of second order (in certain sense; see Remark 2.6). There are several works for first order master equations corresponding to the deterministic setting; see, for example, Gangbo and Swiech [21, 22] and Bensoussan and Yam [6].

We shall propose a new notion of viscosity solutions, motivated from our previous works Ekren, Keller, Touzi and Zhang [17] and Ekren, Touzi and Zhang [18, 19] for viscosity solutions of path dependent PDEs. Our main innovation is to modify the set of test functions so as to ensure certain desired compactness. To be precise, let $V(t,\mu)$ be a candidate solution, where μ is a probability measure, and φ be a smooth (in certain sense) test function at (t,μ), we shall require $[\varphi - V]$ achieves maximum/minimum at (t,μ) only over the set $[t, t+\delta] \times \mathcal{P}_L(t,\mu)$, where $\mathcal{P}_L(t,\mu)$ is a compact set of semimartingale measures with drift and diffusion characteristics bounded by a constant L. We note that, if we replace the above $\mathcal{P}_L(t,\mu)$ with the δ-neighborhood of μ under the Wasserstein distance, as in [10], then the latter set is not compact under the Wasserstein distance and we will encounter serious difficulties for establishing the comparison principle. We should also note that, if the underlying state space (on which the probability measures are defined) is a torus $\mathbb{T}^d$ instead of $\mathbb{R}^d$, then in the state dependent case the δ-neighborhood of μ under the Wasserstein distance is compact and thus the theory is quite hopeful. However, for the applications in our mind it is more natural to consider $\mathbb{R}^d$ as the underlying state space, and in the mean time we are interested in the path dependent case for which the δ-neighborhood wouldn't work for the torus either.

Our choice of $\mathcal{P}_L(t,\mu)$ is large enough so that, in many applications we are interested in, the value function will be a viscosity solution to the corresponding master equation. On the other hand, the compactness of $\mathcal{P}_L(t,\mu)$ enables us to establish the basic properties of viscosity solutions following rather standard arguments: consistency with classical solutions, equivalence to the alternative definition through semi-jets, stability, and partial comparison principle. The comparison principle is of course the main challenge. We nevertheless establish some partial results in the general case and prove the full comparison principle completely in some special cases. To our best knowledge this is the first uniqueness result in the literature for an intrinsic notion of viscosity solutions for second order master equations.

As far as we know, all works on master equations in the existing literature consider only the state dependent case, where the measures are defined on the finite dimensional space $\mathbb{R}^d$ (or the torus $\mathbb{T}^d$). However, in many applications the problem can be path dependent, for example, lookback options, variance swap, rough volatility, delayed SDEs, to mention a few. In particular, Saporito and Zhang [39] studied control problems with information delay, which naturally induces a path dependent master equation. The second goal of this paper is to establish the whole theory in the path dependent setting, namely the involved probability measure μ is the distribution of the stopped underlying process $X_{\cdot\wedge t}$, rather than the distribution of the current state X_t. The main result in this regard is a functional Itô formula in the McKean–Vlasov setting, extending the well-known result of Dupire [16] in the standard setting. To establish this, we require a special structure of the path derivative with respect to the measure; see (2.16) below. In the state dependent case, such structure was established by Lions [30] (see also Cardaliaguet [8] and Gangbo and Tudorascu [23]) by using quite advanced tools. We provided an elementary proof for this well-known result, which was reported separately in the short note [42], and the same arguments work well in our path dependent framework here. We emphasize that, while this paper is in the path dependent setting, our results on viscosity solutions of master equations are new even in the state dependent case.

Our third goal is to study McKean–Vlasov SDEs with closed-loop controls, whose value function is a viscosity solution to the HJB type master equation. We note that in many applications closed-loop controls (i.e., the control depends on the state process) are more appropriate than open-loop controls (i.e., the control depends on the noise), especially when games are considered; see, for example, Zhang [43], Section 9.1 for detailed discussions. For McKean–Vlasov SDEs, the two types of controls have very subtle differences even for control problems (and more subtle for games), and under closed-loop controls, the regularity of the value function becomes rather technical. By choosing the admissible controls carefully and by using some sophisticated approximations, we manage to prove the desired regularity and then verify the viscosity solution property. Again, while we are in the path dependent setting, our result is new even in the state dependent case, and we believe our approximations will be quite useful for more thorough analysis on functions of probability measure.

Finally, we emphasize that our master equation is parabolic, which mainly corresponds to control problems or zero-sum game problems in the McKean–Vlasov setting, and the solution takes the form $V(t, \mu)$. The master equation induced by mean field games involves functions in the form $V(t, x, \mu)$, and in the path dependent setting this becomes $V(t, \omega, \mu)$. The two types of equations have some fundamental differences. On one hand, our master equation could be nonlinear in $\partial_\mu V$, the derivative of V with respect to the probability measure μ, while mean field game master equation is typically linear in $\partial_\mu V$ (but could be nonlinear in $\partial_x V$). On the other hand, mean field game master equation is nonlocal in $\partial_x V$, which destroys certain crucial monotonicity property and thus the comparison principle does not hold (even for classical solutions). In fact, due to these differences, in many works master equations refer only to the equations arising from mean field games, while those from mean field control problems are called HJB equations in Wasserstein space. We nevertheless call both master equations, since they share many properties and require similar technical tools. So this paper studies mean field control master equations, and we refer to the recent work Mou and Zhang [32] for weak solutions (instead of viscosity solutions) to mean field game master equations.

The rest of the paper is organized as follows. In Section 2 we establish the functional Itô calculus in the Wasserstein space. In Section 3 we introduce parabolic master equations and present several examples, which in particular show some applications of master equations. In Section 4 we introduce our notion of viscosity solutions and establish its wellposedness. In Section 5 we study the McKean–Vlasov SDE with closed-loop controls and show its value function is a viscosity solution to the HJB master equation.

2. Functional Itô calculus in the Wasserstein space.

2.1. *A brief overview in the state dependent setting*. We first recall the Wasserstein metric on the space of probability measures. Let $(\Omega, \mathcal{F})$ be an arbitrary measurable space equipped with a metric $\|\cdot\|$. For any probability measures μ, ν on $\mathcal{F}$, let $\mathcal{P}(\mu, \nu)$ denote the space of probability measures $\mathbb{P}_{\mu,\nu}$ on the product space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ with marginal measures μ and ν. Then the 2-Wasserstein distance of μ and ν is defined as (assuming $(\Omega, \mathcal{F})$ is rich enough)

$$(2.1) \qquad \mathcal{W}_2(\mu, \nu) := \inf_{\mathbb{P}_{\mu,\nu} \in \mathcal{P}(\mu,\nu)} \left(\int_{\Omega \times \Omega} \|\omega_1 - \omega_2\|^2 \, d\mathbb{P}_{\mu,\nu}(\omega_1, \omega_2) \right)^{\frac{1}{2}}.$$

In the state dependent setting, one may set the measurable space as $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ (or the torus $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d))$ as in some works). Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the set of square integrable measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, equipped with the metric $\mathcal{W}_2$. For an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbb{L}^2(\mathcal{F}, \mathbb{P})$ denote the Hilbert space of $\mathbb{P}$-square integrable $\mathcal{F}$-measurable $\mathbb{R}^d$-valued random variables. Given a function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we may lift f to a function on $\mathbb{L}^2(\mathcal{F}, \mathbb{P})$: $F(\xi) := f(\mathbb{P}_\xi)$, where $\mathbb{P}_\xi$ is the $\mathbb{P}$-distribution of $\xi \in \mathbb{L}^2(\mathcal{F}, \mathbb{P})$. Assume F is continuously Fréchet differentiable, Lions [30] showed that the Fréchet derivative DF takes the following form: for some deterministic function $h : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$,

$$(2.2) \qquad \mathrm{DF}(\xi) = h(\mathbb{P}_\xi, \xi);$$

see also Cardaliaguet [8], Gangbo and Tudorascu [23] and Wu and Zhang [42]. Thus naturally we may define $\partial_\mu f := h$. Note that $\partial_\mu f$ is essentially equivalent to the Wasserstein gradient in the optimal transportation theory; see, for example, Carmona and Delarue [11]. Assume further that $\partial_\mu f$ is continuously differentiable with respect to the second variable x, then we have the following Itô formula, due to Buckdahn, Li, Peng and Rainer [7] and Chassagneux, Crisan and Delarue [13]:

$$\begin{aligned} f(\mathbb{P}_{X_t}) = f(\mathbb{P}_{X_0}) + \mathbb{E}^{\mathbb{P}}\bigg[\int_0^t \partial_\mu f(\mathbb{P}_{X_s}, X_s)\cdot dX_s \\ + \frac{1}{2}\int_0^t \partial_x\partial_\mu f(\mathbb{P}_{X_s}, X_s) : d\langle X\rangle_s\bigg], \end{aligned} \tag{2.3}$$

for any $\mathbb{P}$-semimartingale X satisfying certain technical conditions, where $\cdot$ and $:$ denote inner product and trace, respectively.

Our goal of this section is to extend both (2.2) and (2.3) to the path dependent setting. We remark that path dependence appears naturally in many applications. For example, in option pricing theory, many exotic options like lookback options and Asian options are path dependent, then their prices would satisfy certain path dependent PDEs. Another interesting example is the rough volatility model, where the state process is non-Markovian and a path dependent PDE is induced naturally even in state dependent models; see Viens and Zhang [41]. All these models will naturally lead to path dependent master equations when extended to the mean field framework. A more interesting example is the stochastic optimization in standard framework but with constant controls, where a state dependent model will naturally induce a path dependent master equation; see Theorem 3.5 below.

Throughout the paper, for an arbitrary process X, we use the notation

$$X_{s,t} := X_t - X_s, \quad 0 \le s \le t \le T. \tag{2.4}$$

2.2. *The canonical setup in the path dependent setting*. Throughout this paper, we shall fix the canonical space $\Omega := C([0,T], \mathbb{R}^d)$, equipped with the uniform norm $\|\omega\| := \sup_{t\in[0,T]}|\omega_t|$. Let X denote the canonical process, namely $X_t(\omega) := \omega_t$, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T} := \mathbb{F}^X$ the natural filtration generated by X, $\mathcal{P}_2$ the set of probability measures μ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{E}^\mu[\|X\|^2] < \infty$, equipped with the Wasserstein distance $\mathcal{W}_2$ defined by (2.1). Note that $(\Omega, \|\cdot\|)$ and $(\mathcal{P}_2, \mathcal{W}_2)$ are Polish spaces, namely they are complete and separable. We may also use the notation $\mathbb{P}$ to denote probability measures. Quite often we shall use μ when viewing it as a variable of functions, and use $\mathbb{P}$ when considering the distribution of some random variables or processes. Moreover, given a random variable or a stochastic process ξ under certain probability measure $\mathbb{P}$, we also use $\mathbb{P}_\xi := \mathbb{P}\circ\xi^{-1}$ to denote its distribution under $\mathbb{P}$. When the measure $\mathbb{P}$ is clear from the context, we may also use the notation $\mathcal{L}_\xi := \mathbb{P}_\xi$.

The state space of our master equation is $\Theta := [0,T]\times\mathcal{P}_2$. For each $(t,\mu)\in\Theta$, let $\mu_{[0,t]}\in\mathcal{P}_2$ be the distribution of the stopped process $X_{t\wedge\cdot}$ under μ. Since $\mathcal{F}_T^{X_{t\wedge\cdot}} = \mathcal{F}_t$, $\mu_{[0,t]}$ is completely determined by the restriction of μ on $\mathcal{F}_t$. For $(t,\mu), (t',\mu')\in\Theta$, by abusing the notation $\mathcal{W}_2$ we define the 2-Wasserstein pseudometric on Θ as

$$\mathcal{W}_2\big((t,\mu),(t',\mu')\big) := \big(|t-t'| + \mathcal{W}_2^2(\mu_{[0,t]}, \mu'_{[0,t']})\big)^{\frac{1}{2}}. \tag{2.5}$$

If a function $f:\Theta\to\mathbb{R}$ is Borel measurable, with respect to the topology induced by $\mathcal{W}_2$, then it must be $\mathbb{F}$-adapted in the sense that $f(t,\mu) = f(t,\mu_{[0,t]})$ for any $(t,\mu)\in\Theta$. In particular, if f is continuous, then it is $\mathbb{F}$-adapted. Moreover, for $(t,\mu)\in\Theta$, let $\mu_t := \mu\circ X_t^{-1}$ denote the distribution of the random variable X_t. We say f is state dependent if $f(t,\mu)$ depends only on μ_t, and in this case we may abuse the notation and denote $f(t,\mu_t) = f(t,\mu)$.

In order to establish the functional Itô formula on Θ, as in Dupire [16] we extend the canonical space to the càdlàg space $\widehat{\Omega} := \mathbb{D}([0,T), \mathbb{R}^d)$ (we use $\widehat{\ }$ to denote the extensions to the càdlàg space), equipped with the Skorohod distance

$$d_{\mathrm{SK}}(\widehat{\omega}, \widehat{\omega}') := \inf_{\lambda} \sup_{0\le t\le T} \big[|t-\lambda(t)| + |\widehat{\omega}_t - \widehat{\omega}'_{\lambda(t)}|\big], \tag{2.6}$$

where $\lambda : [0,T] \to [0,T]$ is continuous, strictly increasing, with $\lambda(0) = 0$ and $\lambda(T) = T$. Extend the notation $\widehat{X}$, $\widehat{\mathbb{F}}$, $\widehat{\mathcal{P}}_2$, $\widehat{\Theta}$, as well as the 2-Wasserstein pseudometric on $\widehat{\Theta}$ in an obvious way, in particular, in (2.1) the metric $\|\omega^1 - \omega^2\|$ should be replaced with $d_{\mathrm{SK}}(\widehat{\omega}^1, \widehat{\omega}^2)$. Then $(\widehat{\Omega}, d_{\mathrm{SK}})$ and $(\widehat{\mathcal{P}}_2, \mathcal{W}_2)$ are also Polish spaces.

2.3. *Pathwise derivatives in the Wasserstein space.* Let $f : \widehat{\Theta} \to \mathbb{R}$ be continuous (and thus $\widehat{\mathbb{F}}$-adapted). We define its time derivative as

$$\partial_t f(t, \widehat{\mu}) := \lim_{\delta\downarrow 0} \frac{f(t+\delta, \widehat{\mu}_{[0,t]}) - f(t, \widehat{\mu})}{\delta}, \tag{2.7}$$

provided the limit in the right-hand side above exists.

REMARK 2.1. The $\partial_t f$ in (2.7) is actually the right time derivative. Due to the adaptedness requirement, similar to the pathwise analysis in Dupire [16], the left time derivative is not convenient to define. Nevertheless, for the theory which we will develop in the paper, in particular for the functional Itô formula, the right time derivative is sufficient.

The spatial derivative is much more involved. Consider an arbitrary atomless Polish probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Let $\mathbb{L}^2(\tilde{\Omega}; \mathbb{R}^d)$ and $\mathbb{L}^2(\tilde{\Omega}; \widehat{\Omega})$ denote the sets of $\tilde{\mathbb{P}}$-square integrable $\tilde{\mathcal{F}}$-measurable mappings $\xi : \tilde{\Omega} \to \mathbb{R}^d$ and $\tilde{X} : \tilde{\Omega} \to \widehat{\Omega}$, respectively. We first lift f to a function $F : [0,T] \times \mathbb{L}^2(\tilde{\Omega}; \widehat{\Omega}) \to \mathbb{R}$:

$$F(t, \tilde{X}) := f(t, \tilde{\mathbb{P}}_{\tilde{X}}) = f(t, \tilde{\mathbb{P}}_{\tilde{X}_{t\wedge\cdot}}). \tag{2.8}$$

We say F is Fréchet differentiable at $(t, \tilde{X})$ with derivative $\mathrm{DF}(t, \tilde{X}) \in \mathbb{L}^2(\tilde{\Omega}; \mathbb{R}^d)$ if

$$F(t, \tilde{X} + \xi \mathbf{1}_{[t,T]}) - F(t, \tilde{X}) = \mathbb{E}^{\tilde{\mathbb{P}}}\big[\mathrm{DF}(t, \tilde{X}) \cdot \xi\big] + o(\|\xi\|_2) \tag{2.9}$$

for all $\xi \in \mathbb{L}^2(\tilde{\Omega}; \mathbb{R}^d)$, where $\|\xi\|_2^2 := \mathbb{E}^{\tilde{\mathbb{P}}}[|\xi|^2]$. In particular, this implies that $\mathrm{DF}(t, \tilde{X})$ is the Gâteux derivative: for all $\xi \in \mathbb{L}^2(\tilde{\Omega}; \mathbb{R}^d)$,

$$\mathbb{E}^{\tilde{\mathbb{P}}}\big[\mathrm{DF}(t, \tilde{X}) \cdot \xi\big] = \lim_{\varepsilon\to 0} \frac{F(t, \tilde{X} + \varepsilon\xi \mathbf{1}_{[t,T]}) - F(t, \tilde{X})}{\varepsilon}. \tag{2.10}$$

We emphasize that the above derivative involves only the perturbation of $\tilde{X}$ on $[t,T]$, but not on $[0,t)$. Moreover, since f is $\widehat{\mathbb{F}}$-adapted, so $\mathrm{DF}(t, \tilde{X})$ actually involves only the perturbation of $\tilde{X}$ at t. Our main result in this subsection is the following.

THEOREM 2.2. *Let $f : \widehat{\Theta} \to \mathbb{R}$ be continuous. Assume the lifted function F defined by* (2.8) *is Fréchet differentiable and* DF *is continuous in the sense that*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\big[|\mathrm{DF}(t, \tilde{X}^n) - \mathrm{DF}(t, \tilde{X})|^2\big] = 0 \\ &\quad \textit{whenever} \lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\big[d^2_{\mathrm{SK}}(\tilde{X}^n, \tilde{X})\big] = 0. \end{aligned} \tag{2.11}$$

Then there exists an $\widehat{\mathcal{F}}_t$-measurable function $\psi : \widehat{\Omega} \to \mathbb{R}^d$ such that

$$\mathrm{DF}(t, \tilde{X}) = \psi(\tilde{X}_{t\wedge\cdot}), \quad \tilde{\mathbb{P}}\text{-}a.s. \tag{2.12}$$

Moreover, ψ is determined by f and $\tilde{\mathbb{P}}_{\tilde{X}}$, and is unique $\tilde{\mathbb{P}}_{\tilde{X}}$-a.s.

PROOF. The uniqueness of ψ follows from (2.12) and the uniqueness of the Fréchet derivative $\mathrm{DF}(t,\tilde{X})$. Moreover, by the $\widehat{\mathbb{F}}$-adaptedness of f, clearly $\mathrm{DF}(t,\tilde{X})$ is determined by $\tilde{X}_{t\wedge\cdot}$, and thus so is ψ. We prove the rest of the theorem in two steps.

Step 1. We first construct ψ in the case that $\tilde{X}$ is discrete: there exist $\widehat{\omega}_i\in\widehat{\Omega}$, $i\ge 1$, such that $\sum_{i\ge 1}p_i=1$, where $p_i:=\tilde{\mathbb{P}}(\tilde{X}=\widehat{\omega}_i)>0$. For any $x\in\mathbb{R}^d\backslash\{0\}$, $E\subset E_i:=\{\tilde{X}=\widehat{\omega}_i\}$, and $\varepsilon>0$, denote $\widehat{\omega}_i^\varepsilon:=\widehat{\omega}_i+\varepsilon x\mathbf{1}_{[t,T]}$ and $\tilde{X}^\varepsilon:=\tilde{X}+\varepsilon x\mathbf{1}_E\mathbf{1}_{[t,T]}$. Note that

$$\tilde{X}^\varepsilon(\tilde{\omega})=\sum_{j\neq i}\widehat{\omega}_j\mathbf{1}_{E_j}(\tilde{\omega})+\widehat{\omega}_i\mathbf{1}_{E_i\backslash E}(\tilde{\omega})+\widehat{\omega}_i^\varepsilon\mathbf{1}_E(\tilde{\omega}),\qquad \tilde{\omega}\in\tilde{\Omega}.$$

Then, denoting by $\delta.$ the Dirac-measure,

$$\mathcal{L}_{\tilde{X}^\varepsilon}=\sum_{j\neq i}p_j\delta_{\{\widehat{\omega}_j\}}+\big[p_i-\tilde{\mathbb{P}}(E)\big]\delta_{\{\widehat{\omega}_i\}}+\tilde{\mathbb{P}}(E)\delta_{\{\widehat{\omega}_i^\varepsilon\}},$$

and thus

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\mathrm{DF}(t,\tilde{X})\cdot x\mathbf{1}_E\big]&=\lim_{\varepsilon\to 0}\frac{F(t,\tilde{X}+\varepsilon x\mathbf{1}_E\mathbf{1}_{[t,T]})-F(t,\tilde{X})}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\bigg[f\bigg(t,\sum_{j\neq i}p_j\delta_{\{\widehat{\omega}_j\}}+\big[p_i-\tilde{\mathbb{P}}(E)\big]\delta_{\{\widehat{\omega}_i\}}+\tilde{\mathbb{P}}(E)\delta_{\{\widehat{\omega}_i^\varepsilon\}}\bigg)\\&\qquad-f\bigg(t,\sum_{j\ge 1}p_j\delta_{\{\widehat{\omega}_j\}}\bigg)\bigg].\end{aligned}$$

This implies that $\mathbb{E}^{\tilde{\mathbb{P}}}[\mathrm{DF}(t,\tilde{X})\cdot x\mathbf{1}_E]=\mathbb{E}^{\tilde{\mathbb{P}}}[\mathrm{DF}(t,\tilde{X})\cdot x\mathbf{1}_{E'}]$ for any $E,E'\subset E_i$ such that $\tilde{\mathbb{P}}(E)=\tilde{\mathbb{P}}(E')$. By Wu and Zhang ([42], Lemma 2), we see that $\mathrm{DF}(t,\tilde{X})\cdot x$ is a constant on E_i: by setting $E=E_i$,

$$\begin{aligned}\mathrm{DF}(t,\tilde{X})\cdot x=&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon p_i}\bigg[f\bigg(t,\sum_{j\neq i}p_j\delta_{\{\widehat{\omega}_j\}}+p_i\delta_{\{\widehat{\omega}_i+\varepsilon x\mathbf{1}_{[t,T]}\}}\bigg)\\&\qquad-f\bigg(t,\sum_{j\ge 1}p_j\delta_{\{\widehat{\omega}_j\}}\bigg)\bigg].\end{aligned}\tag{2.13}$$

Since $x\in\mathbb{R}^d$ is arbitrary, $\mathrm{DF}(t,\tilde{X})=y_i\in\mathbb{R}^d$, $\tilde{\mathbb{P}}$-a.s. on E_i. Clearly there exists a Borel-measurable function $\psi:\widehat{\Omega}\to\mathbb{R}^d$ such that $\psi(\widehat{\omega}_i)=y_i$, $i\ge 1$, and thus $\mathrm{DF}(t,\tilde{X})=\psi(\tilde{X})$, $\tilde{\mathbb{P}}$-a.s. Note that ψ is unique in $\tilde{\mathbb{P}}_{\tilde{X}}$-a.s. sense, and is determined by f and $\tilde{\mathbb{P}}_{\tilde{X}}$.

Step 2. We now consider the general distribution of $\tilde{X}$. For each $n\ge 1$, since $(\widehat{\Omega},d_{\mathrm{SK}})$ is separable, there exists a partition $\{O_i^n,i\ge 1\}\subset\widehat{\Omega}$ such that $d_{\mathrm{SK}}(\widehat{\omega},\widehat{\omega}_i^n)\le 2^{-n}$ for all $\widehat{\omega}\in O_i^n$, where $\widehat{\omega}_i^n\in O_i^n$ is fixed. Denote $\tilde{X}^n:=\sum_{i\ge 1}\widehat{\omega}_i^n\mathbf{1}_{O_i^n}(\tilde{X})$. We remark that $\tilde{X}^n$ may not be $\mathbb{F}^{\tilde{X}}$-adapted, but such adaptedness is not needed here. Since $\tilde{X}^n$ is discrete, by *Step 1* we have $\mathrm{DF}(t,\tilde{X}^n)=\psi_n(\tilde{X}^n)=\tilde{\psi}_n(\tilde{X})$, where ψ_n is defined by *Step 1* corresponding to $\tilde{X}^n$, and $\tilde{\psi}_n(\widehat{\omega}):=\sum_{i\ge 1}\psi_n(\widehat{\omega}_i^n)\mathbf{1}_{O_i^n}(\widehat{\omega})$, $\widehat{\omega}\in\widehat{\Omega}$. Clearly $\mathbb{E}^{\tilde{\mathbb{P}}}[d^2_{\mathrm{SK}}(\tilde{X}^n,\tilde{X})]\le 2^{-2n}$, then by (2.11) we have

$$\lim_{n\to\infty}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\big|\tilde{\psi}_n(\tilde{X})-\mathrm{DF}(t,\tilde{X})\big|^2\big]=0.\tag{2.14}$$

Thus there exists a subsequence $\{n_k\}_{k\ge 1}$ such that $\tilde{\psi}_{n_k}(\tilde{X})\to\mathrm{DF}(t,\tilde{X})$, $\tilde{\mathbb{P}}$-a.s. Define

$$\begin{aligned}&\psi(\widehat{\omega}):=\varlimsup_{k\to\infty}\tilde{\psi}_{n_k}(\widehat{\omega})\\&\quad\text{where } K:=\Big\{\widehat{\omega}\in\widehat{\Omega}:\varlimsup_{k\to\infty}\tilde{\psi}_{n_k}(\widehat{\omega})=\varliminf_{k\to\infty}\tilde{\psi}_{n_k}(\widehat{\omega})\Big\}.\end{aligned}\tag{2.15}$$

Then $\tilde{\mathbb{P}}(\tilde{X}\in K)=1$ and $\mathrm{DF}(t,\tilde{X})=\psi(\tilde{X})$, $\tilde{\mathbb{P}}$-a.s.

Moreover, let $\tilde{X}' \in \mathbb{L}^2(\tilde{\Omega}; \widehat{\Omega})$ be another process such that $\tilde{\mathbb{P}}_{\tilde{X}'} = \tilde{\mathbb{P}}_{\tilde{X}}$, and define $\tilde{X}'^n$ similarly by using the same $\{O_i^n, \widehat{\omega}_i^n, i \ge 1\}$. Then $\mathrm{DF}(t, \tilde{X}'^n) = \tilde{\psi}_n(\tilde{X}')$ for the same function $\tilde{\psi}_n$. Note that $\tilde{\mathbb{P}}(\tilde{X}' \in K) = \tilde{\mathbb{P}}(\tilde{X} \in K) = 1$, then $\lim_{k\to\infty} \tilde{\psi}_{n_k}(\tilde{X}') = \psi(\tilde{X}')$, $\tilde{\mathbb{P}}$-a.s. On the other hand, $\mathrm{DF}(t, \tilde{X}'^{n_k}) \to \mathrm{DF}(t, \tilde{X}')$ in $\mathbb{L}^2$. So $\mathrm{DF}(t, \tilde{X}') = \psi(\tilde{X}')$, $\tilde{\mathbb{P}}$-a.s., and thus ψ does not depend on the choice of $\tilde{X}$. $\square$

Given the above theorem, particularly the fact that ψ is determined by $\tilde{\mathbb{P}}_{\tilde{X}}$, we may introduce a function $\partial_\mu f : \widehat{\Theta} \times \widehat{\Omega} \to \mathbb{R}^d$ such that $\partial_\mu f(t, \tilde{\mathbb{P}}_{\tilde{X}}, \widehat{\omega}) = \psi(\widehat{\omega})$. In particular, this implies: for any $\widehat{\mathcal{F}}_t$-measurable μ-square integrable random variable $\xi : \widehat{\Omega} \to \mathbb{R}^d$,

$$\mathbb{E}^{\widehat{\mu}}\big[\partial_\mu f(t, \widehat{\mu}, \widehat{X}) \cdot \xi\big] = \lim_{\varepsilon\to 0} \frac{f(t, \widehat{\mu} \circ (\widehat{X} + \varepsilon\xi \mathbf{1}_{[t,T]})^{-1}) - f(t, \widehat{\mu})}{\varepsilon}. \tag{2.16}$$

COROLLARY 2.3. *Let all the conditions in Theorem 2.2 hold true. Assume further that the continuity of* DF *in* (2.11) *is uniform. Then there exists a jointly Borel-measurable function* $\partial_\mu f : \widehat{\Theta} \times \widehat{\Omega} \to \mathbb{R}^d$ *such that*

$$\mathrm{DF}(t, \tilde{X}) = \partial_\mu f(t, \tilde{\mathbb{P}}_{\tilde{X}_{t\wedge\cdot}}, \tilde{X}_{t\wedge\cdot}), \quad \tilde{\mathbb{P}}\text{-}a.s. \tag{2.17}$$

Moreover, if $\partial_\mu f(t, \cdot)$ *is jointly continuous in* $\widehat{\mathcal{P}}_2 \times \widehat{\Omega}$ *for all* t, *then* $\partial_\mu f$ *is unique.*

PROOF. In Theorem 2.2, Step 1, noting that f is Borel measurable, then by (2.13) one can easily see that $\partial_\mu f(t, \sum_{j\ge 1} p_j \delta_{\{\widehat{\omega}_i\}}, \widehat{\omega}_i) := \psi(\widehat{\omega}_i)$ is jointly measurable. Now consider the notation in Theorem 2.2 *Step 2*, and denote $\tilde{\psi}_n(t, \widehat{\mu}, \widehat{\omega}) := \tilde{\psi}_n(\widehat{\omega})$ which is jointly measurable in $(t, \widehat{\mu}, \widehat{\omega})$. By the uniform continuity of DF, one can choose a common subsequence $\{n_k, k \ge 1\}$ such that $\tilde{\psi}_{n_k}(\tilde{X}) \to \mathrm{DF}(t, \tilde{X})$, $\tilde{\mathbb{P}}$-a.s. for all $\tilde{X}$. Denote $\partial_\mu f(t, \widehat{\mu}, \widehat{\omega}) := \overline{\lim}_{k\to\infty} \tilde{\psi}_{n_k}(t, \widehat{\mu}, \widehat{\omega})$. Then $\partial_\mu f$ is jointly measurable and (2.17) holds true.

We now assume $\partial_\mu f(t, \cdot)$ is jointly continuous in $\widehat{\mathcal{P}}_2 \times \widehat{\Omega}$ for all t. Notice again that $\partial_\mu f(t, \widehat{\mu}, \cdot)$ is unique, $\widehat{\mu}$-a.s. Then, when $\mathrm{supp}(\widehat{\mu}) = \widehat{\Omega}$, by the continuity of $\partial_\mu f(t, \widehat{\mu}, \cdot)$ we see that $\partial_\mu f(t, \widehat{\mu}, \cdot)$ is pointwise unique. Finally, for any $\widehat{\mu} \in \widehat{\mathcal{P}}_2$, there exist $\widehat{\mu}_n \in \widehat{\mathcal{P}}_2$ such that $\mathrm{supp}(\widehat{\mu}_n) = \widehat{\Omega}$ for each n and $\lim_{n\to\infty} \widehat{\mathcal{W}}_2(\widehat{\mu}_n, \widehat{\mu}) = 0$. Then $\partial_\mu f(t, \widehat{\mu}_n, \cdot)$ is unique and $\lim_{n\to\infty} \partial_\mu f(t, \widehat{\mu}_n, \widehat{\omega}) = \partial_\mu f(t, \widehat{\mu}, \widehat{\omega})$. This clearly implies the uniqueness of $\partial_\mu f(t, \widehat{\mu}, \widehat{\omega})$. $\square$

Now given $\partial_\mu f : \widehat{\Theta} \times \widehat{\Omega} \to \mathbb{R}^d$, assume $\partial_\mu f(t, \cdot)$ is continuous and thus is unique. In the spirit of Dupire [16] we may define further the derivative function $\partial_\omega \partial_\mu f : \widehat{\Theta} \times \widehat{\Omega} \to \mathbb{R}^{d\times d}$ determined by: for all $x \in \mathbb{R}^d$,

$$\partial_\omega \partial_\mu f(t, \widehat{\mu}, \widehat{\omega})x := \lim_{\varepsilon\to 0} \frac{\partial_\mu f(t, \widehat{\mu}, \widehat{\omega} + \varepsilon x \mathbf{1}_{[t,T]}) - \partial_\mu f(t, \widehat{\mu}, \widehat{\omega})}{\varepsilon}. \tag{2.18}$$

EXAMPLE 2.4. Let $f(t, \widehat{\mu}) := \mathbb{E}^{\widehat{\mu}}[\widehat{X}_t \int_0^t \widehat{X}_s\, ds] - \mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]\mathbb{E}^{\widehat{\mu}}[\int_0^t \widehat{X}_s\, ds]$ with $d = 1$. Then

$$\partial_t f(t, \widehat{\mu}) = \mathbb{E}^{\widehat{\mu}}\big[\widehat{X}_t^2\big] - \mathbb{E}^{\widehat{\mu}}\big[\widehat{X}_t^2\big]\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t],$$

$$\partial_\mu f(t, \widehat{\mu}, \widehat{\omega}) = \int_0^t \widehat{\omega}_s\, ds - 2\widehat{\omega}_t \mathbb{E}^{\widehat{\mu}}\bigg[\int_0^t \widehat{X}_s\, ds\bigg],$$

$$\partial_\omega \partial_\mu f(t, \widehat{\mu}, \widehat{\omega}) = -2\mathbb{E}^{\widehat{\mu}}\bigg[\int_0^t \widehat{X}_s\, ds\bigg].$$

PROOF. First, note that

$$f(t+\delta,\widehat{\mu}_{[0,t]})=\mathbb{E}^{\widehat{\mu}}\Big[\widehat{X}_t\int_0^{t+\delta}\widehat{X}_{t\wedge s}\,ds\Big]-\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]\mathbb{E}^{\widehat{\mu}}\Big[\int_0^{t+\delta}\widehat{X}_{t\wedge s}\,ds\Big]$$
$$=f(t,\widehat{\mu})+\delta\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]-\delta\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t].$$

Then by (2.7) one can easily see that $\partial_t f(t,\widehat{\mu})=\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]-\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t^2]\mathbb{E}^{\widehat{\mu}}[\widehat{X}_t]$.

Next, for any appropriate $\tilde{\mathbb{P}}$ and $\tilde{X}$ on $(\tilde{\Omega},\tilde{\mathcal{F}})$, we have

$$F(t,\tilde{X})=\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\tilde{X}_t\int_0^t\tilde{X}_s\,ds\Big]-\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{X}_t^2]\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^t\tilde{X}_s\,ds\Big].$$

Then,

$$\begin{aligned}&F(t,\tilde{X}+\xi\mathbf{1}_{[t,T]})\\&\quad=\mathbb{E}^{\tilde{\mathbb{P}}}\Big[[\tilde{X}_t+\xi]\int_0^t\tilde{X}_s\,ds\Big]-\mathbb{E}^{\tilde{\mathbb{P}}}[[\tilde{X}_t+\xi]^2]\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^t\tilde{X}_s\,ds\Big]\\&\quad=F(t,\tilde{X})+\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\xi\int_0^t\tilde{X}_s\,ds\Big]-\mathbb{E}^{\tilde{\mathbb{P}}}[2\xi\tilde{X}_t+\xi^2]\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^t\tilde{X}_s\,ds\Big].\end{aligned}$$

This implies

$$\mathrm{DF}(t,\tilde{X})=\int_0^t\tilde{X}_s\,ds-2\tilde{X}_t\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^t\tilde{X}_s\,ds\Big],$$

and thus

$$\partial_\mu f(t,\widehat{\mu},\widehat{\omega})=\int_0^t\widehat{\omega}_s\,ds-2\widehat{\omega}_t\mathbb{E}^{\widehat{\mu}}\Big[\int_0^t\widehat{X}_s\,ds\Big].$$

Finally, by (2.18) it is clear that $\partial_\omega\partial_\mu f(t,\widehat{\mu},\widehat{\omega})=-2\mathbb{E}^{\widehat{\mu}}[\int_0^t\widehat{X}_s\,ds]$. □

DEFINITION 2.5. Let $C^{1,1,1}(\widehat{\Theta})$ be the set of continuous mappings $f:\widehat{\Theta}\to\mathbb{R}$ such that there exist continuous functions $\partial_t f:\widehat{\Theta}\to\mathbb{R}$, $\partial_\mu f:\widehat{\Theta}\times\widehat{\Omega}\to\mathbb{R}^d$, and $\partial_\omega\partial_\mu f:\widehat{\Theta}\times\widehat{\Omega}\to\mathbb{R}^{d\times d}$.

Moreover, let $C_b^{1,1,1}(\widehat{\Theta})\subset C^{1,1,1}(\widehat{\Theta})$ denote the subset such that $\partial_t f$ is bounded, and $\partial_\mu f$, $\partial_\omega\partial_\mu f$ have linear growth in $\widehat{\omega}$: for all $(t,\widehat{\mu},\widehat{\omega})\in\widehat{\Theta}\times\widehat{\Omega}$,

$$|\partial_\mu f(t,\widehat{\mu},\widehat{\omega})|+|\partial_\omega\partial_\mu f(t,\widehat{\mu},\widehat{\omega})|\le C[1+\|\widehat{\omega}\|]. \tag{2.19}$$

REMARK 2.6. Our master equation (3.1) below will involve the derivatives $\partial_t f$, $\partial_\mu f$, $\partial_\omega\partial_\mu f$, but not $\partial_\mu\partial_\mu f$ which can be defined in a natural way. The existence of $\partial_\omega\partial_\mu f$ is of course a stronger requirement than that of $\partial_\mu f$, but roughly speaking it is weaker than the existence of $\partial_\mu\partial_\mu f$. In the literature people call master equations involving $\partial_\mu\partial_\mu f$ second order, so our master equation is somewhat between first order and second order.

2.4. *The functional Itô formula.* For any $L>0$, denote by $\widehat{\mathcal{P}}_L$ be the subset of $\mu\in\widehat{\mathcal{P}}_2$ such that μ is a semimartingale measure with both the drift and diffusion characteristics bounded by L. To be precise, $\mu=\tilde{\mathbb{P}}\circ\tilde{X}^{-1}$, where $(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ is a filtered probability space, $\tilde{X}_t=\tilde{X}_0+\int_0^t\tilde{b}_s\,ds+\int_0^t\tilde{\sigma}_s\,d\tilde{B}_s$, $\tilde{X}_0\in\mathbb{L}^2(\tilde{\mathcal{F}}_0,\tilde{\mathbb{P}};\mathbb{R}^d)$, $\tilde{b}:[0,T]\times\tilde{\Omega}\to\mathbb{R}^d$ and $\tilde{\sigma}:[0,T]\times\tilde{\Omega}\to\mathbb{R}^{d\times d}$ are $\tilde{\mathbb{F}}$-progressively measurable with $|\tilde{b}|$, $\frac{1}{2}|\tilde{\sigma}|^2\le L$, and $\tilde{B}$ is a d-dimensional $(\tilde{\mathbb{F}},\tilde{\mathbb{P}})$-Brownian motion. Note that, in particular, $\widehat{X}$ is continuous in t, μ-a.s., namely $\mathrm{supp}(\mu)\subset\Omega\subset\widehat{\Omega}$. So μ can actually be viewed as a measure on Ω and thus we use the notation μ instead of $\widehat{\mu}$ here.

THEOREM 2.7. *Let $f \in C_b^{1,1,1}(\widehat{\Theta})$ and $\mu \in \widehat{\mathcal{P}}_L$ for some $L > 0$. Then*

$$
\begin{aligned}
f(t,\mu) &= f(0,\mu) + \int_0^t \partial_t f(s,\mu)\,ds \\
&\quad + \mathbb{E}^{\mu}\Big[\int_0^t \partial_\mu f(s,\mu,\widehat{X}) \cdot d\widehat{X}_s + \frac{1}{2}\int_0^t \partial_\omega \partial_\mu f(s,\mu,\widehat{X}) : d\langle \widehat{X}\rangle_s\Big].
\end{aligned} \tag{2.20}
$$

PROOF. For notational simplicity, assume $d = 1$ and $t = T$. The general case can be proved without any additional difficulty. Fix $\mu \in \widehat{\mathcal{P}}_L$ and let $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{X})$ be the desired setting so that $\mu = \tilde{\mathbb{P}} \circ \tilde{X}^{-1}$. Fix $n \geq 1$ and let $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a uniform partition of $[0, T]$. Recall (2.4) and denote

$$
\begin{aligned}
\tilde{X}^n &:= \sum_{i=0}^{n-1} \tilde{X}_{t_i} 1_{[t_i,t_{i+1})} + \tilde{X}_T 1_{\{T\}}, \qquad \mu^n := \tilde{\mathbb{P}} \circ (\tilde{X}^n)^{-1}; \\
\tilde{X}^{n,\theta} &:= \tilde{X}^n_{t_i\wedge\cdot} + \theta \tilde{X}_{t_i,t_{i+1}} \mathbf{1}_{[t_{i+1},T]}, \qquad \mu^{n,\theta} := \tilde{\mathbb{P}} \circ (\tilde{X}^{n,\theta})^{-1}, \qquad \theta \in [0,1].
\end{aligned}
$$

Note that $\tilde{X}^n_{t_{i+1}\wedge\cdot} = \tilde{X}^n_{t_i\wedge\cdot} + \tilde{X}_{t_i,t_{i+1}} \mathbf{1}_{[t_{i+1},T]}$. Then

$$
\begin{aligned}
& f(T,\mu^n) - f(0,\mu^n) \\
&= \sum_{i=0}^{n-1} \big[f(t_{i+1}, \mu^n_{[0,t_{i+1}]}) - f(t_i, \mu^n_{[0,t_i]})\big] \\
&= \sum_{i=0}^{n-1} \big[\big[f(t_{i+1}, \mu^n_{[0,t_i]}) - f(t_i, \mu^n_{[0,t_i]})\big] \\
&\qquad + \big[f(t_{i+1}, \mu^n_{[0,t_{i+1}]}) - f(t_{i+1}, \mu^n_{[0,t_i]})\big]\big] \\
&= \sum_{i=0}^{n-1} \Big[\int_{t_i}^{t_{i+1}} \partial_t f(t, \mu^n_{[0,t_i]})\,dt \\
&\qquad + \int_0^1 \mathbb{E}^{\tilde{\mathbb{P}}}\big[\partial_\mu f(t_{i+1}, \mu^{n,\theta}, \tilde{X}^{n,\theta}) \tilde{X}_{t_i,t_{i+1}}\big]\,d\theta\Big] \\
&= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \partial_t f(t, \mu^n_{[0,t_i]})\,dt \\
&\qquad + \sum_{i=0}^{n-1} \int_0^1 \mathbb{E}^{\tilde{\mathbb{P}}}\big[\partial_\mu f(t_{i+1}, \mu^{n,\theta}, \tilde{X}^n_{t_i\wedge\cdot}) \tilde{X}_{t_i,t_{i+1}}\big]\,d\theta \\
&\qquad + \sum_{i=0}^{n-1} \int_0^1 \int_0^1 \mathbb{E}^{\tilde{\mathbb{P}}}\big[\partial_\omega \partial_\mu f(t_{i+1}, \mu^{n,\theta}, \tilde{X}^{n,\tilde{\theta}\theta}) \theta |\tilde{X}_{t_i,t_{i+1}}|^2\big]\,d\tilde{\theta}\,d\theta \\
&=: I_1^n + I_2^n + I_3^n,
\end{aligned} \tag{2.21}
$$

where I_i^n, $i = 1, 2, 3$, are defined in an obvious way.

We now send $n \to \infty$. Since $\tilde{X}$ is continuous, $\tilde{\mathbb{P}}$-a.s., then, for any $t \in [0, T]$ and $\theta \in [0, 1]$,

$$
d_{\mathrm{SK}}(\tilde{X}^n, \tilde{X}) + d_{\mathrm{SK}}(\tilde{X}^n_{t_i\wedge\cdot}, \tilde{X}_{t\wedge\cdot}) + d_{\mathrm{SK}}(\tilde{X}^{n,\theta}_{t_{i+1}\wedge\cdot}, \tilde{X}_{t\wedge\cdot}) \to 0, \quad \tilde{\mathbb{P}}\text{-a.s.}, \tag{2.22}
$$

where we always choose i such that $t_i \leq t < t_{i+1}$. Since $\|\tilde{X}^n\| \leq \|\tilde{X}\|$, $\|\tilde{X}^{n,\theta}\| \leq \|\tilde{X}\|$, by the dominated convergence theorem we have

$$
\mathcal{W}_2(\mu^n_{[0,t_i]}, \mu_{[0,t]}) + \mathcal{W}_2(\mu^{n,\theta}_{[0,t_{i+1}]}, \mu_{[0,t]}) \to 0.
$$

Then, by the desired regularity of f, together with the boundedness of $\partial_t f$, (2.19), and the fact that the $\tilde{b}$ and $\tilde{\sigma}$ associated with $\tilde{X}$ are bounded, we can easily have

$$\lim_{n\to\infty}[f(T,\mu^n)-f(0,\mu^n)]=f(T,\mu)-f(0,\mu),$$

$$\lim_{n\to\infty} I_1^n=\int_0^T \partial_t f(t,\mu)\,dt;$$

$$\lim_{n\to\infty}\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\Big|\int_0^1 \partial_\mu f(t_{i+1},\mu^{n,\theta},\tilde{X}^n_{t_i\wedge\cdot})\,d\theta-\partial_\mu f(t,\mu,\tilde{X})\Big|^2\Big]=0;$$

$$\lim_{n\to\infty}\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\Big|\int_0^1\int_0^1 \partial_\omega\partial_\mu f(t_{i+1},\mu^{n,\theta},\tilde{X}^{n,\tilde{\theta}\theta})\theta\,d\tilde{\theta}\,d\theta-\frac{1}{2}\partial_\omega\partial_\mu f(t,\mu,\tilde{X})\Big|^2\Big]=0.$$

Plug all these into (2.21), and recall that $\tilde{\mathbb{P}}\circ\tilde{X}^{-1}=\mu$, we can easily obtain (2.20). $\square$

We remark that it is possible to relax the technical conditions required for the functional Itô formula (2.20), in particular we can allow $\widehat{\mu}\in\widehat{\mathcal{P}}_2$ to be semimartingale measures with $\operatorname{supp}(\widehat{\mu})$ not within Ω. We also remark that, since $\langle\widehat{X}\rangle$ is symmetric, in the last term of (2.20) we may replace $\partial_\omega\partial_\mu f(s,\mu,\widehat{X}_\cdot)$ with

$$\partial_\omega^{\mathrm{sym}}\partial_\mu f(s,\mu,\widehat{X}):=\frac{1}{2}\big[\partial_\omega\partial_\mu f(s,\mu,\widehat{X})+\big[\partial_\omega\partial_\mu f(s,\mu,\widehat{X})\big]^\top\big]. \tag{2.23}$$

2.5. *The restriction on the space of continuous paths.*

DEFINITION 2.8.

(i) Let $C^{1,1,1}(\Theta)$ denote the set of $f:\Theta\to\mathbb{R}$ such that there exists $\widehat{f}\in C^{1,1,1}(\widehat{\Theta})$ satisfying $\widehat{f}=f$ on Θ, and define, for all $(t,\mu,\omega)\in\Theta\times\Omega$,

$$\begin{aligned}
&\partial_t f(t,\mu):=\partial_t\widehat{f}(t,\mu),\qquad \partial_\mu f(t,\mu,\omega):=\partial_\mu\widehat{f}(t,\mu,\omega),\\
&\partial_\omega\partial_\mu f(t,\mu,\omega):=\partial_\omega\partial_\mu\widehat{f}(t,\mu,\omega),\\
&\partial_\omega^{\mathrm{sym}}\partial_\mu f(t,\mu,\omega):=\partial_\omega^{\mathrm{sym}}\partial_\mu\widehat{f}(t,\mu,\omega).
\end{aligned}\tag{2.24}$$

Moreover, we say $f\in C_b^{1,1,1}(\Theta)$ if the extension $\widehat{f}\in C_b^{1,1,1}(\widehat{\Theta})$.

(ii) Let $\mathcal{P}_L$ denote the subset of $\mu\in\mathcal{P}_2$ such that μ is a semimartingale measure with both the drift and diffusion characteristics bounded by L.

The following result is a direct consequence of Theorem 2.7.

THEOREM 2.9. *Let $f\in C_b^{1,1,1}(\Theta)$.*

(i) *The derivatives $\partial_t f$, $\partial_\mu f$, $\partial_\omega^{\mathrm{sym}}\partial_\mu f$ do not depend on the choices of $\widehat{f}$.*

(ii) *For any $L>0$ and $\mu\in\mathcal{P}_L$, we have*

$$\begin{aligned}
f(t,\mu)&=f(0,\mu)+\int_0^t\partial_t f(s,\mu)\,ds\\
&\quad+\mathbb{E}^\mu\Big[\int_0^t\partial_\mu f(s,\mu,X)\cdot dX_s+\frac{1}{2}\int_0^t\partial_\omega^{\mathrm{sym}}\partial_\mu f(s,\mu,X):d\langle X\rangle_s\Big].
\end{aligned}\tag{2.25}$$

PROOF. (ii) follows directly from Theorem 2.7 and (2.24). To see (i), the uniqueness of $\partial_t f$ is obvious. Now fix $(t,\mu)\in\Theta$ and let $\widehat{f}$ be an arbitrary extension. For any bounded $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variable b_t, let $\tilde{\mu}\in\mathcal{P}_2$ be such that $\tilde{\mu}=\mu$ on $\mathcal{F}_t$ and

$X_s - X_t = b_t[s-t]$, $t \le s \le T$, $\tilde{\mu}$-a.s. Following the same arguments as in Theorem 2.7, for any $\delta > 0$ we have

$$f(t+\delta, \tilde{\mu}) - f(t, \mu) = \int_t^{t+\delta} \partial_t f(s, \tilde{\mu})\, ds + \mathbb{E}^{\mu}\left[\int_t^{t+\delta} \partial_\mu \widehat{f}(s, \tilde{\mu}, X) \cdot b_t\, ds\right].$$

Divide both sides by δ and send $\delta \to 0$, we obtain the uniqueness of $\mathbb{E}^{\mu}[\partial_\mu \widehat{f}(t, \mu, X) \cdot b_t]$. Since b_t is arbitrary, we see that $\partial_\mu \widehat{f}(t, \mu, X)$ is unique, μ-a.s. Similarly, for any bounded $\mathcal{F}_t$-measurable $\mathbb{R}^{d\times d}$-valued random variable σ_t, let $\tilde{\mu} \in \mathcal{P}_2$ be such that $\tilde{\mu} = \mu$ on $\mathcal{F}_t$ and X is a $\tilde{\mu}$-martingale on $[t, T]$ with diffusion coefficient σ_t. Then similarly we can show that $\mathbb{E}^{\mu}[\partial_\omega^{\mathrm{sym}} \partial_\mu \widehat{f}(t, \mu, X) : \sigma_t \sigma_t^\top]$ is unique, which implies the μ-a.s. uniqueness of $\partial_\omega^{\mathrm{sym}} \partial_\mu \widehat{f}(t, \mu, X)$. $\square$

We remark that, under some stronger technical conditions, as in Cont and Fournie [14] one can show that $\partial_\omega \partial_\mu f$ also does not depend on the choices of $\widehat{f}$. However, the analysis below will depend only on $\partial_\omega^{\mathrm{sym}} \partial_\mu f$, so we do not pursue such generality here.

REMARK 2.10. Let $V \in C^{1,1,1}(\Theta)$. If $V(t, \mu) = V(t, \mu_t)$ is state dependent, it is clear that $\partial_\mu V(t, \mu, \omega) = \partial_\mu V(t, \mu, \omega_t)$ also depends only on the current state ω_t. Then naturally we may consider $\partial_x \partial_\mu V$ instead of $\partial_\omega \partial_\mu V$. Throughout the paper we shall take this convention in the state dependent case.

3. Parabolic master equations and some applications. In this paper we are interested in the following so called master equation:

$$\begin{aligned} &\mathbb{L}V(t, \mu) = 0, \quad (t, \mu) \in \Theta \text{ where} \\ &\mathbb{L}V(t, \mu) := \partial_t V(t, \mu) + G\big(t, \mu, V(t, \mu), \partial_\mu V(t, \mu, \cdot), \partial_\omega \partial_\mu V(t, \mu, \cdot)\big), \end{aligned} \tag{3.1}$$

$G(t, \mu, y, Z, \Gamma) \in \mathbb{R}$ is defined in the domain where $(t, \mu, y) \in \Theta \times \mathbb{R}$, and $(Z, \Gamma) \in C^0(\Omega; \mathbb{R}^d) \times C^0(\Omega; \mathbb{R}^{d\times d})$ are $\mathcal{F}_t$-measurable. We remark that G depends on the whole random variables Z and Γ, rather than their values. Such dependence is typically through $\mathbb{E}^{\mu}$ in the form:

$$G = G_1\big(t, \mu, y, \mathbb{E}^{\mu}\big[G_2(t, \mu, X, y, Z, \Gamma)\big]\big)$$

for some deterministic functions $G_1 : \Theta \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ and $G_2 : \Theta \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^k$ for some dimension k.

ASSUMPTION 3.1.

(i) G is continuous in (t, μ) and uniformly Lipschitz continuous in y with a Lipschitz constant L_0.

(ii) G is uniformly Lipschitz continuous in (Z, Γ) with a Lipschitz constant L_0 in the following sense: for any (t, μ, y) and any $\mathcal{F}_t$-measurable random variables $Z_1, \Gamma_1, Z_2, \Gamma_2$, there exist $\mathcal{F}_t$-measurable random variables b_t, σ_t such that $|b_t|, \frac{1}{2}|\sigma_t|^2 \le L_0$, and

$$\begin{aligned} &G(t, \mu, y, Z_1, \Gamma_1) - G(t, \mu, y, Z_2, \Gamma_2) \\ &\quad = \mathbb{E}^{\mu}\left[b_t \cdot [Z_1 - Z_2] + \frac{1}{2}\sigma_t \sigma_t^\top : [\Gamma_1 - \Gamma_2]\right]. \end{aligned} \tag{3.2}$$

We remark that, while (3.2) may look a little less natural, one can easily verify it for all the examples in this paper. Moreover, when μ is degenerate and thus Z, Γ becomes deterministic numbers rather than random variables, (3.2) is equivalent to the standard Lipschitz continuity.

REMARK 3.2. By (3.2), it is clear that G depends on Γ only through $\Gamma^{\mathrm{sym}} := \frac{1}{2}[\Gamma + \Gamma^\top]$, and G is increasing in Γ^{sym}. So (3.1) depends on $\partial_\omega \partial_\mu V$ only through $\partial_\omega^{\mathrm{sym}} \partial_\mu V$, which is unique (or, say, well defined) by Theorem 2.9(i).

DEFINITION 3.3. Let $V \in C^{1,1,1}(\Theta)$. We say V is a classical solution (resp. classical subsolution, classical supersolution) of the master equation (3.1) if

$$\mathbb{L}V(t,\mu) = \text{ (resp. } \geq, \leq) \, 0 \quad \text{for all } (t,\mu) \in \Theta.$$

In the rest of this section we show several examples, which can be viewed as some typical applications of our parabolic master equations. We remark that the smooth differentiability of the involved value functions are often very challenging (and in general may not be true), and thus the main focus of this paper is the viscosity solution. However, for illustration purpose, in this section we shall assume the value functions are smooth and verify they are classical solutions of the corresponding master equations. We shall also show in some special cases that the value functions under consideration are indeed smooth.

3.1. *Stochastic optimization with deterministic controls.* While the value function of a control problem will automatically be path dependent if the coefficients are path dependent, in this subsection we present a state dependent example which endogenously induces a path dependent master equation. Consider a standard control problem:

$$\begin{aligned} & V_0 = \sup_{\alpha \in \mathcal{A}} Y_0^\alpha \quad \text{where} \\ & X_t^\alpha = x_0 + \int_0^t b(s, X_s^\alpha, \alpha_s)\, ds + \int_0^t \sigma(s, X_s^\alpha, \alpha_s)\, dB_s, \\ & Y_t^\alpha = g(X_T^\alpha) + \int_t^T f(s, X_s^\alpha, Y_s^\alpha, Z_s^\alpha, \alpha_s)\, ds - \int_t^T Z_s^\alpha\, dB_s. \end{aligned} \tag{3.3}$$

Here B is a $\mathbb{P}_0$-Brownian motion, the control α takes values in an appropriate set A, and the coefficients b, σ, f, g satisfy standard technical conditions which we shall not specify. When $\mathcal{A}$ is the set of $\mathbb{F}^B$ or $\mathbb{F}^{X^\alpha}$-progressively measurable processes, it is a classical result that $V_0 = u(0, x_0)$, where u is the solution to an HJB equation, and the optimal control α^*, if it exists, typically is feedback type: $\alpha_t^* = I(t, X_t^*)$ for some deterministic function I.

In practice, quite often one needs some time to analyze the information (including the time for numerical computation), and in operations management, one needs to place orders some time before the parts are actually used. Mathematically, this amounts to requiring α_t to be $\mathcal{F}_{t-\delta}$-measurable, for some information delay parameter δ. For simplicity let us assume $T \leq \delta$, then α becomes deterministic. In the rest of this subsection, we shall consider the problem (3.3) where

$$\text{the admissible controls } \alpha \in \mathcal{A} \text{ are deterministic.} \tag{3.4}$$

This seemingly simple problem is actually more involved, and to our best knowledge is not covered by the existing methods in the literature. The main difficulty is the time inconsistency. Indeed, if one naively defines $u(t,x)$ as the value of the optimization problem on $[t,T]$ with initial condition $X_t = x$, then u does not satisfy the dynamic programming principle and consequently it does not satisfy any PDE.

In Saporito and Zhang [39] we investigated this problem in the case $f = f(t, X_t^\alpha, \alpha_t)$. It turns out that in this case the optimal α^* takes the form: $\alpha_t^* = I(t, \mathcal{L}_{X_t^*})$, which is deterministic. To be precise, for any $(t,\mu) \in \Theta$ and $\alpha \in \mathcal{A}$ (deterministic), let $\mathbb{P}^{t,\mu,\alpha}$ be the unique solution satisfying $\mathbb{P}^{t,\mu,\alpha}_{[0,t]} = \mu_{[0,t]}$ and, for some $\mathbb{P}^{t,\mu,\alpha}$-Brownian motion B^α,

$$X_s = X_t + \int_t^s b(r, X_r, \alpha_r)\, dr + \int_t^s \sigma(r, X_r, \alpha_r)\, dB_r^\alpha, \quad s \in [t,T], \tag{3.5}$$

$\mathbb{P}^{t,\mu,\alpha}$-a.s. Define

$$V(t,\mu) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{t,\mu,\alpha}}\Big[g(X_T) + \int_t^T f(s, X_s, \alpha_s)\,ds\Big]. \tag{3.6}$$

Then by [39] we have the following result.

PROPOSITION 3.4. *Assume $f = f(t,x,a)$, b, σ, f, g satisfy standard technical conditions, and define V by* (3.6) *under* (3.4). *Then*:

(i) *$V(t,\mu) = V(t,\mu_t)$ is state dependent and the dynamic programming principle holds: for any $t_1 < t_2$,*

$$V(t_1,\mu_{t_1}) = \sup_{\alpha\in\mathcal{A}}\Big[V\big(t_2, \mathbb{P}^{t_1,\mu,\alpha}_{t_2}\big) + \int_{t_1}^{t_2} \mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\big[f(s,X_s,\alpha_s)\big]\,ds\Big]. \tag{3.7}$$

(ii) *Assume $V \in C^{1,1,1}(\Theta)$ (more precisely $C^{1,1,1}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ here, and also recalling Remark* 2.10), *then V is the classical solution to the following master equation*:

$$\begin{aligned}
&\partial_t V(t,\mu) + \sup_{a\in A}\mathbb{E}^{\mu}\big[G_2\big(t,\mu,X_t,\partial_\mu V(t,\mu,X_t),\partial_x\partial_\mu V(t,\mu,X_t),a\big)\big] = 0,\\
&V(T,\mu) = \mathbb{E}^{\mu}\big[g(X_T)\big]\\
&\qquad \text{where } G_2(t,\mu,x,z,\gamma,a) := \frac{1}{2}\gamma : \sigma\sigma^\top(t,x,a)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + z\cdot b(t,x,a) + f(t,x,a).
\end{aligned} \tag{3.8}$$

(iii) *Assume further that the Hamiltonian in* (3.8) *has an optimal argument $a^* = I(t,\mu_t)$, and the following McKean–Vlasov SDE has a solution*:

$$X^*_t = x_0 + \int_0^t b\big(s, X^*_s, I(s,\mathcal{L}_{X^*_s})\big)\,ds + \int_0^t \sigma\big(s, X^*_s, I(s,\mathcal{L}_{X^*_s})\big)\,dB_s, \tag{3.9}$$

$\mathbb{P}_0$-a.s. Then $\alpha^*_t := I(t,\mathcal{L}_{X^*_t})$ is an optimal control to the problem (3.3).

We remark that the expectation involved in (3.8) is a function of (t,μ_t,a), so the optimal control a^* takes the form $I(t,\mu_t)$ in (iii).

While induced endogenously, the master equation (3.8) is still state dependent. We now consider (3.3) with nonlinear f, again with deterministic α. The general case is quite involved, and we consider only a special case here: $f = f(t,X_t,Y_t)$. Given $(t,\mu)\in\Theta$ and $\alpha\in\mathcal{A}$, let $\mathbb{P}^{t,\mu,\alpha}$ be defined by (3.5), and consider the following BSDE:

$$Y^{t,\mu,\alpha}_s = g(X_T) + \int_s^T f\big(r, X_r, Y^{t,\mu,\alpha}_r\big)\,dr + M^{t,\mu,\alpha}_T - M^{t,\mu,\alpha}_s, \tag{3.10}$$

$s\in[0,T]$, $\mathbb{P}^{t,\mu,\alpha}$-a.s. Here the component M of the solution pair (Y,M) is a $\mathbb{P}^{t,\mu,\alpha}$-martingale. If we set $V(t,\mu) := \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mu}[Y^{t,\mu,\alpha}_t]$ as in (3.6), then V will still be state dependent, but in general the DPP in the spirit of (3.7) does not hold, because of the nonlinearity of f. To keep the time consistency, in this case we shall define the value function as:

$$V(t,\mu) := \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mu}\big[Y^{t,\mu,\alpha}_0\big]. \tag{3.11}$$

Note that $V(t,\mu)$ is path dependent, in particular, $V(T,\mu) = Y^{T,\mu}_0$, where $Y^{T,\mu}$ is the solution to BSDE (3.10) under μ. Then we can extend Proposition 3.4 to this case.

THEOREM 3.5. *Assume $f = f(t, x, y)$, b, σ, f, g satisfy standard technical conditions, and define V by (3.11) under (3.4). Then*:

(i) *The following dynamic programming principle holds*:

$$V(t_1, \mu) = \sup_{\alpha \in \mathcal{A}} V\big(t_2, \mathbb{P}^{t_1,\mu,\alpha}\big), \quad t_1 < t_2. \tag{3.12}$$

(ii) *Assume $V \in C^{1,1,1}(\Theta)$, then V satisfies the path dependent master equation*:

$$\begin{aligned} &\partial_t V(t, \mu) + \sup_{a \in A} \mathbb{E}^{\mu}\big[G_2\big(t, \mu, X_t, \partial_\mu V(t, \mu, X), \partial_\omega \partial_\mu V(t, \mu, X), a\big)\big] = 0, \\ &V(T, \mu) = Y_0^{T,\mu} \end{aligned} \tag{3.13}$$

$$\text{where } G_2(t, \mu, x, z, \gamma, a) := \frac{1}{2}\gamma : \sigma\sigma^\top(t, x, a) + z \cdot b(t, x, a).$$

(iii) *Assume further that the Hamiltonian in (3.13) has an optimal argument $a^* = I(t, \mu_{[0,t]})$, and the following McKean–Vlasov SDE has a solution*:

$$\begin{aligned} X_t^* = x_0 &+ \int_0^t b\big(s, X_s^*, I(s, \mathcal{L}_{X^*_{s\wedge\cdot}})\big)\, ds \\ &+ \int_0^t \sigma\big(s, X_s^*, I(s, \mathcal{L}_{X^*_{s\wedge\cdot}})\big)\, dB_s, \quad \mathbb{P}_0\text{-}a.s. \end{aligned} \tag{3.14}$$

Then $\alpha_t^* := I(t, \mathcal{L}_{X^*_{t\wedge\cdot}})$ is an optimal control to the problem (3.3).

PROOF. (i) We emphasize that, since the $\mathbb{P}^{t_1,\mu,\alpha}$ inside $V(t_2, \cdot)$ is deterministic, the DPP (3.12) does not require any regularity or even measurability of V. Indeed, denote the right side of (3.12) as $\tilde{V}(t_1, \mu)$. For any $\alpha \in \mathcal{A}$, by the flow property of SDEs and BSDEs we have

$$\mathbb{P}^{t_1,\mu,\alpha} = \mathbb{P}^{t_2,\mathbb{P}^{t_1,\mu,\alpha},\alpha}, \quad \text{and thus} \quad Y_0^{t_1,\mu,\alpha} = Y_0^{t_2,\mathbb{P}^{t_1,\mu,\alpha},\alpha}.$$

Note that $\mathbb{P}^{t_1,\mu,\alpha} = \mu$ on $\mathcal{F}_0$. This implies that

$$\mathbb{E}^{\mu}\big[Y_0^{t_1,\mu,\alpha}\big] = \mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\big[Y_0^{t_2,\mathbb{P}^{t_1,\mu,\alpha},\alpha}\big] \le V\big(t_2, \mathbb{P}^{t_1,\mu,\alpha}\big).$$

Then by (3.11) we see that $V(t_1, \mu) \le \tilde{V}(t_1, \mu)$. To see the opposite inequality, for any $\alpha \in \mathcal{A}$ and any $\varepsilon > 0$, there exists $\alpha^\varepsilon \in \mathcal{A}$ such that

$$V\big(t_2, \mathbb{P}^{t_1,\mu,\alpha}\big) \le \mathbb{E}^{\mu}\big[Y_0^{t_2,\mathbb{P}^{t_1,\mu,\alpha},\alpha^\varepsilon}\big] + \varepsilon.$$

Denote $\tilde{\alpha}_s^\varepsilon := \alpha_s \mathbf{1}_{[0,t_2)}(s) + \alpha_s^\varepsilon \mathbf{1}_{[t_2,T]}(s)$. Then clearly $\tilde{\alpha}^\varepsilon \in \mathcal{A}$, $\mathbb{P}^{t_1,\mu,\tilde{\alpha}^\varepsilon}_{[0,t_2]} = \mathbb{P}^{t_1,\mu,\alpha}_{[0,t_2]}$, and

$$\mathbb{E}^{\mu}\big[Y_0^{t_2,\mathbb{P}^{t_1,\mu,\alpha},\alpha^\varepsilon}\big] = \mathbb{E}^{\mu}\big[Y_0^{t_2,\mathbb{P}^{t_1,\mu,\tilde{\alpha}^\varepsilon},\alpha^\varepsilon}\big] = \mathbb{E}^{\mu}\big[Y_0^{t_1,\mu,\tilde{\alpha}^\varepsilon}\big] \le V(t_1, \mu).$$

This implies that $V(t_2, \mathbb{P}^{t_1,\mu,\alpha}) \le V(t_1, \mu) + \varepsilon$. Then it follows from the arbitrariness of α and ε that $\tilde{V}(t_1, \mu) \le V(t_1, \mu)$.

(ii) By applying the functional Itô formula (2.20) on the right-hand side of (3.12) we obtain the master equation (3.13) immediately. The terminal condition follows from the definitions.

(iii) Denote $\mu^* := \mathcal{L}_{X^*}$ and $\alpha_t^* := I(t, \mu^*_{[0,t]})$. Apply the functional Itô formula (2.20) on $V(t, \mu^*)$ we obtain, for the G_2 defined in (3.13),

$$\begin{aligned} &\frac{d}{dt} V\big(t, \mu^*\big) \\ &\quad = \partial_t V\big(t, \mu^*\big) + \mathbb{E}^{\mu^*}\big[G_2\big(t, \mu^*, X_t, \partial_\mu V\big(t, \mu^*, X\big), \partial_\omega \partial_\mu V\big(t, \mu^*, X\big), \alpha_t^*\big)\big] \\ &\quad = \partial_t V\big(t, \mu^*\big) + \sup_{a \in A} \mathbb{E}^{\mu^*}\big[G_2\big(t, \mu^*, X_t, \partial_\mu V\big(t, \mu^*, X\big), \partial_\omega \partial_\mu V\big(t, \mu^*, X\big), a\big)\big], \end{aligned}$$

where the last equality thanks to the fact that α^* is an optimal argument of the Hamiltonian. By the master equation (3.13) we obtain $\frac{d}{dt}V(t,\mu^*)=0$. Thus, noting that $\mu^*=\mathbb{P}^{0,\delta_{\{x_0\}},\alpha^*}$,

$$V_0=V(0,\delta_{\{x_0\}})=V(T,\mu^*)=Y_0^{0,\delta_{\{x_0\}},\alpha^*}.$$

That is, α^* is an optimal control. □

3.2. *Mean field control problems*. The mean field control problem is one major application of the master equations, and will be studied in more detail in Section 5 below. Consider a system of N controlled interacting particle system: $i=1,\dots,N$,

$$\begin{aligned} X_t^{\alpha,i} &= x_i+\int_0^t b\big(s,X_s^{\alpha,i},\mu_s^N,\alpha_s(X^{\alpha,i})\big)\,ds \\ &\quad +\int_0^t \sigma\big(s,X_s^{\alpha,i},\mu_s^N,\alpha_s(X^{\alpha,i})\big)\,dB_s^i \\ &\text{where } \mu_s^N:=\frac{1}{N}\sum_{i=1}^N \delta_{\{X_s^{\alpha,i}\}}. \end{aligned} \tag{3.15}$$

Here B^i are independent Brownian motions, the control α is a closed loop control and is chosen by a central planner (and thus the same α for all i), and the interaction is through the empirical measure μ^N. Assume $\mu_0^N:=\frac{1}{N}\sum_{i=1}^N\delta_{\{x_i\}}\to\mu_0$, while highly nontrivial, under appropriate conditions one can show that; see, for example, Lacker [25] (for relaxed controls), the above system converges to the following controlled McKean–Vlasov SDE with initial distribution $\mathcal{L}_{X_0}=\mu_0$:

$$\begin{aligned} X_t^\alpha &= X_0+\int_0^t b\big(s,X_s^\alpha,\mathcal{L}_{X_s^\alpha},\alpha_s(X^\alpha)\big)\,ds \\ &\quad +\int_0^t \sigma\big(s,X_s^\alpha,\mathcal{L}_{X_s^\alpha},\alpha_s(X^\alpha)\big)\,dB_s, \quad \mathbb{P}_0\text{-a.s.} \end{aligned} \tag{3.16}$$

In many applications, the dynamics could be path dependent (e.g., SDEs with delays), so below we extend (3.16) to the path dependent equation. Moreover, we shall consider a dynamic setting. To be precise, fix t and a process ξ on $[0,t]$, for a control α, let $X_s^\alpha:=\xi_s$ for $s\in[0,t]$ and consider the following equation on $[t,T]$ under $\mathbb{P}_0$:

$$\begin{aligned} X_s^\alpha &= \xi_t+\int_t^s b\big(r,X_{r\wedge\cdot}^\alpha,\mathcal{L}_{X_{r\wedge\cdot}^\alpha},\alpha_r(X_{r\wedge\cdot}^\alpha)\big)\,dr \\ &\quad +\int_t^s \sigma\big(r,X_{r\wedge\cdot}^\alpha,\mathcal{L}_{X_{r\wedge\cdot}^\alpha},\alpha_r(X_{r\wedge\cdot}^\alpha)\big)\,dB_r. \end{aligned} \tag{3.17}$$

Since we will only care about the law of X^α, it is more convenient to use the weak formulation in the canonical setting. That is, instead of fix $\mathbb{P}_0$ and consider the controlled process X^α, we fix the canonical process X and consider the controlled probability $\mathbb{P}^\alpha$. Now given $(t,\mu)\in\Theta$ and a control α, let $\mathbb{P}^{t,\mu,\alpha}\in\mathcal{P}_2$ be such that $\mathbb{P}^{t,\mu,\alpha}_{[0,t]}=\mu_{[0,t]}$ and, for $s\in[t,T]$ and for some $\mathbb{P}^{t,\mu,\alpha}$-Brownian motion B^α, the following holds:

$$\begin{aligned} X_s &= X_t+\int_t^s b\big(r,X_{r\wedge\cdot},\mathbb{P}^{t,\mu,\alpha}_{[0,r]},\alpha_r(X_{r\wedge\cdot})\big)\,dr \\ &\quad +\int_t^s \sigma\big(r,X_{r\wedge\cdot},\mathbb{P}^{t,\mu,\alpha}_{[0,r]},\alpha_r(X_{r\wedge\cdot})\big)\,dB_r^\alpha, \quad \mathbb{P}^{t,\mu,\alpha}\text{-a.s.} \end{aligned} \tag{3.18}$$

Note that α becomes a standard $\mathbb{F}$-progressively measurable process now. Our admissible controls are: for some appropriate set A and for any t_0,

$$\mathcal{A}_{t_0} := \{\alpha : [t_0, T] \times \Omega \to A : \text{for any } t \in [t_0, T] \text{ and any } \mathbb{P}_0\text{-square integrable process } \xi, \text{SDE (3.17) has a unique weak solution}\}. \tag{3.19}$$

Then (3.18) has a unique solution $\mathbb{P}^{t,\mu,\alpha}$ for any $\alpha \in \mathcal{A}_t$.

We are now ready to define our value function:

$$\begin{aligned} &V(t,\mu) := \sup_{\alpha \in \mathcal{A}_t} J(t,\mu,\alpha) \quad \text{where} \\ &J(t,\mu,\alpha) := \mathbb{E}^{\mathbb{P}^{t,\mu,\alpha}}\Big[g\big(X, \mathbb{P}^{t,\mu,\alpha}\big) + \int_t^T f\big(s, X, \mathbb{P}^{t,\mu,\alpha}, \alpha_s\big)\,ds\Big]. \end{aligned} \tag{3.20}$$

Similar to Theorem 3.5, we have the following result.

THEOREM 3.6. *Assume b, σ, f, g satisfy standard technical conditions, in particular they are $\mathbb{F}$-adapted both in X and in μ, and define V by (3.18)–(3.20). Then:*

(i) *The following dynamic programming principle holds: for any $t_1 < t_2$,*

$$V(t_1,\mu) = \sup_{\alpha \in \mathcal{A}_{t_1}} \Big[V\big(t_2, \mathbb{P}^{t_1,\mu,\alpha}\big) + \int_{t_1}^{t_2} \mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\big[f\big(s, X, \mathbb{P}^{t_1,\mu,\alpha}, \alpha_s\big)\big]\,ds\Big]. \tag{3.21}$$

(ii) *Assume $V \in C^{1,1,1}(\Theta)$, then V satisfies the path dependent master equation:*

$$\begin{aligned} &\partial_t V(t,\mu) + \mathbb{E}^{\mu}\Big[\sup_{a \in A} G_2\big(t,\mu,X,\partial_\mu V(t,\mu,X), \partial_\omega\partial_\mu V(t,\mu,X), a\big)\Big] = 0, \\ &V(T,\mu) = \mathbb{E}^{\mu}\big[g(X,\mu)\big] \quad \textit{where} \\ &G_2(t,\mu,\omega,z,\gamma,a) := \frac{1}{2}\gamma : \sigma\sigma^\top(t,\omega,\mu,a) \\ &\qquad\qquad + z \cdot b(t,\omega,\mu,a) + f(t,\omega,\mu,a). \end{aligned} \tag{3.22}$$

(iii) *Assume further that the Hamiltonian in (3.22) has an optimal argument $a^* = I(t,\omega,\mu)$, and the following McKean–Vlasov SDE has a solution:*

$$\begin{aligned} X_t^* = x_0 &+ \int_0^t b\big(s, X^*, \mathcal{L}_{X^*}, I\big(s, X^*, \mathcal{L}_{X^*}\big)\big)\,ds \\ &+ \int_0^t \sigma\big(s, X^*, \mathcal{L}_{X^*}, I\big(s, X^*, \mathcal{L}_{X^*}\big)\big)\,dB_s, \quad \mathbb{P}_0\text{-a.s.} \end{aligned} \tag{3.23}$$

Denote $\alpha_t^* := I(t, X^*, \mathcal{L}_{X^*})$. If $\alpha^* \in \mathcal{A}_0$, then it is an optimal control to the problem $V(0, \delta_{\{x_0\}})$ in (3.20).

We remark that, since the control α is deterministic in Theorem 3.5, in (3.13) the $\sup_{a\in A}$ is outside of the expectation $\mathbb{E}^{\mu}$ and thus the optimal control depends only on μ, but not on X. Here, in (3.22) the $\sup_{a\in A}$ is inside of the expectation $\mathbb{E}^{\mu}$ and thus the optimal control depends on X as well.

PROOF OF THEOREM 3.6. The proof of (ii) and (iii) are almost the same as that of Theorem 3.5, we thus omit it. The proof of (i) is also similar, but since the involvement of $\mathcal{A}_t$ is quite subtle, as we will discuss in more detail in Section 5, we provide a detailed proof again. We emphasize that, even though α is random here, the $\mathbb{P}^{t_1,\mu,\alpha}$ inside $V(t_2,\cdot)$ is still deterministic and the DPP (3.21) does not require the measurability of V.

The proof relies on the following two compatibility properties of $\mathcal{A}_t$: for any $t_1 < t_2$:

$$
\begin{aligned}
&\text{for any } \alpha \in \mathcal{A}_{t_1}, \text{ we have } \alpha_{[t_2,T]} \in \mathcal{A}_{t_2};\\
&\text{for any } \alpha^1 \in \mathcal{A}_{t_1}, \alpha^2 \in \mathcal{A}_{t_2}, \text{ we have } \alpha := \alpha^1 \mathbf{1}_{[t_1,t_2)} + \alpha^2 \mathbf{1}_{[t_2,T]} \in \mathcal{A}_{t_1}.
\end{aligned} \tag{3.24}
$$

Now denote the right-hand side of (3.21) as $\tilde{V}(t_1,\mu)$. On one hand, for any $\alpha \in \mathcal{A}_{t_1}$, denote $\tilde{\alpha} := \alpha_{[t_2,T]}$ and $\tilde{\mu} := \mathbb{P}^{t_1,\mu,\alpha}$. Note that $\tilde{\alpha} \in \mathcal{A}_{t_2}$, thanks to the first line of (3.24). Then $\mathbb{P}^{t_1,\mu,\alpha} = \mathbb{P}^{t_2,\tilde{\mu},\tilde{\alpha}}$, and thus

$$
\begin{aligned}
J(t_1,\mu,\alpha) &= J(t_2,\tilde{\mu},\tilde{\alpha}) + \mathbb{E}^{\tilde{\mu}}\left[\int_{t_1}^{t_2} f(s,X,\tilde{\mu},\alpha_s)\,ds\right]\\
&\le V(t_2,\tilde{\mu}) + \mathbb{E}^{\tilde{\mu}}\left[\int_{t_1}^{t_2} f(s,X,\tilde{\mu},\alpha_s)\,ds\right] \le \tilde{V}(t_1,\mu).
\end{aligned}
$$

This implies that $V(t_1,\mu) \le \tilde{V}(t_1,\mu)$. One the other hand, for any $\alpha \in \mathcal{A}_{t_1}$ and any $\varepsilon > 0$, there exists $\tilde{\alpha} \in \mathcal{A}_{t_2}$ such that: again denoting $\tilde{\mu} := \mathbb{P}^{t_1,\mu,\alpha}$,

$$
V(t_2,\tilde{\mu}) \le J(t_2,\tilde{\mu},\tilde{\alpha}) + \varepsilon.
$$

Now denote $\hat{\alpha} := \alpha \mathbf{1}_{[t_1,t_2)} + \tilde{\alpha}\mathbf{1}_{[t_2,T]} \in \mathcal{A}_{t_1}$, thanks to the second line of (3.24). Then $\mathbb{P}^{t_1,\mu,\hat{\alpha}} = \mathbb{P}^{t_2,\tilde{\mu},\tilde{\alpha}}$, and thus

$$
\begin{aligned}
&V(t_2,\tilde{\mu}) + \mathbb{E}^{\tilde{\mu}}\left[\int_{t_1}^{t_2} f(s,X,\tilde{\mu},\alpha_s)\,ds\right]\\
&\quad\le J(t_2,\tilde{\mu},\tilde{\alpha}) + \mathbb{E}^{\tilde{\mu}}\left[\int_{t_1}^{t_2} f(s,X,\tilde{\mu},\alpha_s)\,ds\right] + \varepsilon\\
&\quad= J(t_1,\mu,\hat{\alpha}) + \varepsilon \le V(t_1,\mu) + \varepsilon.
\end{aligned}
$$

This implies $\tilde{V}(t_1,\mu) \le V(t_1,\mu)$. $\square$

For illustration purposes, in the rest of this subsection we show that V is indeed smooth when there is no control, and hence the master equation is linear. For simplicity we assume $d = 1$, $b = 0$, $\sigma = 1$, and f, g do not depend on μ and thus the path dependence is only through X. For this purpose, let $(t,\mu) \in \Theta$, denote by $\mathbb{P}_0^{t,\mu} \in \mathcal{P}_2$ be such that $\mathbb{P}_0^{t,\mu} = \mu$ on $\mathcal{F}_t$ and $X_{t,\cdot}$ is a $\mathbb{P}_0^{t,\mu}$-Brownian motion on $[t,T]$ independent of $\mathcal{F}_t$. For $g : \widehat{\Omega} \to \mathbb{R}$, define $D_t g : \widehat{\Omega} \to \mathbb{R}$ by

$$
D_t g(\widehat{\omega}) := \lim_{\varepsilon \to 0} \frac{g(\widehat{\omega} + \varepsilon \mathbf{1}_{[t,T]}) - g(\widehat{\omega})}{\varepsilon}, \tag{3.25}
$$

and define $D_t^2 g : \widehat{\Omega} \to \mathbb{R}$ similarly. We note that $D_t g$ is essentially the Malliavin derivative, and in particular $D_t g = 0$ if g is $\mathcal{F}_s$-measurable for some $s < t$.

EXAMPLE 3.7. Let $g \in C_b^0(\widehat{\Omega};\mathbb{R})$ and $f \in C_b^0([0,T]\times\widehat{\Omega};\mathbb{R})$. Assume $D_t g$, $D_t^2 g$, $D_t f$, $D_t^2 f$ exist and are bounded, and $D_t g(\widehat{\omega})$, $D_t^2 g(\widehat{\omega})$ are jointly continuous in $(t,\widehat{\omega})$ under the distance $d((t,\widehat{\omega}),(t',\widehat{\omega}')) := |t-t'| + \|\widehat{\omega}-\widehat{\omega}'\|$, $D_t f(s,\widehat{\omega})$, $D_t^2 f(s,\widehat{\omega})$ are jointly continuous in $(t,s,\widehat{\omega})$ under the distance $d((t,s,\widehat{\omega}),(t',s',\widehat{\omega}')) := |t-t'| + |s-s'| + \|\widehat{\omega}_{s\wedge\cdot} - \widehat{\omega}'_{s'\wedge\cdot}\|$. Define

$$
V(t,\mu) := \mathbb{E}^{\mathbb{P}_0^{t,\mu}}\left[g(X) + \int_t^T f(s,X)\,ds\right]. \tag{3.26}
$$

Then $V \in C_b^{1,1,1}(\Theta)$ and satisfies the following linear master equation:

$$
\begin{aligned}
&\partial_t V(t,\mu) + \mathbb{E}^{\mu}\Big[\frac{1}{2}\partial_\omega \partial_\mu V(t,\mu,X) + f(t,X)\Big] = 0, \\
&V(T,\mu) = \mathbb{E}^{\mu}\big[g(X)\big].
\end{aligned}
\tag{3.27}
$$

PROOF. The proof follows similar arguments as in Peng and Wang [33], which deals with semilinear path dependent PDEs, so we shall only sketch it. We remark that the continuity of f implies its $\mathbb{F}$-adaptedness.

First it is clear that we can extend (3.26) to all $(t,\widehat{\mu}) \in \widehat{\Theta}$ in an obvious way. Denote $(\widehat{\omega} \otimes_t \omega)_s := \widehat{\omega}_s \mathbf{1}_{[0,t]}(s) + [\widehat{\omega}_t + \omega_s - \omega_t]\mathbf{1}_{(t,T]}(s)$ for all $\widehat{\omega} \in \widehat{\Omega}$ and $\omega \in \Omega$. Then

$$
\begin{aligned}
&V(t,\widehat{\mu}) = \mathbb{E}^{\widehat{\mu}}\big[u(t,\widehat{X})\big] \\
&\qquad \text{where } u(t,\widehat{\omega}) := \mathbb{E}^{\mathbb{P}_0}\Big[g(\widehat{\omega} \otimes_t X) + \int_t^T f(s,\widehat{\omega} \otimes_t X)\,ds\Big].
\end{aligned}
$$

By straightforward computation, we have

$$
\partial_\mu V(t,\widehat{\mu},\widehat{\omega}) = \partial_\omega u(t,\widehat{\omega}) = \mathbb{E}^{\mathbb{P}_0}\Big[D_t g(\widehat{\omega} \otimes_t X) + \int_t^T D_t f(s,\widehat{\omega} \otimes_t X)\,ds\Big],
$$

where $\partial_\omega u$ is Dupire's path derivative as in (2.18). We note that in this particular case $\partial_\mu V$ actually does not depend on μ. Then

$$
\partial_\omega \partial_\mu V(t,\widehat{\mu},\widehat{\omega}) = \partial_\omega \partial_\omega u(t,\widehat{\omega}) = \mathbb{E}^{\mathbb{P}_0}\Big[D_t^2 g(\widehat{\omega} \otimes_t X) + \int_t^T D_t^2 f(s,\omega \otimes_t X)\,ds\Big].
$$

By our conditions, it is quite obvious that V, $\partial_\mu V$, $\partial_\omega \partial_\mu V$ are continuous.

On the other hand, note that

$$
V(t+\delta,\widehat{\mu}_{[0,t]}) - V(t,\widehat{\mu}_{[0,t]}) = \mathbb{E}^{\widehat{\mu}}\big[u(t+\delta,\widehat{X}_{t\wedge\cdot}) - u(t,\widehat{X})\big].
$$

Fix t and $t+\delta$, let $t = t_0 < \cdots < t_n = t+\delta$. Recall (2.4) and denote, for $0 \le m \le n$,

$$
X^{n,m} := \widehat{\omega}_{t\wedge\cdot} + \sum_{i=1}^{m} X_{t_{i-1},t_i}\mathbf{1}_{[t_i,T]} + X_{t_n,\cdot}\mathbf{1}_{[t_n,T]}.
$$

Note that

$$
\begin{aligned}
&\widehat{\omega} \otimes_t X = \lim_{n\to\infty}\Big[\widehat{\omega}_{t\wedge\cdot} + \sum_{i=1}^{n-1} X_{t,t_i}\mathbf{1}_{[t_i,t_{i+1})} + X_{t,\cdot}\mathbf{1}_{[t_n,T]}\Big] = \lim_{n\to\infty} X^{n,n}; \\
&\widehat{\omega}_{t\wedge\cdot} \otimes_{t+\delta} X = \widehat{\omega}_{t\wedge\cdot} + X_{t_n,\cdot}\mathbf{1}_{[t_n,T]} = X^{n,0}.
\end{aligned}
$$

Then, denoting $X^{n,m,\theta} := X^{n,m} + \theta X_{t_m,t_{m+1}}\mathbf{1}_{[t_{m+1},T)}$,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\big[g(\widehat{\omega} \otimes_t X) - g(\widehat{\omega}_{t\wedge\cdot} \otimes_{t+\delta} X)\big] \\
&\quad = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[g(X^{n,n}) - g(X^{n,0})\big] = \lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}_0}\big[g(X^{n,m}) - g(X^{n,m-1})\big] \\
&\quad = \lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}_0}\big[g(X^{n,m-1} + X_{t_{m-1},t_m}\mathbf{1}_{[t_m,T]}) - g(X^{n,m-1})\big] \\
&\quad = \lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}_0}\Big[D_{t_m} g(X^{n,m-1})X_{t_{m-1},t_m} + \frac{1}{2}D_{t_m}^2 g(X^{n,m-1})X_{t_{m-1},t_m}^2 \\
&\qquad + \frac{1}{2}\big[D_{t_m}^2 g(X^{n,m-1,\theta_m}) - D_{t_m}^2 g(X^{n,m-1})\big]X_{t_{m-1},t_m}^2\Big],
\end{aligned}
$$

for some random variable θ_m taking values in [0, 1]. Note that, under $\mathbb{P}^0$, X_{t_{m-1},t_m} and $X^{n,m-1}$ are independent. Then

$$\mathbb{E}^{\mathbb{P}_0}\big[D_{t_m} g(X^{n,m-1}) X_{t_{m-1},t_m}\big] = 0,$$
$$\mathbb{E}^{\mathbb{P}_0}\big[D^2_{t_m} g(X^{n,m-1}) X^2_{t_{m-1},t_m}\big] = \mathbb{E}^{\mathbb{P}_0}\big[D^2_{t_m} g(X^{n,m-1})\big][t_m - t_{m-1}]$$

and

$$\begin{aligned}
&\big|\mathbb{E}^{\mathbb{P}_0}\big[\big[D^2_{t_m} g(X^{n,m-1,\theta_m}) - D^2_{t_m} g(X^{n,m-1})\big] X^2_{t_{m-1},t_m}\big]\big| \\
&\quad \le C\Big(\mathbb{E}^{\mathbb{P}_0}\Big[\sup_{0\le\theta\le 1} \big|D^2_{t_m} g(X^{n,m-1,\theta}) - D^2_{t_m} g(X^{n,m-1})\big|^2\Big]\mathbb{E}^{\mathbb{P}_0}\big[|X_{t_{m-1},t_m}|^4\big]\Big)^{\frac{1}{2}} \\
&\quad \le C\Big(\mathbb{E}^{\mathbb{P}_0}\Big[\sup_{0\le\theta\le 1} \big|D^2_{t_m} g(X^{n,m-1,\theta}) - D^2_{t_m} g(X^{n,m-1})\big|^2\Big]\Big)^{\frac{1}{2}}[t_m - t_{m-1}].
\end{aligned}$$

Then, by the assumed regularity and the dominated convergence theorem, we can easily show that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\big[g(\widehat{\omega} \otimes_t X) - g(\widehat{\omega}_{t\wedge\cdot} \otimes_{t+\delta} X)\big] \\
&\quad = \frac{1}{2}\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}_0}\big[D^2_{t_m} g(X^{n,m-1})\big][t_m - t_{m-1}] \\
&\quad = \frac{1}{2}\int_t^{t+\delta} \mathbb{E}^{\mathbb{P}_0}\big[D^2_s g(\widehat{\omega}_{t\wedge\cdot} + X_{t,\cdot}\mathbf{1}_{[t,s]} + X_{t,s}\mathbf{1}_{[s,T]} + X_{t+\delta,\cdot}\mathbf{1}_{[t+\delta,T]})\big]\,ds.
\end{aligned}$$

This implies

$$\lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}^{\mathbb{P}_0}\big[g(\widehat{\omega} \otimes_t X) - g(\widehat{\omega}_{t\wedge\cdot} \otimes_{t+\delta} X)\big] = \frac{1}{2}\mathbb{E}^{\mathbb{P}_0}\big[D^2_t g(\widehat{\omega} \otimes_t X)\big].$$

Similar results hold for f. Then

$$\begin{aligned}
\partial_t u(t,\widehat{\omega}) &:= \lim_{\delta\to 0}\frac{u(t+\delta, \widehat{\omega}_{\cdot\wedge t}) - u(t,\widehat{\omega})}{\delta} \\
&= -\mathbb{E}^{\mathbb{P}_0}\Big[\frac{1}{2}D^2_t g(\widehat{\omega} \otimes_t X) + \frac{1}{2}\int_t^T D^2_t f(s, \widehat{\omega} \otimes_t X)\,ds + f(t, \widehat{\omega} \otimes_t X)\Big].
\end{aligned}$$

Since $\partial_t V(t,\widehat{\mu}) = \mathbb{E}^{\widehat{\mu}}[\partial_t u(t,\widehat{X})]$, then one can easily verify the result. $\square$

3.3. *Stochastic control under probability distortion.* In this subsection we study another application of the parabolic master equation. Probability distortion is an important tool in behavioral finance, in particular the prospect theory; see the survey paper Zhou [45] and the references therein. We say a function $\kappa : [0, 1] \to [0, 1]$ is a *probability distortion function* if κ is continuous, strictly increasing, and $\kappa(0) = 0$, $\kappa(1) = 1$. Given a random variable $\xi \ge 0$, introduce a nonlinear expectation:

$$\mathcal{E}[\xi] := \int_0^\infty \kappa\big(\mathbb{P}(\xi \ge x)\big)\,dx. \tag{3.28}$$

The following properties are straightforward:

- If $\kappa(p) = p$, then $\mathcal{E}[\xi] = \mathbb{E}[\xi]$.
- In general, $\mathcal{E}$ is nonlinear: $\mathcal{E}[\xi_1 + \xi_2] \ne \mathcal{E}[\xi_1] + \mathcal{E}[\xi_2]$.
- $\mathcal{E}$ is law invariant: if $\mathcal{L}_{\xi_1} = \mathcal{L}_{\xi_2}$, then $\mathcal{E}[\xi_1] = \mathcal{E}[\xi_2]$.

In prospect theory, typically κ is in reverse S-shape, namely concave around 0 and convex around 1. Indeed, assume κ is smooth and ξ has density $f(x)$, then it follows from the integration by parts formula that

$$\mathcal{E}[\xi] = \int_0^\infty \kappa'\big(\mathbb{P}(\xi \geq x)\big) f(x) x\, dx.$$

Note that $\kappa'(p)$ is large for p around 0 and 1, so at above integration the probability density $f(x)$ is amplified by κ' when x is around 0 and ∞, which is referred as probability distortion.

Mathematically, the main challenge in this framework is the time inconsistency in the following sense. Assume X is a Markovian process, g is a positive function, and denote

$$u(t,x) := \mathcal{E}\big[g\big(X_T^{t,x}\big)\big] := \int_0^\infty \kappa\big(\mathbb{P}\big(g(X_T) \geq y | X_t = x\big)\big)\, dy. \tag{3.29}$$

Then the flow property (hence the DPP when controls are involved) fails:

$$\mathcal{E}\big[g(X_T)\big] \neq \mathcal{E}\big[u(t, X_t)\big].$$

In particular, the above function u does not satisfy any PDE.

One remedy for the above time inconsistency is to consider $\mathcal{L}_{X_t}$, instead of X_t, as the state variable. Then the expected PDE becomes a master equation. To be precise, assume $d = 1$ and recall the $\mathbb{P}_0^{t,\mu}$ in Example 3.7 and recall Remark 2.10.

EXAMPLE 3.8. Assume the distortion function $\kappa \in C^1([0,1])$ and $g \in C_b^0(\mathbb{R}; \mathbb{R}_+)$. Define

$$V(t,\mu) := \int_0^\infty \kappa\big(\mathbb{P}_0^{t,\mu}\big(g(X_T) \geq y\big)\big)\, dy, \quad (t,\mu) \in \Theta. \tag{3.30}$$

Then V is state dependent: $V(t,\mu) = V(t,\mu_t)$, and $V \in C^{1,1,1}([0,T] \times \mathcal{P}_2(\mathbb{R}))$ satisfies the following master equation:

$$\begin{aligned} &\partial_t V(t,\mu) + \frac{1}{2}\mathbb{E}^\mu\big[\partial_x \partial_\mu V(t,\mu,X_t))\big] = 0, \\ &V(T,\mu) = \int_0^\infty \kappa\big(\mu\big(g(X_T) \geq y\big)\big)\, dy. \end{aligned} \tag{3.31}$$

PROOF. It is clear that

$$V(t,\mu) = \int_0^\infty \kappa\big(\mathbb{E}^\mu\big[I(t, X_t, y)\big]\big)\, dy$$
$$\text{where } I(t,x,y) := \int_{g^{-1}([y,\infty))} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-z)^2}{2(T-t)}}\, dz.$$

One can easily check that

$$\begin{aligned} \partial_t V(t,\mu) &= \int_0^\infty \kappa'(\mathbb{E}^\mu\big[I(t,X_t,y)\big])\mathbb{E}^\mu\big[\partial_t I(t,X_t,y)\big]\, dy; \\ \partial_\mu V(t,\mu,x) &= \int_0^\infty \kappa'(\mathbb{E}^\mu\big[I(t,X_t,y)\big])\partial_x I(t,x,y)\, dy; \\ \partial_x\partial_\mu V(t,\mu,x) &= \int_0^\infty \kappa'(\mathbb{E}^\mu\big[I(t,X_t,y)\big])\partial_{xx} I(t,x,y)\, dy. \end{aligned}$$

It is clear that $\partial_t I(t,x,y) + \frac{1}{2}\partial_{xx} I(t,x,y) = 0$. This implies (3.31) straightforwardly. □

REMARK 3.9.

(i) While $\mathcal{E}$ is a nonlinear function, the master equation (3.31) is actually linear. The nonlinearity is only in the terminal condition: the mapping $\mu \mapsto V(T,\mu)$ is nonlinear in the sense that $V(T,\mathcal{L}_{\xi_1+\xi_2}) \neq V(T,\mathcal{L}_{\xi_1}) + V(T,\mathcal{L}_{\xi_2})$.

(ii) In Ma, Wong and Zhang [31], we introduced a dynamic distortion function $\kappa(t,x,p)$ to recover the flow property for the corresponding u in (3.29) in some special cases. In Example 3.8, we instead raise the "dimension" of the state space from $\mathbb{R}$ to $\mathcal{P}_2(\mathbb{R})$ so as to recover the flow property. We remark that this approach works for many time inconsistent problems, including those in Section 3.1. However, in practice it may not be reasonable to use $V(t,\mu)$ as one's utility at time t, because by that time one observes a path of $X_{t\wedge\cdot}$, then it is not reasonable to consider the whole distribution of $X_{t\wedge\cdot}$ which involves other paths. Nevertheless, when one observes the value $X_0 = x_0$ at time $t=0$, the master equation (3.31) provides a nice characterization for the value $V(0,\delta_{\{x_0\}})$.

We next extend the above discussion to control problems under probability distortion, which to our best knowledge is new in the literature. Recall the $\mathcal{A}$ in (3.19), and similarly as (3.18) we determine $\mathbb{P}^{t,\mu,\alpha}$ by the following controlled SDE on $[t,T]$:

$$X_s = X_t + \int_t^s b\big(r,X,\alpha_r(X)\big)\,dr + \int_t^s \sigma\big(r,X,\alpha_r(X)\big)\,dB_r^\alpha, \quad \mathbb{P}^{t,\mu,\alpha}\text{-a.s.}, \tag{3.32}$$

where b,σ,α are all $\mathbb{F}$-adapted. Our value function is: given $g:\Omega\to[0,\infty)$,

$$V(t,\mu) := \sup_{\alpha\in\mathcal{A}} \int_0^\infty \kappa\big(\mathbb{P}^{t,\mu,\alpha}\big(g(X)\geq y\big)\big)\,dy, \quad (t,\mu)\in\Theta. \tag{3.33}$$

Note that $\tilde{g}(\mu) := \int_0^\infty \kappa(\mu(g(X_\cdot)\geq y))\,dy$ is actually a deterministic function of μ. Then by considering $f=0$ and terminal condition $\tilde{g}$ in Theorem 3.6, we obtain the following.

COROLLARY 3.10. *Assume b,σ,g satisfy standard technical conditions, κ is a probability distortion function, and define V by* (3.32) *and* (3.33). *Then*:

(i) *The following dynamic programming principle holds*:

$$V(t_1,\mu) = \sup_{\alpha\in\mathcal{A}} V\big(t_2,\mathbb{P}^{t_1,\mu,\alpha}\big), \quad t_1<t_2. \tag{3.34}$$

(ii) *Assume $V\in C^{1,1,1}(\Theta)$, then V satisfies the path dependent master equation*:

$$\begin{aligned} &\partial_t V(t,\mu) + \mathbb{E}^\mu\Big[\sup_{a\in A} G_2\big(t,X,\partial_\mu V(t,\mu,X),\partial_\omega\partial_\mu V(t,\mu,X),a\big)\Big] = 0, \\ &V(T,\mu) = \int_0^\infty \kappa\big(\mu\big(g(X)\geq y\big)\big)\,dy \end{aligned} \tag{3.35}$$

$$\text{where } G_2(t,\omega,z,\gamma,a) := \frac{1}{2}\gamma\sigma^2(t,\omega,a) + zb(t,\omega,a).$$

(iii) *Assume further that the Hamiltonian in* (3.35) *has an optimal argument $a^* = I(t,\omega,\mu)$, where I is uniformly Lipschitz continuous in ω, and the following McKean–Vlasov SDE has a solution*:

$$\begin{aligned} X_t^* = x_0 &+ \int_0^t b\big(s,X^*,I\big(s,X^*,\mathcal{L}_{X^*}\big)\big)\,ds \\ &+ \int_0^t \sigma\big(s,X^*,I\big(s,X^*,\mathcal{L}_{X^*}\big)\big)\,dB_s. \end{aligned} \tag{3.36}$$

Then $\alpha_t^* := I(t,X^*,\mathcal{L}_{X^*})$ is an optimal control to the problem $V(0,\delta_{\{x_0\}})$ in (3.33).

4. Viscosity solution of master equations. We emphasize again that the smoothness of V required in Theorem 3.6 is very difficult to verify. In this section we propose a notion of viscosity solution for master equation (3.1), which requires less regularity, and establish its basic properties.

4.1. *Definition of viscosity solutions.* For $(t,\mu)\in\Theta$ and constant $L>0$, let $\mathcal{P}_L(t,\mu)$ denote the set of $\mathbb{P}\in\mathcal{P}_2$ such that $\mathbb{P}_{[0,t]}=\mu_{[0,t]}$ and $X_{[t,T]}$ is a $\mathbb{P}$-semimartingale with drift and diffusion characteristics bounded by L, in the spirit of the $\widehat{\mathcal{P}}_L$ introduced in the beginning of Section 2.4. Note that we do not require X to be a μ-semimartingale on $[0,t]$. The following simple estimates will be used frequently in the paper: for any $(t,\mu)\in\Theta$, $\delta\in[0,T-t]$, and $L>0$, $p\geq 1$,

$$\sup_{\mathbb{P}\in\mathcal{P}_L(t,\mu)}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\leq s\leq t+\delta}|X_{t,s}|^p\Big]\leq C_{p,L}\delta^{\frac{p}{2}}. \tag{4.1}$$

The following compactness result is the key for our viscosity theory.

LEMMA 4.1. *For any $(t,\mu)\in\Theta$ and $L>0$, the set $[t,T]\times\mathcal{P}_L(t,\mu)$ is compact under $\mathcal{W}_2$.*

PROOF. We first show that $\mathcal{P}_L(t,\mu)$ is compact. Let $\{\mathbb{P}^n\}_{n\geq1}\subset\mathcal{P}_L(t,\mu)$. By Zheng [44], Theorem 3, $\mathcal{P}_L(t,\mu)$ is weakly compact, then there exist a convergent subsequence, and without loss of generality we assume $\mathbb{P}^n\to\mathbb{P}\in\mathcal{P}_L(t,\mu)$ weakly. Note that

$$\|X\|\leq\|X_{t\wedge\cdot}\|+\sup_{t\leq s\leq T}|X_{t,s}|\leq 2\Big[\big[\|X_{t\wedge\cdot}\|\big]\vee\Big[\sup_{t\leq s\leq T}|X_{t,s}|\Big]\Big].$$

Since $\mathbb{P}^n=\mu$ on $\mathcal{F}_t$, $\|X_{t\wedge\cdot}\|$ has the same distribution under $\mathbb{P}^n$ and μ. Moreover, since $\mathbb{P}_n\in\mathcal{P}_L(t,\mu)$, for any $R>0$ and any n, by (4.1) (with $\delta=T-t$) we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_n}\big[\|X\|^2\mathbf{1}_{\{\|X\|\geq R\}}\big]\\
&\leq 4\mathbb{E}^{\mathbb{P}_n}\Big[\|X_{t\wedge\cdot}\|^2\mathbf{1}_{\{\|X_{t\wedge\cdot}\|\geq\frac{R}{2}\}}+\sup_{t\leq s\leq T}|X_{t,s}|^2\mathbf{1}_{\{\sup_{t\leq s\leq T}|X_{t,s}|\geq\frac{R}{2}\}}\Big]\\
&=4\mathbb{E}^{\mu}\big[\|X_{t\wedge\cdot}\|^2\mathbf{1}_{\{\|X_{t\wedge\cdot}\|\geq\frac{R}{2}\}}\big]+4\mathbb{E}^{\mathbb{P}_n}\Big[\sup_{t\leq s\leq T}|X_{t,s}|^2\mathbf{1}_{\{\sup_{t\leq s\leq T}|X_{t,s}|\geq\frac{R}{2}\}}\Big]\\
&\leq 4\mathbb{E}^{\mu}\big[\|X\|^2\mathbf{1}_{\{\|X\|\geq\frac{R}{2}\}}\big]+\frac{8}{R}\mathbb{E}^{\mathbb{P}_n}\Big[\sup_{t\leq s\leq T}|X_{t,s}|^3\Big]\\
&\leq 4\mathbb{E}^{\mu}\big[\|X\|^2\mathbf{1}_{\{\|X\|\geq\frac{R}{2}\}}\big]+\frac{C_L}{R}.
\end{aligned}$$

Thus, by the dominated convergence theorem under μ,

$$\lim_{R\to\infty}\sup_{n\geq1}\mathbb{E}^{\mathbb{P}_n}\big[\|X\|^2\mathbf{1}_{\{\|X\|\geq R\}}\big]\leq 4\lim_{R\to\infty}\mathbb{E}^{\mu}\big[\|X\|^2\mathbf{1}_{\{\|X\|\geq\frac{R}{2}\}}\big]=0.$$

Then it follows from Carmona and Delarue ([11], Theorem 5.5) that

$$\lim_{n\to\infty}\mathcal{W}_2(\mathbb{P}^n,\mathbb{P})=0.$$

Next, let $(t_n,\mathbb{P}_n)\in[t,T]\times\mathcal{P}_L(t,\mu)$. By the compactness of $[t,T]$ and $\mathcal{P}_L(t,\mu)$, we may assume without loss of generality that $t_n\to t^*$ and $\mathbb{P}_n\to\mathbb{P}$. Then

$$\begin{aligned}
\mathcal{W}_2\big((t_n,\mathbb{P}_n),(t^*,\mathbb{P})\big)&\leq\mathcal{W}_2\big((t_n,\mathbb{P}_n),(t^*,\mathbb{P}_n)\big)+\mathcal{W}_2\big((t^*,\mathbb{P}_n),(t^*,\mathbb{P})\big)\\
&\leq\big(|t_n-t^*|+\mathbb{E}^{\mathbb{P}_n}\big[\|X_{t_n\wedge\cdot}-X_{t^*\wedge\cdot}\|^2\big]\big)^{\frac12}+\mathcal{W}_2(\mathbb{P}_n,\mathbb{P})\\
&\leq C|t_n-t^*|^{\frac12}+\mathcal{W}_2(\mathbb{P}_n,\mathbb{P})\to0\quad\text{as } n\to\infty.
\end{aligned}$$

This implies that $[t,T]\times\mathcal{P}_L(t,\mu)$ is also compact. $\square$

For the viscosity theory, another crucial thing is the functional Itô formula (2.25). For this purpose, we shall weaken the regularity requirement for the test functions, which will make the theory more convenient.

DEFINITION 4.2. Let $0 \le t_1 < t_2 \le T$ and $\mathcal{P} \subset \mathcal{P}_2$ such that X is a semimartingale on $[t_1, t_2]$ under each $\mathbb{P} \in \mathcal{P}$. We say $V \in C^{1,1,1}([t_1, t_2] \times \mathcal{P})$ if $V \in C^0([t_1, t_2] \times \mathcal{P})$ and there exist $\partial_t V \in C^0([t_1, t_2] \times \mathcal{P})$, $\partial_\mu V, \partial_\omega \partial_\mu V \in C^0([t_1, t_2] \times \mathcal{P} \times \Omega)$ with appropriate dimensions, such that the functional Itô formula (2.25) holds true on $[t_1, t_2]$ under every $\mathbb{P} \in \mathcal{P}$.

Moreover, let $C_b^{1,1,1}([t_1, t_2] \times \mathcal{P})$ denote the subset of $C^{1,1,1}([t_1, t_2] \times \mathcal{P})$ such that $\partial_t V$ is bounded and, for some constant $C \ge 0$,

$$\left|\partial_\mu V(t, \mu, \omega)\right| + \left|\partial_\omega \partial_\mu V(t, \mu, \omega)\right| \le C\left[1 + \|\omega\|\right] \quad \text{for } \mathbb{P}\text{-a.e. } \omega \text{ and all } \mathbb{P} \in \mathcal{P}.$$

REMARK 4.3.

(i) By Theorem 2.9, $C_b^{1,1,1}(\Theta) \subset C^{1,1,1}([t_1, t_2] \times \mathcal{P}_L(t_1, \mu))$ for all (t_1, t_2), L, and $\mu \in \mathcal{P}_2$, and the derivatives $\partial_t V$, $\partial_\mu V$, $\partial_\omega^{\mathrm{sym}} \partial_\mu V$ are consistent.

(ii) For $V \in C^{1,1,1}([t_1, t_2] \times \mathcal{P}_L(t_1, \mu))$, following the same arguments as in Theorem 2.9(i), $\partial_t V$, $\partial_\mu V$, $\partial_\omega^{\mathrm{sym}} \partial_\mu V$ are unique. Since by Remark 3.2, G depends on Γ only through Γ^{sym}, so the uniqueness of $\partial_\omega^{\mathrm{sym}} \partial_\mu V$ is sufficient for our purpose.

(iii) When $\mathcal{P}$ is compact, for example, $\mathcal{P} = \mathcal{P}_L(t, \mu)$, the continuity implies uniform continuity as well as boundedness. In particular, in this case V and $\partial_t V$ are automatically bounded and the linear growth of $\partial_\mu V$, $\partial_\omega \partial_\mu V$ in ω is also a mild requirement.

For a function $V : \Theta \to \mathbb{R}$, we now introduce the following set of test functions:

$$\begin{aligned}
&\mathcal{P}_{L,\delta}^{t,\mu} := [t, t+\delta] \times \mathcal{P}_L(t, \mu), \\
&\mathcal{A}_\delta^L V(t, \mu) := \left\{\varphi \in C_b^{1,1,1}(\mathcal{P}_{L,\delta}^{t,\mu}) : (\varphi - V)(t, \mu) = 0\right\}; \\
&\underline{\mathcal{A}}^L V(t, \mu) := \bigcup_{0<\delta\le T-t} \left\{\varphi \in \mathcal{A}_\delta^L V(t, \mu) : \inf_{(s,\mathbb{P}) \in \mathcal{P}_{L,\delta}^{t,\mu}} (\varphi - V)(s, \mathbb{P}) = 0\right\}; \\
&\overline{\mathcal{A}}^L V(t, \mu) := \bigcup_{0<\delta\le T-t} \left\{\varphi \in \mathcal{A}_\delta^L V(t, \mu) : \sup_{(s,\mathbb{P}) \in \mathcal{P}_{L,\delta}^{t,\mu}} (\varphi - V)(s, \mathbb{P}) = 0\right\}.
\end{aligned} \tag{4.2}$$

DEFINITION 4.4. Let $V \in C^0(\Theta)$.

(i) We say V is an L-viscosity subsolution (resp. supersolution) of (3.1) if $\mathbb{L}\varphi(t, \mu) \ge$ (resp. $\le$) 0 for all $(t, \mu) \in \Theta$ and all $\varphi \in \underline{\mathcal{A}}^L V(t, \mu)$ (resp. $\overline{\mathcal{A}}^L V(t, \mu)$).

(ii) We say V is an L-viscosity solution of (3.1) if it is both an L-viscosity subsolution and an L-viscosity supersolution, and V is a viscosity solution if it is an L-viscosity solution for some $L > 0$.

REMARK 4.5.

(i) Our main idea here is to use $\mathcal{P}_L(t, \mu)$ in (4.2), which by Lemma 4.1 is compact under $\mathcal{W}_2$ and in the meantime is large enough in most applications we are interested in. This is in the same spirit as our notion of viscosity solutions for path dependent PDEs; see Ekren, Keller, Touzi and Zhang [17] and Ekren, Touzi and Zhang [18, 19].

(ii) When V is state dependent: $V(t, \mu) = V(t, \mu_t)$, the above definition still works. However, in this case it is more convenient to change the test functions φ to be state dependent only. In particular, we shall revise (4.2) as follows:

- $\underline{\mathcal{A}}^L V(t,\mu)$ and $\overline{\mathcal{A}}^L V(t,\mu)$ become $\underline{\mathcal{A}}^L V(t,\mu_t)$ and $\overline{\mathcal{A}}^L V(t,\mu_t)$;
- $\mathcal{P}_L(t,\mu)$ becomes $\mathcal{P}_L(t,\mu_t)$ where the initial constraint is relaxed to $\mathbb{P}_t = \mu_t$;
- the extremum is about $[\varphi - V](s,\mathbb{P}_s)$ for $(s,\mathbb{P}) \in [t,t+\delta] \times \mathcal{P}_L(t,\mu_t)$.

(iii) In the state dependent case, if we work on torus $\mathbb{T}^d$ instead of $\mathbb{R}^d$ (namely the state process X takes values in $\mathbb{T}^d$), then the following δ-neighborhood is compact under $\mathcal{W}_2$:

$$D_\delta(t,\mu_t) := \{(s,\mathbb{P}_s) \in [t,t+\delta] \times \mathcal{P}_2(\mathbb{T}^d) : \mathcal{W}_2(\mathbb{P}_s,\mu_t) \le \delta\}, \tag{4.3}$$

and we expect the main results in this paper will remain true by replacing $\mathcal{P}_L(t,\mu_t)$ with $D_\delta(t,\mu_t)$. However, we lose such compactness on $\mathcal{P}_2(\mathbb{R}^d)$, for example, $\mu_n := \frac{1}{n}\delta_{\{n^2\}} + [1 - \frac{1}{n}]\delta_{\{0\}} \in \mathcal{P}_2(\mathbb{R})$ converges to $\delta_{\{0\}}$ weakly, but not under $\mathcal{W}_2$. So our definition of viscosity solution is novel even in the state dependent case.

4.2. *Some equivalence results.*

THEOREM 4.6 (Consistency). *Let Assumption 3.1 hold and $V \in C_b^{1,1,1}(\Theta)$. Then V is a viscosity solution* (*resp. subsolution*, *supersolution*) *of master equation* (3.1) *if and only if it is a classical solution* (*resp. subsolution*, *supersolution*) *of master equation* (3.1).

PROOF. We shall only prove the equivalence of the subolution property. If V is a viscosity subsolution, note that V itself is in $\underline{\mathcal{A}}^L V(t,\mu)$, then clearly $\mathbb{L}V(t,\mu) \ge 0$ and thus is a classical subsolution. Now assume V is a classical subsolution. Fix $(t,\mu) \in \Theta$ and $\varphi \in \underline{\mathcal{A}}^L V(t,\mu)$ for some $L \ge L_0$, where L_0 is the Lipschitz constant in Assumption 3.1. Given $\mathcal{F}_t$-measurable random variables b_t, σ_t with $|b_t|, \frac{1}{2}|\sigma_t|^2 \le L$, let $\mathbb{P} \in \mathcal{P}_L(t,\mu)$ be such that $X_{t,\cdot}$ is a $\mathbb{P}$-semimartingale with drift b_t and volatility σ_t. Then, denoting $\psi := \varphi - V$,

$$\begin{aligned}
0 &\le \psi(t+\delta,\mathbb{P}) - \psi(t,\mu) \\
&= \int_t^{t+\delta} \Big[\partial_t\psi(s,\mathbb{P}) + \mathbb{E}^{\mathbb{P}}\Big[b_t \cdot \partial_\mu\psi(s,\mathbb{P},X) + \frac{1}{2}\sigma_t\sigma_t^\top : \partial_\omega\partial_\mu\psi(s,\mathbb{P},X)\Big]\Big]\,ds.
\end{aligned}$$

Divide both sides by δ and send $\delta \to 0$, we obtain

$$0 \le \partial_t\psi(t,\mu) + \mathbb{E}^\mu\Big[b_t \cdot \partial_\mu\psi(t,\mu,X) + \frac{1}{2}\sigma_t\sigma_t^\top : \partial_\omega\partial_\mu\psi(t,\mu,X)\Big].$$

Set $y := V(t,\mu) = \varphi(t,\mu)$, $Z_1 := \partial_\mu\varphi(t,\mu,\cdot)$, $Z_2 := \partial_\mu V(t,\mu,\cdot)$, $\Gamma_1 := \partial_\omega\partial_\mu\varphi(t,\mu,\cdot)$, and $\Gamma_2 := \partial_\omega\partial_\mu V(t,\mu,\cdot)$. Let b_t and σ_t be as in (3.2), then

$$\begin{aligned}
0 &\le \partial_t\varphi(t,\mu) - \partial_t V(t,\mu) + G(t,\mu,y,Z_1,\Gamma_1) - G(t,\mu,y,Z_2,\Gamma_2) \\
&= \mathbb{L}\varphi(t,\mu) - \mathbb{L}V(t,\mu),
\end{aligned}$$

and thus $\mathbb{L}\varphi(t,\mu) \ge \mathbb{L}V(t,\mu) \ge 0$. That is, V is a viscosity subsolution. $\square$

As in the standard viscosity theory, we may alternatively define viscosity solutions via semi-jets. For $t \in [0,T]$, $y \in \mathbb{R}$, $v \in \mathbb{R}$ and $\mathcal{F}_t$-measurable $Z, \Gamma \in C^0(\Omega;\mathbb{R}^d) \times C^0(\Omega;\mathbb{R}^{d\times d})$ with $|Z(\omega)| + |\Gamma(\omega)| \le C[1 + \|\omega\|]$ for some $C > 0$, define paraboloids as follows: for all $(s,\mathbb{P}) \in [t,T] \times \mathcal{P}_2$,

$$\phi^{t,y,v,Z,\Gamma}(s,\mathbb{P}) := y + v[s-t] + \mathbb{E}^{\mathbb{P}}\Big[Z \cdot X_{t,s} + \frac{1}{2}\Gamma : [X_{t,s}X_{t,s}^\top]\Big]. \tag{4.4}$$

For any $(t,\mu) \in \Theta$, it is clear that $\phi^{t,y,v,Z,\Gamma} \in C_b^{1,1,1}([t,T] \times \mathcal{P}_L(t,\mu))$ with:

$$\partial_t\varphi(t,\mu) = v, \qquad \partial_\mu\varphi(t,\mu,\cdot) = Z, \qquad \partial_\omega^{\mathrm{sym}}\partial_\mu\varphi(t,\mu,\cdot) = \Gamma^{\mathrm{sym}}, \tag{4.5}$$

for $\varphi := \phi^{t,V(t,\mu),v,Z,\Gamma}$. We then introduce the corresponding subjets and superjets: for $L > 0$,

$$\begin{aligned}\overline{\mathcal{J}}^L V(t,\mu) &:= \{(v,Z,\Gamma) : \phi^{t,V(t,\mu),v,Z,\Gamma} \in \overline{\mathcal{A}}^L V(t,\mu)\};\\ \underline{\mathcal{J}}^L V(t,\mu) &:= \{(v,Z,\Gamma) : \phi^{t,V(t,\mu),v,Z,\Gamma} \in \underline{\mathcal{A}}^L V(t,\mu)\}.\end{aligned} \tag{4.6}$$

THEOREM 4.7. *Let Assumption 3.1 hold and $V \in C^0(\Theta)$. Then V is an L-viscosity supersolution (resp. subsolution) of master equation (3.1) if and only if, for any $(t,\mu) \in \Theta$,*

$$\begin{gathered} v + G(t,\mu,V(t,\mu),Z,\Gamma) \le \ (\textit{resp.} \ \ge)\ 0\\ \forall (v,Z,\Gamma) \in \overline{\mathcal{J}}^L V(t,\mu)\ (\textit{resp.}\ \underline{\mathcal{J}}^L V(t,\mu)).\end{gathered} \tag{4.7}$$

PROOF. "$\Longrightarrow$" Assume V is an L-viscosity supersolution at (t,μ). For any $(v,Z,\Gamma) \in \overline{\mathcal{J}}^L V(t,\mu)$, since $\phi^{t,V(t,\mu),v,Z,\Gamma} \in \overline{\mathcal{A}}^L V(t,\mu)$, then it follows from the viscosity property of V and (4.5) that $0 \ge \mathbb{L}\varphi(t,\mu) = v + G(t,\mu,V(t,\mu),Z,\Gamma)$.

"$\Longleftarrow$" Assume (4.7) holds at (t,μ) and $\varphi \in \overline{\mathcal{A}}^L V(t,\mu)$ with corresponding δ. Denote

$$\begin{aligned} &v := \partial_t V(t,\mu), \qquad Z := \partial_\mu \varphi(t,\mu,\cdot), \qquad \Gamma := \partial_\omega \partial_\mu \varphi(t,\mu,\cdot),\\ &v_\varepsilon := v - \varepsilon(1+2L) \quad \forall \varepsilon > 0.\end{aligned} \tag{4.8}$$

Then, for any $(s,\mathbb{P}) \in [t,t+\delta] \times \mathcal{P}_L(t,\mu)$,

$$\begin{aligned} &\phi^{t,V(t,\mu),v_\varepsilon,Z,\Gamma}(s,\mathbb{P}) - \varphi(s,\mathbb{P})\\ &\quad = \int_t^s [v_\varepsilon - \partial_t\varphi(r,\mathbb{P})]\,dr + \mathbb{E}^{\mathbb{P}}\biggl[\int_t^s [Z + \Gamma X_{t,r} - \partial_\mu\varphi(r,\mathbb{P},\cdot)] \cdot dX_r\\ &\qquad + \frac{1}{2}\int_t^s [\Gamma - \partial_\omega\partial_\mu\varphi(r,\mathbb{P},\cdot)] : d\langle X\rangle_r\biggr].\end{aligned}$$

By choosing $\delta > 0$ small, we may assume without loss of generality that

$$\begin{aligned} &\qquad |\partial_t\varphi(s,\mathbb{P}) - v| \le \varepsilon, \qquad \mathbb{E}^{\mathbb{P}}[|\partial_\mu\varphi(s,\mathbb{P}) - Z - \Gamma X_{t,s}|] \le \varepsilon,\\ &\mathbb{E}^{\mathbb{P}}[|\partial_\omega\partial_\mu\varphi(s,\mathbb{P}) - \Gamma|] \le \varepsilon,\end{aligned} \tag{4.9}$$

for all $(s,\mathbb{P}) \in [t,t+\delta] \times \mathcal{P}_L(t,\mu)$. Then,

$$\phi^{t,V(t,\mu),v_\varepsilon,Z,\Gamma}(s,\mathbb{P}) - \varphi(s,\mathbb{P}) \le [s-t][v_\varepsilon - v + \varepsilon + L\varepsilon + L\varepsilon] = 0.$$

Since $\varphi \in \overline{\mathcal{A}}^L V(t,\mu)$, this implies immediately that $(v_\varepsilon,Z,\Gamma) \in \overline{\mathcal{J}}^L V(t,\mu)$. By our assumption we have $v_\varepsilon + G(t,\mu,V(t,\mu),Z,\Gamma) \le 0$. Send $\varepsilon \to 0$, we obtain $\mathbb{L}\varphi(t,\mu) = v + G(t,\mu,V(t,\mu),Z,\Gamma) \le 0$. That is, V is an L-viscosity supersolution at (t,μ). $\square$

REMARK 4.8. Technically speaking, since we can use the semi-jets to define viscosity solutions, our viscosity theory does not require the functional Itô formula. Instead, it is sufficient to have the Itô formula for the paraboloids in (4.4). But nevertheless the functional Itô formula is crucial for classical solutions and is interesting in its own right.

Finally, the following change variable formula is also important for comparison principle.

THEOREM 4.9. *Let Assumption 3.1 hold and $V \in C^0(\Theta)$. For any constant $\lambda \in \mathbb{R}$, define*

$$\tilde{V}(t,\mu) := e^{\lambda t} V(t,\mu), \qquad \tilde{G}(t,\mu,y,Z,\Gamma) := e^{\lambda t} G(t,\mu,e^{-\lambda t}y, e^{-\lambda t}Z, e^{-\lambda t}\Gamma).$$

Then V is an L-viscosity solution (*resp. subsolution*, *supersolution*) *of master equation* (3.1) *if and only if $\tilde{V}$ is an L-viscosity solution* (*resp. subsolution*, *supersolution*) *of the following master equation*:

$$\partial_t \tilde{V}(t,\mu) - \lambda \tilde{V}(t,\mu) + \tilde{G}(t,\mu,\tilde{V},\partial_\mu \tilde{V},\partial_\omega\partial_\mu \tilde{V}) = 0. \tag{4.10}$$

PROOF. We shall only prove that the viscosity subsolution property of V implies the viscosity subsolution property of $\tilde{V}$. The other implications follow the same arguments.

Assume V is an L-viscosity subsolution of (3.1). Let $(\tilde{v},\tilde{Z},\tilde{\Gamma}) \in \underline{\mathcal{J}}^L \tilde{V}(t,\mu)$ with corresponding $\delta_0 > 0$. Then, for any $(s,\mathbb{P}) \in [t,t+\delta_0] \times \mathcal{P}_L(t,\mu)$,

$$\tilde{V}(t,\mu) + \tilde{v}[s-t] + \mathbb{E}^{\mathbb{P}}\Big[\tilde{Z}\cdot X_{t,s} + \frac{1}{2}\tilde{\Gamma} : [X_{t,s}X_{t,s}^\top]\Big] \ge \tilde{V}(s,\mathbb{P}).$$

Thus

$$V(t,\mu) + v[s-t] + \mathbb{E}^{\mathbb{P}}\Big[Z\cdot X_{t,s} + \frac{1}{2}\Gamma : [X_{t,s}X_{t,s}^\top]\Big] \ge e^{\lambda(s-t)}V(s,\mathbb{P})$$
$$\text{where } v := e^{-\lambda t}\tilde{v},\, Z := e^{-\lambda t}\tilde{Z},\, \Gamma := e^{-\lambda t}\tilde{\Gamma}.$$

Note that V is continuous and $[t,t+\delta_0] \times \mathcal{P}_L(t,\mu)$ is compact, then V is bounded and uniformly continuous. Thus

$$\begin{aligned} e^{\lambda(s-t)}V(s,\mathbb{P}) &= [1+\lambda(s-t)+o(s-t)]V(s,\mathbb{P}) \\ &= V(s,\mathbb{P}) + \lambda V(t,\mu)[s-t] + o(s-t). \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $\delta \in (0,\delta_0)$ such that, for $(s,\mathbb{P}) \in [t,t+\delta] \times \mathbb{P}_L(t,\mu)$,

$$V(t,\mu) + [v - \lambda V(t,\mu) + \varepsilon][s-t] + \mathbb{E}^{\mathbb{P}}\Big[Z\cdot X_{t,s} + \frac{1}{2}\Gamma : [X_{t,s}X_{t,s}^\top]\Big] \ge V(s,\mathbb{P}).$$

This implies that $(v - \lambda V(t,\mu) + \varepsilon, Z, \Gamma) \in \underline{\mathcal{J}}^L V(t,\mu)$, and thus

$$v - \lambda V(t,\mu) + \varepsilon + G(t,\mu,V(t,\mu),Z,\Gamma) \ge 0.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$v - \lambda V(t,\mu) + G(t,\mu,V(t,\mu),Z,\Gamma) \ge 0.$$

This implies immediately that

$$\tilde{v} - \lambda \tilde{V}(t,\mu) + \tilde{G}(t,\mu,\tilde{V}(t,\mu),\tilde{Z},\tilde{\Gamma}) \ge 0.$$

That is, $\tilde{V}$ is an L-viscosity subsolution of (4.10). □

4.3. *Stability.* For any (t,μ,y,Z,Γ) and $\delta > 0$, denote

$$\begin{aligned} O_\delta^L(t,\mu,y,Z,\Gamma) := \{&(s,\mathbb{P},\tilde{y},\tilde{Z},\tilde{\Gamma}) : (s,\mathbb{P}) \in [t,t+\delta] \times \mathcal{P}_L(t,\mu), \\ &|\tilde{y}-y| \le \delta, \mathbb{E}^{\mathbb{P}}[|\tilde{Z}-Z|^2 + |\tilde{G}-G|^2] \le \delta^2\}. \end{aligned} \tag{4.11}$$

THEOREM 4.10. *Let $L > 0$, G satisfy Assumption* 3.1, *and $V \in C^0(\Theta)$. Assume*

(i) *for any $\varepsilon > 0$, there exist G^ε and $V^\varepsilon \in C^0(\Theta)$ such that G^ε satisfies Assumption* 3.1 *and V^ε is an L-viscosity subsolution of master equation* (3.1) *with generator G^ε*;

(ii) *as* $\varepsilon \to 0$, $(G^\varepsilon, V^\varepsilon)$ *converge to* (G, V) *locally uniformly in the following sense*: *for any* (t, μ, y, Z, Γ), *there exists* $\delta > 0$ *such that*,

$$\sup_{(s,\mathbb{P},\tilde{y},\tilde{Z},\tilde{\Gamma})\in O^L_\delta(t,\mu,y,Z,\Gamma)} \big[\big|[G^\varepsilon - G](s,\mathbb{P},\tilde{y},\tilde{Z},\tilde{\Gamma})\big| + \big|[V^\varepsilon - V](s,\mathbb{P})\big|\big] \to 0, \tag{4.12}$$

as $\varepsilon \to 0$. *Then* V *is an* L-*viscosity subsolution of master equation* (3.1) *with generator* G.

PROOF. Let $\varphi \in \overline{\mathcal{A}}^L V(t,\mu)$ with corresponding δ_0. By (4.12) we may choose $\delta_0 > 0$ small enough such that

$$\lim_{\varepsilon\to 0} \rho(\varepsilon, \delta_0) = 0, \tag{4.13}$$

where, denoting $(y_0, Z_0, \Gamma_0) := (\varphi(t,\mu), \partial_\mu\varphi(t,\mu,\cdot), \partial_\omega\partial_\mu\varphi(t,\mu,\cdot))$,

$$\rho(\varepsilon,\delta) := \sup_{(s,\mathbb{P},y,Z,\Gamma)\in O^L_\delta(t,\mu,y_0,Z_0,\Gamma_0)} \big[\big|[G^\varepsilon - G](s,\mathbb{P},y,Z,\Gamma)\big| + \big|[V^\varepsilon - V](s,\mathbb{P})\big|\big].$$

For $0 < \delta \le \delta_0$, denote $\varphi_\delta(s,\mathbb{P}) := \varphi(s,\mathbb{P}) + \delta[s-t]$. Then

$$\begin{aligned}[\varphi_\delta - V](t,\mu) &= [\varphi - V](t,\mu) = 0\\ &\le \inf_{\mathbb{P}\in\mathcal{P}_L(t,\mu)} [\varphi - V](t+\delta,\mathbb{P}) < \inf_{\mathbb{P}\in\mathcal{P}_L(t,\mu)} [\varphi_\delta - V](t+\delta,\mathbb{P}).\end{aligned}$$

By (4.13), there exists $\varepsilon_\delta > 0$ small enough such that, for any $\varepsilon \le \varepsilon_\delta$,

$$[\varphi_\delta - V^\varepsilon](t,\mu) < \inf_{\mathbb{P}\in\mathcal{P}_L(t,\mu)} [\varphi_\delta - V^\varepsilon](t+\delta,\mathbb{P}). \tag{4.14}$$

Then there exists $(t^*, \mathbb{P}^*) \in [t, t+\delta) \times \mathcal{P}_L(t,\mu)$, which may depend on (ε,δ), such that

$$c^* := \inf_{(s,\mathbb{P})\in[t,t+\delta]\times\mathcal{P}_L(t,\mu)} [\varphi_\delta - V^\varepsilon](s,\mathbb{P}) = [\varphi_\delta - V^\varepsilon](t^*,\mathbb{P}^*).$$

This implies immediately that

$$\varphi^\varepsilon_\delta := \varphi_\delta - c^* \in \underline{\mathcal{A}}^L V^\varepsilon(t^*,\mathbb{P}^*).$$

Since V^ε is a viscosity L-subsolution of master equation (3.1) with generator G^ε, we have

$$\begin{aligned}0 &\le \big[\partial_t\varphi^\varepsilon_\delta + G^\varepsilon(\cdot, \varphi^\varepsilon_\delta, \partial_\mu\varphi^\varepsilon_\delta, \partial_\omega\partial_\mu\varphi^\varepsilon_\delta)\big](t^*,\mathbb{P}^*)\\ &= \big[\partial_t\varphi + \delta + G^\varepsilon(\cdot, V^\varepsilon, \partial_\mu\varphi, \partial_\omega\partial_\mu\varphi)\big](t^*,\mathbb{P}^*)\\ &\le \big[\partial_t\varphi + G(\cdot, V^\varepsilon, \partial_\mu\varphi, \partial_\omega\partial_\mu\varphi)\big](t^*,\mathbb{P}^*) + \delta + \rho(\varepsilon,\delta_0),\end{aligned} \tag{4.15}$$

for ε and δ small enough. Now send $\delta \to 0$, we get

$$0 \le \big[\partial_t\varphi + G(\cdot, V^\varepsilon, \partial_\mu\varphi, \partial_\omega\partial_\mu\varphi)\big](t,\mu) + \rho(\varepsilon,\delta_0).$$

Send further $\varepsilon \to 0$ and then $\delta_0 \to 0$, we obtain the desired viscosity subsolution property of V at (t,μ). $\square$

4.4. *Partial comparison principle.*

THEOREM 4.11 (Partial comparison principle). *Let Assumption 3.1 hold, V^1 be a viscosity subsolution and V^2 a viscosity supersolution of (3.1). If $V^1(T,\cdot) \le V^2(T,\cdot)$ and either $V^1 \in C_b^{1,1,1}(\Theta)$ or $V^2 \in C_b^{1,1,1}(\Theta)$, then $V^1 \le V^2$.*

PROOF. We shall prove by contradiction. Denote $\Delta V := V^1 - V^2$. Assume without loss of generality that $V^2 \in C_b^{1,1,1}(\Theta)$ and that $c := \Delta V(t,\mu) > 0$ for some $(t,\mu) \in \Theta$. Define

$$c^* := \sup_{(s,\mathbb{P}) \in [t,T]\times \mathcal{P}_L(t,\mu)} \left[\Delta V(s,\mathbb{P}) - \frac{c}{2(T-t)}(T-s)\right]. \tag{4.16}$$

Note that ΔV is continuous and $[t,T] \times \mathcal{P}_L(t,\mu)$ is compact, then there exists $(t^*,\mathbb{P}^*) \in [t,T] \times \mathcal{P}_L(t,\mu)$ such that

$$\Delta V(t^*,\mathbb{P}^*) - \frac{c}{2(T-t)}(T-t^*) = c^*.$$

By considering $s = t$ in (4.16) it is clear that $c^* \ge \frac{c}{2} > 0$. Moreover, by the boundary condition that $\Delta V(T,\cdot) \le 0$, we see that $t^* < T$. Define

$$\varphi(s,\mathbb{P}) := V^2(s,\mathbb{P}) + c^* + \frac{c}{2(T-t)}(T-s).$$

Then $\varphi(t^*,\mathbb{P}^*) = V^1(t^*,\mathbb{P}^*)$. Since $\mathcal{P}_L(t^*,\mathbb{P}^*) \subseteq \mathcal{P}_L(t,\mu)$, for any $s \ge t^*$ and $\mathbb{P} \in \mathcal{P}_L(t^*,\mathbb{P}^*)$, we have $\varphi(s,\mathbb{P}) \ge V^1(s,\mathbb{P})$. This implies that $\varphi \in \underline{\mathcal{A}}^L V^1(t^*,\mathbb{P}^*)$, and thus

$$\begin{aligned}
0 &\le \mathbb{L}\varphi(t^*,\mathbb{P}^*)\\
&= \partial_t\varphi(t^*,\mathbb{P}^*) + G(t^*,\mathbb{P}^*,\varphi(t^*,\mathbb{P}^*),\partial_\mu\varphi(t^*,\mathbb{P}^*,\cdot),\partial_\omega\partial_\mu\varphi(t^*,\mathbb{P}^*,\cdot))\\
&= \partial_t V^2(t^*,\mathbb{P}^*) - \frac{c}{2(T-t)}\\
&\quad + G(t^*,\mathbb{P}^*,\varphi(t^*,\mathbb{P}^*),\partial_\mu V^2(t^*,\mathbb{P}^*,\cdot),\partial_\omega\partial_\mu V^2(t^*,\mathbb{P}^*,\cdot)).
\end{aligned}$$

By Theorem 4.9, we can assume without loss of generality that G is decreasing in y. Then, since $\varphi(t^*,\mathbb{P}^*) > V^2(t^*,\mathbb{P}^*) + c^* > V^2(t^*,\mathbb{P}^*)$, we have

$$\begin{aligned}
0 &\le \partial_t V^2(t^*,\mathbb{P}^*) - \frac{c}{2(T-t)}\\
&\quad + G(t^*,\mathbb{P}^*,V^2(t^*,\mathbb{P}^*),\partial_\mu V^2(t^*,\mathbb{P}^*,\cdot),\partial_\omega\partial_\mu V^2(t^*,\mathbb{P}^*,\cdot))\\
&= \mathbb{L}V^2(t^*,\mathbb{P}^*) - \frac{c}{2(T-t)} \le -\frac{c}{2(T-t)},
\end{aligned}$$

thanks to the classical supersolution property of V^2. This is a desired contradiction. □

4.5. *Comparison principle.* Given $g \in C^0(\mathcal{P}_2,\mathbb{R})$, define

$$\begin{aligned}
&\overline{V}(t,\mu) := \inf\{\psi(t,\mu) : \psi \in \overline{\mathcal{U}}_g\},\\
&\underline{V}(t,\mu) := \sup\{\psi(t,\mu) : \psi \in \underline{\mathcal{U}}_g\},
\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}
\mathcal{U} := \{\psi : \Theta \to \mathbb{R} &\text{ adapted, continuous in } \mu, \text{ càdlàg in } t,\\
&\text{and } \exists\, 0 = t_0 < \cdots < t_n = T \text{ such that}\\
&\psi \in C_b^{1,1,1}([t_i,t_{i+1}) \times \mathcal{P}_L(t_i,\mu)) \text{ for any } t_i, \mu \in \mathcal{P}_2, L > 0\};
\end{aligned}$$

$$\begin{aligned}\overline{\mathcal{U}}_g := \big\{\psi \in \mathcal{U} : \psi(T,\cdot) \geq g, &\text{ and for the corresponding } \{t_i\},\\ &\psi_{t_i} \leq \psi_{t_i-}, \text{ and on each}[t_{i-1},t_i),\\ &\psi \text{ is a classical supersolution of master equation (3.1)}\big\};\\ \underline{\mathcal{U}}_g := \big\{\psi \in \mathcal{U} : \psi(T,\cdot) \leq g, &\text{ and for the corresponding } \{t_i\},\\ &\psi_{t_i} \geq \psi_{t_i-}, \text{ and on each } [t_{i-1},t_i),\\ &\psi \text{ is a classical subsolution of master equation (3.1)}\big\}.\end{aligned} \tag{4.18}$$

Under mild conditions, for example, when g and $G(t,\mu,0,0,0)$ are bounded, one can easily see that $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$ are not empty.

PROPOSITION 4.12. *Let Assumption 3.1 hold, $g \in C^0(\mathcal{P}_2,\mathbb{R})$, and $\underline{\mathcal{U}}_g \neq \varnothing$. If $\underline{V} \in C^0(\Theta)$, then $\underline{V}$ is a viscosity subsolution of master equation (3.1).*

PROOF. Fix $(t,\mu) \in \Theta$. Let $\varphi \in \underline{\mathcal{A}}^L \underline{V}(t,\mu)$ with corresponding $\delta > 0$. For any $\varepsilon > 0$, let $\psi^\varepsilon \in \underline{\mathcal{U}}_g$ be such that $\psi^\varepsilon(t,\mu) \geq \underline{V}(t,\mu) - \varepsilon$. It is clear that $\psi^\varepsilon(s,\mathbb{P}) \leq \underline{V}(s,\mathbb{P})$ for all $(s,\mathbb{P}) \in [t,T] \times \mathcal{P}_L(t,\mu)$. Denote $\varphi_\delta(s,\mathbb{P}) := \varphi(s,\mathbb{P}) + \delta[s-t]$. For $\varepsilon < \delta^2$ and any $\mathbb{P} \in \mathcal{P}_L(t,\mu)$, we have

$$\begin{aligned}[\varphi_\delta - \psi^\varepsilon](t,\mu) = \big[\underline{V} - \psi^\varepsilon\big](t,\mu) \leq \varepsilon < \delta^2 &= [\varphi_\delta - \varphi](t+\delta,\mathbb{P})\\ &\leq [\varphi_\delta - \underline{V}](t+\delta,\mathbb{P}) \leq \big[\varphi_\delta - \psi^\varepsilon\big](t+\delta,\mathbb{P}).\end{aligned}$$

Then there exists $(t^*,\mathbb{P}^*) \in [t,t+\delta) \times \mathcal{P}_L(t,\mu)$ such that

$$\big(t^*,\mathbb{P}^*\big) = c^* := \inf_{(s,\mathbb{P})\in[t,t+\delta]\times\mathcal{P}_L(t,\mu)} \big[\varphi_\delta - \psi^\varepsilon\big](s,\mathbb{P}).$$

This implies that $\varphi_\delta^\varepsilon := \varphi_\delta + c^* \in \underline{\mathcal{A}}^L \psi^\varepsilon(t^*,\mathbb{P}^*)$. By Theorem 4.9, we may assume without loss of generality that G is increasing in y. Then by Theorem 4.6 we have

$$\begin{aligned}0 &\leq \mathbb{L}\varphi_\delta^\varepsilon\big(t^*,\mathbb{P}^*\big)\\ &= \partial_t\varphi\big(t^*,\mathbb{P}^*\big) + \delta + G\big(t^*,\mathbb{P}^*,\psi^\varepsilon\big(t^*,\mathbb{P}^*\big), \partial_\mu\varphi\big(t^*,\mathbb{P}^*,\cdot\big), \partial_\omega\partial_\mu\varphi\big(t^*,\mathbb{P}^*,\cdot\big)\big)\\ &\leq \partial_t\varphi\big(t^*,\mathbb{P}^*\big) + \delta + G\big(t^*,\mathbb{P}^*,\underline{V}\big(t^*,\mathbb{P}^*\big), \partial_\mu\varphi\big(t^*,\mathbb{P}^*,\cdot\big), \partial_\omega\partial_\mu\varphi\big(t^*,\mathbb{P}^*,\cdot\big)\big).\end{aligned}$$

Send $\delta \to 0$, we have $(t^*,\mathbb{P}^*) \to (t,\mu)$. Then the above inequality implies $\mathbb{L}\varphi(t,\mu) \geq 0$. $\square$

THEOREM 4.13. *Let Assumption 3.1 hold and $g \in C^0(\mathcal{P}_2;\mathbb{R})$. Assume V_1 and V_2 are viscosity subsolution and viscosity supersolution of master equation (3.1) with $V_1(T,\cdot) \leq g \leq V_2(T,\cdot)$. Assume further that $\underline{\mathcal{U}}_g$ and $\overline{\mathcal{U}}_g$ are not empty and*

$$\overline{V} = \underline{V} =: V. \tag{4.19}$$

Then $V_1 \leq V \leq V_2$ and V is the unique viscosity solution of master equation (3.1).

PROOF. First one can easily show that $\overline{V}$ is lower semicontinuous and $\underline{V}$ is upper semicontinuous. Then by (4.19) V is continuous, and thus it follows from Proposition 4.12 that V is a viscosity solution of master equation (3.1).

To see the comparison principle, which implies immediately the uniqueness, we fix an arbitrary $\psi \in \overline{\mathcal{U}}_g$. First notice that $V_1(T,\cdot) \leq g \leq \psi(T,\cdot)$. Since V_1 is continuous and $\psi(T,\cdot) \leq \psi(T-,\cdot)$, we have $V_1(T-,\cdot) = V_1(T,\cdot) \leq \psi(T,\cdot) \leq \psi(T-,\cdot)$. Now apply the partial comparison principle Theorem 4.11, one can easily see that $V_1(t,\cdot) \leq \psi(t,\cdot)$ for $t \in [t_{n-1},t_n)$. Repeat the arguments backwardly in time we can prove $V_1 \leq \psi$ on Θ. Since

$\psi \in \overline{\mathcal{U}}_g$ is arbitrary, we have $V_1 \le \overline{V}$. Similarly, one can show that $V_2 \ge \underline{V}$. Then it follows from (4.19) that $V_1 \le V \le V_2$. $\square$

The following result is a direct consequence of the above theorem.

THEOREM 4.14. *Let Assumption 3.1 hold and $g \in C^0(\mathcal{P}_2;\mathbb{R})$. Assume there exist $(\overline{G}^n, \overline{g}^n)$ and $(\underline{G}^n, \underline{g}^n)$ such that, for each n:*

(i) *$\overline{G}^n$, $\underline{G}^n$ satisfy Assumption 3.1 and $\overline{g}^n, \underline{g}^n \in C^0(\mathcal{P}_2;\mathbb{R})$;*
(ii) *the master equation (3.1) with generator $\overline{G}^n$ (resp. $\underline{G}^n$) and terminal condition $\overline{g}^n$ (resp. $\underline{g}^n$) has a classical solution $\overline{V}^n$ (resp. $\underline{V}^n$);*
(iii) *$\underline{G}_n \le G \le \overline{G}_n$, $\underline{g}_n \le g \le \overline{g}_n$;*
(iv) *$\lim_{n\to\infty} \overline{V}^n = \lim_{n\to\infty} \underline{V}^n =: V$.*

Then comparison principle holds for master equation (3.1) with generator G and terminal condition g, and V is its unique viscosity solution.

PROOF. Clearly $\overline{V}^n$ is a classical supersolution of master equation (3.1) with generator G and terminal condition g, and it satisfies $\overline{V}^n \ge g$. Then $\overline{V}^n \ge \overline{V}$. Similarly $\underline{V}^n \le \underline{V}$. Then (iv) implies (4.19) and thus the statements follow from Theorem 4.13. $\square$

4.6. *Some examples.* In this subsection we provide two examples for which we have the complete result for the comparison principle. While only for these special cases, the results are new in the literature, to our best knowledge. The comparison principle for more general master equations, especially the verification of condition (4.19), is very challenging and we shall leave it for future research.

EXAMPLE 4.15. Consider the setting in Example 3.8, but relax the regularity of κ to be only continuous. Then the V defined by (3.30) is in $C^0([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$ and is the unique viscosity solution of the master equation (3.31).

PROOF. (i) One can easily verify that V is continuous and the DPP (3.21) becomes:

$$V(t,\mu_t) = V\big(t+\delta, (\mathbb{P}_0^{t,\mu})_{t+\delta}\big), \quad (t,\mu) \in \Theta. \tag{4.20}$$

Denote $\nu := \mu_t$. Now let $L \ge 1$ and $\varphi \in \underline{\mathcal{A}}^L V(t,\nu)$. Clearly $\mathbb{P}_0^{t,\mu} \in \mathcal{P}_L(t,\nu)$. Then

$$\varphi(t,\nu) = V(t,\nu) = V\big(t+\delta, (\mathbb{P}_0^{t,\mu})_{t+\delta}\big) \le \varphi\big(t+\delta, (\mathbb{P}_0^{t,\mu})_{t+\delta}\big).$$

Apply the Itô formula, this implies

$$\begin{aligned}
0 &\le \int_t^{t+\delta} \partial_t \varphi\big(t+\delta, (\mathbb{P}_0^{t,\mu})_s\big)\,ds + \mathbb{E}^{\mathbb{P}_0^{t,\mu}}\bigg[\int_t^{t+\delta} \partial_\mu \varphi\big(s, (\mathbb{P}_0^{t,\mu})_s, X_s\big)\,dX_s \\
&\quad + \frac{1}{2}\int_t^{t+\delta} \partial_x\partial_\mu \varphi\big(s, (\mathbb{P}_0^{t,\mu})_s, X_s\big)\,d\langle X\rangle_s\bigg] \\
&= \int_t^{t+\delta} \Big[\partial_t \varphi\big(t+\delta, (\mathbb{P}_0^{t,\mu})_s\big) + \frac{1}{2}\mathbb{E}^{\mathbb{P}_0^{t,\mu}}\big[\partial_x\partial_\mu \varphi\big(s, (\mathbb{P}_0^{t,\mu})_s, X_s\big)\big]\Big]\,ds.
\end{aligned}$$

Divide both sides by δ and send $\delta \to 0$, we obtain

$$\partial_t \varphi(t,\nu) + \frac{1}{2}\mathbb{E}^{\nu}\big[\partial_x\partial_\mu \varphi(t,\nu,X_t)\big] \ge 0.$$

That is, V is a viscosity subsolution at (t,ν). Similarly one can show that V is a viscosity supersolution at (t,ν), hence a viscosity solution.

(ii) We next prove the comparison principle, which implies the uniqueness. Assume $|g|\le C_0$. Then (3.30) can be rewritten as

$$V(t,\mu):=\int_0^{C_0}\kappa\big(\mathbb{P}_0^{t,\mu}\big(g(X_T)\ge y\big)\big)\,dy.$$

Since κ is continuous on $[0,1]$, it is uniformly continuous, then there exists a smooth molifier κ_n such that κ_n is strictly increasing and $|\kappa_n-\kappa|\le\frac{1}{n}$. Denote $\overline{\kappa}_n:=\kappa_n+\frac{1}{n}$, $\underline{\kappa}_n:=\kappa_n-\frac{1}{n}$, and define

$$\overline{V}_n(t,\mu):=\int_0^{C_0}\overline{\kappa}_n\big(\mathbb{P}_0^{t,\mu}\big(g(X_T)\ge y\big)\big)\,dy,$$
$$\underline{V}_n(t,\mu):=\int_0^{C_0}\underline{\kappa}_n\big(\mathbb{P}_0^{t,\mu}\big(g(X_T)\ge y\big)\big)\,dy.$$

We remark that $\overline{\kappa}_n$ and $\underline{\kappa}_n$ does not satisfy the boundary conditions: $\kappa(0)=0$, $\kappa(1)=1$. Nevertheless, following the same arguments in Example 3.8, one can easily see that $\overline{V}_n$ and $\underline{V}_n$ are classical solutions of master type heat equation (3.31), with terminal conditions

$$\overline{V}_n(T,\mu):=\int_0^{C_0}\overline{\kappa}_n\big(\mu\big(g(X_T)\ge y\big)\big)\,dy,$$
$$\underline{V}_n(T,\mu):=\int_0^{C_0}\underline{\kappa}_n\big(\mu\big(g(X_T)\ge y\big)\big)\,dy,$$

respectively. It is clear that $\underline{V}_n\le V\le\overline{V}_n$ and $\lim_{n\to\infty}\overline{V}_n=\lim_{n\to\infty}\underline{V}_n=V$. Then the result follows from Theorem 4.14 immediately. □

The next example considers the following nonlinear (state dependent) master equation, which can be viewed as a special case of (3.8) (see [39]):

$$\begin{aligned}&\partial_t V(t,\mu)+\frac{1}{2}\mathbb{E}^\mu\big[\partial_x\partial_\mu V(t,\mu,X_t)\big]+G_1\big(\mathbb{E}^\mu\big[\partial_\mu V(t,\mu,X_t)\big]\big)=0,\\&V(T,\mu)=\mathbb{E}^\mu\big[g(X_T)\big].\end{aligned}\tag{4.21}$$

EXAMPLE 4.16. Assume:

(i) g is Lipschitz continuous with a Lipschitz constant L_0, and $G_1\in C^0([-L_0,L_0])$;
(ii) Either g and $-G_1$ are convex, or g and $-G_1$ are concave.

Then the master equation (4.21) has a unique viscosity solution $V\in C^0([0,T]\times\mathcal{P}_2(\mathbb{R}))$.

PROOF. Let G_1^n and g_n be smooth mollifiers of G_1 and g, respectively, such that $|G_1^n-G_1|\le\frac{1}{n}$, $|g_n-g|\le\frac{1}{n}$. Denote $\overline{G}_1^n:=G_1^n+\frac{1}{n}$, $\underline{G}_1^n:=G_1^n-\frac{1}{n}$, $\overline{g}_n:=g_n+\frac{1}{n}$, $\underline{g}_n:=g_n-\frac{1}{n}$. Then $(\overline{G}_1^n,\overline{g}_n)$ and $(\underline{G}_1^n,\underline{g}_n)$ are smooth and still satisfy (i) and (ii) with the same L_0. By Saporito and Zhang [39], Theorem 3.1, the corresponding master equations (4.21) have a classical solution $\overline{V}_n$ and $\underline{V}_n$, respectively.

Now by Theorem 4.14 it suffices to show that $\overline{V}_n$ and $\underline{V}_n$ converge to the same limit. Without loss of generality, we assume G_1 is convex (and g is concave). Denote

$$b(a):=\sup_{y\in[-L_0,L_0]}\big[ay-G_1(y)\big],\qquad b_n(a):=\sup_{y\in[-L_0,L_0]}\big[ay-G_1^n(y)\big],\quad a\in\mathbb{R}.$$

By [39] (or following similar arguments as in Section 5 below), we have

$$\overline{V}_n(t,\mu) = \sup_{a\in\mathbb{R}} \mathbb{E}^{\mathbb{P}_0^{t,\mu}}\Big[g\Big(X_T + \Big[b_n(a) - \frac{1}{n}\Big][T-t]\Big)\Big],$$
$$\underline{V}_n(t,\mu) = \sup_{a\in\mathbb{R}} \mathbb{E}^{\mathbb{P}_0^{t,\mu}}\Big[g\Big(X_T + \Big[b_n(a) + \frac{1}{n}\Big][T-t]\Big)\Big].$$

It is clear that $|b_n - b| \le \frac{1}{n}$. Then it is straightforward to show that

$$\lim_{n\to\infty} \overline{V}_n(t,\mu) = \lim_{n\to\infty} \overline{V}_n(t,\mu) = V(t,\mu) := \sup_{a\in\mathbb{R}} \mathbb{E}^{\mathbb{P}_0^{t,\mu}}\big[g\big(X_T + b(a)[T-t]\big)\big].$$

Now the result follows directly from Theorem 4.14. □

5. McKean–Vlasov SDEs with closed-loop controls. In this section we apply our viscosity theory to the mean field control problem introduced in Section 3.2. Recall (3.17), (3.18) and (3.20), we shall assume the following.

ASSUMPTION 5.1. b, σ, f are $\mathbb{F}$-progressively measurable in all variables $(t, \omega, \mu, a) \in [0,T] \times \Omega \times \mathcal{P}_2 \times A$ (and in particular $\mathbb{F}$-adapted in both ω and μ), and g is progressively measurable in $(\omega, \mu) \in \Omega \times \mathcal{P}_2$. Moreover:

(i) b, σ are bounded by a constant C_0, continuous in a, and uniform Lipschitz continuous in (ω,μ) with a Lipschitz constant L_0:

$$\big|(b,\sigma)(t,\omega,\mu,a) - (b,\sigma)\big(t,\omega',\mu',a\big)\big| \le L_0\big[\|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\| + \mathcal{W}_2\big(\mu_{[0,t]}, \mu'_{[0,t]}\big)\big];$$

(ii) $f(t,0,\delta_{\{0\}},a)$ is bounded by a constant C_0, f is continuous in a, and f and g are uniformly continuous in (ω,μ) with a modulus of continuity function ρ_0:

$$\big|f(t,\omega,\mu,a) - f\big(t,\omega',\mu',a\big)\big| \le \rho_0\big(\|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\| + \mathcal{W}_2\big((t,\mu),(t,\mu')\big)\big);$$
$$\big|g(\omega,\mu) - g\big(\omega',\mu'\big)\big| \le \rho_0\big(\|\omega - \omega'\| + \mathcal{W}_2\big((T,\mu),(T,\mu')\big)\big).$$

(iii) $\varphi = b, \sigma, f$ is locally uniformly continuous in t in the following sense:

$$\begin{aligned}&\big|\varphi(s, \omega_{t\wedge\cdot}, \mu_{[0,t]}, a) - \varphi(t,\omega,\mu,a)\big| \\ &\quad \le C\big[1 + \|\omega_{t\wedge\cdot}\| + \mathcal{W}_2(\mu_{[0,t]}, \delta_{\{0\}})\big]\rho_0(s-t), \quad t < s.\end{aligned}$$

(iv) $\sigma\sigma^\top$ is positive definite.

We remark that one sufficient condition for (iii) is that A is compact, and the nondegeneracy of σ in (iv) is used in Lemma 5.13 below, but we do not need uniform nondegeneracy.

The choice of the admissible controls is very subtle, with (3.19) as one example. We shall discuss alternative choices in details at below. One basic requirement is that the corresponding value function should satisfy the DPP.

5.1. *Open-loop controls.* In this subsection, we consider open-loop controls, namely $\alpha_t = \alpha_t(B_\cdot)$ depending on B, where B is a Brownian motion in a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$. There are two natural choices: (i) $\mathcal{A}_t^1$, where $\alpha_s = \alpha(s, (B_{t,r})_{t\le r\le s})$ is adapted to the shifted filtration of B; and (ii) $\mathcal{A}_t^2$, where $\alpha_s = \alpha(s, (B_r)_{0\le r\le s})$ is adapted to the full filtration of B. For the standard control problems, they would induce the same value function. However, in our setting the issue is quite subtle. To be precise, for an $\mathbb{F}^B$-progressively measurable process ξ on $[0,t]$ and a control α, let $X_s^{t,\xi,\alpha} := \xi_s$, $s \in [0,t]$, and, for $\mathbb{P}_0$-a.s.

$$X_s^{t,\xi,\alpha} = \xi_t + \int_t^s b\big(r, X^{t,\xi,\alpha}, \mathcal{L}_{X^{t,\xi,\alpha}}, \alpha_r\big)\,dr + \int_t^s \sigma\big(r, X^{t,\xi,\alpha}, \mathcal{L}_{X^{t,\xi,\alpha}}, \alpha_r\big)\,dB_r,$$

which has a unique strong solution under Assumption 5.1. Introduce the value functions: for $i = 1, 2$,

$$V_i(t,\xi) := \sup_{\alpha\in\mathcal{A}_t^i} \mathbb{E}^{\mathbb{P}_0}\Big[g\big(X^{t,\xi,\alpha}, \mathcal{L}_{X^{t,\xi,\alpha}}\big) + \int_t^T f\big(s, X^{t,\xi,\alpha}, \mathcal{L}_{X^{t,\xi,\alpha}}, \alpha_s\big)\,ds\Big]. \tag{5.1}$$

The following example shows that $\mathcal{A}_t^1$ is not a good choice.

EXAMPLE 5.2. Let $d = 1$, $A = [-1, 1]$, $b(t, \omega, \mu, a) = a$, $\sigma \equiv 1$, $f \equiv 0$, and $g(\omega, \mu) = g(\mu) = -\operatorname{Var}^{\mu}(X_T)$.

(i) $V_1(t, \xi) < V_2(t, \xi)$ when $\xi_t = (T - t)\operatorname{sign}(B_t)$ and $T - t > 1$.
(ii) V_1 does not satisfy the DPP: $V_1(t_1, \xi) \neq \sup_{\alpha\in\mathcal{A}_{t_1}^1} V_1(t_2, X^{t_1,\xi,\alpha})$.

PROOF. (i) For any (t, ξ) and $\alpha \in \mathcal{A}_t^1$, notice that ξ_t is independent of α and thus is also independent of $X_T^{t,\xi,\alpha} - \xi_t$. Then

$$\operatorname{Var}\big(X_T^{t,\xi,\alpha}\big) = \operatorname{Var}(\xi_t) + \operatorname{Var}\big(X_T^{t,\xi,\alpha} - \xi_t\big), \tag{5.2}$$

thus $V_1(t, \xi) \le -\operatorname{Var}(\xi_t) = -[T - t]^2$. On the other hand, set $\alpha_s := -\operatorname{sign}(B_t)$, $s \in [t, T]$. Then $\alpha \in \mathcal{A}_t^2$, $X_T^{t,\xi,\alpha} = B_{t,T}$, and thus

$$V_2(t,\xi) \ge \operatorname{Var}(B_{t,T}) = -[T-t] > -[T-t]^2 = V_1(t,\xi).$$

(ii) Denote $h(t) := \sup_{\alpha\in\mathcal{A}_0^1}[-\operatorname{Var}(\int_0^t \alpha_s\,ds + B_t)]$. Then by (5.2) one can easily see that

$$V_1(t,\xi) = h(T-t) - \operatorname{Var}(\xi_t).$$

Assume by contradiction that DPP holds. Then, for any $0 < t < T$,

$$\begin{aligned} h(T) = V_1(0, \delta_{\{0\}}) &= \sup_{\alpha\in\mathcal{A}_0^1} V_1\big(t, X_t^{0,0,\alpha}\big) \\ &= \sup_{\alpha\in\mathcal{A}_0^1} \big[h(T-t) - \operatorname{Var}\big(X_t^{0,0,\alpha}\big)\big] = h(t) + h(T-t). \end{aligned}$$

Following the same arguments we see that h is linear in t. Since $|\alpha| \le 1$, it is clear that

$$\begin{aligned} \operatorname{Var}\Big(\int_0^t \alpha_s\,ds + B_t\Big) &= \mathbb{E}\Big[\Big(\int_0^t \alpha_s\,ds + B_t - \mathbb{E}\Big[\int_0^t \alpha_s\,ds\Big]\Big)^2\Big] \\ &= \mathbb{E}\big[\big(B_t + O(t)\big])^2\big] = \mathbb{E}\big[|B_t|^2\big] + o(t) = t + o(t). \end{aligned}$$

Then

$$\lim_{t\to 0} \frac{h(t)}{t} = -1, \quad \text{and thus} \quad h(t) = -t.$$

On the other hand, fix $t \in (0, T)$ and set $\alpha_s := [(-1) \vee (-\frac{B_t}{T-t}) \wedge 1]1_{[t,T]}(s)$. Then

$$\int_0^T \alpha_s\,ds + B_T = (t-T) \vee (-B_t) \wedge (T-t) + B_t + B_{t,T}.$$

Thus

$$\begin{aligned} -h(T) &\le \operatorname{Var}\Big(\int_0^T \alpha_s\,ds + B_T\Big) \\ &= \operatorname{Var}\big((t-T) \vee (-B_t) \wedge (T-t) + B_t\big) + T - t \\ &= \mathbb{E}^{\mathbb{P}_0}\big[\big([|B_t| - [T-t]]^+\big)^2\big] + T - t \\ &< \mathbb{E}^{\mathbb{P}_0}\big[|B_t|^2\big] + T - t = t + T - t = T. \end{aligned}$$

This is a desired contradiction. □

Technically, the choice of $\mathcal{A}_t^2$ would work; see, for example, Bayraktar, Cosso and Pham [3]. The following results can be proved easily, in particular, the viscosity property in (iii) follows similar arguments as in Theorem 5.8 below, and thus we omit the proofs.

PROPOSITION 5.3. *Let Assumption 5.1 hold and define $V_2(t,\xi)$ by (5.1). Then:*

(i) *V_2 satisfies the following DPP: for any $t_1 < t_2$,*

$$V_2(t_1,\xi) = \sup_{\alpha\in\mathcal{A}_{t_1}^2}\biggl[V_2\bigl(t_2, X^{t_1,\xi,\alpha}\bigr) + \int_{t_1}^{t_2}\mathbb{E}^{\mathbb{P}_0}\bigl[f\bigl(s, X^{t,\xi,\alpha}, \mathcal{L}_{X^{t,\xi,\alpha}}, \alpha_s\bigr)\bigr]\,ds\biggr].$$

(ii) *$V_2(t,\xi)$ is law invariant and thus we may define $V_2'(t,\mu)$ by $V_2(t,\xi) = V_2'(t,\mathcal{L}_\xi)$.*

(iii) *$V_2' \in C^0(\Theta)$ and is a viscosity solution of the HJB type of master equation (3.22).*

Despite the above nice properties, in many applications the state process X is observable while the Brownian motion B is used to model the distribution of X and may not be observable. Then it is not reasonable to have the controls relying on B. The issue becomes more serious when one considers games instead of control problems. We refer to Zhang [43], Section 9.1, for detailed discussions on these issues. Therefore, in the next subsection we shall turn to closed-loop controls.

5.2. *Closed-loop controls.* We now assume α depends on the state process $X^{t,\xi,\alpha}$. One choice is to use the (state dependent) feedback controls: $\alpha_s = \alpha(s, X_s^{t,\xi,\alpha})$; see, for example, Pham and Wei [34]. However, we prefer not to use this for several reasons:

- In practice it is not natural to assume the players cannot use past information;
- It seems difficult to have regularity of $V(t,\mu)$ without strong constraint on α;
- It fails to work in non-Markovian models, which are important in applications.

We shall assume α is $\mathbb{F}^{X^{t,\xi,\alpha}}$-measurable, namely $\alpha_s = \alpha(s, (X_r^{t,\xi,\alpha})_{0\le r\le s})$, and thus we are considering (3.17). As mentioned in Section 3.2, in this case it is more convenient to use weak formulation. That is, we shall use the canonical setting in Section 2.2, and consider the optimization problem (3.18) and (3.20). However, under closed-loop controls, the regularity of $V(t,\mu)$ is rather technical. In this section we content ourselves with the following piecewise constant control process:

$$\mathcal{A}_t := \biggl\{\alpha : \exists\, n \text{ and } t = t_0 < \cdots < t_n = T \text{ such that} \tag{5.3}$$
$$\alpha_s = \sum_{i=0}^{n-1} h_i \mathbf{1}_{[t_i,t_{i+1})}(s), \text{ where } h_i : \Omega \to A \text{ is } \mathcal{F}_{t_i}\text{-measurable}\biggr\}.$$

We emphasize that here we are abusing the notation $\mathcal{A}$ with (3.19). So throughout this section, our optimization problem will always be (3.18)–(5.3)–(3.20).

REMARK 5.4.

(i) Each $\alpha \in \mathcal{A}_t$ here also satisfies the requirement in (3.19), and thus (3.18) has a unique (strong) solution $\mathbb{P}^{t,\mu,\alpha} \in \mathcal{P}_L(t,\mu)$, where $L \ge C_0 \vee [\frac{1}{2}C_0^2]$ for the bound C_0 in Assumption 5.1(i). In particular, $\mathbb{P}^{t,\mu,\alpha}$ satisfies the uniform estimate (4.1).

(ii) Obviously the $\mathcal{A}_t$ in (5.3) also satisfies (3.24). Then following the same arguments as in Theorem 3.6(i) we see that, under Assumption 5.1, V satisfies the DPP (3.21).

REMARK 5.5.

(i) Although (3.17) (and (3.18)) has a strong solution, the formulation (3.20) is still different from the $V_2(t,\xi)$ in (5.1). Indeed, by the piecewise constant structure, one can easily see that $\mathbb{F}^{X^{t,\xi,\alpha}}$ is the same as the filtration generated by the process $\tilde{B}_s := \xi_s 1_{[0,t]}(s) + [\xi_t + B_{t,s}]1_{(t,T]}(s)$, and thus one may rewrite $\alpha(s, X^{t,\xi,\alpha}_{[0,s]})$ as $\tilde{\alpha}(s, \tilde{B}_{[0,s]})$ for some measurable function $\tilde{\alpha}$. However, note that $\tilde{B}_{[0,t]} = \xi_{[0,t]} \neq B_{[0,t]}$, so this control is still not in $\mathcal{A}^2_t$. Indeed, in many practical situations, at time t, one can observe the state process $\xi_{\cdot\wedge t}$, but not necessarily observe an underlying Brownian motion path in the past. That is the main reason we consider the closed-loop controls in this paper.

(ii) The regularity of V_2 and V_2' in Proposition 5.3(iii) is straightforward. However, the above subtle difference makes the regularity of V in (3.20) quite involved, as we will see in Example 5.6 and Section 5.3 below.

EXAMPLE 5.6. Let $d = 1$, $A = [-1, 1]$, $T = 1$, $b \equiv 0$, $\sigma(t,\mu,a) = 1 + a^2$, $f \equiv 0$, $g(\omega,\mu) = g(\mu) = \frac{1}{3}\mathbb{E}^\mu[X_1^4] - (\mathbb{E}^\mu[X_1^2])^2$, and $\mathcal{A}^0_0$ consist of constant controls: $\alpha_t \equiv \alpha_0(X_0)$, $\forall t \in [0,1]$. Then $V^0_0(\mu) := \sup_{\alpha\in\mathcal{A}^0_0} g(\mathbb{P}^{0,\mu,\alpha})$ is discontinuous in $\mu \in \mathcal{P}_2$.

PROOF. Let $\mu_0 := \delta_{\{0\}}$ and $\mu_\varepsilon := \frac{1}{2}[\delta_{\{\varepsilon\}} + \delta_{\{-\varepsilon\}}]$. It is obvious that $\lim_{\varepsilon\to 0}\mathcal{W}_2(\mu_\varepsilon,\mu_0) = 0$. For any $\alpha \in \mathcal{A}^0_0$, we have $\alpha_t = \alpha_0(0)$ and $X_1 = [1 + |\alpha_0(0)|^2]B^\alpha_1$, $\mathbb{P}^{0,\mu,\alpha}$-a.s. Then, denoting $c := 1 + |\alpha_0(0)|^2$ and $\mathbb{P}^\alpha := \mathbb{P}^{0,\mu,\alpha}$, we have

$$g(\mathbb{P}^{0,\mu,\alpha}) = \frac{1}{3}\mathbb{E}^{\mathbb{P}^\alpha}[c^4|B^\alpha_1|^4] - (\mathbb{E}^{\mathbb{P}^\alpha}[c^2|B^\alpha_1|^2])^2 = 0, \quad \text{and thus} \quad V^0_0(\mu_0) = 0.$$

On the other hand, for each $\varepsilon > 0$, set $\alpha_t := \alpha_0(X_0) := \mathbf{1}_{\{X_0>0\}}$. Then

$$X_1 = [\varepsilon + 2B^\alpha_1]1_{\{X_0=\varepsilon\}} + [-\varepsilon + B^\alpha_1]1_{\{X_0=-\varepsilon\}}, \quad \mathbb{P}^{0,\mu_\varepsilon,\alpha}\text{-a.s.}$$

Thus, denoting $\mathbb{P}^\varepsilon := \mathbb{P}^{0,\mu_\varepsilon,\alpha}$,

$$\begin{aligned} g&(\mathbb{P}^{0,\mu_\varepsilon,\alpha}) \\ &= \frac{1}{6}\mathbb{E}^{\mathbb{P}^\varepsilon}[(2B^\alpha_1+\varepsilon)^4 + (B^\alpha_1-\varepsilon)^4] - \Big(\frac{1}{2}\mathbb{E}^{\mathbb{P}^\varepsilon}[(2B^\alpha_1+\varepsilon)^2 + (B^\alpha_1-\varepsilon)^2]\Big)^2 \\ &= \frac{1}{6}[51 + 18\varepsilon^2 + 2\varepsilon^4] - \Big(\frac{1}{2}[5 + 2\varepsilon^2]\Big)^2 = \frac{9}{4} - 2\varepsilon^2 - \frac{2}{3}\varepsilon^4. \end{aligned}$$

Therefore, for all $\varepsilon > 0$ small,

$$V^0_0(\mu_\varepsilon) \geq \frac{9}{4} - 2\varepsilon^2 - \frac{2}{3}\varepsilon^4 \geq 2 > 0 = V^0_0(\mu_0).$$

This implies that V^0_0 is discontinuous at μ_0. □

Nevertheless, by using piecewise constant controls $\mathcal{A}_t$, we have the following theorem.

THEOREM 5.7. *Under Assumption 5.1, there exists a modulus of continuity function ρ such that*

$$\begin{aligned} |V(t_1,\mu) - V(t_2,\nu)| \leq{}& C\rho(\mathcal{W}_2((t_1,\mu),(t_2,\nu)) \\ &+ C[1 + \mathcal{W}_2(\mu_{[0,t_1]},\delta_{\{0\}})][t_2 - t_1]. \end{aligned} \tag{5.4}$$

Assume further that f is bounded, then V is uniformly continuous in (t,μ).

The proof of this theorem is quite involved, so we defer it to the next subsection.

Given the above regularity, we can easily verify the viscosity property.

THEOREM 5.8. *Under Assumption 5.1, V is a viscosity solution of the HJB type of master equation (3.22).*

PROOF. Fix $L > 0$ such that $|b|, \frac{1}{2}|\sigma|^2 \leq L$. We shall show that V is an L-viscosity solution.

Step 1. We first verify its the viscosity subsolution property. Assume by contradiction that V is not an L-viscosity subsolution at (t,μ), then there exists $(v, Z, \Gamma) \in \underline{\mathcal{J}}^L V(t,\mu)$ with corresponding δ, such that

$$-c := \mathbb{L}\varphi(t,\mu) = v + \mathbb{E}^{\mu}\Big[\sup_{a\in A} G_2(t,\mu,X,Z,\Gamma,a)\Big] < 0, \tag{5.5}$$

where $\varphi := \phi^{t,V(t,\mu),v,Z,\Gamma}$. For any $\alpha \in \mathcal{A}_t$, applying the functional Itô formula we have

$$\varphi\big(t+\delta, \mathbb{P}^{t,\mu,\alpha}\big) - \varphi(t,\mu) = \int_t^{t+\delta} \mathbb{L}^{\alpha}\varphi\big(s, \mathbb{P}^{t,\mu,\alpha}\big)\,ds, \tag{5.6}$$

where, abbreviating $\mathbb{P}^{\alpha} := \mathbb{P}^{t,\mu,\alpha}$,

$$\begin{aligned} \mathbb{L}^{\alpha}\varphi\big(s,\mathbb{P}^{\alpha}\big) = a + \mathbb{E}^{\mathbb{P}^{\alpha}}\Big[\Big[& b\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big)\cdot[Z+\Gamma X_{t,s}] \\ &+ \frac{1}{2}\Gamma : \sigma\sigma^{\top}\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big)\Big]\Big]. \end{aligned} \tag{5.7}$$

Note that

$$\mathbb{L}^{\alpha}\varphi\big(s,\mathbb{P}^{\alpha}\big) - \mathbb{L}\varphi(t,\mu) = I_1(s) + I_2(s) - \mathbb{E}^{\mathbb{P}^{\alpha}}\big[f\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big)\big], \tag{5.8}$$

where

$$\begin{aligned} I_1(s) &:= \mathbb{E}^{\mathbb{P}^{\alpha}}\Big[G_2(t,\mu,X,Z,\Gamma,\alpha_s) - \sup_{a\in A} G_2(t,\mu,X,Z,\Gamma,a)\Big]; \\ I_2(s) &:= \mathbb{E}^{\mathbb{P}^{\alpha}}\Big[Z\cdot\big[b\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big) - b(t,X,\mu,\alpha_s)\big] + b\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big)\cdot\Gamma X_{t,s} \\ &\quad + \frac{1}{2}\Gamma : \big[\sigma\sigma^{\top}\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big) - \sigma\sigma^{\top}(t,X,\mu,\alpha_s)\big] \\ &\quad + \big[f\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big) - f(t,X,\mu,\alpha_s)\big]\Big]. \end{aligned}$$

It is clear that $I_1(s) \leq 0$. By Assumption 5.1(ii) and (iii), we have, for $s \in [t, t+\delta]$,

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{\alpha}}\big[f\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big) - f(t,X,\mu,\alpha_s)\big] \\ &\quad = \mathbb{E}^{\mathbb{P}^{\alpha}}\big[\big[f\big(s,X,\mathbb{P}^{\alpha},\alpha_s\big) - f(s,X_{t\wedge\cdot},\mu_{[0,t]},\alpha_s)\big] \\ &\qquad + \big[f(s,X_{t\wedge\cdot},\mu_{[0,t]},\alpha_s) - f(t,X,\mu,\alpha_s)\big]\big] \\ &\quad \leq C\mathbb{E}^{\mathbb{P}^{\alpha}}\big[\rho_0\big(\|X_{s\wedge\cdot} - X_{t\wedge\cdot}\| + \mathcal{W}_2\big(\mathbb{P}^{\alpha}_{[0,s]},\mu_{[0,t]}\big)\big)\big] \\ &\qquad + C\big(\mathbb{E}^{\mu}\big[1+\|X_{t\wedge\cdot}\|^2\big]\big)^{\frac{1}{2}}\rho_0(\delta). \end{aligned}$$

Since b, σ are bounded, by (4.1) one can easily see that

$$\lim_{\delta\to 0}\sup_{\alpha\in\mathcal{A}_t} \mathbb{E}^{\mathbb{P}^{\alpha}}\big[\rho_0\big(\|X_{s\wedge\cdot} - X_{t\wedge\cdot}\| + \mathcal{W}_2\big(\mathbb{P}^{\alpha}_{[0,s]},\mu_{[0,t]}\big)\big)\big] = 0.$$

Then, for some $\delta = \delta(t, \mu) > 0$ small enough, we have

$$\mathbb{E}^{\mathbb{P}^\alpha}\big[f\big(s, X, \mathbb{P}^\alpha, \alpha_s\big) - f(t, X, \mu, \alpha_s)\big] \le \frac{c}{8},$$

for all $s \in [t, t+\delta]$ and all $\alpha \in \mathcal{A}_t$. Similarly, recalling that by definition Z, Γ have linear growth in ω, we may have the desired estimates for the other terms in $I_2(s)$, and thus $I_2(s) \le \frac{c}{2}$. Therefore, (5.8) implies that

$$\mathbb{L}^\alpha\varphi\big(s, \mathbb{P}^\alpha\big) + \mathbb{E}^{\mathbb{P}^\alpha}\big[f\big(s, X, \mathbb{P}^\alpha, \alpha_s\big)\big] = \mathbb{L}\varphi(t, \mu) + I_1(s) + I_2(s) \le -\frac{c}{2}.$$

Plug this into (5.6) and recall (4.2), we get

$$\begin{aligned} &V\big(t+\delta, \mathbb{P}^{t,\mu,\alpha}\big) + \int_t^{t+\delta} \mathbb{E}^{\mathbb{P}^\alpha}\big[f\big(s, X, \mathbb{P}^\alpha, \alpha_s\big)\big]\,ds - V(t, \mu) \\ &\quad\le \varphi\big(t+\delta, \mathbb{P}^{t,\mu,\alpha}\big) - \varphi(t, \mu) + \int_t^{t+\delta} \mathbb{E}^{\mathbb{P}^\alpha}\big[f\big(s, X, \mathbb{P}^\alpha, \alpha_s\big)\big]\,ds \\ &\quad= \int_t^{t+\delta} \big[\mathbb{L}^\alpha\varphi\big(s, \mathbb{P}^\alpha\big) + \mathbb{E}^{\mathbb{P}^\alpha}\big[f\big(s, X, \mathbb{P}^\alpha, \alpha_s\big)\big]\big]\,ds \le -\frac{c\delta}{2} \quad \forall \alpha \in \mathcal{A}_t. \end{aligned}$$

Take supremum over $\alpha \in \mathcal{A}_t$, this contradicts with the DPP (3.21); see Remark 5.4(ii).

Step 2. We next verify its viscosity supersolution property. Assume by contradiction that V is not an L-viscosity supersolution at (t, μ), then there exists $(v, Z, \Gamma) \in \overline{\mathcal{J}}^L V(t, \mu)$ with corresponding δ, such that

$$c := \mathbb{L}\varphi(t, \mu) = v + \mathbb{E}^\mu\Big[\sup_{a \in A} G_2(t, \mu, X, Z, \Gamma, a)\Big] > 0, \tag{5.9}$$

where $\varphi := \phi^{t, V(t,\mu), v, Z, \Gamma}$. Note that $G_2(t, \mu, X, Z, \Gamma, a)$ is $\mathcal{F}_t$-measurable, there exists an $\mathcal{F}_t$-measurable A-valued random variable α_t such that

$$v + \mathbb{E}^\mu\big[G_2(t, \mu, X, Z, \Gamma, \alpha_t)\big] \ge \frac{c}{2}. \tag{5.10}$$

Now let $\alpha_s := \alpha_t$, $s \in [t, t+\delta]$ and denote $\mathbb{P} := \mathbb{P}^{t,\mu,\alpha}$. Clearly $a \in \mathcal{A}_t$. Applying the functional Itô formula we have

$$\varphi(t+\delta, \mathbb{P}) - \varphi(t, \mu) = \int_t^{t+\delta} \mathbb{L}^\alpha\varphi(s, \mathbb{P})\,ds, \tag{5.11}$$

where $\mathbb{L}^\alpha$ is the same as (5.7). Similar to the estimate of $I_2(s)$ in Step 1, for $\delta > 0$ small enough we have

$$v + \mathbb{E}^\mu\big[G_2(t, \mu, X, Z, \Gamma, \alpha_t)\big] - \big[\mathbb{L}^\alpha\varphi(s, \mathbb{P}) + \mathbb{E}^{\mathbb{P}}\big[f(s, X, \mathbb{P}, \alpha_t)\big]\big] \le \frac{c}{4}.$$

Then, by (5.10),

$$\mathbb{L}^\alpha\varphi(s, \mathbb{P}) + \mathbb{E}^{\mathbb{P}}\big[f(s, X, \mathbb{P}, \alpha_t)\big] \ge \frac{c}{4}, \quad s \in [t, t+\delta].$$

This implies

$$\begin{aligned} &V(t+\delta, \mathbb{P}) + \int_t^{t+\delta} \mathbb{E}^{\mathbb{P}}\big[f(s, X, \mathbb{P}, \alpha_t)\big]\,ds - V(t, \mu) \\ &\quad\ge \varphi(t+\delta, \mathbb{P}) + \int_t^{t+\delta} \mathbb{E}^{\mathbb{P}}\big[f(s, X, \mathbb{P}, \alpha_t)\big]\,ds - \varphi(t, \mu) \\ &\quad= \int_t^{t+\delta} \big[\mathbb{L}^\alpha\varphi(s, \mathbb{P}) + \mathbb{E}^{\mathbb{P}}\big[f(s, X, \mathbb{P}, \alpha_t)\big]\big]\,ds \ge \frac{c\delta}{4}. \end{aligned}$$

Again this contradicts the DPP (3.21). $\square$

We remark again that the comparison principle for HJB master equation (3.22) is quite challenging and we shall leave it for future research.

REMARK 5.9. Under nice conditions, in particular when the comparison principle for the master equation (3.22) holds, by Proposition 5.3 and Theorem 5.8 we see that $V = V_2'$, for the V_2' defined by (5.1) and Proposition 5.3(ii). This is well known for standard control problems, and is also known in state dependent McKean–Vlasov setting; see Lacker [25].

However, for zero-sum games, the open-loop controls and closed-loop controls are quite different; see, for example, Pham and Zhang [35], Sirbu [40] and Possamai, Touzi and Zhang [36] in the standard setting. While in this paper we consider only the control problem, we expect our arguments will work for zero-sum game problems with closed-loop controls in McKean–Vlasov setting. We note that such game problem is studied in recent work Cosso and Pham [15] by using strategy versus open-loop controls.

REMARK 5.10. The restriction to piecewise constant controls makes it essentially impossible to obtain optimal controls. As we understand such restriction is mainly for the regularity of the value function V. In Possamai, Touzi and Zhang [36], we studied the zero sum games under general closed-loop controls (but without involving the measures) and faced similar regularity issues. However, in [36] we obtained the desired regularity when b and σ do not depend on the path and then proved the verification theorem for optimal controls. It will be interesting to remove the piecewise constant constraint in this framework when b and σ do not depend on μ.

5.3. *Regularity of V.* In this subsection we prove Theorem 5.7. To simplify the notation, in this subsection we assume $d = 1$. But the proof can be easily extended to the multidimensional case. Introduce

$$V_0(t,\mu) := \sup_{\alpha\in\mathcal{A}_t^0} J(t,\mu,\alpha) \quad \text{where } J \text{ is defined in (3.20) and}$$

$$\mathcal{A}_t^0 := \Big\{\alpha = \sum_{i=0}^{n-1} h_i 1_{[t_i,t_{i+1})} \in \mathcal{A}_t : \text{there exist } 0 \le s_1 < \cdots < s_m \le t \\ \text{such that } h_i = h_i(X_{s_1},\dots,X_{s_m},X_{[t,t_i]}) \text{ for } i = 0,\dots,n-1\Big\}. \tag{5.12}$$

That is, h_i depends on $X_{[0,t]}$ only discretely. Since $\mathcal{A}_t^0 \subset \mathcal{A}_t$, clearly $V_0(t,\mu) \le V(t,\mu)$. We will actually prove $V_0 = V$, then it suffices to establish the regularity of V_0.

To see the idea, let us first observe the following simple fact. Given an arbitrary probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a random variable ζ with continuous distribution, then for any other random variable $\tilde{\zeta}$, there exists a deterministic function φ such that

$$\mathcal{L}_{\varphi(\zeta)} = \mathcal{L}_{\tilde{\zeta}}, \tag{5.13}$$

where $\mathcal{L}$ denotes the distribution under $\tilde{\mathbb{P}}$. Indeed, denoting by F the cumulative distribution function, then $\varphi := F_{\tilde{\zeta}}^{-1} \circ F_\zeta$ serves our purpose. In Example 5.6, assume $\mathcal{L}_\zeta = \mu_0$ and $\mathcal{L}_{\tilde{\zeta}} = \mu_\varepsilon$. The discontinuity of V_0^0 at μ_0 is exactly because there is no function φ such that (5.13) holds. The next lemma is crucial for overcoming such difficulty. Recall the $\mathcal{P}(\mu,\nu)$ and the product space $(\Omega\times\Omega,\mathcal{F}\times\mathcal{F})$ in (2.1), and denote the canonical process as (X,X'). Moreover, for a partition π: $0 \le s_1 < \cdots < s_m \le t$, $\mu \in \mathcal{P}_2$, and two processes ξ, η on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, we introduce the notation

$$\mu_\pi := \mu \circ (X_{s_1}, \dots, X_{s_m})^{-1}, \qquad \xi_\pi := (\xi_{s_1}, \dots, \xi_{s_m}),$$

$$\|\xi - \eta\|_{\tilde{\mathbb{P}},\pi} := \|\xi_\pi - \eta_\pi\|_{\tilde{\mathbb{P}}} := \Big(\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\le j\le n} |\xi_{s_j} - \eta_{s_j}|^2\Big]\Big)^{\frac{1}{2}}. \tag{5.14}$$

LEMMA 5.11. *Let $0 < t < T$, $\mu, \nu \in \mathcal{P}_2$, $\overline{\mathbb{P}} \in \mathcal{P}(\mu, \nu)$. Then for any $\varepsilon > 0$, $\delta > 0$, and any partition $\pi : 0 \le s_1 < \cdots < s_m \le t$, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, two continuous processes (ξ, η), and a Brownian motion $\tilde{B}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that*:

(i) *$\mathcal{L}_\xi = \mu$, $\mathcal{L}_\eta = \nu$, and η is independent of $\tilde{B}$*;
(ii) *ξ_π is measurable to the σ-algebra $\sigma(\eta_\pi, \tilde{B}_{[0,\delta]})$.*
(iii) *$\|\xi - \eta\|_{\tilde{\mathbb{P}},\pi} \le \|X - X'\|_{\overline{\mathbb{P}},\pi} + \varepsilon$.*

PROOF. We prove the lemma in several cases, depending on the joint distribution ν_π. Fix an arbitrary process η with $\mathcal{L}_\eta = \nu$. Note that we shall extend the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ whenever needed, and we still denote this process as η.

Case 1. ν_π is degenerate, namely $\nu_\pi = \delta_{\{(x_1,\dots,x_m)\}}$ for some $(x_1, \dots, x_m) \in \mathbb{R}^m$, and thus $\eta_{s_j} = x_j$, $\tilde{\mathbb{P}}$-a.s. Pick a Brownian motion $\{\tilde{B}_s\}_{s\in[0,\delta]}$ independent of η (which is always doable by extending the probability space if necessary). In the spirit of (5.13), one can easily construct a m-dimensional random vector $\tilde{\xi}_\pi = (\tilde{\xi}_1, \dots, \tilde{\xi}_m)$ such that $\mathcal{L}_{\tilde{\xi}_\pi} = \mu_\pi$ and $\tilde{\xi}_\pi$ is measurable to the σ-algebra $\sigma(\tilde{B}_{\frac{(j-1)\delta}{m}, \frac{j\delta}{m}}, j = 1, \dots, m) \subset \sigma(\tilde{B}_{[0,\delta]})$. Moreover, by otherwise extending the probability space further, it is straightforward to extend $\tilde{\xi}_\pi$ to a continuous process ξ such that $\mathcal{L}_\xi = \mu$ and $\xi_{s_j} = \tilde{\xi}_{s_j}$, $j = 1, \dots, m$, $\tilde{\mathbb{P}}$-a.s. Finally, since ν_π is degenerate, we have

$$\|\xi - \eta\|^2_{\tilde{\mathbb{P}},\pi} = \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\le j\le n} |\xi_{s_j} - x_j|^2\Big] = \mathbb{E}^{\mu}\Big[\max_{1\le j\le n} |X_{s_j} - x_j|^2\Big] = \|X - X'\|^2_{\overline{\mathbb{P}},\pi}.$$

This verifies all the requirements in (i)–(iii).

Case 2. ν_π is discrete, namely $\nu_\pi = \sum_{i\ge1} p_i \delta_{\{(x_1^i,\dots,x_m^i)\}}$, with $p_i > 0$ and $\sum_{i\ge1} p_i = 1$. Fix a partition $\{O_i\}_{i\ge1} \subset \mathcal{B}(\mathbb{R}^m)$ of $\mathbb{R}^m$ such that $(x_1^i, \dots, x_m^i) \in O_i$. Let $\tilde{B}^i_{[0,\delta]}$ be a sequence of independent Brownian motions such that they are all independent of η. For each i, define a conditional probability:

$$\mu^i(E) := \frac{1}{p_i}\overline{\mathbb{P}}\big(X_\pi \in E, X'_\pi \in O_i\big), \quad E \in \mathcal{B}(\mathbb{R}^m).$$

Then by Case 1, one may construct a random vector $\tilde{\xi}^i_\pi = \varphi_i(\tilde{B}^i_{[0,\delta]})$ measurable to $\sigma(\tilde{B}^i_{[0,\delta]})$ such that $\mathcal{L}_{\tilde{\xi}^i_\pi} = \mu^i$. Define

$$\tilde{B} := \sum_{i\ge1} \tilde{B}^i 1_{O_i}(\eta_\pi), \qquad \tilde{\xi}_\pi := \sum_{i\ge1} \tilde{\xi}^i_\pi 1_{O_i}(\eta_\pi).$$

We now verify the desired properties. First, since all $\tilde{B}^i$ are independent of η, then $\tilde{B}$ is also a $\tilde{\mathbb{P}}$-Brownian motion. Moreover, for any $\tilde{\pi} : 0 = t_0 < \cdots < t_n = \delta$ and any $E, \tilde{E} \in \mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}\tilde{\mathbb{P}}(\tilde{B}_{\tilde{\pi}} \in E, \eta_{\tilde{\pi}} \in \tilde{E}) &= \sum_{i\ge1} \tilde{\mathbb{P}}\big(\tilde{B}^i_{\tilde{\pi}} \in E, \eta_{\tilde{\pi}} \in \tilde{E}, \eta_\pi \in O_i\big) \\ &= \sum_{i\ge1} \tilde{\mathbb{P}}\big(\tilde{B}^i_{\tilde{\pi}} \in E\big)\tilde{\mathbb{P}}(\eta_{\tilde{\pi}} \in \tilde{E}, \eta_\pi \in O_i)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\geq 1} \tilde{\mathbb{P}}(\tilde{B}_{\tilde{\pi}} \in E)\tilde{\mathbb{P}}(\eta_{\tilde{\pi}} \in \tilde{E}, \eta_\pi \in O_i) \\
&= \tilde{\mathbb{P}}(\tilde{B}_{\tilde{\pi}} \in E)\tilde{\mathbb{P}}(\eta_{\tilde{\pi}} \in \tilde{E}).
\end{aligned}
$$

That is, $\tilde{B}$ is also independent of η. Next, since O_i is a partition, we see that $\tilde{\xi}_\pi := \sum_{i\geq 1} \varphi_i(\tilde{B}^i_{[0,\delta]})1_{O_i}(\eta_\pi) = \sum_{i\geq 1} \varphi_i(\tilde{B}_{[0,\delta]})1_{O_i}(\eta_\pi)$ and thus $\tilde{\xi}_\pi$ is measurable to $\sigma(\eta_\pi, \tilde{B}_{[0,\delta]})$. Moreover, note that $\tilde{\xi}^i$s are also independent of η, then

$$
\begin{aligned}
\tilde{\mathbb{P}}(\tilde{\xi}_\pi \in E) &= \sum_{i\geq 1} \tilde{\mathbb{P}}(\tilde{\xi}^i_\pi \in E, \eta_\pi \in O_i) = \sum_{i\geq 1} \tilde{\mathbb{P}}(\tilde{\xi}^i_\pi \in E)\tilde{\mathbb{P}}(\eta_\pi \in O_i) \\
&= \sum_{i\geq 1} \mu^i(E)p_i = \sum_{i\geq 1} \overline{\mathbb{P}}(X_\pi \in E, X'_\pi \in O_i) \\
&= \overline{\mathbb{P}}(X_\pi \in E) = \mu(X_\pi \in E).
\end{aligned}
$$

That is, $\mathcal{L}_{\tilde{\xi}_\pi} = \mu_\pi$. Then similar to Case 1, by extending the space if necessary, we may construct ξ such that $\mathcal{L}_\xi = \mu$ and $\xi_\pi = \tilde{\xi}_\pi$, $\tilde{\mathbb{P}}$-a.s. Finally,

$$
\begin{aligned}
\|\xi - \eta\|^2_{\tilde{\mathbb{P}},\pi} &= \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\xi_{s_j} - \eta_{s_j}|^2\Big] = \sum_{i\geq 1} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\xi_{s_j} - \eta_{s_j}|^2 1_{O_i}(\eta_\pi)\Big] \\
&= \sum_{i\geq 1} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\tilde{\xi}^i_{s_j} - x^i_j|^2 1_{O_i}(\eta_\pi)\Big] \\
&= \sum_{i\geq 1} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\tilde{\xi}^i_{s_j} - x^i_j|^2\Big]\tilde{\mathbb{P}}(\eta_\pi \in O_i) \\
&= \sum_{i\geq 1} \mathbb{E}^{\mu^i}\Big[\max_{1\leq j\leq n} |X_{s_j} - x^i_j|^2\Big]p_i = \sum_{i\geq 1} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |X_{s_j} - x^i_j|^2 1_{O_i}(X'_\pi)\Big] \\
&= \sum_{i\geq 1} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |X_{s_j} - X'_{s_j}|^2 1_{O_i}(X'_\pi)\Big] = \mathbb{E}^{\overline{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |X_{s_j} - X'_{s_j}|^2\Big] \\
&= \|X - X'\|^2_{\overline{\mathbb{P}},\pi}.
\end{aligned}
$$

Case 3. We now consider the general case. Let $\{O_i\}$ be a countable partition of $\mathbb{R}^m$ such that for each i, the diameter of O_i is less than $\varepsilon/2$. For each i, fix an arbitrary $x^i \in O_i$ and denote $p_i := \nu_\pi(O_i)$. By otherwise eliminating some i, we may assume $p_i > 0$ for all i. Denote $\tilde{\eta}_\pi := \sum_{i\geq 1} x^i 1_{O_i}(\eta_\pi)$ and $\tilde{X}'_\pi := \sum_{i\geq 1} x^i 1_{O_i}(X'_\pi)$. By Case 2, there exist a $\tilde{\mathbb{P}}$-Brownian motion $\tilde{B}_{[0,\delta]}$ and a continuous process ξ such that:

- $\mathcal{L}_\xi = \mu$ and $\tilde{B}$ is independent of $\tilde{\eta}_\pi$. Moreover, from the arguments we may assume further that $\tilde{B}$ is independent of η;
- Each ξ_{s_j} is measurable to $\sigma(\tilde{\eta}_\pi, \tilde{B}_{[0,\delta]}) \subset \sigma(\eta_\pi, \tilde{B}_{[0,\delta]})$;
- $\mathbb{E}^{\tilde{\mathbb{P}}}[\max_{1\leq j\leq m} |\xi_{s_j} - \tilde{\eta}_{s_j}|^2] = \mathbb{E}^{\overline{\mathbb{P}}}[\max_{1\leq j\leq m} |X_{s_j} - \tilde{X}'_{s_j}|^2]$.

This verifies (i) and (ii). To see (iii), note that $|\eta_{s_j} - \tilde{\eta}_{s_j}| \leq \frac{\varepsilon}{2}$, $|X'_{s_j} - \tilde{X}'_{s_j}| \leq \frac{\varepsilon}{2}$. Then

$$
\begin{aligned}
\|\xi - \eta\|_{\tilde{\mathbb{P}},\pi} &= \Big(\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\xi_{s_j} - \eta_{s_j}|^2\Big]\Big)^{\frac{1}{2}} \\
&\leq \Big(\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\max_{1\leq j\leq n} |\xi_{s_j} - \tilde{\eta}_{s_j}|^2\Big]\Big)^{\frac{1}{2}} + \frac{\varepsilon}{2}
\end{aligned}
$$

$$= \Big(\mathbb{E}^{\overline{\mathbb{P}}}\Big[\max_{1\le j\le m}|X_{s_j} - \tilde{X}'_{s_j}|^2\Big]\Big)^{\frac{1}{2}} + \frac{\varepsilon}{2}$$

$$\le \Big(\mathbb{E}^{\overline{\mathbb{P}}}\Big[\max_{1\le j\le m}|X_{s_j} - X'_{s_j}|^2\Big]\Big)^{\frac{1}{2}} + \varepsilon = \|X - X'\|_{\overline{\mathbb{P}},\pi} + \varepsilon.$$

This completes the proof. □

REMARK 5.12.

(i) As mentioned right before the lemma, the main difficulty of establishing the regularity of V at ν is due to the possible degeneracy of ν, and thus in the above lemma one may not be able to write ξ as a function of η. Our trick here is to introduce the independent Brownian motion $\tilde{B}_{[0,\delta]}$ (which always has continuous distribution) and then Lemma 5.11(ii) holds.

(ii) The construction of ξ, which relies on (5.13), works only for finite dimensional random vectors. It is not clear to us how to generalize this result to the case where the m-tuple $(s_1,\dots,s_m)$ is replaced by the uncountable interval $[0,t]$. This is why we need to consider value function V_0 first.

LEMMA 5.13. *Under Assumption 5.1, V_0 is uniformly continuous in μ, uniformly in t. That is, there exists a modulus of continuity function ρ such that, for all $t\in[0,T]$, $\mu,\nu\in\mathcal{P}_2$,*

$$|V_0(t,\mu) - V_0(t,\nu)| \le \rho\big(\mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]})\big). \tag{5.15}$$

PROOF. Let us fix $t\in[0,T]$, $\mu,\nu\in\mathcal{P}_2$, $\alpha\in\mathcal{A}^0_t$, and $\varepsilon,\delta>0$. Choose $\overline{\mathbb{P}}\in\mathcal{P}(\mu,\nu)$ such that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X_{t\wedge\cdot} - X'_{t\wedge\cdot}\|^2\big] \le c_0^2 + \varepsilon^2 \quad \text{where } c_0 := \mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]}). \tag{5.16}$$

Our idea is to construct some $\tilde{\alpha}\in\mathcal{A}^0_t$ such that $\mathbb{P}^{t,\nu,\tilde{\alpha}}$ is close to $\mathbb{P}^{t,\mu,\alpha}$ in certain way.

By (5.12), we assume $\alpha = \sum_{i=0}^{n-1} h^0_i(X_{\pi_0}, X_{[t,t_i]})\mathbf{1}_{[t_i,t_{i+1}]}$, where $\pi_0: 0\le s^0_1<\dots<s^0_{m_0} = t$ and $t = t_0<\dots<t_n = T$. We shall fix n, and assume $\delta<\min_{1\le i\le n}[t_i - t_{i-1}]$. But to obtain a desired approximation, we shall consider finer partitions $\pi: 0\le s_1<\dots<s_m = t$ such that $\pi_0\subset\pi$. Clearly, we may rewrite $\alpha = \sum_{i=0}^{n-1} h_i(X_\pi, X_{[t,t_i]})\mathbf{1}_{[t_i,t_{i+1}]}$. Let $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, $\tilde{B}$, ξ and η be as in Lemma 5.11, corresponding to $(t,\mu,\nu,\pi,\varepsilon,\delta,\overline{\mathbb{P}})$. Denote $B'_s := \tilde{B}_{s-t}$, $B^\delta_s := \tilde{B}_{\delta,s-t+\delta}$, $s\in[t,T]$. It is clear that $\mathbb{P}^{t,\mu,\alpha} = \tilde{\mathbb{P}}\circ(X^\alpha)^{-1}$, where $X^\alpha_{[0,t]} = \xi_{[0,t]}$ and, for $i = 0,\dots,n-1$ and $s\in(t_i,t_{i+1}]$,

$$\begin{aligned} X^\alpha_s = X^\alpha_{t_i} &+ \int_{t_i}^s b\big(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,dr \\ &+ \int_{t_i}^s \sigma\big(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,dB^\delta_r. \end{aligned} \tag{5.17}$$

Step 1. We first construct $\tilde{\alpha}\in\mathcal{A}^0_t$ and $\tilde{X} := X^{t,\eta,\tilde{\alpha}}$ with $\tilde{X}_{[0,t]} := \eta_{[0,t]}$ and

$$d\tilde{X}_s := b(s,\tilde{X},\mathcal{L}_{\tilde{X}},\tilde{\alpha}_s)\,ds + \sigma(s,\tilde{X},\mathcal{L}_{\tilde{X}},\tilde{\alpha}_s)\,dB'_s, \quad s\ge t, \tilde{\mathbb{P}}\text{-a.s.} \tag{5.18}$$

The corresponding partitions for $\tilde{\alpha}$ will be π and $t = t_0 < t_0+\delta < t_1+\delta<\dots< t_{n-1}+\delta < t_n = T$. First, fix an arbitrary $a_0\in A$ and set $\tilde{\alpha}_s := a_0$ for $s\in[t_0,t_0+\delta)$. Then we may determine $\tilde{X}$ on $[t_0,t_0+\delta]$ by (5.18) with initial condition $\tilde{X}_{t_0} = \eta_{t_0}$. Since the SDE (5.18) has a strong solution and σ is nondegenerate, we know the σ-algebras $\sigma(\eta_\pi,\tilde{B}_{[0,\delta]}) = \sigma(\eta_\pi,\tilde{X}_{[t,t_0+\delta]})$ (abusing the notation σ here!). Then, by Lemma 5.11(ii), $\xi_\pi = \varphi(\eta_\pi,\tilde{X}_{[t,t_0+\delta]})$ for some function φ. Set $\tilde{h}_0(\eta_\pi,\tilde{X}_{[t,t_0+\delta]}) := h_0(\varphi(\eta_\pi,\tilde{X}_{[t,t_0+\delta]})) =$

$h_0(\xi_\pi)$. Then, for $s \in [t_0+\delta, t_1+\delta)$, setting $\tilde{\alpha}_s := \tilde{h}_0(\eta_\pi, \tilde{X}_{[t,t_0+\delta]}) = h_0(\xi_\pi)$, we may determine $\tilde{X}$ further on $[t_0+\delta, t_1+\delta]$ by (5.18). Next, again since σ is nondegenerate, we see that $X^\alpha_{[t,t_1]}$ is measurable to

$$\sigma\big(\xi_\pi, B^\delta_{[t_0,t_1]}\big) \subset \sigma\big(\eta_\pi, \tilde{B}_{[0,\delta+t_1-t_0]}\big) \subset \sigma\big(\eta_\pi, \tilde{X}_{[t,t_1+\delta]}\big).$$

Then $h_1(\xi_\pi, X^\alpha_{[t,t_1]}) = \tilde{h}_1(\eta_\pi, \tilde{X}_{[t,t_1+\delta]})$ for some function $\tilde{h}_1$. For $s \in (t_1+\delta, t_2+\delta]$, set $\tilde{\alpha}_s := \tilde{h}_1(\eta_\pi, \tilde{X}_{[t,t_1+\delta]}) = h_1(\xi_\pi, X^\alpha_{[t,t_1]})$. Repeat the arguments, we may construct $\tilde{\alpha} \in \mathcal{A}^0_t$ such that, for the corresponding $\tilde{X}$ determined by (5.18),

$$\tilde{\alpha}_s = \tilde{h}_i(\eta_\pi, \tilde{X}_{[t,t_i+\delta]}) = h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big), \quad s \in [t_i+\delta, t_{i+1}+\delta). \tag{5.19}$$

Step 2. We next estimate the difference between X^α and $\tilde{X}$. Denote

$$\begin{aligned} &\Delta X_s := \tilde{X}_s - X^\alpha_s, \qquad \Delta_\delta X_s := \tilde{X}_{s+\delta} - X^\alpha_s, \\ &\mathrm{OSC}_\delta(\tilde{X}) := \sup_{t \le r_1 < r_2 \le T, r_2 - r_1 \le \delta} |\tilde{X}_{r_1,r_2}|. \end{aligned} \tag{5.20}$$

Note that, for $s \in [t, T-\delta]$,

$$\|\tilde{X}_{(s+\delta)\wedge\cdot} - X^\alpha_{s\wedge\cdot}\| \le \|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\| + \sup_{t\le r\le s} |\Delta_\delta X_r| + \mathrm{OSC}_\delta(\tilde{X}).$$

By Assumption 5.1(i) and (iii), for $\varphi = b, \sigma, f$, and $s \in [t_i, t_{i+1}]$,

$$\begin{aligned} &\mathbb{E}^{\tilde{\mathbb{P}}}\big[\big|\varphi\big(s+\delta, \tilde{X}, \mathcal{L}_{\tilde{X}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big) - \varphi\big(s, X^\alpha, \mathcal{L}_{X^\alpha}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\big|^2\big] \\ &\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\big|\varphi\big(s+\delta, \tilde{X}, \mathcal{L}_{\tilde{X}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big) \\ &\qquad - \varphi\big(s+\delta, X^\alpha_{s\wedge\cdot}, \mathcal{L}_{X^\alpha_{[0,s]}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\big|^2\big] \\ &\qquad + C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\big|\varphi\big(s+\delta, X^\alpha_{s\wedge\cdot}, \mathcal{L}_{X^\alpha_{[0,s]}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big) \\ &\qquad - \varphi\big(s, X^\alpha, \mathcal{L}_{X^\alpha}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\big|^2\big] \\ &\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\tilde{X}_{(s+\delta)\wedge\cdot} - X^\alpha_{s\wedge\cdot}\|^2 + \big[1 + \|X^\alpha_{s\wedge\cdot}\|^2\big]\rho_0^2(\delta)\big] \\ &\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\|^2 + \sup_{t\le r\le s} |\Delta_\delta X_r|^2 + \mathrm{OSC}^2_\delta(\tilde{X})\Big] + C_\mu \rho_0^2(\delta). \end{aligned} \tag{5.21}$$

Note that we may rewrite (5.18) as, for $s \in [t_i, t_{i+1}]$,

$$\begin{aligned} \tilde{X}_{s+\delta} = \tilde{X}_{t_i+\delta} &+ \int_{t_i}^s b\big(r+\delta, \tilde{X}, \mathcal{L}_{\tilde{X}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,dr \\ &+ \int_{t_i}^s \sigma\big(r+\delta, \tilde{X}, \mathcal{L}_{\tilde{X}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,dB^\delta_r. \end{aligned}$$

Compare this with (5.17), then it follows from standard arguments that

$$\begin{aligned} &\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{t\le s\le T-\delta} |\Delta_\delta X_s|^2\Big] \\ &\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[|\Delta_\delta X_t|^2 + \|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\|^2 + \mathrm{OSC}^2_\delta(\tilde{X})\big] + C_\mu \rho_0^2(\delta). \end{aligned} \tag{5.22}$$

Note that, for $s \in [t, T-\delta]$,

$$|\Delta_\delta X_t| = |\tilde{X}_{t+\delta} - \tilde{X}_t| + |\xi_t - \eta_t| \le \|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\| + \mathrm{OSC}_\delta(\tilde{X});$$

$$
\begin{aligned}
|\Delta X_s| &\le |\Delta_\delta X_s| + \mathrm{OSC}_\delta(\tilde{X}), \quad s \in [t, T-\delta]; \\
|\Delta X_s| &\le |\Delta_\delta X_{T-\delta}| + |\tilde{X}_s - \tilde{X}_T| + \big|X^\alpha_s - X^\alpha_{T-\delta}\big| \\
&\le |\Delta_\delta X_{T-\delta}| + \mathrm{OSC}_\delta(\tilde{X}) + \mathrm{OSC}_\delta\big(X^\alpha\big), \quad s \in [T-\delta, T],
\end{aligned}
$$

where $\mathrm{OSC}_\delta(X^\alpha)$ is defined similar to (5.20). Then (5.22) leads to

$$
\begin{aligned}
&\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{t\le s\le T} |\Delta X_s|^2\Big] \\
&\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\|^2 + \mathrm{OSC}^2_\delta(\tilde{X}) + \mathrm{OSC}^2_\delta\big(X^\alpha\big)\big] + C_\mu \rho_0^2(\delta).
\end{aligned}
$$

Since $|b|, |\sigma| \le C_0$, by Revuz and Yor ([38], Chapter I, Theorem 2.1) one can easily see that

$$
\mathbb{E}^{\tilde{\mathbb{P}}}\big[\big|\mathrm{OSC}_\delta(\tilde{X})\big|^2 + \big|\mathrm{OSC}_\delta\big(X^\alpha\big)\big|^2\big] \le C\sqrt{\delta} \le C\rho_0^2(\delta). \tag{5.23}
$$

Here we assume without loss of generality that $\rho_0(\delta) \ge \delta^{\frac14}$ (otherwise replace ρ_0 with $\rho_0(\delta) \vee \delta^{\frac14}$). Then, noting that $\|\Delta X\| \le \sup_{t\le s\le T} |\Delta X_s| + \|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\|$,

$$
\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\Delta X\|^2\big] \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\|^2\big] + C_\mu \rho_0^2(\delta). \tag{5.24}
$$

Step 3. We now estimate $V_0(t,\mu) - V_0(t,\nu)$. By Assumption 5.1(ii) and (iii), we have

$$
\begin{aligned}
&J(t,\mu,\alpha) - V_0(t,\nu) \\
&\quad \le J(t,\mu,\alpha) - J(t,\nu,\tilde{\alpha}) \\
&\quad = \mathbb{E}^{\tilde{\mathbb{P}}}\Bigg[g\big(X^\alpha, \mathcal{L}_{X^\alpha}\big) - g(\tilde{X}, \mathcal{L}_{\tilde{X}}) \\
&\qquad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f\big(s, X^\alpha, \mathcal{L}_{X^\alpha}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,ds \\
&\qquad - \sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}} f\big(s+\delta, \tilde{X}, \mathcal{L}_{\tilde{X}}, h_i\big(\xi_\pi, X^\alpha_{[t,t_i]}\big)\big)\,ds \\
&\qquad - \int_t^{t+\delta} f(s, \tilde{X}, \mathcal{L}_{\tilde{X}}, a_0)\,ds - \int_{T-\delta}^{T} f(s, \tilde{X}, \mathcal{L}_{\tilde{X}}, \tilde{\alpha}_s)\,ds\Bigg] \\
&\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\rho_0\big(\|\Delta X\| + \mathcal{W}_2(\mathcal{L}_{X^\alpha}, \mathcal{L}_{\tilde{X}})\big) \\
&\qquad + \big[1 + \|\tilde{X}\| + \big\|X^\alpha\big\| + \mathcal{W}_2(\mathcal{L}_{\tilde{X}}, \delta_{\{0\}}) + \mathcal{W}_2(\mathcal{L}_{X^\alpha}, \delta_{\{0\}})\big]\rho_0(\delta)\big] \\
&\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\rho_0\big(\|\Delta X\| + \mathcal{W}_2(\mathcal{L}_{X^\alpha}, \mathcal{L}_{\tilde{X}})\big)\big] + C_{\mu,\nu}\rho_0(\delta).
\end{aligned} \tag{5.25}
$$

Note that we may assume without loss of generality that ρ_0 has linear growth. Then,

$$
\begin{aligned}
&J(t,\mu,\alpha) - V_0(t,\nu) \\
&\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[\rho_0\big(\|\Delta X\| + \mathcal{W}_2(\mathcal{L}_{X^\alpha}, \mathcal{L}_{\tilde{X}})\big)\mathbf{1}_{\{\|\Delta X\| > c_0\}}\big] \\
&\qquad + C\rho_0\big(c_0 + \mathcal{W}_2(\mathcal{L}_{X^\alpha}, \mathcal{L}_{\tilde{X}})\big) + C_{\mu,\nu}\rho_0(\delta) \\
&\quad \le \frac{C}{c_0}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\Delta X\|^2\big] + C\rho_0\big(c_0 + \big(\mathbb{E}^{\tilde{\mathbb{P}}}\big[\|\Delta X\|^2\big]\big)^{\frac12}\big) + C_{\mu,\nu}\rho_0(\delta),
\end{aligned} \tag{5.26}
$$

where c_0 is defined by (5.16). Note further that, denoting $|\pi| := \min_{1\le j\le m} |s_j - s_{j-1}|$,

$$
\|\xi_{t\wedge\cdot} - \eta_{t\wedge\cdot}\| \le \mathrm{OSC}_{|\pi|}(\xi_{[0,t]}) + \mathrm{OSC}_{|\pi|}(\eta_{[0,t]}) + \max_{0\le j\le m} |\xi_{s_j} - \eta_{s_j}|.
$$

Plug this into (5.24); by Lemma 5.11(iii) and (5.16) we have

$$
\begin{aligned}
&\mathbb{E}^{\tilde{\mathbb{P}}}[\|\Delta X\|^2] \\
&\quad \le C\mathbb{E}^{\tilde{\mathbb{P}}}[\mathrm{OSC}^2_{|\pi|}(\xi_{[0,t]}) + \mathrm{OSC}^2_{|\pi|}(\eta_{[0,t]})] + \|\xi - \eta\|^2_{\tilde{\mathbb{P}},\pi} + C_\mu \rho_0^2(\delta) \\
&\quad \le C\big[\mathbb{E}^\mu[\mathrm{OSC}^2_{|\pi|}(X_{[0,t]})] + \mathbb{E}^\nu[\mathrm{OSC}^2_{|\pi|}(X_{[0,t]})] + \|X_{t\wedge\cdot} - X'_{t\wedge\cdot}\|^2_{\tilde{\mathbb{P}},\pi} + \varepsilon^2\big] \\
&\qquad + C_\mu \rho_0^2(\delta) \\
&\quad \le C\big[\mathbb{E}^\mu[\mathrm{OSC}^2_{|\pi|}(X_{[0,t]})] + \mathbb{E}^\nu[\mathrm{OSC}^2_{|\pi|}(X_{[0,t]})] + c_0^2 + \varepsilon^2\big] + C_\mu \rho_0^2(\delta).
\end{aligned}
$$

Plug this into (5.26), and note that α depends on π_0, but not π. Then, by sending $\delta \to 0$, $\varepsilon \to 0$ and $|\pi| \to 0$, we obtain

$$
\begin{aligned}
J(t,\mu,\alpha) - V_0(t,\nu) &\le \frac{C}{c_0}\mathcal{W}_2^2(\mu_{[0,t]},\nu_{[0,t]}) + C\rho_0\big(c_0 + \mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]})\big) \\
&= C\mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]}) + C\rho_0\big(2\mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]})\big).
\end{aligned}
$$

Now by the arbitrariness of $\alpha \in \mathcal{A}_t^0$, we obtain

$$
V_0(t,\mu) - V_0(t,\nu) \le C\mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]}) + C\rho_0\big(2\mathcal{W}_2(\mu_{[0,t]},\nu_{[0,t]})\big).
$$

Following the same arguments we also have the estimate for $V_0(t,\nu) - V_0(t,\mu)$, and thus complete the proof. $\square$

LEMMA 5.14. *Under Assumption 5.1, we have $V = V_0$.*

PROOF. By definition, it is clear that $V_0 \le V$. To prove the opposite inequality, we fix $(t,\mu) \in \Theta$ and $\alpha := \sum_{i=0}^{n-1} h_i(X_{[0,t_i]})1_{[t_i,t_{i+1})} \in \mathcal{A}_t$ as in (5.3). Fix $\mathbb{P}_0$, B, and ξ being such that $\mathbb{P}_0 \circ (\xi_{[0,t]})^{-1} = \mu$, and let $X^\alpha = X^{t,\xi,\alpha}$ be defined by (3.17). We shall prove $J(t,\mu,\alpha) \le V_0(t,\mu)$ in two steps.

Step 1. We first assume all the functions $h_i : C([0,t_i]) \to \mathbb{R}$ are continuous. For each $m \ge 1$, consider the partition $\pi_m : 0 = s_0^m < \cdots < s_m^m = t$ be such that $s_i^m = \frac{i}{m}t$. Define

$$
\begin{aligned}
&h_i^m(\eta_{\pi_m}, \eta_{[t,t_i]}) := h_i\big(\eta^m_{[0,t_i]}\big) \\
&\qquad \text{where } \eta^m_{[0,t]} \text{ is the linear interpolation of } \eta_{\pi_m} \text{ and } \eta^m_{[t,t_i]} := \eta_{[t,t_i]}.
\end{aligned}
\tag{5.27}
$$

Denote $\alpha^m := \sum_{i=0}^{n-1} h_i^m(X_{\pi_m}, X_{[t,t_i]})1_{[t_i,t_{i+1})} \in \mathcal{A}_t^0$, and define $X^m := X^{t,\xi,\alpha^m}$ in an obvious way. We shall estimate $\Delta X^m := X^m - X^\alpha$.

Clearly $\Delta X_s^m = 0$ for $s \in [0,t]$. For $s \in [t_0,t_1]$, we have

$$
\begin{aligned}
X_s^\alpha &= \xi_t + \int_{t_0}^s b(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_0(\xi_{[0,t]})\,dr + \int_{t_0}^s \sigma(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_0(\xi_{[0,t]})\,dB_r; \\
X_s^m &= \xi_t + \int_{t_0}^s b\big(r, X^m, \mathcal{L}_{X^m}, h_0^m(\xi_{\pi_m}, \xi_t)\big)\,dr \\
&\quad + \int_{t_0}^s \sigma\big(r, X^m, \mathcal{L}_{X^m}, h_0^m(\xi_{\pi_m}, \xi_t)\big)\,dB_r.
\end{aligned}
$$

Since h_0 is continuous, it is clear that

$$
\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\big|h_0^m(\xi_{\pi_m}, \xi_t) - h_0(\xi_{[0,t]})\big|^2 \wedge 1\big] = 0.
$$

By Assumption 5.1(i) and (iii), it follows from standard arguments that

$$
\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X^m_{t_1\wedge\cdot}\|^2\big] = 0. \tag{5.28}
$$

Next, for $s \in [t_1, t_2]$, we have

$$\begin{aligned}X_s^\alpha &= X_{t_1}^\alpha + \int_{t_1}^s b\big(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_1\big(\xi_{[0,t]}, X_{[t,t_1]}^\alpha\big)\big)\,dr \\ &\quad + \int_{t_1}^s \sigma\big(r, X^\alpha, \mathcal{L}_{X^\alpha}, h_1\big(\xi_{[0,t]}, X_{[t,t_1]}^\alpha\big)\,dB_r; \\ X_s^m &= X_{t_1}^m + \int_{t_1}^s b\big(r, X^m, \mathcal{L}_{X^m}, h_1^m\big(\xi_{\pi_m}, X_{[t,t_1]}^m\big)\big)\,dr \\ &\quad + \int_{t_0}^s \sigma\big(r, X^m, \mathcal{L}_{X^m}, h_1^m\big(\xi_{\pi_m}, X_{[t,t_1]}^m\big)\big)\,dB_r.\end{aligned}$$

Since h_1 is continuous, by (5.27) and (5.28) we have

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\big|h_1^m\big(\xi_{\pi_m}, X_{[t,t_1]}^m\big) - h_1\big(\xi_{[0,t]}, X_{[t,t_1]}^\alpha\big)\big|^2 \wedge 1\big] = 0.$$

Then, similar to (5.28) we have $\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}[\|\Delta X_{t_2\wedge\cdot}^m\|^2] = 0$. Repeating the arguments we obtain

$$\begin{aligned}&\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X^m\|^2\big] = 0, \\ &\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\big|h_i^m\big(\xi_{\pi_m}, X_{[t,t_i]}^m\big) - h_i\big(\xi_{[0,t]}, X_{[t,t_i]}^\alpha\big)\big|^2 \wedge 1\big] = 0, \quad i < n.\end{aligned}$$

Now by the regularity of f and g in Assumption 5.1(ii) and (iii), we have

$$J(t,\mu,\alpha) = \lim_{m\to\infty} J\big(t,\mu,\alpha^m\big) \le V_0(t,\mu). \tag{5.29}$$

Step 2. We now consider the general Borel measurable functions h_i. We shall construct $\alpha^m = \sum_{i=0}^{n-1} h_i^m(X_{[0,t_i]})\mathbf{1}_{[t_i,t_{i+1})} \in \mathcal{A}_t$ such that each h_i^m is continuous and, for the corresponding $X^m := X^{t,\xi,\alpha^m}$,

$$\begin{aligned}&\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\big|h_i^m\big(X_{[0,t_i]}^m\big) - h_i\big(X_{[0,t_i]}^\alpha\big)\big|^2 \wedge 1\big] = 0, \\ &\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X^m\|^2\big] = 0 \quad \text{where } \Delta X^m := X^m - X^\alpha.\end{aligned} \tag{5.30}$$

Then by Step 1 we have $J(t,x,\alpha^m) \le V_0(t,\mu)$, and similar to (5.29) we can easily show that $J(t,\mu,\alpha) = \lim_{m\to\infty} J(t,\mu,\alpha^m) \le V_0(t,\mu)$.

We now construct h_i^m recursively in i. First, denote $X_{[0,t]}^m := \xi_{[0,t]}$. Then $\|\Delta X_{t_0\wedge}^m\| = 0$. Assume by induction that we have constructed $X_{[0,t_i]}^m$ such that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X_{t_i\wedge\cdot}^m\|^2\big] = 0. \tag{5.31}$$

For $h_i : C([0,t_0]) \to A$, by Lusin's lemma, there exist continuous functions $\tilde{h}_i^m : C([0,t_i]) \to A$ and closed sets $K_i^m \subset C([0,t_i])$ such that

$$\tilde{h}_i^m = h_i \quad \text{on } K_i^m \quad \text{and} \quad \lim_{m\to\infty} \mathbb{P}_0\big(X_{[0,t_i]}^\alpha \notin K_i^m\big) = 0. \tag{5.32}$$

For each m, since $\tilde{h}_i^m$ is continuous, by (5.32) we have

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\big|\tilde{h}_i^m\big(X_{[0,t_i]}^k\big) - \tilde{h}_i^m\big(X_{[0,t_i]}^\alpha\big)\big|^2 \wedge 1\big] = 0.$$

Then there exists k_m such that

$$\mathbb{E}^{\mathbb{P}_0}\big[\big|\tilde{h}_i^m\big(X_{[0,t_i]}^{k_m}\big) - \tilde{h}_i^m\big(X_{[0,t_i]}^\alpha\big)\big|^2 \wedge 1\big] \le \frac{1}{m} \quad \forall m.$$

This implies that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\big[\big|\tilde h_i^m\big(X^{k_m}_{[0,t_i]}\big)-h_i\big(X^{\alpha}_{[0,t_i]}\big)\big|^2\wedge 1\big]\\
&\quad\le C\mathbb{E}^{\mathbb{P}_0}\big[\big|\tilde h_i^m\big(X^{k_m}_{[0,t_i]}\big)-\tilde h_i^m\big(X^{\alpha}_{[0,t_i]}\big)\big|^2\wedge 1\big]\\
&\qquad+C\mathbb{E}^{\mathbb{P}_0}\big[\big|\tilde h_i^m\big(X^{\alpha}_{[0,t_i]}\big)-h_i\big(X^{\alpha}_{[0,t_i]}\big)\big|^2\wedge 1\big]\\
&\quad\le \frac{C}{m}+C\mathbb{P}_0\big(X^{\alpha}_{[0,t_i]}\notin K_i^m\big)\to 0\quad \text{as } m\to\infty.
\end{aligned}$$

By considering the subsequence k_m and set $h_i^{k_m}:=\tilde h_i^m$, we obtain

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_0}\big[\big|h_i^{k_m}\big(X^{k_m}_{[0,t_i]}\big)-h_i\big(X^{\alpha}_{[0,t_i]}\big)\big|^2\wedge 1\big]=0. \tag{5.33}$$

By choosing the subsequence k_m, and for notational simplicity, we assume $k_m=m$, then we constructed the desired h_i^m under assumption (5.31).

Next, for $s\in[t_i,t_{i+1}]$ and for $\varphi=b,\sigma$, denote

$$\Delta\varphi_s^m:=\varphi\big(s,X^\alpha,\mathcal{L}_{X^\alpha},h_i^m\big(X^m_{[0,t_i]}\big)\big)-\varphi\big(s,X^\alpha,\mathcal{L}_{X^\alpha},h_i\big(X^\alpha_{[0,t_i]}\big)\big).$$

Since b and σ are bounded and uniformly Lipschitz continuous in (ω,μ), we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\big[\big|\varphi\big(s,X^m,\mathcal{L}_{X^m},h_i^m\big(X^m_{[0,t_i]}\big)\big)-\varphi\big(s,X^\alpha,\mathcal{L}_{X^\alpha},h_i\big(X^\alpha_{[0,t_i]}\big)\big|^2\big]\\
&\quad=\mathbb{E}^{\mathbb{P}_0}\big[\big|\big[\varphi\big(s,X^m,\mathcal{L}_{X^m},h_i^m\big(X^m_{[0,t_i]}\big)\big)\\
&\qquad-\varphi\big(s,X^\alpha,\mathcal{L}_{X^\alpha},h_i^m\big(X^m_{[0,t_i]}\big)\big)\big]+\Delta\varphi_s^m\big|^2\big]\\
&\quad\le C\mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X^m_{s\wedge\cdot}\|^2+|\Delta\varphi_s^m|^2\big].
\end{aligned}$$

Note that

$$\begin{aligned}
X_s^\alpha&=X_{t_i}^\alpha+\int_{t_i}^s b\big(r,X^\alpha,\mathcal{L}_{X^\alpha},h_i\big(X^\alpha_{[0,t_i]}\big)\big)\,dr\\
&\quad+\int_{t_i}^s\sigma\big(r,X^\alpha,\mathcal{L}_{X^\alpha},h_i\big(X^\alpha_{[0,t_i]}\big)\big)\,dB_r;\\
X_s^m&=X_{t_i}^m+\int_{t_i}^s b\big(r,X^m,\mathcal{L}_{X^m},h_i^m\big(X^m_{[0,t_i]}\big)\big)\,dr\\
&\quad+\int_{t_i}^s\sigma\big(r,X^m,\mathcal{L}_{X^m},h_i^m\big(X^m_{[0,t_i]}\big)\big)\,dB_r.
\end{aligned}$$

By standard arguments one can easily see that

$$\mathbb{E}^{\mathbb{P}_0}\big[\|\Delta X^m_{t_{i+1}\wedge\cdot}\|^2\big]\le C\mathbb{E}^{\mathbb{P}_0}\bigg[\|\Delta X^m_{t_i\wedge\cdot}\|^2+\int_{t_i}^{t_{i+1}}\big[|\Delta b_s^m|^2+|\Delta\sigma_s^m|^2\big]\,ds\bigg].$$

By (5.33) (with $k_m=m$) and the dominated convergence theorem, we have

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_0}\bigg[\int_{t_i}^{t_{i+1}}\big[|\Delta b_s^m|^2+|\Delta\sigma_s^m|^2\big]\,ds\bigg]=0.$$

This, together with (5.31), implies that $\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_0}[\|\Delta X^m_{t_{i+1}\wedge\cdot}\|^2]=0$. Then the induction procedure can continue, and by possibly choosing a subsequence, we construct the desired h_i^m satisfying (5.30) for all i, hence completing the proof. $\square$

PROOF OF THEOREM 5.7. First, by Lemmas 5.13 and 5.14, we see that V is uniformly continuous in μ with certain modulus of continuity function ρ. Now let $t_1 < t_2$ and $\mu, \nu \in \mathcal{P}_2$. By DPP (3.21) (see Remark 5.4(ii)) and noting that f has linear growth in (ω, μ), we have

$$\begin{aligned}
|V(t_1,\mu) - V(t_2,\nu)| &\le \sup_{\alpha\in\mathcal{A}_{t_1}} \Big[|V(t_2, \mathbb{P}^{t_1,\mu,\alpha}) - V(t_2,\nu)| \\
&\qquad + \int_{t_1}^{t_2} \mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\big[|f(s, X, \mathbb{P}^{t_1,\mu,\alpha}, \alpha_s)|\big]\,ds\Big] \\
&\le \rho\big(\mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,t_2]}, \nu_{[0,t_2]})\big) \\
&\qquad + C_0 \int_{t_1}^{t_2} \big[1 + \mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}[\|X_{s\wedge\cdot}\|] + \mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,s]}, \delta_{\{0\}})\big]\,ds.
\end{aligned} \tag{5.34}$$

Note that, since b and σ are bounded, for $s \in [t_1.t_2]$,

$$\begin{aligned}
\mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,t_2]}, \nu_{[0,t_2]}) &\le \mathcal{W}_2(\mu_{[0,t_1]}, \nu_{[0,t_2]}) + \mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,t_2]}, \mu_{[0,t_1]}) \\
&\le \mathcal{W}_2(\mu_{[0,t_1]}, \nu_{[0,t_2]}) + \Big(\mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\Big[\sup_{t_1\le s\le t_2} |X_s - X_{t_1}|^2\Big]\Big)^{\frac{1}{2}} \\
&\le \mathcal{W}_2(\mu_{[0,t_1]}, \nu_{[0,t_2]}) + C(t_2 - t_1)^{\frac{1}{2}} \le C\mathcal{W}_2((t_1,\mu),(t_2,\nu);
\end{aligned}$$

$$\begin{aligned}
&\big(\mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}[\|X_{s\wedge\cdot}\|] + \mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,s]}, \delta_{\{0\}})\big)^2 \\
&\quad \le C\mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\big[\|X_{s\wedge\cdot}\|^2\big] \\
&\quad \le C\mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\Big[\|X_{t_1\wedge\cdot}\|^2 + \sup_{t_1\le r\le s} |X_{t_1,r}|^2\Big] \\
&\quad = C\mathbb{E}^{\mu}\big[\|X_{t_1\wedge\cdot}\|^2\big] + C\mathbb{E}^{\mathbb{P}^{t_1,\mu,\alpha}}\Big[\sup_{t_1\le r\le s} |X_{t_1,r}|^2\Big] \\
&\quad \le C\mathbb{E}^{\mu}\big[\|X_{t_1\wedge\cdot}\|^2\big] + C[t_2 - t_1] \le C\mathbb{E}^{\mu}\big[\|X_{t_1\wedge\cdot}\|^2\big] + C.
\end{aligned}$$

Then

$$\begin{aligned}
|V(t_1,\mu) - V(t_2,\nu)| &\le \rho(C\mathcal{W}_2\big((t_1,\mu),(t_2,\nu)\big) \\
&\qquad + C\big(1 + \mathbb{E}^{\mu}[\|X_{t_1\wedge\cdot}\|^2]\big)^{\frac{1}{2}}[t_2 - t_1].
\end{aligned}$$

This proves (5.4).

Moreover, if f is bounded, then (5.34) implies that

$$\begin{aligned}
|V(t_1,\mu) - V(t_2,\nu)| &\le \rho\big(\mathcal{W}_2(\mathbb{P}^{t_1,\mu,\alpha}_{[0,t_2]}, \nu_{[0,t_2]})\big) + C[t_2 - t_1] \\
&\le \rho(C\mathcal{W}_2\big((t_1,\mu),(t_2,\nu)\big) + C[t_2 - t_1].
\end{aligned}$$

This implies that V is uniformly continuous in (t,μ). $\square$

5.4. *A state dependent property.* We conclude this section with the following state dependent property.

THEOREM 5.15. *Let Assumption 5.1 hold. Assume further that b, σ, f, g are state dependent, namely $(b,\sigma,f)(t,\omega,\mu,a) = (b,\sigma,f)(t,\omega_t,\mu_t,a)$ and $g(\omega,\mu) = g(\omega_T,\mu_T)$, then $V(t,\mu) = V(t,\mu_t)$ is also state dependent.*

PROOF. By Lemma 5.14, it suffices to show that $V_0(t,\mu)=V_0(t,\nu)$ for all t, μ, ν such that $\mu_t=\nu_t$. We proceed in three steps.

Step 1. First, one may construct $\overline{\mathbb{P}}\in\mathcal{P}(\mu,\nu)$ such that $\overline{\mathbb{P}}(X_t=X'_t)=1$. Indeed, one may construct it such that the conditional distributions are independent: for any $\xi,\xi'\in C_b^0(\Omega)$,

$$\mathbb{E}^{\overline{\mathbb{P}}}[\xi(X_{t\wedge\cdot})\xi'(X'_{t\wedge\cdot})]:=\mathbb{E}^{\mu_t}\big[\mathbb{E}^{\mu}[\xi(X_{t\wedge\cdot})|X_t]\mathbb{E}^{\nu}[\xi'(X'_{t\wedge\cdot})|X'_t=X_t]\big].$$

Step 2. For any $\pi:0=s_0<\cdots<s_m=t$ and $\varepsilon>0$, $\delta>0$, we may mimic the arguments in Lemma 5.11 and construct $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde B,\xi,\eta)$ such that:

- $\mathcal{L}_\xi=\mu$, $\mathcal{L}_\eta=\nu$, and η is independent of $\tilde B$;
- ξ_π is measurable to the σ-algebra $\sigma(\eta_\pi,\tilde B_{[0,\delta]})$.
- $\mathbb{E}^{\tilde{\mathbb{P}}}[|\xi_t-\eta_t|^2]\le\varepsilon^2$.

Indeed, since $\overline{\mathbb{P}}(X_t=X'_t)=1$, in Cases 1 and 2 in Lemma 5.14, it is obvious that $\xi_t=\eta_t$. In Case 3, we can show that $\mathbb{E}^{\tilde{\mathbb{P}}}[|\xi_t-\eta_t|^2]\le\varepsilon^2$.

Step 3. We now mimic the arguments in Lemma 5.13 to prove $V_0(t,\mu)=V_0(t,\nu)$. Fix an arbitrary $\alpha=\sum_{i=0}^{n-1}h_i(X_\pi,X_{[t,t_i]})1_{[t_i,t_{i+1})}\in\mathcal{A}_t^0$ with the corresponding partition $\pi:0\le s_1<\cdots<s_m=t$. Consider the notation in Steps 1 and 2 in this proof, and introduce $\tilde{\mathbb{P}}$, B', B^δ, X^α, $\tilde X$, δ as in Lemma 5.13. Similar to (5.22) we can prove

$$\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{t\le s\le T}|\tilde X_s-X_s^\alpha|^2\Big]\le C\mathbb{E}^{\tilde{\mathbb{P}}}\big[|\xi_t-\eta_t|^2\big]+C_\mu\rho_0^2(\delta)\le C\varepsilon^2+C_\mu\rho_0^2(\delta).$$

Moreover, following the arguments in (5.25) and (5.26), we can show that

$$\begin{aligned}&J(t,\mu,\alpha)-V_0(t,\nu)\\&\quad\le C\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\rho_0\Big(\sup_{t\le s\le T}\big[|\tilde X_s-X_s^\alpha|+\mathcal{W}_2(\mathcal{L}_{\tilde X_s},\mathcal{L}_{X_s^\alpha})\big]\Big)\Big]+C_{\mu,\nu}\rho_0(\delta)\\&\quad\le C\rho\big(C\varepsilon+C_\mu\rho_0(\delta)\big)+C_{\mu,\nu}\rho_0(\delta),\end{aligned}$$

for some modulus of continuity function ρ. Send $\varepsilon,\delta\to0$, we obtain: $J(t,\mu,\alpha)-V_0(t,\nu)\le 0$. Since α is arbitrary, this implies that $V_0(t,\mu)\le V_0(t,\nu)$. The opposite inequality can be proved similarly, and thus $V_0(t,\mu)=V_0(t,\nu)$. □

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