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TWO PERSON ZERO-SUM GAME IN WEAK FORMULATION AND PATH DEPENDENT BELLMAN–ISAACS EQUATION*

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Abstract. In this paper we study a two person zero sum stochastic differential game in weak formulation. Unlike the standard literature, which uses strategy type controls, the weak formulation allows us to consider the game with control against control. We shall prove the existence of game value under natural conditions. Another main feature of the paper is that we allow for non-Markovian structure, and thus the game value is a random process. We characterize the value process as the unique viscosity solution of the corresponding path dependent Bellman–Isaacs equation, a notion recently introduced by Ekren et al. [*Ann. Probab.*, 42 (2014), pp. 204–236] and Ekren, Touzi, and Zhang [*Stochastic Process.*, to appear; preprint, arXiv:1210.0006v2; preprint, arXiv:1210.0007v2].

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1. Introduction.

Since the seminal paper Fleming and Souganidis [21], two person zero sum stochastic differential games have been studied extensively in the literature; see, e.g., [2, 3, 4, 5, 6, 8, 14, 20, 23, 24, 25, 28, 35], to mention a few. There are typically two approaches. One is to use dynamical programming, particularly to show that the value function of the game is the unique viscosity solution of the associated Bellman–Isaacs equation, and the other is to use the stochastic maximum principle, which characterizes the value process as the solution to a related backward SDE (BSDE).

To be precise, let u and v denote the controls of the two players, B a Brownian motion, $X^{S,u,v}$ the controlled state process in the strong formulation

$$X_t^{S,u,v} = x + \int_0^t b(s, u_s, v_s)ds + \int_0^t \sigma(s, u_s, v_s)dB_s, \tag{1.1}$$

and $J(u, v)$ the corresponding value (utility or cost) which is determined by $X^{S,u,v}$, B, and (u, v). The lower and upper values of the game are defined as

$$\underline{V}_0 := \sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}} J(u, v), \quad \overline{V}_0 := \inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} J(u, v),$$

where $\mathcal{U}$ and $\mathcal{V}$ are appropriate sets of admissible controls. It is clear that $\underline{V}_0 \le \overline{V}_0$. Two central problems in the game literature are as follows:

(i) When does the game value exist, namely, $V_0 := \underline{V}_0 = \overline{V}_0$?

(ii) Given the existence of the game value, is there a saddle point? That is, we want to find $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that $V_0 = J(u^*, v^*) = \inf_{v\in\mathcal{V}} J(u^*, v) = \sup_{u\in\mathcal{U}} J(u, v^*)$.

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However, even under reasonable assumptions, the game value may not exist. We shall provide a counterexample, see Example E.1 below, which is due to Buckdahn.

To overcome the difficulty, Fleming and Souganidis [21] introduced strategy types of controls:

$$\underline{V}'_0 := \sup_{\alpha\in\mathcal{A}} \inf_{v\in\mathcal{V}} J(\alpha(v), v), \quad \overline{V}'_0 := \inf_{\beta\in\mathcal{B}} \sup_{u\in\mathcal{U}} J(u, \beta(u)).$$

Here $\alpha : \mathcal{V} \to \mathcal{U}$ and $\beta : \mathcal{U} \to \mathcal{V}$ are so-called strategies and $\mathcal{A}$, $\mathcal{B}$ are appropriate sets of admissible strategies. Under the Isaacs condition and assuming the comparison principle for the viscosity solution of the corresponding Bellman–Isaacs equation, [21] showed that $\underline{V}'_0 = \overline{V}'_0$. This work has been extended by many authors in various aspects. In particular, Buckdahn and Li [6] defined $J(u, v)$ via BSDEs, and very recently Bayraktar and Yao [2] used doubly reflected BSDEs. The main drawback of this approach, however, is that the two players have asymmetric information, and for $\underline{V}'_0$ and $\overline{V}'_0$, the roles of two players are switched. Consequently, it is less convenient to study the saddle point in this setting.

We propose to attack the problem in weak formulation, which is more convenient for proving the dynamic programming principle. Note that in (1.1) the controls (u, v) mean $u(B_\cdot), v(B_\cdot)$. Our weak formulation is equivalent to the following feedback type controls:

$$X^{W,u,v}_t = x + \int_0^t b(s, u_s(X^{W,u,v}_\cdot), v_s(X^{W,u,v}_\cdot))ds + \int_0^t \sigma(s, u_s(X^{W,u,v}_\cdot), v_s(X^{W,u,v}_\cdot))dB_s. \tag{1.2}$$

Here, $X^{W,u,v}_\cdot$ denotes the path of $X^{W,u,v}$, and the superscript ^{W} stands for weak formulation. Under natural assumptions, we show that the game value does exist. The advantage of the weak formulation setting is that we are using control against control, and thus one can define the saddle point naturally. When there is only drift control, namely, σ is independent of (u, v), one can prove the existence of saddle point under mild conditions. However, when there is diffusion control, the problem is much more involved. We shall obtain some approximate saddle point.

We remark that the weak formulation is also more natural in many practical problems. Indeed, in practice a player of a zero sum game may not want to share his or her control to the other player, and then the strategy cannot be implemented. In a weak formulation setting, however, both players decide their control based on the state process X, which is typically public information in practice.

When there is only drift control, the weak formulation has already been used in the literature; see Bensoussan and Lions [3] for the Markovian case and Hamadene and Lepetier [23] for the non-Markovian case. The former one relies on PDE arguments, and the latter one uses BSDEs. The advantage in this case is that one can easily obtain the weak solution of SDE (1.2) by applying the Girsanov theorem. Our general case with diffusion control has a different nature. Roughly speaking, the drift control is associated with semilinear PDEs, while the diffusion control is associated with fully nonlinear PDEs. We also note that, in a Markovian model but also with optimal stopping problem, Karatzas and Sudderth [25] studied the game problem with diffusion control in weak formulation, under certain strong conditions.

Another main feature of our paper is that we study the game in a non-Markovian framework, or, say, in a path dependent manner. The standard approach in the literature, e.g., [21] and [6], is to prove that the lower value and the upper value are a viscosity solution (or viscosity semisolution) of the corresponding Bellman–Isaacs equation, and then by assuming the comparison principle for the viscosity solution of the PDE, one obtains the existence of the game value. These works rely on the PDE arguments and thus work only in a Markovian setting. In a series of papers, Ekren et al. [16] and Ekren, Touzi, and Zhang [17, 18, 19] introduced a notion of viscosity solution for the so-called path dependent PDEs (PPDEs) and established its wellposedness. This enables us to extend the above approach to a path dependent setting. Indeed, based on the dynamic programming principle we establish, we show that the lower value and the upper value of the game are viscosity solutions of the corresponding path dependent Bellman–Isaacs equations. Then, under the Isaacs condition and assuming the uniqueness of viscosity solutions, we characterize the game value as the unique viscosity solution of the path dependent Bellman–Isaacs equation.

Finally we remark that, due to weak formulation with diffusion control, this paper is by nature closely related to the second order BSDEs (2BSDEs) introduced by Cheridito et al. [9] and Soner, Touzi, and Zhang [32, 33], and the G-expectation introduced by Peng [30]. While more involved here, our arguments for the dynamic programming principle follow the idea in [32, 33] and Peng [29]. However, G-expectations and 2BSDEs involve only stochastic optimization and thus the generator is convex in terms of the hessian. Consequently, the dynamic value process is a supermartingale under each associated probability measure. For our game problem, the Bellman–Isaacs equation is nonconvex, and the value process is not a supermartingale anymore. Under additional technical conditions, we conjecture that our value process will be a semimartingale. This requires one to develop a semimartingale theory under nonlinear expectation and to generalize the 2BSDE theory to nonconvex generators. We established some norm estimates for semimartingales in another paper, Pham and Zhang [31], and will leave the general 2BSDE theory for future research.

The rest of the paper is organized as follows. In section 2 we present some preliminaries. The game problem is introduced in section 3. In sections 4 and 5 we prove the dynamic programming principle and the viscosity property, respectively. In section 6 we study the comparison principle for PPDEs, and in section 7 we investigate approximate saddle points. Finally, some technical proofs are presented in the appendix.

2. Preliminaries.

2.1. The canonical space. Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = \mathbf{0}\}$, the set of continuous paths starting from the origin, B be the canonical process, $\mathbb{F}$ be the filtration generated by B, $\mathbb{P}_0$ be the Wiener measure, and $\Lambda := [0,T]\times\Omega$. Here and in what follows, for notational simplicity we use $\mathbf{0}$ to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. Let $\mathbb{S}^d$ denote the set of $d\times d$ matrices, $\mathbb{S}^d_+ := \{\sigma\in\mathbb{S}^d : \sigma \geq \mathbf{0}\}$, and

$$x\cdot x' := \sum_{i=1}^d x_i x_i' \quad\text{for any}\quad x,x'\in\mathbb{R}^d,\quad \gamma:\gamma' := \text{Trace}[\gamma\gamma'] \quad\text{for any}\quad \gamma,\gamma'\in\mathbb{S}^d.$$

We define a seminorm on Ω and a pseudometric on Λ as follows: for any $(t,\omega),(t',\omega')\in\Lambda$,

$$\|\omega\|_t := \sup_{0\leq s\leq t}|\omega_s|,\quad \mathbf{d}_\infty\big((t,\omega),(t',\omega')\big) := |t-t'| + \big\|\omega_{\cdot\wedge t} - \omega'_{\cdot\wedge t'}\big\|_T. \tag{2.1}$$

Then $(\Omega, \|\cdot\|_T)$ is a Banach space and $(\Lambda, \mathbf{d}_\infty)$ is a complete pseudometric space. In fact, the subspace $\{(t, \omega_{\cdot\wedge t}) : (t,\omega) \in \Lambda\}$ is a complete metric space under $\mathbf{d}_\infty$. We shall denote by $\mathbb{L}^0(\mathcal{F}_T)$ and $\mathbb{L}^0(\Lambda)$ the collection of all $\mathcal{F}_T$-measurable random variables and $\mathbb{F}$-progressively measurable processes, respectively. Let $C^0(\Lambda)$ (resp., $UC(\Lambda)$) be the subset of $\mathbb{L}^0(\Lambda)$ whose elements are continuous (resp., uniformly continuous) in (t,ω) under $\mathbf{d}_\infty$. The corresponding subsets of bounded processes are denoted by $\mathbb{L}^\infty(\Lambda)$, $C_b^0(\Lambda)$, and $UC_b(\Lambda)$. Finally, $\mathbb{L}^0(\Lambda, \mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued processes with entries in $\mathbb{L}^0(\Lambda)$, and we define similar notation for the spaces $\mathbb{L}^\infty$, C^0, C_b^0, UC, and UC_b.

Definition 2.1. *By $\underline{\mathcal{U}}$, we denote the collection of all processes $Y \in \mathbb{L}^0(\Lambda)$ such that*

- *Y is bounded from above and right continuous in t for all ω;*
- *there exists a modulus of continuity function ρ such that for any $(t,\omega), (t',\omega') \in \Lambda$:*

$$Y(t,\omega) - Y(t',\omega') \le \rho\Big(\mathbf{d}_\infty\big((t,\omega),(t',\omega')\big)\Big) \text{ whenever } t \le t'. \tag{2.2}$$

By $\overline{\mathcal{U}}$ we denote the set of all processes Y such that $-Y \in \underline{\mathcal{U}}$.

It is clear that $\underline{\mathcal{U}} \cap \overline{\mathcal{U}} = UC_b(\Lambda)$.

We next introduce the shifted spaces. Let $0 \le s \le t \le T$.

- Let $\Omega^t := \{\omega \in C([t,T], \mathbb{R}^d) : \omega_t = \mathbf{0}\}$ be the shifted canonical space, B^t the shifted canonical process on Ω^t, $\mathbb{F}^t$ the shifted filtration generated by B^t, $\mathbb{P}_0^t$ the Wiener measure on Ω^t, and $\Lambda^t := [t,T] \times \Omega^t$.
- Define $\|\cdot\|_s$ on Ω^t and $\mathbf{d}_\infty$ on $\Lambda^t \times \Lambda^t$ in the spirit of (2.1), and the sets $\mathbb{L}^0(\Lambda^t)$, etc., in an obvious way.
- For $\omega \in \Omega$, denote ω^t as the natural mapping from Ω to Ω^t, where

$$\omega_s^t := \omega_s - \omega_t, \ t \le s \le T.$$

- For $\omega \in \Omega^s$ and $\omega' \in \Omega^t$, define the concatenation path $\omega \otimes_t \omega' \in \Omega^s$ by

$$(\omega \otimes_t \omega')(r) := \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \omega'_r)\mathbf{1}_{[t,T]}(r) \text{ for all } r \in [s,T].$$

- Let $s \in [0,T)$, $\xi \in \mathbb{L}^0(\mathcal{F}_T^s)$, and $X \in \mathbb{L}^0(\Lambda^s)$. For $(t,\omega) \in \Lambda^s$, define $\xi^{t,\omega} \in \mathbb{L}^0(\mathcal{F}_T^t)$ and $X^{t,\omega} \in \mathbb{L}^0(\Lambda^t)$ by

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega') \text{ for all } \omega' \in \Omega^t.$$

It is clear that, for any $(t,\omega) \in \Lambda$ and any $Y \in \mathbb{L}^\infty(\Lambda)$, we have $Y^{t,\omega} \in \mathbb{L}^\infty(\Lambda^t)$. Similarly, this property holds for other spaces defined in Definition 2.1.

2.2. Probability measures. In this subsection we introduce the probability measures on Ω^t in different formulations. First, let $\sigma \in \mathbb{L}^\infty(\Lambda, \mathbb{S}_+^d)$, $b \in \mathbb{L}^\infty(\Lambda, \mathbb{R}^d)$. Define

$$\mathbb{P}^{S,\sigma,b} := \mathbb{P}_0 \circ (X^{S,\sigma,b})^{-1}, \text{ where } X_t^{S,\sigma,b} := \int_0^t b_s ds + \int_0^t \sigma_s dB_s, \quad \mathbb{P}_0\text{-a.s.} \tag{2.3}$$

Here the superscript ^{S} stands for strong formulation. We next introduce the corresponding weak formulation. We denote a probability measure $\mathbb{P}$ on Ω as $\mathbb{P}^{W,\sigma,b}$ if

$$M_t^b := B_t - \int_0^t b_s ds \text{ is a } \mathbb{P}\text{-martingale and } \langle M^b \rangle_t = \int_0^t \sigma_s^2 ds \ \ \mathbb{P}\text{-a.s.} \tag{2.4}$$

Here the quadratic variation $\langle M^b \rangle$ is under $\mathbb{P}$. We remark that $\mathbb{P}^{W,\sigma,b} := \mathbb{P}_0 \circ (X^{W,\sigma,b})^{-1}$, where $X^{W,\sigma,b}$ is a weak solution of the following (path dependent) SDE:

$$X_t^{W,\sigma,b} := \int_0^t b_s(X_\cdot^{W,\sigma,b})ds + \int_0^t \sigma_s(X_\cdot^{W,\sigma,b})dB_s, \quad \mathbb{P}_0\text{-a.s.} \tag{2.5}$$

In other words, we are considering (non-Markovian) feedback type controls.

In this paper we shall use the weak formulation, which is more convenient for proving the dynamical programming principle. As mentioned in the introduction, in many practical problems where the state process X is a public information, the weak formulation is also more natural since it will require the controls to depend only on X. We note that, for arbitrarily given (σ, b), the SDE (2.5) may not have a weak solution, namely, there is no $\mathbb{P}$ such that $\mathbb{P} = \mathbb{P}^{W,\sigma,b}$. Let

$$\begin{aligned} \overline{\Xi}^W &:= \Big\{(\sigma, b) \in \mathbb{L}^\infty(\Lambda, \mathbb{S}_+^d) \times \mathbb{L}^\infty(\Lambda, \mathbb{R}^d) : \\ &\qquad \text{SDE (2.5) has a unique weak solution}\Big\}; \\ \overline{\Xi}^S &:= \Big\{(\sigma, b) \in \mathbb{L}^\infty(\Lambda, \mathbb{S}_+^d) \times \mathbb{L}^\infty(\Lambda, \mathbb{R}^d) : \\ &\qquad \text{SDE (2.5) has a unique strong solution}\Big\}. \end{aligned} \tag{2.6}$$

For probability measures on the shifted space Ω^t, we define $\mathbb{P}^{S,t,\sigma,b}$, $\mathbb{P}^{W,t,\sigma,b}$, and $\overline{\Xi}^{W,t}$, $\overline{\Xi}^{S,t}$, etc., similarly.

We next introduce the regular conditional probability distribution (r.c.p.d.) due to Stroock and Varadhan [34]. We shall follow the presentation in Soner, Touzi, and Zhang [32]. Let $\mathbb{P}$ be an arbitrary probability measure on Ω and τ be an $\mathbb{F}$- stopping time. The r.c.p.d. $\{\mathbb{P}^{\tau,\omega}, \omega \in \Omega\}$ satisfies the following:

- For each ω, $\mathbb{P}^{\tau,\omega}$ is a probability measure on $\mathcal{F}_T^{\tau(\omega)}$.
- For every bounded $\mathcal{F}_T$-measurable random variable ξ,

$$\mathbb{E}^{\mathbb{P}}\big[\xi|\mathcal{F}_\tau\big](\omega) = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}\big[\xi^{\tau(\omega),\omega}\big], \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

The following simple lemma will be important for the proof of dynamic programming principle in section 4 below. Its proof is postponed to the appendix.

Lemma 2.2. *Let $(\sigma, b) \in \overline{\Xi}^S$ (resp., $\overline{\Xi}^W$), $t \in [0, T]$, $\{E_i, 1 \le i \le n\} \subset \mathcal{F}_t$ be a partition of Ω, and $(\sigma^i, b^i) \in \overline{\Xi}^{S,t}$ (resp., $\overline{\Xi}^{W,t}$). Define*

$$\bar\sigma(\omega) := \sigma(\omega)\mathbf{1}_{[0,t)} + \sum_{i=1}^n \sigma^i(\omega^t)\mathbf{1}_{E_i}\mathbf{1}_{[t,T]}, \quad \bar b(\omega) := b(\omega)\mathbf{1}_{[0,t)} + \sum_{i=1}^n b^i(\omega^t)\mathbf{1}_{E_i}\mathbf{1}_{[t,T]}.$$

Then $(\bar\sigma, \bar b) \in \overline{\Xi}^S$ (resp., $\overline{\Xi}^W$), and, for $i = 1, \dots, n$,

$$\mathbb{P}^{\bar\sigma,\bar b} = \mathbb{P}^{\sigma,b} \text{ on } \mathcal{F}_t \text{ and } \big(\mathbb{P}^{\bar\sigma,\bar b}\big)^{t,\omega} = \mathbb{P}^{t,\sigma^i,b^i} \quad \text{for } \mathbb{P}^{\sigma,b}\text{-a.e. } \omega \in E_i.$$

2.3. Viscosity solutions of PPDEs. Our notion of viscosity solutions of PPDEs is introduced by Ekren et al. [16] for semilinear PPDE and Ekren, Touzi, and Zhang [18, 19] for fully nonlinear PPDE. We follow the presentation in [18, 19] here.

For any constant $L > 0$, denote by $\mathcal{P}_L$ the collection of all continuous semimartingale measures $\mathbb{P}$ on Ω whose drift and diffusion characteristics are bounded by

L and $\sqrt{2L}$, respectively. To be precise, let $\tilde\Omega := \Omega^3$ be an enlarged canonical space, $\tilde B := (B, A, M)$ be the canonical processes, and $\tilde\omega = (\omega, a, m) \in \tilde\Omega$ be the paths. For any $\mathbb{P} \in \mathcal{P}_L$, there exists an extension Q on $\tilde\Omega$ such that

$$(2.8)\qquad \begin{array}{l} B = A + M, \quad A \text{ is absolutely continuous, } M \text{ is a martingale,} \\ |\alpha^{\mathbb{P}}| \le L, \;\; \frac{1}{2}\mathrm{tr}\,((\beta^{\mathbb{P}})^2) \le L, \quad \text{where } \alpha^{\mathbb{P}}_t := \frac{dA_t}{dt},\; \beta^{\mathbb{P}}_t := \sqrt{\frac{d\langle M\rangle_t}{dt}}, \end{array} \quad Q\text{-a.s.}$$

It is clear that

$$(2.9)\qquad \left\{\mathbb{P}^{S,\sigma,b} : |b| \le L, |\sigma| \le \sqrt{2L}\right\} \cup \left\{\mathbb{P}^{W,\sigma,b} : |b| \le L, |\sigma| \le \sqrt{2L}\right\} \subset \mathcal{P}_L.$$

Denote $\mathcal{P}_\infty := \cup_{L>0}\mathcal{P}_L$, and simlarly we may define $\mathcal{P}^t_L$ and $\mathcal{P}^t_\infty$ on the shifted space Ω^t.

Let $Y \in C^0(\Lambda)$. For $t \in [0, T)$, we define the right time-derivative, if it exists, as in Dupire [13] and Cont and Fournie [10]:

$$(2.10)\qquad \partial_t Y(t,\omega) := \lim_{h\downarrow 0} \frac{1}{h}\left[Y\big(t+h, \omega_{\cdot\wedge t}\big) - Y\big(t,\omega\big)\right].$$

For the final time T, we define $\partial_t Y(T,\omega) := \lim_{t\uparrow T} \partial_t Y(t,\omega)$ whenever the limit exists.

Definition 2.3.

(i) *We say* $Y \in C^{1,2}(\Lambda)$ *if* $Y \in C^0(\Lambda)$, $\partial_t Y \in C^0(\Lambda)$, *and there exist* $\partial_\omega Y \in C^0(\Lambda, \mathbb{R}^d)$, $\partial^2_{\omega\omega} Y \in C^0(\Lambda, \mathbb{S}^d)$ *such that, for any* $\mathbb{P} \in \mathcal{P}^0_\infty$, Y *is a local* $\mathbb{P}$-*semimartingale and it holds that*

$$(2.11)\; dY_t = \partial_t Y_t dt + \partial_\omega Y_t \cdot dB_t + \frac{1}{2}\partial^2_{\omega\omega} Y_t : d\langle B\rangle_t, \quad 0 \le t \le T, \;\mathbb{P}\text{-}a.s.$$

(ii) *We say* $Y \in C^{1,2}_b(\Lambda)$ *if* $Y \in UC_b(\Lambda)$, $\partial_t Y \in C^0_b(\Lambda)$, *and the above* $\partial_\omega Y$ *and* $\partial^2_{\omega\omega} Y$ *exist and are in* $C^0_b(\Lambda, \mathbb{R}^d)$ *and* $C^0_b(\Lambda, \mathbb{S}^d)$, *respectively.*

Next, let $\mathcal{T}$ denote the set of $\mathbb{F}$-stopping times, and $\mathcal{H} \subset \mathcal{T}$ the subset of those hitting times $\textsc{h}$ taking the following form: for some open and convex set $O \subset \mathbb{R}^d$ containing $\mathbf{0}$ and some $0 < t_0 \le T$,

$$(2.12)\qquad \textsc{h} := \inf\{t : B_t \in O^c\} \wedge t_0 = \inf\{t : d(\omega_t, O^c) = 0\} \wedge t_0.$$

We may define $C^{1,2}(\Lambda^t)$, $C^{1,2}_b(\Lambda^t)$, $\mathcal{T}^t$, and $\mathcal{H}^t$ similarly. It is clear that, for any (t,ω) and $Y \in C^{1,2}(\Lambda)$ (resp., $Y \in C^{1,2}_b(\Lambda)$), we have $Y^{t,\omega} \in C^{1,2}(\Lambda^t)$ (resp., $Y^{t,\omega} \in C^{1,2}_b(\Lambda)$), and for any $\textsc{h} \in \mathcal{H}$ such that $\textsc{h}(\omega) > t$, we have $\textsc{h}^{t,\omega} \in \mathcal{H}^t$.

For any $L > 0$, $(t,\omega) \in \Lambda$ with $t < T$, and $\mathbb{F}$-adapted process Y, define

$$(2.13)\qquad \begin{aligned} \underline{\mathcal{A}}^L Y(t,\omega) := \Big\{&\varphi \in C^{1,2}_b(\Lambda^t) : \text{ for some } \textsc{h} \in \mathcal{H}^t, \\ &(\varphi - Y^{t,\omega})(t,\mathbf{0}) = 0 = \inf_{\tau\in\mathcal{T}^t} \inf_{\mathbb{P}\in\mathcal{P}^t_L} \mathbb{E}^{\mathbb{P}}\big[(\varphi - Y^{t,\omega})_{\tau\wedge\textsc{h}}\big]\Big\}, \\ \overline{\mathcal{A}}^L Y(t,\omega) := \Big\{&\varphi \in C^{1,2}_b(\Lambda^t) : \text{ for some } \textsc{h} \in \mathcal{H}^t, \\ &(\varphi - Y^{t,\omega})(t,\mathbf{0}) = 0 = \sup_{\tau\in\mathcal{T}^t} \sup_{\mathbb{P}\in\mathcal{P}^t_L} \mathbb{E}^{\mathbb{P}}\big[(\varphi - Y^{t,\omega})_{\tau\wedge\textsc{h}}\big]\Big\}. \end{aligned}$$

We are now ready to introduce the viscosity solution of PPDEs. Consider the following PPDE with generator G:

$$(2.14)\qquad \mathcal{L}Y_t := -\partial_t Y_t - G(t,\omega, Y_t, \partial_\omega Y_t, \partial^2_{\omega\omega} Y_t) = 0.$$

DEFINITION 2.4.

(i) *Let $L > 0$. We say $Y \in \underline{\mathcal{U}}$ (resp., $\overline{\mathcal{U}}$) is a viscosity L-subsolution (resp., L-supersolution) of PPDE* (2.14) *if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \underline{\mathcal{A}}^L Y(t, \omega)$ (resp., $\varphi \in \overline{\mathcal{A}}^L Y(t, \omega)$),*

$$\mathcal{L}^{t,\omega}\varphi_t := \Big(-\partial_t\varphi - G^{t,\omega}(.,\varphi,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi)\Big)(t,\mathbf{0}) \leq \quad (\textit{resp.}, \geq)\ 0.$$

(ii) *We say $Y \in \underline{\mathcal{U}}$ (resp., $\overline{\mathcal{U}}$) is a viscosity subsolution (resp., supersolution) of PPDE* (2.14) *if Y is viscosity L-subsolution (resp., L-supersolution) of PPDE* (2.14) *for some $L > 0$.*

(iii) *We say $Y \in UC_b(\Lambda)$ is a viscosity solution of PPDE* (2.14) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 2.5. For $0 < L_1 < L_2$, obviously $\mathcal{P}^t_{L_1} \subseteq \mathcal{P}^t_{L_2}$ and $\underline{\mathcal{A}}^{L_2}Y(t,\omega) \subseteq \underline{\mathcal{A}}^{L_1}Y(t,\omega)$. Then one can easily check that a viscosity L_1-subsolution must be a viscosity L_2-subsolution. Consequently, Y is a viscosity subsolution of PPDE (2.14) if and only if there exists an $L \geq 1$ such that for all $\tilde{L} \geq L$, Y is a viscosity $\tilde{L}$-subsolution. However, we require the same L for all (t, ω). A similar statement holds for the viscosity supersolution.

Remark 2.6.

(i) In the Markovian case, namely, $Y(t, \omega) = Y(t, \omega_t)$ and $G = g(t, \omega_t, y, z, \gamma)$, our definition of viscosity solution is stronger than the standard viscosity solution in PDE literature. That is, that Y is a viscosity solution in our sense implies that it is a viscosity solution in the standard sense as in Crandall, Ishii, and Lions [11].

(ii) The state space Λ of PPDEs is not locally compact, and thus the standard arguments by using Ishii's lemma do not work in the path dependent case. The main idea of [16, 17, 18] is to transform the definition to an optimal stopping problem in (2.13), which helps one to obtain the comparison and hence the uniqueness of viscosity solutions.

3. The zero-sum game.

3.1. The admissible controls. Let $\mathbb{U}$ and $\mathbb{V}$ be two Borel measurable spaces equipped with some topology. From now on we shall fix two measurable mappings:

$$\sigma : [0,T] \times \mathbb{U} \times \mathbb{V} \to \mathbb{S}^d_+, b : [0,T] \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^d.$$

We shall always assume the following.

Assumption 3.1. σ and b are bounded by a constant C_0 and uniformly continuous in t, uniformly in u, v.

For $t \in [0, T]$, let $\overline{\mathcal{U}}_t$ (resp., $\overline{\mathcal{V}}_t$) denote the set of $\mathbb{U}$-valued (resp., $\mathbb{V}$-valued), $\mathbb{F}^t$-progressively measurable processes u (resp., v) on Λ^t. Throughout the paper, when $(u, v) \in \overline{\mathcal{U}}_t \times \overline{\mathcal{V}}_t$ is given, for any process φ on Λ^t with appropriate dimension, we denote

$$\widehat{\varphi}_s := \widehat{\varphi}^{t,u,v}_s := \varphi_s \sigma(s, u_s, v_s). \tag{3.1}$$

Define

$$\begin{aligned} \Xi_t &:= \Big\{(u,v) \in \overline{\mathcal{U}}_t \times \overline{\mathcal{V}}_t : (\sigma(\cdot,u,v), \widehat{b}(\cdot,u,v)) \in \overline{\Xi}^{S,t}\Big\}, \\ X^{t,u,v} &:= X^{W,t,\sigma(u,v),\widehat{b}(u,v)}, \quad \mathbb{P}^{t,u,v} := \mathbb{P}^{W,t,\sigma(u,v),\widehat{b}(u,v)} \quad \text{for } (u,v) \in \Xi_t. \end{aligned} \tag{3.2}$$

We note that, from now on, the σ, b in the previous section will actually be $\sigma(t, u_t, v_t)$ and $\widehat{b}(t, u_t, v_t)$ for some $(u, v) \in \Xi_0$. In particular, for the convenience of studying the

BSDE later, we are considering SDE in the form

$$X_s^{t,u,v} = \int_t^s \sigma(r, u_r(X_\cdot^{t,u,v}), v_r(X_\cdot^{t,u,v}))\Big[dB_r^t + b(r, u_r(X_\cdot^{t,u,v}), v_r(X_\cdot^{t,u,v}))dr\Big], \quad \mathbb{P}_0^t\text{-a.s.} \tag{3.3}$$

Moreover, one can easily check that there exists a $\mathbb{P}^{t,u,v}$-Brownian motion $W^{t,u,v}$ such that

$$dB_s^t = \sigma(s, u_s, v_s)\Big[dW_s^{t,u,v} + b(s, u_s, v_s)ds\Big], \quad \mathbb{P}^{t,u,v}\text{-a.s.} \tag{3.4}$$

To formulate the game problem, we shall restrict the controls to subsets $\mathcal{U}_t \subset \overline{\mathcal{U}}_t$ and $\mathcal{V}_t \subset \overline{\mathcal{V}}_t$, whose elements u and v take the following form:

$$u = \sum_{i=0}^{m-1}\sum_{j=1}^{n_i} u_{ij}\mathbf{1}_{E_j^i}\mathbf{1}_{[t_i,t_{i+1})}, \quad v = \sum_{i=0}^{m-1}\sum_{j=1}^{n_i} v_{ij}\mathbf{1}_{E_j^i}\mathbf{1}_{[t_i,t_{i+1})}, \text{ where} \tag{3.5}$$
$$t = t_0 < \cdots < t_m = T, \ \{E_j^i\}_{1\le j\le n_i} \subset \mathcal{F}_{t_i}^t \text{ is a partition, and } u_{ij},\ v_{ij} \text{ are constants.}$$

It is clear that, for $u \in \overline{\mathcal{U}}_t$,

$$u \in \mathcal{U}_t \text{ if and only if } u \text{ takes finitely many values with finitely many jump times.} \tag{3.6}$$

We have the following simple lemma.

LEMMA 3.2.

(i) *$\mathcal{U}_0$ is closed under pasting. That is, for $u \in \mathcal{U}_0$, $t \in [0,T]$, $u^i \in \mathcal{U}_t$, $i = 1,\dots,n$, and disjoint $\{E_i, i = 1,\dots,n\} \subset \mathcal{F}_t$, the following $\overline{u}$ is also in $\mathcal{U}_0$:*

$$\overline{u} := u\mathbf{1}_{[0,t)} + \left[\sum_{i=1}^n u^i(\omega^t)\mathbf{1}_{E_i} + u\mathbf{1}_{\cap_{i=1}^n E_i^c}\right]\mathbf{1}_{[t,T]}.$$

(ii) *Under Assumption* 3.1, *it holds that $\mathcal{U}_t \times \mathcal{V}_t \subset \Xi_t$.*

Proof. In light of (3.6), (i) is obvious. To see (ii), we notice that any pair of constant processes (u, v) is obviously in Ξ_t. In particular, for simplicity we show that the following SDE has a strong solution:

$$X_t = \int_0^t u_t(X_\cdot)dB_t, \ \mathbb{P}_0 \ a.s., \text{ where } u_t(X_\cdot) = \sum_{i=0}^{m-1}\sum_{j=1}^{n_i} u_{ij}\mathbf{1}_{E_j^i(X_\cdot)}\mathbf{1}_{[t_i,t_{i+1})}. \tag{3.7}$$

Let $X_t^0 = u_0 B_t, 0 \le t \le t_1$. Then clearly $\mathcal{F}_t^{X^0} \subseteq \mathcal{F}_t^B$. In particular $\mathbf{1}_{E_j^1(X_{t_1}^0)} \in \mathcal{F}_{t_1}$ for all j. Let

$$X_t^1 = \sum_{j=1}^{n_1} u_{1j}\mathbf{1}_{E_j^1(X_{t_1}^0)}(B_t - B_{t_1}),$$

and continue in the same manner. It is clear that $X_t = \sum_{i=0}^{m-1} X_t^i \mathbf{1}_{[t_i,t_{i+1})}$ is a strong solution to (3.7). ∎

3.2. The BSDEs. Let $f(t,\omega,y,z,u,v):\Lambda\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{U}\times\mathbb{V}\to\mathbb{R}$ be an $\mathbb{F}$-progressively measurable nonlinear generator. Throughout the paper, we shall assume the following.

Assumption 3.3.

(i) $f(t,\omega,0,\mathbf{0},u,v)$ is bounded by a constant C_0, and continuous in (t,ω) with a modulus of continuity function ρ_0, uniformly in (u,v).

(ii) f is Lipschitz in (y,z) with a Lipschitz constant L_0, uniformly in (t,ω,u,v).

Now for any $(t,\omega)\in\Lambda$, $(u,v)\in\mathcal{U}_t\times\mathcal{V}_t$, $\tau\in\mathcal{T}^t$, and $\mathcal{F}^t_\tau$-measurable terminal condition η, recall the notation (3.1) and consider the following BSDE on $[t,\tau]$:

$$\mathcal{Y}_s=\eta+\int_s^\tau f^{t,\omega}(r,B^t_\cdot,\mathcal{Y}_r,\widehat{\mathcal{Z}}_r,u_r,v_r)dr-\int_s^\tau \mathcal{Z}_r dB^t_r,\ \mathbb{P}^{t,u,v}\text{-a.s.} \tag{3.8}$$

We have the following simple lemma whose proof is presented in the appendix for completeness.

Lemma 3.4. *Let Assumptions* 3.1 *and* 3.3(ii) *hold, and*

$$I_0^2(t,\omega,u,v):=\mathbb{E}^{\mathbb{P}^{t,u,v}}\left[|\eta|^2+\int_t^\tau |f^{t,\omega}(s,B^t_\cdot,0,\mathbf{0},u_s,v_s)|^2ds\right]<\infty.$$

Then BSDE (3.8) *has a unique solution, denoted as* $(\mathcal{Y}^{t,\omega,u,v}[\tau,\eta],\mathcal{Z}^{t,\omega,u,v}[\tau,\eta])$, *and there exists a constant C, depending only on C_0, L_0, T, and the dimension d, such that*

$$\mathbb{E}^{\mathbb{P}^{t,u,v}}\left[\sup_{t\le s\le\tau}|\mathcal{Y}^{t,\omega,u,v}_s[\tau,\eta]|^2+\int_t^\tau|\widehat{\mathcal{Z}}^{t,\omega,u,v}_s[\tau,\eta]|^2ds\right]\le CI_0^2(t,\omega,u,v). \tag{3.9}$$

Moreover, if $\tau\le t+\delta$, then

$$\left|\mathcal{Y}^{t,\omega,u,v}_t[\tau,\eta]\right|\le C\Big(\mathbb{E}^{\mathbb{P}^{t,u,v}}[|\eta|^2]\Big)^{\frac12}+C\delta^{\frac12}I_0(t,\omega,u,v). \tag{3.10}$$

Throughout the paper, we shall use the generic constant C which depends only on C_0, L_0, T, and the dimension d, and may vary from line to line.

3.3. The value processes. We now fix an $\mathcal{F}_T$-measurable terminal condition ξ and assume the following throughout the paper.

Assumption 3.5. ξ is bounded by a constant C_0 and is uniformly continuous in ω with a modulus of continuity function ρ_0.

We then define the lower value and upper value of the game as follows:

$$\underline{Y}(t,\omega):=\sup_{u\in\mathcal{U}_t}\inf_{v\in\mathcal{V}_t}\mathcal{Y}^{t,\omega,u,v}_t[T,\xi^{t,\omega}];\overline{Y}(t,\omega):=\inf_{v\in\mathcal{V}_t}\sup_{u\in\mathcal{U}_t}\mathcal{Y}^{t,\omega,u,v}_t[T,\xi^{t,\omega}]. \tag{3.11}$$

As a direct consequence of Lemma 3.4, we have

$$-C\ \le\ \underline{Y}\le\overline{Y}\ \le\ C. \tag{3.12}$$

When there is no confusion, we will simplify the notation:

$$\begin{aligned}\big(\mathcal{Y}^{t,\omega,u,v},\mathcal{Z}^{t,\omega,u,v}\big)&:=\big(\mathcal{Y}^{t,\omega,u,v}[T,\xi^{t,\omega}],\mathcal{Z}^{t,\omega,u,v}[T,\xi^{t,\omega}]\big),\\ (\mathcal{Y}^{u,v},\mathcal{Z}^{u,v})&:=(\mathcal{Y}^{0,\mathbf{0},u,v},\mathcal{Z}^{0,\mathbf{0},u,v}).\end{aligned} \tag{3.13}$$

Our goal of this paper is to show, under certain additional assumptions, that $\underline{Y}=\overline{Y}$ and it is the unique viscosity solution of certain PPDE. See Theorem 5.1 below.

Remark 3.6.

(i) In this paper we restrict our controls to $\mathcal{U}_t \times \mathcal{V}_t \subset \Xi_t$. Therefore, the solution to the SDE (2.5) is actually a strong solution. We note that in general $\overline{\mathcal{U}}_t \times \overline{\mathcal{V}}_t$ is not in Ξ_t. We may, though, study the problem

$$\underline{Y}'(t,\omega) := \sup_{u\in\overline{\mathcal{U}}_t} \inf_{v\in\overline{\mathcal{V}}_t(u)} \mathcal{Y}_t^{t,\omega,u,v}, \overline{Y}'(t,\omega) := \inf_{v\in\overline{\mathcal{V}}_t} \sup_{u\in\overline{\mathcal{U}}_t(v)} \mathcal{Y}_t^{t,\omega,u,v},$$

where

$$\overline{\mathcal{V}}_t(u) := \Big\{v \in \overline{\mathcal{V}}_t : (u,v) \in \Xi_t\Big\}, \quad \overline{\mathcal{U}}_t(v) := \Big\{u \in \overline{\mathcal{U}}_t : (u,v) \in \Xi_t\Big\},$$

and we take the convention that, for the empty set ϕ, $\sup_\phi[\cdot] = -\infty$ and $\inf_\phi[\cdot] = \infty$. However, we will be able to prove only the partial dynamic programming principle in this formulation.

(ii) Another important constraint we impose is that σ and b are independent of ω. When σ and b are random, given $(t,\omega) \in \Lambda$, the solution $X^{t,\omega,u,v}$ of SDE (3.3) and its distribution $\mathbb{P}^{t,\omega,u,v}$ will depend on ω as well. This has some subtle consequences, e.g., in Lemma 4.1 concerning the regularity of the value processes. The main difficulty is that we do not have a good stability result for feedback type SDEs (3.3). We hope to address this issue in future research.

(iii) Note that we may get rid of the drift b by using Girsanov transformation. After some slight modifications, our results hold true when b is random, given that σ is independent of ω. However, to simplify the presentation we assume that b is independent of ω as well.

Remark 3.7. For each $(u,v) \in \mathcal{U}_0 \times \mathcal{V}_0$, denote $\tilde{u}_t := u_t(X^{u,v})$ and $\tilde{v}_t := v_t(X^{u,v})$. Notice that $X^{u,v}$ is a strong solution, and then for $E \in \mathcal{F}_s^t$, the set $(X^{t,u,v})^{-1}(E) = \{\omega : X^{u,v}(\omega) \in E\}$ is also $\mathcal{F}_s^t$ measurable. Hence, $(\tilde{u},\tilde{v}) \in \mathcal{U}_0 \times \mathcal{V}_0$ and $\mathbb{P}^{u,v} = \mathbb{P}^{S,\tilde{u},\tilde{v}} := \mathbb{P}^{S,\sigma(\tilde{u},\tilde{v}),\widehat{b}(\tilde{u},\tilde{v})}$. Thus we have $(\mathcal{Y}_0^{0,\mathbf{0},u,v}, \mathcal{Z}_0^{0,\mathbf{0},u,v}) = (\mathcal{Y}_0^{S,\tilde{u},\tilde{v}}, \mathcal{Z}_0^{S,\tilde{u},\tilde{v}})$, where

$$\begin{aligned} X_t^{u,v} &= \int_0^t \sigma(s,\tilde{u}_s,\tilde{v}_s)\Big[dB_s + b(s,\tilde{u}_s,\tilde{v}_s)ds\Big]; \\ \mathcal{Y}_t^{S,\tilde{u},\tilde{v}} &= \xi(X^{u,v}) + \int_t^T \big[f(s,X_\cdot^{\tilde{u},\tilde{v}},\mathcal{Y}_s^{S,\tilde{u},\tilde{v}},\widehat{\mathcal{Z}}_s^{S,\tilde{u},\tilde{v}},\tilde{u}_s,\tilde{v}_s)ds - \hat{\mathcal{Z}}_s^{S,\tilde{u},\tilde{v}} b(s,\tilde{u}_s,\tilde{v}_s)\big] \\ &\quad - \int_t^T \hat{\mathcal{Z}}_s^{S,\tilde{u},\tilde{v}} dB_s, \quad \mathbb{P}_0\text{-a.s.} \end{aligned}$$

However, we shall emphasize that the mapping from (u,v) to $(\tilde{u},\tilde{v})$ is in pairs, and it does not induce a mapping from u to $\tilde{u}$ (or from v to $\tilde{v}$). Consequently, the game values defined below in strong formulation are different from the $\underline{Y}_0$ and $\overline{Y}_0$ in (3.11):

$$\underline{Y}_0^S := \sup_{\tilde{u}\in\mathcal{U}_0} \inf_{\tilde{v}\in\mathcal{V}_0} \mathcal{Y}_0^{S,\tilde{u},\tilde{v}}, \overline{Y}_0^S := \inf_{\tilde{v}\in\mathcal{V}_0} \sup_{\tilde{u}\in\mathcal{U}_0} \mathcal{Y}_0^{S,\tilde{u},\tilde{v}}.$$

Indeed, in strong formulation the above game with control against control may not have the game value, namely, $\underline{Y}_0^S < \overline{Y}_0^S$, even if the Isaacs condition and the comparison principle for the viscosity solutions of the corresponding Bellman–Isaacs equation hold. See the counterexample Example E.1 below.

Remark 3.8.

(i) In standard literature, see, e.g., [21] and [6], one transforms the problem into a game with strategy controls. That is, let $\alpha : \mathcal{V}_t \to \mathcal{U}_t$ and $\beta : \mathcal{U}_t \to \mathcal{V}_t$ be appropriate strategies. One considers the following:

$$\underline{Y}''(t,\omega) := \sup_\alpha \inf_v \mathcal{Y}_t^{t,\omega,\alpha(v),v}, \overline{Y}''(t,\omega) := \inf_\beta \sup_u \mathcal{Y}_t^{t,\omega,u,\beta(u)}.$$

This type control problem is in fact a principal-agent problem; see, e.g., Cvitanic and Zhang [12]. In the Markovian framework and under appropriate conditions, one can show that $\underline{Y}'' = \overline{Y}''$ and is the unique solution of the corresponding Bellman–Isaacs equation. However, in this formulation the two players have asymmetric informations, and the lower and upper values are defined using different information settings. In particular, it is less convenient to define saddle point in this formulation.

(ii) Our weak formulation actually has the feature of strategy type controls. Indeed, consider the $(\tilde{u}, \tilde{v})$ in Remark 3.7 again. We observe that given u, $X^{u,v}$ is unique strong solution of the following SDE:

$$X_t^{u,v} = \int_0^t \sigma(s, u_s(X_\cdot^{u,v}), \tilde{v}_s(B_\cdot))[dB_s + b(s, u_s(X_\cdot^{u,v}), \tilde{v}_s(B_\cdot))ds].$$

That is, $X^{u,v}$ is uniquely determined by $\tilde{v}$ and B. In particular, for some deterministic mapping ϕ^u,

$$\tilde{u} = u(X^{u,v}) = \phi^u(\tilde{v}, B).$$

Thus u can be viewed as a (random) strategy α which maps $\tilde{v}$ (and B) to $\tilde{u}$. Similarly, v can be viewed as a strategy β which maps $\tilde{u}$ (and B) to $\tilde{v}$. Compared to the strategy against control, the advantage of weak formulation is that it is control against control and the two players have symmetric information.

Remark 3.9. When there is only drift control, namely, σ is independent of (u, v), our formulation reduces to the work Hamadene and Lepeltier [23]. Under the Isaacs condition, by using Girsanov transformation and comparison for BSDEs, they proved $\underline{Y} = \overline{Y}$ and the existence of saddle point. We allow for both diffusion and drift control, and we shall prove $\underline{Y} = \overline{Y}$. However, when there is diffusion control, the comparison used in [23] fails. Consequently, we are not able to follow the arguments in [23] to establish the existence of saddle point. We shall instead obtain some approximate saddle point in section 7 below.

4. Dynamic programming principle.

We start with the regularity of $\underline{Y}$ and $\overline{Y}$ in ω. This property is straightforward in strong formulation. Our proof here relies heavily on our assumption that σ and b are independent of ω. As pointed out in Remark 3.6(ii), the problem is very subtle in the general case, and we hope to address it in some future research.

LEMMA 4.1. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold. Then* $\underline{Y}$ *and* $\overline{Y}$ *are uniformly continuous in* ω *with modulus of continuity function* $C\rho_0$ *for some constant* $C > 0$. *Consequently,* $\underline{Y}$ *and* $\overline{Y}$ *are* $\mathbb{F}$*-adapted.*

Proof. Let $t \in [0,T], \omega, \omega' \in \Omega$. For any $(u,v) \in \mathcal{U}_t \times \mathcal{V}_t$, denote $\Delta\mathcal{Y} := \mathcal{Y}^{t,\omega,u,v} - \mathcal{Y}^{t,\omega',u,v}, \Delta\mathcal{Z} := \mathcal{Z}^{t,\omega,u,v} - \mathcal{Z}^{t,\omega',u,v}$. Then, $\mathbb{P}^{t,u,v}$-a.s.

$$\Delta\mathcal{Y}_s = \xi^{t,\omega}(B^t_\cdot) - \xi^{t,\omega'}(B^t_\cdot) - \int_s^T \Delta\mathcal{Z}_r dB_r + \int_s^T \Big[\alpha_r \Delta\mathcal{Y}_r + \Delta\widehat{\mathcal{Z}}_r \beta_r + [f^{t,\omega}(r, B^t, \mathcal{Y}^{t,\omega,u,v}_r, \widehat{\mathcal{Z}}^{t,\omega,u,v}_r, u_r, v_r) - f^{t,\omega'}(r, B^t, \mathcal{Y}^{t,\omega,u,v}_r, \widehat{\mathcal{Z}}^{t,\omega,u,v}_r, u_r, v_r)]\Big] dr,$$

where α and β are bounded. Applying an analogy of (3.9) on the above BSDE, one obtains

$$|\mathcal{Y}^{t,\omega,u,v}_t - \mathcal{Y}^{t,\omega',u,v}_t| \le C\rho_0(\|\omega - \omega'\|_t). \tag{4.1}$$

Thus

$$|\underline{Y}_t(\omega) - \underline{Y}_t(\omega')| \le \sup_{(u,v)\in\mathcal{U}_t\times\mathcal{V}_t} |\mathcal{Y}^{t,\omega,u,v}_t - \mathcal{Y}^{t,\omega',u,v}_t| \le C\rho_0(\|\omega - \omega'\|_t).$$

Similarly, one can prove the estimate for $\overline{Y}$. ∎

The following dynamical programming principle is important for us.

LEMMA 4.2. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold true. For any* $0 \le s \le t \le T$ *and* $\omega \in \Omega$, *we have*

$$\underline{Y}_s(\omega) = \sup_{u\in\mathcal{U}_s} \inf_{v\in\mathcal{V}_s} \mathcal{Y}^{s,\omega,u,v}_s\big[t, \underline{Y}^{s,\omega}_t\big]; \overline{Y}_s(\omega) = \inf_{v\in\mathcal{V}_s} \sup_{u\in\mathcal{U}_s} \mathcal{Y}^{s,\omega,u,v}_s\big[t, \overline{Y}^{s,\omega}_t\big].$$

To prove this, we need a technical lemma. Its proof is standard but lengthy and is postponed to the appendix in order to not distract from our main arguments.

LEMMA 4.3. *For any* $\varepsilon > 0$ *and* $t \in (0,T)$, *there exist disjoint sets* $\{E_i, i = 1,\dots,n\} \subseteq \mathcal{F}_t$ *such that*

$$\|\omega - \omega'\|_t \le \varepsilon \text{ for all } \omega, \omega' \in E_i, i = 1,\dots,n, \text{ and } \sup_{(u,v)\in\mathcal{U}_0\times\mathcal{V}_0} \mathbb{E}^{\mathbb{P}^{0,u,v}}\big(\cap_{i=1}^n E_i^c\big) \le \varepsilon.$$

Proof of Lemma 4.2. We shall prove the result only for $\underline{Y}$. The proof for $\overline{Y}$ is similar. Without loss of generality, we assume $s = 0$. That is, we shall prove the following: recalling (3.13),

$$\underline{Y}_0 = \sup_{u\in\mathcal{U}_0} \inf_{v\in\mathcal{V}_0} \mathcal{Y}^{u,v}_0[t, \underline{Y}_t]. \tag{4.2}$$

Step 1. We first prove "≥." Fix arbitrary $\varepsilon > 0$ and $u \in \mathcal{U}_0$. Let $\{E_i, i = 1,\dots,n\} \subset \mathcal{F}_t$ be given by Lemma 4.3, and fix an $\omega^i \in E_i$ for each i. For any $\omega \in E_i$, by Lemma 4.1 and (4.1) we have

$$|\underline{Y}_t(\omega) - \underline{Y}_t(\omega^i)| \le C\rho_0(\varepsilon) \text{ and } \sup_{(u,v)\in\mathcal{U}_t\times\mathcal{V}_t} \big|\mathcal{Y}^{t,\omega,u,v}_t - \mathcal{Y}^{t,\omega^i,u,v}_t\big| \le C\rho_0(\varepsilon). \tag{4.3}$$

Let $u^i \in \mathcal{U}_t$ be an ε-optimizer of $\underline{Y}_t(\omega^i)$, that is,

$$\inf_{v\in\mathcal{V}_t} \mathcal{Y}^{t,\omega^i,u^i,v}_t + \varepsilon \ge \underline{Y}_t(\omega^i). \tag{4.4}$$

Denote $\hat{E}_n := \cap_{i=1}^n (E_i)^c$. By Lemma 3.2(i), we define $u^\varepsilon \in \mathcal{U}_0$ by

$$u_s^\varepsilon(\omega) := u_s(\omega)\mathbf{1}_{[0,t)}(s) + \left[\sum_{i=1}^n u_s^i(\omega^t)\mathbf{1}_{E_i}(\omega) + u_s(\omega)\mathbf{1}_{\hat{E}_n}(\omega)\right]\mathbf{1}_{[t,T]}(s). \tag{4.5}$$

Now for any $v \in \mathcal{V}_0$, we have

$$\mathcal{Y}_0^{u^\varepsilon,v} = \mathcal{Y}_0^{u^\varepsilon,v}\big(t, \mathcal{Y}_t^{u^\varepsilon,v}\big) = \mathcal{Y}_0^{u^\varepsilon,v}\left[t, \sum_{i=1}^n \mathcal{Y}_t^{u^\varepsilon,v}\mathbf{1}_{E_i} + \mathcal{Y}_t^{u^\varepsilon,v}\mathbf{1}_{\hat{E}^n}\right].$$

Since solutions of BSDEs can be constructed via Picard iteration, one can easily check that, for any $(u, v) \in \mathcal{U}_0 \times \mathcal{V}_0$,

$$\mathcal{Y}_t^{u,v}(\omega) = \mathcal{Y}_t^{t,\omega,u^{t,\omega},v^{t,\omega}}, \quad \mathbb{P}^{0,u,v}\text{-a.e. } \omega \in \Omega.$$

Then it follows from (4.4) and Lemma 2.2 that, for $\mathbb{P}^{0,u^\varepsilon,v}$-a.e. $\omega \in E_i$,

$$\begin{aligned}\mathcal{Y}_t^{u^\varepsilon,v}(\omega) &= \mathcal{Y}_t^{t,\omega,(u^\varepsilon)^{t,\omega},v^{t,\omega}} = \mathcal{Y}_t^{t,\omega,u^i,v^{t,\omega}} \geq \inf_{v\in\mathcal{V}_t} \mathcal{Y}_t^{t,\omega,u^i,v} \\ &\geq \inf_{v\in\mathcal{V}_t} \mathcal{Y}_t^{t,\omega^i,u^i,v} - C\rho_0(\varepsilon) \geq \underline{Y}_t(\omega^i) - \varepsilon - C\rho_0(\varepsilon) \geq \underline{Y}_t(\omega) - \varepsilon - C\rho_0(\varepsilon).\end{aligned}$$

Therefore, by the comparison principle of BSDEs and (3.12), we have

$$\begin{aligned}\mathcal{Y}_0^{u^\varepsilon,v} &\geq \mathcal{Y}_0^{u^\varepsilon,v}\left[t, \sum_{i=1}^n \underline{Y}_t\mathbf{1}_{E_i} - (\varepsilon + C\rho_0(\varepsilon)) + \mathcal{Y}_t^{u^\varepsilon,v}\mathbf{1}_{\hat{E}^n}\right] \\ &\geq \mathcal{Y}_0^{u^\varepsilon,v}\left[t, \underline{Y}_t - (\varepsilon + C\rho_0(\varepsilon)) - C\mathbf{1}_{\hat{E}^n}\right].\end{aligned}$$

Recall that $\sup_{(u,v)\in\mathcal{U}_0\times\mathcal{V}_0} \mathbb{P}^{0,u,v}\big(\hat{E}_n\big) \leq \varepsilon$. Applying (3.10), we get

$$\mathcal{Y}_0^{u^\varepsilon,v} \geq \mathcal{Y}_0^{u^\varepsilon,v}\big[t, \underline{Y}_t\big] - C(\varepsilon + \rho_0(\varepsilon))^{\frac{1}{2}} = \mathcal{Y}_0^{u,v}\big[t, \underline{Y}_t\big] - C(\varepsilon + \rho_0(\varepsilon))^{\frac{1}{2}}.$$

Since v is arbitrary, this implies that

$$\inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{u^\varepsilon,v} \geq \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{u,v}\big[t, \underline{Y}_t\big] - C(\varepsilon + \rho_0(\varepsilon))^{\frac{1}{2}}.$$

Then

$$\underline{Y}_0 \geq \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{u,v}\big[t, \underline{Y}_t\big] - C(\varepsilon + \rho_0(\varepsilon))^{\frac{1}{2}}.$$

Sending $\varepsilon \to 0$ and by the arbitrariness of $u \in \mathcal{U}_0$, we obtain $\underline{Y}_0 \geq \sup_{u\in\mathcal{U}_0} \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{u,v}$ $[t, \underline{Y}_t]$.

Step 2. We now prove "$\leq$." Fix $\overline{u} \in \mathcal{U}_0$ in the form of (3.5) with u_{ij} being replaced by $\overline{u}_{ij}$. It suffices to prove that

$$\inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{t,\mathbf{0},\overline{u},v} \leq \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{\overline{u},v}[t, \underline{Y}_t].$$

Without loss of generality, assume $t = t_{i_0}$ for some i_0. Since $\underline{Y}_{t_m} = \xi$, it suffices to prove

$$\inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{\overline{u},v}[t_{i+1}, \underline{Y}_{t_{i+1}}] \leq \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{\overline{u},v}[t_i, \underline{Y}_{t_i}] \text{ for all } i. \tag{4.6}$$

We now fix i and recall that $\overline{u}_t = \sum_{j=1}^{n_i} \overline{u}_{ij} \mathbf{1}_{E_j^i}$ for $t \in [t_i, t_{i+1})$. For any $\varepsilon > 0$, let $\{E_k, k = 1, \dots, K\} \subset \mathcal{F}_{t_i}$ be given by Lemma 4.3. Denote $E_{jk}^i := E_j^i \cap E_k$ and fix an $\omega^{jk} \in E_{jk}^i$ for each (j,k). For any $\overline{v} \in \mathcal{V}_0$, as in Step 1 we have

$$\begin{aligned}\mathcal{Y}_0^{\overline{u},\overline{v}}[t_i, \underline{Y}_{t_i}] &= \mathcal{Y}_0^{\overline{u},\overline{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \underline{Y}_{t_i} \mathbf{1}_{E_{jk}^i} + \underline{Y}_{t_i} \mathbf{1}_{\cap_{k=1}^K E_k^c}\right] \\ &\geq \mathcal{Y}_0^{\overline{u},\overline{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \underline{Y}_{t_i}(\omega^{jk}) \mathbf{1}_{E_{jk}^i}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}}.\end{aligned}$$

By Step 1, we see that

$$\underline{Y}_{t_i}(\omega^{jk}) \geq \sup_{u \in \mathcal{U}_{t_i}} \inf_{v \in \mathcal{V}_{t_i}} \mathcal{Y}_{t_i}^{t_i,\omega^{jk},u,v}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega^{jk}}\big] \geq \inf_{v \in \mathcal{V}_{t_i}} \mathcal{Y}_{t_i}^{t_i,\omega^{jk},\overline{u}_{ij},v}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega^{jk}}\big].$$

Here the constant $\overline{u}_{ij}$ denotes the constant process. Then there exists $v^{jk} \in \mathcal{V}_{t_i}$ such that

$$\underline{Y}_{t_i}(\omega^{jk}) \geq \mathcal{Y}_{t_i}^{t_i,\omega^{jk},\overline{u}_{ij},v^{jk}}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega^{jk}}\big] - \varepsilon.$$

Now define

$$\hat{v} := \bar{v}\mathbf{1}_{[0,t_i)} + \left[\sum_{j,k=1}^{n_i,K} v^{jk}(B^{t_i}) \mathbf{1}_{E_{jk}^i} + \bar{v}\mathbf{1}_{\cap_{k=1}^K E_k^c}\right] \mathbf{1}_{[t_i,T]}.$$

By Lemma 3.2, we have $\hat{v} \in \mathcal{V}_0$. Then, noting that $\overline{u}_t^{t_i,\omega} = \overline{u}_{ij}$ for $\omega \in E_{jk}^i$ and $t \in [t_i, t_{i+1})$,

$$\begin{aligned}\mathcal{Y}_0^{\overline{u},\overline{v}}[t_i, \underline{Y}_{t_i}] &\geq \mathcal{Y}_0^{\overline{u},\overline{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \mathcal{Y}_{t_i}^{t_i,\omega^{jk},\overline{u}_{ij},v^{jk}}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega^{jk}}\big] \mathbf{1}_{E_{jk}^i}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &= \mathcal{Y}_0^{\overline{u},\overline{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \mathcal{Y}_{t_i}^{t_i,\omega^{jk},\overline{u}^{t_i,\omega},\hat{v}^{t_i,\omega}}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega^{jk}}\big] \mathbf{1}_{E_{jk}^i}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &\geq \mathcal{Y}_0^{\overline{u},\overline{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \mathcal{Y}_{t_i}^{t_i,\omega,\overline{u}^{t_i,\omega},\hat{v}^{t_i,\omega}}\big[t_{i+1}, \underline{Y}_{t_{i+1}}^{t_i,\omega}\big] \mathbf{1}_{E_{jk}^i}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &= \mathcal{Y}_0^{\overline{u},\hat{v}}\left[t_i, \sum_{j,k=1}^{n_i,K} \mathcal{Y}_{t_i}^{\overline{u},\hat{v}}[t_{i+1}, \underline{Y}_{t_{i+1}}] \mathbf{1}_{E_{jk}^i}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &= \mathcal{Y}_0^{\overline{u},\hat{v}}\left[t_i, \mathcal{Y}_{t_i}^{\overline{u},\hat{v}}[t_{i+1}, \underline{Y}_{t_{i+1}}] \mathbf{1}_{\cap_{k=1}^K E_k^c}(\omega)\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &\geq \mathcal{Y}_0^{\overline{u},\hat{v}}\left[t_i, \mathcal{Y}_{t_i}^{\overline{u},\hat{v}}[t_{i+1}, \underline{Y}_{t_{i+1}}]\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &\geq \mathcal{Y}_0^{\overline{u},\hat{v}}\left[t_{i+1}, \underline{Y}_{t_{i+1}}\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}} \\ &\geq \inf_{v \in \mathcal{V}_0} \mathcal{Y}_0^{\overline{u},v}\left[t_{i+1}, \underline{Y}_{t_{i+1}}\right] - C(\rho_0(\varepsilon) + \varepsilon)^{\frac{1}{2}}.\end{aligned}$$

Send $\varepsilon \to 0$, and by the arbitrariness of $\overline{v} \in \mathcal{V}_0$ we prove (4.6). ∎

Remark 4.4. If we use the strong formulation with control against control as in Remark 3.7, we can only prove the following partial dynamic programming principle:

$$\underline{Y}^S_s(\omega) \le \sup_{u\in\mathcal{U}_s}\inf_{v\in\mathcal{V}_s} \mathcal{Y}^{s,\omega,\mathbb{P}^{S,s,u,v}}\big[t,(\underline{Y}^S_t)^{s,\omega}\big],$$
$$\overline{Y}^S_s(\omega) \ge \inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s} \mathcal{Y}^{s,\omega,\mathbb{P}^{S,s,u,v}}\big[t,(\overline{Y}^S_t)^{s,\omega}\big],$$

where for $\xi \in \mathcal{F}_t$, $\mathcal{Y}^{s,\omega,\mathbb{P}^{S,s,u,v}}_\cdot[t,\xi]$ denotes the solution to the BSDE (3.8) on $[s,t]$ under $\mathbb{P}^{S,s,u,v}$. Clearly, this does not lead to the desired viscosity property. That is why we use weak formulation instead of strong formulation.

We now turn to the regularity of $\underline{Y}$ and $\overline{Y}$ in t, which is required for studying their viscosity property.

LEMMA 4.5. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold. Then, for any* $0 \le t_1 < t_2 \le T$ *and* $\omega \in \Omega$,

$$|\underline{Y}_{t_1}(\omega) - \underline{Y}_{t_2}(\omega)| + |\overline{Y}_{t_1}(\omega) - \overline{Y}_{t_2}(\omega)| \le C\rho_1\Big(d_\infty((t_1,\omega),(t_2,\omega))\Big), \tag{4.7}$$

where ρ_1 *is a modulus of continuity function defined by*

$$\rho_1(\delta) := \rho_0(\delta + \delta^{\frac14}) + \delta^{\frac12} + \delta^{\frac14}. \tag{4.8}$$

Consequently, $\overline{Y}$ *and* $\underline{Y}$ *are* $\mathbb{F}$*-progressively measurable.*

Proof. We shall only prove the regularity of $\underline{Y}$ in t. The estimate for $\overline{Y}$ can be proved similarly. Denote $\delta := d_\infty((t_1,\omega),(t_2,\omega))$.

By Theorem 4.6 and Lemma 4.1, we have

$$\begin{aligned}\Big|\underline{Y}_{t_1}(\omega) - \underline{Y}_{t_2}(\omega)\Big| &= \Big|\sup_{u\in\mathcal{U}_{t_1}}\inf_{v\in\mathcal{V}_{t_1}} \mathcal{Y}^{t_1,\omega,u,v}_{t_1}\big[t_2,\underline{Y}^{t_1,\omega}_{t_2}\big] - \underline{Y}_{t_2}(\omega)\Big|\\ &\le \sup_{u\in\mathcal{U}_{t_1},v\in\mathcal{V}_{t_1}} \Big|\mathcal{Y}^{t_1,\omega,u,v}_{t_1}\big[t_2,\underline{Y}^{t_1,\omega}_{t_2}\big] - \underline{Y}_{t_2}(\omega)\Big|.\end{aligned} \tag{4.9}$$

Denote

$$\mathcal{Y}_t := \mathcal{Y}^{t_1,\omega,u,v}_t\big[t_2,\underline{Y}^{t_1,\omega}_{t_2}\big] - \underline{Y}_{t_2}(\omega), \mathcal{Z}_t := \mathcal{Z}^{t_1,\omega,u,v}_t\big[t_2,\underline{Y}^{t_1,\omega}_{t_2}\big].$$

Then, $\mathbb{P}^{t_1,u,v}$-a.s.

$$\begin{aligned}\mathcal{Y}_t = \underline{Y}^{t_1,\omega}_{t_2} - \underline{Y}_{t_2}(\omega) + \int_t^{t_2} f^{t_1,\omega}(s,B^{t_1},\mathcal{Y}_s + \underline{Y}_{t_2}(\omega),\widehat{\mathcal{Z}}_s,u_s,v_s)ds\\ - \int_t^{t_2} \mathcal{Z}_s dB^{t_1}_s,\ t\in[t_1,t_2].\end{aligned}$$

Recall from (3.12) that $\underline{Y}$ is bounded. Applying (3.10) and Lemma 4.1, we get

$$\begin{aligned}|\mathcal{Y}_{t_1}| &\le C\Big(\mathbb{E}^{\mathbb{P}^{t_1,u,v}}\big[|\underline{Y}^{t_1,\omega}_{t_2} - \underline{Y}_{t_2}(\omega)|^2\big]\Big)^{\frac12} + C\delta^{\frac12}\\ &\le C\Big(\mathbb{E}^{\mathbb{P}^{t_1,u,v}}\big[\rho_0^2(d_\infty((t_2,\omega),(t_2,\omega\otimes_{t_1}B^{t_1})))\big]\Big)^{\frac12} + C\delta^{\frac12}.\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{t_1,u,v}}\big[\rho_0^2(d_\infty((t_2,\omega),(t_2,\omega\otimes_{t_1}B^{t_1})))\big]\\
&\quad\le \mathbb{E}^{\mathbb{P}^{t_1,u,v}}\Big[\rho_0^2(\delta+\|B^{t_1}\|_{t_2})\Big]\\
&\quad\le \rho_0^2(\delta+\delta^{\frac14})+C\mathbb{P}^{t_1,u,v}\Big[\|B^{t_1}\|_{t_2}\ge\delta^{\frac14}\Big]\\
&\quad\le \rho_0^2(\delta+\delta^{\frac14})+C\delta^{-\frac12}\mathbb{E}^{\mathbb{P}^{t_1,u,v}}[\|B^{t_1}\|_{t_2}^2]\\
&\quad\le \rho_0^2(\delta+\delta^{\frac14})+C\delta^{\frac12}.
\end{aligned}$$

Then

$$|\mathcal{Y}_{t_1}|\le C\Big[\rho_0(\delta+\delta^{\frac14})+\delta^{\frac14}+\delta^{\frac12}\Big]=C\rho_1(\delta).$$

Plug this into (4.9), and we complete the proof. ∎

Combining Lemmas 4.2 and 4.5, we have the following dynamic programming principle for stopping times.

THEOREM 4.6. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold true. For any* $(t,\omega)\in\Lambda$ *and* $\tau\in\mathcal{T}^t$, *we have*

$$\underline{Y}_t(\omega)=\sup_{u\in\mathcal{U}_t}\inf_{v\in\mathcal{V}_t}\mathcal{Y}_t^{t,\omega,u,v}\big[\tau,\underline{Y}_\tau^{t,\omega}\big];\overline{Y}_t(\omega)=\inf_{v\in\mathcal{V}_t}\sup_{u\in\mathcal{U}_t}\mathcal{Y}_t^{t,\omega,u,v}\big[\tau,\overline{Y}_\tau^{t,\omega}\big].$$

Proof. Again we shall only prove the result for $\underline{Y}$. Without loss of generality, we assume $t=0$ and recall (3.13). That is, we shall prove

$$\underline{Y}_0=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\big[\tau,\underline{Y}_\tau\big]. \tag{4.10}$$

First, suppose that $\tau\in\mathcal{T}$ takes only two values $0<t_1<t_2\le T$. Then we have

$$\mathcal{Y}_0^{u,v}\big[\tau,\underline{Y}_\tau\big]=\mathcal{Y}_0^{u,v}\big[t_1,\mathcal{Y}_{t_1}^{y,v}[\tau,\underline{Y}_\tau]\big]=\mathcal{Y}_0^{u,v}\Big[t_1,\underline{Y}_{t_1}\mathbf{1}_{\{\tau=t_1\}}+\mathcal{Y}_{t_1}^{u,v}\big[t_2,\underline{Y}_{t_2}\big]\mathbf{1}_{\{\tau>t_1\}}\Big].$$

Following a similar argument to the proof of Lemma 4.2 and noting the fact that $\mathbf{1}_{\{\tau>t_1\}}\in\mathcal{F}_{t_1}$, we have

$$\begin{aligned}
\underline{Y}_0&=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\Big[t_1,\underline{Y}_{t_1}\mathbf{1}_{\{\tau=t_1\}}+\underline{Y}_{t_1}\mathbf{1}_{\{\tau>t_1\}}\Big]\\
&=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\Big[t_1,\underline{Y}_{t_1}\mathbf{1}_{\{\tau=t_1\}}+\sup_{\tilde u\in\mathcal{U}_{t_1}}\inf_{\tilde v\in\mathcal{V}_{t_1}}\mathcal{Y}_{t_1}^{t_1,\omega,\tilde u,\tilde v}\big[t_2,\underline{Y}_{t_2}^{t_1,\omega}\big]\mathbf{1}_{\{\tau>t_1\}}\Big]\\
&=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\Big[t_1,\underline{Y}_{t_1}\mathbf{1}_{\{\tau=t_1\}}+\mathcal{Y}_{t_1}^{t_1,\omega,\tilde u,\tilde v}\big[t_2,\underline{Y}_{t_2}^{t_1,\omega}\big]\mathbf{1}_{\{\tau>t_1\}}\Big]\\
&=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\Big[t_1,\mathcal{Y}_{t_1}^{u,v}\big[\tau,\underline{Y}_\tau\big]\Big]=\sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}\big[\tau,\underline{Y}_\tau\big].
\end{aligned}$$

By induction, the result is true when τ takes finitely many values. In general, there exists a sequence of stopping times $\tau_k\downarrow\tau$ such that τ_k takes only finitely many values and $\tau_k-\tau\le\frac1k$. By Lemma 4.5 and standard BSDE estimates, we see that the dynamic programming principle holds for stopping time τ. ∎

5. Viscosity solution properties. Define

$$\underline{G}(t,\omega,y,z,\gamma) := \sup_{u\in\mathbb{U}}\inf_{v\in\mathbb{V}}\left[\frac{1}{2}\sigma^2(t,u,v):\gamma + b\sigma(t,u,v)z + f(t,\omega,y,z\sigma(t,u,v),u,v)\right]$$
$$\overline{G}(t,\omega,y,z,\gamma) := \inf_{v\in\mathbb{V}}\sup_{u\in\mathbb{U}}\left[\frac{1}{2}\sigma^2(t,u,v):\gamma + b\sigma(t,u,v)z + f(t,\omega,y,z\sigma(t,u,v),u,v)\right], \tag{5.1}$$

and consider the following PPDEs:

$$\underline{\mathcal{L}}Y_t := -\partial_t Y_t - \underline{G}(t,\omega,Y_t,\partial_\omega Y_t,\partial^2_{\omega\omega}Y_t) = 0, \tag{5.2}$$
$$\overline{\mathcal{L}}Y_t := -\partial_t Y_t - \overline{G}(t,\omega,Y_t,\partial_\omega Y_t,\partial^2_{\omega\omega}Y_t) = 0. \tag{5.3}$$

THEOREM 5.1. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold. Then* $\underline{Y}$ *(resp.,* $\overline{Y}$*) is a viscosity solution of PPDE* (5.2) *(resp.,* (5.3)*).*

Proof. We shall only prove that $\underline{Y}$ satisfies viscosity property of the PPDE (5.2) at $(0,\mathbf{0})$. The other statements can be proved similarly.

Step 1. We first prove the viscosity supersolution property. Assume by contradiction that there exists $\varphi\in\overline{\mathcal{A}}^L\underline{Y}(0,\mathbf{0})$ such that by denoting $\varphi_0 := \varphi(0,\mathbf{0})$,

$$c := \partial_t\varphi_0 + \sup_{u\in\mathbb{U}}\inf_{v\in\mathbb{V}}\left\{\frac{1}{2}\sigma^2(0,u,v):\partial^2_{\omega\omega}\varphi_0 + b\sigma(0,u,v)\partial_\omega\varphi_0 + f(0,\mathbf{0},\underline{Y}_0,\partial_\omega\varphi_0\sigma(0,u,v),u,v)\right\} > 0.$$

By Remark 2.5, we can assume that L is large enough, as we will see later. Then there exists $\tilde u\in\mathbb{U}$ such that, for all $v\in\mathbb{V}$

$$\partial_t\varphi_0 + \frac{1}{2}\sigma^2(t,\tilde u,v):\partial^2_{\omega\omega}\varphi_0 + b\sigma(t,\tilde u,v)\partial_\omega\varphi_0 + f(0,\mathbf{0},\underline{Y}_0,\partial_\omega\varphi_0\sigma(0,\tilde u,v),\tilde u,v) \ge \frac{c}{2}. \tag{5.4}$$

Let $\textsc{h}\in\mathcal{H}^t$ be the hitting time corresponding to φ in (2.13). For any $\varepsilon>0$, set

$$\textsc{h}_\varepsilon := \inf\{t : t + |B_t| = \varepsilon\}.$$

By choosing $\varepsilon>0$ small enough, we have $\textsc{h}_\varepsilon \le \textsc{h}$. Since $\varphi\in C^{1,2}(\Lambda)$, there exist a constant $C_\varphi\ge C_0$ and a modulus of continuity function $\rho_\varphi\ge\rho_1$, which may depend on φ, such that

$$|\psi(t,B)\le C_\varphi,\ |\psi(t,B)-\psi_0|\le\rho_\varphi(\varepsilon),\ \text{for } 0\le t\le\textsc{h}_\varepsilon,\ \psi=\varphi,\partial_t\varphi,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi. \tag{5.5}$$

Now set $u := \tilde u\in\mathcal{U}_0$ to be a constant process and let $v\in\mathcal{V}_0$ be arbitrary. Fix $\delta>0$ and denote $\textsc{h}^\delta_\varepsilon := \textsc{h}_\varepsilon\wedge\delta$, and

$$\mathcal{Y} := \mathcal{Y}^{u,v}\big[\textsc{h}^\delta_\varepsilon, \underline{Y}_{\textsc{h}^\delta_\varepsilon}\big],\quad \mathcal{Z} := \mathcal{Z}^{u,v}\big[\textsc{h}^\delta_\varepsilon, \underline{Y}_{\textsc{h}^\delta_\varepsilon}\big],$$
$$\Delta Y_t := \varphi(t,B) - \mathcal{Y}_t,\quad \Delta Z_t := \partial_\omega\varphi(t,B) - \mathcal{Z}_t.$$

Then, applying the functional Itô's formula, we obtain

$$\begin{aligned} d\Delta Y_t &= \Big[\partial_t\varphi + \frac{1}{2}\partial^2_{\omega\omega}\varphi : \sigma^2(t,u_t,v_t) + f(\cdot,\mathcal{Y}_t,\widehat{\mathcal{Z}}_t,u_t,v_t)\Big](t,B)dt + \Delta Z_t dB_t \\ &= \Big[\partial_t\varphi + \frac{1}{2}\partial^2_{\omega\omega}\varphi : \sigma^2(t,u_t,v_t) + f(\cdot,\underline{Y}_0,\partial_\omega\varphi(\cdot)\sigma(t,u_t,v_t),u_t,v_t)\Big](t,B)dt \\ &\quad + \Big[\alpha_t(\mathcal{Y}_t - \underline{Y}_0) + \Delta\widehat{Z}_t\beta_t\Big]dt + \Delta Z_t dB_t, \end{aligned}$$

where $|\alpha|,|\beta| \le L_0$. By (5.4) and (5.5), we have for $0 \le t \le \mathrm{H}^\delta_\varepsilon$

$$d\Delta Y_t \ge \Big[\frac{c}{2} - C_\varphi\rho_\varphi(\varepsilon) - L_0|\mathcal{Y}_t - \underline{Y}_0| + \Delta\widehat{Z}_t\beta_t\Big]dt + \Delta Z_t dB_t. \tag{5.6}$$

Recall (3.4) and define $d\overline{\mathbb{P}} := M_{\mathrm{H}^\delta_\varepsilon} d\mathbb{P}^{u,v}$, where

$$M_t := \exp\left(\int_0^t [b(s,u_s,v_s) + \beta_s]dW^{u,v}_s - \frac{1}{2}\int_0^t |b(s,u_s,v_s)+\beta_s|^2 ds\right).$$

Then $\Delta Z_t dB_t + \Delta\widehat{Z}_t\beta_t dt$ is a $\overline{\mathbb{P}}$-martingale, and thus

$$\Delta Y_0 \le \mathbb{E}^{\overline{\mathbb{P}}}\left[\Delta Y_{\mathrm{H}^\delta_\varepsilon} - \int_0^{\mathrm{H}^\delta_\varepsilon}\Big[\frac{c}{2} - C_\varphi\rho_\varphi(\varepsilon) - L_0|\mathcal{Y}_t - \underline{Y}_0|\Big]dt\right].$$

By choosing L large enough, we see that $\overline{\mathbb{P}} \in \mathcal{P}^0_L$. Then it follows from the definition of $\overline{\mathcal{A}}^L\underline{Y}(0,\mathbf{0})$ that

$$\mathbb{E}^{\overline{\mathbb{P}}}[\Delta Y_{\mathrm{H}^\delta_\varepsilon}] = \mathbb{E}^{\overline{\mathbb{P}}}\Big[\varphi(\mathrm{H}^\delta_\varepsilon, B) - \underline{Y}_{\mathrm{H}^\delta_\varepsilon}\Big] \le 0.$$

Therefore, since b and β are bounded,

$$\begin{aligned} \underline{Y}_0 - \mathcal{Y}_0 &\le \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_0^{\mathrm{H}^\delta_\varepsilon}\Big[-\frac{c}{2} + C_\varphi\rho_\varphi(\varepsilon) + L_0|\mathcal{Y}_t - \underline{Y}_0|\Big]dt\right] \\ &= \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_0^{\delta}\Big[-\frac{c}{2} + C_\varphi\rho_\varphi(\varepsilon) + L_0|\mathcal{Y}_t - \underline{Y}_0|\Big]dt\right] \\ &\quad + \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_{\mathrm{H}^\delta_\varepsilon}^{\delta}\Big[\frac{c}{2} - C_\varphi\rho_\varphi(\varepsilon) - L_0|\mathcal{Y}_t - \underline{Y}_0|\Big]dt\right] \\ &\le \Big[-\frac{c}{2} + C_\varphi\rho_\varphi(\varepsilon)\Big]\delta + \frac{c}{2}\delta\overline{\mathbb{P}}(\mathrm{H}_\varepsilon \le \delta) + L_0\delta\mathbb{E}^{\overline{\mathbb{P}}}\Big[\|\mathcal{Y}_\cdot - \underline{Y}_0\|_{\mathrm{H}^\delta_\varepsilon}\Big] \\ &\le \Big[-\frac{c}{2} + C_\varphi\rho_\varphi(\varepsilon)\Big]\delta + \frac{c}{2}\delta\Big(\mathbb{P}^{u,v}(\mathrm{H}_\varepsilon \le \delta)\Big)^{\frac{1}{2}} + L_0\delta\Big(\mathbb{E}^{\mathbb{P}^{u,v}}\Big[\|\mathcal{Y}_\cdot - \underline{Y}_0\|^2_{\mathrm{H}^\delta_\varepsilon}\Big]\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.7}$$

Note that, for $\delta \le \frac{\varepsilon}{2}$,

$$\begin{aligned} \mathbb{P}^{u,v}\Big(\mathrm{H}_\varepsilon \le \delta\Big) &\le \mathbb{P}^{u,v}\Big(\delta + \|B\|_\delta \ge \varepsilon\Big) \le \mathbb{P}^{u,v}\Big(\|B\|_\delta \ge \frac{\varepsilon}{2}\Big) \\ &\le \frac{C}{\varepsilon^2}\mathbb{E}^{\mathbb{P}^{u,v}}\big[\|B\|^2_\delta\big] \le \frac{C\delta}{\varepsilon^2}. \end{aligned} \tag{5.8}$$

Moreover, denote $\tilde{\mathcal{Y}} := \mathcal{Y} - \underline{Y}_0$. Then

$$\tilde{\mathcal{Y}}_t = \underline{Y}_{\mathrm{H}_\varepsilon^\delta} - \underline{Y}_0 + \int_t^{\mathrm{H}_\varepsilon^\delta} f(s, B, \tilde{\mathcal{Y}}_s + \underline{Y}_0, \widehat{\mathcal{Z}}_s, u_s, v_s)ds - \int_t^{\mathrm{H}_\varepsilon^\delta} \mathcal{Z}_s dB_s.$$

By (3.9) and applying Lemma 4.5, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{u,v}}\Big[\|\tilde{\mathcal{Y}}\|^2_{\mathrm{H}_\varepsilon^\delta}\Big] &\le C\mathbb{E}^{\mathbb{P}^{u,v}}\Big[|\underline{Y}_{\mathrm{H}_\varepsilon^\delta} - \underline{Y}_0|^2\Big] + C\delta \\ &\le C\delta + C\mathbb{E}^{\mathbb{P}^{u,v}}\Big[\rho_1^2\big(\delta + \|B\|_\delta\big)\Big] \le C\rho_2(\delta), \end{aligned} \tag{5.9}$$

where

$$\rho_2(\delta) := \delta + \sup_{(u,v)\in\mathcal{U}_0\times\mathcal{V}_0} \mathbb{E}^{\mathbb{P}^{u,v}}\Big[\rho_1^2\big(\delta + \|B\|_\delta\big)\Big]. \tag{5.10}$$

Plug (5.8) and (5.9) into (5.7), we have

$$\underline{Y}_0 - \mathcal{Y}_0 \le \delta\left[-\frac{c}{2} + C_\varphi\rho_\varphi(\varepsilon) + \frac{C\delta^{\frac{1}{2}}}{\varepsilon} + C\rho_2^{\frac{1}{2}}(\delta)\right].$$

It is clear that $\lim_{\delta\to 0}\rho_2(\delta) = 0$. Then by first choosing ε small and then choosing δ small enough, we have

$$\underline{Y}_0 - \mathcal{Y}_0^{u,v}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \le -\frac{c}{4}\delta.$$

Since v is arbitrary, we get

$$\underline{Y}_0 - \inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \le -\frac{c}{4}\delta,$$

which implies further that

$$\underline{Y}_0 - \sup_{u\in\mathcal{U}_0}\inf_{v\in\mathcal{V}_0}\mathcal{Y}_0^{u,v}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \le -\frac{c}{4}\delta < 0.$$

This contradicts the dynamic programming principle Theorem 4.6. Therefore, $\underline{Y}$ satisfies the viscosity supersolution property of PPDE (5.2) at $(0, \mathbf{0})$.

Step 2. We now prove the viscosity subsolution property. Assume by contradiction that, for some L large enough, there exists $\varphi \in \underline{\mathcal{A}}^L\underline{Y}(0, \mathbf{0})$ such that

$$\begin{aligned} -c := \partial_t\varphi_0 + \sup_{u\in\mathbb{U}}\inf_{v\in\mathbb{V}}\Big\{&\frac{1}{2}\sigma^2(0,u,v) : \partial^2_{\omega\omega}\varphi_0 + b(0,u,v)\partial_\omega\varphi_0 \\ &+ f(0,\mathbf{0},\underline{Y}_0, \partial_\omega\varphi_0\sigma(0,u,v),u,v)\Big\} < 0. \end{aligned}$$

Then there exists a mapping (no measurability is involved!) $\psi : \mathbb{U} \to \mathbb{V}$ such that, for any $u \in \mathbb{U}$,

$$\begin{aligned} &\partial_t\varphi_0 + \frac{1}{2}\sigma^2(0,u,\psi(u)) : \partial^2_{\omega\omega}\varphi_0 + b(0,u,\psi(u))\partial_\omega\varphi_0 \\ &\quad + f(0,\mathbf{0},\underline{Y}_0, \partial_\omega\varphi_0\sigma(0,u,v),u,\psi(u)) \le -\frac{c}{2}. \end{aligned} \tag{5.11}$$

For any $u \in \mathcal{U}_0$, by the structure (3.5) one can easily see that $v := \psi(u) \in \mathcal{V}_0$. Introduce the same notation as in Step 1 and follow almost the same arguments, and we obtain

$$\underline{Y}_0 - \mathcal{Y}_0 \geq \delta \left[\frac{c}{2} - C_\varphi \rho_\varphi(\varepsilon) - \frac{C\delta^{\frac{1}{2}}}{\varepsilon} - C\rho_2^{\frac{1}{2}}(\delta) \right].$$

Again, by first choosing ε small and then choosing δ small enough, we have

$$\underline{Y}_0 - \mathcal{Y}_0^{u,\psi(u)}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \geq \frac{c}{4}\delta.$$

This implies

$$\underline{Y}_0 - \inf_{v \in \mathcal{V}_0} \mathcal{Y}_0^{u,v}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \geq \frac{c}{4}\delta.$$

Since u is arbitrary,

$$\underline{Y}_0 - \sup_{u \in \mathcal{U}_0} \inf_{v \in \mathcal{V}_0} \mathcal{Y}_0^{u,v}[\mathrm{H}_\varepsilon^\delta, \underline{Y}_{\mathrm{H}_\varepsilon^\delta}] \geq \frac{c}{4}\delta > 0.$$

This contradicts the dynamic programming principle Theorem 4.6. Therefore, $\underline{Y}$ satisfies the viscosity subsolution property of PPDE (5.2) at $(0, \mathbf{0})$. ∎

We now assume the Isaacs condition

$$\underline{G}(t, \omega, y, z, \gamma) = \overline{G}(t, \omega, y, z, \gamma) =: G(t, \omega, y, z, \gamma) \tag{5.12}$$

and consider the following path dependent Isaacs equation:

$$\mathcal{L}Y_t := -\partial_t Y_t - G(t, \omega, Y_t, \partial_\omega Y_t, \partial^2_{\omega\omega} Y_t) = 0. \tag{5.13}$$

Our main result of the paper is as follows.

THEOREM 5.2. *Let Assumptions* 3.1, 3.3, *and* 3.5 *hold. Assume further that the Isaacs condition* (5.12) *and, given terminal condition* ξ*, the uniqueness for viscosity solutions of the PPDE* (5.13) *hold. Then* $\underline{Y} = \overline{Y} =: Y$ *and it is the unique viscosity solution of PPDE* (5.13) *with terminal condition* ξ.

Proof. Applying Theorem 5.1 and by the uniqueness of viscosity solutions, we see immediately that $\underline{Y} = \overline{Y}$ and it is the unique viscosity solution of PPDE (5.13) with terminal condition ξ. ∎

Remark 5.1.

(i) For the comparison principle of viscosity solutions of PPDEs, we refer to Ekren, Touzi, and Zhang [19] for general results. We shall also provide a sufficient condition for PPDE (5.13) in section 6 below.

(ii) In the Markovian framework, the PPDE (5.13) becomes a standard PDE. Note that a viscosity solution (resp., supersolution, subsolution) in the sense of Definition 2.4 is a viscosity solution (resp., supersolution, subsolution) in the standard literature. Then, assuming that the comparison principle for standard viscosity solution of PDEs holds true, $Y := \underline{Y} = \overline{Y}$ and it is the unique viscosity solution of the Bellman–Isaacs PDE with terminal condition $Y(T, x) = \xi(x)$.

(iii) If we use the strong formulation, but with strategy against control as in Remark 3.8, we believe the path dependent game will still have a value characterized as the unique viscosity solution of the PPDE. However, as explained in the introduction, we think our weak formulation with control against control is more natural.

6. Comparison principle for viscosity solutions of PPDEs. In this section we study the comparison principle of PPDE (5.13), which clearly implies the uniqueness required in Theorem 5.1. We remark that our technical conditions, in particular (6.1) below, are somewhat strong. It is our great interest to weaken these conditions in our future reseach. Our main result is as follows.

THEOREM 6.1. *Let Assumptions* 3.1, 3.3*, and* 3.5 *and the Isaacs condition* (5.12) *hold. Assume further that*

$$\sigma \geq c_0 I_d \text{ for some } c_0 > 0 \text{ and the dimension } d \leq 2. \tag{6.1}$$

Then comparison principle holds for PPDE (5.13)*. Consequently* $\underline{Y} = \overline{Y} =: Y$ *and it is the unique viscosity solution of PPDE* (5.13).

The proof follows from the analysis in [19] and [18, section 7]. We begin with introducing the path frozen PDEs. For any $(t, \omega) \in \Lambda$, denote the following deterministic function with parameter (t, ω) on $[t, \infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$:

$$g^{t,\omega}(s, y, z, \gamma) := G(s \wedge T, \omega_{\cdot\wedge t}, y, z, \gamma). \tag{6.2}$$

For any $\varepsilon > 0$ and $\eta \geq 0$, we denote $T_\eta := (1 + \eta)T$, $\varepsilon_\eta := (1 + \eta)\varepsilon$, and

$$\begin{aligned} O_\varepsilon &:= \{x \in \mathbb{R}^d : |x| < \varepsilon\}, \quad \overline{O}_\varepsilon := \{x \in \mathbb{R}^d : |x| \leq \varepsilon\}, \\ \partial O_\varepsilon &:= \{x \in \mathbb{R}^d : |x| = \varepsilon\}, \\ Q_t^{\varepsilon,\eta} &:= [t, T_\eta) \times O_{\varepsilon_\eta}, \overline{Q}_t^{\varepsilon,\eta} := [t, T_\eta] \times \overline{O}_{\varepsilon_\eta}, \\ \partial Q_t^{\varepsilon,\eta} &:= \big([t, T_\eta] \times \partial O_{\varepsilon_\eta}\big) \cup \big(\{T_\eta\} \times O_{\varepsilon_\eta}\big). \end{aligned} \tag{6.3}$$

The following localized and path-frozen PDE is crucial: for every $(t, \omega) \in \Lambda$,

$$\mathbf{L}^{t,\omega}\theta := -\partial_t\theta - g^{t,\omega}(s, \theta, D\theta, D^2\theta) = 0. \tag{6.4}$$

Next, for $\varepsilon > 0$, denote

$$\Pi_n^\varepsilon := \Big\{\pi_n = (t_i, x_i)_{1\leq i\leq n} : 0 < t_1 < \cdots < t_n < T \text{ and } |x_i| \leq \varepsilon \text{ for all } i\Big\},$$

and for any $\pi_n \in \Pi_n^\varepsilon$ and $(t, x) \in Q_{t_n}^\varepsilon$, define $\mathrm{H}_i^{t,x,\varepsilon} := \mathrm{H}^{\pi_n,t,x,\varepsilon}$ by

$$\begin{aligned} \mathrm{H}_1^{t,x,\varepsilon} &:= T \wedge \inf\{s \geq t : |B_s^t + x| = \varepsilon\}, \\ \mathrm{H}_{i+1}^{t,x,\varepsilon} &:= T \wedge \inf\big\{s \geq \mathrm{H}_i^{t,x,\varepsilon} : |B_s^t - B_{\mathrm{H}_i^{t,x,\varepsilon}}^t| = \varepsilon\big\}, \ i \geq 1. \end{aligned}$$

LEMMA 6.2. *Assume Assumption* 3.1 *and* $\sigma \geq c_0 I_d$ *for some* $c_0 > 0$*. Then for any* $\varepsilon' < \varepsilon$ *and any* n*, there exists a modulus of continuity function* ρ_i*, depending on* ε' *but independent of* t*, such that*

$$\sup_{u\in\mathcal{U}_t, v\in\mathcal{V}_t} \mathbb{E}^{\mathbb{P}^{t,u,v}}|\mathrm{H}_i^{t,x,\varepsilon} - \mathrm{H}_i^{t,x',\varepsilon}| \leq \rho_i(|x - x'|) \quad \text{for all } x, x' \in O_{\varepsilon'},$$

$$\sup_{u\in\mathcal{U}_t, v\in\mathcal{V}_t} \mathbb{P}^{t,u,v}\Big\{\mathrm{H}_i^{t,x,\varepsilon} < T\Big\} \leq \frac{C}{i\varepsilon^2} \quad \text{for all } x \in O_\varepsilon.$$

Proof. The first estimate follows from the arguments in [15, Appendix B]. Note that

$$\Big\{\mathrm{H}_i^{t,x,\varepsilon} < T\Big\} \subset \left\{\sum_{j=1}^i |B_{\mathrm{H}_j^{t,x,\varepsilon}}^t - B_{\mathrm{H}_{j-1}^{t,x,\varepsilon}}^t|^2 \geq (i-1)\varepsilon^2\right\}.$$

Then the second estimate follows immediately from Assumption 3.1. ∎

Denote by $\hat{B}^{\varepsilon,\pi_n,t,x}(\omega)$ the linear interpolation $(0,\mathbf{0})$, $(t_i, \sum_{j=0}^{i} x_j)_{1\le i\le n}$, and $(\mathrm{H}_i^{t,x,\varepsilon}(\omega), \sum_{j=0}^{n} x_j + x + B^t_{\mathrm{H}_i^{t,x,\varepsilon}}(\omega))_{i\ge 1}$, and define

$$\theta_n^\varepsilon(\pi_n; t, x) := \sup_{u\in\mathcal{U}_t} \inf_{v\in\mathcal{V}_t} \mathcal{Y}_t^{\varepsilon,\pi_n,t,x,u,v}, \tag{6.5}$$

where, denoting $\mathrm{H}_0^{t,x,\varepsilon} := t$ and omitting the superscripts ${}^{\varepsilon,\pi_n,t,x,u,v}$,

$$\mathcal{Y}_s = \xi(\hat{B}) + \int_s^T f\left(r, \sum_{i\ge 0} \hat{B}_{\cdot\wedge \mathrm{H}_i^{t,x,\varepsilon}} \mathbf{1}_{[\mathrm{H}_i^{t,x,\varepsilon}, \mathrm{H}_{i+1}^{t,x,\varepsilon})}(r), \mathcal{Y}_r, \hat{\mathcal{Z}}_r, u_r, v_r\right) dr$$
$$- \int_s^T \mathcal{Z}_r \cdot dB_r, \ \mathbb{P}^{t,u,v}\text{-a.s.}$$

Following the arguments in sections 4 and 5, together with Lemma 6.2, we have the following result which verifies Lemma 6.2 of [19]. The proof is standard but rather lengthy, and we omit it.

LEMMA 6.3. *Let Assumptions* 3.1, 3.3, *and* 3.5 *and the Isaacs condition* (5.12) *hold. Assume further that* $\sigma \ge c_0 I_d$ *for some* $c_0 > 0$. *Then the following hold:*

(i) $|\theta_n^\varepsilon| \le C_1$ *for some constant* C_1, $\theta_n^\varepsilon \in C^0(\hat{\Pi}_n^\varepsilon)$, *where* $\hat{\Pi}_n^\varepsilon := \{(\pi_n, t, x) : \pi_n \in \Pi_n^\varepsilon, (t,x) \times Q_{t_n}^\varepsilon\}$, *and*

$$\begin{aligned} &\lim\nolimits_{t\uparrow T} \theta_n^\varepsilon(\pi_n; t, x) = \xi(\omega^{\pi_n,(T,x)}),\ |x| \le \varepsilon,\\ &\lim\nolimits_{(t',x')\in O_\varepsilon,(t',x')\to(t,x)} \theta_n^\varepsilon(\pi_n; t', x') = \theta_{n+1}^\varepsilon(\pi_n, (t,x),\\ &\qquad t, \mathbf{0}),\ t \in (t_n, T), x \in \partial O_\varepsilon. \end{aligned} \tag{6.6}$$

(ii) $\theta_n^\varepsilon(\pi_n; \cdot)$ *is a viscosity solution of PDE* (6.4) *in* $Q_{t_n}^\varepsilon$.

Remark 6.4. The above lemma is slightly weaker than [19, Lemma 6.2]. In particular, $\theta_n^\varepsilon(\pi_n; \cdot)$ may be discontinuous on $\{t_n\} \times \partial O_\varepsilon$. If ξ satisfies an additional regularity in [19, Assumption 3.4], then by [19, Lemma 6.2] we know θ_n^ε is uniformly continuous in $\hat{\Pi}_n^\varepsilon$.

We next verify Assumption 3.6 of [19]. For that purpose, we first need a PDE result.

LEMMA 6.5. *Assume that all the conditions in Theorem* 6.1 *hold. Let* Q *be a domain such that* $\overline{Q}_t^{\varepsilon,\frac{\eta}{2}} \subset Q \subset \overline{Q} \subset Q_t^{\varepsilon,\eta}$ *and* ∂Q *is smooth enough. Assume* $h \in C^{1+\frac{\alpha}{2},2+\alpha}(\overline{Q})$ *for some* $\alpha > 0$, *and let* $g := g_\delta^{t,\omega}$ *be a smooth molifier of* $g^{t,\omega}$ *such that* $\|g - g^{t,\omega}\|_\infty \le \delta$. *Then the following PDE has a classical solution* $\theta \in C^{1,2}(Q) \cap C^0(\overline{Q})$*:*

$$-\partial_t\theta - g(s, \theta, D\theta, D^2\theta) = 0 \ \text{in } Q, \quad \theta = h \ \text{on } \partial Q. \tag{6.7}$$

Moreover, there exists a constant $C_{\eta,\delta}$, *which may depend on* η, δ, *the bound of* h, *and the parameters in the assumptions of Theorem* 6.1 *(including the* c_0 *in* (6.1)*), but is independent of* (t, x, ω), *such that*

$$|D\theta| + |D^2\theta| \le C_{\eta,\delta}\, on \overline{Q}_t^\varepsilon. \tag{6.8}$$

The case $d = 1$ can be found in [26, Theorem 14.24]. The case $d = 2$ is communicated to us by Lihe Wang. The arguments are standard in the PDE literature, and nevertheless we sketch a proof in the appendix for completeness.

The following result amounts to verifying Assumption 3.6 of [19].

PROPOSITION 6.6. *Assume that all the conditions in Theorem* 6.1 *hold. Let* $\varepsilon > 0, \eta > 0$, $(t, \omega) \in \Lambda$, *and* Q *be as in Lemma* 6.5. *Then for any* $h \in C^0(\overline{Q})$, *we* *have* $\overline{\theta} = \underline{\theta}$ *in* Q*, where*

$$
\begin{aligned}
&\overline{\theta}(s,x) := \inf\Big\{w(s,x) : w \textit{ classical supersolution} \\
&\qquad\qquad \textit{of } (6.7) \textit{ in } Q \textit{ and } w \geq h \textit{ on } \partial Q\Big\}, \\
&\underline{\theta}(s,x) := \sup\Big\{w(s,x) : w \textit{ classical subsolution} \\
&\qquad\qquad \textit{of } (6.7) \textit{ in } Q \textit{ and } w \leq h \textit{ on } \partial Q\Big\}.
\end{aligned} \tag{6.9}
$$

Proof. By [18, Proposition 3.14] we may assume without loss of generality that

$$G(\cdot, y_1, \cdot) - G(\cdot, y_2, \cdot) \leq y_2 - y_1 \text{for any} y_1 \geq y_2. \tag{6.10}$$

For any $\delta > 0$, let h_δ and $g := g_\delta^{t,\omega}$ be a smooth mollifier of h and $g^{t,\omega}$ such that $\|h_\delta - h\|_\infty \leq \delta$, $\|g_\delta^{t,\omega} - g\|_\infty \leq \delta$. Apply Lemma 6.5, the PDE (6.7) has a classical solution $\theta_\delta \in C^{1,2}(Q) \cap C^0(\overline{Q})$. Denote $\overline{\theta}_\delta := \theta_\delta + \delta$. Then clearly $\overline{\theta}_\delta \in C^{1,2}(Q) \cap C^0(\overline{Q})$, $\overline{\theta}_\delta \geq h$ on ∂Q, and, by (6.10)

$$
\begin{aligned}
\mathbf{L}^{t,\omega}\overline{\theta}_\delta &= -\partial_t\theta_\delta - g^{t,\omega}(s, \theta_\delta + \delta, D\theta_\delta, D^2\theta_\delta) \\
&\geq -\partial_t\theta_\delta - g^{t,\omega}(s, \theta_\delta, D\theta_\delta, D^2\theta_\delta) + \delta \\
&= g_\delta^{t,\omega}(s, \theta_\delta, D\theta_\delta, D^2\theta_\delta) - g^{t,\omega}(s, \theta_\delta, D\theta_\delta, D^2\theta_\delta) + \delta \geq 0.
\end{aligned}
$$

Then $\overline{\theta}_\delta$ is a classical supersolution of PDE (6.7), and thus $\overline{\theta} \leq \overline{\theta}_\delta$. Similarly, $\underline{\theta}_\delta := \theta_\delta - \delta \leq \underline{\theta}$. Then $0 \leq \overline{\theta} - \underline{\theta} \leq \overline{\theta}_\delta - \underline{\theta}_\delta = 2\delta$. Since $\delta > 0$ is arbitrary, we conclude that $\overline{\theta} = \underline{\theta}$. ☐

We now sketch a proof for Theorem 6.1, following the arguments in [19, sections 6 and 9.1].

Proof of Theorem 6.1. We first remark that our G is in general not uniformly continuous in t, due to the dependence of σ and b on t. However, it is clear that our assumptions implies Assumption 9.1 of [19]. Let Y^1, Y^2 be a viscosity subsolution and viscosity supersolution of PPDE (2.14), respectively, and $Y_T^1 \leq \xi \leq Y_T^2$. Without loss of generality, we shall prove only $Y_0^1 \leq Y_0^2$ under the additional monotone assumption (6.10). Denote $\textsc{h}_0 := 0$, $\textsc{h}_i := \textsc{h}_i^{0,\mathbf{0},\varepsilon}$ for $i \geq 1$, and let $\hat{\pi}_n$ be the linear interpolation of $(\textsc{h}_i, \omega_{\textsc{h}_i})_{0 \leq i \leq n}$. We proceed in several steps.

Step 1. Let $\eta > 0$, $\lambda \geq 0$, and recall the constant C_1 in Lemma 6.3. Denote

$$h^{\eta,\lambda}(t,x) = C_1 - \left(\frac{t}{\eta} \wedge 1\right)\left[C_1 - \theta_0^\varepsilon\left(t \wedge T, \frac{\varepsilon x}{|x| \wedge \varepsilon}\right)\right] + \lambda, \quad (t,x) \in \overline{Q}_0^{\varepsilon,\eta}.$$

Then, clearly, $h \in C^0(\overline{Q}_0^{\varepsilon,\eta})$, $h(t,x) = \theta_0^\varepsilon(t \wedge T, \frac{\varepsilon x}{|x| \wedge \varepsilon})] + \lambda$ for $t \geq \eta$, and $h(t,x) \geq \theta_0^\varepsilon(t \wedge T, \frac{\varepsilon x}{|x| \wedge \varepsilon})] + \lambda$ for $t < \eta$. Now let Q be a domain such that $\overline{Q}_0^{\varepsilon,\frac{\eta}{2}} \subset Q \subset \overline{Q} \subset Q_0^{\varepsilon,\eta}$ such that ∂Q is smooth. Under our conditions, one can easily see that PDE (6.7) on Q with boundary condition $h^{\eta,\lambda}$ has a unique viscosity solution $\theta_0^{\eta,\lambda}$. One can easily see that $0 \leq \theta_0^{\eta,\lambda} - \theta_0^{\eta,0} \leq C\lambda$ for some constant C. Moreover, since $h^{\eta,0}(t,x) \downarrow \theta_0^\varepsilon(t,x)$ as $\eta \downarrow 0$, by Lemma 6.3 one can show that $|\theta_0^{\eta,0}(0,\mathbf{0}) - \theta_0(0,\mathbf{0})| \leq \rho(\eta)$ for some modulus of continuity function ρ, which is independent of the choice of Q. Now applying Lemma 6.6, we may find $w_0 \in C^{1,2}(Q) \cap C^0(\overline{Q})$ such that

$$w_0(0,\mathbf{0}) < \theta_0^{\eta,0}(0,\mathbf{0}) + \frac{\varepsilon}{2}, \quad \mathbf{L}^{0,\mathbf{0}} w_0 \geq 0 \text{ in } Q, \quad w_0(t,x) \geq h^{\eta,\lambda}(t,x) \text{ on } \partial Q.$$

By choosing $\lambda = \lambda_0$ small enough and then $\eta = \eta_0$ small enough, and by modifying w outside of Q_0^ε, we may have

$$\begin{aligned} &w_0 \in C^{1,2}([0,T]\times\mathbb{R}^d), \quad \mathbf{L}^{0,\mathbf{0}} w_0 \geq 0 \text{ in } Q_0^\varepsilon, \quad |Dw_0|, \\ &|D^2 w_0| \leq C_{\eta_0,\lambda_0,\frac{\varepsilon}{2}} \text{ on } \overline{Q}_0^\varepsilon; w_0(0,\mathbf{0}) \leq \theta_0(0,\mathbf{0}) + \tfrac{\varepsilon}{2}, \\ &w_0(t,x) \geq \theta_0^\varepsilon(t,x) = \theta_1^\varepsilon(t,x;t,\mathbf{0}) \text{ on } \partial Q_0^\varepsilon \backslash (\{0\}\times\partial O_\varepsilon). \end{aligned} \tag{6.11}$$

Here the estimate of $|Dw_0|, |D^2 w_0|$ is due to (6.8), while the δ there is related to λ_0 and $\frac{\varepsilon}{2}$ here. We now define

$$\psi(t,\omega) := w_0(t,\omega_t) + \frac{\varepsilon}{2} + \rho_0(2\varepsilon)(T-t), \quad t\in[0,\mathrm{H}_1].$$

Then, following [19, (6.11)], we have

$$\psi(0,\mathbf{0}) \leq \theta_0(0,\mathbf{0}) + \varepsilon + \rho_0(2\varepsilon)T, \ \mathcal{L}\psi \geq 0, \ 0\leq t<\mathrm{H}_1, \qquad \psi(\mathrm{H}_1,\omega) \geq \theta_1^\varepsilon(\hat{\pi}_1;\mathrm{H}_1,\mathbf{0}). \tag{6.12}$$

Step 2. Let η, λ, δ be small positive numbers which will be decided later. Set $s_i := (1-\delta)^i T$, $i\geq 0$. Since $\overline{O}_\varepsilon$ is compact, there exist a partition $D_1,\dots,D_n$ such that $|y-\tilde{y}| \leq T\delta$ for any $y,\tilde{y}\in D_j$, $j=1,\dots,n$. For each j, fix a point $y_j\in D_j$. Following the arguments in Step 1, there exist λ_1, η_1 small enough such that, for each (i,j), there exists $w_1^{i,j}(s_i,y_j;\cdot)$ satisfying the following: denoting by $\omega^{(s_i,y_j)}$ the linear interpolation of $(0,\mathbf{0}),(s_i,y_j),(T,y_j)$,

$$\begin{aligned} &w_1^{i,j} \in C^{1,2}([s_i,T]\times\mathbb{R}^d), \ \mathbf{L}^{s_i,\omega^{s_i,y_j}} w_1^{i,j} \geq 0 \text{ in } Q_{s_i}^\varepsilon, \\ &|Dw_1^{i,j}|, |D^2 w_1^{i,j}| \leq C_{\eta_1,\lambda_1,\frac{\varepsilon}{4}} \text{ on } \overline{Q}_{s_i}^\varepsilon, \\ &w_1^{i,j}(s_i,y_j;s_i,\mathbf{0}) \leq \theta_1(s_i,y_j;s_i,\mathbf{0}) + \frac{\varepsilon}{4}, \\ &w_1^{i,j}(t,x) \geq \theta_1^\varepsilon(s_i,y_j;t,x) = \theta_2^\varepsilon((s_i,y_j),(t,x);t,\mathbf{0}) \text{ on } \partial Q_{s_i}^\varepsilon\backslash(\{s_i\}\times\partial O_\varepsilon). \end{aligned} \tag{6.13}$$

Denote

$$E_{ij}^1 := \{s_{i+1} < \mathrm{H}_1 \leq s_i\} \cap \{B_{\mathrm{H}_1}\in D_j\} \in \mathcal{F}_{\mathrm{H}_1},$$

and define the following: for $t\in(\mathrm{H}_1,\mathrm{H}_2]$,

$$\psi(t,\omega) := \sum_{i,j}\Big[w_0(\hat{\pi}_1) + w_1^{ij}(s_i+t-\mathrm{H}_1, B_t - B_{\mathrm{H}_1}) - w_1^{ij}(s_i,\mathbf{0}) + \frac{\varepsilon}{2}\Big]\mathbf{1}_{E_{ij}^1} + \rho_0(2\varepsilon)(T-t).$$

Note that the estimate $|Dw_1^{i,j}|, |D^2 w_1^{i,j}| \leq C_{\eta_1}$ in (6.13) implies (9.1) in [19]. Following the arguments in [19, section 9.1], we see that, for δ small enough,

$$\psi \text{ is continuous at } \mathrm{H}_1, \quad \mathcal{L}\psi\geq 0, \ 0\leq t<\mathrm{H}_2, \quad \psi(\hat{\pi}_1;\mathrm{H}_2,\omega^{\mathrm{H}_1}) \geq \theta_2^\varepsilon(\hat{\pi}_2;\mathrm{H}_2,\mathbf{0}). \tag{6.14}$$

Step 3. Repeating the arguments, we may construct ψ over $[0,\mathrm{H}_n]$ for all n. Since $\mathrm{H}_n = T$ when n is large enough, ψ is well defined on $[0,T]$ and it is clear that $\psi(T,\omega) \geq \xi(\omega^\varepsilon)$, where ω^ε is the linear interpolation of $(\mathrm{H}_i,\omega_{\mathrm{H}_i})_{i\geq 0}$. Define $\overline{\psi} := \psi + \rho_0(2\varepsilon)$. We see that $\overline{\psi}(T,\omega)\geq\xi(\omega)$, and by (6.10), we still have $\mathcal{L}\overline{\psi}\geq 0$. By the partial comparison principle Proposition 4.1 of [19], for the viscosity subsolution Y^1 we have

$$Y_0^1 \leq \overline{\psi}(0,\mathbf{0}) \leq \psi(0,\mathbf{0}) + \rho_0(2\varepsilon) \leq \theta_0(0,\mathbf{0}) + \varepsilon + \rho_0(2\varepsilon)(T+1).$$

Similarly we have $Y_0^2 \geq \theta_0(0,\mathbf{0}) - \varepsilon - \rho_0(2\varepsilon)(T+1)$. Then $Y_0^2 - Y_0^1 \leq 2\varepsilon + 2(T+1)\rho_0(2\varepsilon)$. Send $\varepsilon\to 0$, and we obtain $Y_0^1\leq Y_0^2$. ∎

7. Approximate saddle point. In this section we briefly discuss saddle points of the game, assuming the game value exists. In our setting, it is natural to define the following.

DEFINITION 7.1. *We call $(u^*, v^*) \in \mathcal{U}_0 \times \mathcal{V}_0$ a saddle point of the game if*

$$\mathcal{Y}_0^{0,\mathbf{0},u,v^*} \leq \mathcal{Y}_0^{0,\mathbf{0},u^*,v^*} \leq \mathcal{Y}_0^{0,\mathbf{0},u^*,v} \text{ for all } u \in \mathcal{U}_0, v \in \mathcal{V}_0.$$

We remark that, if a saddle point (u^*, v^*) exists, then it is straightforward to check that the game has a value $Y_0 := \mathcal{Y}_0^{0,\mathbf{0},u^*,v^*}$. However, in the setting with diffusion control, the existence of saddle point is a very challenging problem and we shall leave it for future research. In this section we study approximate saddle points only.

DEFINITION 7.2. *For any $\varepsilon > 0$, we call $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_0 \times \mathcal{V}_0$ an ε-saddle point of the game if*

$$\mathcal{Y}_0^{0,\mathbf{0},u,v^\varepsilon} - \varepsilon \leq \mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v^\varepsilon} \leq \mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v} + \varepsilon \text{ for all } u \in \mathcal{U}_0, v \in \mathcal{V}_0.$$

We have the following simple observation.

PROPOSITION 7.3. *Assume the game has a value, and then it has an ε-saddle point $(u^\varepsilon, v^\varepsilon)$ for any $\varepsilon > 0$.*

Proof. Let $\underline{Y}_0 = Y_0 = \overline{Y}_0$ be the game value. Then for any $\varepsilon > 0$, there exist $u^\varepsilon \in \mathcal{U}_0, v^\varepsilon \in \mathcal{V}_0$ such that

$$Y_0 - \frac{\varepsilon}{2} < \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v} \leq Y_0 \leq \sup_{u\in\mathcal{U}_0} \mathcal{Y}_0^{0,\mathbf{0},u,v^\varepsilon} \leq Y_0 + \frac{\varepsilon}{2}.$$

In particular, this implies that

$$\mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v^\varepsilon} \leq \sup_{u\in\mathcal{U}_0} \mathcal{Y}_0^{0,\mathbf{0},u,v^\varepsilon} \leq \inf_{v\in\mathcal{V}_0} \mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v} + \varepsilon \leq \mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v^\varepsilon} + \varepsilon.$$

That is, $(u^\varepsilon, v^\varepsilon)$ is an ε-saddle point. Moreover, we observe that $|\mathcal{Y}_0^{0,\mathbf{0},u^\varepsilon,v^\varepsilon} - Y_0| \leq \varepsilon$. ∎

Appendix A. Proof of Lemma 2.2.

Proof. (i) We first prove the lemma for the case $\overline{\Xi}^S$. Let X be the unique strong solution to SDE (2.5) with coefficients (σ, b), and X^i be the unique strong solution to SDE (2.5) on $[t, T]$ with coefficients (σ^i, b^i).

First, denote

$$\bar{X}_s = X_s \mathbf{1}_{[0,t)}(s) + \left[X_t + \sum_{i=1}^n \mathbf{1}_{E_i}(X) X_s^i(B^t)\right] \mathbf{1}_{[t,T]}(s), \quad 0 \leq s \leq T.$$

One can check straightforwardly that $\bar{X}$ is a strong solution to SDE (2.5) with coefficients $(\bar{\sigma}, \bar{b})$. On the other hand, let $\tilde{X}$ be an arbitrary strong solution to SDE (2.5) with coefficients $(\bar{\sigma}, \bar{b})$. Then both $\bar{X}$ and $\tilde{X}$ satisfy SDE (2.5) on $[0, t]$ with coefficients (σ, b). By the uniqueness assumption of (σ, b), we see that $\bar{X} = \tilde{X}$ on $[0, t]$, $\mathbb{P}_0$-a.s. In particular, this implies $\mathbf{1}_{E_i}(\bar{X}) = \mathbf{1}_{E_i}(\tilde{X})$. Then for $\mathbb{P}_0$-a.e. $\omega \in \Omega$, there exists unique i such that $\mathbf{1}_{E_i}(\bar{X}) = \mathbf{1}_{E_i}(\tilde{X}) = 1$. Thus both $\bar{X}^{t,\omega} - \bar{X}_t(\omega)$ and $\tilde{X}^{t,\omega} - \tilde{X}_t(\omega)$ satisfy SDE (2.5) on $[t, T]$ with coefficients (σ^i, b^i). By the uniqueness assumption of (σ^i, b^i), we see that $\bar{X}^{t,\omega} = \tilde{X}^{t,\omega}$, $\mathbb{P}_0^t$-a.s. This implies that $\bar{X} = \tilde{X}$, $\mathbb{P}_0$-a.s., and, therefore, $(\bar{\sigma}, \bar{b}) \in \overline{\Xi}^S$.

Finally, since $\bar{X} = X$ on $[0,t]$, we have $\mathbb{P}^{\bar{\sigma},\bar{b}} = \mathbb{P}^{\sigma,b}$ on $\mathcal{F}_t$. Moreover, since $\bar{X}^{t,\omega}(B^t) = X_t(\omega) + X^i(B^t)$ whenever $\mathbf{1}_{E_i}(X) = 1$, by the definition of r.c.p.d. we see that $(\mathbb{P}^{\bar{\sigma},\bar{b}})^{t,\omega} = \mathbb{P}^{t,\sigma^i,b^i}$ for $\mathbb{P}^{\sigma,b}$-a.e. $\omega \in E_i$.

(ii) The case $\overline{\Xi}^W$ follows from similar arguments. Indeed, we note that a weak solution to (2.5) is equivalent to a probability measure $\mathbb{P}$ and a $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$ such that

$$dB_t = b_t(B_\cdot)dt + \sigma_t(B_\cdot)dW_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \tag{A.1}$$

Let $(\mathbb{P}, W^{\mathbb{P}})$ be the unique weak solution to the SDE (A.1) corresponding to (σ, b) on $(\Omega, \mathcal{F}_t)$ and let $(\mathbb{P}^i, W^i)$ be the unique weak solution to the SDE (A.1) corresponding to (σ^i, b^i) on $(\Omega^t, \mathcal{F}_T^t)$. Denote

$$\bar{\mathbb{P}} := \mathbb{P} \otimes_t \sum_{i=1}^n \mathbb{P}^i \mathbf{1}_{E_i} \text{ and } \bar{W}(\omega) = W_s^{\mathbb{P}}(\omega)\mathbf{1}_{[0,t)} + \sum_{i=1}^n W_s^i(\omega^t)\mathbf{1}_{E_i}(\omega)\mathbf{1}_{[t,T]}(s).$$

That is, $\mathbb{E}^{\bar{\mathbb{P}}}(\xi) = \mathbb{E}^{\mathbb{P}}\big[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}^i}(\xi^{t,\omega})\big]$ for any $\xi \in \mathcal{F}_T$. Then following the arguments in (i), one can easily check that $(\bar{\mathbb{P}}, \bar{W})$ is the unique weak solution to SDE (A.1) corresponding to $(\bar{\sigma}, \bar{b})$. ∎

Appendix B. Proof of Lemma 3.4.

Proof. Recall the $\mathbb{P}^{t,u,v}$-Brownian motion $W^{t,u,v}$ defined in (3.4). One may rewrite BSDE (3.8) as

$$\mathcal{Y}_s = \eta + \int_s^\tau \Big[f^{t,\omega}(r, B_\cdot^t, \mathcal{Y}_r, \widehat{\mathcal{Z}}_r, u_r, v_r) - \widehat{\mathcal{Z}}_r b(r, u_r, v_r)\Big]dr - \int_s^\tau \widehat{\mathcal{Z}}_r dW_r^{t,u,v}, \ \mathbb{P}^{t,u,v}\text{-a.s.}$$

Then (3.9) follows from standard BSDE arguments. Moreover, note that

$$\mathcal{Y}_s = \eta + \int_s^\tau \Big[f^{t,\omega}(r, B_\cdot^t, 0, \mathbf{0}, u_r, v_r) + \alpha_r \mathcal{Y}_r + \widehat{\mathcal{Z}}_r \beta_r\Big]dr - \int_s^\tau \widehat{\mathcal{Z}}_r dW_r^{t,u,v}, \ \mathbb{P}^{t,u,v}\text{-a.s.},$$

where α, β are bounded. Denote

$$\Gamma_r := \exp\left(\int_t^r \beta_s dW_s^{t,u,v} + \int_t^r [\alpha_r - \frac{1}{2}|\beta_r|^2]dr\right).$$

Then

$$\mathcal{Y}_t = \Gamma_\tau \eta + \int_t^\tau \Gamma_r f^{t,\omega}(r, B_\cdot^t, 0, \mathbf{0}, u_r, v_r)dr - \int_t^\tau [\cdots]dW_r^{t,u,v}, \ \mathbb{P}^{t,u,v}\text{-a.s.}$$

Thus

$$\begin{aligned}
|\mathcal{Y}_t| &= \left|\mathbb{E}^{\mathbb{P}^{t,u,v}}\left[\Gamma_\tau \eta + \int_t^\tau \Gamma_r f^{t,\omega}(r, B_\cdot^t, 0, \mathbf{0}, u_r, v_r)dr\right]\right| \\
&\le \left(\mathbb{E}^{\mathbb{P}^{t,u,v}}[\Gamma_\tau^2]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{P}^{t,u,v}}[|\eta|^2]\right)^{\frac{1}{2}} \\
&\quad + \delta^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{P}^{t,u,v}}[\|\Gamma\|_\tau^2]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{P}^{t,u,v}}\left[\int_t^\tau |f^{t,\omega}(r, B_\cdot^t, 0, \mathbf{0}, u_r, v_r)|^2 dr\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

It is clear that $\mathbb{E}^{\mathbb{P}^{t,u,v}}[\|\Gamma\|_\tau^2] \le C$. Then (3.10) follows immediately. ∎

Appendix C. Proof of Lemma 4.3.

Proof. We introduce the following capacity $\mathcal{C}$:

$$\mathcal{C}(A) := \sup_{(u,v)\in\mathcal{U}_0\times\mathcal{V}_0} \mathbb{P}^{0,u,v}(A) \text{ for all } A \in \mathcal{F}_T. \tag{C.1}$$

In this proof, we abuse the notation a little bit by denoting $B^s_r := B_r - B_s$ for $0 \le s \le r \le t$.

Step 1. We first show that, for any $c, \delta > 0$ and $R > 0$,

$$\mathcal{C}\Big(\|B\|_t > R\Big) \le \frac{C}{R^4} \text{and} \mathcal{C}\left(\sup_{0\le s\le t} \|B^s\|_{(s+\delta)\wedge t} \ge c\right) \le \frac{C\delta}{c^4}. \tag{C.2}$$

Indeed, for any $(u, v) \in \mathcal{U}_0 \times \mathcal{V}_0$ and any $0 \le t_1 < t_2$, since σ and b are bounded, by (3.4) and applying the Burkholder–Davis–Gundy inequality we get

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{0,u,v}}\Big[\|B^{t_1}\|^4_{t_2}\Big] &= \mathbb{E}^{\mathbb{P}^{0,u,v}}\left[\sup_{t_1\le s\le t_2}\left|\int_{t_1}^s \sigma b(r,u_r,v_r)dr + \int_{t_1}^s \sigma(r,u_r,v_r)dW^{u,v})_r\right|^4\right]\\ &\le C\mathbb{E}^{\mathbb{P}^{0,u,v}}\left[\left(\int_{t_1}^{t_2} |\sigma b(r,u_r,v_r)|dr\right)^4 + \left(\int_{t_1}^{t_2} |\sigma(r,u_r,v_r)|^2 d_r\right)^2\right]\\ &\le C(t_2-t_1)^2.\end{aligned} \tag{C.3}$$

Then

$$\mathbb{P}^{0,u,v}\Big(\|B\|_t > R\Big) \le \frac{1}{R^4}\mathbb{E}^{\mathbb{P}^{0,u,v}}\Big[\|B\|^4_t\Big] \le \frac{C}{R^4}.$$

By the definiton of $\mathcal{C}$, this implies the first estimate in (C.2).

Next, let $0 = t_1 < \cdots < t_m = t$ such that $\delta \le \Delta t_i < 2\delta$ for all i. Then

$$\begin{aligned}\sup_{0\le s\le t} \|B^s\|_{(s+\delta)\wedge t} &= \max_{0\le i\le m-1}\ \sup_{t_i\le s\le t_{i+1}}\ \sup_{s\le r\le (s+\delta)\wedge t} |B_r - B_s|\\ &\le \max_{0\le i\le m-1}\ \sup_{t_i\le s\le t_{i+1}}\ \sup_{s\le r\le (s+\delta)\wedge t} \Big[|B_r - B_{t_i}| + |B_s - B_{t_i}|\Big]\\ &\le 2 \max_{0\le i\le m-1} \|B^{t_i}\|_{t_i+3\delta}.\end{aligned}$$

Then, noting that $m \le \frac{T}{\delta}$, by (C.3) we have

$$\begin{aligned}\mathbb{P}^{0,u,v}\left(\sup_{0\le s\le t}\|B^s\|_{(s+\delta)\wedge t} \ge c\right) &\le \frac{1}{c^4}\mathbb{E}^{\mathbb{P}^{0,u,v}}\left[\sup_{0\le s\le t}\|B^s\|^4_{(s+\delta)\wedge t}\right]\\ &\le \frac{C}{c^4}\sum_{i=0}^{m-1}\mathbb{E}^{\mathbb{P}^{0,u,v}}\Big[\|B^{t_i}|^4_{t_i+3\delta}\Big] \le \frac{C}{c^4}m\delta^2 \le \frac{C\delta}{c^4}.\end{aligned}$$

By the definition of $\mathcal{C}$, we obtain the second estimate in (C.2).

Step 2. We now fix $\varepsilon > 0$. For the constant C in (C.2), set

$$c := \frac{\varepsilon}{3}, \quad \delta := \frac{c^4\varepsilon}{2C} \wedge t = \frac{\varepsilon^5}{162C} \wedge t, \quad R := \left(\frac{2C}{\varepsilon}\right)^{\frac{1}{4}}.$$

Let $0 = t_0 < \cdots < t_m = t$ such that $\delta \le \Delta t_i \le 2\delta$, $i = 1, \ldots, m$. Clearly, there exists disjoint sets $\{\tilde{E}_j, 1 \le j \le n\} \subset \mathcal{F}_t$ such that

$$\cup_{j=1}^n \tilde{E}_j = \left\{\max_{0\le i\le m} |B_{t_i}| \le R + c\right\} \text{ and } \max_{0\le i\le m} |\omega_{t_i} - \omega'_{t_i}| \le \frac{\varepsilon}{3} \text{ for all } \omega, \omega' \in \tilde{E}_i.$$

Now set

$$E_j := \tilde{E}_j \cap A, \text{ where } A := \left\{\sup_{0\le s\le t} \|B^s\|_{(s+\delta)\wedge t} \le c\right\} \in \mathcal{F}_t.$$

Then for any $\omega, \omega' \in E_j$,

$$\begin{aligned}
\|\omega - \omega'\|_t &= \max_{0\le i\le m-1} \sup_{t_i\le s\le t_{i+1}} |\omega_s - \omega'_s| \\
&\le \max_{0\le i\le m-1} \sup_{t_i\le s\le t_{i+1}} \Big[|\omega_s - \omega_{t_i}| + |\omega'_s - \omega'_{t_i}| + |\omega_{t_i} - \omega'_{t_i}|\Big] \\
&\le \max_{0\le i\le m-1} \sup_{t_i\le s\le t_{i+1}} \Big[\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\Big] = \varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\cap_{j=1}^n E_j^c &= \Big(\cup_{j=1}^n \tilde{E}_j\Big)^c \cup A^c = \Big\{\max_{0\le i\le m} |B_{t_i}| > R + c\Big\} \cup A^c \\
&\subset \Big(\Big\{\max_{0\le i\le m} |B_{t_i}| > R + c\Big\} \cap A\Big) \cup A^c.
\end{aligned}$$

For each $\omega \in \{\max_{0\le i\le m} |B_{t_i}| > R + c\} \cap A$, we have

$$\|\omega\|_t = \max_{0\le i\le m-1} \sup_{t_i\le s\le t_{i+1}} |\omega_s| \ge \max_{0\le i\le m-1} \sup_{t_i\le s\le t_{i+1}} \Big[|\omega_{t_i}| - |\omega_s - \omega_{t_i}|\Big] > (R + c) - c = R,$$

that is,

$$\Big\{\max_{0\le i\le m} |B_{t_i}| > R + c\Big\} \cap A \subset \Big\{\|B\|_t > R\Big\},$$

and therefore,

$$\cap_{j=1}^n E_j^c \subset \Big\{\|B\|_t > R\Big\} \cup A^c.$$

Now it follows from (C.2) that $\mathcal{C}(\cap_{j=1}^n E_j^c) \le \varepsilon$. ∎

Appendix D. Proof of Lemma 6.5.

Proof. We now sketch a proof in the case $d = 2$. Without loss of generality, we assume $t = 0$. As standard in PDE literature, it suffices to provide a priori estimates. That is, we assume that θ is smooth enough and satisfies PDE (6.7), and we shall provide estimates which depend only on the parameters in our assumptions. By (6.1) and Assumption 3.1, it is clear that the $c_0|\gamma_1 - \gamma_2| \le G(\cdot, \gamma_1) - G(\cdot, \gamma_2) \le \frac{C_0^2}{2}|\gamma_1 - \gamma_2|$ for any $\gamma_2 \le \gamma_1$. Then

$$c_0 I_d \le \partial_\gamma g \le \frac{C_0^2}{2} I_d.$$

We first cite an a priori estimate for fully nonlinear parabolic PDE in the case $d = 2$; see, e.g., [1, Theorem 5]. Let $\theta \in C^3(\overline{Q})$ be a solution to PDE (6.7). Then for any $Q' \subset\subset Q$

$$|\theta|_{2+\alpha,Q'} \le C(1 + |\theta|_{2,Q}),$$

where $\alpha > 0$ depends on c_0, C_0 and C depends on $c_0, C_0, d(Q', \partial Q)$ and the bounds on the first derivatives of g in y, z, γ. For $a > 0$, the norm $|\theta|_{a,Q}$ is the standard Holder norm in parabolic distance; see, e.g., [26, Chapter IV, section 1].

Next, by the standard interpolation inequality of Holder spaces, see, e.g., [26, Proposition 4.2] and [22, Lemma 6.35], the above estimate can be rewritten as

$$|\theta|_{2+\alpha,Q'} \le C\big[1 + \|\theta\|_{L^\infty(Q)} + \|D^2\theta\|_{L^\infty(Q)}\big].$$

Then, by [36, Lemma 1], we conclude that, for any $\beta > 0$,

$$\|D^2\theta\|_{L^\infty(Q)} \le C(1 + |\theta|_{1+\beta,Q}).$$

Now by [26, Theorems 14.12 and 14.14], for any $Q' \subset\subset Q$ we have

$$|\theta|_{1+\beta,Q'} \le C,$$

where C depends on $d(\overline{Q}', \partial Q)$, c_0, the bounds on the first derivatives of g in y, z, γ, the bound of $g(\cdot, 0, \mathbf{0}, \mathbf{0})$ on $\overline{Q}$, and the bound of h on ∂Q. Thus the estimate (6.8) is established. We also note that the above argument implies

$$|\theta|_{2+\alpha,Q'} \le C(1 + \|\theta\|_{L^\infty(Q)}).$$

Finally, this a priori estimate, together with the method of continuity, implies the existence of a classical solution $C^{1,2}(Q) \cap C^0(\overline{Q})$; see, e.g., [7, section 5] for the main arguments in the elliptic case. ∎

Appendix E. Buckdahn's counterexample. As pointed out in Remark 3.7, a game with control against control in strong formulation may not have the game value, even if the Isaacs condition and the comparison principle for the associate Bellman–Isaacs equation hold. The following counterexample is communicated to us by Rainer Buckdahn.

Example E.1. Let $d = 2$, $\mathbb{U} := \{x \in \mathbb{R} : |x| \le 1\}$, $\mathbb{V} := \{x \in \mathbb{R} : |x| \le 2\}$, and $\mathcal{U}$ (resp., $\mathcal{V}$) be the set of $\mathbb{F}$-progressively measurable $\mathbb{U}$-valued (resp., $\mathbb{V}$-valued) processes. Write $B = (B^1, B^2)$. Given $(u, v) \in \mathcal{U} \times \mathcal{V}$, the controlled state process $X^{u,v} = (X^{1,u}, X^{2,v})$ is determined by

$$X^{1,u}_t := \alpha B^1_t + \int_0^t u_s ds, X^{2,v}_t := \alpha B^2_t + \int_0^t v_s ds,$$

where $\alpha \ge 0$ is a constant. Define, for some $a \in \mathbb{R}$,

$$J(u, v) := \mathbb{E}^{\mathbb{P}_0}\Big[|a + X^{1,u}_T - X^{2,v}_T|\Big], \quad \underline{Y}_0 := \sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}} J(u, v), \quad \overline{Y}_0 := \inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} J(u, v).$$

Then, for $0 \le \alpha < \sqrt{\frac{T}{2}}$ and $|a| \le T$, we have $\underline{Y}_0 < \overline{Y}_0$.

Proof. For any $u \in \mathcal{U}$, set $v_t := u_t + \frac{a}{T}$. Then $v \in \mathcal{V}$ and

$$a + X^{1,u}_T - X^{2,v}_T = a + \alpha B^1_T + \int_0^T u_t dt - \alpha B^2_T - \int_0^T [u_t + \frac{a}{T}] dt = \alpha[B^1_T - B^2_T].$$

Thus

$$J(u,v) = \alpha\mathbb{E}^{\mathbb{P}_0}\Big[|B^1_T - B^2_T|\Big] \le \alpha\sqrt{2T}.$$

This implies that $\inf_{v\in\mathcal{V}} J(u,v) \le \alpha\sqrt{2T}$. Since u is arbitrary, we get

$$\underline{Y}_0 \le \alpha\sqrt{2T}. \tag{E.1}$$

On the other hand, for any $v \in \mathcal{V}$, set

$$u_t := u_0 := \frac{a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]}{|a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]|}\mathbf{1}_{\{a-\mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]\neq 0\}} + \mathbf{1}_{\{a-\mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]=0\}}. \tag{E.2}$$

That is, u is a constant process. One can easily check that

$$u \in \mathcal{U}, \quad |u_0| = 1, \quad a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T] = u_0|a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]|.$$

Then

$$\mathbb{E}^{\mathbb{P}_0}\Big[a + X^{1,u}_T - X^{2,v}_T\Big] = a + u_0T - \mathbb{E}^{\mathbb{P}_0}[X^{2,v}_T] = u_0\Big[T + |a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]|\Big].$$

Thus,

$$\begin{aligned} J(u,v) &\ge \Big|\mathbb{E}^{\mathbb{P}_0}\Big[a + X^{1,u}_T - X^{2,v}_T\Big]\Big| = |u_0|\Big[T + |a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]|\Big] \\ &= T + |a - \mathbb{E}^{\mathbb{P}^0}[X^{2,v}_T]| \ge T. \end{aligned}$$

This implies $\sup_{u\in\mathcal{U}} J(u,v) \ge T$. Since v is arbitrary, we have $\overline{Y}_0 \ge T$. This, together with E.1, implies that $\underline{Y}_0 < \overline{Y}_0$ when $0 \le \alpha < \sqrt{\frac{T}{2}}$.

Moreover, note that in this case the system is Markovian and $\partial_\omega Y = DY$. The Hamiltonians in (5.1) become the following: for $(t,x,y,z,\gamma) \in [0,T]\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{S}^2$,

$$\begin{aligned} \underline{G}(t,x,y,z,\gamma) &:= \sup_{u\in\mathbb{U}}\inf_{v\in\mathbb{V}}\left[\frac{1}{2}\alpha\mathrm{tr}\,(\gamma) + uz_1 + vz_2\right] = \frac{1}{2}\alpha\mathrm{tr}\,(\gamma) + |z_1| - 2|z_2|, \\ \overline{G}(t,x,y,z,\gamma) &:= \inf_{v\in\mathbb{V}}\sup_{u\in\mathbb{U}}\left[\frac{1}{2}\alpha\mathrm{tr}\,(\gamma) + uz_1 + vz_2\right] = \frac{1}{2}\alpha\mathrm{tr}\,(\gamma) + |z_1| - 2|z_2|. \end{aligned}$$

Then the Isaacs condition holds, and the corresponding Bellman–Isaacs equation becomes

$$-\partial_t Y_t - \frac{1}{2}\alpha\Big[D^2_{x_1x_1}Y_t + D^2_{x_2x_2}Y_t\Big] - |D_{x_1}Y_t| + 2|D_{x_2}Y_t| = 0.$$

It is clear that the comparison principle for the viscosity solutions of above PDE holds.

Remark E.2.

(i) The above counterexample stays valid when $\alpha = 0$, and thus the game is deterministic. We note that, even in deterministic case, our weak formulation is different from strong formulation. Indeed, the corresponding state process $X^{W,u,v}$ in weak formulation is

$$\begin{aligned} X^{W,1,u,v}_t &= \int_0^t u(s, X^{W,1,u,v}_\cdot, X^{W,2,u,v}_\cdot)ds, \\ X^{W,2,u,v}_t &= \int_0^t v(s, X^{W,1,u,v}_\cdot, X^{W,2,u,v}_\cdot)ds. \end{aligned}$$

In particular, $X^{W,2,u,v}$ depends on u as well. Consequently, given v, one cannot define u through (E.2).

(ii) In this paper the drift coefficient is $b\sigma$, see (3.3), so the above deterministic example is not covered in our current framework. However, this assumption is mainly to ensure the wellposedness of the BSDE (3.8). When $f = 0$, one may define the value processes via conditional expectations, instead of $\mathcal{Y}$. Then we may consider X in the form of (1.2) and all our results, after appropriate modifications, will still hold true. In particular, the above deterministic game in weak formulation has a value.

(iii) We shall remark, though, that when $\alpha = 0$, the above deterministic game corresponds to a first order PPDE which violates the condition (6.1), and thus we cannot apply Theorem 6.1 to obtain the uniqueness. However, for first order PPDEs the uniqueness can be proved following the compactness arguments in Lukoyanov [27]; see also [18, section 8].

(iv) We note again that, even assuming further Markovian structure, to the best of our knowledge, our existence of game value under control against control is new in the literature.

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