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ON WELL-POSEDNESS OF FORWARD–BACKWARD SDES—A UNIFIED APPROACH

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In this paper, we study the well-posedness of the Forward–Backward Stochastic Differential Equations (FBSDE) in a general non-Markovian framework. The main purpose is to find a unified scheme which combines all existing methodology in the literature, and to address some fundamental longstanding problems for non-Markovian FBSDEs. An important device is a *decoupling random field* that is *regular* (uniformly Lipschitz in its spatial variable). We show that the regulariy of such decoupling field is closely related to the bounded solution to an associated *characteristic BSDE*, a backward stochastic Riccati-type equation with superlinear growth in both components Y and Z. We establish various sufficient conditions for the well-posedness of an ODE that dominates the characteristic BSDE, which leads to the existence of the desired regular decoupling random field, whence the solvability of the original FBSDE. A synthetic analysis of the solvability is given, as a "User's Guide," for a large class of FBSDEs that are not covered by the existing methods. Some of them have important implications in applications.

1. Introduction. The theory of Backward Stochastic Differential Equations (BSDEs) and Forward–Backward Stochastic Differential Equations (FBSDEs) have been studied extensively for the past two decades, and its applications have been found in many branches of applied mathematics, especially the stochastic control theory and mathematical finance. It has been noted, however, that while in many situations the solvability of the original (applied) problems is essentially equivalent to the solvability of certain type of FBSDEs, these FBSDEs are often beyond the scope of any existing frameworks, especially when they are outside the Markovian paradigm, where the PDE tool becomes powerless. In fact, the balance between the regularity of the coefficients and the time duration, as well as the nondegeneracy (of the forward diffusion), has been a longstanding problem

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in the FBSDE literature, especially in a general non-Markovian framework. It has become increasingly clear that the theory now calls for new insights and ideas that can lead to a better understanding of the problem and hopefully to a unified solution scheme for the general FBSDEs.

A strongly coupled FBSDE takes the following form:

$$
\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)\,ds \\ \qquad\quad + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,dB_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dB_s, \end{cases} \qquad t \in [0, T], \tag{1.1}
$$

where b, f and σ are (progressively) measurable functions defined on appropriate spaces, B is a standard Brownian motion and g is a (possibly random) function that is defined on $\mathbb{R}^n \times \Omega$ such that $g(x, \cdot)$ is $\mathcal{F}_T$-measurable for each fixed x.

There have been three main methods to solve FBSDE (1.1). First, the *Method of Contraction Mapping*. This method, first used by Antonelli [1] and later detailed by Pardoux and Tang [17], works well when the duration T is relatively small. Second, the *Four Step Scheme*. This was the first solution method that removed restriction on the time duration for Markovian FBSDEs, initiated by Ma, Protter and Yong [12]. The trade-off is the requirement on the regularity of the coefficients so that a "decoupling" quasi-linear PDE has a classical solution. Third, the *Method of Continuation*. This was a method that can treat non-Markovian FBSDEs with arbitrary duration, initiated by Hu and Peng [7] and Peng and Wu [18], and later developed by Yong [24] and recently in [26]. The main assumption for this method is the so-called "monotonicity conditions" on the coefficients, which is restrictive in a different way. This method has been used widely in applications (see, e.g., [21, 23, 27]) because of its pure probabilistic nature. We refer to the book of Ma and Yong [16] for the detailed accounts for all three methods. It is worth noting that these three methods do not cover each other.

To make our motivation clearer, let us take a quick look at some main difficulties in the FBSDE theory. For example, consider the following simple FBSDE:

$$
X_t = x + \int_0^t \sigma Z_s\,dW_s, \qquad Y_t = X_T - \int_t^T Z_s\,dW_s. \tag{1.2}
$$

Clearly, the FBSDE has infinitely many solutions when $\sigma = 1$, and is well-posed when $\sigma = 0$. But more or less surprisingly, for $\sigma \neq 0, 1$, none of the three standard methods works for this FBSDE when T is arbitrarily large. The FBSDE with such a feature has been encountered in many stochastic control problems when diffusion contains control, which is often the case in the optimal investment problems in finance. Understanding its solvability is therefore extremely desirable, especially when seeking the closed-loop optimal control via Pontrygin's maximum principle.

Another simple example, appeared in an earlier works of the fourth author [4] where the idea of *method of optimal control* (cf., e.g., [16]) was adopted to study a Monte Carlo method for FBSDEs, is of the following form:

$$
\begin{aligned}
X_t &= x + \int_0^t [a_s X_s + b_s Z_s]\,ds + \int_0^t \sigma_s\,dB_s; \\
Y_t &= hX_T + \int_t^T [c_s X_s + d_s Z_s]\,ds - \int_t^T Z_s\,dB_s,
\end{aligned}
\tag{1.3}
$$

where a, b, c, d and σ are stochastic processes, and h is an $\mathcal{F}_T$-random variable. Again, this FBSDE is not covered by any existing method. However, as we will see in Section 7 that the solvability of (1.2) and (1.3), including an crucial estimate in [4] regarding the solution to (1.3), will be the easy consequences of our general results. In fact, the work [4] was the motivation for [28], which in turn motivated this paper.

The main goal of this paper is to develop a strategy to construct a decoupling random field which will be the key to the solvability of general non-Markovian FBSDEs. Our starting point is the work of Delarue [5], in a Markovian framework with $\sigma = \sigma(t, x, y)$ being uniformly nondegenerate. In that case, an FBSDE over arbitrary time duration was solved under only Lipschitz conditions on the coefficients, by combining nicely the Method of Contraction Mapping, the Four Step Scheme, and some delicate PDE arguments. The idea was later extended by Zhang [28] to the non-Markovian cases [again in the case $\sigma = \sigma(t, x, y)$], by using mainly probabilistic arguments, and with the help of some compatibility conditions. The main point is still, as in the Four Step Scheme, around finding a function u such that

$$
Y_t = u(t, X_t), \qquad t \in [0, T]. \tag{1.4}
$$

Clearly, if the FBSDE (1.1) is non-Markovian, then u should be a random field. The key issue here, as we shall argue, is the existence of such a *decoupling* random field that is uniformly Lipschitz in its spatial variable. We will show that the existence of such a random field is closely related to the solvability of an associated BSDE (called the *characteristic BSDE* in this paper), and will ultimately lead to the well-posedness of the original FBSDEs. We shall provide a set of sufficient conditions for the existence of such decoupling field, and show that most of the existing frameworks in the literature could be analyzed by using our criteria. Furthermore, we note that in the case when the FBSDE is linear with constant coefficients, some of our conditions are actually necessary. In other words, these conditions *cannot* be improved.

A brief description of our plan is as follows. Assume that the decoupling field u exists and the FBSDE is well-posed. Denote (X^x, Y^x, Z^x) to be the solution to FBSDE (1.1) with initial value x. Then we argue that the derivative of (X^x, Y^x, Z^x) with respect to x, denoted by $(\nabla X, \nabla Y, \nabla Z)$, would satisfy a linear "variational FBSDE" [see (3.5) below]. Since $Y_t^x = u(t, X_t^x)$ by (1.4), we must have $\nabla Y_t = u_x(t, X_t)\nabla X_t$, and thus $u_x(t, X_t) = \nabla Y_t(\nabla X_t)^{-1} \stackrel{\triangle}{=} \hat{Y}_t$. In other words, proving u is uniformly Lipschitz continuous amounts to finding solutions to the linear FBSDE (3.5) such that $\hat{Y}$ is uniformly bounded. Furthermore, one can check that $\hat{Y}$ actually satisfies a BSDE [see (3.8) below] which will be called the *characteristic BSDE* in this paper. We note that this BSDE has superlinear growth in both components of the solutions, thus it is itself a novel subject in BSDE theory, and thus is interesting in its own right.

Seeking the bounded solution to the characteristic BSDE over an arbitrary time duration is by no means trivial, due to its superlinear growth behavior. We shall accomplish this by studying two dominating ODEs [see (3.13) below], which bound $\hat{Y}$ from above and below, respectively. Although the ODEs also have the combined complexity from its nonlinearity, superlinear growth, and the singularity, it is much more tractable. We shall give a set of sufficient conditions to guarantee the existence of the solutions to the ODEs, which in turn guarantees the solvability of the original FBSDE (1.1). Our results extend those of [28] in many ways, and we believe they are by far the most general criteria for the solvability of FBSDEs. As a byproduct, we also prove a comparison theorem for the decoupling random field over all time, thus confirming a common belief (see, e.g., [16, 20, 22]).

There are several technical aspects in this paper that are worth emphasizing. First, unlike the linear FBSDEs studied in [25], where conditions were made so that the associated characteristic BSDE is linear in $\hat{Y}$, or the so-called backward stochastic Riccati equation, often seen in the linear-quadratic stochastic control literature (see, e.g., [11] and [19]) in which the growth condition is quadratic in $\hat{Y}$ but linear in Z, in the present case the generator has at least quadratic growth on both components. To our best knowledge, such a case has not been investigated in the literature. Second, our method requires the minimum assumptions on the coefficients, and covers both Markovian and non-Markovian cases, without having to go through the quasilinear PDEs and backward SPDEs (see, e.g., [5, 6, 12, 14, 15]). In an accompanying paper [13], however, we show that the FBSDE has a uniformly Lipschitz continuous decoupling field (and thus is well-posed) if and only if the corresponding quasi-linear BSPDE has a uniformly Lipschitz continuous Sobolev type weak solution. We hope that this connection can enhance further understanding on both FBSDEs and BSPDEs. Third, the method in this paper is particularly effective for the cases where the forward diffusion coefficient σ depends on Z, which has been avoided in many existing works, as it brings in some extra complications for the solvability analysis (see, e.g., [5, 16]). Finally, in this paper we content ourselves for one-dimensional FBSDEs. In fact, the characteristic BSDE becomes much more subtle in high-dimensional cases, as it involves the combination of high-dimensional BSDEs with quadratic growth (in Z) and high-dimensional backward stochastic Riccati equations, each of which is very challenging. We hope to be able to address this issue in our future publications.

The rest of the paper is organized as follows. In Section 2, we introduce the decoupling field and show how it leads to the well-posedness of FBSDEs. In Section 3, we heuristically discuss our strategy for obtaining the uniformly Lipschitz continuity of the decoupling field. In Section 4, we study the relation between the solvability of the linear variational FBSDE and its characteristic BSDE, and in Section 5 we investigate the global solutions of the dominant ODEs. In Section 6, we investigate the well-posedness of FBSDEs over small time duration, and in Section 7 we conclude our well-posedness result for general FBSDEs over arbitrary time interval. In Section 8, we prove several further properties of FBSDEs. Finally, in the Appendix, we complete some technical proofs.

2. The decoupling field. Throughout this paper, we denote $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ to be a filtered probability space on which is defined a Brownian motion $B = (B_t)_{t \geq 0}$. We assume that $\mathbb{F} \stackrel{\triangle}{=} \mathbb{F}^B \stackrel{\triangle}{=} \{\mathcal{F}_t^B\}_{t \geq 0}$, the natural filtration generated by B, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. For any sub-σ-filed $\mathcal{G} \subseteq \mathcal{F}$, and $0 \leq p \leq \infty$, we denote $L^p(\mathcal{G})$ to be the spaces of all $\mathcal{G}$-measurable, L^p-integrable random variables. In what follows, we assume that *all processes involved are one-dimensional.*

Let $T > 0$ be a fixed time horizon. We consider the general FBSDEs (1.1), where the coefficients b, σ, f, g are measurable functions, and are allowed to be random in general. For technical clarity, we shall make use of the following *standing assumptions* throughout the paper.

ASSUMPTION 2.1. (i) The coefficients $b, \sigma, f : [0, T] \times \Omega \times \mathbb{R}^3 \mapsto \mathbb{R}$ are $\mathbb{F}$-progressively measurable, for fixed $(x, y, z) \in \mathbb{R}^3$; and the function $g : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is $\mathcal{F}_T$-measurable, for fixed $x \in \mathbb{R}$. Moreover, the following integrability condition holds:

$$I_0^2 \stackrel{\triangle}{=} \mathbb{E}\left\{\left(\int_0^T [|b| + |f|](t, 0, 0, 0)\, dt\right)^2 + \int_0^T |\sigma|^2(t, 0, 0, 0)\, dt + |g(0)|^2\right\} < \infty. \tag{2.1}$$

(ii) The coefficients b, σ, f, g are uniformly Lipschitz continuous in the spatial variable $(x, y, z) \in \mathbb{R}^3$, uniformly in $\omega \in \Omega$, and with a common Lipschitz constant $K_0 > 0$.

To simplify notation, throughout the paper we denote $\Theta \stackrel{\triangle}{=} (X, Y, Z)$. Our purpose is to find $\mathbb{F}$-progressively measurable, square-integrable processes Θ, such that (1.1) holds for all $t \in [0, T]$, $\mathbb{P}$-a.s. However, to facilitate the discussion, in what follows we often consider the FBSDE on a subinterval $[t_1, t_2]$:

$$\begin{cases} X_t = \eta + \displaystyle\int_{t_1}^t b(s, \Theta_s)\, ds + \int_{t_1}^t \sigma(s, \Theta_s)\, dB_s; \\ Y_t = \varphi(X_{t_2}) + \displaystyle\int_t^{t_2} f(s, \Theta_s)\, ds - \int_t^{t_2} Z_s\, dB_s, \end{cases} \qquad t \in [t_1, t_2], \tag{2.2}$$

where $\eta \in L^2(\mathcal{F}_{t_1})$ and $\varphi(x,\cdot) \in L^2(\mathcal{F}_{t_2})$, for each fixed x. We denote the solution to FBSDE (2.2), if exists, by $\Theta^{t_1,t_2,\eta,\varphi}$. In particular, we denote $\Theta^{t,x} := \Theta^{t,T,x,g}$.

A well understood technique for solving an FBSDE, initiated in [12], is to find a "decoupling function" u so that the solution Θ to the FBSDE satisfies the relation (1.2). In Markovian cases, especially when $\sigma = \sigma(t,x,y)$, it was shown that u is related to the solution to a quasilinear PDE, either in classical sense or in viscosity sense (cf., e.g., [5, 12] or [17]). When the coefficients are allowed to be random, special cases were also studied and the function u was found either as the solution to certain backward stochastic PDEs (see, [14, 15]), or as a random field constructed by extending the localization technique of [5] under certain compatibility conditions of the coefficients (see, [28]). In the sequel, we call such random function u the *decoupling random field* or simply *decoupling field* of the FBSDE (1.1). More precisely, we have the following definition.

DEFINITION 2.2. An $\mathbb{F}$-progressively measurable random field $u:[0,T] \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ with $u(T,x) = g(x)$ is said to be a "decoupling field" of FBSDE (1.1) if there exists a constant $\delta > 0$ such that, for any $0 = t_1 < t_2 \le T$ with $t_2 - t_1 \le \delta$ and any $\eta \in L^2(\mathcal{F}_{t_1})$, the FBSDE (2.2) with initial value η and terminal condition $u(t_2,\cdot)$ has a unique solution that satisfies (1.4) for $t \in [t_1,t_2]$, $\mathbb{P}$-a.s.

A decoupling field u is called *regular* if it is uniformly Lipschitz continuous in x.

By a slight abuse of notation, we shall denote the solution in Definition 2.2 by $\Theta^{t_1,t_2,\eta,u}$. One should note that the existence of the (regular) decoupling field implies the well-posedness of the FBSDE over a small interval, which is usually guaranteed by the Method of Contraction Mapping given the Assumption 2.1. The following result shows the significance of the existence of the decoupling field for the well-posedness for FBSDEs over an arbitrary duration.

THEOREM 2.3. *Assume that Assumption* 2.1 *holds, and that there exists a decoupling field u for FBSDE* (1.1). *Then FBSDE* (1.1) *has a unique solution* Θ *and* (1.4) *holds over an arbitrary duration* $[0,T]$.

PROOF. Let $T > 0$ be given. Consider a partition: $0 = t_0 < \cdots < t_n = T$ of $[0,T]$ such that $t_{i+1} - t_i \le \delta$, $i = 0,\dots,n-1$, where δ is the constant in Definition 2.2.

Define $X_{t_0} \stackrel{\triangle}{=} x$, $Y_{t_0} \stackrel{\triangle}{=} u(0,x)$, and for $i = 0,\dots,n-1$, define recursively

$$\Theta_t \stackrel{\triangle}{=} \Theta_t^{t_i,t_{i+1},X_{t_i},u}, \qquad t \in (t_i,t_{i+1}].$$

Then Θ would solve FBSDE (1.1) if they could be "patched" together. But note that

$$X_{t_i+} = X_{t_i+}^{t_i,t_{i+1},X_{t_i},u} = X_{t_i}^{t_i,t_{i+1},X_{t_i},u} = X_{t_i};$$

$$Y_{t_i+} = Y_{t_i+}^{t_i,t_{i+1},X_{t_i},u} = Y_{t_i}^{t_i,t_{i+1},X_{t_i},u} = u(t_i,X_{t_i}) = Y_{t_i}^{t_{i-1},t_i,X_{t_{i-1}},u} = Y_{t_i}.$$

That is, (X, Y) is continuous on $[0, T]$. Moreover, $u(T, x) = g(x)$, then by (2.2) one can check straightforwardly that Θ satisfies FBSDE (1.1) on $[0, T]$, proving the existence. Furthermore, from our construction it is clear that (1.4) holds.

We now prove the uniqueness. Let $\tilde{\Theta}$ be an arbitrary solution to FBSDE (1.1). Note that $\tilde{\Theta}$ satisfies FBSDE (2.2) on $[t_{n-1}, t_n]$ with initial condition $\tilde{X}_{t_{n-1}}$. Then by the definition of the decoupling field, we have $\tilde{Y}_{t_{n-1}} = u(t_{n-1}, \tilde{X}_{t_{n-1}})$. This implies that $\tilde{\Theta}$ satisfies FBSDE (2.2) on $[t_{n-2}, t_{n-1}]$ with initial condition $\tilde{X}_{t_{n-2}}$. Then we have $\tilde{Y}_{t_{n-2}} = u(t_{n-2}, \tilde{X}_{t_{n-2}})$. Repeating the arguments backwardly in time, we obtain that $\tilde{Y}_{t_i} = u(t_i, \tilde{X}_{t_i})$, $i = n, \dots, 0$. Now consider FBSDE (2.2) on $[t_0, t_1]$. Since $\tilde{X}_{t_0} = x = X_{t_0}$, by the uniqueness of solutions we know that $\tilde{\Theta} = \Theta$ on $[t_0, t_1]$. In particular, $\tilde{X}_{t_1} = X_{t_1}$, and thus the corresponding FBSDEs (2.2) on $[t_1, t_2]$ have the same initial condition. Repeating the arguments, this time forwardly for $i = 1, \dots, n$, we see that $\tilde{\Theta} = \Theta$ on $[0, T]$, and thus the solution is unique. $\square$

We conclude this section by making the following observations.

REMARK 2.4. (i) Definition 2.2 and Theorem 2.3 can be extended to higher-dimensional cases (with the constant δ possibly depending on the dimensions as well), and the proof stays exactly the same.

(ii) By the uniqueness in Theorem 2.3, it is obvious that the decoupling field, if exists, is also unique. In fact, it is clear that $u(t, x) = Y_t^{t,x}$.

REMARK 2.5. A typical condition for well-posedness of FBSDEs over small time interval is the uniform Lipschitz continuity of the terminal condition. Therefore, the main goal of this paper is to provide sufficient conditions which guarantee the existence of the regular decoupling field u. Such a feature was also observed from a different angle in [13], in which we characterize the regular decoupling field u as a Sobolev type weak solution to certain *backward stochastic PDE* that is Lipschitz in x. We note that the idea of "decoupling device" was also used for linear FBSDEs in [25]. But in that work the uniform Lipschitz continuity was not studied.

3. Some Heuristic analysis. From Theorem 2.3 and Remark 2.5, it is easy to see that the issue of the well-posedness of FBSDE (1.1) can be decomposed into two parts. First, the well-posedness on small time interval, and second, finding a decoupling field u that is uniformly Lipschitz continuous in its spatial variable. The first issue was more or less "classical" (see, e.g., [1]), but we will fine-tune it in Section 6 to suit our purpose. The second issue, however, is much more subtle, and is the main focus of this paper. In this section, we first give a heuristic analysis, from which several fundamental problems will be formulated, and their proofs will be carried out in Sections 4 and 5 below. A synthetic analysis will then be given in Section 7.

We first introduce some notation: for $\theta_j := (x_j, y_j, z_j)$, $j = 1, 2$, and for $\varphi = b, \sigma, f$, denote

$$
\begin{aligned}
&\tilde{h}(x_1, x_2) \stackrel{\triangle}{=} [g(x_1) - g(x_2)]/[x_1 - x_2];\\
&\tilde{\varphi}_1(t, \theta_1, \theta_2) \stackrel{\triangle}{=} [\varphi(t, x_1, y_1, z_1) - \varphi(t, x_2, y_1, z_1)]/[x_1 - x_2];\\
&\tilde{\varphi}_2(t, \theta_1, \theta_2) \stackrel{\triangle}{=} [\varphi(t, x_2, y_1, z_1) - \varphi(t, x_2, y_2, z_1)]/[y_1 - y_2];\\
&\tilde{\varphi}_3(t, \theta_1, \theta_2) \stackrel{\triangle}{=} [\varphi(t, x_2, y_2, z_1) - \varphi(t, x_2, y_2, z_2)]/[z_1 - z_2].
\end{aligned}
\tag{3.1}
$$

Here and in the sequel, for any Lipschitz continuous function $\varphi(x)$, when $x_1 = x_2$ we will always take the convention that

$$
\frac{\varphi(x) - \varphi(x)}{x - x} := \lim_{\tilde{x} \to x} \frac{\varphi(\tilde{x}) - \varphi(x)}{\tilde{x} - x}.
\tag{3.2}
$$

Our main idea to decouple the FBSDE (1.1) is as follows. Assume that there exists a decoupling field $u = u(t, x)$ that is uniformly Lipschitz continuous in x and (1.4) holds. Assume also that (1.1) is well-posed on $[0, T]$, with $X_0 = x$ for any x. Given x_i, $i = 1, 2$, let Θ^i denote the unique solution to (1.1) with initial condition x_i, and. By slightly

$$
\nabla \Theta \stackrel{\triangle}{=} \frac{\Theta^1 - \Theta^2}{x_1 - x_2}, \qquad \nabla u(t) \stackrel{\triangle}{=} \frac{u(t, X_t^1) - u(t, X_t^2)}{X_t^1 - X_t^2}.
\tag{3.3}
$$

Since $Y_t^i = u(t, X_t^i)$, $i = 1, 2$, one must have

$$
\nabla Y_t = \nabla u(t) \nabla X_t,
\tag{3.4}
$$

and one can check immediately that $\nabla\Theta$ satisfies the following "variational FBSDE:"

$$
\begin{cases}
\nabla X_t = 1 + \displaystyle\int_0^t (b_1 \nabla X_s + b_2 \nabla Y_s + b_3 \nabla Z_s)\, ds \\
\qquad\qquad + \displaystyle\int_0^t (\sigma_1 \nabla X_s + \sigma_2 \nabla Y_s + \sigma_3 \nabla Z_s)\, dB_s; \\
\nabla Y_t = h \nabla X_T + \displaystyle\int_t^T (f_1 \nabla X_s + f_2 \nabla Y_s + f_3 \nabla Z_s)\, ds \\
\qquad\qquad - \displaystyle\int_t^T \nabla Z_s\, dB_s,
\end{cases}
\qquad t \in [0, T],
\tag{3.5}
$$

where $h \stackrel{\triangle}{=} \tilde{h}(X_T^1, X_T^2)$ and $\varphi_i(t) \stackrel{\triangle}{=} \tilde{\varphi}_i(t, \Theta_t^1, \Theta_t^2)$, $i = 1, 2, 3$, $\varphi = b, \sigma, f$, respectively. We note here that b_i, σ_i, f_i, $i = 1, 2, 3$, are $\mathbb{F}$-adapted processes and h is a $\mathcal{F}_T$-measurable random variable, and they are all bounded, thanks to Assumption 2.1.

Furthermore, in light of (3.4) we see that a decoupling field u being regular (i.e., uniformly Lipschitz continuous in x) is essentially equivalent to $\hat{Y}_t \stackrel{\triangle}{=} \nabla Y(\nabla X)^{-1}$ being uniformly bounded. Thus, let us assume $\nabla X \neq 0$ and denote

$$\begin{aligned} &\hat{Y}_t \stackrel{\triangle}{=} \nabla Y_t / \nabla X_t \quad \text{and} \\ &\hat{Z}_t \stackrel{\triangle}{=} [\nabla Z_t - \hat{Y}_t(\sigma_1 \nabla X_t + \sigma_2 \nabla Y_t + \sigma_3 \nabla Z_t)]/\nabla X_t, \end{aligned} \tag{3.6}$$

or equivalently,

$$\nabla Y_t = \hat{Y}_t \nabla X_t, \qquad \nabla Z_t = \frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2 \hat{Y}_t)}{1 - \sigma_3 \hat{Y}_t} \nabla X_t. \tag{3.7}$$

A simple application of Itô's formula to $\hat{Y}_t$, assuming $\sigma_3 \hat{Y} \neq 1$, yields that

$$\hat{Y}_t = h + \int_t^T [F_s(\hat{Y}_s) + G_s(\hat{Y}_s)\hat{Z}_s + \Lambda_s(\hat{Y}_s)|\hat{Z}_s|^2]\, ds - \int_t^T \hat{Z}_s\, dB_s, \tag{3.8}$$

where

$$\begin{aligned} F_s(y) &\stackrel{\triangle}{=} f_1 + f_2 y + y(b_1 + b_2 y) + \frac{(f_3 + b_3 y) y(\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y}; \\ G_s(y) &\stackrel{\triangle}{=} \sigma_1 + \sigma_2 y + \frac{f_3 + b_3 y + \sigma_3 y(\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y} \\ &= \frac{(\sigma_1 + f_3) + (\sigma_2 + b_3) y}{1 - \sigma_3 y}; \\ \Lambda_s(y) &\stackrel{\triangle}{=} \frac{\sigma_3}{1 - \sigma_3 y}. \end{aligned} \tag{3.9}$$

Equation (3.8) is clearly a legitimate BSDE, even without assuming $\nabla X \neq 0$. We shall call this BSDE the "*Characteristic BSDE*" of the linear variational FBSDE (3.5) [or of the original FBSDE (1.1)], and their connection will be studied rigorously in the next section. We note that the identities in (3.7) and the desired Lipschitz property of the decoupling field u tell us that we should look for conditions under which the BSDE (3.8) has a solution $(\hat{Y}, \hat{Z})$ such that

$$\text{both } \hat{Y} \text{ and } (1 - \sigma_3 \hat{Y})^{-1} \text{ are bounded.} \tag{3.10}$$

REMARK 3.1. It is worth noting that the BSDE (3.8) is nonstandard in several aspects. Most notable is that its generator has at least quadratic growth in both Y and Z, thus it can be thought of as a Backward Stochastic Riccati Equations (BSRE) with quadratic growth in Z, which to our best knowledge, has not been studied in literature.

Besides the commonly cited reference of BSDEs with quadratic growth in Z (e.g., [2, 10]), the following special cases of (3.8) are worth mentioning. In [19], the BSRE with linear growth in Z was studied in the context of stochastic LQ (linear-quadratic) problem, in which the FBSDE is a natural consequence of the stochastic maximum principle. The characteristic BSDE (3.8) was also observed in [25], where the linear FBSDEs were considered. But some special assumptions were made so that the BSDE has linear growth in Y. Finally, in [28] certain compatibility conditions were also added so that (3.8) becomes a standard BSDE, and thus its well-posedness was not an issue. Our results will contain those of [19, 25] and [28] as special cases.

We conclude this section by outlining the strategy for obtaining the a priori uniform estimate of $\hat{Y}$, which is crucial for finding the solution of (3.8) satisfying (3.10). To begin with, for any bounded random variable ξ, define its deterministic upper and lower bounds by

$$\begin{aligned} \overline{\xi} &\stackrel{\triangle}{=} \operatorname{esssup} \xi \stackrel{\triangle}{=} \inf\{a \in \mathbb{R} : \xi \le a, \text{ a.s.}\}, \\ \underline{\xi} &\stackrel{\triangle}{=} \operatorname{essinf} \xi \stackrel{\triangle}{=} \sup\{a \in \mathbb{R} : \xi \ge a, \text{ a.s.}\}. \end{aligned} \tag{3.11}$$

For any $\theta_j \stackrel{\triangle}{=} (x_j, y_j, z_j)$, $j = 1, 2$, we define $F(\theta_1, \theta_2; t, y)$ by replacing the coefficients φ_i in (3.9) with $\tilde{\varphi}_i(t, \theta_1, \theta_2)$ defined in (3.1), $i = 1, 2, 3$, $\varphi = b, \sigma, f$. We then define

$$\begin{aligned} \overline{h} &\stackrel{\triangle}{=} \operatorname{esssup}\Big(\sup_{x_1 \ne x_2} \tilde{h}(x_1, x_2)\Big), \\ \underline{h} &\stackrel{\triangle}{=} \operatorname{essinf}\Big(\inf_{x_1 \ne x_2} \tilde{h}(x_1, x_2)\Big), \\ \overline{F}(t, y) &\stackrel{\triangle}{=} \operatorname{esssup}\Big(\sup_{x_1 \ne x_2, y_1 \ne y_2, z_1 \ne z_2} F(\theta_1, \theta_2; t, y)\Big), \\ \underline{F}(t, y) &\stackrel{\triangle}{=} \operatorname{essinf}\Big(\inf_{x_1 \ne x_2, y_1 \ne y_2, z_1 \ne z_2} F(\theta_1, \theta_2; t, y)\Big). \end{aligned} \tag{3.12}$$

Here, we should remark that $\overline{F}(t, y)$ is a *deterministic* function, and we should note its notational difference from the possibly random processes, for example, $F_t(y)$, $G_t(y)$, etc., appeared previously. We have the following a priori estimate of $\hat{Y}$.

LEMMA 3.2. *Let Assumption 2.1 hold. Assume that the BSDE (3.8) has a solution $(\hat{Y}, \hat{Z})$, and the following ordinary differential equations (ODEs) admit solutions $\overline{\mathbf{y}}$, $\underline{\mathbf{y}}$:*

$$\overline{\mathbf{y}}_t = \overline{h} + \int_t^T \overline{F}(s, \overline{\mathbf{y}}_s)\, ds, \qquad \underline{\mathbf{y}}_t = \underline{h} + \int_t^T \underline{F}(s, \underline{\mathbf{y}}_s)\, ds. \tag{3.13}$$

Assume further that $\hat{Y}$, $\overline{\mathbf{y}}$ *and* $\underline{\mathbf{y}}$ *all satisfy* (3.10). *Then* $\underline{\mathbf{y}}_t \le \hat{Y}_t \le \overline{\mathbf{y}}_t$, *for all* $t \in [0, T]$, $\mathbb{P}$-*a.s.*

PROOF. Denote $\tilde{G}_t(z) \triangleq G_t(\hat{Y}_t)z + \Lambda_t(\hat{Y}_t)z^2$. Note that $(\hat{Y}, \hat{Z})$ satisfies the following BSDE:

$$Y_t = h + \int_t^T [F_s(Y_s) + \tilde{G}_s(Z_s)]\,ds - \int_t^T Z_s\,dB_s$$

and $(\overline{\mathbf{y}}, 0)$ satisfy the following BSDE:

$$Y_t = \overline{h} + \int_t^T [\overline{F}(s, Y_s) + \tilde{G}_s(Z_s)]\,ds - \int_t^T Z_s\,dB_s.$$

Let $C > 0$ be the common upbound of $|\hat{Y}|$, $|1 - \sigma_3\hat{Y}|^{-1}$, $|\overline{\mathbf{y}}|$, $|1 - \sigma_3\overline{\mathbf{y}}|^{-1}$, $|\underline{\mathbf{y}}|$, and $|1 - \sigma_3\underline{\mathbf{y}}|^{-1}$. Note that F is uniformly Lipschitz continuous in y in the set $\{y : |y| \le C, |1 - \sigma_3 y|^{-1} \le C\}$. It then follows from the comparison theorem for quadratic BSDEs (see, e.g., [10]) that $\hat{Y} \le \overline{\mathbf{y}}$. Similarly, we have $\hat{Y} \ge \underline{\mathbf{y}}$. $\square$

Combining the discussions in Sections 2 and 3, especially Lemma 3.2, it is now clear that finding the uniform Lipschitz decoupling random field u will eventually come down to finding conditions so that the ODEs in (3.13) admit nonexplosive solutions over the arbitrarily prescribed duration $[0, T]$. In the rest of the paper, we shall call the ODEs in (3.13) the "*dominating ODEs*" of BSDE (3.8), whose well-posedness will be the main subject of Section 5.

4. The characteristic BSDE. In this section, we study the connection between well-posedness of the linear variational FBSDE (3.5) and the corresponding characteristic BSDE (3.8). Such a relation is not only interesting in its own right, but also important for us to construct the desired regular decoupling field in Sections 6 and 7 below. We should note that the variational FBSDE (3.5) coincides with the original FBSDE if (1.1) is actually linear.

For notational simplicity, we denote $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) := (\nabla X, \nabla Y, \nabla Z)$ and then the variational FBSDE (3.5) becomes the following linear FBSDE with random coefficients:

$$\begin{cases} \mathcal{X}_t = 1 + \displaystyle\int_0^t (b_1\mathcal{X}_s + b_2\mathcal{Y}_s + b_3\mathcal{Z}_s)\,ds \\ \qquad\quad + \displaystyle\int_0^t (\sigma_1\mathcal{X}_s + \sigma_2\mathcal{Y}_s + \sigma_3\mathcal{Z}_s)\,dB_s; \\ \mathcal{Y}_t = h\mathcal{X}_T + \displaystyle\int_t^T (f_1\mathcal{X}_s + f_2\mathcal{Y}_s + f_3\mathcal{Z}_s)\,ds - \int_t^T \mathcal{Z}_s\,dB_s. \end{cases} \tag{4.1}$$

In this case, (3.6) and (3.7) become

$$\hat{Y} \triangleq \mathcal{Y}/\mathcal{X}, \qquad \hat{Z} \triangleq [\mathcal{Z} - \hat{Y}(\sigma_1\mathcal{X} + \sigma_2\mathcal{Y} + \sigma_3\mathcal{Z})]/\mathcal{X}, \tag{4.2}$$

$$\mathcal{Y} = \hat{Y}\mathcal{X}, \qquad \mathcal{Z} = [\hat{Z} + \hat{Y}(\sigma_1 + \sigma_2\hat{Y})]\mathcal{X}/[1 - \sigma_3\hat{Y}]. \tag{4.3}$$

The original Assumption 2.1 can be translated into the following assumption.

ASSUMPTION 4.1. Assume b_i, σ_i, f_i, $i = 1, 2, 3$, are $\mathbb{F}$-adapted processes, h is a $\mathcal{F}_T$-measurable random variable, and they are all bounded.

The following spaces are important in our discussion. For $p \geq 1$, denote

$$\begin{aligned} &\mathbb{L}^p \triangleq \left\{\Theta : \|\Theta\|_{\mathbb{L}^p}^p \triangleq \mathbb{E}\left\{\sup_{0\leq t\leq T}[|X_t|^p + |Y_t|^p] + \left(\int_0^T |Z_t|^2\,dt\right)^{p/2}\right\} < \infty\right\}; \\ &\widehat{\mathbb{L}}_p \triangleq \bigcup_{q>p} \mathbb{L}^q. \end{aligned} \tag{4.4}$$

We begin our discussion with the following observation. For any $\mathbb{F}$-adapted process u such that $\int_0^T |u_t|^2\,dt < \infty$, $\mathbb{P}$-a.s., we define

$$M_t^u \triangleq \exp\left\{\int_0^t u_s\,dB_s - \frac{1}{2}\int_0^t |u_s|^2\,ds\right\}. \tag{4.5}$$

Consider the following simplified form of (3.8):

$$\hat{Y}_t = h + \int_t^T [\alpha_s + \beta_s \hat{Y}_s + \gamma_s \hat{Z}_s + \lambda_s |\hat{Z}_s|^2]\,ds - \int_t^T \hat{Z}_s\,dB_s, \quad t \in [0, T], \tag{4.6}$$

where $\alpha, \beta, \gamma, \lambda$ are $\mathbb{F}$-adapted processes and h is an $\mathcal{F}_T$-measurable random variable, all bounded. Then it is well known (see, e.g., [2]) that the BSDE (4.6) admits a unique solution $(\hat{Y}, \hat{Z})$ such that, for some constant C depending on the bounds of $\alpha, \beta, \gamma, \lambda, h$ and T,

$$|\hat{Y}_t| \leq C \quad \text{and} \quad \mathbb{E}_t\left\{\int_t^T |\hat{Z}_s|^2\,ds\right\} \leq C.$$

Furthermore, applying some BMO analysis (cf. [9], Lemma 4 and Theorem 1), one shows that there exists a constant $\varepsilon > 0$, depending also on the bounds of the coefficients and T, such that

$$\mathbb{E}\left\{\exp\left(\varepsilon \int_0^T |\hat{Z}_t|^2\,dt\right) + |M_T^{\lambda\hat{Z}}|^{1+\varepsilon}\right\} < \infty. \tag{4.7}$$

Consequently, $M^{\lambda\hat{Z}}$ is a true martingale.

Bearing this observation in mind, we now give the main result of this section.

THEOREM 4.2. *Assume Assumption 4.1 holds.*

(i) *If the BSDE (3.8) has a solution $(\hat{Y}, \hat{Z})$ such that (3.10) holds, then the FBSDE (4.1) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \widehat{\mathbb{L}}_1$ such that $\mathcal{X} \neq 0$ and (4.3) holds.*

(ii) *Conversely, if the FBSDE (4.1) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \widehat{\mathbb{L}}_1$ such that*

$$|\mathcal{Y}_t| \leq C|\mathcal{X}_t|, \qquad |\mathcal{X}_t| \leq C|\mathcal{X}_t - \sigma_3 \mathcal{Y}_t|, \tag{4.8}$$

then $\mathcal{X} \neq 0$, *and the processes* $(\hat{Y}, \hat{Z})$ *defined by* (4.2) *satisfies BSDE* (3.8) *and* (3.10).

PROOF. (i) In light of (4.3), we consider the following SDE:

$$
\begin{aligned}
d\mathcal{X}_t &= \mathcal{X}_t\left[b_1 + b_2\hat{Y}_t + b_3\frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2\hat{Y}_t)}{1 - \sigma_3\hat{Y}_t}\right]dt \\
&\quad + \mathcal{X}_t\left[\sigma_1 + \sigma_2\hat{Y}_t + \sigma_3\frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2\hat{Y}_t)}{1 - \sigma_3\hat{Y}_t}\right]dB_t \qquad (4.9) \\
&= \mathcal{X}_t\{H_t(\hat{Y}_t, \hat{Z}_t)\,dt + [I_t(\hat{Y}_t) + \Lambda_t(\hat{Y}_t)\hat{Z}_t]\,dB_t\},
\end{aligned}
$$

where

$$
H_t(y, z) \triangleq \left[b_1 + b_2 y + b_3\frac{z + y(\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y}\right]; \qquad I_t(y) \triangleq \frac{\sigma_1 + \sigma_2 y}{1 - \sigma_3 y}.
$$

It is then easy to check that

$$
\begin{aligned}
\mathcal{X}_t &= \exp\left\{\int_0^t [I_s(\hat{Y}_s) + \Lambda_s(\hat{Y}_s)\hat{Z}_s]\,dB_s \right. \\
&\quad \left. + \int_0^t \left[H_s(\hat{Y}_s, \hat{Z}_s) - \frac{1}{2}[I_s(\hat{Y}_s) + \Lambda_s(\hat{Y}_s)\hat{Z}_s]^2\right]ds\right\} \qquad (4.10) \\
&= M_t^{\Lambda(\hat{Y})\hat{Z}} M_t^{I(\hat{Y})} \exp\left\{\int_0^t [H_s(\hat{Y}_s, \hat{Z}_s) - I_s(\hat{Y}_s)\Lambda_s(\hat{Y}_s)\hat{Z}_s]\,ds\right\}.
\end{aligned}
$$

Clearly, $\mathcal{X} > 0$. Furthermore, since (3.10) implies that in (4.9) $\Lambda(\hat{Y})$, $I(\hat{Y})$ are bounded and $H(\hat{Y}, \hat{Z})$ has a linear growth in $\hat{Z}$, and (4.7) implies

$$
\mathbb{E}\left\{\sup_{0\le t\le T}|M_t^{I(\hat{Y})}|^p + \exp\left(p\int_0^T [1 + |\hat{Z}_t|]\,dt\right)\right\} < \infty \qquad (4.11)
$$
$$
\text{for any } p > 1,
$$

we deduce from (4.10) that, for ε in (4.7) [noting that $(\frac{2(1+\varepsilon)}{2+\varepsilon}, \frac{2(1+\varepsilon)}{\varepsilon})$ are conjugates],

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{0\le t\le T}|\mathcal{X}_t|^{1+\varepsilon/2}\right\} \\
&\quad \le \left(E\left\{\sup_{0\le t\le T}|M_t^{\Lambda(\hat{Y})\hat{Z}}|^{1+\varepsilon}\right\}\right)^{(2+\varepsilon)/(2(1+\varepsilon))} \qquad (4.12) \\
&\qquad \times \left(E\left\{\sup_{0\le t\le T}|M_t^{I(\hat{Y})}|^{(2+\varepsilon)(1+\varepsilon)/\varepsilon} e^{C(2+\varepsilon)(1+\varepsilon)/\varepsilon\int_0^T [1+|\hat{Z}_t|]\,dt}\right\}\right)^{\varepsilon/(2(1+\varepsilon))} \\
&\quad < \infty.
\end{aligned}
$$

Now if we define $(\mathcal{Y}, \mathcal{Z})$ by (4.3), then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ satisfy (4.1) and, by (4.7) again,

$$\mathbb{E}\left\{\sup_{0\le t\le T} |\mathcal{Y}_t|^{1+\varepsilon/2} + \left(\int_0^T |\mathcal{Z}_t|^2\,dt\right)^{1+\varepsilon/4}\right\} < \infty. \tag{4.13}$$

That is, $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbb{L}^{1+\varepsilon/4} \subset \widehat{\mathbb{L}}_1$, proving (i).

(ii) We now assume that FBSDE (4.1) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \widehat{\mathbb{L}}_1$ such that (4.8) holds. Denote $\tau_n \stackrel{\triangle}{=} \inf\{t : \mathcal{X}_t = \frac{1}{n}\} \wedge T$, $\tau \stackrel{\triangle}{=} \inf\{t : \mathcal{X}_t = 0\} \wedge T$, and define $\hat{Y}$, $\hat{Z}$ by (4.2). Clearly, the assumption (4.8) implies that $\hat{Y}$ satisfies (3.10) in $[0, \tau)$, and applying Itô's formula we see that $(\hat{Y}, \hat{Z})$ satisfies

$$d\hat{Y}_t = -\left[F_t(\hat{Y}_t) + G_t(\hat{Y}_t)\hat{Z}_t + \Lambda_t(\hat{Y}_t)|\hat{Z}_t|^2\right]dt + \hat{Z}_t\,dB_t, \qquad t \in [0, \tau).$$

Note that the boundedness of $\hat{Y}$ implies that the above SDE is actually of the form of (4.6), and at least on $[0, \tau_n)$ the stochastic integral $\int_0^\cdot \hat{Z}_s\,dB_s$ is a true martingale. Thus, we can apply the same argument there to obtain the bound (4.7) on $[0, \tau_n)$:

$$\mathbb{E}\left\{\exp\left(\varepsilon \int_0^{\tau_n} |\hat{Z}_t|^2\,dt\right) + \left|M_{\tau_n}^{\Lambda(\hat{Y})\hat{Z}}\right|^{1+\varepsilon}\right\} \le C < \infty.$$

Note that the constants ε and C above depend on the coefficients, which depend only on the bound of $\hat{Y}$ and is independent of n, thanks to (4.8). Thus, letting $n \to \infty$ we have

$$\mathbb{E}\left\{\exp\left(\varepsilon \int_0^{\tau} |\hat{Z}_t|^2\,dt\right) + \left|M_{\tau}^{\Lambda(\hat{Y})\hat{Z}}\right|^{1+\varepsilon}\right\} \le C < \infty.$$

On the other hand, since $\mathcal{X}$ satisfies (4.10) on $[0, \tau)$, we see that the estimate above implies that $\mathcal{X}_\tau > 0$, a.s. Thus, $\tau = T$ a.s. In other words, $(\hat{Y}, \hat{Z})$ satisfies (3.8) over $[0, T]$, and (3.10) holds. The proof is now complete. $\square$

REMARK 4.3. We should point out that Theorems 4.2 only indicates an a priori relationship between the characteristic BSDE and the "derivative" of the decoupling field, whenever exists, via the variational FBSDE (4.1). The boundedness requirement (3.10), or equivalently, the "regularity" of the decoupling field, is crucial for the solution scheme to be effective (recall the inductive procedure in Theorem 2.3). The actual construction of the decoupling field, however, depends on the well-posedness of the dominating ODEs to be analyzed in details in next section, which is motivated by but independent of the results in this section. In fact, only a localized version (in small time duration) of Theorem 4.2(ii) will be used in the proof of Theorem 6.1(iii) below.

We conclude this section by presenting a result regarding the uniqueness of the solutions to FBSDE (4.1) and its characteristic BSDE (3.8), which might be of independent interest. We should note that this result will not be used in our future discussion, but its arguments will be useful whenever a linearized FBSDE is encountered (e.g., the proof of Theorem 8.6 below). To this end, we make use of an additional condition on $(\hat{Y}, \hat{Z})$ that strengthen the estimate (4.7):

$$\mathbb{E}\Big\{\sup_{0\le t\le T}\big|M_t^{\Lambda(\hat{Y})\hat{Z}}\big|^{2+\varepsilon}\Big\}<\infty \qquad \text{for some } \varepsilon>0. \tag{4.14}$$

THEOREM 4.4. *Let Assumption 4.1 hold. Then the BSDE (3.8) has a solution $(\hat{Y}, \hat{Z})$ satisfying (3.10) and (4.14) if and only if the FBSDE (4.1) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \widehat{\mathbb{L}}_2$ satisfying (4.8).*

Moreover, in such a case the uniqueness holds for solutions to BSDE (3.8) satisfying (3.10) and (4.14) with $\varepsilon = 0$ and for solutions to FBSDE (4.1) in $\mathbb{L}^2$ satisfying (4.8).

PROOF. We proceed in three steps. To make the presentation more precise, we denote:

- $\mathcal{A}_0 \stackrel{\triangle}{=}$ {all solutions $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbb{L}^2$ to FBSDE (4.1) satisfying (4.8)};
- $\mathcal{A} \stackrel{\triangle}{=} \mathcal{A}_0 \cap \widehat{\mathbb{L}}^2$;
- $\mathcal{B}_0 \stackrel{\triangle}{=}$ {all solutions $(\hat{Y}, \hat{Z})$ to BSDE (3.8) satisfying (3.10) and (4.14) with $\varepsilon = 0$}; and
- $\mathcal{B} \stackrel{\triangle}{=}$ {all solutions $(\hat{Y}, \hat{Z})$ in $\mathcal{B}_0$ satisfying (4.14)}.

Step 1. We first prove the equivalence of the existence of desired solutions in $\mathcal{A}$ and $\mathcal{B}$. First, assume there exists $(\hat{Y}, \hat{Z}) \in \mathcal{B}$. Then by Theorem 4.2, the FBSDE (4.1) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \widehat{\mathbb{L}}_1$. Furthermore, using condition (4.14) we can actually improve the estimates (4.12) and (4.13) to $\mathbb{L}^{2+\varepsilon/2}$, and thus $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}$.

Conversely, there exists $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A} \subseteq \widehat{\mathbb{L}}_1$, then by Theorem 4.2(ii), the $(\hat{Y}, \hat{Z})$ defined by (4.2) satisfy (3.8) and (3.10), and $\mathcal{X}$ satisfy (4.10). Thus,

$$M_t^{\Lambda(\hat{Y})\hat{Z}} = \mathcal{X}_t\big[M_t^{I(\hat{Y})}\big]^{-1}\exp\Big\{-\int_0^t \big[H_s(\hat{Y}_s, \hat{Z}_s) - I_s(\hat{Y}_s)\Lambda_s(\hat{Y}_s)\hat{Z}_s\big]\,ds\Big\}.$$

If $\mathbb{E}\{\sup_{0\le t\le T}|\mathcal{X}_t|^p\} < \infty$ for some $p > 2$, by estimates similar to (4.12) we obtain (4.14).

Step 2. We next turn to the uniqueness. We claim that:

(4.15) For any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}_0$ and $(\hat{Y}, \hat{Z}) \in \mathcal{B}_0$, if either $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}$ or $(\hat{Y}, \hat{Z}) \in \mathcal{B}$, then relation (4.2) and equivalently (4.3) must hold.

Now fix an $(\mathcal{X}^0, \mathcal{Y}^0, \mathcal{Z}^0) \in \mathcal{A}$ and $(\hat{Y}^0, \hat{Z}^0) \in \mathcal{B}$ which satisfy (4.2) and (4.3). For any $(\hat{Y}, \hat{Z}) \in \mathcal{B}_0$, apply (4.15) on $(\mathcal{X}^0, \mathcal{Y}^0, \mathcal{Z}^0)$ and $(\hat{Y}, \hat{Z})$, we see that they satisfy (4.2), and thus $(\hat{Y}, \hat{Z}) = (\hat{Y}^0, \hat{Z}^0)$. On the other hand, for any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}_0$, apply (4.15) on $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and $(\hat{Y}^0, \hat{Z}^0)$. By (4.3), we see that $\mathcal{X}$ must satisfy (4.10) with $(\hat{Y}^0, \hat{Z}^0)$ in the right-hand side, and thus $\mathcal{X} = \mathcal{X}^0$. Moreover, it follows from (4.3) that $(\mathcal{Y}, \mathcal{Z}) = (\mathcal{Y}^0, \mathcal{Z}^0)$.

Step 3. We now prove claim (4.15). Given $(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathcal{A}_0$ and $(\hat Y,\hat Z)\in\mathcal{B}_0$, denote

$$
\begin{aligned}
\delta Y_t &\stackrel{\triangle}{=} \mathcal{Y}_t-\hat Y_t\mathcal{X}_t,\\
\delta Z_t &\stackrel{\triangle}{=} \mathcal{Z}_t-\big[\mathcal{X}_t\hat Z_t+\hat Y_t(\sigma_1\mathcal{X}_t+\sigma_2\mathcal{Y}_t+\sigma_3\mathcal{Z}_t)\big].
\end{aligned}
\tag{4.16}
$$

Applying Itô's formula to δY_t, we have

$$
\begin{aligned}
d(\delta Y_t) &= -\big[f_1\mathcal{X}_t+f_2\mathcal{Y}_t+f_3\mathcal{Z}_t-\mathcal{X}_t\big[F_t(\hat Y_t)+G_t(\hat Y_t)\hat Z_t+\Lambda_t(\hat Y_t)|\hat Z_t|^2\big]\\
&\qquad +\hat Y_t(b_1\mathcal{X}_t+b_2\mathcal{Y}_t+b_3\mathcal{Z}_t)+\hat Z_t(\sigma_1\mathcal{X}_t+\sigma_2\mathcal{Y}_t+\sigma_3\mathcal{Z}_t)\big]\,dt\\
&\quad +\delta Z_t\,dB_t\\
&= -\big[\mathcal{X}_t\big[f_1-F_t(\hat Y_t)-G_t(\hat Y_t)\hat Z_t-\Lambda_t(\hat Y_t)|\hat Z_t|^2+b_1\hat Y_t+\sigma_1\hat Z_t\big]\\
&\qquad +\mathcal{Y}_t[f_2+b_2\hat Y_t+\sigma_2\hat Z_t]+\mathcal{Z}_t[f_3+b_3\hat Y_t+\sigma_3\hat Z_t]\big]\,dt\\
&\quad +\delta Z_t\,dB_t.
\end{aligned}
\tag{4.17}
$$

By (4.16), one can easily check that

$$
\begin{aligned}
\mathcal{Y} &= \delta Y+\hat Y\mathcal{X},\\
\mathcal{Z} &= \frac{\delta Z+\mathcal{X}\hat Z+\hat Y(\sigma_1\mathcal{X}+\sigma_2\mathcal{Y})}{1-\sigma_3\hat Y}\\
&= \frac{\delta Z+\sigma_2\hat Y\delta Y+\mathcal{X}[\hat Z+(\sigma_1+\sigma_2\hat Y)\hat Y]}{1-\sigma_3\hat Y}.
\end{aligned}
$$

Plugging these into (4.17), we obtain

$$
d(\delta Y_t) = -[\alpha_t\mathcal{X}_t+\beta_t\delta Y_t+\gamma_t\delta Z_t]\,dt+\delta Z_t\,dB_t,
$$

where

$$
\begin{aligned}
\gamma_t &\stackrel{\triangle}{=} \frac{f_3+b_3\hat Y_t+\sigma_3\hat Z_t}{1-\sigma_3\hat Y_t} = \frac{f_3+b_3\hat Y_t}{1-\sigma_3\hat Y_t}+\Lambda_t(\hat Y_t)\hat Z_t;\\
\beta_t &\stackrel{\triangle}{=} f_2+b_2\hat Y_t+\sigma_2\hat Z_t+\frac{\sigma_2\hat Y_t[f_3+b_3\hat Y_t+\sigma_3\hat Z_t]}{1-\sigma_3\hat Y_t}
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
\alpha_t &\stackrel{\triangle}{=} f_1-F_t(\hat Y_t)-G_t(\hat Y_t)\hat Z_t-\Lambda_t(\hat Y_t)|\hat Z_t|^2+b_1\hat Y_t+\sigma_1\hat Z_t\\
&\quad +[f_2+b_2\hat Y_t+\sigma_2\hat Z_t]\hat Y_t+[f_3+b_3\hat Y_t+\sigma_3\hat Z_t]\frac{\hat Z_t+(\sigma_1+\sigma_2\hat Y_t)\hat Y_t}{1-\sigma_3\hat Y_t}\\
&= 0,
\end{aligned}
$$

thanks to (3.9). Denote

$$
\begin{aligned}
\Gamma_t &\triangleq M_t^{\gamma} \exp\left(\int_0^t \beta_s\, ds\right) \\
&= M_t^{\Lambda(\hat{Y})\hat{Z}} M_t^{(f_3+b_3\hat{Y})/(1-\sigma_3\hat{Y})} \\
&\qquad \times \exp\left(\int_0^t \left[\beta_s - \frac{f_3+b_3\hat{Y}_s}{1-\sigma_3\hat{Y}_s}\Lambda_s(\hat{Y}_s)\hat{Z}_s\right] ds\right).
\end{aligned}
\tag{4.19}
$$

Then by applying Itô's formula, one obtains immediately

$$
d(\Gamma_t \delta Y_t) = \Gamma_t[\gamma_t \delta Y_t + \delta Z_t]\, dB_t. \tag{4.20}
$$

We claim that

$$
\begin{aligned}
&\mathbb{E}\left\{\left(\int_0^T |\Gamma_t|^2[\gamma_t\delta Y_t + \delta Z_t]^2\, dt\right)^{1/2}\right\} \\
&\quad \le \mathbb{E}\left\{\sup_{0\le t\le T}|\Gamma_t|\left(\int_0^T [\gamma_t\delta Y_t + \delta Z_t]^2\, dt\right)^{1/2}\right\} < \infty,
\end{aligned}
\tag{4.21}
$$

so that $\int_0^\cdot \Gamma_s[\gamma_t\delta Y_t + \delta Z_s]\, dB_s$ is a true martingale. Since $\delta Y_T = 0$ and $\Gamma_0 = 1$, it follows from (4.20) that $\delta Y = 0$, and hence $\delta Z = 0$. Then (4.16) leads to (4.2) immediately.

It remains to prove (4.21). Note that

$$
\begin{aligned}
&|\gamma_t| \le C\big[1 + |\hat{Z}_t|\big], \qquad |\delta Y_t| \le C\big[|\mathcal{X}_t| + |\mathcal{Y}_t|\big], \\
&|\delta Z_t| \le C\big[|\mathcal{X}_t| + |\mathcal{Y}_t| + |\mathcal{Z}_t| + |\mathcal{X}_t||\hat{Z}_t|\big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_0^T [\gamma_t\delta Y_t + \delta Z_t]^2\, dt \\
&\quad \le C\Big[1 + \sup_{0\le t\le T}[|\mathcal{X}_t|^2 + |\mathcal{Y}_t|^2] \\
&\qquad + \int_0^T [|\mathcal{Z}_t|^2 + |\hat{Z}_t|^2]\, dt + \sup_{0\le t\le T}|\mathcal{X}_t|^2 \int_0^T |\hat{Z}_t|^2\, dt\Big].
\end{aligned}
\tag{4.22}
$$

Since $(\hat{Y}, \hat{Z})$ satisfies (3.10), by (4.7) we have

$$
\mathbb{E}\left\{\left(\int_0^T |\hat{Z}_t|^2\, dt\right)^p\right\} < \infty \qquad \text{for any } p \ge 1. \tag{4.23}
$$

We now verify (4.21) in the two cases:

Case 1. $(\hat{Y}, \hat{Z}) \in \mathcal{B}$, namely (4.14) holds with some $\varepsilon > 0$. Following the arguments for (4.12), we have

$$\mathbb{E}\Big\{\sup_{t\in[0,T]} |\Gamma_t|^{2+\varepsilon/2}\Big\} < \infty. \tag{4.24}$$

Then for any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}_0$, plugging (4.23) and (4.24) into (4.22) we have (4.21) immediately.

Case 2. $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathcal{A}$, namely $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbb{L}^{2+\varepsilon}$ for some $\varepsilon > 0$. Note that

$$\frac{1}{2+\varepsilon} + \frac{3+2\varepsilon}{6+3\varepsilon} + \frac{\varepsilon}{6+3\varepsilon} = 1 \quad \text{and} \quad \frac{6+3\varepsilon}{3+2\varepsilon} < 2.$$

Since (4.14) holds with $\varepsilon = 0$, following the arguments for (4.12) we have $\mathbb{E}\{\sup_{t\in[0,T]} |\Gamma_t|^{(6+3\varepsilon)/(3+2\varepsilon)}\} < \infty$. This implies that

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T} |\Gamma_t| \sup_{0\le t\le T} |\mathcal{X}_t| \int_0^T |\hat{Z}_t|^2\,dt\Big\} \\
&\quad\le \Big(\mathbb{E}\Big\{\sup_{0\le t\le T} |\Gamma_t|^{(6+3\varepsilon)/(3+2\varepsilon)}\Big\}\Big)^{(3+2\varepsilon)/(6+3\varepsilon)} \\
&\qquad\times \Big(\mathbb{E}\Big\{\sup_{0\le t\le T} |\mathcal{X}_t|^{2+\varepsilon}\Big\}\Big)^{1/(2+\varepsilon)} \Big(\mathbb{E}\Big\{\Big(\int_0^T |\hat{Z}_t|^2\,dt\Big)^{(6+3\varepsilon)/\varepsilon}\Big\}\Big)^{\varepsilon/(6+3\varepsilon)} \\
&\quad< \infty.
\end{aligned}$$

Then one can easily prove (4.21) again. □

5. Well-posedness of the dominating equations. We note that Theorems 4.2 and 4.4 only established the relations of the well-posedness between the characteristic BSDEs and the original FBSDE, it does not provide the well-posedness result for either one of them. In this section, we take a closer look at the dominating ODEs (3.13). Since the existence of bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ to the dominating ODEs will be essential in constructing the desired regular decoupling field, which will eventually lead to well-posedness of the FBSDE (1.1), the results in this section will be the blueprint of a *user's guide* in the end.

We begin with a special form of comparison theorem among the solutions to ODEs. Consider the following "backward ODEs" on $[0, T]$:

$$\mathbf{y}_t^0 = h^0 + \int_t^T F^0(s, \mathbf{y}_s^0)\,ds \tag{5.1}$$

and

$$\begin{aligned}
\mathbf{y}_t^1 &= h^1 - C^1 + \int_t^T [F^1(s, \mathbf{y}_s^1) + c_s^1]\,ds, \\
\mathbf{y}_t^2 &= h^2 + C^2 + \int_t^T [F^2(s, \mathbf{y}_s^2) - c_s^2]\,ds,
\end{aligned} \tag{5.2}$$

where $F^0, F^1, F^2 : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are (deterministic) measurable functions. The following simple lemma will be useful in our discussion. Its proof is rather elementary and we defer it to the Appendix.

LEMMA 5.1. *Assume that*:

(i) $h^1 \le h^0 \le h^2$, *and* $F^1 \le F^0 \le F^2$.
(ii) *Both ODEs in* (5.2) *admit bounded solutions* $\mathbf{y}^1$ *and* $\mathbf{y}^2$ *on* $[0, T]$.
(iii) *For each* $t \in [0, T]$, *the functions* $y \mapsto F^i(t, y)$, $i = 0, 1, 2$, *are uniformly Lipschitz continuous for* $y \in [\mathbf{y}^1_t, \mathbf{y}^2_t]$, *with a common Lipschitz constant* L.
(iv) $C^i \ge \int_t^T e^{-\int_s^T \alpha_r\, dr} c^i_s\, ds$, *for all* $t \in [0, T]$ *and all* α *satisfying* $|\alpha| \le L$.

Then (5.1) *has a unique solution* $\mathbf{y}^0$ *satisfying* $\mathbf{y}^1 \le \mathbf{y}^0 \le \mathbf{y}^2$.

REMARK 5.2. A typical sufficient condition for the above (iv) is: $C^i \ge \int_0^T e^{L(T-t)}(c^i_t)^+\, dt$. In particular, this is satisfied if $C^i = 0$ and $c^i \le 0$.

5.1. *Linear FBSDE with constant coefficients*. We first investigate the linear FBSDE (4.1) where all the coefficients are constants. We shall show that in such a case some "sharp" (sufficient and necessary) conditions regarding well-posedness can be obtained. These results, to our best knowledge, are novel in the literature; and at the same time, they more or less set the "limits" for the solvability of general FBSDE (1.1).

We carry out our analysis in two cases.

Case 1: $\sigma_3 = 0$. In this case, $\overline{h} = \underline{h} = h$, $\overline{F}(t, y) = \underline{F}(t, y) = F(y)$, and two ODEs in (3.13) become the same:

$$\mathbf{y}_t = h + \int_t^T F(\mathbf{y}_s)\, ds, \tag{5.3}$$

where

$$F(y) = f_1 + [f_2 + b_1 + \sigma_1 f_3]y + [b_2 + f_3\sigma_2 + b_3\sigma_1]y^2 + \sigma_2 b_3 y^3. \tag{5.4}$$

We have the following theorem.

THEOREM 5.3. *Assume that in the linear FBSDE* (4.1) *all coefficients are constants*, *and* $\sigma_3 = 0$. *Then the corresponding dominating ODE* (5.3) *with* F *defined by* (5.4) *has a bounded solution for arbitrary* T *if and only if one of the following three cases hold true*:

(i) $F(h) \ge 0$ *and* F *has a zero point in* $[h, \infty)$.
(ii) $F(h) \le 0$ *and* F *has a zero point in* $(-\infty, h]$.
(iii) $\sigma_2 b_3 = 0$ *and* $b_2 + f_3\sigma_2 + b_3\sigma_1 = 0$.

PROOF. We first prove the sufficiency part. In case (i), there exists $\lambda \geq h$ such that $F(\lambda) = 0$. Note that F is locally Lipschitz continuous in y and

$$h = h + \int_t^T \big[F(h) - F(h)\big]\,ds, \qquad \lambda = \lambda + \int_t^T F(\lambda)\,ds.$$

Then it follows from Lemma 5.1 and in particular Remark 5.2 that $\mathbf{y}_t \in [h, \lambda]$, $t \in [0, T]$. Similarly, in case (ii), one has $\mathbf{y}_t \in [\lambda, h]$, for some $\lambda \leq h$ such that $F(\lambda) = 0$. Finally, in case (iii) the ODE (5.3) becomes linear:

$$\mathbf{y}_t = h + \int_t^T \big[f_1 + (f_2 + b_1 + \sigma_1 f_3)\mathbf{y}_s\big]\,ds. \tag{5.5}$$

Thus, it is obviously bounded.

The proof of necessity is elementary but lengthy, we postpone it to the Appendix. $\square$

When the terminal time T is fixed, we have the following slightly weaker sufficient conditions:

THEOREM 5.4. *For any given* $T > 0$, *the ODE* (5.3) *with* F *given in* (5.4) *has a bounded solution on* $[0, T]$ *if one of the following three cases hold true*:

(i) $\sigma_2 b_3 < 0$ *or* $F(h) = 0$.

(ii) $F(h) > 0$, *and there exists a constant* $\varepsilon = \varepsilon(T) > 0$ *small enough*, *such that*

$$\sigma_2 b_3 \leq \varepsilon \quad \text{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \leq \varepsilon.$$

(iii) $F(h) < 0$, *and there exists a constant* $\varepsilon = \varepsilon(T) > 0$ *small enough such that*

$$\sigma_2 b_3 \leq \varepsilon \quad \text{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \geq -\varepsilon.$$

PROOF. (i) In this case clearly, the result follows from either (i) or (ii) of Theorem 5.3.

(ii) In this case we have, for some small constant $\varepsilon > 0$ which will be specified later and for some constants C_1, C_0 independent of ε,

$$F(y) \leq \varepsilon y^3 + \varepsilon y^2 + C_1 y + C \leq 2\varepsilon y^3 + C_1 y + C_0 \qquad \text{for all } y \geq 0. \tag{5.6}$$

We first solve

$$\tilde{\mathbf{y}}_t = h^+ + \int_t^T [C_1 \tilde{\mathbf{y}}_s + C_0 + 1]\,ds$$

and obtain

$$\begin{aligned} \tilde{\mathbf{y}}_t &= e^{C_1(T-t)} h^+ + \frac{C_0 + 1}{C_1}\big[e^{C_1(T-t)} - 1\big] \\ &\leq C_2 := e^{C_1 T} h^+ + \frac{C_0 + 1}{C_1}\big[e^{C_1 T} - 1\big]. \end{aligned} \tag{5.7}$$

Set $\varepsilon \stackrel{\triangle}{=} \frac{1}{2C_2^3}$ so that $2\varepsilon\tilde{\mathbf{y}}_t^3 \le 1$. Note that

$$\tilde{\mathbf{y}}_t = h^+ + \int_t^T [2\varepsilon\tilde{\mathbf{y}}_s^3 + C_1\tilde{\mathbf{y}}_s + C_0 + (1 - 2\varepsilon\tilde{\mathbf{y}}_s^3)]\,ds.$$

By (5.6), applying Lemma 5.1 and in particular Remark 5.2 we see that ODE (5.3) has a solution $\mathbf{y} \in [h, \tilde{\mathbf{y}}] \subset [h, C_2]$.

(iii) can be proved similarly. □

Case 2: $\sigma_3 \ne 0$. In this case, we still have $\overline{h} = \underline{h} = h$, $\overline{F}(t, y) = \underline{F}(t, y) = F(y)$, where the deterministic function F in (5.4) can be rewritten as

$$F(y) = \frac{\alpha_0}{1/\sigma_3 - y} + \alpha_1 + \alpha_2 y + \left[b_2 - \frac{b_3\sigma_2}{\sigma_3}\right]y^2, \tag{5.8}$$

for some constants $\alpha_0, \alpha_1, \alpha_2$. In this case, the two ODEs in (3.13) also become the same one (5.3) and, in light of (3.10), we want to find its solution satisfying that

$$\text{both } \mathbf{y} \text{ and } (1 - \sigma_3\mathbf{y})^{-1} \text{ are bounded.} \tag{5.9}$$

REMARK 5.5. We note that (5.9) amounts to saying that $\sigma_3 h \ne 1$ since $\mathbf{y}_T = h$. In fact, if $\sigma_3 h = 1$, there are counter examples in both existence and uniqueness of the linear FBSDE (4.1) (cf., e.g., [16]).

We now have the following theorem.

THEOREM 5.6. *Assume the FBSDE is the linear one* (4.1) *and all the coefficients are constants. Assume also that $\sigma_3 \ne 0$ and $h\sigma_3 \ne 1$. Then the ODE* (5.3) *has a solution satisfying* (5.9) *for arbitrary T if and only if one of the following four cases holds*:

(i) $h < \frac{1}{\sigma_3}$, $F(h) \le 0$, *and either F has a zero point in $(-\infty, h]$ or* $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$.

(ii) $h > \frac{1}{\sigma_3}$, $F(h) \ge 0$, *and either F has a zero point in $[h, \infty)$ or $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$.*

(iii) $h < \frac{1}{\sigma_3}$, $F(h) \ge 0$, *and F has a zero point in $[h, \frac{1}{\sigma_3})$.*

(iv) $h > \frac{1}{\sigma_3}$, $F(h) \le 0$, *and F has a zero point in $(\frac{1}{\sigma_3}, h]$.*

PROOF. We prove the sufficiency here and again postpone the necessary part to the Appendix.

(i) If $F(\lambda) = 0$, for some $\lambda \in (-\infty, h]$, then as in Theorem 5.3 we see that ODE (5.3) has a solution $\mathbf{y} \in [\lambda, h]$. Thus (5.9) holds. We now assume instead that $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$. Then from (5.8), we see that $F(y) = \alpha_0(\frac{1}{\sigma_3} - y)^{-1} + \alpha_1 + \alpha_2 y$.

Consider

$$\tilde{\mathbf{y}}_t = h + \int_t^T \left[-|\alpha_0| \left(\frac{1}{\sigma_3} - h \right)^{-1} + \alpha_1 + \alpha_2 \tilde{\mathbf{y}}_s \right] ds.$$

Since $F(h) \le 0$, clearly the above SDE has a bounded solution $\tilde{\mathbf{y}} \le h$. Applying Lemma 5.1, one can easily see that (5.3) has a solution $\mathbf{y} \in [\tilde{\mathbf{y}}, h]$. Thus, (5.9) holds.

(iii) Let $\lambda \in [h, \frac{1}{\sigma_3})$ be such that $F(\lambda) = 0$. Note that $\mathbf{y}_t^1 \overset{\triangle}{=} h$ and $\mathbf{y}_t^2 \overset{\triangle}{=} \lambda$ are (constant) solutions of the following ODEs, respectively:

$$\mathbf{y}_t^1 = h + \int_t^T [F(\mathbf{y}_s^1) - F(h)] \, ds, \qquad \mathbf{y}_t^2 = \lambda + \int_t^T F(\mathbf{y}_s^2) \, ds.$$

Comparing these two equations with (5.3) and applying Lemma 5.1, we have $h \le \mathbf{y}_t \le \lambda$, for any $t \in [0, T]$. This implies (5.9) immediately.

(ii) and (iv) can be proved similarly as (i) and (iii), respectively. $\square$

When T is fixed, we may also have some slightly weaker sufficient conditions. However, these conditions are more involved, so we omit them here and will discuss directly for the general case in next subsection; see Theorems 5.9 and 5.10 below.

5.2. *The nonlinear case.* Again we consider the case that $\sigma_3 = 0$ first.

Case 1: $\sigma = \sigma(t, x, y)$. We recall that in this case F takes the form (5.4), where $b_i, \sigma_i, f_i, i = 1, 2, 3$, are bounded, adapted processes defined by (3.1), and thus F is also random and may depend on t. Now recall the definition of the functions $\overline{F}$ and $\underline{F}$ in (3.12). Again, by a slight abuse of notation we replace Θ^j, $j = 1, 2$ in (3.1) by θ_j, $j = 1, 2$, and still denote them by b_i, σ_i, f_i, $i = 1, 2, 3$. In what follows, all assumptions involving coefficients in (5.4) will be in the sense that they hold uniformly for all θ_j, $j = 1, 2$. In analogy to Theorem 5.4, we have the following result.

THEOREM 5.7. *Assume Assumption* 2.1 *holds and $\sigma = \sigma(t, x, y)$. Then, for any $T > 0$, the ODEs* (3.13) *have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ on $[0, T]$ if one of the following three cases holds true*:

(i) *There exists a constant $\varepsilon > 0$ such that*

$$\sigma_2 b_3 \le -\varepsilon |b_2 + f_3 \sigma_2 + b_3 \sigma_1|. \tag{5.10}$$

(ii) *There exists a constant $\lambda \le \underline{h}$, and a constant $\varepsilon > 0$ small enough such that*

$$\underline{F}(t, \lambda) \ge 0, \qquad \sigma_2 b_3 \le \varepsilon \quad \textit{and} \quad b_2 + f_3 \sigma_2 + b_3 \sigma_1 \le \varepsilon. \tag{5.11}$$

(iii) *There exists a constant $\lambda \ge \overline{h}$, and a constant $\varepsilon > 0$ small enough such that*

$$\overline{F}(t, \lambda) \le 0, \qquad \sigma_2 b_3 \le \varepsilon \quad \textit{and} \quad b_2 + f_3 \sigma_2 + b_3 \sigma_1 \ge -\varepsilon. \tag{5.12}$$

PROOF. (i) In this case, we have

$$\overline{F}(t,y) \le C[y+1] \qquad \text{for all } y \ge \frac{1}{\varepsilon} \quad \text{and}$$

$$\underline{F}(t,y) \ge C[y-1] \qquad \text{for all } y \le -\frac{1}{\varepsilon}.$$

Following the arguments in Theorem 5.4(ii), one can easily prove the result.

(ii) In this case, similar to (5.6) we have

$$\overline{F}(t,\lambda) \ge \underline{F}(t,\lambda) \ge 0 \quad \text{and} \quad \underline{F}(t,y) \le \overline{F}(t,y) \le 2\varepsilon y^3 + C_1 y + C_0 \qquad \text{for all } y \ge 0.$$

Let C_2 be defined by (5.7) and set $\varepsilon := \frac{1}{2C_2^3}$. Following the arguments in Theorem 5.4(ii), we see that the ODEs in (3.13) have bounded solutions $\lambda \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le C_2$.

(iii) can be proved similarly. □

Case 2: $\sigma = \sigma(t,x,y,z)$. This case has been avoided in many of the existing literature, especially when one uses the decoupling strategy. A well-known sufficient condition for the existence is, roughly speaking, that $|\sigma_3 h| < 1$. As we will see below, the condition we need is essentially $\sigma_3 h \ne 1$. In particular, we shall discuss three different cases:

(2-a) $|\sigma_3 h| < 1$;
(2-b) $|\sigma_3 h| > 1$ and both σ_3 and h do not change sign;
(2-c) $\sigma_3 h < 1$ and either σ_3 or h does not change sign.

REMARK 5.8. We remark that, if all the coefficients are constants, the above three cases (actually the latter two) cover all possible cases of $\sigma_3 h \ne 1$. However, for general nonlinear FBSDEs with random coefficients, we need them to hold uniformly in certain sense.

To be more precise, let $T > 0$ be given. We begin by fixing three constants c_1, c_2, c_3 satisfying

$$c_1 > 0, \qquad 0 < c_2 < c_3, \qquad c_1 c_3 < 1. \tag{5.13}$$

The following result gives the answer to case (2-a).

THEOREM 5.9. *Assume that Assumption 2.1 and (5.13) are in force. Assume also that there exists a constant $\varepsilon = \varepsilon(T) > 0$ small enough such that*

$$|\sigma_3| \le c_1, \qquad |h| \le c_2 \quad \textit{and} \quad \overline{F}(t,c_3) \le \varepsilon, \qquad \underline{F}(t,-c_3) \ge -\varepsilon. \tag{5.14}$$

Then the ODEs in (3.13) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfying

$$-c_3 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3 \quad \textit{and hence} \quad \textit{both } \overline{\mathbf{y}} \textit{ and } \underline{\mathbf{y}} \textit{ satisfy } (5.9).$$

PROOF. Note that $1-\sigma_3 y \geq 1-c_1c_3 > 0$ for $y \in [-c_3, c_3]$, then $\overline{F}$ and $\underline{F}$ are uniformly Lipschitz continuous in y for $y \in [-c_3, c_3]$, and we denote by L their uniform Lipschitz constant. Clearly, $\tilde{\mathbf{y}}^1_t \stackrel{\triangle}{=} -c_3$ and $\tilde{\mathbf{y}}^2_t \stackrel{\triangle}{=} c_3$ satisfy the following ODEs:

$$\tilde{\mathbf{y}}^1_t = -c_2 - (c_3 - c_2) + \int_t^T \big[\underline{F}(s, \tilde{\mathbf{y}}^1_s) - \underline{F}(s, -c_3)\big]\, ds,$$

$$\tilde{\mathbf{y}}^2_t = c_2 + (c_3 - c_2) + \int_t^T \big[\overline{F}(s, \tilde{\mathbf{y}}^2_s) - \overline{F}(s, c_3)\big]\, ds.$$

Now set $\varepsilon > 0$ small enough such that $c_3 - c_2 > \int_0^T e^{L(T-t)}\varepsilon\, dt$. Then it follows from Lemma 5.1 and in particular Remark 5.2 we obtain the result. $\square$

We next consider case (2-b).

THEOREM 5.10. *Let Assumption 2.1 and (5.13) hold. Assume that there exists a constant $\varepsilon > 0$ small enough such that one of the following four cases holds true*:

$$\begin{aligned} &\sigma_3 \geq c_1^{-1}, \qquad h \geq c_2^{-1} \quad \textit{and} \\ &\underline{F}(t, c_3^{-1}) \geq -\varepsilon, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \leq \varepsilon; \end{aligned} \tag{5.15}$$

$$\begin{aligned} &\sigma_3 \leq -c_1^{-1}, \qquad h \geq c_2^{-1} \quad \textit{and} \\ &\underline{F}(t, c_3^{-1}) \geq -\varepsilon, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \leq \varepsilon; \end{aligned} \tag{5.16}$$

$$\begin{aligned} &\sigma_3 \geq c_1^{-1}, \qquad h \leq -c_2^{-1} \quad \textit{and} \\ &\overline{F}(t, -c_3^{-1}) \leq \varepsilon, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \geq -\varepsilon; \end{aligned} \tag{5.17}$$

$$\begin{aligned} &\sigma_3 \leq -c_1^{-1}, \qquad h \leq -c_2^{-1} \quad \textit{and} \\ &\overline{F}(t, -c_3^{-1}) \leq \varepsilon, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \geq -\varepsilon. \end{aligned} \tag{5.18}$$

Then the ODEs in (3.13) have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that they satisfy the corresponding property of h in the above conditions with c_2 being replaced by c_3. In particular, both $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfy (5.9).

PROOF. We prove only the case (5.15). The other cases can be proved similarly.

In this case, we have

$$\overline{F}(t, y) \leq \frac{C}{c_3^{-1} - c_1} + C_1 y + \varepsilon y^2 = C_0 + C_1 y + \varepsilon y^2 \qquad \text{for all } y \geq c_3^{-1}.$$

Let $\tilde{\mathbf{y}}$ denote the bounded solution to the following ODE:

$$\tilde{\mathbf{y}}_t = \overline{h} + \int_t^T [C_1 \tilde{\mathbf{y}}_s + C_0 + 1]\,ds \quad \text{and} \quad C_2 := \tilde{\mathbf{y}}_0 = \sup_{0 \le t \le T} \tilde{\mathbf{y}}_t.$$

Let L denote the uniform Lipschitz constant of $\underline{F}$ and $\overline{F}$ for $y \in [c_3^{-1}, C_2]$. Note that $\underline{F}(t, c_3^{-1}) \ge -\varepsilon$. Now follow the arguments in Theorem 5.9 for the lower bound and those in Theorem 5.4(ii) for the upper bound, one can easily show that, for ε sufficiently small, the ODEs in (3.13) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $c_3^{-1} \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le C_2$. $\square$

We finally present the result for case (2-c).

THEOREM 5.11. *Let Assumption 2.1 and (5.13) hold. Assume there exists a constant $\varepsilon > 0$ small enough such that one of the following four cases holds true*:

$$\begin{aligned} &\sigma_3 \le c_1, \qquad 0 \le h \le c_2 \quad \textit{and} \\ &\overline{F}(t, c_3) \le \varepsilon, \qquad f_1 \ge 0; \end{aligned} \tag{5.19}$$

$$\begin{aligned} &0 \le \sigma_3 \le c_1, \qquad h \le c_2 \quad \textit{and} \\ &\overline{F}(t, c_3) \le \varepsilon, \qquad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \ge -\varepsilon; \end{aligned} \tag{5.20}$$

$$\begin{aligned} &\sigma_3 \ge -c_1, \qquad 0 \ge h \ge -c_2 \quad \textit{and} \\ &\underline{F}(t, -c_3) \ge -\varepsilon, \qquad f_1 \le 0; \end{aligned} \tag{5.21}$$

$$\begin{aligned} &0 \ge \sigma_3 \ge -c_1, \qquad h \ge -c_2 \quad \textit{and} \\ &\underline{F}(t, -c_3) \ge -\varepsilon, \qquad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \le \varepsilon. \end{aligned} \tag{5.22}$$

Then the ODEs in (3.13) have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that they satisfy the corresponding property of h in the above conditions with c_2 being replaced by c_3. In particular, both $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfy (5.9).

PROOF. If (5.19) holds, then $\underline{F}(t, 0) \ge 0$ and $F(t, c_3) \le \varepsilon$. Following the arguments in Theorem 5.3 for the lower bound and those in Theorem 5.7 for the upper bound, one can easily show that, for ε sufficiently small, the ODEs in (3.13) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $0 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3$.

If (5.20) holds, follow the arguments in Theorem 5.4(ii) for the lower bound and those in Theorem 5.7 for the upper bound, one can easily show that, for ε sufficiently small, the ODEs in (3.13) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $-C_2 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3$ for some $C_2 > 0$.

The other two cases can be proved similarly. $\square$

6. Small duration case revisited. In this and the next section, we shall argue that the well-posedness of the dominating ODEs will lead to the desired regular decoupling field. Our starting point will be the "local existence" result for FBSDE, or more precisely, the well-posedness of FBSDE (1.1) over small time interval. We note that this seemingly well-understood problem still contains many interesting issues that have not been completely observed, especially in the case when σ depends on z (i.e., $\sigma_3 \neq 0$), which we now describe.

Let us first fix some constants $c_1, c_2 > 0$ such that

$$c_1 c_2 < 1. \tag{6.1}$$

Set $\tilde{c}_2 \triangleq \frac{c_2 + c_1^{-1}}{2}$, so that $c_2 < \tilde{c}_2 < c_1^{-1}$. Furthermore, recall $b_i, \sigma_i, f_i, i = 1, 2, 3$ in (3.1). In what follows, all assumptions involving coefficients in (5.4) will be in the sense that they hold uniformly for all θ_j, $j = 1, 2$.

Recall again that it is essential to have $\sigma_3 h \neq 1$. We shall establish the results for the cases (2-a)–(2-c) listed in Section 5.2. Our first result corresponds to case (2-a) and Theorem 5.9. We remark that the case $\sigma = \sigma(t, x, y)$ satisfies case (2-a) with arbitrary small $c_1 > 0$.

THEOREM 6.1. *Suppose that Assumption 2.1 and (6.1) are in force, and assume that $|\sigma_3| \leq c_1$ and $|h| \leq c_2$. Then there exists a constant $\delta > 0$, which depends only on c_1, c_2, and the Lipschitz constants in Assumption 2.1, such that whenever $T \leq \delta$, it holds that*:

(i) *the FBSDE (1.1) has a unique solution $\Theta \in \mathbb{L}^2$*;

(ii) *the ODEs in (3.13) have solutions $\overline{\mathbf{y}}, \underline{\mathbf{y}}$ such that*

$$-\tilde{c}_2 \leq \underline{\mathbf{y}}_t \leq \overline{\mathbf{y}}_t \leq \tilde{c}_2 \qquad \forall t \in [0, T]; \tag{6.2}$$

(iii) *there exists a random field u such that, for all $t \in [0, T]$, $Y_t = u(t, X_t)$ and*

$$\underline{\mathbf{y}}_t \leq \frac{u(t, x_1) - u(t, x_2)}{x_1 - x_2} \leq \overline{\mathbf{y}}_t \qquad \textit{for any } x_1 \neq x_2. \tag{6.3}$$

PROOF. (i) follows directly from [16] Theorem I.5.1. To see (ii), we notice that $\underline{F}$ and $\overline{F}$ are uniformly Lipschitz continuous in y for $y \in [-\tilde{c}_2, \tilde{c}_2]$ and denote by L the uniform Lipschitz constant. We assume that (i) holds for some $\delta > 0$. Modifying δ if necessary we may assume that

$$\left[\int_0^\delta e^{Lt}\, dt\right]\left[\sup_{|y| \leq \tilde{c}_2} \sup_{t \in [0,T]} \left[|\overline{F}(t, y)| + |\underline{F}(t, y)|\right]\right] \leq \tilde{c}_2 - c_2.$$

Now for any $T < \delta$, note that $\tilde{\mathbf{y}}^1 \triangleq -\tilde{c}_2$ and $\tilde{\mathbf{y}}^2 \triangleq \tilde{c}_2$ satisfy the following ODEs:

$$\tilde{\mathbf{y}}_t^1 = -c_2 - [\tilde{c}_2 - c_2] + \int_t^T \left[\underline{F}(s, \tilde{\mathbf{y}}_s^1) - \underline{F}(s, -\tilde{c}_2)\right] ds,$$

$$\tilde{\mathbf{y}}_t^2 = c_2 + [\tilde{c}_2 - c_2] + \int_t^T \left[\overline{F}(s, \tilde{\mathbf{y}}_s^2) - \overline{F}(s, \tilde{c}_2)\right] ds.$$

Following the arguments in Theorem 5.9, we prove (ii).

It remains to prove (iii). Let $\delta > 0$ be small enough so that both (i) and (ii) hold. For any (t, x), denote the (unique) solution to FBSDE (1.1) starting from (t, x) by $\Theta^{t,x}$, and define a random field $u(t, x) \triangleq Y_t^{t,x}$. The uniqueness of the solution to FBSDE then leads to that $Y_s^{t,x} = u(s, X_s^{t,x})$, for all $s \in [t, T]$, $\mathbb{P}$-a.s. In particular, denoting $\Theta_t = \Theta_t^{0,x}$, we have $Y_t = u(t, X_t)$, $t \in [0, T]$.

Now let $x_1 \neq x_2$ be given, and recall (3.3) and (4.1). Following standard arguments, see, for example, [16] Theorem I.5.1, for a smaller δ if necessary, one can easily see that $|\nabla Y_t| \le \tilde{c}_2 |\nabla X_t|$. This also implies that

$$|\nabla X_t| \le \frac{1}{1 - c_1 \tilde{c}_2} |\nabla X_t - \sigma_3 \nabla Y_t|.$$

Applying Theorem 4.2 we see that $\nabla X \neq 0$ and $\hat{Y} \triangleq \nabla Y / \nabla X$ satisfies the BSDE (3.8) and (3.10). Then (6.3) follows from Lemma 3.2. $\square$

Our next result corresponds to case (2-b) and Theorem 5.10.

THEOREM 6.2. *Suppose that Assumption* 2.1 *and* (6.1) *are in force, and assume that σ_3 and h satisfy one of the conditions in* (5.15)–(5.18). *Then there exists a constant $\delta > 0$, depending only on c_1, c_2, and the Lipschitz constants in Assumption* 2.1, *such that when $T \le \delta$, all the results in Theorem* 6.1 *hold true, except that* (6.2) *should be replaced by the following*:

$$\bar{\mathbf{y}} \ge \underline{\mathbf{y}} \ge \tilde{c}_2^{-1} \qquad \textit{in cases } (5.15) \textit{ and } (5.16) \quad \textit{and}$$

$$\underline{\mathbf{y}} \le \bar{\mathbf{y}} \le -\tilde{c}_2^{-1} \qquad \textit{in cases } (5.17) \textit{ and } (5.18).$$

PROOF. We shall argue that the assertions (i)–(iii) in Theorem 6.1 all remain true under the current assumptions. Without loss of generality, we prove the result only for the case (5.15). The other cases can be proved similarly.

We first assume (i) holds. Note that $c_2^{-1} \le h \le L$, where L is the uniform Lipschitz constant in Assumption 2.1. By similar arguments as those in Theorem 6.1(ii), for δ small enough one can easily show that the ODEs in (3.13) have solutions $\bar{\mathbf{y}}, \underline{\mathbf{y}}$ such that

$$\tilde{c}_2^{-1} \le \underline{\mathbf{y}}_t \le \bar{\mathbf{y}}_t \le 2L \qquad \text{for all } t \in [0, T]. \tag{6.4}$$

This proves (ii). (iii) follows from (i) and similar arguments as those in Theorem 6.1(iii).

So it remains to prove (i). Our main idea is to reverse the roles of forward and backward components and then apply Theorem 6.1. To this end, we consider a simple transformation: $\tilde{X} \triangleq Y$ and $\tilde{Y} \triangleq X$. In other words, we define the coordinate change:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and, correspondingly,} \quad \tilde{z} \triangleq \sigma(t, x, y, z).$$

Note that, under (5.15), both functions $z \mapsto \sigma(t,x,y,z)$ and $x \mapsto g(x)$ are invertible, that is, there exist functions $\hat\sigma$ and $\hat g$ such that

$$\hat\sigma\big(t,x,y,\sigma(t,x,y,z)\big) = z, \qquad \hat g\big(g(x)\big) = x. \tag{6.5}$$

Define

$$\begin{aligned} &\tilde\sigma(t,\tilde\theta) \stackrel{\triangle}{=} \hat\sigma(t,\tilde y,\tilde x,\tilde z), \qquad \tilde g(\tilde x) \stackrel{\triangle}{=} \hat g(\tilde x);\\ &\tilde b(t,\tilde\theta) \stackrel{\triangle}{=} -f\big(t,\tilde y,\tilde x,\tilde\sigma(t,\tilde\theta)\big), \qquad \tilde f(t,\tilde\theta) \stackrel{\triangle}{=} -b\big(t,\tilde y,\tilde x,\tilde\sigma(t,\tilde\theta)\big) \end{aligned}$$

and consider a new FBSDE:

$$\begin{cases} \tilde X_t = \tilde x + \int_0^t \tilde b(s,\tilde\Theta_s)\,ds + \int_0^t \tilde\sigma(s,\tilde\Theta_s)\,dB_s;\\ \tilde Y_t = \tilde g(\tilde X_T) + \int_t^T \tilde f(s,\tilde\Theta_s)\,ds - \int_t^T \tilde Z_s\,dB_s, \end{cases} \qquad t\in[0,T]. \tag{6.6}$$

We now show that FBSDE (6.6) satisfies the conditions in Theorem 6.1. First, by definition of inverse functions and by (5.15), we have

$$\hat\sigma_1 + \hat\sigma_3\sigma_1 = 0, \qquad \hat\sigma_2 + \hat\sigma_3\sigma_2 = 0, \qquad \hat\sigma_3\sigma_3 = 1 \quad \text{and} \quad \hat h h = 1,$$

where $\hat\sigma_i, \hat h$ and more notation below are defined in the spirit of (3.1) for the functions $\hat\sigma, \hat g$. Note that $\tilde\sigma_3 = \hat\sigma_3 = (\sigma_3)^{-1}$ and $\tilde h = \hat h = h^{-1}$. This implies that, by (5.15),

$$L^{-1} \le \tilde\sigma_3 \le c_1, \qquad L^{-1} \le \tilde h \le c_2. \tag{6.7}$$

Next, since

$$\tilde b_1 = -f_2 - f_3\tilde\sigma_1 = -f_2 - f_3\hat\sigma_2 = -f_2 - f_3\sigma_2(\sigma_3)^{-1},$$

we see that $|\tilde b_1| \le C$. Similarly, $|\tilde\varphi_j| \le C$ for $\varphi = b, \sigma, f$ and $j = 1,2,3$. Moreover, note that

$$\begin{aligned} \big|\tilde g(0)\big| &= \big|\tilde g(0) - \tilde g\big(g(0)\big)\big| \le L\big|g(0)\big|;\\ \big|\tilde\sigma(t,0,0,0)\big| &= \big|\hat\sigma(t,0,0,0)\big| = \big|\hat\sigma(t,0,0,0) - \hat\sigma\big(t,0,0,\sigma(t,0,0,0)\big)\big|\\ &\le C\big|\sigma(t,0,0,0)\big|;\\ \big|\tilde b(t,0,0,0)\big| &\le \big|f(t,0,0,0)\big| + C\big|\sigma(t,0,0,0)\big|,\\ \big|\tilde f(t,0,0,0)\big| &\le \big|b(t,0,0,0)\big| + C\big|\sigma(t,0,0,0)\big|. \end{aligned}$$

Thus (2.1) holds for FBSDE (6.6).

We can now apply Theorem 6.1 to conclude that for some $\delta > 0$, the FBSDE (6.6) admits a unique solution $\tilde\Theta \in \mathbb{L}^2$ for all $T \le \delta$, and $\tilde Y_t = \tilde u(t,\tilde X_t)$ for some decoupling random field $\tilde u$. Moreover, by (6.7) and modifying the arguments in Theorem 6.1 slightly, we see that $\tilde u$ satisfies

$$\frac{1}{2L} \le \frac{\tilde u(t,\tilde x_1) - \tilde u(t,\tilde x_2)}{\tilde x_1 - \tilde x_2} \le \tilde c_2.$$

Then $\tilde{u}(t,\tilde{x})$ has an inverse function $u(t,x)$ in terms of x. Now for any x, let $\tilde{x} \stackrel{\triangle}{=} u(0,x)$ and let $\tilde{\Theta}$ be the unique solution to FBSDE (6.6) with initial value $\tilde{X}_0 = \tilde{x}$. Then it is straightforward to check that

$$X_t \stackrel{\triangle}{=} \tilde{Y}_t, \qquad Y_t \stackrel{\triangle}{=} \tilde{X}_t, \qquad Z_t \stackrel{\triangle}{=} \tilde{\sigma}(t, \tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)$$

satisfy FBSDE (1.1) with initial value $X_0 = x$.

Finally, note that $|\tilde{Z}| \le |\tilde{\sigma}(t,0,0,0)| + C[|\tilde{X}| + |\tilde{Y}| + |\tilde{Z}|]$, it is clear that $(X,Y,Z) \in \mathbb{L}^2$. The proof is now complete. $\square$

Our final result corresponds to case (2-c) and Theorem 5.11.

THEOREM 6.3. *Suppose that Assumption 2.1 and (6.1) are in force, and assume that σ_3 and h satisfy one of the conditions in (5.19)–(5.22). Then there exists a constant $\delta > 0$, depending only on c_1, c_2, and the Lipschitz constants in Assumption 2.1, such that when $T \le \delta$, all the results in Theorem 6.1 hold true, except that (6.2) should be replaced by the following*:

$$\begin{aligned} &0 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le \tilde{c}_2, \qquad \textit{in case of } (5.19);\\ &\underline{\mathbf{y}} \le \overline{\mathbf{y}} \le \tilde{c}_2, \qquad \textit{in case of } (5.20);\\ &0 \ge \overline{\mathbf{y}} \ge \underline{\mathbf{y}} \ge -\tilde{c}_2, \qquad \textit{in case of } (5.21);\\ &\overline{\mathbf{y}} \ge \underline{\mathbf{y}} \ge -\tilde{c}_2, \qquad \textit{in case of } (5.22). \end{aligned}$$

PROOF. Again we consider only the case (5.19), and the other cases can be argued similarly. Following similar arguments as in Theorem 6.2, we shall only prove (i).

Slightly different from the proof of Theorem 6.2 we consider a slightly more complicated transformation: $(\tilde{x}, \tilde{y}, \tilde{z}) \stackrel{\triangle}{=} \Phi[\varepsilon](x,y,z)$, where

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \stackrel{\triangle}{=} \begin{bmatrix} 2\varepsilon & 1 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \tilde{z} \stackrel{\triangle}{=} \varepsilon\sigma(t,x,y,z) + z. \tag{6.8}$$

Note that

$$-L \le \sigma_3 \le c_1, \qquad 0 \le h \le c_2. \tag{6.9}$$

By choosing $\varepsilon > 0$ small enough, we see that the mappings

$$z \mapsto \tilde{z} = \varepsilon\sigma(t,x,y,z) + z \quad \text{and} \quad x \mapsto 2\varepsilon x + g(x)$$

are both strictly increasing and thus both are invertible. Denote the corresponding inverse functions by $\hat{\sigma}$ and $\hat{g}$, respectively. Namely,

$$\hat{\sigma}\big(t,x,y,\varepsilon\sigma(t,x,y,z) + z\big) = z, \qquad \hat{g}\big(2\varepsilon x + g(x)\big) = x. \tag{6.10}$$

Furthermore, from (6.8) we can solve $(x, y) = (\frac{\tilde{x}-\tilde{y}}{\varepsilon}, 2\tilde{y} - \tilde{x})$, the inverse transformation of $\Phi[\varepsilon]$ is thus

$$(x, y, z) = \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z}) \stackrel{\triangle}{=} \left(\frac{\tilde{x} - \tilde{y}}{\varepsilon}, 2\tilde{y} - \tilde{x}, \hat{\sigma}\left(t, \frac{\tilde{x} - \tilde{y}}{\varepsilon}, 2\tilde{y} - \tilde{x}, \tilde{z}\right)\right).$$

We now consider the FBSDE (6.6) with the following new coefficients:

$$\begin{aligned}
&\tilde{b}(t, \tilde{x}, \tilde{y}, \tilde{z}) = 2\varepsilon b(t, \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z})) - f(t, \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z})),\\
&\tilde{f}(t, \tilde{x}, \tilde{y}, \tilde{z}) = -\varepsilon b(t, \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z})) + f(t, \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z})),\\
&\tilde{\sigma}(t, \tilde{x}, \tilde{y}, \tilde{z}) = 2\varepsilon\sigma(t, \Psi[\varepsilon](\tilde{x}, \tilde{y}, \tilde{z})) + \hat{\sigma}\left(t, \frac{\tilde{x} - \tilde{y}}{\varepsilon}, 2\tilde{y} - \tilde{x}, \tilde{z}\right),\\
&\tilde{g}(\tilde{x}) = \varepsilon\hat{g}(\tilde{x}) + g(\hat{g}(\tilde{x})).
\end{aligned} \tag{6.11}$$

Our idea is again to apply Theorem 6.1. Note that $\hat{\sigma}_3[\varepsilon\sigma_3 + 1] = 1$ and $\hat{h}[2\varepsilon + h] = 1$, we have

$$\tilde{\sigma}_3 = 2\varepsilon\sigma_3\hat{\sigma}_3 + \hat{\sigma}_3 = \frac{2\varepsilon\sigma_3 + 1}{\varepsilon\sigma_3 + 1}, \qquad \tilde{h} = \varepsilon\hat{h} + h\hat{h} = \frac{\varepsilon + h}{2\varepsilon + h}.$$

By (6.9) and for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
&0 < \frac{1 - 2L\varepsilon}{1 - \varepsilon L} \le \tilde{\sigma}_3 \le \frac{1 + 2c_1\varepsilon}{1 + c_1\varepsilon} \stackrel{\triangle}{=} \overline{c}_1;\\
&0 < \frac{1}{2} \le \tilde{h} \le \frac{\varepsilon + c_2}{2\varepsilon + c_2} \stackrel{\triangle}{=} \overline{c}_2.
\end{aligned} \tag{6.12}$$

Since $c_1 c_2 < 1$, we obtain

$$\overline{c}_1\overline{c}_2 = \frac{1 + 2c_1\varepsilon}{1 + c_1\varepsilon} \cdot \frac{\varepsilon + c_2}{2\varepsilon + c_2} < 1. \tag{6.13}$$

Moreover, note that $\hat{\sigma}_1 + \varepsilon\hat{\sigma}_3\sigma_1 = 0$ and $\hat{\sigma}_2 + \varepsilon\hat{\sigma}_3\sigma_2 = 0$, we see that $\hat{\sigma}_1 = \frac{-\varepsilon\sigma_1}{1+\varepsilon\sigma_3}$, $\hat{\sigma}_2 = \frac{-\varepsilon\sigma_2}{1+\varepsilon\sigma_3}$ are bounded and, therefore,

$$\tilde{b}_1 = 2\varepsilon\left[b_1\varepsilon^{-1} - b_2 + b_3\left[\hat{\sigma}_1\varepsilon^{-1} - \hat{\sigma}_2\right]\right] - \left[f_1\varepsilon^{-1} - f_2 + f_3\left[\hat{\sigma}_1\varepsilon^{-1} - \hat{\sigma}_2\right]\right]$$

is bounded. Similarly, one can check that all other coefficients are all uniformly Lipschitz continuous and (2.1) also holds for FBSDE (1.1). Then we can apply Theorem 6.1, with c_1, c_2 being replaced by $\overline{c}_1, \overline{c}_2$ here, to conclude that (6.6) with coefficients given by (6.11) admits a unique solution $\tilde{\Theta} \in \mathbb{L}^2$, for $T \le \delta$ and δ small enough. Furthermore, by (6.12) and following similar arguments as in Theorem 6.1, it holds that $\tilde{Y}_t = \tilde{u}(t, \tilde{X}_t)$ for some decoupling random field $\tilde{u}$, which satisfies, for $\tilde{x}_1 \ne \tilde{x}_2$, and $\overline{c}_3 \stackrel{\triangle}{=} \frac{\overline{c}_1^{-1} + \overline{c}_2}{2}$,

$$\frac{1}{4} \le \frac{\tilde{u}(t, \tilde{x}_1) - \tilde{u}(t, \tilde{x}_2)}{\tilde{x}_1 - \tilde{x}_2} \le \overline{c}_3.$$

This then implies that $\tilde{x} \mapsto \tilde{u}(t, \tilde{x})$ has an inverse, denoted by $u(t, x)$.

Now for any x, let $\tilde{x} \stackrel{\triangle}{=} 2\varepsilon x + u(0, x)$ and $\tilde{\Theta}$ be the unique solution to FBSDE (6.6) starting from $\tilde{X}_0 = \tilde{x}$. Then one can easily check that $\Theta := \Psi[\varepsilon](\tilde{\Theta})$ satisfies all the requirement. $\square$

7. Synthetic analysis. In this section, we summarize all the results proved in the previous sections and give a synthetic analysis for the solvability of FBSDE (1.1) over an arbitrary duration $[0, T]$, which in a sense could serve as a *User's Guide* for solving general FBSDEs. We should note that all the cases listed below cannot be covered by the existing methods, therefore, they are all new.

7.1. *Linear case.* We first consider the linear FBSDE (4.1). Bearing Remarks 5.5 and 5.8 in mind, then combining Theorems 6.1, 6.2 and 6.3, we have the following "local" well-posedness result. We note that since σ is allowed to depend on z, and the condition is both necessary and sufficient, this result is already new.

THEOREM 7.1. *Assume that the linear FBSDE* (4.1) *has constant coefficients. Then there exists a constant* $\delta > 0$, *such that it is well-posed on* $[0, T]$, *whenever* $T \le \delta$, *if and only if*

$$\sigma_3 h \neq 1. \tag{7.1}$$

REMARK 7.2. If the duration T is arbitrarily given, then even in the case when FBSDE is linear with constant coefficients the necessary and sufficient conditions become slightly more complicated. The reader should use Theorem 5.4 or 5.6 as a benchmark.

If the coefficients of FBSDE (4.1) are random, then the analysis becomes more involved. In fact, the degree of difficulty is no less than that of general Lipschitz coefficient case. We therefore do not discuss them separately.

7.2. *The case* $\sigma = \sigma(t, x, y)$. We remark that the work [28] is a special case of the following result.

THEOREM 7.3. *Assume all the conditions in Theorem* 5.7 *hold, and let* $\underline{\mathbf{y}}, \overline{\mathbf{y}}$ *be the bounded solutions of ODEs* (3.13). *Then*:

(i) *FBSDE* (1.1) *possesses a decoupling field* u *satisfying* (6.3).
(ii) *FBSDE* (1.1) *admits a unique solution* $\Theta \in \mathbb{L}^2$, *such that*

$$\|\Theta\|_{\mathbb{L}^2}^2 \le C\big[|x|^2 + I_0^2\big]. \tag{7.2}$$

Here, the constant $C > 0$ *depends only on* T, *the Lipschitz constant in Assumption* 2.1, *and the bound of* $\underline{\mathbf{y}}, \overline{\mathbf{y}}$.

PROOF. (i) First, applying Theorem 5.7, there exists a constant $c_2 > 0$ such that

$$-c_2 \le \underline{\mathbf{y}}_t \le \overline{\mathbf{y}}_t \le c_2 \qquad \text{for all } 0 \le t \le T. \tag{7.3}$$

Notice that in this case $\sigma_3 = 0$, thus we may set arbitrarily small $c_1 > 0$ in Theorem 6.1.

Let $\delta > 0$ be the constant determined by (c_1, c_2) in Theorem 6.1, and $0 = t_0 < \cdots < t_n = T$ be a partition of $[0, T]$ such that $t_i - t_{i-1} \le \delta$, $i = 1, \dots, n$. We first consider FBSDE (1.1) on $[t_{n-1}, t_n]$. Since $\underline{\mathbf{y}}_T \le h \le \overline{\mathbf{y}}_T$, we see that the Lipschitz constant of the terminal condition g is less than c_2, then by Theorem 6.1 there exists a random field $u(t, x)$ for $t \in [t_{n-1}, t_n]$ such that (6.3) holds for all $t \in [t_{n-1}, t_n]$. In particular, the estimate (6.3) at t_{n-1} and (7.3) imply that c_2 is also a Lipschitz constant of $u(t_{n-1}, \cdot)$. Next, consider FBSDE (1.1) on $[t_{n-2}, t_{n-1}]$ with terminal condition $u(t_{n-1}, \cdot)$. Applying Theorem 6.1 again, we find u on $[t_{n-2}, t_{n-1}]$ such that (6.3) holds for $t \in [t_{n-2}, t_{n-1}]$. Repeating this procedure backwardly finitely many times, we extend the random field u to the whole interval $[0, T]$. Clearly, it is a decoupling field satisfying (6.3).

(ii) We first note that the above n is fixed. Since u is uniformly Lipschitz continuous in x, applying Theorem 6.1 on each interval $[t_i, t_{i+1}]$ with initial value $X_{t_i} = 0$, we see that there exists a constant C such that

$$\mathbb{E}\{|u(t_i, 0)|^2\} = \mathbb{E}\{|Y_{t_i}^{t_i,0}|^2\} \le C\mathbb{E}\{|u(t_{i+1}, 0)|^2\} + CI_0^2.$$

Note that $u(t_n, 0) = g(0)$, we see that, for a larger C, $\max_{0\le i\le n} \mathbb{E}\{|u(t_i, 0)|^2\} \le CI_0^2$.

Next, by Theorem 2.3 FBSDE (1.1) admits a unique global solution Θ. Applying Theorem 6.1 on each interval $[t_i, t_{i+1}]$ again, we obtain

$$\begin{aligned} &\mathbb{E}\left\{\sup_{t_i \le t \le t_{i+1}} [|X_t|^2 + |Y_t|^2] + \int_{t_i}^{t_{i+1}} Z_t^2\, dt\right\} \\ &\qquad \le C\mathbb{E}\{|X_{t_i}|^2 + |u(t_{i+1}, 0)|^2\} + CI_0^2. \end{aligned} \tag{7.4}$$

This implies that

$$\mathbb{E}\{|X_{t_{i+1}}|^2\} \le C\mathbb{E}\{|X_{t_i}|^2 + |u(t_{i+1}, 0)|^2\} + CI_0^2 \le C\mathbb{E}\{|X_{t_i}|^2\} + CI_0^2,$$

thus $\max_i \mathbb{E}\{|X_{t_i}|^2\} \le C[|x|^2 + I_0^2]$. Plugging into (7.4) and summing over i, we obtain (7.2). □

In Table 1 below we list a few classes of FBSDEs whose coefficients (b, σ, f) satisfy condition (5.10), and thus are well-posed for arbitrary T under standard Lipschitz conditions. We note that all coefficients are allowed to be random, and $Y_T = g(X_T)$.

TABLE 1
Cases satisfying (5.10)

Assumption	***b***	***σ***	***f***
	$b(t,x,z)$	$\sigma(t)$	$f(t,x,y,z)$
	$b(t,x)$	$\sigma(t,x,y)$	$f(t,x,y)$
$\sigma_2 b_3 \le 0$, $\beta_t \ge c$	$b(t,x,z)$	$\sigma(t,\beta_t x + y)$	$f(t,x,y)$
$\sigma_2 b_3 \le 0$, $\beta_t \ge c$	$b(t,x,z)$	$\sigma(t,y)$	$f_0(t,x,y) + \beta_t b(t,x,z)$

7.3. *The general case* $\sigma = \sigma(t,x,y,z)$. We now turn to the general case. We assume that the standing Assumption 2.1, (5.13), and one of the assumptions (5.14), (5.15)–(5.18) and (5.19)–(5.22) hold. For the convenience of the reader, we tabulate these conditions so that the nature of these assumptions are more explicit. Let $\varepsilon > 0$ be given as that in Theorems 5.9, 5.10, 5.11 and $\alpha_3 \stackrel{\triangle}{=} b_2 - \frac{b_3\sigma_2}{\sigma_3}$.

Case I. $|\sigma_3| \le c_1$, $|h| \le c_2$; and $\overline{F}(t,c_3) \le \varepsilon$, $\underline{F}(t,-c_3) \ge -\varepsilon$.

Case II. $|\sigma_3| \ge c_1^{-1}$, $|h| \ge c_2^{-1}$, and both of them keep the same sign (see Table 2).

Case III. $\sigma_3 h \le c_1 c_2$, and one of them keeps the same sign (see Table 3).

Our main result is the following.

THEOREM 7.4. *Suppose that Assumption* 2.1 *and* (5.13) *are in force, and for* c_1, c_2, c_3 *in* (5.13), *either one of the conditions listed in cases* I–III *holds. Then*:

(i) *FBSDE* (1.1) *possesses a decoupling field u such that* $\frac{u(t,x_1)-u(t,x_2)}{x_1-x_2}$ *satisfies the corresponding property of h with* c_2 *being replaced by* c_3.

(ii) *FBSDE* (1.1) *admits a unique solution* $\Theta \in \mathbb{L}^2$, *and there exists a constant* $C > 0$, *depending only on* T, *the Lipschitz constant in Assumption* 2.1, *and* c_1, c_2, c_3, *such that* (7.2) *holds.*

PROOF. The proof is similar to that of Theorem 7.3 and is thus omitted. However, we emphasize that when one applies Theorems 6.1, 6.2 or 6.3, the constant δ should be determined by c_1, c_3, not by c_1, c_2. □

The following special case deserves special attention.

TABLE 2
$\sigma_3 \ne 0$, *Case* II

	$h \ge c_2^{-1}$	$h \le -c_2^{-1}$
$\sigma_3 \ge c_1^{-1}$	$\underline{F}(t,c_3^{-1}) \ge -\varepsilon, \alpha_3 \le \varepsilon$	$\overline{F}(t,c_3^{-1}) \le \varepsilon, \alpha_3 \ge -\varepsilon$
$\sigma_3 \le -c_1^{-1}$	$\underline{F}(t,c_3^{-1}) \ge -\varepsilon, \alpha_3 \le \varepsilon$	$\overline{F}(t,c_3^{-1}) \le \varepsilon, \alpha_3 \ge -\varepsilon$

TABLE 3
$\sigma_3 \neq 0$, *Case* III

	$h \geq 0$	$h \leq 0$	$\sigma_3 \geq 0$	$\sigma_3 \leq 0$
$\sigma_3 \leq c_1$, $h \leq c_2$	$\overline{F}(t, c_3) \leq \varepsilon$, $f_1 \geq 0$		$\overline{F}(t, c_3) \leq \varepsilon$, $\alpha_3 \geq -\varepsilon$	
$\sigma_3 \geq -c_1$, $h \geq -c_2$		$\underline{F}(t, -c_3) \geq -\varepsilon$, $f_1 \leq 0$		$\underline{F}(t, -c_3) \geq -\varepsilon$, $\alpha_3 \leq \varepsilon$

COROLLARY 7.5. *Assume that Assumption* 2.1 *hold. If the coefficients in the variational FBSDE* (3.5), *defined by* (3.1), *satisfy either*

$$\sigma_3 \geq 0, \qquad h \leq 0, \qquad f_1 \leq 0, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \geq 0 \tag{7.5}$$

or

$$\sigma_3 \leq 0, \qquad h \geq 0, \qquad f_1 \geq 0, \qquad b_2 - \frac{b_3\sigma_2}{\sigma_3} \leq 0; \tag{7.6}$$

then the FBSDE (1.1) *is well-posed over arbitrary duration* $[0, T]$.

PROOF. We assume (7.5) holds. Let c_1 be the Lipschitz constant of σ with respect to z, and let $0 < c_2 < c_3 < \delta$ for some δ small enough. One can easily check that (5.20) holds. $\square$

7.4. *Comparison to the existing methods*. We now compare our conditions to those of the three well-known existing methods.

1. *Method of Contraction Mapping*. It has been understood that the fundamental assumptions for this method are $|\sigma_3 g_1| < 1$ and that T is small enough (see, e.g., [16], Theorem I.5.1). In fact, [16], Example I.5.2, shows that the FBSDE could be unsolvable if $\sigma_3 g_1 = 1$. Therefore, Theorem 7.1 in this paper indeed presents the sharpest result in the linear case.

For the general case, we note that in Antonelli [1] $\sigma_3 = 0$. To compare with the work of Pardoux and Tang [17], we recall (3.9). Then it is easy to see that in [17] it is essentially assumed, besides σ_3 and h satisfying condition (5.14), that one of the following conditions holds:

(i) either $b_2, b_3, \sigma_2, \sigma_3$ or f_1, h are small ("*weak coupling*");
(ii) either b_1 or f_2 is very negative ("*strong monotone*").

But for fixed T, (i) implies that the coefficients of y^2 and y^3 is small enough, and thus the ODEs (3.13) has desired solutions on $[0, T]$, and (ii) implies that the coefficient of y is very negative, which ensures that the solution to ODEs (3.13) does not blow up before T.

2. *Method of Continuation.* The "monotonicity condition" in Hu and Peng [7], Peng and Wu [18], Yong [24] states

$$\begin{aligned} &\Delta b \Delta y + \Delta \sigma \Delta z - \Delta f \Delta x \geq \beta\big[|\Delta x|^2 + |\Delta y|^2 + |\Delta z|^2\big], \\ &\Delta g \Delta x \leq -\beta |\Delta x|^2, \end{aligned} \tag{7.7}$$

for some constant $\beta > 0$. By some simple analysis, one sees immediately that (7.7) implies

$$b_2 \geq \beta, \qquad \sigma_3 \geq \beta, \qquad f_1 \leq -\beta \leq 0, \qquad h \leq -\beta \leq 0.$$

Moreover, by setting $\Delta x = 0$, we see that

$$b_2|\Delta y|^2 + \sigma_3|\Delta z|^2 + (b_3 + \sigma_2)\Delta y \Delta z \geq 0 \qquad \text{for any } \Delta y, \Delta z.$$

Then it must hold that $(b_3 + \sigma_2)^2 - 4b_2\sigma_3 \leq 0$, and thus $b_2\sigma_3 \geq \frac{1}{4}(b_3 + \sigma_2)^2 \geq b_3\sigma_2$. These lead exactly to (7.5), and thus the FBSDE is well-posed. Clearly, the monotonicity condition can be easily further weakened in our framework.

3. *Four Step Scheme.* We should note that our solvability conditions (5.14), (5.15)–(5.18), (5.19)–(5.22) do not cover the results in [12] and [5]. This is because the generality of the FBSDE that we are pursuing in this paper, especially the non-Markovian structure (i.e., random coefficients) and the possible degeneracy of σ, essentially inhibits us from taking advantage of the special features of nondegenerate PDEs. We nevertheless observe that in both [12] and [5], the solution of the PDE, which serves as a deterministic decoupling function, is indeed uniformly Lipschitz continuous, and thus falls into the framework of Theorem 2.3. In fact, our definition of regular decoupling fields is strongly motivated by these works.

7.5. *Regarding examples* (1.2) *and* (1.3). We now return to the two examples (1.2) and (1.3) mentioned in the Introduction. Note that in (1.2) we actually have $F(h) = 0$ and $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$. Then, for $\sigma \neq 0, 1$, either (i) or (ii) of Theorem 5.6 will hold, and thus the FBSDE is well-posed. Since the equation is trivial for $\sigma = 0$, we can thus conclude that *the FBSDE* (1.2) *is well-posed if and only if* $\sigma \neq 1$.

We now turn attention to example (1.3). To understand the problem, we briefly describe its origin (see [4] for more details). Consider the following FBSDE:

$$\begin{aligned} &dX_t = \sigma(t, X_t, Y_t)\,dB_t, \qquad dY_t = f(t, X_t, Y_t)\,dt - Z_t\,dB_t; \\ &X_0 = x, \qquad Y_T = g(X_T), \end{aligned} \tag{7.8}$$

where the coefficients are all deterministic. The purpose is to find a Monte Carlo method for the numerical solution, without using PDEs. Following the idea of "method of optimal control" (cf. [16]), one can consider (7.8) as a controlled diffusion starting from (x, y), and try to find the "control" $(y, Z) \in \mathbb{R} \times L^2_{\mathbb{F}}([0, T])$ so that

$$0 = \inf_{y,Z} V(x, y; Z) \stackrel{\triangle}{=} \inf_{y,Z} \frac{1}{2}\mathbb{E}\big[\big|Y_T^{x,y,Z} - g\big(X_T^{x,y,Z}\big)\big|^2\big].$$

Since the existence of the optimal control is known (as the FBSDE is solvable), the main task here is to numerically compute the optimal control and trajectory. We proceed iteratively: given some initial control (y_0, Z^0) and we find the approximating sequence (y_n, Z^n) that converges to the true solution (Y_0, Z) of the FBSDE (7.8). The so-called "steepest descent method" proposed in [4] suggests that at each step one should set $(y_n, Z^n) := (y_{n-1}, Z^{n-1}) - \lambda(\overline{Y}_0^n, \overline{Z}^n)$ for some small constant $\lambda > 0$, where $(\overline{Y}^n, \overline{Z}^n, \tilde{Y}^n, \tilde{Z}^n)$ solves a certain BSDE which can be rewritten as

$$\begin{aligned} \overline{Y}_t^n &= \overline{Y}_0 + \int_0^t [f_y \overline{Y}_s^n + \sigma_y \tilde{Z}_s^n]\, ds + \int_0^t \overline{Z}_s^n\, dB_s; \\ \tilde{Y}_t^n &= g_x \overline{Y}_T^n + \int_t^T [f_x \overline{Y}_s^n + \sigma_x \tilde{Z}_s^n]\, ds - \int_t^T \tilde{Z}_s^n\, dB_s. \end{aligned} \tag{7.9}$$

If we view $\overline{Z}^n$ as a given random coefficient, $\overline{Y}^n$ the forward component, and $(\tilde{Y}^n, \tilde{Z}^n)$ the backward one, then equations (7.9) is an FBSDE same as (1.3). This FBSDE cannot be covered by any existing method, but it satisfies condition (5.10), and thus falls into our framework. Furthermore applying Corollary 8.4 below we can derive an important estimate in [4]. We refer the interested reader to [4] for details.

8. Properties of the solution.

In this section, we establish some further properties of the solution to the FBSDE (1.1). These will include a stability result, an $\mathbb{L}^p$-estimate for $p > 2$, and a comparison theorem for FBSDE.

We first prove the stability result.

THEOREM 8.1 (Stability). *Assume both (b, σ, f, g) and $(\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})$ satisfy the same conditions (i.e., they belong to the same case) in Theorem 7.4 (or Theorem 7.3). Let $u, \tilde{u}$ be the corresponding random fields and, for any (t, x), $\Theta^{t,x}$ and $\tilde{\Theta}^{t,x}$ the solutions to the corresponding FBSDEs. For $\varphi = b, \sigma, f, g$, denote $\Delta\varphi \stackrel{\triangle}{=} \tilde{\varphi} - \varphi$. Then*

$$\begin{aligned} &\|\tilde{\Theta}^{0,\tilde{x}} - \Theta^{0,x}\|_{\mathbb{L}^2}^2 \\ &\quad \le C\mathbb{E}\biggl\{|\tilde{x} - x|^2 + \big|\Delta g\big(X_T^{0,x}\big)\big|^2 \\ &\qquad + \biggl(\int_0^T [|\Delta b| + |\Delta f|]\big(t, \Theta_t^{0,x}\big)\, dt\biggr)^2 + \int_0^T |\Delta\sigma|^2\big(t, \Theta_t^{0,x}\big)\, dt\biggr\}, \end{aligned} \tag{8.1}$$

$$
\begin{aligned}
&\left|\tilde{u}(t,x)-u(t,x)\right|^2 \\
&\quad \le C\mathbb{E}_t\biggl\{\left|\Delta g(X_T^{t,x})\right|^2+\biggl(\int_t^T[|\Delta b|+|\Delta f|](s,\Theta_s^{t,x})\,ds\biggr)^2 \\
&\qquad\qquad +\int_t^T|\Delta\sigma|^2(s,\Theta_s^{t,x})\,ds\biggr\} \qquad a.s.
\end{aligned}
\tag{8.2}
$$

PROOF. Note that $\tilde{u}(t,x)-u(t,x)=\tilde{Y}_t^{t,x}-Y_t^{t,x}$, and Consider the FBSDEs on $[t,T]$ and replace $\mathbb{E}$ with $\mathbb{E}_t$, (8.2) follows directly from (8.1).

To show (8.1), denote $\Delta\Theta \stackrel{\triangle}{=} \tilde{\Theta}^{0,\tilde{x}}-\Theta^{0,x}$ and $\Delta x \stackrel{\triangle}{=} \tilde{x}-x$. Then

$$
\begin{aligned}
\Delta X_t &= \Delta x+\int_0^t\bigl[\tilde{b}_1\Delta X_s+\tilde{b}_2\Delta Y_s+\tilde{b}_3\Delta Z_s+\Delta b(s,\Theta_s^{0,x})\bigr]\,ds \\
&\quad +\int_0^t\bigl[\tilde{\sigma}_1\Delta X_s+\tilde{\sigma}_2\Delta Y_s+\tilde{\sigma}_3\Delta Z_s+\Delta\sigma(s,\Theta_s^{0,x})\bigr]\,dB_s; \\
\Delta Y_t &= \tilde{h}\Delta X_T+\Delta g(X_T^{0,x}) \\
&\quad +\int_t^T\bigl[\tilde{f}_1\Delta X_s+\tilde{f}_2\Delta Y_s+\tilde{f}_3\Delta Z_s+\Delta f(s,\Theta_s^{0,x})\bigr]\,ds-\int_t^T\Delta Z_s\,dB_s.
\end{aligned}
$$

Here, the notation $\tilde{b}_1$, etc., are defined similar to (3.1). One can easily check that the above linear FBSDE (with solution $\Delta\Theta$) satisfies the corresponding conditions in Theorem 7.4 (or Theorem 7.3). Then applying the theorem we obtain the estimate immediately. □

We next establish the L^p-estimates for some $p>2$. First, following Karatzas and Shreve [8] (cases 2 and 4, page 164), one can easily prove the following lemma.

LEMMA 8.2. *For any $p\ge 2$ and $Z\in L^{2,p}$, that is, $E[(\int_0^T|Z_t|^2\,dt)^{p/2}]<\infty$, we have*

$$
\begin{aligned}
\left|\psi_1(p)\right|^{-p}E\biggl[\biggl|\int_0^t Z_s\,dB_s\biggr|^p\biggr] &\le E\biggl[\biggl(\int_0^t|Z_s|^2\,ds\biggr)^{p/2}\biggr] \\
&\le \left|\psi_2(p)\right|^pE\biggl[\biggl|\int_0^t Z_s\,dB_s\biggr|^p\biggr],
\end{aligned}
\tag{8.3}
$$

where

$$
\begin{aligned}
\psi_1(p) &\stackrel{\triangle}{=} 2^{-1/p}p^{1/2}\biggl(\frac{2^{p/2}-2}{p-2}\biggr)^{1/2-1/p}, \\
\psi_2(p) &\stackrel{\triangle}{=} \biggl(\frac{p-1}{2}\biggr)^{1/p}p^{1/2}\Bigl[2^{p/2}+2^{p/2}-2/(p-2)\Bigr]^{1/2-1/p}.
\end{aligned}
\tag{8.4}
$$

Moreover, for $i = 1, 2$, ψ_i is continuous, strictly increasing on $[2, \infty)$ and $\psi_i(2) = 1$, $\psi_i(\infty) = \infty$.

We now give the L^p-estimate of the solutions.

THEOREM 8.3 (L^p-estimates). *Let (b, σ, f, g) satisfy the conditions in Theorem 7.4. Assume*

$$2 \le p < \psi^{-1}\left(\frac{1}{c_1 c_3}\right), \tag{8.5}$$

where $\psi \triangleq \psi_1\psi_2$ and ψ^{-1} denote the inverse function of ψ; and

$$\begin{aligned} I_p^p \triangleq \mathbb{E}\bigg\{&\left(\int_0^T [|b| + |f|](t, 0, 0, 0)\,dt\right)^p \\ &+ \left(\int_0^T |\sigma|^2(t, 0, 0, 0)\,dt\right)^{p/2} + |g(0)|^p\bigg\} < \infty. \end{aligned} \tag{8.6}$$

Then the unique solution Θ of FBSDE (1.1) is in L^p and satisfies

$$\|\Theta\|_{L^p} \le C_p\big[|x| + I_p\big]. \tag{8.7}$$

Consequently, the corresponding characteristic BSDE (3.8) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.10) and (4.14).

PROOF. By Theorem 7.4 and following its arguments, we may assume $p > 2$ and shall only prove the theorem under (5.14) and for $T \le \delta$, where δ is a constant which depends on c_1, c_3, the Lipschitz constants, and p and will be specified later. Moreover, by using the standard stopping arguments, we can assume without loss of generality that

$$\|\Theta\|_{w,p}^p \triangleq \mathbb{E}\left[\int_0^T [|X_t|^p + |Y_t|^p]\,dt + \left(\int_0^T |Z_t|^2\,dt\right)^{p/2}\right] < \infty. \tag{8.8}$$

For any $0 < \varepsilon \le 1$ and $a, b > 0$, note that $(a + b)^p \le C_{p,\varepsilon}a^p + (1 + \varepsilon)b^p$, for some generic constant $C_{p,\varepsilon} \ge 1$ which may depend on p and ε. Then, for any $0 \le t \le T \le \delta$, we have [denoting $\varphi_s = \varphi(s, \Theta_s)$, $\varphi = b, \sigma$, for simplicity]

$$\begin{aligned} \mathbb{E}\big[|X_t|^p\big] &\le C_{p,\varepsilon}\mathbb{E}\left[|x|^p + \left(\int_0^t |b_s|\,ds\right)^p\right] + (1 + \varepsilon)\mathbb{E}\left[\left|\int_0^t \sigma_s\,dB_s\right|^p\right] \\ &\le C_{p,\varepsilon}\mathbb{E}\left[|x|^p + \left(\int_0^t |b_s|\,ds\right)^p\right] \\ &\qquad + (1 + \varepsilon)\psi_1(p)^p\mathbb{E}\left[\left(\int_0^t |\sigma_s|^2\,ds\right)^{p/2}\right], \end{aligned}$$

where the second inequality thanks to Lemma 8.2. Note that

$$
\begin{aligned}
&\bigg[\int_0^t |b_s|\,ds\bigg]^p \\
&\quad \le C_p\bigg\{\int_0^T \big[\big|b(s,0)\big| + |X_s| + |Y_s| + |Z_s|\big]\,ds\bigg\}^p \\
&\quad \le C_p\bigg\{\bigg[\int_0^T \big|b(s,0)\big|\,ds\bigg]^p \\
&\qquad + T^{p-1}\int_0^T \big[|X_s|^p + |Y_s|^p\big]\,ds + T^{p/2}\bigg[\int_0^T |Z_s|^2\,ds\bigg]^{p/2}\bigg\}, \\
&\bigg[\int_0^t |\sigma_s|^2\,ds\bigg]^{p/2} \\
&\quad \le \bigg(\int_0^T \big[C_\varepsilon\big[\big|\sigma(s,0)\big|^2 + |X_s|^2 + |Y_s|^2\big] + (1+\varepsilon)c_1^2|Z_s|^2\big]\,ds\bigg)^{p/2} \\
&\quad \le C_{p,\varepsilon}\bigg\{\int_0^T C_\varepsilon\big[\big|\sigma(s,0)\big|^2 + |X_s|^2 + |Y_s|^2\big]\,ds\bigg\}^{p/2} \\
&\qquad + (1+\varepsilon)\bigg[\int_0^T (1+\varepsilon)c_1^2|Z_s|^2\,ds\bigg]^{p/2} \\
&\quad \le C_{p,\varepsilon}\bigg\{\int_0^T \big[\big|\sigma(s,0)\big|^2 + |X_s|^2 + |Y_s|^2\big]\,ds\bigg\}^{p/2} \\
&\qquad + (1+\varepsilon)^{p/2+1}c_1^p\bigg[\int_0^T |Z_s|^2\,ds\bigg]^{p/2} \\
&\quad \le C_{p,\varepsilon}\bigg\{\bigg[\int_0^T \big|\sigma(s,0)\big|^2\,ds\bigg]^{p/2} + T^{p/2-1}\int_0^T \big[|X_s|^p + |Y_s|^p\big]\,ds\bigg\} \\
&\qquad + (1+\varepsilon)^{p/2+1}c_1^p\bigg[\int_0^T |Z_s|^2\,ds\bigg]^{p/2}.
\end{aligned}
$$

In the above, $\varphi(s,0) \stackrel{\triangle}{=} \varphi(s,0,0,0)$, for $\varphi = b,\sigma$, respectively. Then

$$
\begin{aligned}
\mathbb{E}\big[|X_t|^p\big] &\le C_{p,\varepsilon}\big[|x|^p + I_p^p + \delta^{p/2}\|\Theta\|_{w,p}^p\big] \\
&\qquad + (1+\varepsilon)\psi_1(p)^p\big[C_{p,\varepsilon}\big[I_p^p + \delta^{p/2-1}\|\Theta\|_{w,p}^p\big] \\
&\qquad + (1+\varepsilon)^{p/2+1}c_1^p\|\Theta\|_{w,p}^p\big] \\
&\le C_{p,\varepsilon}\big[|x|^p + I_p^p + \delta^{p/2-1}\|\Theta\|_{w,p}^p\big] \\
&\qquad + (1+\varepsilon)^{p/2+2}\psi_1(p)^p c_1^p\|\Theta\|_{w,p}^p.
\end{aligned}
\tag{8.9}
$$

Next, by Theorem 7.4 we have

$$|Y_t|^2 \le C\mathbb{E}_t\left[|X_t|^2 + |g(0)|^2 + \left(\int_t^T [|b| + |f|](s,0)\,ds\right)^2 + \int_t^T |\sigma(s,0)|^2\,dt\right].$$

This implies that

$$\mathbb{E}[|Y_t|^p] \le C_p\mathbb{E}[|X_t|^p] + C_p I_p^p. \tag{8.10}$$

In particular,

$$|Y_0|^p \le C_p[|x|^p + I_p^p]. \tag{8.11}$$

Moreover, following standard arguments

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T Z_t\,dB_t\right|^p\right] \\
&\quad = \mathbb{E}\left[\left|g(X_T) - g(0) + g(0) - Y_0 + \int_0^T f(t,\Theta_t)\,dt\right|^p\right] \\
&\quad \le (1+\varepsilon)\mathbb{E}[|g(X_T) - g(0)|^p] \\
&\qquad + C_{p,\varepsilon}\mathbb{E}\left[|g(0)|^p + |Y_0|^p + \left|\int_0^T f(t,\Theta_t)\,dt\right|^p\right] \\
&\quad \le (1+\varepsilon)c_3^p\mathbb{E}[|X_T|^p] + C_{p,\varepsilon}[|x|^p + I_p^p + \delta^{p/2}\|\Theta\|_{w,p}^p].
\end{aligned}$$

Now by the second inequality in (8.3) and (8.9), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |Z_t|^2\,dt\right)^{p/2}\right] \\
&\quad \le (1+\varepsilon)c_3^p|\psi_2(p)|^p\mathbb{E}[|X_T|^p] + C_{p,\varepsilon}[|x|^p + I_p^p + \delta^{p/2}\|\Theta\|_{w,p}^p] \\
&\quad \le (1+\varepsilon)^{p/2+3}[\psi(p)c_1c_3]^p\|\Theta\|_{w,p}^p \\
&\qquad + C_{p,\varepsilon}[|x|^p + I_p^p + \delta^{p/2-1}\|\Theta\|_{w,p}^p].
\end{aligned} \tag{8.12}$$

Set $\varepsilon = 1$ in (8.9), and plug (8.9), (8.10), (8.12) into (8.8), we get

$$\begin{aligned}
\|\Theta\|_{w,p}^p &\le \mathbb{E}\left[\left(\int_0^T |Z_t|^2\,dt\right)^{p/2}\right] + \delta\sup_{0\le t\le T}\mathbb{E}[|X_t|^p + |Y_t|^p] \\
&\le [(1+\varepsilon)^{p/2+3}[\psi(p)c_1c_3]^p + C_{p,\varepsilon}\delta^{p/2-1} \\
&\qquad + C_p\delta]\|\Theta\|_{w,p}^p + C_{p,\varepsilon}[|x|^p + I_p^p].
\end{aligned} \tag{8.13}$$

Denote

$$c_p \triangleq [\psi(p)c_1c_3]^p < 1.$$

We may first choose ε such that $(1+\varepsilon)^{p/2+3}[\psi(p)c_1c_3]^p = \frac{2c_p+1}{3}$, and then choose δ such that $C_{p,\varepsilon}\delta^{p/2-1} + C_p\delta = \frac{1-c_p}{6}$. Then (8.13) implies that

$$\|\Theta\|_{w,p}^p \le \frac{c_p+1}{2}\|\Theta\|_{w,p}^p + C_p\big[|x|^p + I_p^p\big].$$

Since $\frac{c_p+1}{2} < 1$, we obtain $\|\Theta\|_{w,p}^p \le C_p[|x|^p + I_p^p]$. Now following standard arguments we can prove (8.7) straightforwardly.

Finally, the claim on $(\hat{Y}, \hat{Z})$ follows from Theorem 4.4 immediately. □

We note that if $\sigma = \sigma(t, x, y)$, then we could simply take $c_1 = 0$. Note that $\psi^{-1}(\infty) = \infty$, by combining the arguments in Theorems 7.3 and 8.3 [noting (8.5)], we obtain the following result immediately.

COROLLARY 8.4. *Let (b, σ, f, g) satisfy the conditions in Theorem 7.3. For any $p \ge 2$, if $I_p < \infty$, then the unique solution Θ of FBSDE (1.1) is in L^p and satisfies (8.7). Consequently, the corresponding characteristic BSDE (3.8) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.10) and (4.14).*

For FBSDE (4.1), we have $I_p = 0$ for all $p \ge 2$, which leads to the following result.

COROLLARY 8.5. *Assume the linear FBSDE (4.1) satisfy the conditions in Theorem 7.4 (or Theorem 7.3). Then any $2 \le p < \psi^{-1}(\frac{1}{c_1c_3})$, the unique solution Θ of FBSDE (4.1) is in L^p. Consequently, the corresponding characteristic BSDE (3.8) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.10) and (4.14).*

Finally, as an application of Corollary 8.5, we prove the comparison theorem.

THEOREM 8.6 (Comparison). *Assume both (b, σ, f, g) and $(b, \sigma, \tilde{f}, \tilde{g})$ satisfy the same conditions (i.e., they belong to the same case) in Theorem 7.4 (or Theorem 7.3), and let $u, \tilde{u}$ be the corresponding random fields. If $f \le \tilde{f}, g \le \tilde{g}$, then $u \le \tilde{u}$.*

PROOF. Without loss of generality, we shall prove the result only at $t = 0$. Let $\Theta, \tilde{\Theta} \in L^2$ be the corresponding solutions to the FBSDE (1.1) associated to (b, σ, f, g) and $(b, \sigma, \tilde{f}, \tilde{g})$, respectively. Denote $\Delta\Theta_t \stackrel{\triangle}{=} \Theta_t - \tilde{\Theta}_t$, and define φ_i similar to (3.1) for $\varphi = b, \sigma, f$, respectively. Then $\Delta\Theta$ would be the unique solution to the following linear FBSDE:

$$
(8.14)\qquad \begin{cases} \Delta X_t = \int_0^t (b_1 \Delta X_s + b_2 \Delta Y_s + b_3 \Delta Z_s)\, ds \\ \qquad\qquad + \int_0^t (\sigma_1 \Delta X_s + \sigma_2 \Delta Y_s + \sigma_3 \Delta Z_s)\, dB_s; \\ \Delta Y_t = h \Delta X_T + \Delta g(\tilde{X}_T) \\ \qquad\qquad + \int_t^T \big(f_1 \Delta X_s + f_2 \Delta Y_s + f_3 \Delta Z_s + \Delta f(t, \tilde{\Theta}_t)\big)\, ds \\ \qquad\qquad - \int_t^T \Delta Z_s\, dB_s. \end{cases}
$$

Let $(\hat{Y}, \hat{Z})$ denote the unique solution to BSDE (3.8) which, by Corollary 8.5, satisfies (3.10) and (4.14). Denote

$$
\delta Y \stackrel{\triangle}{=} \Delta Y - \hat{Y} \Delta X,
$$

$$
\delta Z \stackrel{\triangle}{=} \Delta Z - \hat{Z} \Delta X - \hat{Y}[\sigma_1 \Delta X + \sigma_2 \Delta Y + \sigma_3 \Delta Z],
$$

and define β, γ and Γ by (4.18) and (4.19). Applying Itô's formula, we have

$$
\delta Y_0 = \Gamma_0 \delta Y_0 = \Gamma_T \Delta g(\tilde{X}_T) + \int_0^T \Gamma_t \Delta f(t, \tilde{\Theta}_t)\, dt - \int_0^T \Gamma_t [\gamma_t \delta Y_t + \delta Z_t]\, dB_t.
$$

Now by (4.14) and following similar arguments as in Theorem 4.4 one can easily show that $\int_0^t \Gamma_s [\gamma_s \delta Y_s + \delta Z_s]\, dB_s$ is a true martingale. Then by our assumptions we see that

$$
u(0, x) - \tilde{u}(0, x) = \Delta Y_0 = \delta Y_0 = \mathbb{E}\left\{\Gamma_T \Delta g(\tilde{X}_T) + \int_0^T \Gamma_t \Delta f(t, \tilde{\Theta}_t)\, dt\right\} \le 0.
$$

This proves the theorem. $\square$

REMARK 8.7. We notice that we cannot get $\Delta Y_t \ge 0$ even $\Gamma_t \ge 0$, $0 \le t \le T$, in the above proof. This coincides with the results in Wu and Xu [22] (Theorem 3.2 and Counterexample 3.1). However, for the corresponding random decoupling field, the comparison theorem holds over all time which coincides with Theorem 4.1 in Cvitanic and Ma [3] by virtue of PDE method under Markovian frame work.

APPENDIX

In this Appendix, we complete the technical proofs for some results in Section 5.

PROOF OF LEMMA 5.1. We first show the existence. Define a truncation function

$$
\tilde{F}(t, y) \stackrel{\triangle}{=} F\big(t, \mathbf{y}_t^1 \vee y \wedge \mathbf{y}_t^2\big),
$$

then by assumption (iii) $\tilde{F}$ is uniformly Lipschitz continuous in y with a Lipschitz constant L, and thus the following ODE has a unique solution $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \tilde{F}(s, \tilde{\mathbf{y}}_s)\, ds, \qquad t \in [0, T]. \tag{A.1}$$

We claim that

$$\mathbf{y}^1 \le \tilde{\mathbf{y}} \le \mathbf{y}^2. \tag{A.2}$$

This would lead to that $\tilde{F}(t, \tilde{\mathbf{y}}_t) = F(t, \tilde{\mathbf{y}}_t)$. Thus, $\tilde{\mathbf{y}}$ is a solution to ODE (5.1) and (A.2) holds.

In fact, denote $\Delta \mathbf{y}^2 \stackrel{\triangle}{=} \mathbf{y}^2 - \tilde{\mathbf{y}}$, $\Delta h^2 \stackrel{\triangle}{=} h^2 - h$, $\Delta F^2 \stackrel{\triangle}{=} F^2 - F$. Note that $F(t, \mathbf{y}_t^2) = \tilde{F}(t, \mathbf{y}_t^2)$, we have

$$\begin{aligned} \Delta \mathbf{y}_t^2 &= \Delta h^2 + C^2 + \int_t^T \big[F^2(s, \mathbf{y}_s^2) - \tilde{F}(s, \tilde{\mathbf{y}}_s) - c_s^2\big]\, ds \\ &= \Delta h^2 + C^2 + \int_t^T \big[\Delta F^2(s, \mathbf{y}_s^2) + \alpha_s \Delta \mathbf{y}_s^2 - c_s^2\big]\, ds, \end{aligned}$$

where $\alpha_s \stackrel{\triangle}{=} \frac{\tilde{F}(s, \mathbf{y}_s^2) - \tilde{F}(s, \tilde{\mathbf{y}}_s)}{\Delta \mathbf{y}_s^2} \mathbf{1}_{\{\Delta \mathbf{y}_s^2 \neq 0\}}$ satisfies $|\alpha| \le L$. Now define $\gamma_t \stackrel{\triangle}{=} \exp(\int_0^t \alpha_s\, ds) > 0$. Then

$$\begin{aligned} \gamma_t \Delta \mathbf{y}_t^2 &= \gamma_T \big[\Delta h^2 + C^2\big] + \int_t^T \gamma_s \big[\Delta F^2(s, \mathbf{y}_s^2) - c_s^2\big]\, ds \\ &= \gamma_T \Delta h^2 + \int_t^T \gamma_s \Delta F^2(s, \mathbf{y}_s^2)\, ds + \gamma_T \bigg[C^2 - \int_t^T \gamma_T^{-1} \gamma_s c_s^2\, ds\bigg] \ge 0. \end{aligned}$$

This implies that $\tilde{\mathbf{y}} \le \mathbf{y}^2$. Similarly, we have $\tilde{\mathbf{y}} \ge \mathbf{y}^1$.

It remains to prove the uniqueness. Let $\mathbf{y}$ be an arbitrary solution to ODE (5.1) satisfying (A.2). Then $\tilde{F}(t, \mathbf{y}_t) = F(t, \mathbf{y}_t)$, and thus $\mathbf{y}$ satisfies ODE (A.1). By the uniqueness of ODE (A.1) we have $\mathbf{y} = \tilde{\mathbf{y}}$, and thus uniqueness follows. □

PROOF OF THEOREM 5.3. (*Necessity*). For simplicity, let us rewrite (5.4) as

$$F(y) = f_1 + a_1 y + a_2 y^2 + a_3 y^3, \tag{A.3}$$

where $a_3 = \sigma_2 b_3$, $a_2 = b_2 + f_3 \sigma_2 + b_3 \sigma_1$, $a_1 = f_2 + b_1 + \sigma_1 f_3$.

We shall show that if none of (i)–(iii) holds, then the solution of ODE (5.3) will blow-up in finite time, which would complete the proof. To this end, we assume without loss of generality that $F(h) \ge 0$. [The case when $F(h) \le 0$ can be argued in the same way but using the conditions (ii) and (iii).] Since (i) does not hold, F has no zero point in $[h, \infty)$, and hence $F(h) > 0$. Now since (iii) does not hold, $|a_3| + |a_2| \neq 0$. Note that if $a_3 < 0$ or $a_3 = 0$ but $a_2 < 0$, then $\lim_{y \to \infty} F(y) = -\infty$ which, together with $F(h) > 0$, will imply that F has a zero point in $[h, \infty)$, a contradiction. Thus, we need only check the case where *either* "$a_3 > 0$" *or* "$a_3 = 0$, $a_2 > 0$." We investigate the two cases separately.

Case 1. Assume $a_3 > 0$. We claim that there exist $\varepsilon > 0$ and $y_1 < h$ such that

$$F(y) \geq \varepsilon (y - y_1)^3 \qquad \text{for all } y \geq h. \tag{A.4}$$

Indeed, in this case $F(y)$ is a polynomial of degree 3, it must have at least one real zero point. By our assumption, F has no zero point after h, then all real zero points must be in $(-\infty, h)$. If there are three real zero points (possibly equal), we list them as $-\infty < y_1 \leq y_2 \leq y_3 < h$. Then for any $y \geq h$, one has

$$F(y) = a_3 \prod_{i=1}^{3} (y - y_i) \geq a_3 (y - y_1)^3. \tag{A.5}$$

On the other hand, if F has only one real zero point, denoted as y_1, then we may write

$$F(y) = a_3 (y - y_1)\big[(y - y_2)^2 + c\big] \qquad \text{for some } c > 0.$$

Note that the function $\tilde{F}(y) \stackrel{\triangle}{=} a_3[(y - y_2)^2 + c](y - y_1)^{-2}$ is continuous for $y \in [h, \infty)$, $\tilde{F}(y) > 0$ and $\lim_{y\to\infty} \tilde{F}(y) = a_3 > 0$. Then

$$\varepsilon \stackrel{\triangle}{=} \inf_{y \geq h} \frac{a_3[(y - y_2)^2 + c]}{(y - y_1)^2} > 0.$$

Thus, noting that $y - y_1 > 0$ for $y \geq h$,

$$F(y) = a_3 (y - y_1)\big[(y - y_2)^2 + c\big] \geq \varepsilon (y - y_1)^3 \qquad \text{for all } y \geq h.$$

This, together with (A.5), proves (A.4).

Now consider the following ODE:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \varepsilon (\tilde{\mathbf{y}}_t - y_1)^3 \, dt. \tag{A.6}$$

Solving this ODE, we have $\tilde{\mathbf{y}}_t - y_1 = \frac{1}{\sqrt{2\varepsilon(t-T) + (h-y_1)^{-2}}}$. Thus, if $T > \frac{1}{2\varepsilon(h-y_1)^2}$, then the solution $\tilde{\mathbf{y}}_t$ blows up at $t = T - \frac{1}{2\varepsilon(h-y_1)^2} \in (0, T)$. On the other hand, by comparison theorem we can easily show that $\mathbf{y}_t \geq \tilde{\mathbf{y}}_t$. Thus, the solution of (5.3) will blow-up at finite time as well.

Case 2. Assume $a_3 = 0$ and $a_2 > 0$. Following similar arguments, in this case we have $F(y) \geq \varepsilon(y - y_1)^2$, for all $y \geq h$, and similarly $\mathbf{y}$ will blow up if T is large enough. $\square$

PROOF OF THEOREM 5.6. (*Necessity*):

(i) Assume $h < \sigma_3^{-1}$, $F(h) \leq 0$, and $\alpha_3 \stackrel{\triangle}{=} b_2 - b_3\sigma_2\sigma_3^{-1} \neq 0$. We show that either F has a zero in $(-\infty, h]$ or $\mathbf{y}$ blows up when T is large enough.

Indeed, if $\alpha_3 > 0$, then $\lim_{y\to-\infty} F(y) = \infty$. Note that F is continuous for $y \in (-\infty, h]$. These, together with $F(h) \leq 0$, imply that F has a zero point in

$(-\infty, h]$. We now assume $\alpha_3 < 0$. Denote $\tilde{F}(y) \overset{\triangle}{=} -\frac{F(y)}{(h+1-y)^2}$. In $(-\infty, h]$, if F has no zero point, then $\tilde{F}$ is continuous, has no zero point, and $\lim_{y\to-\infty} \tilde{F}(y) = -\alpha_3 > 0$. Denote $\varepsilon \overset{\triangle}{=} \inf_{y\le h} \tilde{F}(y) > 0$. Then we have

$$F(y) \le -\varepsilon(h+1-y)^2 \qquad \text{for all } y \le h.$$

Following the arguments for the proof of the necessary part of Theorem 5.3, we prove that $\mathbf{y}$ blows up when T is large.

(ii) Assume $h > \sigma_3^{-1}$, $F(h) \ge 0$, and $\alpha_3 \ne 0$. Similarly, we can show that either F has a zero point in $[h, \infty)$ or $\mathbf{y}$ blows up when T is large enough.

(iii) Assume $h < \sigma_3^{-1}$ and $F(h) \ge 0$. We show that either F has a zero point in $[h, \sigma_3^{-1})$ or $\mathbf{y}$ violates (5.9) when T is large enough.

Indeed, recall the α_0 in (5.8). If $\alpha_0 < 0$, then $\lim_{y\uparrow\sigma_3^{-1}} F(y) = -\infty$. This implies that F has a zero point in $[h, \sigma_3^{-1})$.

If $\alpha_0 > 0$ and F has no zero point in $[h, \sigma_3^{-1})$. Denote $\tilde{F}(y) \overset{\triangle}{=} F(y)[\sigma_3^{-1} - y]$. Then in $[h, \sigma_3^{-1})$, $\tilde{F}$ is continuous, $\tilde{F} > 0$, and $\lim_{y\uparrow\sigma_3^{-1}} \tilde{F}(y) = \alpha_0 > 0$. Denote

$$\varepsilon \overset{\triangle}{=} \inf_{y\in[h,\sigma_3^{-1})} \tilde{F}(y) > 0 \quad \text{and thus} \quad F(y) \ge \varepsilon(\sigma_3^{-1} - y)^{-1} \qquad \text{for } y \in [h, \sigma_3^{-1}).$$

Let $\tilde{\mathbf{y}}$ solve the following ODE:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \varepsilon(\sigma_3^{-1} - \tilde{\mathbf{y}}_s)^{-1}\, ds,$$

we obtain explicitly $(\sigma_3^{-1} - \tilde{\mathbf{y}}_t)^2 = (\sigma_3^{-1} - h)^2 - 2\varepsilon(T-t)$. Let $T \ge \frac{1}{2\varepsilon}(\sigma_3^{-1} - h)^2$. Then for $t = T - \frac{1}{2\varepsilon}(\sigma_3^{-1} - h)^2 \in [0, T]$, we have $\tilde{\mathbf{y}}_t = \sigma_3^{-1}$. By comparison, we see that $(1 - \sigma_3\mathbf{y})^{-1}$ would blow up.

Finally, if $\alpha_0 = 0$ and F has no zero point in $[h, \sigma_3^{-1})$. Then F is continuous and positive on $[h, \sigma_3^{-1}]$. Denote $\varepsilon \overset{\triangle}{=} \inf_{y\in[h,\sigma_3^{-1}]} F(y) > 0$, and define $\tilde{\mathbf{y}}_t \overset{\triangle}{=} h + \int_t^T \varepsilon\, ds = h + \varepsilon(T-t)$, $t \in [0, T]$. Thus, if $T \ge \varepsilon^{-1}[\sigma_3^{-1} - h]$, then $\tilde{\mathbf{y}}_t = \sigma_3^{-1}$ at $t = T - \varepsilon^{-1}[\sigma_3^{-1} - h]$. By comparison again, we see that $(1 - \sigma_3\mathbf{y})^{-1}$ would blow up.

(iv) Assume $h > \sigma_3^{-1}$ and $F(h) \le 0$. We can similarly show that either F has a zero point in $(\sigma_3^{-1}, h]$ or $\mathbf{y}$ violates (5.9) when T is large enough. $\square$

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