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MEAN FIELD GAMES MASTER EQUATIONS WITH NONSEPARABLE HAMILTONIANS AND DISPLACEMENT MONOTONICITY

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In this manuscript we propose a structural condition on nonseparable Hamiltonians, which we term *displacement monotonicity* condition, to study second-order mean field games master equations. A rate of dissipation of a bilinear form is brought to bear a global (in time) well-posedness theory, based on a priori uniform Lipschitz estimates on the solution in the measure variable. Displacement monotonicity being sometimes in dichotomy with the widely used Lasry–Lions monotonicity condition, the novelties of this work persist even when restricted to separable Hamiltonians.

1. Introduction. In this manuscript, $T > 0$ is a given arbitrary time horizon and $\beta \geq 0$. We consider evolutive equations, which represent games where the players are in motion in the space $\mathbb{R}^d$ and their distributions at each time are represented by elements of $\mathcal{P}_2(\mathbb{R}^d)$, the set of Borel probability measures on $\mathbb{R}^d$, with finite second moments. The data governing the game are a Hamiltonian H and a terminal cost function G such that

$$H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R} \quad \text{and} \quad G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}.$$

For our description we assume to be given a rich enough underlying probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$. The problem at hand is to find a real valued function V which depends on the time variable t, the space variable x and the probability measure variable μ such that

$$\begin{cases} -\partial_t V - \dfrac{\widehat{\beta}^2}{2} \operatorname{tr}(\partial_{xx} V) + H(x, \mu, \partial_x V) - \mathcal{N} V = 0 & \text{in } (0, T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \\ V(T, x, \mu) = G(x, \mu) & \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d). \end{cases} \tag{1.1}$$

This second-order equation is called the master equation in mean field games, in presence of both idiosyncratic and common noise (if $\beta > 0$), where $\mathcal{N}$ is the nonlocal operator defined by

$$\begin{aligned} &\mathcal{N} V(t, x, \mu) \\ &:= \operatorname{tr}\biggl(\bar{\tilde{\mathbb{E}}}\biggl[\frac{\widehat{\beta}^2}{2} \partial_{\tilde{x}} \partial_\mu V(t, x, \mu, \tilde{\xi}) + \frac{\beta^2}{2} \partial_{\mu\mu} V(t, x, \mu, \bar{\xi}, \tilde{\xi}) \\ &\quad + \beta^2 \partial_x \partial_\mu V(t, x, \mu, \tilde{\xi}) - \partial_\mu V(t, x, \mu, \tilde{\xi}) (\partial_p H)^\top (\tilde{\xi}, \mu, \partial_x V(t, \tilde{\xi}, \mu))\biggr]\biggr). \end{aligned} \tag{1.2}$$

Above, β stands for the intensity of the common noise; the idiosyncratic noise is supposed to be nondegenerate (for simplicity, its intensity is set to be 1), and we use the notation

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$\widehat{\beta}^2 := 1 + \beta^2$. We always assume $H(x, \mu, \cdot)$ to be convex; however, we emphasize already at this point the fact that, in general, it can have a "nonseparable structure," that is, we *do not* assume to have a decomposition of the form

$$H(x, \mu, p) = H_0(x, p) - F(x, \mu). \tag{1.3}$$

In (1.1)–(1.2), ∂_t stands for the time derivative while ∂_x stands for the gradient operator on $\mathbb{R}^d$. We postpone to Section 2, comments on the W_2-Wasserstein gradient ∂_μ and the W_2-Wasserstein second gradient $\partial_{\mu\mu}$. Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\tilde{\xi}$ and $\bar{\xi}$ are independent random variables with the same law μ, and $\bar{\tilde{\mathbb{E}}}$ is the expectation with respect to their joint law.

First introduced by Lions in lectures [36], the master equation appeared in the context of the theory of mean field games, a theory initiated independently by Lasry–Lions [33–35] and Huang–Malhamé–Caines [32]. It is a time dependent equation which serves to describe the interaction between an individual agent and a continuum of other agents. The master equation characterizes the equilibrium cost of a representative agent within a continuum of players, provided there is a unique mean field equilibrium. Roughly speaking, it plays the role of the Hamilton–Jacobi–Bellman equation in the stochastic control theory. We refer the reader to [19, 24, 25] for a comprehensive exposition on the subject.

The master equation (1.1) is known to admit a local (in time) classical solution when the data H and G are sufficiently smooth, even when the noises are absent (cf. [13, 29, 37]). Local solutions are known to be unique (cf. [20, 25]), including cases where the Hamiltonians are local functions of the measure variable (cf. [6]). Nevertheless, it is much more challenging to obtain global classical solutions, as they are expected to exist only under additional structural assumptions on the data. Such a sufficient condition is typically a sort of *monotonicity condition* that provides uniqueness of solutions to the underlying mean field game system (a phenomenon that heuristically corresponds to the noncrossing of generalized characteristics of the master equation). For a nonexhaustible list of results on the global in time well-posedness theory of mean field games master equations in various settings, we refer the reader to [21, 24–26] and in the realm of potential mean field games to [11, 12, 28]. We also refer to [15, 22, 39] for global existence and uniqueness of weak solutions and to [8–10, 14, 16, 17] for finite state mean field games master equations. All the above global well-posedness results require the Hamiltonian H to be separable in μ and p; that is, it is of the form (1.3), for some H_0 and F. Moreover, as highlighted above, F and G need to satisfy a certain monotonicity condition which, in particular, ensures the uniqueness of mean field equilibria of the corresponding mean field games. We remark that nonseparable Hamiltonians appear naturally in applications (such as economical models, problems involving congestions effects, etc.; see, e.g., [1, 2, 5, 31]). We shall also note that [18] establishes the global in time well-posedness result for a linear master equation, without requiring separability or monotonicity conditions. However, since the Hamiltonian H is linear in p, there is no underlying game involved in [18].

A typical condition, extensively used in the literature [10, 16, 21, 24–26, 39], is the so-called *Lasry–Lions monotonicity* condition. For a function $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, this can be formulated as

$$\mathbb{E}\big[G(\xi_1, \mathcal{L}_{\xi_1}) + G(\xi_2, \mathcal{L}_{\xi_2}) - G(\xi_1, \mathcal{L}_{\xi_2}) - G(\xi_2, \mathcal{L}_{\xi_1})\big] \geq 0 \tag{1.4}$$

for any random variables ξ_1, ξ_2 with appropriate integrability assumptions. Here, $\mathcal{L}_\xi := \xi_\# \mathbb{P}$ stands for the law of the random variable ξ.

In this manuscript we turn to a different condition. The main condition we impose here on G is what we term the *displacement monotonicity* condition which can be formulated as

$$\mathbb{E}\big[\big[\partial_x G(\xi_1, \mathcal{L}_{\xi_1}) - \partial_x G(\xi_2, \mathcal{L}_{\xi_2})\big][\xi_1 - \xi_2]\big] \geq 0. \tag{1.5}$$

When G is sufficiently smooth, displacement monotonicity means that the bilinear form

$$(1.6)\quad (\eta_1,\eta_2)\mapsto (d_x d)_\xi G(\eta_1,\eta_2) := \tilde{\mathbb{E}}\big[\langle \partial_{x\mu} G(\xi,\mu,\tilde{\xi})\tilde{\eta}_1,\eta_2\rangle\big] + \mathbb{E}\big[\langle \partial_{xx} G(\xi,\mu)\eta_1,\eta_2\rangle\big]$$

is nonnegative definite for all square integrable random variables ξ. Here, $(\tilde{\xi},\tilde{\eta}_1)$ is an independent copy of (ξ,η_1), and $\tilde{\mathbb{E}}$ is the expectation with respect to the joint law of $(\xi,\eta_1,\eta_2,\tilde{\xi},\tilde{\eta}_1)$.

Our terminology is inspired by the so–called *displacement convexity* condition, a popular notion in the theory of optimal transport theory (cf. [38]). Indeed, when G is derived from a potential, that is, there exists $g:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ such that $\partial_x G = \partial_\mu g$, then (1.5) is equivalent to the displacement convexity of g. Let us underline that, in the current study, we never need to require that G is derived from a potential.

Displacement convexity and monotonicity have some sparse history in the framework of mean field games and control problems of McKean–Vlasov type. In the context of mean field game systems, the first work using this seems to be the one of Ahuja [3] (see also [4]), whose weak monotonicity condition is essentially equivalent to the displacement monotonicity. In the context of control problems of McKean–Vlasov type, displacement convexity assumptions appeared first in [23] and [26]. It seems that [26] is the first work that relied on displacement convexity in the study of well posedness of a master equation arising in a McKean–Vlasov control problem. However, let us emphasize that this master equation is different from the master equation appearing in the theory of mean field games, and the techniques developed in [26] are not applicable in our setting. In the framework of potential master equations and, in particular, in more classical infinite dimensional control problems on Hilbert spaces, the displacement convexity condition has been used in [11, 12][1] and [28].

Our main contribution in this manuscript is the discovery of a condition on H which allows a global well-posedness theory of classical solution for the master equation (1.1). This condition, which we continue to term *displacement monotonicity condition for Hamiltonians*, amounts to impose that the bilinear form

$$\begin{aligned}(\eta_1,\eta_2)\mapsto &\big(\mathrm{displ}_\xi^\varphi H\big)(\eta_1,\eta_2)\\ &:= (d_x d)_\xi H\big(\cdot,\varphi(\xi)\big)(\eta_1,\eta_2)\\ &\quad + \frac{1}{4}\tilde{\mathbb{E}}\big[\big\langle\big(\partial_{pp}H\big(\xi,\mu,\varphi(\xi)\big)\big)^{-1}\partial_{p\mu}H\big(\xi,\mu,\tilde{\xi},\varphi(\xi)\big)\tilde{\eta}_1,\partial_{p\mu}H\big(\xi,\mu,\tilde{\xi},\varphi(\xi)\big)\tilde{\eta}_2\big\rangle\big]\end{aligned}$$

is nonpositive definite for all $\mu\in\mathcal{P}_2$, $\xi\in\mathbb{L}^2(\mathcal{F},\mu)$ and all appropriate $\varphi\in C^1(\mathbb{R}^d;\mathbb{R}^d)$; see Definition 3.4 below for the precise condition. In the previous formula we clearly assume strict convexity on H the p variable. This condition is instrumental for our global well-posedness theory of classical solutions to the master equation (1.1). To the best of our knowledge, this is the first global well-posedness result in the literature of mean field games master equations with nonseparable Hamiltonians. When H is separable (i.e., of the form (1.3)) and $H_0 = H_0(p)$, the nonpositive definiteness assumption on $\mathrm{displ}_\xi^\varphi H$, is equivalent to (1.5) for F. For certain Hamiltonians, displacement monotonicity is in dichotomy with the Lasry–Lions monotonicity. Thus, not only are our well-posedness results new for a wide class of data functions but we shall also soon see that the novelty in our results extends to a class of separable Hamiltonians. For discussions on displacement monotone functions that fail to be Lasry–Lions monotone, we refer to [3, 28] and to Section 2.3 below.

[1]These references essentially used a notion of λ-convexity in displacement sense and obtained local in time classical solutions for the master equation. However, it is clear from their results that the solution is global when the data are actually displacement convex.

We show, at the heart of our analysis, that under the displacement monotonicity condition on H and (1.5) on G, $V(t,\cdot,\cdot)$, the solution to the master equation (1.1), which has sufficient a priori regularity, also satisfies (1.5) for all $t \in [0,T]$. Let us recall that when H is separable and both G and F satisfy the Lasry–Lions monotonicity condition (1.4), then $V(t,\cdot,\cdot)$ inherits (1.4) as well for all $t \in [0,T]$. However, when H is nonseparable, it remains a challenge to find an appropriate condition on H which ensures that if G satisfies the Lasry–Lions monotonicity so does $V(t,\cdot,\cdot)$ for all $t \in [0,T]$ (see Remark 4.2(iii) below).

For separable H the Lasry–Lions monotonicity of $V(t,\cdot,\cdot)$ is typically proven through the mean field game system, the corresponding coupled system of forward backward (stochastic) PDEs or SDEs for which V serves as the decoupling field (see Remark 2.10 below). We instead follow a different route and derive the displacement monotonicity of $V(t,\cdot,\cdot)$ by using the master equation itself. We show that if V is a smooth solution to the master equation, then for any $\mu \in \mathcal{P}_2$ and $\eta \in \mathbb{L}^2(\mathcal{F}_0)$ there exists a path $t \mapsto (X_t, \delta X_t)$ of random variables starting at (ξ, η), with $\mu = \mathcal{L}_\xi$, such that

$$(d_x d)_{X_T} V_T(\delta X_T, \delta X_T) - \int_0^T \big(\mathrm{displ}^{\varphi_t}_{X_t} H\big)(\delta X_t, \delta X_t)\,dt \le (d_x d)_{X_0} V_0(\eta,\eta). \tag{1.7}$$

Here (see Remark 4.2 for a more accurate formulation),

$$\varphi_t = \partial_x V(t,\cdot,\mu_t), \quad \mu_t = X_{t\#}\mathbb{P}.$$

Note that (1.7) provides us an explicit "rate of dissipation of displacement monotonicity" of the bilinear form $(d_x d)V(t,\cdot,\cdot)$, from smaller to larger times. This favors our terminal value problem, as we are provided with a "rate at which the displacement monotonicity is built in" from larger to smaller times.

Our approach seems new, even when restricted to separable H. We are also able to obtain a variant of (1.7) that is applicable to the Lasry–Lions monotonicity case, but only for separable H. One trade-off in our approach is that, since we apply Itô's formula on the derivatives of V, we need higher-order a priori regularity estimates on V and, consequently, require regularity of the data slightly higher than what is needed for the existence of local classical solutions (cf. [25]). We believe that, thanks to the smooth mollification technique developed in [39], one could relax these regularity requirements. In fact, we expect a well-posedness theory of weak solutions in the sense of [39]. In this work our main goal is to overcome the challenge of dealing with nonseparable Hamiltonians, and so this manuscript postpones the optimal regularity issue to future studies.

The displacement monotonicity of $V(t,\cdot,\cdot)$ has a noticeable implication: it yields an a priori uniform W_2-Lipschitz continuity estimate for V in the μ variable. Here is the main principle to emphasize: any possible alternative condition to the nonpositive definiteness assumption on $\mathrm{displ}^{\varphi}_{\xi} H$, which ensures the monotonity of V (either in Lasry–Lions sense or in displacement sense), will also provide the uniform Lipschitz continuity of V in μ (with respect to either W_1 or W_2). As a consequence, this yields the global well posedness of the master equation. We shall next elaborate on this observation which seems to be new in the literature and interesting on its own right.

Uniform W_1-Lipschitz continuity of V is known to be the key ingredient for constructing even local in time classical solutions of the master equation (cf. [25, 39]) in mean field games with common noise. The uniform W_2-Lipschitz property we obtain is not final. We complement this in light of a crucial observation: when the data H and G are uniformly W_1-Lipschitz continuous in μ, we can show that the uniform W_2-Lipschitz continuity of V actually implies its uniform W_1-Lipschitz continuity in the μ variable. We achieve this by a delicate analysis on the pointwise representation formula for $\partial_\mu V$, developed in [39], tailored to our setting.

In our final step to establish the global well-posedness of the master equation, we follow the by now standard approach in [25, 26, 39]. That is, based on the a priori uniform Lipschitz continuity property of V in the μ variable (with respect to W_1), we construct the local classical solution and then extend it backwardly in time. Another important point in our argument is that the length of the time intervals, used for the local solutions, depends only on the W_2-Lipschitz constants of the data.

The rest of the paper is organized as follows. Section 2 contains the setting of our problem and some preliminary results. In Section 3 we present our technical assumptions and introduce the new notion of displacement monotonicity for nonseparable H. In Section 4 we show that any solution of the master equation, which is regular enough, preserves the displacement monotonicity property. Section 5 is devoted to uniform a priori W_2-Lipschitz estimates on V. In Section 6 we derive the uniform W_1-Lipschitz estimates and establish the global well posedness of the master equation (1.1).

2. Preliminaries.

2.1. *The product probability space.* In this paper we shall use a probabilistic approach. In order to reach out to the largest community of people working on mean field games, in this subsection we present our probabilistic setting in details, which we think will facilitate the reading of those who are not experts in stochastic analysis.

Throughout the paper we fix $T > 0$ to be a given arbitrary time horizon. Let $(\Omega_0, \mathbb{F}^0, \mathbb{P}_0)$ and $(\Omega_1, \mathbb{F}^1, \mathbb{P}_1)$ be two filtered probability spaces on which there are defined d-dimensional Brownian motions B^0 and B, respectively. For $\mathbb{F}^i = \{\mathcal{F}^i_t\}_{0\le t\le T}$, $i = 0, 1$, we assume $\mathcal{F}^0_t = \mathcal{F}^{B^0}_t$, $\mathcal{F}^1_t = \mathcal{F}^1_0 \vee \mathcal{F}^B_t$ and $\mathbb{P}_1$ has no atom in $\mathcal{F}^1_0$, so it can support any measure on $\mathbb{R}^d$ with finite second moment. Consider the product spaces

$$
\begin{aligned}
&\Omega := \Omega_0 \times \Omega_1, \qquad \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T} := \{\mathcal{F}^0_t \otimes \mathcal{F}^1_t\}_{0\le t\le T},\\
&\mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1, \qquad \mathbb{E} := \mathbb{E}^{\mathbb{P}}.
\end{aligned} \tag{2.1}
$$

In particular, $\mathcal{F}_t := \sigma(A_0 \times A_1 : A_0 \in \mathcal{F}^0_t, A_1 \in \mathcal{F}^1_t\}$ and $\mathbb{P}(A_0 \times A_1) = \mathbb{P}_0(A_0)\mathbb{P}_1(A_1)$. We shall automatically extend B^0, B, $\mathbb{F}^0$, $\mathbb{F}^1$ to the product space in the obvious sense but using the same notation. For example, $B^0(\omega) = B^0(\omega^0)$ for $\omega = (\omega^0, \omega^1) \in \Omega$, and $\mathcal{F}^0_t = \{A_0 \times \Omega_1 : A_0 \in \mathcal{F}^0_t\}$. In particular, this implies that B^0 and B^1 are independent $\mathbb{P}$-Brownian motions and are independent of $\mathcal{F}_0$.

It is convenient to introduce another filtered probability space $(\tilde{\Omega}_1, \tilde{\mathbb{F}}^1, \tilde{B}, \tilde{\mathbb{P}}_1)$ in the same manner as $(\Omega_1, \mathbb{F}^1, B, \mathbb{P}_1)$ and to consider the larger filtered probability space given by

$$
\begin{aligned}
&\tilde{\Omega} := \Omega \times \tilde{\Omega}_1, \qquad \tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{0\le t\le T} := \{\mathcal{F}_t \otimes \tilde{\mathcal{F}}^1_t\}_{0\le t\le T},\\
&\tilde{\mathbb{P}} := \mathbb{P} \otimes \tilde{\mathbb{P}}_1, \qquad \tilde{\mathbb{E}} := \mathbb{E}^{\tilde{\mathbb{P}}}.
\end{aligned} \tag{2.2}
$$

Given an $\mathcal{F}_t$-measurable random variable $\xi = \xi(\omega^0, \omega^1)$, we say $\tilde{\xi} = \tilde{\xi}(\omega^0, \tilde{\omega}^1)$ is a conditionally independent copy of ξ if, for each ω^0; the $\mathbb{P}_1$-distribution of $\xi(\omega^0, \cdot)$ is equal to the $\tilde{\mathbb{P}}_1$-distribution of $\tilde{\xi}(\omega^0, \cdot)$. That is, conditional on $\mathcal{F}^0_t$, by extending to $\tilde{\Omega}$ the random variables ξ and $\tilde{\xi}$ are conditionally independent and have the same conditional distribution under $\tilde{\mathbb{P}}$. Note that, for any appropriate deterministic function φ,

$$
\begin{aligned}
&\tilde{\mathbb{E}}_{\mathcal{F}^0_t}[\varphi(\xi, \tilde{\xi})](\omega^0) = \mathbb{E}^{\mathbb{P}_1\otimes\tilde{\mathbb{P}}_1}[\varphi(\xi(\omega^0, \cdot), \tilde{\xi}(\omega^0, \tilde{\cdot}))], \quad \mathbb{P}_0\text{-a.e. } \omega^0;\\
&\tilde{\mathbb{E}}_{\mathcal{F}_t}[\varphi(\xi, \tilde{\xi})](\omega^0, \omega^1) = \mathbb{E}^{\tilde{\mathbb{P}}_1}[\varphi(\xi(\omega^0, \omega^1), \tilde{\xi}(\omega^0, \tilde{\cdot}))], \quad \mathbb{P}\text{-a.e. } (\omega^0, \omega^1).
\end{aligned} \tag{2.3}
$$

Here, $\mathbb{E}^{\tilde{\mathbb{P}}_1}$ is the expectation on $\tilde{\omega}^1$, and $\mathbb{E}^{\mathbb{P}_1\times\tilde{\mathbb{P}}_1}$ is on $(\omega^1,\tilde{\omega}^1)$. Throughout the paper we will use the probability space $(\Omega,\mathbb{F},\mathbb{P})$. However, when conditionally independent copies of random variables or processes are needed, we will tacitly use the extension to the larger space $(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ without mentioning.

When we need two conditionally independent copies, we introduce further $(\bar{\Omega}_1,\bar{\mathbb{F}}^1,\bar{B},\bar{\mathbb{P}}_1)$ and the product space $(\bar{\Omega},\bar{\mathbb{F}},\bar{\mathbb{P}},\bar{\mathbb{E}})$, as in (2.2), and set the joint product space

$$\bar{\tilde{\Omega}}:=\Omega\times\tilde{\Omega}_1\times\bar{\Omega}_1,\qquad \bar{\tilde{\mathbb{F}}}=\{\bar{\tilde{\mathcal{F}}}_t\}_{0\le t\le T}:=\{\mathcal{F}_t\otimes\tilde{\mathcal{F}}^1_t\otimes\bar{\mathcal{F}}^1_t\}_{0\le t\le T},$$
$$\bar{\tilde{\mathbb{P}}}:=\mathbb{P}\otimes\tilde{\mathbb{P}}_1\otimes\bar{\mathbb{P}}_1,\qquad \bar{\tilde{\mathbb{E}}}:=\mathbb{E}^{\bar{\tilde{\mathbb{P}}}}. \tag{2.4}$$

Then, given $\mathcal{F}_t$-measurable $\xi=\xi(\omega^0,\omega^1)$, we may have two conditionally independent copies under $\bar{\tilde{\mathbb{P}}}$: $\tilde{\xi}(\bar{\tilde{\omega}})=\tilde{\xi}(\omega^0,\tilde{\omega}^1)$ and $\bar{\xi}(\bar{\tilde{\omega}})=\bar{\xi}(\omega^0,\bar{\omega}^1)$, $\bar{\tilde{\omega}}=(\omega^0,\omega^1,\tilde{\omega}^1,\bar{\omega}^1)\in\bar{\tilde{\Omega}}$.

To avoid possible notation confusion, we emphasize that

$$\begin{aligned}&\text{when } \xi=\xi(\omega^1) \text{ is } \mathcal{F}^1_t\text{-measurable, then } \tilde{\xi},\bar{\xi} \text{ are independent copies of } \xi \text{ under } \bar{\tilde{\mathbb{P}}};\\ &\text{the expectation } \tilde{\mathbb{E}} \text{ is on } \tilde{\omega}=(\omega^0,\omega^1,\tilde{\omega}^1), \text{ not just on } \tilde{\omega}^1; \text{ similarly, } \bar{\mathbb{E}} \text{ is acting on}\\ &\bar{\omega}=(\omega^0,\omega^1,\bar{\omega}^1), \text{ not just on } \bar{\omega}^1, \text{ and } \bar{\tilde{\mathbb{E}}} \text{ is an expectation on } \bar{\tilde{\omega}}=(\omega^0,\omega^1,\tilde{\omega}^1,\bar{\omega}^1).\end{aligned} \tag{2.5}$$

2.2. *Preliminary analysis on the Wasserstein space.* Let $\mathcal{P}:=\mathcal{P}(\mathbb{R}^d)$ be the set of all probability measures on $\mathbb{R}^d$, and $\delta_x\in\mathcal{P}$ denotes the Dirac mass at $x\in\mathbb{R}^d$. For any $q\ge1$ and any measure $\mu\in\mathcal{P}$, we set

$$M_q(\mu):=\left(\int_{\mathbb{R}^d}|x|^q\mu(dx)\right)^{\frac{1}{q}}\quad\text{and}\quad \mathcal{P}_q:=\mathcal{P}_q(\mathbb{R}^d):=\{\mu\in\mathcal{P}:M_q(\mu)<\infty\}. \tag{2.6}$$

For any sub-σ-field $\mathcal{G}\subset\mathcal{F}_T$ and $\mu\in\mathcal{P}_q$, denote by $\mathbb{L}^q(\mathcal{G})$ the set of $\mathbb{R}^d$-valued, $\mathcal{G}$-measurable and q-integrable random variables ξ, and $\mathbb{L}^q(\mathcal{G};\mu)$ the set of $\xi\in\mathbb{L}^q(\mathcal{G})$ such that $\mathcal{L}_\xi=\mu$. Here, $\mathcal{L}_\xi=\xi_{\#}\mathbb{P}$ is the law of ξ, obtained as the push–forward of $\mathbb{P}$ by ξ. Also, for $\mu\in\mathcal{P}_q$, let $\mathbb{L}^q_\mu(\mathbb{R}^d;\mathbb{R}^d)$ denote the set of Borel measurable functions $v:\mathbb{R}^d\to\mathbb{R}^d$ such that $\|v\|^q_{\mathbb{L}^q_\mu}:=\int_{\mathbb{R}^d}|v(x)|^q\mu(dx)<\infty$. Moreover, for any $\mu,\nu\in\mathcal{P}_q$, their W_q-Wasserstein distance is defined as follows:

$$W_q(\mu,\nu):=\inf\{(\mathbb{E}[|\xi-\eta|^q])^{\frac{1}{q}}:\text{for all }\xi\in\mathbb{L}^q(\mathcal{F}_T;\mu),\eta\in\mathbb{L}^q(\mathcal{F}_T;\nu)\}. \tag{2.7}$$

According to the terminology in [7], the Wasserstein gradient of a function $U:\mathcal{P}_2\to\mathbb{R}$ at μ is an element $\partial_\mu U(\mu,\cdot)$ of $\overline{\nabla C^\infty_c(\mathbb{R}^d)}^{\mathbb{L}^2_\mu}$ (the closure of gradients of C^∞_c functions in $\mathbb{L}^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$), and so it is a priori defined μ-almost everywhere. The theory developed in [19, 30, 36, 40] shows that $\partial_\mu U(\mu,\cdot)$ can be characterized by the property

$$U(\mathcal{L}_{\xi+\eta})-U(\mu)=\mathbb{E}[\langle\partial_\mu U(\mu,\xi),\eta\rangle]+o(\|\eta\|_2)\quad\forall\xi,\eta,\text{ with }\mathcal{L}_\xi=\mu. \tag{2.8}$$

Let $\mathcal{C}^0(\mathcal{P}_2)$ denote the set of W_2-continuous functions $U:\mathcal{P}_2\to\mathbb{R}$. For $k\in\{1,2\}$, we next define a subset of $\mathcal{C}^k(\mathcal{P}_2)$, referred to as functions of *full* $\mathcal{C}^k$ regularity in [24], Chapter 5, as follows. By $\mathcal{C}^1(\mathcal{P}_2)$, we mean the space of functions $U\in C^0(\mathcal{P}_2)$ such that $\partial_\mu U$ exists for all $\mu\in\mathcal{P}_2$, and it has a unique jointly continuous extension to $\mathcal{P}_2\times\mathbb{R}^d$, which we continue to denote by

$$\mathbb{R}^d\times\mathcal{P}_2\ni(\tilde{x},\mu)\mapsto\partial_\mu U(\mu,\tilde{x})\in\mathbb{R}^d.$$

We sometimes refer to the extension as the global version, and we note that our requirement of pointwise continuity property of this global version is stronger than the $\mathbb{L}^2$-continuity requirement made in some of the mean field game literature (cf., e.g., [26]). Similarly, $\mathcal{C}^2(\mathcal{P}_2)$ stands for the set of functions $U \in \mathcal{C}^1(\mathcal{P}_2)$ such that the global version of $\partial_\mu U$ is differentiable in the sense that the following maps exist and have unique jointly continuous extensions:

$$\mathbb{R}^d \times \mathcal{P}_2 \ni (\tilde{x}, \mu) \mapsto \partial_{\tilde{x}\mu} U(\mu, \tilde{x}) \in \mathbb{R}^d \quad \text{and}$$
$$\mathbb{R}^{2d} \times \mathcal{P}_2 \ni (\tilde{x}, \bar{x}, \mu) \mapsto \partial_{\mu\mu} U(\mu, \tilde{x}, \bar{x}) \in \mathbb{R}^{d\times d}.$$

$\mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$ is the set of continuous functions $U : \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}$ satisfying the following:

(i) $\partial_x U$, $\partial_{xx} U$ exist and are jointly continuous on $\mathbb{R}^d \times \mathcal{P}_2$;

(ii) The following maps exist and have unique jointly continuous extensions:

$$\mathbb{R}^{2d} \times \mathcal{P}_2 \ni (x, \tilde{x}, \mu) \mapsto \partial_\mu U(x, \mu, \tilde{x}) \in \mathbb{R}^d \quad \text{and}$$
$$\mathbb{R}^{2d} \times \mathcal{P}_2 \ni (x, \tilde{x}, \mu) \mapsto \partial_{x\mu} U(x, \mu, \tilde{x}) \in \mathbb{R}^{d\times d};$$

(iii) Finally, the following maps exist and have unique jointly continuous extensions:

$$\mathbb{R}^{2d} \times \mathcal{P}_2 \ni (x, \tilde{x}, \mu) \mapsto \partial_{\tilde{x}\mu} U(x, \mu, \tilde{x}) \in \mathbb{R}^{d\times d} \quad \text{and}$$
$$\mathbb{R}^{3d} \times \mathcal{P}_2 \ni (x, \tilde{x}, \bar{x}, \mu) \mapsto \partial_{\mu\mu} U(x, \mu, \tilde{x}, \bar{x}) \in \mathbb{R}^{d\times d}.$$

Lastly, we fix the state space for our master equation as

$$\Theta := [0, T] \times \mathbb{R}^d \times \mathcal{P}_2$$

and let $\mathcal{C}^{1,2,2}(\Theta)$ denote the set of $U \in C^0(\Theta; \mathbb{R})$ such that the following maps exist and have a unique jointly continuous extensions, as previously described: $\partial_t U$, $\partial_x U$, $\partial_{xx} U$, $\partial_\mu U$, $\partial_x \partial_\mu U$, $\partial_{\tilde{x}} \partial_\mu U$, $\partial_{\mu\mu} U$.

We underline that, for notational conventions, we always denote the "new spacial variables," appearing in Wasserstein derivatives with "tilde" symbols (for first-order Wasserstein derivatives), with "bar" symbols (for second-order Wasserstein derivatives) and so on, and we place them right after the corresponding measures variables. For example, when $U : \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R}$ is typically evaluated as $U(x, \mu, p)$, we use the notations $\partial_\mu U(x, \mu, \tilde{x}, p)$, $\partial_{\tilde{x}} \partial_\mu U(x, \mu, \tilde{x}, p)$, $\partial_\mu \partial_\mu U(x, \mu, \tilde{x}, \bar{x}, p)$, and so on. This convention will be carried through to compositions with random variables too, for example $\partial_\mu U(x, \mu, \tilde{\xi}, p)$, when $\tilde{\xi}$ is an $\mathbb{R}^d$-valued random variable.

Throughout the paper, we shall also use the following notations: for any $R > 0$,

$$B_R^o := \{p \in \mathbb{R}^d : |p| < R\}, \qquad B_R := \{p \in \mathbb{R}^d : |p| \le R\},$$
$$D_R := \mathbb{R}^d \times \mathcal{P}_2 \times B_R. \tag{2.9}$$

The following simple technical lemma (not to confuse with [25], Remark 4.16) is useful.

LEMMA 2.1. *For any $U \in \mathcal{C}^2(\mathcal{P}_2)$ and $(\mu, \tilde{x}) \in \mathcal{P}_2 \times \mathbb{R}^d$, $\partial_{\tilde{x}\mu} U(\mu, \tilde{x})$ is a symmetric matrix.*

PROOF. Since U is of class $\mathcal{C}^2(\mathcal{P}_2)$, we may assume without loss of generality that μ is supported by a closed ball B_R, it is absolutely continuous and has a smooth density ρ with $c := \inf_{x\in B_R} \rho(x) > 0$. By the fact that $\partial_\mu U(\mu, \cdot) \in \overline{\nabla C_c^\infty(\mathbb{R}^d)}^{\mathbb{L}^2_\mu}$, there exists a sequence $(\varphi_n)_n \subset \nabla C_c^\infty(\mathbb{R}^d)$ such that

$$0 = \lim_n \|\partial_{\tilde{x}} \varphi_n - \partial_\mu U(\mu, \cdot)\|_{\mathbb{L}^2_\mu} \ge c \lim_n \|\partial_{\tilde{x}} \varphi_n - \partial_\mu U(\mu, \cdot)\|_{L^2(B_R)}, \tag{2.10}$$

where $L^2(B_R)$ stands for the standard Lebesgue space. Set

$$\bar{\varphi}_n := \varphi_n - \frac{1}{\mathcal{L}^d(B_R)} \int_{B_R} \varphi_n(x)\, dx.$$

By the Poincaré–Wirtinger inequality there exists a universal constant c_d such that

$$\|\bar{\varphi}_n\|_{L^2(B_R)} \le c_d \|\partial_{\tilde{x}} \bar{\varphi}_n\|_{L^2(B_R)}.$$

Thanks to the Sobolev embedding theorem and the strong convergence of $(\partial_{\tilde{x}}\bar{\varphi}_n)_n$ in $L^2(B_R)$, we conclude that there exists φ in the Sobolev space $H^1(B_R)$ such that $(\bar{\varphi}_n)_n$ converges to φ in $H^1(B_R)$. By (2.10) we have

$$\partial_\mu U(\mu, \cdot) = \partial_{\tilde{x}} \varphi, \qquad \int_{B_R} \varphi(x)\, dx = 0. \tag{2.11}$$

Since $\partial_\mu U(\mu, \cdot)$ is continuously differentiable, the representation formula

$$\varphi(\tilde{x}) = \varphi(0) + \int_0^1 \partial_\mu U(\mu, t\tilde{x}) \cdot \tilde{x}\, dt$$

implies that φ is continuously differentiable. Since $\partial_{\tilde{x}}\varphi = \partial_\mu U(\mu, \cdot)$ is continuously differentiable, we conclude that φ is twice continuously differentiable. Thus, $\partial_{\tilde{x}}\partial_\mu U(\mu, \cdot) = \partial_{\tilde{x}\tilde{x}}\varphi$ is symmetric. □

An interesting property of $\mathcal{C}^{1,2,2}(\Theta)$ functions is their use in a general Itô formula. Let $U \in C^{1,2,2}(\Theta)$ be such that, for any compact subset $K \subset \mathbb{R}^d \times \mathcal{P}_2$,

$$\sup_{(t,x,\mu)\in[0,T]\times K} \biggl[\int_{\mathbb{R}^d} \bigl(|\partial_\mu U(t,x,\mu,\tilde{x})|^2 + |\partial_{\tilde{x}}\partial_\mu U(t,x,\mu,\tilde{x})|^2 + |\partial_x \partial_\mu U(t,x,\mu,\tilde{x})|^2\bigr)\mu(d\tilde{x})$$
$$+ \int_{\mathbb{R}^{2d}} |\partial_{\mu\mu} U(t,x,\mu,\tilde{x},\bar{x})|^2 \mu(\tilde{x})\mu(\bar{x})\biggr] < +\infty.$$

For $i = 1, 2$, consider $\mathbb{F}$-progressively measurable and bounded processes

$$b^i : [0,T] \times \Omega \to \mathbb{R}^d \quad \text{and} \quad \sigma^i, \sigma^{i,0} : [0,T] \times \Omega \to \mathbb{R}^{d\times d}.$$

Set

$$dX_t^i := b_t^i\, dt + \sigma_t^i\, dB_t + \sigma_t^{i,0}\, dB_t^0 \quad \text{and} \quad \text{introduce the conditional law } \rho_t := \mathcal{L}_{X_t^2|\mathcal{F}_t^0}.$$

Then (cf., e.g., [25], Theorem 4.17, [18, 26]), recalling the notations for conditionally independent copies and (2.5),

$$\begin{aligned}
&dU(t, X_t^1, \rho_t) \\
&\quad = \Bigl[\partial_t U + \partial_x U \cdot b_t^1 + \frac{1}{2}\operatorname{tr}\bigl(\partial_{xx} U[\sigma_t^1(\sigma_t^1)^\top + \sigma_t^{1,0}(\sigma_t^{1,0})^\top]\bigr)\Bigr](t, X_t^1, \rho_t)\, dt \\
&\qquad + \partial_x U(t, X_t^1, \rho_t) \cdot \sigma_t^1\, dB_t \\
&\qquad + [(\sigma_t^{1,0})^\top \partial_x U + \tilde{\mathbb{E}}_{\mathcal{F}_t}[(\tilde{\sigma}_t^{2,0})^\top \partial_\mu U(\cdot, \tilde{X}_t^2)]](t, X_t^1, \rho_t) \cdot dB_t^0 \\
&\qquad + \operatorname{tr}\Bigl(\bar{\tilde{\mathbb{E}}}_{\mathcal{F}_t}\Bigl[\partial_\mu U(\cdot, \tilde{X}_t^2)(\tilde{b}_t^2)^\top + \frac{1}{2}\partial_{\tilde{x}}\partial_\mu U(\cdot, \tilde{X}_t^2)[\tilde{\sigma}_t^2(\tilde{\sigma}_t^2)^\top + \tilde{\sigma}_t^{2,0}(\tilde{\sigma}_t^{2,0})^\top] \\
&\qquad + \partial_x \partial_\mu U(\cdot, \tilde{X}_t^2)\sigma_t^{1,0}(\tilde{\sigma}_t^{2,0})^\top + \frac{1}{2}\partial_{\mu\mu} U(\cdot, \tilde{X}_t^2, \bar{X}_t^2)\tilde{\sigma}_t^{2,0}(\bar{\sigma}_t^{2,0})^\top\Bigr](t, X_t^1, \rho_t)\Bigr)\, dt.
\end{aligned} \tag{2.12}$$

Throughout this paper the elements of $\mathbb{R}^d$ are viewed as column vectors; $\partial_x U, \partial_\mu U \in \mathbb{R}^d$ are also column vectors; $\partial_{x\mu} U := \partial_x \partial_\mu U := \partial_x[(\partial_\mu U)^\top] \in \mathbb{R}^{d\times d}$, where $^\top$ denotes the transpose and, similarly, for the other second-order derivatives; both the notations "$\cdot$" and $\langle \cdot, \cdot \rangle$ denote the inner product of column vectors. Moreover, the term $\partial_x U \cdot \sigma_t^1\, dB_t$ means $\partial_x U \cdot (\sigma_t^1\, dB_t)$, but we omit the parentheses for notational simplicity.

2.3. *The Lasry–Lions monotonicity and the displacement monotonicity.* In this subsection we discuss two types of monotonicity conditions and provide more convenient alternative formulations.

DEFINITION 2.2. Let $U : \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}$:

(i) U is called *Lasry–Lions monotone*, if for any $\xi_1, \xi_2 \in \mathbb{L}^2(\mathcal{F}_T^1)$,

$$\mathbb{E}\big[U(\xi_1, \mathcal{L}_{\xi_1}) + U(\xi_2, \mathcal{L}_{\xi_2}) - U(\xi_1, \mathcal{L}_{\xi_2}) - U(\xi_2, \mathcal{L}_{\xi_1})\big] \geq 0. \tag{2.13}$$

(ii) U is called *displacement monotone* if $U(\cdot, \mu) \in C^1(\mathbb{R}^d)$ for all $\mu \in \mathcal{P}_2$ and for any $\xi_1, \xi_2 \in \mathbb{L}^2(\mathcal{F}_T^1)$,

$$\mathbb{E}\big[\langle \partial_x U(\xi_1, \mathcal{L}_{\xi_1}) - \partial_x U(\xi_2, \mathcal{L}_{\xi_2}), \xi_1 - \xi_2 \rangle\big] \geq 0. \tag{2.14}$$

REMARK 2.3. Assume $U \in \mathcal{C}^1(\mathbb{R}^d \times \mathcal{P}_2)$:

(i) If $\partial_\mu U(\cdot, \mu, \tilde{x}) \in C^1(\mathbb{R}^d)$, for all $(\mu, \tilde{x}) \in \mathcal{P}_2 \times \mathbb{R}^d$, then the inequality (2.13) implies

$$\begin{aligned} 0 &\leq \mathbb{E}\big[U(\xi, \mathcal{L}_\xi) + U(\xi + \varepsilon\eta, \mathcal{L}_{\xi+\varepsilon\eta}) - U(\xi, \mathcal{L}_{\xi+\varepsilon\eta}) - U(\xi + \varepsilon\eta, \mathcal{L}_\xi)\big] \\ &= \varepsilon^2 \int_0^1 \int_0^1 \tilde{\mathbb{E}}\big[\langle \partial_{x\mu} U(\xi + \theta_1 \varepsilon\eta, \mathcal{L}_{\xi+\theta_2\varepsilon\eta}, \tilde{\xi} + \theta_2 \varepsilon\tilde{\eta})\tilde{\eta}, \eta \rangle\big]\, d\theta_1\, d\theta_2, \end{aligned}$$

for any $\xi, \eta \in \mathbb{L}^2(\mathcal{F}_T^1)$ and any $\varepsilon > 0$, where $(\tilde{\xi}, \tilde{\eta})$ is an independent copy of (ξ, η). Thus,

$$\tilde{\mathbb{E}}\big[\langle \partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}, \eta \rangle\big] \geq 0 \quad \forall \xi, \eta \in \mathbb{L}^2(\mathcal{F}_T^1). \tag{2.15}$$

(ii) If $\partial_x U \in \mathcal{C}^1(\mathbb{R}^d \times \mathcal{P}_2)$, then the inequality (2.14) implies

$$\begin{aligned} 0 &\leq \mathbb{E}\big[\langle \partial_x U(\xi + \varepsilon\eta, \mathcal{L}_{\xi+\varepsilon\eta}) - \partial_x U(\xi, \mathcal{L}_\xi), \varepsilon\eta \rangle\big] \\ &= \varepsilon^2 \int_0^1 \tilde{\mathbb{E}}\big[\langle \partial_{xx} U(\xi + \theta\varepsilon\eta, \mathcal{L}_{\xi+\theta\varepsilon\eta})\eta, \eta \rangle + \langle \partial_{x\mu} U(\xi + \theta\varepsilon\eta, \mathcal{L}_{\xi+\theta\varepsilon\eta}, \tilde{\xi} + \theta\varepsilon\tilde{\eta})\tilde{\eta}, \eta \rangle\big]\, d\theta \end{aligned}$$

for any $\xi, \eta \in \mathbb{L}^2(\mathcal{F}_T^1)$ and $\varepsilon > 0$, where $(\tilde{\xi}, \tilde{\eta})$ is an independent copy of (ξ, η), and thus,

$$(d_x d)_\xi U(\eta, \eta) := \tilde{\mathbb{E}}\big[\langle \partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}, \eta \rangle + \langle \partial_{xx} U(\xi, \mathcal{L}_\xi)\eta, \eta \rangle\big] \geq 0. \tag{2.16}$$

(iii) Assume $\mathcal{U} \in \mathcal{C}^2(\mathcal{P}_2)$ and $U \in \mathcal{C}^1(\mathbb{R}^d \times \mathcal{P}_2)$ are such that $\partial_\mu \mathcal{U} \equiv \partial_x U(x, \mu)$ on $\mathbb{R}^d \times \mathcal{P}_2$. Then, U is displacement monotone if and only if $\mathcal{U}$ is *displacement convex*, cf. [38].

REMARK 2.4. Throughout this manuscript, given $U \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$, we call (2.15) the Lasry–Lions monotonicity condition and call (2.16) the displacement monotonicity condition. Indeed, it is obvious that (2.14) and (2.16) are equivalent. We prove in the Appendix that (2.13) and (2.15) are also equivalent.

REMARK 2.5. (i) (2.16) implies that U is convex in x, namely, $\partial_{xx} U$ is nonnegative definite. We provide a simple proof in Lemma 2.6 below, and we refer to [28], Proposition B.6, for a more general result. Note that, in particular, (2.15) does not imply (2.16). Indeed, let $U(x, \mu) = U_0(x) + U_1(\mu)$ such that $\partial_{xx} U_0$ is not nonnegative definite. Then, $\partial_{x\mu} U(x, \mu, \tilde{x}) \equiv 0$, and so (2.15) holds while $\partial_{xx} U = \partial_{xx} U_0$ is not nonnegative definite. Thus, (2.16) fails.

(ii) For any function $U \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$ with $|\partial_{xx} U|$ and $|\partial_{x\mu} U|$ bounded above by $C > 0$, the function $\bar{U}(x, \mu) := U(x, \mu) + C|x|^2$ will always satisfy (2.16),

$$\begin{aligned} (d_x d)_\xi \bar{U}(\eta, \eta) &= \tilde{\mathbb{E}}\big[\langle \partial_{x\mu} U(\xi, \mu, \tilde{\xi})\tilde{\eta}, \eta \rangle + \langle \partial_{xx} U(\xi, \mu)\eta, \eta \rangle + 2C|\eta|^2\big] \\ &\geq \tilde{\mathbb{E}}\big[-C|\eta||\tilde{\eta}| - C|\eta|^2 + 2C|\eta|^2\big] = C\big[\mathbb{E}[|\eta|^2] - |\mathbb{E}[\eta]|^2\big] \geq 0. \end{aligned}$$

This means that (2.16) does not imply (2.15) either. Indeed, if U is a function violating (2.15) but having bounded derivatives, then the above $\bar{U}$ satisfies (2.16). But, since $\partial_{x\mu}\bar{U} = \partial_{x\mu}U$, $\bar{U}$ violates (2.15).

(iii) We note that, for the function $\bar{U}$ above, $\partial_x \bar{U}$ is unbounded. For our main results later, we need displacement monotone functions with bounded derivatives. One can construct such an example as follows. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be convex, even and smooth with bounded derivatives. Set $U(x,\mu) := \int_{\mathbb{R}^d} \phi(x-y)\mu(dy)$. Then, U satisfies (2.16), and its derivatives are bounded.

LEMMA 2.6. *Assume $\partial_x U \in \mathcal{C}^1(\mathbb{R}^d \times \mathcal{P}_2)$ and U satisfies (2.16). Then, $\partial_{xx} U$ is nonnegative definite.*

PROOF. Without loss of generality, we assume that μ has a positive and smooth density ρ. For $\xi \in \mathbb{L}^2(\mathcal{F}_T^1, \mu)$, $x_0 \in \mathbb{R}^d$ and $\eta_\varepsilon = v_\varepsilon(\xi)$, where $v \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and for $\varepsilon > 0$, denote $v_\varepsilon(x) := \varepsilon^{-d} v(\frac{x-x_0}{\varepsilon})$. We see that $\tilde{\eta}_\varepsilon = v_\varepsilon(\tilde{\xi})$. Then, straightforward calculation reveals

$$(d_x d)_\xi U(\eta_\varepsilon, \eta_\varepsilon) = \int_{\mathbb{R}^{2d}} \langle \partial_{x\mu} U(x_0 + \varepsilon z, \mu, x_0 + \varepsilon \tilde{z}) v(\tilde{z}), v(z)\rangle \rho(x_0 + \varepsilon z)\rho(x_0 + \varepsilon \tilde{z})\, dz\, d\tilde{z}$$
$$+ \varepsilon^{-d} \int_{\mathbb{R}^d} \langle \partial_{xx} U(x_0 + \varepsilon z, \mu) v(z), v(z)\rangle \rho(x_0 + \varepsilon z)\, dz.$$

Thus, by (2.16) we have

$$0 \le \lim_{\varepsilon \to 0} \big[\varepsilon^d (d_x d)_\xi U(\eta_\varepsilon, \eta_\varepsilon)\big] = \rho(x_0) \int_{\mathbb{R}^d} \langle \partial_{xx} U(x_0, \mu) v(z), v(z)\rangle\, dz.$$

Since $\rho(x_0) > 0$ and v is arbitrary, this implies immediately that $\partial_{xx} U(x_0, \mu)$ is nonnegative definite. □

In Section 5 below, we will also use the following notion of displacement semimonotonicity, inspired by the displacement semiconvexity in potential games (cf. [7, 12]).

DEFINITION 2.7. Assume $U, \partial_x U \in \mathcal{C}^1(\mathbb{R}^d \times \mathcal{P}_2)$. We say U is *displacement semimonotone* if there exists a constant $\lambda \ge 0$ such that, for any $\xi, \eta \in \mathbb{L}^2(\mathcal{F}_T^1)$,

$$(d_x d)_\xi U(\eta, \eta) \ge -\lambda \mathbb{E}\big[|\eta|^2\big]. \tag{2.17}$$

REMARK 2.8. It is obvious that displacement semimonotonicity is weaker than the displacement monotonicity. Moreover, when $\partial_{xx} U$ is bounded, the Lasry–Lions monotonicity (2.13) also implies the displacement semimonotonicity.

2.4. *The master equation and mean field games*. In this subsection we summarize, in an informal and elementary way, the well-known connection between the solutions of the master equation (1.1) and the value functions arising in mean field games (cf., e.g., [24, 25]). We recall $\beta \ge 0$ represents the intensity of the common noise and L, G are two given functions,

$$L : \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R} \quad \text{and} \quad G : \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}$$

that are continuous in all variables. As usual, the Legendre–Fenchel transform of the Lagrangian L with respect to the last variable is the Hamiltonian H defined as

$$H(x, \mu, p) := \sup_{a \in \mathbb{R}^d} \big[-\langle a, p\rangle - L(x, \mu, a)\big], \quad (x, p, \mu) \in \mathbb{R}^{2d} \times \mathcal{P}_2. \tag{2.18}$$

Given $t \in [0, T]$, we set

$$B^t_s := B_s - B_t, \qquad B^{0,t}_s := B^0_s - B^0_t \quad \forall s \in [t, T]$$

and denote by $\mathcal{A}_t$ the set of admissible controls $\alpha : [t, T] \times \mathbb{R}^d \times C([t, T]; \mathbb{R}^d) \to \mathbb{R}^d$ that are uniformly Lipschitz continuous in the second variable, progressively measurable and adapted. For any $\xi \in \mathbb{L}^2(\mathcal{F}_t)$ and $\alpha \in \mathcal{A}_t$, by the Lipschitz continuity property of α, the SDE

$$X^{t,\xi,\alpha}_s = \xi + \int_t^s \alpha_r\big(X^{t,\xi,\alpha}_r, B^{0,t}_\cdot\big)\, dr + B^t_s + \beta B^{0,t}_s, \quad s \in [t, T] \tag{2.19}$$

has a unique strong solution. We note that, by the adaptedness, the control α actually takes the form $\alpha_r(X^{t,\xi,\alpha}_r, B^{0,t}_{[t,r]})$, where, $B^{0,t}_{[t,r]}$ stands for the restriction of $B^{0,t}$ to the interval $[t, r]$. Consider the conditionally expected cost functional for the mean field game,

$$\begin{aligned} J(t, x, \xi; \alpha, \alpha') := \mathbb{E}^{\mathbb{P}}_{\mathcal{F}^0_t}\Big[&G\big(X^{t,x,\alpha'}_T, \mathcal{L}_{X^{t,\xi,\alpha}_T|\mathcal{F}^0_T}\big) \\ &+ \int_t^T L\big(X^{t,x,\alpha'}_s, \mathcal{L}_{X^{t,\xi,\alpha}_s|\mathcal{F}^0_s}, \alpha'_s\big(X^{t,x,\alpha'}_\cdot, B^{0,t}_\cdot\big)\big)\, ds\Big]. \end{aligned} \tag{2.20}$$

Here, ξ represents the initial state of the "other" players, α is the common control of the other players and (x, α') is the initial state and control of the individual player. When $\xi \in \mathbb{L}^2(\mathcal{F}^1_t)$ is independent of $\mathcal{F}^0_t$, it is clear that $J(t, x, \xi; \alpha, \alpha')$ is deterministic. One shows that

$$\xi' \in \mathbb{L}^2(\mathcal{F}^1_t), \quad \mathcal{L}_{\xi'} = \mathcal{L}_\xi \quad \Longrightarrow \quad J(t, x, \xi'; \alpha, \alpha') = J(t, x, \xi; \alpha, \alpha') \quad \forall x, \alpha, \alpha'.$$

Therefore, we may define

$$J(t, x, \mu; \alpha, \alpha') := J(t, x, \xi; \alpha, \alpha'), \quad \xi \in \mathbb{L}^2(\mathcal{F}^1_t, \mu). \tag{2.21}$$

Now, for any $(t, x, \mu) \in \Theta$ and $\alpha \in \mathcal{A}_t$, we consider the infimum

$$V(t, x, \mu; \alpha) := \inf_{\alpha' \in \mathcal{A}_t} J(t, x, \mu; \alpha, \alpha'). \tag{2.22}$$

DEFINITION 2.9. We say $\alpha^* \in \mathcal{A}_t$ is a mean field Nash equilibrium of (2.22) at (t, μ) if

$$V(t, x, \mu; \alpha^*) = J(t, x, \mu; \alpha^*, \alpha^*) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

When there is a unique mean field equilibrium for each (t, μ), denoted as $\alpha^*(t, \mu)$, it makes sense to define

$$V(t, x, \mu) := V\big(t, x, \mu; \alpha^*(t, \mu)\big). \tag{2.23}$$

Using the Itô formula (2.12), one shows that if V is sufficiently regular, then it is a classical solution to the master equation (1.1). However, we would like to point out that the theory of the global well-posedness of (1.1) that we develop will not rely explicitly on this connection.

The master equation (1.1) is also associated to the following forward-backward McKean–Vlasov SDEs on $[t_0, T]$: given t_0 and $\xi \in \mathbb{L}^2(\mathcal{F}_{t_0})$,

$$\begin{cases} X^\xi_t = \xi - \displaystyle\int_{t_0}^t \partial_p H\big(X^\xi_s, \rho_s, Z^\xi_s\big)\, ds + B^{t_0}_t + \beta B^{0,t_0}_t; \\ Y^\xi_t = G\big(X^\xi_T, \rho_T\big) + \displaystyle\int_t^T \widehat{L}\big(X^\xi_s, \rho_s, Z^\xi_s\big)\, ds - \int_t^T Z^\xi_s \cdot dB_s - \int_t^T Z^{0,\xi}_s \cdot dB^0_s, \end{cases} \tag{2.24}$$

where

$$\widehat{L}(x, \mu, p) := L\big(x, \mu, \partial_p H(x, \mu, p)\big) = p \cdot \partial_p H(x, \mu, p) - H(x, \mu, p),$$

$$\rho_t := \rho^\xi_t := \mathcal{L}_{X^\xi_t|\mathcal{F}^0_t}.$$

Given ρ as above and $x \in \mathbb{R}^d$, we consider on $[t_0, T]$ the standard decoupled FBSDE

$$
\begin{cases} X_t^x = x + B_t^{t_0} + \beta B_t^{0,t_0}; \\ Y_t^{x,\xi} = G(X_T^x, \rho_T) - \int_t^T H(X_s^x, \rho_s, Z_s^{x,\xi})\, ds \\ \qquad - \int_t^T Z_s^{x,\xi} \cdot dB_s - \int_t^T Z_s^{0,x,\xi} \cdot dB_s^0. \end{cases} \tag{2.25}
$$

Alternatively, we may consider the coupled FBSDE, instead of the decoupled one (2.25),

$$
\begin{cases} X_t^{\xi,x} = x - \int_{t_0}^t \partial_p H(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x})\, ds + B_t^{t_0} + \beta B_t^{0,t_0}; \\ Y_t^{\xi,x} = G(X_T^{\xi,x}, \rho_T) + \int_t^T \widehat{L}(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x})\, ds \\ \qquad - \int_t^T Z_s^{\xi,x} \cdot dB_s - \int_t^T Z_s^{0,\xi,x} \cdot dB_s^0. \end{cases} \tag{2.26}
$$

These FBSDEs connect to the master equation (1.1) as follows: if V is a classical solution to (1.1) and if the above FBSDEs have strong solution, then

$$
\begin{aligned} &Y_t^{\xi} = V(t, X_t^{\xi}, \rho_t), Y_t^{x,\xi} = V(t, X_t^x, \rho_t), Y_t^{\xi,x} = V(t, X_t^{\xi,x}, \rho_t), \\ &Z_t^{\xi} = \partial_x V(t, X_t^{\xi}, \rho_t), Z_t^{x,\xi} = \partial_x V(t, X_t^x, \rho_t), Z_t^{\xi,x} = \partial_x V(t, X_t^{\xi,x}, \rho_t). \end{aligned} \tag{2.27}
$$

REMARK 2.10. (i) The forward-backward SDE system (2.24)–(2.25) or (2.24)–(2.26) is called the mean field game system. Equivalently, one may also consider the following forward-backward stochastic PDE system as the mean field system on $[t_0, T]$:

$$
\begin{cases} d\rho(t,x) = \Big[\frac{\widehat{\beta}^2}{2} \operatorname{tr}(\partial_{xx}\rho(t,x)) + \operatorname{div}(\rho(t,x)\partial_p H(x, \rho(t,\cdot), \partial_x u(t,x)))\Big] dt \\ \qquad - \beta \partial_x \rho(t,x) \cdot dB_t^0; \\ du(t,x) = v(t,x) \cdot dB_t^0 - \Big[\operatorname{tr}\Big(\frac{\widehat{\beta}^2}{2} \partial_{xx} u(t,x) + \beta \partial_x v^\top(t,x)\Big) \\ \qquad - H(x, \rho(t,\cdot), \partial_x u(t,x))\Big] dt; \\ \rho(t_0, \cdot) = \mathcal{L}_\xi, \qquad u(T,x) = G(x, \rho(T,\cdot)). \end{cases} \tag{2.28}
$$

Here, the solution triple (ρ, u, v) is $\mathbb{F}^0$-progressively measurable, and $\rho(t, \cdot, \omega)$ is a (random) probability measure. The solution V to the master equation also serves as the decoupling field for this forward-backward system, that is,

$$
u(t, x, \omega) = V(t, x, \rho(t, \cdot, \omega)). \tag{2.29}
$$

(ii) In this paper we focus on the well-posedness of the master equation (1.1). It is now a folklore in the literature that, once we obtain a classical solution V (with suitably bounded derivatives), we immediately get existence and uniqueness of a mean field equilibrium α^* in (2.22) in the sense of Definition 2.9. Indeed, given V, in light of (2.27) we may decouple the forward-backward system (2.24) (or, similarly, decouple (2.28)) as

$$
X_t^{\xi} = \xi - \int_{t_0}^t \partial_p H(X_s^{\xi}, \rho_s, \partial_x V(s, X_s^{\xi}, \rho_s))\, ds + B_t^{t_0} + \beta B_t^{0,t_0}, \qquad \rho_t := \mathcal{L}_{X_t^{\xi} | \mathcal{F}_t^0}. \tag{2.30}
$$

If V is sufficiently regular, this SDE has a unique solution (X^{ξ}, ρ), and then we can easily see that

$$
\alpha^*(t, x, \omega) := -\partial_p H(x, \rho_t(\omega), \partial_x V(t, x, \rho_t(\omega)))
$$

is the unique mean field equilibrium of the game.

(iii) Given a classical solution V with bounded derivatives, in particular with bounded $\partial_{\mu\mu} V$, we can show the convergence of the corresponding N-player game. The arguments are more or less standard; see [21, 25], and we leave the details to interested readers.

3. The displacement monotonicity of nonseparable H. In this section we collect all our standing assumptions on the data that are used in this manuscript to prove our main theorems. In particular, we shall introduce our new notion of displacement monotonicity for nonseparable H. Under appropriate condition on H and recalling (1.2) for the nonlocal operator $\mathcal{N}$, it is convenient in the sequel to define the operator

$$\mathscr{L}V(t,x,\mu) := -\partial_t V - \frac{\widehat{\beta}^2}{2}\operatorname{tr}(\partial_{xx}V) + H(x,\mu,\partial_x V) - \mathcal{N}V$$

which acts on the set of smooth functions on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2$.

We first specify the technical conditions on G and H. Recall the B_R and D_R in (2.9).

ASSUMPTION 3.1. We make the following assumptions on G:

(i) $G \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$ with $|\partial_x G|, |\partial_{xx} G| \le L_0^G$ and $|\partial_\mu G|, |\partial_{x\mu} G| \le L_1^G$;

(ii) $G, \partial_x G, \partial_{xx} G \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$, and $\partial_\mu G, \partial_{x\mu} G \in C^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d)$, and the supremum norms of all their derivatives are uniformly bounded.

ASSUMPTION 3.2. We make the following assumptions on H:

(i) $H \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d)$, and, for any $R > 0$, there exists $L^H(R)$ such that

$$|\partial_x H|, |\partial_p H|, |\partial_{xx} H|, |\partial_{xp} H|, |\partial_{pp} H| \le L^H(R) \quad \text{on } D_R;$$
$$|\partial_\mu H|, |\partial_{x\mu} H|, |\partial_{p\mu} H| \le L^H(R) \quad \text{on } \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d \times B_R;$$

(ii) $H \in \mathcal{C}^3(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d)$, and

$$H, \partial_x H, \partial_p H, \partial_{xx} H, \partial_{xp} H, \partial_{pp} H, \partial_{xxp} H, \partial_{xpp} H, \partial_{ppp} H \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d),$$

the supremum norms of all their derivatives are uniformly bounded on D_R and

$$\partial_\mu H, \partial_{x\mu} H, \partial_{p\mu} H, \partial_{xp\mu} H, \partial_{pp\mu} H \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^{2d}),$$

and the supremum norms of all their derivatives are bounded on $\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d \times B_R$;

(iii) There exists $C_0 > 0$ such that

$$|\partial_x H(x,\mu,p)| \le C_0(1+|p|) \quad \text{for any } (x,\mu,p) \in \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d;$$

(iv) H is strictly convex in p, and, for any $R > 0$, there exists $L^H(R)$ such that

$$\left|(\partial_{pp} H(x,\mu,p))^{-\frac{1}{2}} \partial_{p\mu} H(x,\mu,\tilde{x},p)\right| \le L^H(R)$$
$$\text{for any } (x,\mu,\tilde{x},p) \in \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d \times B_R.$$

REMARK 3.3. (i) Given a function $U \in \mathcal{C}^1(\mathcal{P}_2)$, one can easily see that U is uniformly W_1-Lipschitz continuous if and only if $\partial_\mu U$ is bounded.

(ii) Under Assumption 3.1 and by the above remark, we see that G and $\partial_x G$ are uniformly Lipschitz continuous in μ under W_1 on $\mathbb{R}^d \times \mathcal{P}_2$ with Lipschitz constant L_1^G. This implies further the Lipchitz continuity of G, $\partial_x G$ in μ under W_2 on $\mathbb{R}^d \times \mathcal{P}_2$, and we denote the Lipschitz constant by $L_2^G \le L_1^G$,

$$\tilde{\mathbb{E}}\big[|\partial_\mu G(x,\mu,\tilde{\xi})\tilde{\eta}|\big] \le L_2^G\big(\mathbb{E}[|\eta|^2]\big)^{\frac{1}{2}}, \qquad \tilde{\mathbb{E}}\big[|\partial_{x\mu} G(x,\mu,\tilde{\xi})\tilde{\eta}|\big] \le L_2^G\big(\mathbb{E}[|\eta|^2]\big)^{\frac{1}{2}}, \tag{3.1}$$

for all $\xi \in \mathbb{L}^2(\mathcal{F}_T^1, \mu)$, $\eta \in \mathbb{L}^2(\mathcal{F}_T^1)$. Similarly, under Assumption 3.2, H, $\partial_x H$, $\partial_p H$ are uniformly Lipschitz continuous in μ under W_1 (or W_2) on $\mathbb{R}^d \times \mathcal{P}_2 \times B_R$ with Lipschitz constant $L^H(R)$.

We now introduce the crucial notion of displacement monotonicity for nonseparable H.

DEFINITION 3.4. Let H be a Hamiltonian satisfying 3.2(i) and (iv). We say that H is *displacement monotone* if, for any $\mu \in \mathcal{P}_2$, $\xi \in \mathbb{L}^2(\mathcal{F}_T^1, \mu)$ and any bounded Lipschitz continuous function $\varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, the following bilinear form is nonpositive definite on $\eta \in \mathbb{L}^2(\mathcal{F}_T^1)$:

$$
\begin{aligned}
&\big(\mathrm{displ}_\xi^\varphi H\big)(\eta, \eta) := (d_x d)_\xi^\varphi H(\eta, \eta) + Q_\xi^\varphi H(\eta, \eta) \le 0 \quad \text{where} \\
&(d_x d)_\xi^\varphi H(\eta, \eta) := \tilde{\mathbb{E}}\big[\big\langle \partial_{x\mu} H\big(\xi, \mu, \tilde{\xi}, \varphi(\xi)\big)\tilde{\eta} + \partial_{xx} H\big(\xi, \mu, \varphi(\xi)\big)\eta, \eta\big\rangle\big]; \\
&Q_\xi^\varphi H(\eta, \eta) := \frac{1}{4}\mathbb{E}\big[\big|\big(\partial_{pp} H\big(\xi, \mu, \varphi(\xi)\big)\big)^{-\frac{1}{2}} \tilde{\mathbb{E}}_{\mathcal{F}_T^1}\big[\partial_{p\mu} H\big(\xi, \mu, \tilde{\xi}, \varphi(\xi)\big)\tilde{\eta}\big]\big|^2\big].
\end{aligned} \tag{3.2}
$$

The following assumptions are central in our work.

ASSUMPTION 3.5. (i) G satisfies Assumption 3.1(i), and it is displacement monotone; namely, it satisfies (2.16).

(ii) H satisfies Assumptions 3.2(i), (iv) and is displacement monotone; namely, (3.2) holds.

REMARK 3.6. (i) When $H(x, \mu, p) = H_0(p) - F(x, \mu)$, (3.2) reads off

$$
\big(\mathrm{displ}_\xi^\varphi H\big)(\eta, \eta) = -(d_x d)_\xi F(\eta, \eta) \le 0.
$$

This is precisely the displacement monotonicity condition (2.16) on F, and so (3.2) is an extension of the displacement monotonicity to the functions on $\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d$.

(ii) Under Assumptions 3.1 and 3.2, we may weaken the requirement in Definition 3.4 such that (3.2) holds true only for those φ satisfying $|\varphi| \le C_1^x$, $|\partial_x \varphi| \le C_2^x$ for the constants C_1^x, C_2^x determined in (6.2) below. All the results in this paper will remain true under this weaker condition.

(iii) As in Remark 2.5(i), one can easily see that (3.2) implies $\partial_{xx} H$ is nonpositive definite. This will be useful in the proof of Proposition 3.7.

PROPOSITION 3.7. *Under Assumptions* 3.2(i) *and* (iv), H *is displacement monotone if and only if* (3.2) *holds true for* $\sigma(\xi)$*-measurable* η; *namely*, $\eta = v(\xi)$ *for some deterministic function* v. *That is, by writing in integral form,* H *is displacement monotone if and only if, for any* $\mu \in \mathcal{P}_2$, $v \in \mathbb{L}_\mu^2(\mathbb{R}^d; \mathbb{R}^d)$ (*defined in Section* 2.2) *and any bounded Lipschitz continuous function* $\varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, *it holds*

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \big\langle \partial_{x\mu} H\big(x, \mu, \tilde{x}, \varphi(x)\big)v(\tilde{x}) + \partial_{xx} H\big(x, \mu, \varphi(x)\big)v(x), v(x)\big\rangle \mu(dx)\mu(d\tilde{x}) \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^d}\bigg[\bigg|\big(\partial_{pp} H\big(x, \mu, \varphi(x)\big)\big)^{-\frac{1}{2}} \int_{\mathbb{R}^d}\big[\partial_{p\mu} H\big(x, \mu, \tilde{x}, \varphi(x)\big)v(\tilde{x})\big]\mu(d\tilde{x})\bigg|^2\bigg]\mu(dx) \\
&\le 0.
\end{aligned} \tag{3.3}
$$

In particular, when H *is separable, namely,* $\partial_{p\mu} H = 0$ *and hence* $Q_\xi^\varphi H(\eta, \eta) = 0$, *then* (3.3) *reduces to*

$$
\int_{\mathbb{R}^{2d}} \big\langle \partial_{x\mu} H\big(x, \mu, \tilde{x}, \varphi(x)\big)v(\tilde{x}) + \partial_{xx} H\big(x, \mu, \varphi(x)\big)v(x), v(x)\big\rangle \mu(dx)\mu(d\tilde{x}) \le 0. \tag{3.4}
$$

PROOF. First, assume (3.2) holds. For any desired μ, v, φ, let $\xi \in \mathbb{L}^2(\mathcal{F}_T^1, \mu)$ and $\eta := v(\xi)$. Note that $\tilde{\eta} = v(\tilde{\xi})$ for the same function v. Then, (3.3) is exactly the integral form of (3.2).

We now prove the opposite direction. Assume (3.3) holds true. Following the same line of arguments as in the proof of Remark 2.5(i), one first shows that $\partial_{xx} H$ is nonpositive definite. Now, for any $\xi \in \mathbb{L}^2(\mathcal{F}_T^1, \mu)$ and $\eta \in \mathbb{L}^2(\mathcal{F}_T^1)$, denote $\eta' := \mathbb{E}[\eta|\xi]$. Then, there exists $v \in \mathbb{L}^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ such that $\eta' = v(\xi)$. Note that

$$\tilde{\eta}' := \tilde{\mathbb{E}}[\tilde{\eta}|\mathcal{F}_T^1, \tilde{\xi}] = \tilde{\mathbb{E}}[\tilde{\eta}|\tilde{\xi}] = v(\tilde{\xi})$$

for the same function v. Then, (3.3) implies that (3.2) holds for $(\eta', \tilde{\eta}')$. Note that, by the independence of $(\tilde{\xi}, \tilde{\eta})$ and (ξ, η), we have

$$\tilde{\mathbb{E}}[\langle \partial_{x\mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}, \eta\rangle] = \tilde{\mathbb{E}}[\langle \partial_{x\mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}', \eta'\rangle],$$
$$\tilde{\mathbb{E}}_{\mathcal{F}_T^1}[\partial_{p\mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}] = \tilde{\mathbb{E}}_{\mathcal{F}_T^1}[\partial_{p\mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}'].$$

Since $\partial_{xx} H$ is nonpositive definite, we have

$$\mathbb{E}[\langle \partial_{xx} H(\xi, \mu, \varphi(\xi))\eta, \eta\rangle] \le \mathbb{E}[\langle \partial_{xx} H(\xi, \mu, \varphi(\xi))\eta', \eta'\rangle].$$

We combine all these to obtain

$$(\text{displ}_\xi^\varphi H)(\eta, \eta) \le (\text{displ}_\xi^\varphi H)(\eta', \eta') \le 0$$

which completes the proof. $\square$

We next provide an example of nonseparable H which satisfies all of our assumptions. We first note that, similar to Remark 2.5(ii), for any $H \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d)$ with bounded second-order derivates, the function $H(x, \mu, p) - C|x|^2$ always satisfies (3.2) for $C > 0$ large enough. However, this function $H(x, \mu, p) - C|x|^2$ fails to be Lipschitz in x. We thus modify it as follows.

Let $H_0(x, \mu, p)$ be any smooth function with bounded derivatives up to the appropriate order so that H_0 satisfies Assumption 3.2(i)–(ii). Suppose for some constant $R_0 > 0$,

$$H_0(x, \mu, p) = 0 \quad \text{when } |x| > R_0 \quad \text{and} \quad \partial_\mu H_0(x, \mu, \tilde{x}, p) = 0 \quad \text{when } |\tilde{x}| > R_0. \tag{3.5}$$

A particular example of H_0 satisfying both conditions in (3.5) is

$$H_0(x, \mu, p) = h\left(x, p, \int_{\mathbb{R}^d} f(x, \tilde{x}, p)\mu(d\tilde{x})\right),$$

where f and h are smooth, $h(x, p, r) = 0$ for $|x| > R_0$ and $\partial_{\tilde{x}} f(x, \tilde{x}, p) = 0$ for $|\tilde{x}| \ge R_0$. Let $\psi_C : \mathbb{R}^d \to \mathbb{R}$ be a smooth and convex function such that $\psi_C(x) = C|x|^2$ when $|x| \le R_0$ and $\psi_C(x)$ growth linearly when $|x| \ge R_0 + 1$. Then, we have the following result.

LEMMA 3.8. *If C_0 is sufficiently large, then the Hamiltonian*

$$H(x, \mu, p) := H_0(x, \mu, p) + C_0|p|^2 - \psi_{C_0}(x) \tag{3.6}$$

satisfies Assumption 3.2 and is displacement monotone.

PROOF. It is straightforward to verify Assumption 3.2(i), (ii), (iii), and H also satisfies Assumption 3.2(iv) when C_0 is large enough. Then, it remains to prove (3.2). Let $C > 0$ be a bound of $\partial_{x\mu} H_0$, $\partial_{xx} H_0$, $\partial_{p\mu} H_0$, and choose C_0 such that

$$2C_0 > 3C, \partial_{pp} H_0 + 2C_0 I_d \ge I_d.$$

We first note that

$$\partial_{x\mu}H = \partial_{x\mu}H_0, \qquad \partial_{p\mu}H = \partial_{p\mu}H_0, \qquad \partial_{pp}H = \partial_{pp}H_0 + 2C_0 I_d \geq I_d,$$
$$\partial_{xx}H(x,\mu,p) = \partial_{xx}H_0(x,\mu,p) - 2C_0 I_d \mathbf{1}_{\{|x|\leq R_0\}} - \partial_{xx}\psi_{C_0}(x)\mathbf{1}_{\{|x|>R_0\}}.$$

By (3.5) we have

$$\begin{aligned}
(\mathrm{displ}^{\varphi}_{\xi} H)(\eta,\eta) = \mathbb{E}[&\langle \tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{x\mu}H_0(\xi,\mu,\tilde{\xi},\varphi(\xi))\tilde{\eta}],\eta\rangle \\
&+ \mathbf{1}_{\{|\xi|\leq R_0\}}\langle[\partial_{xx}H_0(\xi,\mu,\varphi(\xi)) - 2C_0 I_d]\eta,\eta\rangle \\
&- \mathbf{1}_{\{|\xi|>R_0\}}\Big[\langle \partial_{xx}\psi_{C_0}(\xi)\eta,\eta\rangle \\
&+ \frac{1}{4}\big|[2C_0 I_d + \partial_{pp}H_0(\xi,\mu,\varphi(\xi))]^{-\frac{1}{2}}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{p\mu}H_0(\xi,\mu,\tilde{\xi},\varphi(\xi))\tilde{\eta}]\big|^2\Big].
\end{aligned}$$

We use Jensen's inequality, the assumption on C_0 and by the convexity of ψ_{C_0} to obtain

$$\begin{aligned}
(\mathrm{displ}^{\varphi}_{\xi} H)(\eta,\eta) &\leq \mathbb{E}[C\mathbf{1}_{\{|\xi|\leq R_0\}}|\eta|\tilde{\mathbb{E}}[\mathbf{1}_{\{|\tilde{\xi}|\leq R_0\}}|\tilde{\eta}|] + [C-2C_0]\mathbf{1}_{\{|\xi|\leq R_0\}}|\eta|^2 \\
&\quad + C[\tilde{E}[\mathbf{1}_{\{|\tilde{\xi}|\leq R_0\}}|\tilde{\eta}|]]^2 - \mathbf{1}_{\{|\xi|>R_0\}}\langle[\partial_{xx}\psi_{C_0}(\xi)\eta,\eta\rangle] \\
&\leq [C-2C_0]\mathbb{E}[\mathbf{1}_{\{|\xi|\leq R_0\}}|\eta|^2] + 2C(\mathbb{E}[\mathbf{1}_{\{|\xi|\leq R_0\}}|\eta|])^2 \\
&\quad - \mathbb{E}[\mathbf{1}_{\{|\xi|>R_0\}}\langle[\partial_{xx}\psi_{C_0}(\xi)\eta,\eta\rangle] \leq 0.
\end{aligned}$$

Thus, H satisfies (3.2). □

We next express the displacement monotonicity of H in terms of L, defined through (2.18).

PROPOSITION 3.9. *Let H be such that Assumptions 3.2*(i) *and* (iv) *hold. Let $\mu \in \mathcal{P}_2$:*

(i) *H satisfies* (3.2) *if and only if L satisfies the following*:

$$\begin{aligned}
&\tilde{\mathbb{E}}[\langle \partial_{x\mu}L(\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta},\eta\rangle + \langle \partial_{xx}L(\xi,\mu,\psi(\xi))\eta,\eta\rangle] \\
&\quad \geq \mathbb{E}\Big[\Big|[\partial_{aa}L(\xi,\mu,\psi(\xi))]^{-\frac{1}{2}}\Big[\frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{a\mu}L(\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta}] + \partial_{ax}L(\xi,\mu,\psi(\xi))\eta\Big]\Big|^2\Big]
\end{aligned} \tag{3.7}$$

for all μ, ξ, η, φ, as in Definition 3.4, and $\psi(x) := -\partial_p H(x,\mu,\varphi(x))$.

(ii) *A sufficient condition for L to satisfy* (3.7) *and hence for H to satisfy* (3.2) *is*

$$\Lambda := \frac{d^2}{d\varepsilon\, d\delta}\mathbb{E}[L(\xi + (\varepsilon+\delta)\eta, \mathcal{L}_{\xi+\varepsilon\eta}, \xi' + (\varepsilon+\delta)\eta')]\big|_{(\varepsilon,\delta)=(0,0)} \geq 0, \tag{3.8}$$

for all $\xi, \xi', \eta, \eta' \in \mathbb{L}^2(\mathcal{F}^1_T)$.

PROOF. (i) First, standard convex analysis theory ensures regularity properties of L. The optimal argument $a^* = a^*(x,\mu,p)$ satisfies

$$H(x,\mu,p) = -L(x,\mu,a^*) - \langle a^*, p\rangle, \qquad \partial_a L(x,\mu,a^*) + p = 0, \qquad a^* = -\partial_p H(x,\mu,p).$$

One can easily derive further the following identities (some of them are well known in convex analysis):

$$
(3.9)\quad
\begin{aligned}
&\partial_x H(x,\mu,p) = -\partial_x L(x,\mu,a^*); \qquad \partial_\mu H(x,\mu,\tilde{x},p) = -\partial_\mu L(x,\mu,\tilde{x},a^*);\\
&\partial_{aa} L \geq \frac{1}{L^H(R)} I_d \quad \text{on } D_R \quad \text{and} \quad \partial_{pp} H(x,\mu,p) = \big[\partial_{aa} L(x,\mu,a^*)\big]^{-1};\\
&\partial_{xp} H(x,\mu,p) = \partial_{xa} L(x,\mu,a^*)\big[\partial_{aa} L(x,\mu,a^*)\big]^{-1};\\
&\partial_{xx} H(x,\mu,p) = -\partial_{xx} L(x,\mu,a^*) + \partial_{xp} H(x,\mu,p)\partial_{ax} L(x,\mu,a^*)\\
&\qquad = \big[-\partial_{xx} L + \partial_{xa} L[\partial_{aa} L]^{-1}\partial_{ax} L\big](x,\mu,a^*);\\
&\partial_{x\mu} H(x,\mu,\tilde{x},p) = -\partial_{x\mu} L(x,\mu,\tilde{x},a^*) + \partial_{xp} H(x,\mu,p)\partial_{a\mu} L(x,\mu,\tilde{x},a^*)\\
&\qquad = \big[-\partial_{x\mu} L + \partial_{xa} L[\partial_{aa} L]^{-1}\partial_{a\mu} L\big](x,\mu,\tilde{x},a^*);\\
&\partial_{p\mu} H(x,\mu,\tilde{x},p) = \partial_{pp} H(x,\mu,p)\partial_{a\mu} L(x,\mu,\tilde{x},a^*)\\
&\qquad = \big[\partial_{aa} L(x,\mu,a^*)\big]^{-1}\partial_{a\mu} L(x,\mu,\tilde{x},a^*).
\end{aligned}
$$

Now, let φ be chosen as in Definition 3.4 and $\psi(x) := -\partial_p H(x,\mu,\varphi(x))$, then we have

$$
\begin{aligned}
&-(\mathrm{displ}^{\varphi}_{\xi} H)(\eta,\eta)\\
&= \tilde{\mathbb{E}}\Big[\big\langle\big[\partial_{x\mu} L - \partial_{xa} L[\partial_{aa} L]^{-1}\partial_{a\mu} L\big](\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta},\eta\big\rangle\\
&\quad + \big\langle\big[\partial_{xx} L - \partial_{xa} L[\partial_{aa} L]^{-1}\partial_{ax} L\big](\xi,\mu,\psi(\xi))\eta,\eta\big\rangle\\
&\quad - \frac{1}{4}\big|\big[\partial_{aa} L(\xi,\mu,\psi(\xi))\big]^{-\frac{1}{2}}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}\big[\partial_{a\mu} L(\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta}\big]\big|^2\Big]\\
&= \tilde{\mathbb{E}}\Big[\big\langle\partial_{x\mu} L(\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta},\eta\big\rangle + \big\langle\partial_{xx} L(\xi,\mu,\psi(\xi))\eta,\eta\big\rangle\\
&\quad - \Big|\big[\partial_{aa} L(\xi,\mu,\psi(\xi))\big]^{-\frac{1}{2}}\Big[\frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}\big[\partial_{a\mu} L(\xi,\mu,\tilde{\xi},\psi(\xi))\tilde{\eta}\big] + \partial_{ax} L(\xi,\mu,\psi(\xi))\eta\Big]\Big|^2\Big].
\end{aligned}
$$

Then clearly, (3.2) is equivalent to (3.7).

(ii) Assume (3.8) holds and $\xi, \xi', \eta, \eta' \in \mathbb{L}^2(\mathcal{F}^1_T)$. By straightforward calculations we have

$$
\begin{aligned}
\Lambda &= \frac{d}{d\varepsilon}\mathbb{E}\big[\big\langle\partial_x L(\xi+\varepsilon\eta,\mathcal{L}_{\xi+\varepsilon\eta},\xi'+\varepsilon\eta'),\eta\big\rangle + \big\langle\partial_a L(\xi+\varepsilon\eta,\mathcal{L}_{\xi+\varepsilon\eta},\xi'+\varepsilon\eta'),\eta'\big\rangle\big]\big|_{\varepsilon=0}\\
&= \tilde{\mathbb{E}}\big[\big\langle\partial_{xx} L(\xi,\mathcal{L}_\xi,\xi')\eta,\eta\big\rangle + \big\langle\partial_{x\mu} L(\xi,\mathcal{L}_\xi,\tilde{\xi},\xi')\tilde{\eta},\eta\big\rangle\\
&\quad + 2\big\langle\partial_{ax} L(\xi,\mathcal{L}_\xi,\xi')\eta,\eta'\big\rangle + \big\langle\partial_{a\mu} L(\xi,\mathcal{L}_\xi,\tilde{\xi},\xi')\tilde{\eta},\eta'\big\rangle + \big\langle\partial_{aa} L(\xi,\mathcal{L}_\xi,\xi')\eta',\eta'\big\rangle\big].
\end{aligned}
$$

The expression Λ remains nonnegative in particular when

$$
\begin{aligned}
&\xi' := \psi(\xi) \quad \text{and}\\
&\eta' := -\big(\partial_{aa} L(\xi,\mathcal{L}_\xi,\xi')\big)^{-1}\Big(\frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}\big[\partial_{a\mu} L(\xi,\mathcal{L}_\xi,\tilde{\xi},\xi')\tilde{\eta}\big] + \partial_{ax} L(\xi,\mathcal{L}_\xi,\xi')\eta\Big).
\end{aligned}
$$

Omitting the variables $(\xi,\mathcal{L}_\xi,\tilde{\xi},\xi')$ inside the derivatives of L, we have

$$
\begin{aligned}
0 \leq \Lambda &= \mathbb{E}\Big[\langle\partial_{xx} L\eta,\eta\rangle + \big\langle\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{x\mu} L\tilde{\eta}],\eta\big\rangle + 2\Big\langle\partial_{ax} L\eta + \frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{a\mu} L\tilde{\eta}],\eta'\Big\rangle + \langle\partial_{aa} L\eta',\eta'\rangle\Big]\\
&= \mathbb{E}\Big[\langle\partial_{xx} L\eta,\eta\rangle + \big\langle\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{x\mu} L\tilde{\eta}],\eta\big\rangle + \Big|[\partial_{aa} L]^{\frac{1}{2}}\eta' + [\partial_{aa} L]^{-\frac{1}{2}}\Big[\partial_{ax} L\eta + \frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}^1_T}[\partial_{a\mu} L\tilde{\eta}]\Big]\Big|^2
\end{aligned}
$$

$$- \left| [\partial_{aa} L]^{-\frac{1}{2}} \left[\partial_{ax} L\eta + \frac{1}{2} \tilde{\mathbb{E}}_{\mathcal{F}_T^1} [\partial_{a\mu} L\tilde{\eta}] \right] \right|^2 \Bigg]$$

$$= \mathbb{E}\left[\langle \partial_{xx} L\eta, \eta \rangle + \langle \tilde{\mathbb{E}}_{\mathcal{F}_T^1} [\partial_{x\mu} L\tilde{\eta}], \eta \rangle - \left| [\partial_{aa} L]^{-\frac{1}{2}} \left[\partial_{ax} L\eta + \frac{1}{2} \tilde{\mathbb{E}}_{\mathcal{F}_T^1} [\partial_{a\mu} L\tilde{\eta}] \right] \right|^2 \right].$$

This is exactly (3.7). □

REMARK 3.10. Observe that (3.8) expresses a certain convexity property of L. To illustrate this, consider the separable case, where $L(x, \mu, a) := L_0(x, \mu) + L_1(x, a)$. Then, $\Lambda = \Lambda_0 + \Lambda_1$, where

$$\Lambda_0 := \frac{d^2}{d\varepsilon\, d\delta} \mathbb{E}\big[L_0(\xi + \varepsilon\eta + \delta\eta, \mathcal{L}_{\xi+\varepsilon\eta})\big]\big|_{(\varepsilon,\delta)=(0,0)},$$

$$\Lambda_1 := \frac{d^2}{d\varepsilon\, d\delta} \mathbb{E}\big[L_1\big(\xi + \varepsilon\eta + \delta\eta, \xi' + \varepsilon\eta' + \delta\eta'\big)\big]\big|_{(\varepsilon,\delta)=(0,0)},$$

and so $\Lambda_0 \geq 0$, $\Lambda_1 \geq 0$ implies (3.7). Note that $\Lambda_1 \geq 0$ exactly means L_1 is convex in (x, a). Moreover, consider the potential game case for L_0: $\partial_x L_0(x, \mu) = \partial_\mu \widehat{L}_0(\mu, x)$ for some function $\widehat{L}_0(\mu)$. Then,

$$\Lambda_0 = \frac{d}{d\varepsilon} \mathbb{E}\big[\langle \partial_x L_0(\xi + \varepsilon\eta, \mathcal{L}_{\xi+\varepsilon\eta}), \eta \rangle\big]\big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \mathbb{E}\big[\langle \partial_\mu \widehat{L}_0(\mathcal{L}_{\xi+\varepsilon\eta}, \xi + \varepsilon\eta), \eta \rangle\big]\big|_{\varepsilon=0}$$
$$= \frac{d^2}{d\varepsilon^2} \widehat{L}_0(\mathcal{L}_{\xi+\varepsilon\eta})\big|_{\varepsilon=0}.$$

Thus, $\Lambda_0 \geq 0$ exactly means $\widehat{L}_0$ is displacement convex, namely, the mapping $\xi \mapsto \widehat{L}_0(\mathcal{L}_\xi)$ is convex. These are the same displacement convexity assumptions on the data for potential deterministic mean field master equations imposed in [28]. In particular, (3.8) is reminiscent to the joint convexity assumption on the Lagrangian (Assumption (H7)) in [28].

4. The displacement monotonicity of V. In this section we show that, under our standing assumptions, the displacement monotonicity condition is propagated along any classical solution of the master equation. More precisely, let H and G satisfy our standing assumptions of the previous section, and in particular, suppose that they are displacement monotone in the sense of Assumption 3.5.

THEOREM 4.1. *Let Assumptions* 3.1 *and* 3.2(i), (iv) *and* 3.5 *hold, and* V *be a classical solution of the master equation* (1.1). *Assume further that*

$$V(t, \cdot, \cdot), \partial_x V(t, \cdot, \cdot), \partial_{xx} V(t, \cdot, \cdot) \in \mathcal{C}^2\big(\mathbb{R}^d \times \mathcal{P}_2\big),$$
$$\partial_\mu V(t, \cdot, \cdot, \cdot), \partial_{x\mu} V(t, \cdot, \cdot, \cdot) \in \mathcal{C}^2\big(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d\big),$$

and all their derivatives in the state and probability measure variables are also continuous in the time variable and are uniformly bounded. Then, $V(t, \cdot, \cdot)$ *satisfies* (2.16) *for all* $t \in [0, T]$.

PROOF. Without loss of generality, we shall prove the thesis of the theorem only for $t_0 = 0$, that is, that $V(0, \cdot, \cdot)$ satisfies (2.16).

Fix $\xi, \eta \in \mathbb{L}^2(\mathcal{F}_0)$. Let us consider the following decoupled McKean–Vlasov SDEs:

$$
\begin{aligned}
&X_t = \xi - \int_0^t \partial_p H(X_s, \mu_s, \partial_x V(s, X_s, \mu_s))\,ds + B_t + \beta B_t^0, \qquad \mu_t := \mathcal{L}_{X_t|\mathcal{F}_t^0};\\
&\delta X_t = \eta - \int_0^t \Big[H_{px}(X_s)\delta X_s + \frac{1}{2}\tilde{\mathbb{E}}_{\mathcal{F}_s}[H_{p\mu}(X_s, \tilde{X}_s)\delta\tilde{X}_s] + H_{pp}(X_s)N_s\Big]\,ds \quad \text{where}\\
&N_t := \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t, \tilde{X}_t)\delta\tilde{X}_t] + \partial_{xx}V(X_t)\delta X_t\\
&\qquad + \frac{1}{2}H_{pp}(X_t)^{-1}\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t, \tilde{X}_t)\delta\tilde{X}_t].
\end{aligned} \tag{4.1}
$$

Here and in the sequel, for simplicity of notation we omit the variables (t, μ_t) as well as the dependence on $\partial_x V$ and denote

$$
\begin{aligned}
H_p(X_t) &:= \partial_p H(X_t, \mu_t, \partial_x V(t, X_t, \mu_t)),\\
H_{p\mu}(X_t, \tilde{X}_t) &:= \partial_{p\mu} H(X_t, \mu_t, \tilde{X}_t, \partial_x V(t, X_t, \mu_t)),
\end{aligned} \tag{4.2}
$$

and, similarly, for H_{xp}, H_{pp}, $H_{x\mu}$, $\partial_{xx}V$, $\partial_{x\mu}V$. Since V is assumed to be regular enough with $\partial_x V$, $\partial_{xx}V$, $\partial_{x\mu}V$ uniformly bounded and H satisfies Assumption 3.2(i), the driving vector field is globally Lipschitz continuous. Therefore, classical results imply the existence of unique solutions X_t and δX_t. We also observe that δX_t can be interpreted as $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[X_t^{\xi+\varepsilon\eta} - X_t^{\xi}]$ (cf. [18]).

Below, we shall use the notation, for $\theta \in \mathbb{R}^d$,

$$
\theta^\top \partial_{xx\mu}V(x, \tilde{x}) := \sum_{i=1}^d \theta_i \partial_{x_i x\mu}V(x, \tilde{x}), \qquad \operatorname{tr}(\partial_{\mu\mu})\partial_{x\mu}V := \sum_{i=1}^d \partial_{\mu_i\mu_i}\partial_{x\mu}V \tag{4.3}
$$

and, similarly, for other higher-order derivatives of V. Introduce

$$
I(t) := \tilde{\mathbb{E}}[\langle \partial_{x\mu}V(t, X_t, \mu_t, \tilde{X}_t)\delta\tilde{X}_t, \delta X_t\rangle], \qquad \bar{I}(t) := \mathbb{E}[\langle \partial_{xx}V(t, X_t, \mu_t)\delta X_t, \delta X_t\rangle].
$$

We remark that, since $(\tilde{X}_t, \delta\tilde{X}_t)$ is a conditionally independent copy of $(X_t, \delta X_t)$ and μ_t is $\mathcal{F}_t^0$-measurable, for the notations in Section 2.1 we have

$$
I(t) + \bar{I}(t) = \mathbb{E}[(d_x d)_{X_t(\omega^0,\cdot)}V(t,\cdot)(\delta X_t(\omega^0,\cdot), \delta X_t(\omega^0,\cdot))]. \tag{4.4}
$$

Our plan is to show that

$$
\dot{I}(t) + \dot{\bar{I}}(t) \le 0. \tag{4.5}
$$

Then, recalling $V(T,\cdot) = G$ and applying Assumption 3.5(i),

$$
\begin{aligned}
(d_x d)_\xi V(0,\cdot)(\eta,\eta) &= I(0) + \bar{I}(0) \ge I(T) + \bar{I}(T)\\
&= \mathbb{E}[(d_x d)_{X_T(\omega^0,\cdot)}G(\delta X_T(\omega^0,\cdot), \delta X_T(\omega^0,\cdot))] \ge 0.
\end{aligned}
$$

That is, $V(0,\cdot)$ satisfies (2.16).

To show (4.5), we apply Itô's formula (2.12) to obtain

$$
\dot{I}(t) = I_1 + I_2 + I_3, \tag{4.6}
$$

where, introducing another conditionally independent copy $\hat{X}$ of X and defining $\hat{\tilde{\mathbb{E}}}$ in the manner of (2.4),

$$
\begin{aligned}
I_1 := \hat{\tilde{\mathbb{E}}}\Big[\Big\langle\Big\{&\partial_{tx\mu}V(X_t, \tilde{X}_t) + \frac{\widehat{\beta}^2}{2}((\operatorname{tr}\partial_{xx})\partial_{x\mu}V)(X_t, \tilde{X}_t) - H_p(X_t)^\top \partial_{xx\mu}V(X_t, \tilde{X}_t)\\
&+ \beta^2(\operatorname{tr}(\partial_{x\mu})\partial_{x\mu}V)(X_t, \bar{X}_t, \tilde{X}_t) + \beta^2(\operatorname{tr}(\partial_{\tilde{x}\mu})\partial_{x\mu}V)(X_t, \bar{X}_t, \tilde{X}_t)
\end{aligned}
$$

$$+ \beta^2 \big(\mathrm{tr}(\partial_{\tilde{x}x})\partial_{x\mu} V\big)(X_t, \tilde{X}_t) + \frac{\beta^2}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_{x\mu} V\big)(X_t, \hat{X}_t, \bar{X}_t, \tilde{X}_t)$$
$$+ \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\bar{x}\mu})\partial_{x\mu} V\big)(X_t, \bar{X}_t, \tilde{X}_t) - H_p(\bar{X}_t)^\top \partial_{\mu x\mu} V(X_t, \bar{X}_t, \tilde{X}_t)$$
$$+ \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\tilde{x}\tilde{x}})\partial_{x\mu} V\big)(X_t, \tilde{X}_t) - H_p(\tilde{X}_t)^\top \partial_{\tilde{x}x\mu} V(X_t, \tilde{X}_t)\Big\}\delta\tilde{X}_t, \delta X_t\Big\rangle\Big],$$

and, rewriting $(\tilde{X}, \delta\tilde{X}, \tilde{\mathbb{E}})$ in the expression of N as $(\bar{X}, \delta\bar{X}, \bar{\mathbb{E}})$ (which does not change the value of N),

$$\begin{aligned}
I_2 &:= -\bar{\tilde{\mathbb{E}}}\big[\big\langle \partial_{\mu x} V(X_t, \tilde{X}_t)\big\{\big[H_{px}(X_t) + H_{pp}(X_t)\partial_{xx} V(X_t)\big]\delta X_t + \mathbf{II_2}\big\}, \delta\tilde{X}_t\big\rangle\big],\\
\mathbf{II_2} &:= \big[H_{p\mu}(X_t, \bar{X}_t) + H_{pp}(X_t)\partial_{x\mu} V(X_t, \bar{X}_t)\big]\delta\bar{X}_t,\\
I_3 &:= -\bar{\tilde{\mathbb{E}}}\big[\big\langle \partial_{x\mu} V(X_t, \tilde{X}_t)\big\{\big[H_{px}(\tilde{X}_t) + H_{pp}(\tilde{X}_t)\partial_{xx} V(\tilde{X}_t)\big]\delta\tilde{X}_t + \mathbf{III_3}\big\}, \delta X_t\big\rangle\big],\\
\mathbf{III_3} &:= \big[H_{p\mu}(\tilde{X}_t, \bar{X}_t) + H_{pp}(\tilde{X}_t)\partial_{x\mu} V(\tilde{X}_t, \bar{X}_t)\big]\delta\bar{X}_t.
\end{aligned}$$

We apply $-\partial_{x\mu}$ to (1.1) and rewrite $(\tilde{\xi}, \bar{\xi}, \bar{\tilde{\mathbb{E}}})$ in (1.2) as $(\bar{\xi}, \hat{\xi}, \hat{\bar{\mathbb{E}}})$ to obtain

$$0 = -(\partial_{x\mu}\mathscr{L}V)(t, x, \mu, \tilde{x}) = J_1 + J_2 + J_3. \tag{4.7}$$

Here, we have set, recalling the notation in (4.3),

$$\begin{aligned}
J_1 &:= \partial_{tx\mu} V(x, \tilde{x}) + \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{xx})\partial_{x\mu} V\big)(x, \tilde{x}) - H_{x\mu}(x, \tilde{x}) - \partial_{xx} V(x) H_{p\mu}(x, \tilde{x})\\
&\quad - \big(H_{xp}(x) + \partial_{xx} V(x) H_{pp}(x)\big)\partial_{x\mu} V(x, \tilde{x}) - H_p(x)^\top \partial_{xx\mu} V(x, \tilde{x}),\\
J_2 &:= \frac{\widehat{\beta}^2}{2}\big(\partial_{x\tilde{x}}\,\mathrm{tr}(\partial_{\tilde{x}\mu}) V\big)(x, \tilde{x}) - H_p(\tilde{x})^\top \partial_{\tilde{x}x\mu} V(x, \tilde{x})\\
&\quad - \partial_{x\mu} V(x, \tilde{x})\big(H_{px}(\tilde{x}) + H_{pp}(\tilde{x})\partial_{xx} V(\tilde{x})\big)\\
&\quad + \beta^2\big(\partial_{x\tilde{x}}\,\mathrm{tr}(\partial_{x\mu}) V\big)(x, \tilde{x}) + \beta^2 \bar{\mathbb{E}}\big[\big(\partial_{x\tilde{x}}\,\mathrm{tr}(\partial_{\mu\mu}) V\big)(x, \bar{\xi}, \tilde{x})\big],\\
J_3 &:= \hat{\bar{\mathbb{E}}}\Big[\frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\bar{x}\mu})\partial_{x\mu} V\big)(x, \tilde{x}, \bar{\xi}) - H_p(\bar{\xi})^\top \partial_{\mu x\mu} V(x, \tilde{x}, \bar{\xi})\\
&\quad - \partial_{x\mu} V(x, \bar{\xi})\big[H_{p\mu}(\bar{\xi}, \tilde{x}) + H_{pp}(\bar{\xi})\partial_{x\mu} V(\bar{\xi}, \tilde{x})\big]\\
&\quad + \beta^2\big(\mathrm{tr}(\partial_{x\mu})\partial_{x\mu} V\big)(x, \tilde{x}, \bar{\xi}) + \frac{\beta^2}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_{x\mu} V\big)(x, \tilde{x}, \hat{\xi}, \bar{\xi})\Big].
\end{aligned}$$

Note that we can switch the order of the differentiation in $\partial_x\partial_\mu$, $\partial_x\partial_{\tilde{x}}$, etc. We emphasize that special care is needed when considering $\partial_\mu\partial_{\tilde{x}}$ (since we cannot change here the order of differentiation). For such terms we use their symmetric properties, given in Lemma 2.1. By evaluating (4.7) along $(X_t, \mu_t, \tilde{X}_t)$ and plugging into (4.6), one can cancel many terms and simplify the previous derivation as

$$\begin{aligned}
\dot{I}(t) = \bar{\tilde{\mathbb{E}}}\big[&-\big\langle \partial_{\mu x} V(X_t, \tilde{X}_t)\big[H_{p\mu}(X_t, \bar{X}_t) + H_{pp}(X_t)\partial_{x\mu} V(X_t, \bar{X}_t)\big]\delta\bar{X}_t, \delta\tilde{X}_t\big\rangle\\
&+ \big\langle\big[H_{x\mu}(X_t, \tilde{X}_t) + \partial_{xx} V(X_t) H_{p\mu}(X_t, \tilde{X}_t)\big]\delta\tilde{X}_t, \delta X_t\big\rangle\big].
\end{aligned}$$

Thus, by using the tower property of conditional expectations and the conditional i.i.d. property of $(X, \delta X)$, $(\tilde{X}, \delta\tilde{X})$, $(\bar{X}, \delta\bar{X})$, we have

$$
\begin{aligned}
\dot{I}(t) = \mathbb{E}\big[&-\langle H_{pp}(X_t)\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t], \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]\rangle \\
&- \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t], \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t] - \partial_{xx}V(X_t)\delta X_t\rangle \\
&+ \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t], \delta X_t\rangle\big].
\end{aligned} \tag{4.8}
$$

Similarly as above, we apply Itô formula (2.12) to $\bar{I}(t)$ to obtain

$$\dot{\bar{I}}(t) = \bar{I}_1 + \bar{I}_2 + \bar{I}_3,$$

where

$$
\begin{aligned}
\bar{I}_1 &:= \tilde{\mathbb{E}}\bigg[\bigg\langle\bigg\{\partial_{txx}V(X_t) + \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{xx})\partial_{xx}V\big)(X_t) - H_p(X_t)^\top\partial_{xxx}V(X_t) \\
&\quad + \beta^2\big(\mathrm{tr}(\partial_{x\mu})\partial_{xx}V\big)(X_t,\tilde{X}_t)\bigg\}\delta X_t, \delta X_t\bigg\rangle\bigg], \\
\bar{I}_2 &:= \bar{\tilde{\mathbb{E}}}\bigg[\bigg\langle\bigg\{\frac{\beta^2}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_{xx}V\big)(X_t,\tilde{X}_t,\bar{X}_t) \\
&\quad + \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\tilde{x}\mu})\partial_{xx}V\big)(X_t,\tilde{X}_t) - H_p(\tilde{X}_t)^\top\partial_{\mu xx}V(X_t,\tilde{X}_t)\bigg\}\delta X_t, \delta X_t\bigg\rangle\bigg], \\
\bar{I}_3 &:= \tilde{\mathbb{E}}\big[-2\langle\partial_{xx}V(X_t)\{[H_{px}(X_t) + H_{pp}(X_t)\partial_{xx}V(X_t)]\delta X_t \\
&\quad + [H_{p\mu}(X_t,\tilde{X}_t) + H_{pp}(X_t)\partial_{x\mu}V(X_t,\tilde{X}_t)]\delta\tilde{X}_t\}, \delta X_t\rangle\big].
\end{aligned}
$$

On the other hand, applying $-\partial_{xx}$ to (1.1), we obtain

$$0 = -(\partial_{xx}\mathscr{L}V)(t,x,\mu) = \bar{J}_1 + \bar{J}_2, \tag{4.9}$$

where

$$
\begin{aligned}
\bar{J}_1 &:= \partial_{txx}V + \frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{xx})\partial_{xx}V\big) - H_{xx}(x) - 2H_{xp}(x)\partial_{xx}V(x) \\
&\quad - \partial_{xx}V(x)H_{pp}(x)\partial_{xx}V(x) - H_p(x)^\top\partial_{xxx}V(x), \\
\bar{J}_2 &:= \bar{\tilde{\mathbb{E}}}\bigg[\frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\tilde{x}\mu})\partial_{xx}V\big)(x,\tilde{\xi}) - H_p(\tilde{\xi})^\top\partial_{\mu xx}V(x,\tilde{\xi}) \\
&\quad + \beta^2\big(\mathrm{tr}(\partial_{x\mu})\partial_{xx}V\big)(x,\tilde{\xi}) + \frac{\beta^2}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_{xx}V\big)(x,\bar{\xi},\tilde{\xi})\bigg].
\end{aligned}
$$

We evaluate the previous expression along (X_t, μ_t) to obtain, after a simplification,

$$
\begin{aligned}
\dot{\bar{I}}(t) = \mathbb{E}\big[&-\langle H_{pp}(X_t)\partial_{xx}V(X_t)\delta X_t, \partial_{xx}V(X_t)\delta X_t\rangle \\
&- 2\langle H_{pp}(X_t)\partial_{xx}V(X_t)\delta X_t, \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]\rangle \\
&- 2\langle\partial_{xx}V(X_t)\delta X_t, \tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t]\rangle + \langle H_{xx}(X_t)\delta X_t, \delta X_t\rangle\big].
\end{aligned} \tag{4.10}
$$

We combine (4.8) and (4.10) to deduce that

$$
\begin{aligned}
\dot{I}(t)+\dot{\tilde{I}}(t) &= \mathbb{E}\big[-\big|H_{pp}^{\frac{1}{2}}(X_t)\{\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]+\partial_{xx}V(X_t)\delta X_t\}\big|^2 \\
&\quad -\langle\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t],\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]+\partial_{xx}V(X_t)\delta X_t\rangle \\
&\quad +\langle\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t]+H_{xx}(X_t)\delta X_t,\delta X_t\rangle\big] \\
&= \mathbb{E}\Big[-\Big|H_{pp}^{\frac{1}{2}}(X_t)\{\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]+\partial_{xx}V(X_t)\delta X_t\} \\
&\quad +\frac{1}{2}H_{pp}^{-\frac{1}{2}}(X_t)\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t]\Big|^2+\langle\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t],\delta X_t\rangle \\
&\quad +\langle H_{xx}(X_t)\delta X_t,\delta X_t\rangle+\frac{1}{4}\big|H_{pp}^{-\frac{1}{2}}(X_t)\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t]\big|^2\Big] \\
&= -\mathbb{E}\big[\big|H_{pp}^{\frac{1}{2}}(X_t)N_t\big|^2\big]+\mathbb{E}^{\mathbb{P}_0}\big[\big(\mathrm{displ}^{\varphi}_{X_t(\omega^0,\cdot)}H\big)\big(\delta\tilde{X}_t(\omega^0,\cdot),\delta\tilde{X}_t(\omega^0,\cdot)\big)\big],
\end{aligned}
\tag{4.11}
$$

where we have set $\varphi(x) := \partial_x V(t,x,\mu_t(\omega^0))$ and the last line is in the spirit of (4.4). Applying (3.2), we obtain (4.5) immediately. □

REMARK 4.2. (i) The main trick here is that we may complete the square in (4.11) for the terms involving $\partial_{xx}V$ and, more importantly, $\partial_{x\mu}V$, which is hard to estimate a priori. Since the identity is exact, (3.2) seems essential for not loosing displacement monotonicity.

Moreover, recalling (4.4), we see that

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}^{\mathbb{P}_0}\big[(d_x d)_{X_t(\omega^0,\cdot)}V(t,\cdot,\cdot)\big(\delta X_t(\omega^0,\cdot),\delta X_t(\omega^0,\cdot)\big)\big] \\
&\quad\le \mathbb{E}^{\mathbb{P}_0}\big[\big(\mathrm{displ}^{\partial_x V(t,\cdot,\mu_t(\omega^0))}_{X_t(\omega^0,\cdot)}H\big)\big(\delta\tilde{X}_t(\omega^0,\cdot),\delta\tilde{X}_t(\omega^0,\cdot)\big)\big].
\end{aligned}
\tag{4.12}
$$

So, roughly speaking, $\mathrm{displ}\,H$ measures the rate of dissipation of the displacement monotonicity of V through the bilinear form $(d_x d)V(t,\cdot,\cdot)$.

(ii) In the separable case, that is, $H(x,\mu,p) = H_0(x,p) - F(x,\mu)$ for some H_0 and F, (4.8) becomes

$$
\dot{I}(t) = -\mathbb{E}\big[\big|[(H_0)_{pp}(X_t)]^{\frac{1}{2}}\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t]\big|^2+\tilde{\mathbb{E}}_{\mathcal{F}_t}\langle[\partial_{x\mu}F(X_t,\tilde{X}_t)\delta\tilde{X}_t],\delta X_t\rangle\big].
$$

The term involving $\partial_{x\mu}V$ is again in a complete square, and it is no surprise that $V(t,\cdot,\cdot)$ would satisfy Lasry–Lions monotonicity condition (2.15), provided that the data G and F also satisfy (2.15). So, our arguments provide an alternative proof for the propagation of the Lasry–Lions monotonicity along $V(t,\cdot,\cdot)$ (and for the global well-posedness of the master equation, as a consequence of it, just as in the rest of the paper) in the case of separable Hamiltonians and Lasry–Lions monotone data.

(iii) When H is nonseparable, however, it remains a challenge to find sufficient conditions on H that could ensure the right-hand side of (4.8) being negative (for arbitrary times). This makes the propagation of the Lasry–Lions monotonicity condition along $V(t,\cdot,\cdot)$ hard to envision. In [36] a notion of monotonicity condition for nonseparable Hamiltonians that depend *locally* on the measure variable was proposed (see also [2]). This condition allows to obtain uniqueness of solutions for the corresponding MFG system.

5. The uniform Lipschitz continuity of V under W_2. The main result in this section is that the displacement semimonotone solutions to the master equation (1.1) are always uniformly W_2-Lipschitz continuous. We note in this observation that the Lipschitz continuity of $V(t,\cdot,\cdot)$ in μ (under W_2) is the consequence of the displacement semimonotonicity of $V(t,\cdot,\cdot)$ only (other than the technical conditions), which seems to be new even in the separable case. We remark again that the displacement semimonotonicity is weaker than both the displacement monotonicity and the Lasry–Lions monotonicity (if $\partial_{xx}V$ is uniformly bounded); see Remark 2.8.

THEOREM 5.1. *Let all the conditions in Theorem 4.1 hold, except that we do not require Assumption 3.5(ii). Assume further that $V(t,\cdot,\cdot)$ satisfies the displacement semimonotonicity (2.17) for each $t\in[0,T]$. Then, V and $\partial_x V$ are uniformly Lipschitz continuous in μ under W_2 with Lipschitz constant C_2^μ, where $C_2^\mu>0$ depends only on d, T, $\|\partial_x V\|_{L^\infty}$, $\|\partial_{xx}V\|_{L^\infty}$, the L_2^G in Remark 3.3(ii), the $L^H(\|\partial_x V\|_{L^\infty})$ in Assumption 3.2(i) and the λ in (2.17).*

PROOF. In this proof, $C>0$ denotes a generic constant depending only on quantities mentioned in the statement of the theorem. Without loss of generality, we show the thesis of the theorem only for $t_0=0$. We fix $\xi,\eta\in\mathbb{L}^2(\mathcal{F}_0)$ and continue to use the notation, as in the proof of Theorem 4.1. In particular, δX is defined by (4.1). First, we emphasize that the equality (4.11) does not rely on (3.2). Then, integrating (4.11) over $[0,t]$ we obtain

$$\begin{aligned}\int_0^t \mathbb{E}\big[\big|H_{pp}(X_s)^{\frac12}N_s\big|^2\big]\,ds &= \big[I(0)+\bar I(0)\big]-\big[I(t)+\bar I(t)\big]\\ &\quad+\int_0^t \mathbb{E}\Big[\big\langle \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[H_{x\mu}(X_s,\tilde X_s)\delta\tilde X_s\big],\delta X_s\big\rangle+\big\langle H_{xx}(X_s)\delta X_s,\delta X_s\big\rangle\\ &\quad+\frac14\big|H_{pp}^{-\frac12}(X_s)\tilde{\mathbb{E}}_{\mathcal{F}_s}\big[H_{p\mu}(X_s,\tilde X_s)\delta\tilde X_s\big]\big|^2\Big]\,ds\\ &\le I(0)-\big[I(t)+\bar I(t)\big]+C\mathbb{E}\big[|\eta|^2\big]+C\int_0^t\mathbb{E}\big[|\delta X_s|^2\big]\,ds,\end{aligned}$$

where we used the bound of $\partial_{xx}V$, $H_{x\mu}$, H_{xx}, $H_{p\mu}$. Since $V(t,\cdot,\cdot)$ satisfies (2.17), by (4.4) we have $I(t)+\bar I(t)\ge -\lambda\mathbb{E}[|\delta X_t|^2]$. Then,

$$\int_0^t \mathbb{E}\big[\big|H_{pp}(X_s)^{\frac12}N_s\big|^2\big]\,ds\le I(0)+C\mathbb{E}\big[|\delta X_t|^2\big]+C\mathbb{E}\big[|\eta|^2\big]+C\int_0^t\mathbb{E}\big[|\delta X_s|^2\big]\,ds. \tag{5.1}$$

Next, using (4.1) and Young's inequality, we have, for any $\epsilon>0$,

$$|\delta X_t|^2\le|\eta|^2+C_\epsilon\int_0^t|\delta X_s|^2\,ds+\epsilon\int_0^t\big|H_{pp}(X_s)^{\frac12}N_s\big|^2\,ds.$$

Taking expectations on both sides and choosing $\epsilon>0$ small enough, by (5.1) we obtain

$$\mathbb{E}\big[|\delta X_t|^2\big]\le C\int_0^t\mathbb{E}\big[|\delta X_s|^2\big]\,ds+C\mathbb{E}\big[|\eta|^2\big]+C\big|I(0)\big|.$$

Then, it follows from Grönwall's inequality that

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}\big[|\delta X_t|^2\big]\le C\mathbb{E}\big[|\eta|^2\big]+C\big|I(0)\big|\le C\mathbb{E}\big[|\eta|^2+|\eta||\Upsilon_0|\big]\\ &\quad\text{where } \Upsilon_t:=\tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_{x\mu}V(t,X_t,\mu_t,\tilde X_t)\delta\tilde X_t\big].\end{aligned}\tag{5.2}$$

We shall follow the arguments in Theorem 4.1 to estimate Υ. We first observe that

$$\Upsilon_t=\tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_{x\mu}V(t,X_t,\mu_t,\tilde X_t)\delta\tilde X_t\big].\tag{5.3}$$

So, by applying Itô formula (2.12) on $\partial_{x\mu}V(t, X_t, \mu_t, \tilde{X}_t)\delta\tilde{X}_t$, taking conditional expectation $\tilde{\mathbb{E}}_{\mathcal{F}_T}$ and then changing back to $\tilde{\mathbb{E}}_{\mathcal{F}_t}$ as in (5.3), we obtain

$$d\Upsilon_t = (dB_t)^\top K_1(t) + \beta\big(dB_t^0\big)^\top K_2(t) + \big[K_3(t) - K_4(t)\big]\,dt, \tag{5.4}$$

where, recalling the notation in (4.3) (in particular, the stochastic integral terms above are column vectors),

$$\begin{aligned}
K_1(t) &:= \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_{xx\mu}V(X_t, \tilde{X}_t)\delta\tilde{X}_t\big],\\
K_2(t) &:= K_1(t) + \bar{\tilde{\mathbb{E}}}_{\mathcal{F}_t}\big[\big\{(\partial_{\mu x\mu}V)(X_t, \bar{X}_t, \tilde{X}_t) + \partial_{\tilde{x}x\mu}V(X_t, \tilde{X}_t)\big\}\delta\tilde{X}_t\big],\\
K_3(t) &:= \hat{\bar{\tilde{\mathbb{E}}}}_{\mathcal{F}_t}\bigg[\bigg\{\partial_{tx\mu}V(X_t, \tilde{X}_t) - H_p(X_t)^\top\partial_{xx\mu}V(X_t, \tilde{X}_t)\\
&\qquad - H_p(\tilde{X}_t)^\top\partial_{\tilde{x}x\mu}V(X_t, \tilde{X}_t) - H_p(\bar{X}_t)^\top\partial_{\mu x\mu}V(X_t, \bar{X}_t, \tilde{X}_t)\\
&\qquad + \frac{\widehat{\beta}^2}{2}\big[\big(\mathrm{tr}(\partial_{xx})\partial_{x\mu}V\big)(X_t, \tilde{X}_t)\\
&\qquad + \big(\mathrm{tr}(\partial_{\tilde{x}\tilde{x}})\partial_{x\mu}V\big)(X_t, \tilde{X}_t) + \big(\mathrm{tr}(\partial_{\bar{x}\mu})\partial_{x\mu}V\big)(X_t, \bar{X}_t, \tilde{X}_t)\big]\\
&\qquad + \beta^2\bigg[\big(\mathrm{tr}(\partial_{x\mu})\partial_{x\mu}V\big)(X_t, \bar{X}_t, \tilde{X}_t) + \big(\mathrm{tr}(\partial_{\tilde{x}x})\partial_{x\mu}V\big)(X_t, \tilde{X}_t)\\
&\qquad + \big(\mathrm{tr}(\partial_{\tilde{x}\mu})\partial_{x\mu}V\big)(X_t, \bar{X}_t, \tilde{X}_t) + \frac{1}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_{x\mu}V\big)(X_t, \hat{X}_t, \bar{X}_t, \tilde{X}_t)\bigg]\bigg\}\delta\tilde{X}_t\bigg],\\
K_4(t) &:= \bar{\tilde{\mathbb{E}}}_{\mathcal{F}_t}\big[\partial_{x\mu}V(X_t, \tilde{X}_t)\big\{\big[H_{px}(\tilde{X}_t) + H_{pp}(\tilde{X}_t)\partial_{xx}V(\tilde{X}_t)\big]\delta\tilde{X}_t\\
&\qquad + \big[H_{p\mu}(\tilde{X}_t, \bar{X}_t) + H_{pp}(\tilde{X}_t)\partial_{x\mu}V(\tilde{X}_t, \bar{X}_t)\big]\delta\bar{X}_t\big\}\big].
\end{aligned}$$

In light of (4.7), by straightforward calculation and simplification and setting

$$\begin{aligned}
K_5(t) &:= H_{xp}(X_t) + \partial_{xx}V(X_t)H_{pp}(X_t),\\
K_6(t) &:= \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\big[H_{x\mu}(X_t, \tilde{X}_t) + \partial_{xx}V(X_t)H_{p\mu}(X_t, \tilde{X}_t)\big]\delta\tilde{X}_t\big],
\end{aligned}$$

we derive that

$$d\Upsilon_t = (dB_t)^\top K_1(t) + \beta\big(dB_t^0\big)^\top K_2(t) + \big[K_5(t)\Upsilon_t + K_6(t)\big]\,dt. \tag{5.5}$$

We have

$$\Upsilon_t = \Upsilon_T - \int_t^T (dB_s)^\top K_1(s) - \int_t^T \beta\big(dB_s^0\big)^\top K_2(s) - \int_t^T \big[K_5(s)\Upsilon_s + K_6(s)\big]\,ds.$$

Take conditional expectation $\tilde{\mathbb{E}}_{\mathcal{F}_t}$ and recall (5.2), we have

$$\Upsilon_t = \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_{x\mu}G(X_T, \mu_T, \tilde{X}_T)\delta\tilde{X}_T\big] - \int_t^T \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[K_5(s)\Upsilon_s + K_6(s)\big]\,ds. \tag{5.6}$$

Then, by (5.5) and the required regularity of G, H and V, in particular (3.1), we have

$$|\Upsilon_t|^2 \le C\tilde{\mathbb{E}}_{\mathcal{F}_t}\big[|\delta\tilde{X}_T|^2\big] + C\int_t^T \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[|\Upsilon_s|^2 + |\delta\tilde{X}_s|^2\big]\,ds.$$

Now, take conditional expectation $\tilde{\mathbb{E}}_{\mathcal{F}_0}$; we get

$$\tilde{\mathbb{E}}_{\mathcal{F}_0}\big[|\Upsilon_t|^2\big] \le C\tilde{\mathbb{E}}_{\mathcal{F}_0}\big[|\delta\tilde{X}_T|^2\big] + C\int_t^T \tilde{\mathbb{E}}_{\mathcal{F}_0}\big[|\Upsilon_s|^2 + |\delta\tilde{X}_s|^2\big]\,ds.$$

Thus, by the Grönwall inequality we have

$$|\Upsilon_0|^2 = \tilde{\mathbb{E}}_{\mathcal{F}_0}[|\Upsilon_0|^2] \le C\tilde{\mathbb{E}}_{\mathcal{F}_0}[|\delta\tilde{X}_T|^2] + C\int_0^T \tilde{\mathbb{E}}_{\mathcal{F}_0}[|\delta\tilde{X}_s|^2]\,ds. \tag{5.7}$$

Plug this into (5.2), for any $\varepsilon > 0$, we have

$$\sup_{t\in[0,T]} \mathbb{E}[|\delta X_t|^2] \le C_\varepsilon \mathbb{E}[|\eta|^2] + \varepsilon\mathbb{E}[|\Upsilon_0|^2] \le C_\varepsilon\mathbb{E}[|\eta|^2] + C\varepsilon \sup_{t\in[0,T]} \mathbb{E}[|\delta X_t|^2].$$

Set $\varepsilon = \frac{1}{2C}$ as above; we have

$$\sup_{t\in[0,T]} \mathbb{E}[|\delta X_t|^2] \le C\mathbb{E}[|\eta|^2]. \tag{5.8}$$

Note that, recalling the setting in Section 2.1, $\delta\tilde{X}_t$ is measurable with respect to $\mathcal{F}_t^0 \vee \tilde{\mathcal{F}}_t^1$ which is independent of $\mathcal{F}_0$ under $\tilde{\mathbb{P}}$. Then, the conditional expectation in the right side of (5.7) is actually an expectation. Plug (5.8) into (5.7); we have

$$\big|\tilde{\mathbb{E}}_{\mathcal{F}_0}[\partial_{x\mu}V(0,\xi,\mu,\tilde{\xi})\tilde{\eta}]\big|^2 = |\Upsilon_0|^2 \le C\mathbb{E}[|\eta|^2]. \tag{5.9}$$

This implies

$$\big|\tilde{\mathbb{E}}[\partial_{x\mu}V(0,x,\mu,\tilde{\xi})\tilde{\eta}]\big| \le C\big(\mathbb{E}|\eta|^2\big)^{\frac{1}{2}}, \quad \mu\text{-a.e. } x.$$

Since $\partial_{x\mu}V$ is continuous, we have

$$\big|\mathbb{E}[\partial_{x\mu}V(0,x,\mu,\xi)\eta]\big| \le C\big(\mathbb{E}|\eta|^2\big)^{\frac{1}{2}} \quad \text{for all } x, \mu, \xi, \eta.$$

In particular, this implies that there exists a constant $C_2^\mu > 0$ such that

$$\big|\partial_x V(0,x,\mathcal{L}_{\xi+\eta}) - \partial_x V(0,x,\mathcal{L}_\xi)\big| = \left|\int_0^1 \mathbb{E}[\partial_{x\mu}V(0,x,\mathcal{L}_{\xi+\theta\eta},\xi+\theta\eta)\eta]\,d\theta\right| \le C_2^\mu\big(\mathbb{E}|\eta|^2\big)^{\frac{1}{2}}.$$

Now, taking random variables ξ, η such that $W_2^2(\mathcal{L}_{\xi+\eta},\mathcal{L}_\xi) = \mathbb{E}|\eta|^2$, the above inequality exactly means that $\partial_x V(0,x,\cdot)$ is uniformly Lipschitz continuous in μ under W_2 with uniform Lipschitz constant C_2^μ.

Finally, denote

$$\bar{\Upsilon}_t := \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_\mu V(t,X_t,\mu_t,\tilde{X}_t)\delta\tilde{X}_t].$$

Following similar arguments as in (5.4), we have

$$d\bar{\Upsilon}_t = (dB_t)^\top \bar{K}_1(t) + \beta\big(dB_t^0\big)^\top \bar{K}_2(t) + [\bar{K}_3(t) - \bar{K}_4(t)]\,dt, \tag{5.10}$$

where

$$\begin{aligned}
&\bar{K}_1(t) := \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t],\\
&\bar{K}_2(t) := K_1(t) + \bar{\tilde{\mathbb{E}}}_{\mathcal{F}_t}[\{(\partial_{\mu\mu}V)(X_t,\bar{X}_t,\tilde{X}_t) + \partial_{\tilde{x}\mu}V(X_t,\tilde{X}_t)\}\delta\tilde{X}_t];\\
&\bar{K}_3(t) := \hat{\bar{\tilde{\mathbb{E}}}}_{\mathcal{F}_t}\bigg[\bigg\{\partial_{t\mu}V(X_t,\tilde{X}_t) - H_p(X_t)^\top\partial_{x\mu}V(X_t,\tilde{X}_t)\\
&\qquad - H_p(\tilde{X}_t)^\top\partial_{\tilde{x}\mu}V(X_t,\tilde{X}_t) - H_p(\bar{X}_t)^\top\partial_{\mu\mu}V(X_t,\bar{X}_t,\tilde{X}_t)\\
&\qquad + \frac{\widehat{\beta}^2}{2}\big[\big(\mathrm{tr}(\partial_{xx})\partial_\mu V\big)(X_t,\tilde{X}_t) + \big(\mathrm{tr}(\partial_{\tilde{x}\tilde{x}})\partial_\mu V\big)(X_t,\tilde{X}_t) + \big(\mathrm{tr}(\partial_{\bar{x}\mu})\partial_\mu V\big)(X_t,\bar{X}_t,\tilde{X}_t)\big]\\
&\qquad + \beta^2\Big[\big(\mathrm{tr}(\partial_{x\mu})\partial_\mu V\big)(X_t,\bar{X}_t,\tilde{X}_t) + \big(\mathrm{tr}(\partial_{\tilde{x}x})\partial_\mu V\big)(X_t,\tilde{X}_t)
\end{aligned}$$

$$+\big(\mathrm{tr}(\partial_{\tilde{x}\mu})\partial_\mu V\big)(X_t,\bar{X}_t,\tilde{X}_t)+\frac{1}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_\mu V\big)(X_t,\hat{X}_t,\bar{X}_t,\tilde{X}_t)\Big]\Big\}\delta\tilde{X}_t\Big],$$

$$\bar{K}_4(t):=\bar{\tilde{\mathbb{E}}}_{\mathcal{F}_t}\big[\partial_\mu V(X_t,\tilde{X}_t)\big\{\big[H_{px}(\tilde{X}_t)+H_{pp}(\tilde{X}_t)\partial_{xx}V(\tilde{X}_t)\big]\delta\tilde{X}_t$$
$$+\big[H_{p\mu}(\tilde{X}_t,\bar{X}_t)+H_{pp}(\tilde{X}_t)\partial_{x\mu}V(\tilde{X}_t,\bar{X}_t)\big]\delta\bar{X}_t\big\}\big].$$

On the other hand, by taking $-\partial_\mu$ of (1.1) and omitting the variables (t,μ) we have

$$\begin{aligned}
0&=-\partial_\mu(\mathscr{L}V)(t,x,\mu,\tilde{x})\\
&=\partial_{t\mu}V(x,\tilde{x})+\frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{xx})\partial_\mu V\big)(x,\tilde{x})-H_\mu(x,\tilde{x})-H_p(x)^\top\partial_{x\mu}V(x,\tilde{x})\\
&\quad+\frac{\widehat{\beta}^2}{2}\big(\partial_{\tilde{x}}\,\mathrm{tr}(\partial_{\tilde{x}\mu})V\big)(x,\tilde{x})+\beta^2\big(\partial_{\tilde{x}}\,\mathrm{tr}(\partial_{x\mu})V\big)(x,\tilde{x})+\beta^2\bar{\mathbb{E}}\big[\big(\partial_{\tilde{x}}\,\mathrm{tr}(\partial_{\mu\mu})V\big)(x,\bar{\xi},\tilde{x})\big]\\
&\quad-H_p(\tilde{x})^\top\partial_{\tilde{x}\mu}V(x,\tilde{x})-\partial_\mu V(x,\tilde{x})\big(H_{px}(\tilde{x})+H_{pp}(\tilde{x})\partial_{xx}V(\tilde{x})\big)\\
&\quad+\hat{\bar{\mathbb{E}}}\Big[\frac{\widehat{\beta}^2}{2}\big(\mathrm{tr}(\partial_{\bar{x}\mu})\partial_\mu V\big)(x,\tilde{x},\bar{\xi})+\beta^2\big(\mathrm{tr}(\partial_{x\mu})\partial_\mu V\big)(x,\tilde{x},\bar{\xi})+\frac{\beta^2}{2}\big(\mathrm{tr}(\partial_{\mu\mu})\partial_\mu V\big)(x,\tilde{x},\hat{\xi},\bar{\xi})\\
&\quad-H_p(\bar{\xi})^\top\partial_{\mu\mu}V(x,\tilde{x},\bar{\xi})-\partial_\mu V(x,\bar{\xi})\big(H_{p\mu}(\bar{\xi},\tilde{x})+H_{pp}(\bar{\xi})\partial_{x\mu}V(\bar{\xi},\tilde{x})\big)\Big].
\end{aligned}$$

Plug this into (5.10), we have

$$d\bar{\Upsilon}_t=(dB_t)^\top\bar{K}_1(t)+\beta\big(dB_t^0\big)^\top\bar{K}_2(t)+\tilde{\mathbb{E}}_{\mathcal{F}_t}\big[H_\mu(X_t,\tilde{X}_t)\delta\tilde{X}_t\big]\,dt.$$

Then,

$$\begin{aligned}
\bar{\Upsilon}_0&=\tilde{\mathbb{E}}_{\mathcal{F}_0}\Big[\bar{\Upsilon}_T-\int_0^T H_\mu(X_t,\tilde{X}_t)\delta\tilde{X}_t\,dt\Big]\\
&=\tilde{\mathbb{E}}_{\mathcal{F}_0}\Big[\partial_\mu G(X_T,\tilde{X}_T)\delta\tilde{X}_T-\int_0^T H_\mu(X_t,\tilde{X}_t)\delta\tilde{X}_t\,dt\Big],
\end{aligned}$$

and thus, by (3.1) again,

$$\big|\tilde{\mathbb{E}}_{\mathcal{F}_0}\big[\partial_\mu V(0,\xi,\mu,\tilde{\xi})\tilde{\eta}\big]\big|^2=|\bar{\Upsilon}_0|^2\le C\tilde{\mathbb{E}}_{\mathcal{F}_0}\Big[|\delta\tilde{X}_T|^2+\int_0^T|\delta\tilde{X}_t|^2\,dt\Big].$$

Now, by (5.8), follow the arguments for (5.9) and the analysis afterward; we see that $V(0,x,\cdot)$ is also uniformly Lipschitz continuous in μ under W_2 with uniform Lipschitz constant C_2^μ. □

We note that the a priori W_2-Lipschitz continuity of V in μ is not sufficient for the global well-posedness of the master equation. We shall prove in the next section that, together with other properties, it actually implies the uniform W_1-Lipschitz continuity. The following proposition, which can be viewed as an analogue of Theorem 5.1 for the version of Lasry–Lions monotonicity, obtains the W_1-Lipschtiz estimate directly. Since the proof should be standard for experts and is very similar to that of Theorem 5.1, we only sketch it and focus on the main differences.

PROPOSITION 5.2. *Let all the conditions in Theorem 4.1 hold, except that we do not require Assumption 3.5. Assume further that there exists $\lambda>0$ such that, for any $\xi,\eta\in\mathbb{L}^2(\mathcal{F}_T^1)$,*

$$\mathbb{E}\big[\big\langle\partial_{x\mu}V(t,\xi,\mathcal{L}_\xi,\tilde{\xi})\tilde{\eta},\eta\big\rangle\big]\ge-\lambda\big(\mathbb{E}\big[|\eta|\big]\big)^2\quad\textit{for each } t\in[0,T],\tag{5.11}$$

where $(\tilde{\xi}, \tilde{\eta})$ *is an independent copy of* (ξ, η). *Then*, V *and* $\partial_x V$ *are uniformly Lipschitz continuous in* μ *under* W_1 *with Lipschitz constant* C_1^μ, *where* $C_1^\mu > 0$ *depends only on* d, T, $\|\partial_x V\|_{L^\infty}$, $\|\partial_{xx} V\|_{L^\infty}$, *the* L_1^G *in Assumption* 3.1(*i*), *the* $L^H(\|\partial_x V\|_{L^\infty})$ *in Assumption* 3.2(*i*) *and the* λ *in* (5.11).

PROOF. Denote

$$N_t' := \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu} V(X_t, \tilde{X}_t)\delta\tilde{X}_t] + \frac{1}{2} H_{pp}(X_t)^{-1}\tilde{\mathbb{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t, \tilde{X}_t)\delta\tilde{X}_t].$$

First, by using (4.8) and (5.11), similar to (5.1), we can show

$$\begin{aligned}
&\int_0^t \mathbb{E}[|H_{pp}(X_s)^{\frac{1}{2}} N_s'|^2]\,ds \\
&\quad\le I(0) + C\mathbb{E}[(\mathbb{E}_{\mathcal{F}_t^0}[|\delta X_t|])^2] + C\int_0^t \mathbb{E}[|\delta X_s|\tilde{\mathbb{E}}_{\mathcal{F}_s}[|\delta\tilde{X}_s|]]\,ds \qquad (5.12)\\
&\quad= I(0) + C\mathbb{E}[(\mathbb{E}_{\mathcal{F}_t^0}[|\delta X_t|])^2] + C\int_0^t \mathbb{E}[(\mathbb{E}_{\mathcal{F}_s^0}[|\delta X_s|])^2]\,ds.
\end{aligned}$$

Next, by (4.1), Young's inequality and noting that $\mathcal{F}_0$ is independent of $\mathcal{F}_t^0$, we have

$$(\mathbb{E}_{\mathcal{F}_t^0}[|\delta X_t|])^2 \le (\mathbb{E}[|\eta|])^2 + C_\epsilon \int_0^t (\mathbb{E}_{\mathcal{F}_s^0}[|\delta X_s|])^2\,ds + \epsilon\int_0^t (\mathbb{E}_{\mathcal{F}_s^0}[H_{pp}^{\frac{1}{2}}(X_s)|N_s'|])^2\,ds.$$

Taking expectation on both sides, choosing $\epsilon > 0$ small enough, together with (5.12) and for the same Υ in (5.2), it then follows from Grönwall's inequality that

$$\sup_{t\in[0,T]} \mathbb{E}[(\mathbb{E}_{\mathcal{F}_t^0}[|\delta X_t|])^2] \le C(\mathbb{E}[|\eta|])^2 + C|I(0)| \le C(\mathbb{E}[|\eta|])^2 + C\mathbb{E}[|\eta||\Upsilon_0|]. \qquad (5.13)$$

Now by (5.5) and (5.6) and noting that $|\partial_{x\mu} G| \le L_1^G$, we have

$$\begin{aligned}
|\Upsilon_t| &\le C\tilde{\mathbb{E}}_{\mathcal{F}_t}[|\delta\tilde{X}_T|] + C\int_t^T \tilde{\mathbb{E}}_{\mathcal{F}_t}[|\Upsilon_s| + |\delta\tilde{X}_s|]\,ds \\
&= C\mathbb{E}_{\mathcal{F}_t^0}[|\delta\tilde{X}_T|] + C\int_t^T [\mathbb{E}_{\mathcal{F}_t}[|\Upsilon_s|] + \mathbb{E}_{\mathcal{F}_t^0}[|\delta\tilde{X}_s|]]\,ds.
\end{aligned}$$

Then, since $\mathcal{F}_0^0$ is degenerate, namely, it reduced to $\{\varnothing, \Omega_0\}$

$$|\Upsilon_0| \le C\mathbb{E}[|\delta X_T|] + C\int_0^T \mathbb{E}[|\delta X_s|]\,ds \le C\sup_{0\le t\le T}\mathbb{E}[|\delta X_t|].$$

Combine this with (5.13), we have

$$|\tilde{\mathbb{E}}_{\mathcal{F}_0}[\partial_{x\mu} V(0, \xi, \mu, \tilde{\xi})\tilde{\eta}]|^2 = |\Upsilon_0|^2 \le C\sup_{t\in[0,T]}\mathbb{E}[(\mathbb{E}_{\mathcal{F}_t^0}[|\delta X_t|])^2] \le C(\mathbb{E}[|\eta|])^2. \qquad (5.14)$$

This is the counterpart of (5.9). Then, following similar arguments as after (5.9), we conclude as follows. First, one obtains

$$|\mathbb{E}[\partial_{x\mu} V(0, x, \mu, \xi)\eta]| \le C\mathbb{E}[|\eta|] \quad \forall x, \mu, \xi, \eta.$$

In particular, this implies that there exists a constant $C_1^\mu > 0$ such that

$$|\partial_x V(0, x, \mathcal{L}_{\xi+\eta}) - \partial_x V(0, x, \mathcal{L}_\xi)| = \left|\int_0^1 \mathbb{E}[\partial_{x\mu} V(0, x, \mathcal{L}_{\xi+\theta\eta}, \xi+\theta\eta)\eta]\,d\theta\right| \le C_1^\mu \mathbb{E}[|\eta|].$$

Now, taking random variables ξ, η such that $W_1(\mathcal{L}_{\xi+\eta}, \mathcal{L}_\xi) = \mathbb{E}[|\eta|]$, the above inequality implies that $\partial_x V(0, x, \cdot)$ is uniformly Lipschitz continuous in μ under W_1 with uniform Lipschitz constant C_1^μ.

By analyzing $\tilde{\Upsilon}$ similarly, we show that V is also uniformly Lipschitz continuous in μ under W_1. $\square$

REMARK 5.3. (i) Proposition 5.2 indicates that the a priori W_1-Lipschitz continuity of V and $\partial_x V$ is a consequence of (5.11), even if H is nonseparable. However, we emphasize that, although (5.11) is weaker than the Lasry–Lions monotonicity (2.15), it is stronger than the displacement semimonotonicity condition (2.17). Unfortunately, for nonseparable H, we are not able to find sufficient conditions to ensure (5.11) a priori for V.

(ii) We note that, in general, for displacement monotone Hamiltonians considered in this manuscript, the displacement semimonotonicity of the terminal datum is not propagated in time along the solution of the master equation (cf. [28], Section B.4).

6. The global well-posedness. In this section we establish the global well-posedness of master equation (1.1). As illustrated in [25, 26, 39], the key to extend a local classical solution to a global one is the a priori uniform Lipschitz continuity estimate of the solution. We first investigate the regularity of V with respect to x. The following result is somewhat standard, while our technical conditions could be slightly different from those in the literature. For completeness we provide a proof in the Appendix. We remark that the regularity of G and H in μ is actually not needed in this result.

PROPOSITION 6.1. *Let Assumptions* 3.1(i) *and* 3.2(i), (iii) *hold and* $\rho : [0, T] \times \Omega \to \mathcal{P}_2$ *be* $\mathbb{F}^0$*-progressively measurable* (*not necessarily a solution to* (2.24)) *with* $\sup_{t\in[0,T]} \mathbb{E}[M_2^2(\rho_t)] < +\infty$:

(i) *For any* $x \in \mathbb{R}^d$ *and for the* X^x *in* (2.25), *the following BSDE on* $[t_0, T]$ *has a unique solution with bounded* Z^x:

$$Y_t^x = G(X_T^x, \rho_T) - \int_t^T H(X_s^x, \rho_s, Z_s^x)\, ds - \int_t^T Z_s^x \cdot dB_s - \int_t^T Z_s^{0,x} \cdot dB_s^0. \tag{6.1}$$

(ii) *Denote* $u(t_0, x) := Y_{t_0}^x$, *then there exist* $C_1^x, C_2^x > 0$, *depending only on* d, T, *the* C_0 *in Assumption* 3.2(iii), *the constant* L_0^G *in Assumption* 3.1 *and the function* L^H *in Assumption* 3.2, *such that*

$$\left|\partial_x u(t_0, x)\right| \le C_1^x, \qquad \left|\partial_{xx} u(t_0, x)\right| \le C_2^x. \tag{6.2}$$

Here, the notation C_i^x denotes the bound of the ith-order derivative of u with respect to x, in particular, it is *not* a function of x.

The above result, combined with Theorems 4.1 and 5.1, implies immediately the uniform a priori Lipschitz continuity of V with respect to μ under W_2, with the uniform Lipschitz estimate depending only on the parameters in the assumptions but not on the additional regularities required in Theorem 4.1. However, the existence of local classical solutions to the master equation (1.1) requires the Lipschitz continuity under W_1, cf. [25], Theorem 5.10. To show that, eventually, the W_2-Lipschitz continuity of V together with our standing assumptions on the data imply its Lipschitz continuity under W_1, we rely on a pointwise representation formula for $\partial_\mu V$, developed in [39], tailored to our setting.

For this purpose we fix $t_0 \in [0, T]$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{L}^2(\mathcal{F}_{t_0})$ and let ρ be given in (2.24), provided its well-posedness. We then consider the following FBSDEs on $[t_0, T]$, which can be interpreted as a formal differentiation of (2.26) with respect to x_k,

$$
\begin{cases}
\nabla_k X_t^{\xi,x} = e_k - \int_{t_0}^t \big[(\nabla_k X_s^{\xi,x})^\top \partial_{xp} H(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x}) \\
\qquad\qquad + (\nabla_k Z_s^{\xi,x})^\top \partial_{pp} H(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x})\big]\, ds; \\
\nabla_k Y_t^{\xi,x} = \partial_x G(X_T^{\xi,x}, \rho_T) \cdot \nabla_k X_T^{\xi,x} - \int_t^T \nabla_k Z_s^{\xi,x} \cdot dB_s^{t_0} - \int_t^T \nabla_k Z_s^{0,\xi,x} \cdot dB_s^{0,t_0} \\
\qquad\qquad + \int_t^T \partial_x \widehat{L}(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x}) \cdot \nabla_k X_s^{\xi,x} + \partial_p \widehat{L}(X_s^{\xi,x}, \rho_s, Z_s^{\xi,x}) \cdot \nabla_k Z_s^{\xi,x}\, ds,
\end{cases} \tag{6.3}
$$

the following McKean–Vlasov FBSDE on $[t_0, T]$:

$$
\begin{cases}
\nabla_k \mathcal{X}_t^{\xi,x} = -\int_{t_0}^t \{(\nabla_k \mathcal{X}_s^{\xi,x})^\top \partial_{xp} H(X_s^\xi, \rho_s, Z_s^\xi) + (\nabla_k \mathcal{Z}_s^{\xi,x})^\top \partial_{pp} H(X_s^\xi, \rho_s, Z_s^\xi) \\
\qquad + \tilde{\mathbb{E}}_{\mathcal{F}_s}[(\nabla_k \tilde{X}_s^{\xi,x})^\top (\partial_{\mu p} H)(X_s^\xi, \rho_s, \tilde{X}_s^{\xi,x}, Z_s^\xi) \\
\qquad + (\nabla_k \tilde{\mathcal{X}}_s^{\xi,x})^\top \partial_{\mu p} H(X_s^\xi, \rho_s, \tilde{X}_s^\xi, Z_s^\xi)]\}\, ds; \\
\nabla_k \mathcal{Y}_t^{\xi,x} = \partial_x G(X_T^\xi, \rho_T) \cdot \nabla_k \mathcal{X}_T^{\xi,x} \\
\qquad + \tilde{\mathbb{E}}_{\mathcal{F}_T}[\partial_\mu G(X_T^\xi, \rho_T, \tilde{X}_T^{\xi,x}) \cdot \nabla_k \tilde{X}_T^{\xi,x} + \partial_\mu G(X_T^\xi, \rho_T, \tilde{X}_T^\xi) \cdot \nabla_k \tilde{\mathcal{X}}_T^{\xi,x}] \\
\qquad + \int_t^T \{\partial_x \widehat{L}(X_s^\xi, \rho_s, Z_s^\xi) \cdot \nabla_k \mathcal{X}_s^{\xi,x} + \partial_p \widehat{L}(X_s^\xi, \rho_s, Z_s^\xi) \cdot \nabla_k \mathcal{Z}_s^{\xi,x} \\
\qquad + \tilde{\mathbb{E}}_{\mathcal{F}_s}[\partial_\mu \widehat{L}(X_s^\xi, \rho_s, \tilde{X}_s^{\xi,x}, Z_s^\xi) \cdot \nabla_k \tilde{X}_s^{\xi,x} + \partial_\mu \widehat{L}(X_s^\xi, \rho_s, \tilde{X}_s^\xi, Z_s^\xi) \cdot \nabla_k \tilde{\mathcal{X}}_s^{\xi,x}]\}\, ds \\
\qquad - \int_t^T \nabla_k \mathcal{Z}_s^{\xi,x} \cdot dB_s^{t_0} - \int_t^T \nabla_k \mathcal{Z}_s^{0,\xi,x} \cdot dB_s^{0,t_0}
\end{cases} \tag{6.4}
$$

and the following McKean–Vlasov BSDE on $[t_0, T]$:

$$
\begin{aligned}
\nabla_{\mu_k} Y_t^{x,\xi,\tilde{x}} = {} & \tilde{\mathbb{E}}_{\mathcal{F}_T}[\partial_\mu G(X_T^x, \rho_T, \tilde{X}_T^{\xi,\tilde{x}}) \cdot \nabla_k \tilde{X}_T^{\xi,\tilde{x}} + \partial_\mu G(X_T^x, \rho_T, \tilde{X}_T^\xi) \cdot \nabla_k \tilde{\mathcal{X}}_T^{\xi,\tilde{x}}] \\
& - \int_t^T \{\partial_p H(X_s^x, \rho_s, Z_s^{x,\xi}) \cdot \nabla_{\mu_k} Z_s^{x,\xi,\tilde{x}} \\
& + \tilde{\mathbb{E}}_{\mathcal{F}_s}[\partial_\mu H(X_s^x, \rho_s, \tilde{X}_s^{\xi,\tilde{x}}, Z_s^{x,\xi}) \cdot \nabla_k \tilde{X}_s^{\xi,\tilde{x}} \\
& + \partial_\mu H(X_s^x, \rho_s, \tilde{X}_s^\xi, Z_s^{x,\xi}) \cdot \nabla_k \tilde{\mathcal{X}}_s^{\xi,\tilde{x}}]\}\, ds \\
& - \int_t^T \nabla_{\mu_k} Z_s^{x,\xi,\tilde{x}} \cdot dB_s - \int_t^T \nabla_{\mu_k} Z_s^{0,x,\xi,\tilde{x}} \cdot dB_s^0.
\end{aligned} \tag{6.5}
$$

The following result provides the crucial W_1-Lipschitz continuity of V. In particular, this extends [39], Theorem 9.2, to our setting.

PROPOSITION 6.2. *Let Assumptions 3.1*(i) *and 3.2*(i), (iii) *hold. Recall the constants C_1^x in (6.2), L_0^G, L_1^G in Assumption 3.1, L_2^G in Remark 3.3, and the function L^H in Assumption 3.2. Then, there exists a constant $\delta > 0$, depending only d, L_0^G, L_2^G, $L^H(C_1^x)$, such that, whenever $T - t_0 \le \delta$, the following hold*:

(i) *The McKean–Vlasov FBSDEs* (2.24), (2.25), (2.26), (6.3), (6.4) *and* (6.5) *are well-posed on $[t_0, T]$, for any $\mu \in \mathcal{P}_2$ and $\xi \in \mathbb{L}^2(\mathcal{F}_{t_0}, \mu)$.*

(ii) *Define $V(t_0, x, \mu) := Y_{t_0}^{x,\xi}$. We have the pointwise representation,*

$$
\partial_{\mu_k} V(t_0, x, \mu, \tilde{x}) = \nabla_{\mu_k} Y_{t_0}^{x,\xi,\tilde{x}}. \tag{6.6}
$$

Moreover, there exists a constant $C_1^\mu > 0$, depending only on d, L_0^G, L_1^G, $L^H(C_1^x)$, such that

$$
|\partial_\mu V(t_0, x, \mu, \tilde{x})| \le C_1^\mu, \qquad |\partial_{x\mu} V(t_0, x, \mu, \tilde{x})| \le C_1^\mu. \tag{6.7}
$$

(iii) *Assume further that Assumptions* 3.1(ii) *and* 3.2(ii) *hold true. Then*, *the master equation* (1.1) *has a unique classical solution* V *on* $[t_0, T]$ *and* (2.27) *holds. Moreover*,

$$V(t,\cdot,\cdot), \partial_x V(t,\cdot,\cdot), \partial_{xx} V(t,\cdot,\cdot) \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2),$$
$$\partial_\mu V(t,\cdot,\cdot,\cdot), \partial_{x\mu} V(t,\cdot,\cdot,\cdot) \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d),$$

and all their derivatives in the state and probability measure variables are continuous in the time variable and are uniformly bounded.

PROOF. Since this proof is essentially the same as that in [39], Section 9, we postpone it to the Appendix. □

We emphasize that the δ in the above result depends on L_2^G, but not on L_1^G, while the C_1^μ in (6.7) depends on L_1^G. This observation is crucial. We now establish the main result of the paper.

THEOREM 6.3. *Let Assumptions* 3.1, 3.2 *and* 3.5 *hold. Then*, *the master equation* (1.1) *on* $[0, T]$ *admits a unique classical solution* V *with bounded* $\partial_x V$, $\partial_{xx} V$, $\partial_\mu V$ *and* $\partial_{x\mu} V$.

Moreover, *the McKean–Vlasov FBSDEs* (2.24), (2.25), (2.26), (6.3), (6.4) *and* (6.5) *are also well-posed on* $[0, T]$, *and the representation formula* (6.6) *remains true on* $[0, T]$.

PROOF. Let C_1^x, C_2^x be as in (6.2) and C_2^μ be the a priori (global) uniform Lipschitz estimate of V with respect to μ under W_2, as established by Theorems 4.1 and 5.1. Let $\delta > 0$ be the constant in Proposition 6.2, but with L_0^G replaced with $C_1^x \vee C_2^x$ and L_2^G replaced with C_2^μ. Let $0 = T_0 < \cdots < T_n = T$ be a partition such that $T_{i+1} - T_i \le \frac{\delta}{2}$, $i = 0, \ldots, n-1$. We proceed in three steps:

Step 1. Existence. First, since $T_n - T_{n-2} \le \delta$, by Proposition 6.2 the master equation (1.1) on $[T_{n-2}, T_n]$ with terminal condition G has a unique classical solution V. For each $t \in [T_{n-2}, T_n]$, applying Proposition 6.1, we have $|\partial_x V(T_{n-1},\cdot,\cdot)| \le C_1^x$, $|\partial_{xx} V(T_{n-1},\cdot,\cdot)| \le C_2^x$. Note that, by Proposition 6.2(iii), $V(t,\cdot,\cdot)$ has further regularities; this enables us to apply Theorems 4.1 and 5.1 and obtain that $V(t,\cdot,\cdot)$ is uniform Lipschitz continuous in μ under W_2 with Lipschitz constant C_2^μ. Moreover, by Proposition 6.2(ii) $V(T_{n-1},\cdot,\cdot)$ is also uniformly Lipschitz continuous in μ under W_1.

We next consider the master equation (1.1) on $[T_{n-3}, T_{n-1}]$ with terminal condition $V(T_{n-1},\cdot,\cdot)$. We emphasize that $V(T_{n-1},\cdot,\cdot)$ has the above uniform regularity with the same constants C_1^x, C_2^x, C_2^μ; then, we may apply Proposition 6.2 with the same δ and obtain a classical solution V on $[T_{n-3}, T_{n-1}]$ with the additional regularities specified in Proposition 6.2(iii). Clearly, this extends the classical solution of the master equation to $[T_{n-3}, T_n]$. We emphasize again that, while the bound of $\partial_\mu V(t,\cdot)$, $\partial_{x\mu} V(t,\cdot)$ may become larger for $t \in [T_{n-3}, T_{n-2}]$ because the C_1^μ in (6.7) now depends on $\|\partial_\mu V(T_{n-1},\cdot)\|_{L^\infty}$ instead of $\|\partial_\mu V(T_n,\cdot)\|_{L^\infty}$, by the global a priori estimates in Theorems 4.1 and 5.1 we see that $V(t,\cdot)$ corresponds to the same C_1^x, C_2^x and C_2^μ for all $t \in [T_{n-3}, T_n]$. This enables us to consider the master equation (1.1) on $[T_{n-4}, T_{n-2}]$ with terminal condition $V(T_{n-2},\cdot,\cdot)$, and then we obtain a classical solution on $[T_{n-4}, T_n]$ with the desired uniform estimates and additional regularities.

Now, repeat the arguments backwards in time; we may construct a classical solution V for the original master equation (1.1) on $[0, T]$ with terminal condition G. Moreover, since this procedure is repeated only n times, by applying (6.7) repeatedly we see that (6.7) indeed holds true on $[0, T]$.

Step 2. Uniqueness. This follows directly from the local uniqueness in Proposition 6.2. Indeed, assume V' is another classical solution with bounded $\partial_x V'$, $\partial_{xx} V'$, $\partial_\mu V'$ and $\partial_{x\mu} V'$.

By otherwise choosing larger C_1^x, C_2^x, C_2^μ, we assume $|\partial_x V'| \le C_1^x$, $|\partial_{xx} V'| \le C_2^x$, and C_2^μ also serves as a Lipschitz constant for the W_2-Lipschitz continuity of V' in μ. Then, applying Proposition 6.2 on the master equation on $[T_{n-1}, T_n]$ with terminal condition G, by the uniqueness in Proposition 6.2(iii) (or in (i)) we see that $V'(t, \cdot) = V(t, \cdot)$ for $t \in [T_{n-1}, T_n]$. Next, consider the master equation on $[T_{n-2}, T_{n-1}]$ with terminal condition $V'(T_{n-1}, \cdot) = V(T_{n-1}, \cdot)$, by the uniqueness in Proposition 6.2(iii), again we see that $V'(t, \cdot) = V(t, \cdot)$ for $t \in [T_{n-2}, T_{n-1}]$. Repeat the arguments backwards in time we prove the uniqueness on $[0, T]$.

Step 3. Let V be the unique classical solution to the master equation (1.1) on $[0, T]$ with bounded $\partial_x V$ and $\partial_{xx} V$. Then, for $t_0 \in [0, T]$ and $\xi \in \mathbb{L}^2(\mathcal{F}_{t_0})$, the McKean–Vlasov SDE (2.30) on $[t_0, T]$ has a unique solution X^ξ and ρ. Set

$$Y_t^\xi := V\big(t, X_t^\xi, \rho_t\big), \qquad Z_t^\xi := \partial_x V\big(t, X_t^\xi, \rho_t\big),$$
$$Z_t^{0,\xi} := \beta\big(\partial_x V\big(t, X_t^\xi, \rho_t\big) + \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_\mu V\big(t, X_t^\xi, \rho_t, \tilde{X}_t^\xi\big)\big]\big).$$

By (1.1) and Itô formula (2.12), one verifies that $(X^\xi, Y^\xi, Z^\xi, Z^{0,\xi})$ satisfies FBSDE (2.24). The uniqueness follows from the same arguments, as in Step 2.

Similarly, by the above decoupling technique we can easily see that the other McKean–Vlasov FBSDEs (2.25), (2.26), (6.3), (6.4) and (6.5) are also well-posed. In particular, besides (2.27) we have the following:

$$\begin{aligned}
\nabla_k Y_t^{\xi,x} &= \partial_{x_k} V\big(t, X_t^{\xi,x}, \rho_t\big)\nabla_k X_t^{\xi,x}, \\
\nabla_k \mathcal{Y}_t^{\xi,x} &= \partial_x V\big(t, X_t^\xi, \rho_t\big) \cdot \nabla_k \mathcal{X}_t^{\xi,x} \\
&\quad + \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_\mu V\big(t, X_t^\xi, \rho_t, \tilde{X}_t^{\xi,x}\big) \cdot \nabla_k \tilde{X}_t^{\xi,x} + \partial_\mu V\big(X_t^\xi, \rho_t, \tilde{X}_t^\xi\big) \cdot \nabla_k \tilde{\mathcal{X}}_t^{\xi,x}\big], \\
\nabla_{\mu_k} Y_t^{x,\xi,\tilde{x}} &= \tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_\mu V\big(t, X_t^x, \rho_t, \tilde{X}_t^{\xi,\tilde{x}}\big) \cdot \nabla_k \tilde{X}_t^{\xi,\tilde{x}} + \partial_\mu V\big(t, X_t^x, \rho_t, \tilde{X}_t^\xi\big) \cdot \nabla_k \tilde{\mathcal{X}}_t^{\xi,\tilde{x}}\big].
\end{aligned}$$

Set $t = t_0$, and note that $\nabla_k \tilde{X}_{t_0}^{\xi,\tilde{x}} = e_k$ and $\nabla_k \tilde{\mathcal{X}}_{t_0}^{\xi,\tilde{x}} = 0$; then, the last equation above implies

$$\nabla_{\mu_k} Y_{t_0}^{x,\xi,\tilde{x}} = \tilde{\mathbb{E}}_{\mathcal{F}_{t_0}}\big[\partial_\mu V(t_0, x, \rho_{t_0}, \tilde{x}) \cdot e_k\big] = \partial_{\mu_k} V(t_0, x, \mathcal{L}_\xi, \tilde{x})$$

which is exactly (6.6). □

APPENDIX: PROOFS OF SOME TECHNICAL RESULTS

PROOF OF REMARK 2.4. We show the equivalence of (2.13) and (2.15) for any $U \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{P}_2)$. In fact, by Remark 2.3(i), we only need to prove that (2.15) implies (2.13). We now assume (2.15) holds, and we want to show the following which is equivalent to (2.13): for any $\mu_0, \mu_1 \in \mathcal{P}_2$,

$$J_0 := \int_{\mathbb{R}^d} \big[U(x, \mu_1) - U(x, \mu_0)\big]\big[\mu_1(dx) - \mu_0(dx)\big] \ge 0. \tag{A.1}$$

Recall (2.9). Since U is continuous, by the standard density argument it suffices to show (A.1) for μ_i, $i \in \{0, 1\}$, which have densities $\rho_i \in C^\infty(B_R)$ such that $\min_{B_R} \rho_i > 0$. Consider one of the W_1-geodesic interpolations, such as in [27],

$$\mu_t := \rho_t \mathcal{L}^d, \qquad \rho_t = (1 - t)\rho_0 + t\rho_1 \quad \forall t \in [0, 1].$$

Since ρ is bounded away from 0 on $[0, 1] \times B_R$, then, for each t, there is a unique solution $\phi_t \in H_0^1(B_R^o) \cap C^\infty(B_R)$ to the elliptic equation

$$\nabla \cdot (\rho_t \nabla \phi_t) = \rho_0 - \rho_1 \quad \text{that is} \quad \partial_t \rho_t + \nabla \cdot (\rho_t \nabla \phi_t) = 0.$$

Note that ϕ_t and $\nabla\phi_t$ are continuous in t too. Setting $v_t := \nabla\phi_t$, we see that v is a velocity for $t \to \mu_t$, and thus the following chain rule holds, cf. [29], Lemma 9.8:

$$\frac{d}{dt}U(x,\mu_t) = \int_{\mathbb{R}^d} \langle \partial_\mu U(x,\mu_t,\tilde{x}), v_t(\tilde{x})\rangle \rho_t(\tilde{x})\,d\tilde{x}. \tag{A.2}$$

We now compute J_0,

$$\begin{aligned} J_0 &= \int_{\mathbb{R}^d} [U(x,\mu_1) - U(x,\mu_0)][\rho_1(x) - \rho_0(x)]\,dx = -\int_0^1 \int_{\mathbb{R}^d} \frac{d}{dt}U(x,\mu_t)\nabla\cdot(\rho_t v_t)\,dx\,dt \\ &= -\int_0^1 \int_{\mathbb{R}^{2d}} \langle \partial_\mu U(x,\mu_t,\tilde{x}), v_t(\tilde{x})\rangle \rho_t(\tilde{x})\big[\nabla\cdot\big(\rho_t(x)v_t(x)\big)\big]\,d\tilde{x}\,dx\,dt. \end{aligned}$$

By integration by parts formula and recalling that $\partial_{x\mu}U := \partial_x[\partial_\mu U]^\top$, we have

$$J_0 = \int_0^1 \int_{\mathbb{R}^{2d}} \langle \partial_x\partial_\mu U(x,\mu_t,\tilde{x})v_t(\tilde{x}), v_t(x)\rangle \rho_t(\tilde{x})\rho_t(x)\,d\tilde{x}\,dx\,dt.$$

Choose $\xi \in \mathbb{L}^2(\mathcal{F}_t^1, \mu_t)$, and set $\eta := v_t(\xi)$; we see immediately that

$$J_0 = \tilde{\mathbb{E}}\big[\langle \partial_x\partial_\mu U(\xi,\mu_t,\tilde{\xi})\tilde{\eta}, \eta\rangle\big] \geq 0,$$

where the inequality is due to (2.15). This proves (A.1). □

PROOF OF PROPOSITION 6.1. We proceed in three steps:

Step 1. We first show the well-posedness of the BSDE (6.1) in the case when $[T - t_0]C_0 < 1$, where C_0 is given in Assumption 3.2(iii). We note that H is only locally Lipschitz continuous. For this purpose, let $R > 0$ be a constant which will be specified later. Let $I_R \in C^\infty(\mathbb{R}^d)$ be a truncation function such that $I_R(p) = p$ for $|p| \leq R$, $|\partial_p I_R(p)| = 0$ for $|p| \geq R+1$ and $|\partial_p I_R(p)| \leq 1$ for $p \in \mathbb{R}^d$. Denote $H_R(x,\mu,p) = H(x,\mu,I_R(p))$. Then, clearly, $|\partial_p H_R(x,\mu,p)| \leq L^H(R+1)$ and $|\partial_x H_R(x,\mu,p)| \leq \tilde{L}^H(R+1)$ for all $(x,\mu,p) \in \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{R}^d$, where $\tilde{L}^H(R) := \sup_{(x,\mu,p)\in D_R} |\partial_x H_R(x,\mu,p)|$. Consider the following BSDE on $[t_0, T]$ (abusing the notation here):

$$Y_t^x = G\big(X_T^x, \rho_T\big) - \int_t^T H_R\big(X_s^x, \rho_s, Z_s^x\big)\,ds - \int_t^T Z_s^x \cdot dB_s^{t_0} - \int_t^T Z_s^{0,x} \cdot dB_s^{0,t_0}, \tag{A.3}$$

and denote $u(t_0, x) := Y_{t_0}^x$ which is $\mathcal{F}_{t_0}^0$-measurable. By standard BSDE arguments, clearly the above system is well-posed, and it holds $Y_t^x = u(t, X_t^x)$, $Z_t^x = \partial_x u(t, X_t^x)$. Moreover, we have $\partial_x u(t_0, x) = \nabla Y_{t_0}^x$, where

$$\begin{aligned} \nabla Y_t^x &= \partial_x G\big(X_T^x, \rho_T\big) - \int_t^T \big[\partial_x H_R\big(X_s^x, \rho_s, Z_s^x\big) + \nabla Z_s^x \partial_p H_R\big(X_s^x, \rho_s, Z_s^x\big)\big]\,ds \\ &\quad - \int_t^T \nabla Z_s^x\,dB_s^{t_0} - \int_t^T \nabla Z_s^{0,x}\,dB_s^{0,t_0}, \quad t_0 \leq t \leq T. \end{aligned} \tag{A.4}$$

Note that $|\partial_x G| \leq L_0^G$ and $|\partial_x H_R| \leq \tilde{L}^H(R+1)$; one can easily see that

$$\big|\partial_x u(t_0, x)\big| = \big|\nabla Y_{t_0}^x\big| \leq L_0^G + T\tilde{L}^H(R+1).$$

Note that

$$\overline{\lim_{R\to\infty}} \frac{L_0^G + T\tilde{L}^H(R+1)}{R} = \overline{\lim_{R\to\infty}} \frac{T\tilde{L}^H(R+1)}{R+1} \leq TC_0 < 1.$$

We may choose $R > 0$ large enough such that

$$\big|\partial_x u(t_0, x)\big| \leq L_0^G + T\tilde{L}^H(R+1) \leq R.$$

This proves $|\partial_x u(t,x)| \le C_1^x$ by setting $C_1^x := R$. Moreover, since $|Z_t^x| = |\partial_x u(t, X_t^x)| \le R$, we see that $H_R(X_t^x, \rho_t, Z_t^x) = H(X_t^x, \rho_t, Z_t^x)$. Thus, $(Y^x, Z^x, Z^{0,x})$ actually satisfies (6.1).

On the other hand, for any solution $(Y^x, Z^x, Z^{0,x})$ with bounded Z^x, let $R > 0$ be larger than the bound of Z^x. Then, we see that $(Y^x, Z^x, Z^{0,x})$ satisfies (A.3). Now, the uniqueness follows from the uniqueness of the BSDE (A.3) which has Lipschitz continuous data.

Step 2. We next estimate $\partial_{xx} u$, again in the case $[T - t_0]C_0 < 1$. First, applying standard BSDE estimates on (A.4), we see that

$$\mathbb{E}\Big[\Big(\int_{t_0}^T |\nabla Z_s^x|^2\,ds\Big)^2\Big] \le C \quad \text{a.s.} \tag{A.5}$$

Then, we have $\partial_{xx} u(t_0, x) = \nabla^2 Y_{t_0}^x$, where, by differentiating (A.4) formally in x,

$$\begin{aligned}\nabla^2 Y_t^x &= \partial_{xx} G(X_T^x, \rho_T) - \int_t^T \sum_{i=1}^d [\nabla^2 Z_s^{i,x}\,dB_s^{i,t_0} + \nabla^2 Z_s^{0,i,x}\,dB_s^{0,i,t_0}] \\ &\quad - \int_t^T \Big[\partial_{xx} H_R(\cdot) + 2\nabla Z_s^x \partial_{xp} H_R(\cdot) + \nabla Z_s^x \partial_{pp} H_R(\cdot)[\nabla Z_s^x]^\top \\ &\quad + \sum_{i=1}^d \nabla^2 Z_s^{i,x} \partial_{p_i} H_R(\cdot)\Big](X_s^x, \rho_s, Z_s^x)\,ds.\end{aligned} \tag{A.6}$$

Denote

$$M_T^x := \exp\Big(-\int_{t_0}^T \partial_p H_R(X_s^x, \rho_s, Z_s^x)\cdot dB_s^{t_0} - \frac{1}{2}\int_{t_0}^T |\partial_p H_R(X_s^x, \rho_s, Z_s^x)|^2\,ds\Big).$$

Then,

$$\begin{aligned}\nabla^2 Y_{t_0}^x = \mathbb{E}_{\mathcal{F}_{t_0}}\Big[&M_T^x \partial_{xx} G(X_T^x, \rho_T) - M_T^x \int_{t_0}^T \{\partial_{xx} H_R(\cdot) + 2\nabla Z_s^x \partial_{xp} H_R(\cdot) \\ &+ \nabla Z_s^x \partial_{pp} H_R(\cdot)[\nabla Z_s^x]^\top\}(X_s^x, \rho_s, Z_s^x)\,ds\Big].\end{aligned}$$

Thus, by (A.5) there exists some $C_2^x > 0$ such that

$$\begin{aligned}|\partial_{xx} u(t_0, x)| = |\nabla^2 Y_{t_0}^x| &\le C\mathbb{E}_{\mathcal{F}_{t_0}}\Big[M_T^x + M_T^x \int_{t_0}^T [1 + |\nabla Z_s^x|^2]\,ds\Big] \\ &\le C + C(\mathbb{E}_{\mathcal{F}_{t_0}}[|M_T^x|^2])^{\frac{1}{2}}\Big(\mathbb{E}_{\mathcal{F}_{t_0}}\Big[\Big(\int_{t_0}^T |\nabla Z_s^x|^2\,ds\Big)^2\Big]\Big)^{\frac{1}{2}} \le C_2^x.\end{aligned}$$

Step 3. We now consider the general case. Fix a partition $t_0 < \cdots < t_n = T$ such that $[t_{i+1} - t_i]C_0 < 1$ for all $i = 0, \dots, n-1$. We proceed backwardly in time by induction. Denote $u(t_n, \cdot) := G$. Assume we have defined $u(t_{i+1}, \cdot)$ with bounded first- and second-order derivatives in x. Consider the BSDE (6.1) on $[t_i, t_{i+1}]$ with terminal condition $u(t_{i+1}, \cdot)$. Applying the well-posedness result in Step 1 we obtain $u(t_i, x)$ satisfying (6.2), for a possibly larger C_1^x, C_2^x which depend on the same parameters. Since n is finite, we obtain (6.2) for all i. Now, it follows from standard arguments in FBSDE literature, cf. [41], Theorem 8.3.4, that the BSDE (6.1) on $[t_0, T]$ is well-posed, and (6.2) holds for all $t \in [t_0, T]$. □

PROOF OF PROPOSITION 6.2. (i) We first note that, given another initial value $\xi' \in \mathbb{L}^2(\mathcal{F}_{t_0})$ in (2.24), by (3.1) the following estimate depends on L_0^G, L_2^G but not on L_1^G:

$$\begin{aligned}\mathbb{E}[|G(X_T^\xi, \rho_T^\xi) - G(X_T^{\xi'}, \rho_T^{\xi'})|^2] &\le 2\mathbb{E}[|L_0^G|^2|X_T^\xi - X_T^{\xi'}|^2 + |L_2^G|^2 W_2^2(\rho_T^\xi, \rho_T^{\xi'})] \\ &\le 2[|L_0^G|^2 + |L_2^G|^2]\mathbb{E}[|X_T^\xi - X_T^{\xi'}|^2].\end{aligned} \tag{A.7}$$

By first replacing H with H_R, as in the proof of Proposition 6.1, it follows from the standard contraction mapping argument in FBSDE literature, cf. [41], Theorem 8.2.1, there exists $\delta = \delta_R > 0$ such that the McKean–Vlasov FBSDE (2.24) with H_R is well-posed whenever $T - t_0 \le \delta$. Noticing that, in the contraction mapping argument, G is used exactly in the form of (A.7), here δ_R depends on d, L_0^G, L_2^G, the function L^H and R but not on L_1^G. By Proposition 6.1 we can see that $|Z^\xi| \le C_1^x$, for the C_1^x in (6.2) which does not depend on R. Now, set $R = C_1^x$, and hence δ depends only on d, L_0^G, L_2^G, $L_2^H(C_1^x)$; we see that $H_R(X_s^\xi, \rho_s, Z_s^\xi) = H(X_s^\xi, \rho_s, Z_s^\xi)$, and thus (2.24) is well-posed, which includes existence, uniqueness and, in particular, stability. Similarly in (2.25), (2.26), (6.3), (6.4) and (6.5), the difference of the terminal condition also appears like (A.7), and thus they are also well-posed when $T - t_0 \le \delta$. In particular, we point out that, in the terminal condition of (6.4),

$$\tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^\xi, \rho_T, \tilde{X}_T^{\xi,x}\big) \cdot \nabla_k \tilde{X}_T^{\xi,x} + \partial_\mu G\big(X_T^\xi, \rho_T, X_T^\xi\big) \cdot \nabla_k \tilde{\mathcal{X}}_T^{\xi,x}\big],$$

only $\nabla_k \tilde{\mathcal{X}}_T^{\xi,x}$ is part of the solution, while all other involved random variables are already obtained from the other FBSDEs. Then, in the contraction mapping argument, the random coefficient $\partial_\mu G(X_T^\xi, \rho_T, X_T^\xi)$ of the solution term $\nabla_k \tilde{\mathcal{X}}_T^{\xi,x}$ again appears in $\mathbb{L}^2$-sense,

$$\tilde{\mathbb{E}}\big[\big|\partial_\mu G\big(X_T^\xi, \rho_T, \tilde{X}_T^\xi\big) \cdot \tilde{\zeta}_1 - \partial_\mu G\big(X_T^\xi, \rho_T, \tilde{X}_T^\xi\big) \cdot \tilde{\zeta}_2\big|^2\big] \le \big|L_2^G\big|^2 \mathbb{E}\big[|\zeta_1 - \zeta_2|^2\big].$$

(ii) The proof of this point is rather lengthy but follows almost the same arguments as in [39], Theorem 9.2. Given (6.6), the first estimate of (6.7) follows directly from the estimate for BSDE (6.5). Indeed, by Proposition 6.1 we obtained $|Z^\xi|, |Z^{x,\xi}|, |Z^{\xi,x}| \le C_1^x$. Then, using Assumptions 3.1(i) and 3.2(i), the coefficients in the linear FBSDEs (6.3) and (6.4) and the linear BSDE (6.5) are bounded by L_0^G, L_1^G, $L^H(C_1^x)$ correspondingly. Therefore, there exists $C_1^\mu > 0$, depending on these constants and d. such that

$$\big|\partial_{\mu_k} V(t_0, x, \mu, \tilde{x})\big| = \big|\nabla_{\mu_k} Y_{t_0}^{x,\xi,\tilde{x}}\big| \le C_1^\mu.$$

Moreover, by differentiating (6.5) with respect to x, we can derive the representation formula for $\partial_{x\mu} V$ from (6.6), and then the second estimate of (6.7) also follows directly from the estimate for the differentiated BSDE.

We now prove (6.6) in four steps. Without loss of generality, we prove only the case that $k = 1$ and $t_0 = 0$. Throughout the proof, it is sometimes convenient to use the notation $X^{x,\xi} := X^x$:

Step 1. For any $\xi \in \mathbb{L}^2(\mathcal{F}_0, \mu)$ and any scalar random variable $\eta \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R})$, following standard arguments and by the stability property of the involved systems, we have

$$\lim_{\varepsilon \to 0} \mathbb{E}\bigg[\sup_{0 \le t \le T} \bigg|\frac{1}{\varepsilon}\big[X_t^{\xi + \varepsilon\eta e_1} - X_t^\xi\big] - \delta X_t^{\xi,\eta e_1}\bigg|^2\bigg] = 0, \tag{A.8}$$

where $(\delta X^{\xi,\eta e_1}, \delta Y^{\xi,\eta e_1}, \delta Z^{\xi,\eta e_1}, \delta Z^{0,\xi,\eta e_1})$ satisfies the linear McKean–Vlasov FBSDE,

$$\begin{cases} \delta X_t^{\xi,\eta e_1} = \eta e_1 - \displaystyle\int_0^t \big(\delta X_s^{\xi,\eta e_1}\big)^\top \partial_{xp} H\big(X_s^\xi, \rho_s, Z_s^\xi\big) + \big(\delta Z_s^{\xi,\eta e_1}\big)^\top \partial_{pp} H\big(X_s^\xi, \rho_s, Z_s^\xi\big) \\ \qquad\qquad + \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_{p\mu} H\big(X_s^\xi, \rho_s, \tilde{X}_s^\xi, Z_s^\xi\big) \cdot \delta \tilde{X}_s^{\xi,\eta e_1}\big]\, ds; \\ \delta Y_t^{\xi,\eta e_1} = \partial_x G\big(X_T^\xi, \rho_T\big) \cdot \delta X_T^{\xi,\eta e_1} + \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^\xi, \rho_T, \tilde{X}_T^\xi\big) \cdot \delta \tilde{X}_T^{\xi,\eta e_1}\big] \\ \qquad\qquad + \displaystyle\int_t^T \partial_x \widehat{L}\big(X_s^\xi, \rho_s, Z_s^\xi\big) \cdot \delta X_s^{\xi,\eta e_1} + \partial_p \widehat{L}\big(X_s^\xi, \rho_s, Z_s^\xi\big) \cdot \delta Z_s^{\xi,\eta e_1} \\ \qquad\qquad + \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu \widehat{L}\big(X_s^\xi, \rho_s, \tilde{X}_s^\xi, Z_s^\xi\big) \cdot \delta \tilde{X}_s^{\xi,\eta e_1}\big]\, ds \\ \qquad\qquad - \displaystyle\int_t^T \delta Z_s^{\xi,\eta e_1} \cdot dB_s - \int_t^T \delta Z_s^{0,\xi,\eta e_1} \cdot dB_s^0. \end{cases} \tag{A.9}$$

Similarly, by (A.8) and (2.25), one can show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\frac{1}{\varepsilon}\big[Y_t^{x,\xi+\varepsilon\eta e_1}-Y_t^{x,\xi}\big]-\delta Y_t^{x,\xi,\eta e_1}\Big|^2\Big]=0, \tag{A.10}$$

where $(\delta Y^{x,\xi,\eta e_1},\delta Z^{x,\xi,\eta e_1},\delta Z^{0,x,\xi,\eta e_1})$ satisfies the linear (standard) BSDE,

$$\begin{aligned}\delta Y_t^{x,\xi,\eta e_1}&=\tilde{\mathbb{E}}_{\mathcal{F}_T^0}\big[\partial_\mu G\big(X_T^x,\rho_T,\tilde X_T^\xi\big)\cdot\delta\tilde X_T^{\xi,\eta e_1}\big]\\&\quad-\int_t^T\partial_p H\big(X_s^{x,\xi},\rho_s,Z_s^{x,\xi}\big)\cdot\delta Z_s^{x,\xi,\eta e_1}\\&\quad+\tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu H\big(X_s^{x,\xi},\rho_s,\tilde X_s^\xi,Z_s^{x,\xi}\big)\cdot\delta\tilde X_s^{\xi,\eta e_1}\big]\,ds\\&\quad-\int_t^T\delta Z_s^{x,\xi,\eta e_1}\cdot dB_s-\int_t^T\delta Z_s^{0,x,\xi,\eta e_1}\cdot dB_s^0.\end{aligned} \tag{A.11}$$

In particular, (A.10) implies

$$\lim_{\varepsilon\to 0}\Big|\frac{1}{\varepsilon}\big[V(0,x,\mathcal{L}_{\xi+\varepsilon\eta e_1})-V(0,x,\mathcal{L}_\xi)\big]-\delta Y_0^{x,\xi,\eta e_1}\Big|^2=0. \tag{A.12}$$

Thus, by the definition of $\partial_\mu V$,

$$\mathbb{E}\big[\partial_{\mu_1}V(0,x,\mu,\xi)\eta\big]=\delta Y_0^{x,\xi,\eta e_1}. \tag{A.13}$$

Step 2. In this step we assume ξ (or say, μ) is discrete: $p_i=\mathbb{P}(\xi=x_i)$, $i=1,\dots,n$. Fix i; consider the following system of McKean–Vlasov FBSDEs: for $j=1,\dots,n$,

$$\begin{cases}\nabla_{\mu_1}X_t^{i,j}=\mathbf{1}_{\{i=j\}}e_1-\displaystyle\int_0^t\sum_{k=1}^n p_k\tilde{\mathbb{E}}_{\mathcal{F}_s}\big[(\nabla_{\mu_1}\tilde X_s^{i,k})^\top\partial_{\mu p}H\big(X_s^{\xi,x_j},\rho_s,\tilde X_T^{\xi,x_k},Z_s^{\xi,x_j}\big)\big]\\ \qquad\qquad+(\nabla_{\mu_1}X_s^{i,j})^\top\partial_{xp}H\big(X_s^{\xi,x_j},\rho_s,Z_s^{\xi,x_j}\big)\\ \qquad\qquad+(\nabla_{\mu_1}Z_s^{i,j})^\top\partial_{pp}H\big(X_s^{\xi,x_j},\rho_s,Z_s^{\xi,x_j}\big)\,ds;\\ \nabla_{\mu_1}Y_t^{i,j}=\partial_x G\big(X_T^{\xi,x_j},\rho_T\big)\cdot\nabla_{\mu_1}X_T^{i,j}\\ \qquad\qquad+\displaystyle\sum_{k=1}^n p_k\tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^{\xi,x_j},\rho_T,\tilde X_T^{\xi,x_k}\big)\cdot\nabla_{\mu_1}\tilde X_T^{i,k}\big]\\ \qquad\qquad+\displaystyle\int_t^T\partial_x\widehat L\big(X_s^{\xi,x_j},\rho_s,Z_s^{\xi,x_j}\big)\cdot\nabla_{\mu_1}X_s^{i,j}\\ \qquad\qquad+\partial_p\widehat L\big(X_s^{\xi,x_j},\rho_s,Z_s^{\xi,x_j}\big)\cdot\nabla_{\mu_1}Z_s^{i,j}\\ \qquad\qquad+\displaystyle\sum_{k=1}^n p_k\tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu\widehat L\big(X_s^{\xi,x_j},\rho_s,\tilde X_s^{\xi,x_k},Z_s^{\xi,x_j}\big)\cdot\nabla_{\mu_1}\tilde X_s^{i,k}\,ds\\ \qquad\qquad-\displaystyle\int_t^T\nabla_{\mu_1}Z_s^{i,j}\cdot dB_s-\beta\int_t^T\nabla_{\mu_1}Z_s^{0,i,j}\cdot dB_s^0.\end{cases} \tag{A.14}$$

For any $\Phi\in\{X,Y,Z,Z^0\}$, we define

$$\nabla_1\Phi^{\xi,x_i}:=\nabla_{\mu_1}\Phi^{i,i},\qquad \nabla_1\Phi^{\xi,x_i,*}:=\frac{1}{p_i}\sum_{j\ne i}\nabla_{\mu_1}\Phi^{i,j}\mathbf{1}_{\{\xi=x_j\}}.$$

Note that $\Phi^\xi = \sum_{j=1}^n \Phi^{\xi,x_j}\mathbf{1}_{\{\xi=x_j\}}$. Since (A.14) is linear, one can easily check that

$$\begin{aligned}\nabla_1 X_t^{\xi,x_i} = e_1 - \int_0^t \big\{&(\nabla_1 X_s^{\xi,x_i})^\top \partial_{xp} H(X_s^{\xi,x_i},\rho_s,Z_s^{\xi,x_i})\\ &+ (\nabla_1 Z_s^{\xi,x_i})^\top \partial_{pp} H(X_s^{\xi,x_i},\rho_s,Z_s^{\xi,x_i})\\ &+ p_i \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[(\nabla_1 \tilde{X}_s^{\xi,x_i})^\top \partial_{\mu p} H(X_s^{\xi,x_i},\rho_s,\tilde{X}_s^{\xi,x_i},Z_s^{\xi,x_i})\\ &+ (\nabla_1 \tilde{X}_s^{\xi,x_i,*})^\top \partial_{\mu p} H(X_s^{\xi,x_i},\rho_s,\tilde{X}_s^{\xi},Z_s^{\xi,x_i})\big]\big\}\,ds,\end{aligned} \tag{A.15}$$

$$\begin{aligned}\nabla_1 X_t^{\xi,x_i,*} = - \int_0^t \big\{&(\nabla_1 X_s^{\xi,x_i,*})^\top \partial_{xp} H(X_s^{\xi},\rho_s,Z_s^{\xi})\\ &+ (\nabla_1 Z_s^{\xi,x_i,*})^\top \partial_{pp} H(X_s^{\xi},\rho_s,Z_s^{\xi})\\ &+ \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[(\nabla_1 \tilde{X}_s^{\xi,x_i})^\top \partial_{\mu p} H(X_s^{\xi},\rho_s,\tilde{X}_s^{\xi,x_i},Z_s^{\xi})\\ &+ (\nabla_1 \tilde{X}_s^{\xi,x_i,*})^\top \partial_{\mu p} H(X_s^{\xi},\rho_s,\tilde{X}_s^{\xi},Z_s^{\xi})\big]\mathbf{1}_{\{\xi\neq x_i\}}\big\}\,ds,\end{aligned} \tag{A.16}$$

$$\begin{aligned}\nabla_1 Y_t^{\xi,x_i} = {}&\partial_x G(X_T^{\xi,x_i},\rho_T)\cdot \nabla_1 X_T^{\xi,x_i} - \int_t^T \nabla_1 Z_s^{\xi,x_i}\,dB_s\\ &- \int_t^T \nabla_1 Z_s^{0,\xi,x_i}\,dB_s^0\\ &+ p_i \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G(X_T^{\xi,x_i},\rho_T,\tilde{X}_T^{\xi,x_i})\cdot \nabla_1 \tilde{X}_s^{\xi,x_i}\\ &+ \partial_\mu G(X_T^{\xi,x_i},\rho_T,\tilde{X}_T^{\xi})\cdot \nabla_1 \tilde{X}_T^{\xi,x_i,*}\big]\\ &+ \int_t^T \big\{\partial_x \widehat{L}(X_s^{\xi,x_i},\rho_s,Z_s^{\xi,x_i})\cdot \nabla_1 X_s^{\xi,x_i}\\ &+ \partial_p \widehat{L}(X_s^{\xi,x_i},\rho_s,Z_s^{\xi,x_i})\cdot \nabla_1 Z_s^{\xi,x_i} + p_i\\ &\times \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu \widehat{L}(X_s^{\xi,x_i},\rho_s,\tilde{X}_s^{\xi,x_i},Z_s^{\xi,x_i})\cdot \nabla_1 \tilde{X}_s^{\xi,x_i}\\ &+ \partial_\mu \widehat{L}(X_s^{\xi,x_i},\rho_s,\tilde{X}_s^{\xi},Z_s^{\xi,x_i})\cdot \nabla_1 \tilde{X}_T^{\xi,x_i,*})\big]\big\}\,ds,\end{aligned} \tag{A.17}$$

$$\begin{aligned}\nabla_1 Y_t^{\xi,x_i,*} = {}&\partial_x G(X_T^{\xi},\rho_T)\cdot \nabla_1 X_T^{\xi,x_i,*}\\ &- \int_t^T \nabla_1 Z_s^{\xi,x_i,*}\cdot dB_s - \int_t^T \nabla_1 Z_s^{0,\xi,x_i,*}\cdot dB_s^0\\ &+ \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G(X_T^{\xi},\rho_T,\tilde{X}_T^{\xi,x_i})\cdot \nabla_1 \tilde{X}_s^{\xi,x_i}\\ &+ \partial_\mu G(X_T^{\xi},\rho_T,\tilde{X}_T^{\xi})\cdot \nabla_1 \tilde{X}_T^{\xi,x_i,*}\big]\mathbf{1}_{\{\xi\neq x_i\}}\\ &+ \int_t^T \big\{\partial_x \widehat{L}(X_s^{\xi},\rho_s,Z_s^{\xi})\cdot \nabla_1 X_s^{\xi,x_i,*}\\ &+ \partial_p \widehat{L}(X_s^{\xi},\rho_s,Z_s^{\xi})\cdot \nabla_1 Z_s^{\xi,x_i,*}\big]\\ &+ \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu \widehat{L}(X_s^{\xi},\rho_s,\tilde{X}_s^{\xi,x_i},Z_s^{\xi})\cdot \nabla_1 \tilde{X}_s^{\xi,x_i}\\ &+ \partial_\mu \widehat{L}(X_s^{\xi},\rho_s,\tilde{X}_s^{\xi},Z_s^{\xi})\cdot \nabla_1 \tilde{X}_T^{\xi,x_i-})\big]\mathbf{1}_{\{\xi\neq x_i\}}\big\}\,ds.\end{aligned} \tag{A.18}$$

Since (A.9) is also linear, one can easily check that, for $\Phi \in \{X, Y, Z, Z^0\}$,

$$\delta \Phi^{\xi,\mathbf{1}_{\{\xi=x_i\}}e_1} = \nabla_1 \Phi^{\xi,x_i}\mathbf{1}_{\{\xi=x_i\}} + p_i \nabla_1 \Phi^{\xi,x_i,*}. \tag{A.19}$$

Moreover, note that

$$
\begin{aligned}
&\tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^{x,\xi}, \rho_T, \tilde{X}_T^{\xi}\big) \cdot \delta \tilde{X}_T^{\xi, \mathbf{1}_{\{\xi=x_i\}} e_1}\big] \\
&\quad = \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^{x,\xi}, \rho_T, \tilde{X}_T^{\xi}\big) \cdot \big[\nabla_1 \tilde{X}_T^{\xi, x_i} \mathbf{1}_{\{\tilde{\xi}=x_i\}} + p_i \nabla_1 \tilde{X}_T^{\xi, x_i, *}\big]\big] \\
&\quad = p_i \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^{x,\xi}, \rho_T, \tilde{X}_T^{\xi, x_i}\big) \cdot \nabla_1 \tilde{X}_T^{\xi, x_i} + \partial_\mu G\big(X_T^{x,\xi}, \rho_T, \tilde{X}_T^{\xi}\big) \cdot \nabla_1 \tilde{X}_T^{\xi, x_i, *}\big]
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu H\big(X_s^{x,\xi}, \rho_s, \tilde{X}_s^{\xi}, Z_s^{x,\xi}\big) \cdot \delta \tilde{X}_s^{\xi, \mathbf{1}_{\{\xi=x_i\}} e_1}\big] \\
&\quad = p_i \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu H\big(X_s^{x,\xi}, \rho_s, \tilde{X}_s^{\xi, x_i}, Z_s^{x,\xi}\big) \cdot \nabla_1 \tilde{X}_s^{\xi, x_i} + \partial_\mu H\big(X_s^{x,\xi}, \rho_s, \tilde{X}_s^{\xi}, Z_s^{x,\xi}\big) \cdot \nabla_1 \tilde{X}_s^{\xi, x_i, *}\big)\big].
\end{aligned}
$$

Plug this into (A.11); we obtain

$$
\delta \Phi_t^{x,\xi,\mathbf{1}_{\{\xi=x_i\}} e_1} = p_i \nabla_{\mu_1} \Phi_t^{x,\xi,x_i}, \tag{A.20}
$$

where

$$
\begin{aligned}
\nabla_{\mu_1} Y_t^{x,\xi,x_i} = {} & \tilde{\mathbb{E}}_{\mathcal{F}_T}\big[\partial_\mu G\big(X_T^{x}, \rho_T, \tilde{X}_T^{\xi, x_i}\big) \cdot \nabla_1 \tilde{X}_T^{\xi, x_i} \\
& + \partial_\mu G\big(X_T^{x}, \rho_T, \tilde{X}_T^{\xi}\big) \cdot \nabla_1 \tilde{X}_T^{\xi, x_i, *}\big)\big] \\
& - \int_t^T \big\{\partial_p H\big(X_s^{x,\xi}, \rho_s, Z_s^{x,\xi}\big) \cdot \nabla_{\mu_1} Z_s^{x,\xi,x_i} \\
& + \tilde{\mathbb{E}}_{\mathcal{F}_s}\big[\partial_\mu H\big(X_s^{x,\xi}, \rho_s, \tilde{X}_s^{\xi, x_i}, Z_s^{x,\xi}\big) \cdot \nabla_1 \tilde{X}_s^{\xi, x_i} \\
& + \partial_\mu H\big(X_s^{x,\xi}, \rho_s, \tilde{X}_s^{\xi}, Z_s^{x,\xi}\big) \cdot \nabla_1 \tilde{X}_s^{\xi, x_i, *}\big]\big\} \, ds \\
& - \int_t^T \nabla_{\mu_1} Z_s^{x,\xi,x_i} \cdot dB_s - \int_t^T \nabla_{\mu_1} Z_s^{0,x,\xi,x_i} \cdot dB_s^0.
\end{aligned} \tag{A.21}
$$

In particular, by setting $\eta = \mathbf{1}_{\{\xi=x_i\}}$ in (A.13), we obtain

$$
\partial_{\mu_1} V(0, x, \mu, x_i) = \nabla_{\mu_1} Y_0^{x,\xi,x_i}. \tag{A.22}
$$

We shall note that (A.15)–(A.16), (A.17)–(A.18) is different from (6.3) and (6.4), so (A.22) provides an alternative discrete representation.

Step 3. We now prove (6.6) in the case that μ is absolutely continuous. For each $n \geq 3$, set

$$
x_{\vec{i}}^n := \frac{\vec{i}}{n}, \qquad \Delta_{\vec{i}}^n := \left[\frac{i_1}{n}, \frac{i_1+1}{n}\right) \times \cdots \times \left[\frac{i_d}{n}, \frac{i_d+1}{n}\right), \qquad \vec{i} = (i_1, \dots, i_d)^\top \in \mathbb{Z}^d.
$$

For any $x \in \mathbb{R}^d$, there exists $\vec{i}(x) := (i_1(x), \dots, i_d(x)) \in \mathbb{Z}^d$ such that $x \in \Delta_{\vec{i}(x)}^n$. Let

$$
\vec{i}^n(x) := \big(i_1^n(x), \dots, i_d^n(x)\big) \in \mathbb{Z}^d \quad \text{where } i_l^n(x) := \min\{\max\{i_l, -n^2\}, n^2\}, l = 1, \dots, d.
$$

Denote $Q_n := \{x \in \mathbb{R}^d : |x_i| \leq n, i = 1, \dots, d\}$, $\mathbb{Z}_n^d := \{\vec{i} \in \mathbb{Z}^d : \Delta_{\vec{i}}^n \cap Q_n \neq \varnothing\}$ and

$$
\xi_n := \sum_{\vec{i} \in \mathbb{Z}_n^d} x_{\vec{i}}^n \mathbf{1}_{\Delta_{\vec{i}}^n}(\xi) + \frac{\vec{i}^n(\xi)}{n} \mathbf{1}_{Q_n^c}(\xi). \tag{A.23}
$$

It is clear that $\lim_{n \to +\infty} \mathbb{E}[|\xi_n - \xi|^2] = 0$, and thus $\lim_{n \to \infty} W_2(\mathcal{L}_{\xi_n}, \mathcal{L}_\xi) = 0$. Then, for any scalar random variable η, by stability of FBSDE (A.9) and BSDE (A.11), we derive from (A.13) that

$$
\mathbb{E}\big[\partial_{\mu_1} V(0, x, \mu, \xi)\eta\big] = \delta Y_0^{x,\xi,\eta e_1} = \lim_{n \to \infty} \delta Y_0^{x,\xi_n,\eta e_1}. \tag{A.24}
$$

For each $\tilde{x} \in \mathbb{R}^d$, let $\vec{i}(\tilde{x})$ be the i such that $\tilde{x} \in \Delta^n_{\vec{i}}$ which holds when $n > |\tilde{x}|$. Then, $(\mathcal{L}_{\xi_n}, \frac{\vec{i}(\tilde{x})}{n}) \to (\mu, \tilde{x})$ as $n \to \infty$. By the stability of FBSDEs (2.24)–(2.25), we have $(X^{\xi_n, \frac{\vec{i}(\tilde{x})}{n}}, Z^{\xi_n, \frac{\vec{i}(\tilde{x})}{n}}) \to (X^{\xi,\tilde{x}}, Z^{\xi,\tilde{x}})$ under appropriate norms. Moreover, since μ is absolutely continuous,

$$\mathbb{P}\Big(\xi_n = \frac{\vec{i}(\tilde{x})}{n}\Big) = \mathbb{P}(\xi \in \Delta^n_{\vec{i}}) \to 0 \quad \text{as } n \to \infty.$$

Then, by the stability of (A.15)–(A.16), (A.17)–(A.18) and (A.21) we can check that

$$\lim_{n\to\infty}\big(\nabla_1 \Phi^{\xi_n, \frac{\vec{i}(\tilde{x})}{n}}, \nabla_1 \Phi^{\xi_n, \frac{\vec{i}(\tilde{x})}{n},*}, \nabla_{\mu_1} \Phi^{x,\xi_n, \frac{\vec{i}(\tilde{x})}{n}}\big) = \big(\nabla_1 \Phi^{\xi,\tilde{x}}, \nabla_1 \Phi^{\xi,\tilde{x},*}, \nabla_{\mu_1} \Phi^{x,\xi,\tilde{x}}\big). \tag{A.25}$$

Now, for any bounded function $\varphi \in C(\mathbb{R}^d)$, set $\eta = \varphi(\xi)$ in (A.24); by (A.20) we have

$$\mathbb{E}\big[\partial_{\mu_1} V(0, x, \mu, \xi)\varphi(\xi)\big] = \lim_{n\to\infty} \delta Y_0^{x,\xi_n,\varphi(\xi_n)e_1} = \lim_{n\to\infty} \sum_{\vec{i}\in\mathbb{Z}^d_n} \varphi(x^n_{\vec{i}})\delta Y_0^{x,\xi_n,\mathbf{1}_{\{\xi_n = x^n_{\vec{i}}\}}e_1},$$

and so,

$$\begin{aligned}\mathbb{E}\big[\partial_{\mu_1} V(0, x, \mu, \xi)\varphi(\xi)\big] &= \lim_{n\to\infty} \sum_{\vec{i}\in\mathbb{Z}^d_n} \varphi(x^n_{\vec{i}})\nabla_{\mu_1} Y_0^{x,\xi_n,x^n_{\vec{i}}} \mathbb{P}(\xi \in \Delta_{\vec{i}}) \\ &= \int_{\mathbb{R}^d} \varphi(\tilde{x})\nabla_{\mu_1} Y_0^{x,\xi,\tilde{x}} \mu(d\tilde{x}).\end{aligned}$$

This implies (6.6) immediately.

Step 4. We finally prove the general case. Denote $\psi(x, \mu, \tilde{x}) := \nabla_{\mu_1} Y_0^{x,\xi,\tilde{x}}$. By the stability of FBSDEs, ψ is continuous in all the variables. Fix an arbitrary (μ, ξ). One can easily construct ξ_n such that $\mathcal{L}_{\xi_n}$ is absolutely continuous, and $\lim_{n\to\infty} \mathbb{E}[|\xi_n - \xi|^2] = 0$. Then, for any $\eta = \varphi(\xi)$, as in Step 3, by (A.13) and Step 3 we have

$$\begin{aligned}\mathbb{E}\big[\partial_{\mu_1} V(0, x, \mu, \xi)\varphi(\xi)\big] &= \lim_{n\to\infty} \delta Y_0^{x,\xi_n,\varphi(\xi_n)e_1} = \lim_{n\to\infty} \mathbb{E}\big[\psi(x, \mathcal{L}_{\xi_n}, \xi_n)\varphi(\xi_n)\big] \\ &= \mathbb{E}\big[\psi(x, \mu, \xi)\varphi(\xi)\big]\end{aligned}$$

which implies (6.6) in the general case.

(iii) This result follows immediately from well-known facts. Indeed, given the uniform estimate of $\partial_\mu V$ in (6.7), the well-posedness of the master equation (1.1) on $[t_0, T]$ follows from the arguments in [25], Theorem 5.10. This, together with Itô formula (2.12), will easily lead to (2.27). Under the additional Assumptions 3.1(ii) and 3.2(ii), the representation formulas and the boundedness of higher-order derivatives in state and probability variables can be proved by further differentiating the McKean–Vlasov FBSDEs (2.24)–(2.26), (6.3)–(6.4) and BSDE (6.5) with respect to the state and probability variables. The calculation is lengthy but very similar to that in [39], Section 9.2. We omit the details. □

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