

## Monotone schemes for fully nonlinear parabolic path dependent PDEs

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### Abstract

In this paper, we extend the results of the seminal work Barles and Souganidis (1991) to path dependent case. Based on the viscosity theory of path dependent PDEs, developed by Ekren *et al.* (2012a, 2012b, 2014a and 2014b), we show that a monotone scheme converges to the unique viscosity solution of the (fully nonlinear) parabolic path dependent PDE. An example of such monotone scheme is proposed. Moreover, in the case that the solution is smooth enough, we obtain the rate of convergence of our scheme.

*Keywords:* Monotone scheme; path dependent PDE; viscosity solution; rate of convergence.

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### 1. Introduction

In this paper, we aim to numerically solve the following fully nonlinear path dependent PDE (PPDE) with terminal condition  $u(T, \omega) = g(\omega)$ :

$$\mathcal{L}u(t, \omega) := -\partial_t u(t, \omega) - G(t, \omega, u, \partial_\omega u, \partial_{\omega\omega}^2 u) = 0, \quad 0 \leq t < T. \quad (1)$$

Here  $\omega$  is a continuous path on  $[0, T]$ , and  $G$  is increasing in  $\partial_{\omega\omega}^2 u$  and thus the PPDE is parabolic. Such PPDE provides a convenient tool for non-Markovian models, especially in stochastic control/game with diffusion control and financial models with volatility uncertainty. Its typical examples include: martingales as path dependent heat equations, Backward SDEs of [Pardoux and Peng \(1990\)](#) as

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semilinear PPDEs, and  $G$ -martingales of Peng (2007) and second-order backward SDEs of Soner *et al.* (2012) as path dependent HJB equations. The notion of PPDE was proposed by Peng (2010). Based on the functional Itô calculus, initiated by Dupire (2009) and further developed by Cont and Fournie (2013), Ekren *et al.* (2012a, 2012b, 2014a and 2014b) developed a viscosity theory for PPDEs.

In the Markovian case, namely  $u(t, \omega) = v(t, \omega_t)$ ,  $g(\omega) = f(\omega_T)$ , and  $G(t, \omega, y, z, \gamma) = F(t, \omega_t, y, z, \gamma)$  for some deterministic functions  $v, f, F$ , the PPDE (1) becomes a standard PDE with terminal condition  $v(T, x) = f(x)$ :

$$\mathbb{L}v(t, x) := -\partial_t v(t, x) - F(t, x, v, Dv, D^2v) = 0, \quad 0 \leq t < T. \quad (2)$$

In their seminal work, Barles and Souganidis (1991) proposed some time discretization scheme for the above PDE and showed that, under certain conditions, the discretized approximation converges to the unique viscosity solution of the PDE. Their key assumption is the monotonicity of the scheme, see Theorem 2.7 (ii) below, which can roughly be viewed as the comparison principle for the discretized PDE. This work has been extended by many authors, either by improving the error analysis including the rate of convergence, or by proposing specific algorithms which indeed satisfy the required conditions, see e.g. (Barles and Jakobsen, 2007; Bonnans and Zidani, 2003; Fahim *et al.*, 2011; Guo *et al.*, 2013; Krylov, 1998; Tan, 2013a,b), to mention a few.

Our goal of this paper is to extend the work (Barles and Souganidis, 1991) to PPDE (1). Notice that the viscosity solution in Ekren *et al.*, (2012a, 2012b, 2014a and 2014b) is defined through some optimal stopping problem under nonlinear expectation, which is different from the standard viscosity theory for PDEs. Consequently, our notion of monotonicity for the scheme also involves the nonlinear expectation, see (15) below. This requires some technical estimates for the hitting time involved in the theory. Then, following the arguments in Barles and Souganidis (1991) we show that our monotone scheme converges to the unique viscosity solution of the PPDE.

We next propose a specific scheme which satisfies all the conditions and thus indeed converges. Moreover, when the PPDE has smooth enough classical solution, we obtain the rate of convergence of our scheme.

In the semilinear case, there have been many works on numerical methods for the associated backward SDEs, see e.g., (Briand *et al.*, 2001; Briand and Labart, 2014; Henry-Labordre *et al.*, 2013; Hu *et al.*, 2011; Ma *et al.*, 2002; Peng and Xu, 2011; Zhang, 2004). In particular, Henry-Labordre *et al.* (2013) used the arguments for viscosity theory of PPDEs. Moreover, Tan (2013b) studied certain numerical approximation for path dependent HJB equations, in the language of second-order BSDEs. However, we should point out that most of these works are mainly theoretical studies and are not feasible, especially in high dimensions.

Efficient numerical algorithms for PPDEs, including the implementation of our discretization scheme in the present paper, remains a challenging problem and we shall explore further in our future research.

The rest of the paper is organized as follows. In Sec. 2, we introduce path dependent PDEs and its viscosity solutions, as well as monotone schemes for (standard) PDEs. In Sec. 3, we prove the main theorem, namely the convergence of monotone schemes. In Sec. 4, we propose a scheme which satisfies all the desired conditions. Finally in Sec. 5, we obtain the rate of convergence of our scheme in the case that the solution is smooth enough.

## 2. Preliminaries

### 2.1. Path dependent PDEs and viscosity solutions

In this section, we recall the setup and the notations of Ekren *et al.* (2012a, 2012b and 2014b).

#### 2.1.1. The canonical setting

Let  $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = \mathbf{0}\}$ , the set of continuous paths starting from the origin,  $B$  the canonical process,  $\mathbb{F}$  the natural filtration generated by  $B$ ,  $\mathbb{P}_0$  the Wiener measure, and  $\Lambda := [0, T] \times \Omega$ . Here and in the sequel, for notational simplicity, we use  $\mathbf{0}$  to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. Let  $\mathbb{S}^d$  denote the set of  $d \times d$  symmetric matrices, and

$$x \cdot x' := \sum_{i=1}^d x_i x'_i \quad \text{for any } x, x' \in \mathbb{R}^d,$$

$$\gamma : \gamma' := \text{Trace}[\gamma\gamma'] \quad \text{for any } \gamma, \gamma' \in \mathbb{S}^d.$$

We define a semi-norm on  $\Omega$  and a pseudometric on  $\Lambda$  as follows: for any  $(t, \omega), (t', \omega') \in \Lambda$ ,

$$\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|, \quad \mathbf{d}((t, \omega), (t', \omega')) := \sqrt{|t - t'|} + \|\omega_{\cdot, \Lambda t} - \omega'_{\cdot, \Lambda t'}\|_T. \quad (3)$$

Then  $(\Omega, \|\cdot\|_T)$  is a Banach space and  $(\Lambda, \mathbf{d})$  is a complete pseudometric space.

**Remark 2.1.** In (Ekren *et al.*, 2012a, 2012b and 2014b), following (Dupire, 2009) we used pseudometric:

$$\mathbf{d}_\infty((t, \omega), (t', \omega')) := |t - t'| + \|\omega_{\cdot, \Lambda t} - \omega'_{\cdot, \Lambda t'}\|_T.$$

Clearly  $\mathbf{d}$  and  $\mathbf{d}_\infty$  induce the same topology, and all the results in (Ekren *et al.*, 2012a, 2012b and 2014b) still hold true under  $\mathbf{d}$ . However, when we consider the regularity of viscosity solutions, see (46) below, it is more natural to use  $\mathbf{d}$ . Indeed, since  $B$  is typically a semimartingale, for  $t < t'$  we see that  $\sqrt{t' - t}$  and  $\|B'\|_{t'}$  are roughly in the same order.

We shall denote by  $\mathbb{L}^0(\mathcal{F}_T)$  and  $\mathbb{L}^0(\Lambda)$  the collection of all  $\mathcal{F}_T$ -measurable random variables and  $\mathbb{F}$ -progressively measurable processes, respectively. In particular, for any  $u \in \mathbb{L}^2(\Lambda)$ , the progressive measurability implies that  $u(t, \omega) = u(t, \omega_{\cdot \wedge t})$ . Let  $C^0(\Lambda)$  (resp.  $UC(\Lambda)$ ) be the subset of  $\mathbb{L}^0(\Lambda)$  whose elements are continuous (resp. uniformly continuous) in  $(t, \omega)$  under  $\mathbf{d}$ . The corresponding subsets of bounded processes are denoted as  $C_b^0(\Lambda)$  and  $UC_b(\Lambda)$ . Finally,  $\mathbb{L}^0(\Lambda, \mathbb{R}^d)$  denote the space of  $\mathbb{R}^d$ -valued processes with entries in  $\mathbb{L}^0(\Lambda)$ , and we define similar notations for the spaces  $C^0, C_b^0, UC$ , and  $UC_b$ .

We denote by  $\mathcal{T}$  the set of  $\mathbb{F}$ -stopping times, and  $\mathcal{H} \subset \mathcal{T}$  the subset of those hitting times  $H$  of the form

$$H := \inf\{t : B_t \notin O\} \wedge t_0 = \inf\{t : d(\omega_t, O^c) = 0\} \wedge t_0, \tag{4}$$

for some  $0 < t_0 \leq T$ , and some open and convex set  $O \subset \mathbb{R}^d$  containing  $\mathbf{0}$ .

For all  $L > 0$ , let  $\mathcal{P}_L$  denote the set of semimartingale measures  $\mathbb{P}$  on  $\Omega$  whose drift and diffusion characteristics are bounded by  $L$  and  $\sqrt{2L}$ , respectively, and  $\mathcal{P}_\infty := \bigcup_{L>0} \mathcal{P}_L$ . To be precise, let  $\tilde{\Omega} := \Omega^2$  be an enlarged canonical space with canonical process  $(B, W)$ . For any  $\mathbb{P} \in \mathcal{P}_L$ , there exist an extension of probability measure  $\tilde{\mathbb{P}}$  on  $\tilde{\Omega}$  and  $\alpha^\mathbb{P} \in \mathbb{L}^0([0, T] \times \tilde{\Omega}, \mathbb{R}^d), \mathbf{0} \leq \beta^\mathbb{P} \in \mathbb{L}^0([0, T] \times \tilde{\Omega}, \mathbb{S}^d)$  such that  $W$  is a  $\tilde{\mathbb{P}}$ -Brownian motion and

$$|\alpha^\mathbb{P}| \leq L, \quad |\beta^\mathbb{P}| \leq \sqrt{2L}, \quad dB_t = \alpha_t^\mathbb{P} dt + \beta_t^\mathbb{P} dW_t, \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{5}$$

We remark that, when  $\alpha^\mathbb{P}$  and  $\beta^\mathbb{P}$  are deterministic, especially when they are constants,  $\mathbb{P}$  is uniquely determined by them.

**Definition 2.2.** We say  $u \in C^{1,2}(\Lambda)$  if  $u \in C^0(\Lambda)$  and there exist  $\partial_t u \in C^0(\Lambda), \partial_\omega u \in C^0(\Lambda, \mathbb{R}^d), \partial_{\omega\omega}^2 u \in C^0(\Lambda, \mathbb{S}^d)$  such that, for any  $\mathbb{P} \in \mathcal{P}_\infty, u$  is a local  $\mathbb{P}$ -semimartingale and it holds:

$$du = \partial_t u dt + \partial_\omega u \cdot dB_t + \frac{1}{2} \partial_{\omega\omega}^2 u : d\langle B \rangle_t, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \tag{6}$$

The above  $\partial_t u, \partial_\omega u$  and  $\partial_{\omega\omega}^2 u$ , if they exist, are unique. Consequently, we call them the time derivative, the first order and second order space derivatives of  $u$ , respectively.

**Definition 2.3.** We say  $u \in C^{1,2}(\Lambda)$  is a classical solution (resp. supersolution, subsolution) of PPDE (1) if  $\mathcal{L}u(t, \omega) = (\text{resp. } \geq, \leq) 0$ , for all  $(t, \omega) \in [0, T] \times \Omega$ .

### 2.1.2. The shifted spaces

Fix  $0 \leq t \leq T$ .

- Let  $\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d) : \omega_t = \mathbf{0}\}$  be the shifted canonical space;  $B^t$  the shifted canonical process on  $\Omega^t$ ;  $\mathbb{F}^t$  the shifted filtration generated by  $B^t$ ,  $\mathbb{P}_0^t$  the Wiener measure on  $\Omega^t$ , and  $\Lambda^t := [t, T] \times \Omega^t$ .
- For  $s \in [t, T]$ , define  $\|\cdot\|_s$  on  $\Omega^t$  and  $\mathbf{d}$  on  $\Lambda^t$  in the spirit of (3), and the sets  $\mathbb{L}^0(\Lambda^t)$  etc. in an obvious way.
- For  $s \in [0, t]$ ,  $\omega \in \Omega^s$  and  $\omega' \in \Omega^t$ , define the concatenation path  $\omega \otimes_t \omega' \in \Omega^s$  by:

$$(\omega \otimes_t \omega')(r) := \omega_r \mathbf{1}_{[s, t]}(r) + (\omega_t + \omega'_r) \mathbf{1}_{[t, T]}(r), \quad \text{for all } r \in [s, T].$$

- Let  $s \in [0, T]$ ,  $\xi \in \mathbb{L}^0(\mathcal{F}_T^s)$ , and  $X \in \mathbb{L}^0(\Lambda^s)$ . For  $(t, \omega) \in \Lambda^s$ , define  $\xi^{t, \omega} \in \mathbb{L}^0(\mathcal{F}_T^t)$  and  $X^{t, \omega} \in \mathbb{L}^0(\Lambda^t)$  by:

$$\xi^{t, \omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t, \omega}(\omega') := X(\omega \otimes_t \omega'), \quad \text{for all } \omega' \in \Omega^t.$$

Moreover, for a random time  $\tau$ , we shall use the notation  $\xi^{\tau, \omega} := \xi^{\tau(\omega), \omega}$ .

- Define  $\mathcal{T}^t, \mathcal{H}^t, \mathcal{P}_L^t, \mathcal{P}_\infty^t$ , and  $C^{1,2}(\Lambda^t)$ , etc. in an obvious manner.

It is clear that  $u^{t, \omega} \in C^0(\Lambda^t)$  for any  $u \in C^0(\Lambda)$  and  $(t, \omega) \in \Lambda$ . Similar property holds for other spaces introduced above. Moreover, for any  $\tau \in \mathcal{T}$  (resp.  $H \in \mathcal{H}$ ) and any  $(t, \omega) \in \Lambda$  such that  $t < \tau(\omega)$  (resp.  $t < H(\omega)$ ), it is clear that  $\tau^{t, \omega} \in \mathcal{T}^t$  (resp.  $H^{t, \omega} \in \mathcal{H}^t$ ).

### 2.1.3. Viscosity solutions of PPDEs

We first introduce the spaces for viscosity solutions.

**Definition 2.4.** Let  $u \in \mathbb{L}^0(\Lambda)$ .

- We say  $u$  is right continuous in  $(t, \omega)$  under  $\mathbf{d}$  if: for any  $(t, \omega) \in \Lambda$  and any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for any  $(s, \tilde{\omega}) \in \Lambda^t$  satisfying  $\mathbf{d}((s, \tilde{\omega}), (t, \mathbf{0})) \leq \delta$ , we have  $|u^{t, \omega}(s, \tilde{\omega}) - u(t, \omega)| \leq \varepsilon$ .
- We say  $u \in \underline{\mathcal{U}}$  if  $u$  is bounded from above, right continuous in  $(t, \omega)$  under  $\mathbf{d}$ , and there exists a modulus of continuity function  $\rho$  such that for any  $(t, \omega), (t', \omega') \in \Lambda$ :

$$u(t, \omega) - u(t', \omega') \leq \rho(\mathbf{d}((t, \omega), (t', \omega'))) \quad \text{whenever } t \leq t'. \quad (7)$$

- We say  $u \in \overline{\mathcal{U}}$  if  $-u \in \underline{\mathcal{U}}$ .

It is clear that  $\underline{U} \cap \overline{U} = UC_b(\Lambda)$ . We also recall from (Ekren *et al.*, 2014b) Remark 3.2 that Condition (7) implies that  $u$  has left-limits and positive jumps.

We next introduce the nonlinear expectations. Denote by  $\mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t)$  the set of  $\xi \in \mathbb{L}^0(\mathcal{F}_T^t)$  with  $\sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[|\xi|] < \infty$ , and define, for  $\xi \in \mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t)$ ,

$$\overline{\mathcal{E}}_t^L[\xi] = \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}_t^L[\xi] = \inf_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi] = -\overline{\mathcal{E}}_t^L[-\xi].$$

We now define viscosity solutions. For any  $u \in \mathbb{L}^0(\Lambda)$ ,  $(t, \omega) \in [0, T) \times \Omega$ , and  $L > 0$ , let

$$\begin{aligned} \underline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C^{1,2}(\Lambda^t) : \exists H \in \mathcal{H}^t \text{ s.t. } (\varphi - u^{t,\omega})_t = 0 \right. \\ &\quad \left. = \inf_{\tau \in T^t} \underline{\mathcal{E}}_t^L[(\varphi - u^{t,\omega})_{\tau \wedge H}] \right\}, \\ \overline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C^{1,2}(\Lambda^t) : \exists H \in \mathcal{H}^t \text{ s.t. } (\varphi - u^{t,\omega})_t = 0 \right. \\ &\quad \left. = \sup_{\tau \in T^t} \overline{\mathcal{E}}_t^L[(\varphi - u^{t,\omega})_{\tau \wedge H}] \right\}. \end{aligned} \tag{8}$$

**Definition 2.5.** (i) Let  $L > 0$ . We say  $u \in \underline{U}$  (resp.  $\overline{U}$ ) is a viscosity  $L$ -subsolution (resp.  $L$ -supersolution) of PPDE (1) if, for any  $(t, \omega) \in [0, T) \times \Omega$  and any  $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$  (resp.  $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$ ):

$$\mathcal{L}^{t,\omega} \varphi(t, \mathbf{0}) := [-\partial_t \varphi - G^{t,\omega}(\cdot, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)](t, \mathbf{0}) \leq (\text{resp. } \geq) 0. \tag{9}$$

(ii) We say  $u \in \underline{U}$  (resp.  $\overline{U}$ ) is a viscosity subsolution (resp. supersolution) of PPDE (1) if  $u$  is a viscosity  $L$ -subsolution (resp.  $L$ -supersolution) of PPDE (1) for some  $L > 0$ .

(iii) We say  $u \in UC_b(\Lambda)$  is a viscosity solution of PPDE (1) if it is both a viscosity subsolution and a viscosity supersolution.

As pointed out in (Ekren *et al.*, 2012a) Remark 3.11 (i), without loss of generality in (8) we may always set  $H = H_\varepsilon^t$  for some small  $\varepsilon > 0$ :

$$H_\varepsilon^t := \inf\{s > t : |B_s^t| \geq \varepsilon\} \wedge (t + \varepsilon). \tag{10}$$

## 2.2. Monotone schemes for (standard) PDEs

In this section, we introduce the main result of (Barles and Souganidis, 1991). We shall follow the presentation in (Guo *et al.*, 2013). We first recall the definition of viscosity solutions for PDE (2): an upper (resp. lower) semicontinuous function

$v$  is called a viscosity subsolution (resp. viscosity supersolution) of PDE (2) if  $\mathbb{L}\varphi(t, x) \leq$  (resp.  $\geq$ )  $0$ , for any  $(t, x) \in [0, T) \times \mathbb{R}^d$  and any smooth function  $\varphi$  satisfying:

$$[u - \varphi](t, x) = 0 \geq \text{(resp. } \leq) [u - \varphi](s, y), \quad \text{for all } (s, y) \in [0, T] \times \mathbb{R}^d. \quad (11)$$

For the viscosity theory of PDEs, we refer to the classical references (Crandall *et al.*, 1992; Fleming and Soner, 2006; Yong and Zhou, 1999).

We shall adopt the following standard assumptions:

**Assumption 2.6.** (i)  $F(\cdot, 0, \mathbf{0}, \mathbf{0})$  and  $f$  are bounded.

(ii)  $F$  is continuous in  $t$ , uniformly Lipschitz continuous in  $(x, y, z, \gamma)$ , and  $f$  is uniformly Lipschitz continuous in  $x$ .

(iii) PDE (2) is parabolic, that is,  $F$  is nondecreasing in  $\gamma$ .

(iv) Comparison principle for PDE (2) holds in the class of bounded viscosity solutions. That is, if  $v_1$  and  $v_2$  are bounded viscosity subsolution and viscosity supersolution of PDE (2), respectively, and  $v_1(T, \cdot) \leq f \leq v_2(T, \cdot)$ , then  $v_1 \leq v_2$  on  $[0, T] \times \mathbb{R}^d$ .

For any  $t \in [0, T)$  and  $h \in (0, T - t)$ , let  $\mathbb{T}_h^{t,x}$  be an operator on the set of measurable functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ . For  $n \geq 1$ , denote  $h := \frac{T}{n}$ ,  $t_i := ih$ ,  $i = 0, 1, \dots, n$ , and define:

$$v^h(t_n, x) := f(x), \quad v^h(t, x) := \mathbb{T}_{t_i-t}^{t_i, x}[v^h(t_i, \cdot)], \quad t \in [t_{i-1}, t_i], \quad i = n, \dots, 1. \quad (12)$$

The following convergence result is reported in (Guo *et al.*, 2013) Theorem 2.2, which is based on Barles and Souganidis (1991) and is due to Fahim *et al.* (2011) Theorem 3.6.

**Theorem 2.7.** Let Assumption 2.6 hold. Assume  $\mathbb{T}_h^{t,x}$  satisfies the following conditions:

(i) *Consistency:* for any  $(t, x) \in [0, T) \times \mathbb{R}^d$  and any  $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^d)$ ,

$$\lim_{(t', x', h, c) \rightarrow (t, x, 0, 0)} \frac{[c + \varphi](t', x') - \mathbb{T}_h^{t', x'}[[c + \varphi](t' + h, \cdot)]}{h} = \mathbb{L}\varphi(t, x).$$

(ii) *Monotonicity:*  $\mathbb{T}_h^{t,x}[\varphi] \leq \mathbb{T}_h^{t,x}[\psi]$  whenever  $\varphi \leq \psi$ .

(iii) *Stability:*  $v^h$  is bounded uniformly in  $h$  whenever  $f$  is bounded.

(iv) *Boundary condition:*  $\lim_{(t', x', h) \rightarrow (T, x, 0)} v^h(t', x') = f(x)$  for any  $x \in \mathbb{R}^d$ .

Then PDE (2) with terminal condition  $v(T, \cdot) = f$  has a unique bounded viscosity solution  $v$ , and  $v^h$  converges to  $v$  locally uniformly as  $h \rightarrow 0$ .

### 3. Monotone Scheme for PPDEs

Our goal of this section is to extend Theorem 2.7 to PPDE (1). Similar to Assumption 2.6, we assume:

**Assumption 3.1.** (i)  $G(\cdot, 0, \mathbf{0}, \mathbf{0})$  and  $g$  are bounded.

- (ii)  $G$  is continuous in  $(t, \omega)$ , uniformly Lipschitz continuous in  $(y, z, \gamma)$ , and  $g$  is uniformly continuous in  $\omega$ . Denote by  $L_0$  the Lipschitz constant of  $G$  in  $(z, \gamma)$ .
- (iii) PPDE (1) is parabolic, that is,  $G$  is nondecreasing in  $\gamma$ .
- (iv) Comparison principle for PPDE (1) holds in the class of bounded viscosity solutions. That is, if  $u_1$  and  $u_2$  are bounded viscosity subsolution and viscosity supersolution of PPDE (1), respectively, and  $u_1(T, \cdot) \leq g \leq u_2(T, \cdot)$ , then  $u_1 \leq u_2$  on  $\Lambda$ .

For the comparison principle in (iv) above, we refer to (Ekren *et al.*, 2012b) for some sufficient conditions.

Now for any  $(t, \omega) \in [0, T) \times \Omega$  and  $h \in (0, T - t)$ , let  $\mathbb{T}_h^{t, \omega}$  be an operator on  $\mathbb{L}^0(\mathcal{F}_{t+h}^t)$ . For  $n \geq 1$ , denote  $h := \frac{T}{n}$ ,  $t_i := ih, i = 0, 1, \dots, n$ , and define:

$$u^h(t_n, \omega) := g(\omega), \quad u^h(t, \omega) := \mathbb{T}_{t_i-t}^{t, \omega}[u^h(t_i, \cdot)], \quad t \in [t_{i-1}, t_i], \quad i = n, \dots, 1. \tag{13}$$

where we abuse the notation that:

$$\mathbb{T}_h^{t, \omega}[\varphi] := \mathbb{T}_h^{t, \omega}[\varphi^{t, \omega}], \quad \text{for } \varphi \in \mathbb{L}^0(\mathcal{F}_{t+h}).$$

The following main result is analogous to Theorem 2.7.

**Theorem 3.2.** *Let Assumption 3.1 hold. Assume  $\mathbb{T}_h^{t, \omega}$  satisfies the following conditions:*

- (i) *Consistency: for any  $(t, \omega) \in [0, T) \times \Omega$  and  $\varphi \in C^{1,2}(\Lambda^t)$ ,*

$$\lim_{(t', \omega', h, c) \rightarrow (t, \mathbf{0}, 0, 0)} \frac{[c + \varphi](t', \omega') - \mathbb{T}_h^{t', \omega', \omega'}[[c + \varphi](t' + h, \cdot)]}{h} = \mathcal{L}^{t, \omega} \varphi(t, \mathbf{0}), \tag{14}$$

where  $(t', \omega') \in \Lambda^t, h \in (0, T - t), c \in \mathbb{R}$ , and  $\mathcal{L}^{t, \omega} \varphi$  is defined in (9).

- (ii) *Monotonicity: for some constant  $L \geq L_0$  and any  $\varphi, \psi \in UC_b(\mathcal{F}_{t+h}^t)$ ,*

$$\bar{\mathcal{E}}_t^L[\varphi - \psi] \leq 0 \quad \text{implies} \quad \mathbb{T}_h^{t, \omega}[\varphi] \leq \mathbb{T}_h^{t, \omega}[\psi]. \tag{15}$$

- (iii) *Stability:  $u^h$  is uniformly bounded and uniformly continuous in  $\omega$ , uniformly on  $h$ . Moreover, there exists a modulus of continuity function  $\rho$ , independent*



of  $h$ , such that

$$|u^h(t, \omega) - u^h(t', \omega_{\wedge t})| \leq \rho((t' - t) \vee h), \quad \text{for any } t < t' \text{ and any } \omega \in \Omega. \quad (16)$$

Then PPDE (1) with terminal condition  $u(T, \cdot) = g$  has a unique bounded  $L$ -viscosity solution  $u$ , and  $u_h$  converges to  $u$  locally uniformly as  $h \rightarrow 0$ .

**Remark 3.3.** The conditions in Theorem 3.2 reflect the features of our definition of viscosity solution for PPDEs.

- (i) For the consistency condition (14), we require the convergence only for  $t' \geq t$ .
- (ii) The monotonicity condition in Theorem 2.7 (ii) is due to the maximum condition (11) in the definition of viscosity solutions for PDEs. In our path dependent case, the monotonicity condition (15) is modified in a way to adapt to (8).
- (iii) Due to the uniform continuity required in the definition of viscosity solutions, the stability condition in Theorem 3.2 (iii) is somewhat strong. Note that this condition obviously implies the counterparts of the stability and boundary conditions in Theorem 2.7.

**Remark 3.4.** Following a little more involved arguments, one may weaken the monotonicity condition (15) slightly and all the results in the paper still hold true: for some constant  $L \geq L_0$  and any  $\varphi, \psi \in UC_b(\mathcal{F}_{t+h}^t)$ , there exists a modulus of continuity function  $\rho_{mon}$ , which depends only on  $L, d$ , and the uniform continuity of  $\varphi, \psi$ , but does not depend on the specific  $\varphi, \psi$ , such that

$$\bar{\mathcal{E}}_t^L[\varphi - \psi] \leq 0 \quad \text{implies} \quad \mathbb{T}_h^{t, \omega}[\varphi] \leq \mathbb{T}_h^{t, \omega}[\psi] + h\rho_{mon}(h). \quad (17)$$

To prove the theorem, we need a technical lemma.

**Lemma 3.5.** Let  $L > 0, H \in \mathcal{H}, \tau \in \mathcal{T}, \tau \leq H$ , and  $X \in \underline{\mathcal{U}}$  with modulus of continuity function  $\rho$  in (7). Assume

$$\bar{\mathcal{E}}_0^L[X_\tau] - \bar{\mathcal{E}}_0^L[X_H] \geq c > 0 \quad (18)$$

Then there exist constants  $\delta_0 = \delta_0(c, L, d, \rho) > 0, C = C(L, d) > 0$ , and  $\omega^* \in \Omega$  such that

$$t_* := \tau(\omega^*) < H(\omega^*) \quad \text{and} \quad \sup_{\mathbb{P} \in \mathcal{P}_L^*} \mathbb{P}[H^{t_*, \omega^*} - t_* \leq \delta] \leq C\delta^2 \quad \text{for all } \delta \leq \delta_0. \quad (19)$$

**Proof.** Let  $H$  correspond to  $O$  and  $t_0$  in (4). We first claim there exist  $\delta_0 = \delta_0(c, L, d, \rho)$  and  $\omega^*$  such that

$$t_* := \tau(\omega^*) < t_0 - \delta_0 \quad \text{and} \quad d(\omega_{t_*}^*, O^c) \geq \delta_0^{\frac{1}{6}}. \quad (20)$$

In particular, this implies that  $t_* < H(\omega^*)$ . Then, for any  $\mathbb{P} \in \mathcal{P}_L^{t_*}$  and  $\delta \leq \delta_0$ ,

$$\begin{aligned} \mathbb{P}[H^{t_*, \omega^*} - t_* \leq \delta] &= \mathbb{P}(H^{t_*, \omega^*} - t_* \leq \delta, \omega_{t_*}^* + B_{H^{t_*, \omega^*}}^{t_*} \in O^c) \\ &\leq \mathbb{P}\left(\sup_{t_* \leq s \leq t_* + \delta} |B_s^{t_*}| \geq d(\omega_{t_*}^*, O^c)\right) \leq \mathbb{P}\left(\sup_{t_* \leq s \leq t_* + \delta} |B_s^{t_*}| \geq \delta_0^{\frac{1}{6}}\right) \\ &\leq \delta_0^{-1} \mathbb{E}^{\mathbb{P}} \left[ \sup_{t_* \leq s \leq t_* + \delta} |B_s^{t_*}|^6 \right] \leq C\delta^2, \end{aligned}$$

proving (19).

We now prove (20) by contradiction. Assume (20) is not true, then

$$\tau \geq t_0 - \delta_0 \quad \text{or} \quad d(B_\tau, O^c) < \delta_0^{\frac{1}{6}}, \quad \forall \omega \in \Omega. \quad (21)$$

By definition of  $\bar{\mathcal{E}}_0^L$ , there exists  $\mathbb{P} \in \mathcal{P}_L^0$  such that

$$\bar{\mathcal{E}}_0^L[X_\tau] \leq \mathbb{E}^{\mathbb{P}}[X_\tau] + \frac{c}{2}. \quad (22)$$

Note that  $B_\tau(\omega) \in O$  whenever  $\tau(\omega) < H(\omega)$ . Recall (5) and let  $\eta(\omega)$  denote the unit vector pointing from  $B_\tau(\omega)$  to  $O^c$ . Set  $\eta(\omega)$  be a fixed unit vector when  $\tau(\omega) = H(\omega)$ . Then  $\eta \in \mathcal{F}_\tau$ . Construct  $\hat{\mathbb{P}} \in \mathcal{P}_L^0$  as follows:

$$\alpha_t^{\hat{\mathbb{P}}} := \alpha_t^{\mathbb{P}} \mathbf{1}_{[0, \tau)}(t) + L\eta \mathbf{1}_{[\tau, t_0)}, \quad \beta_t^{\hat{\mathbb{P}}} := \beta_t^{\mathbb{P}} \mathbf{1}_{[0, \tau)}(t).$$

That is,  $\hat{\mathbb{P}} = \mathbb{P}$  on  $\mathcal{F}_\tau$  and  $dB_t^{\tau(\omega)} = L\eta(\omega)dt, t \geq \tau, \hat{\mathbb{P}}^{\tau, \omega}$ -a.s., where  $\hat{\mathbb{P}}^{\tau, \omega}$  is the regular conditional probability distribution of  $\hat{\mathbb{P}}$ . Then, one can easily see that

$$\begin{aligned} |B_t^{\tau(\omega)}| &= L[t - \tau(\omega)], \quad H^{\tau, \omega} - \tau(\omega) = \frac{d(B_\tau(\omega), O^c)}{L} \wedge [t_0 - \tau(\omega)], \\ &\hat{\mathbb{P}}^{\tau, \omega}\text{-a.s. for all } \omega. \end{aligned}$$

This, together with (21), implies

$$\begin{aligned} \mathbf{d}((\tau, \omega), (H, \omega)) &= H - \tau + \sup_{\tau \leq t \leq H} |B_t^\tau| \leq C[H - \tau] \leq C \left[ \frac{\delta_0^{\frac{1}{6}}}{L} + \delta_0 \right] \leq C\delta_0^{\frac{1}{6}}, \\ &\hat{\mathbb{P}}^{\tau, \omega}\text{-a.s.} \quad (23) \end{aligned}$$

Then, by (22), (7), and (23),

$$\begin{aligned} \bar{\mathcal{E}}_0^L[X_\tau] - \bar{\mathcal{E}}_0^L[X_H] &\leq \mathbb{E}^{\mathbb{P}}[X_\tau] - \mathbb{E}^{\hat{\mathbb{P}}}[X_H] + \frac{c}{2} = \mathbb{E}^{\hat{\mathbb{P}}}[X_\tau - X_H] + \frac{c}{2} \\ &\leq \mathbb{E}^{\hat{\mathbb{P}}}[\rho(\mathbf{d}((\tau, \omega), (H, \omega)))] + \frac{c}{2} \leq \rho(C\delta_0^{\frac{1}{6}}) + \frac{c}{2}. \end{aligned}$$

This contradicts with (18) when  $\delta_0$  is small enough, and thus (20) holds true.  $\square$

**Proof of Theorem 3.2.** By the stability,  $u^h$  is bounded. Define

$$\underline{u}(t, \omega) := \liminf_{h \rightarrow 0} u^h(t, \omega), \quad \bar{u}(t, \omega) := \limsup_{h \rightarrow 0} u^h(t, \omega). \quad (24)$$

Clearly  $\underline{u}(T, \omega) = g(\omega) = \bar{u}(T, \omega)$ ,  $\underline{u} \leq \bar{u}$ , and  $\underline{u}, \bar{u}$  are bounded and uniformly continuous. We shall show that  $\underline{u}$  (resp.  $\bar{u}$ ) is a viscosity  $L$ -supersolution (resp.  $L$ -subsolution) of PPDE (1). Then by the comparison principle we see that  $\bar{u} \leq \underline{u}$  and thus  $u := \bar{u} = \underline{u}$  is the unique viscosity solution of PPDE (1). The convergence of  $u^h$  is obvious now, which, together with the uniform regularity of  $u^h$  and  $u$ , implies further the locally uniform convergence.

Without loss of generality, we shall only prove by contradiction that  $\underline{u}$  satisfies the viscosity  $L$ -supersolution property at  $(0, \mathbf{0})$ . Assume not, then there exists  $\varphi^0 \in \bar{\mathcal{A}}^L \underline{u}(0, \mathbf{0})$  with corresponding  $H \in \mathcal{H}$  such that  $-c_0 := \mathcal{L}\varphi^0(0, \mathbf{0}) < 0$ . Denote

$$\varphi(t, \omega) := \varphi^0(t, \omega) - \frac{c_0}{2} t. \quad (25)$$

Then

$$\mathcal{L}\varphi(0, \mathbf{0}) = -\frac{c_0}{2} < 0. \quad (26)$$

Denote  $X^0 := \varphi - \underline{u}$ ,  $X^h := \varphi - u^h$ , and  $\bar{\mathcal{E}} := \bar{\mathcal{E}}_0^L$ ,  $\underline{\mathcal{E}} := \underline{\mathcal{E}}_0^L$ . Recall (10) and denote  $H_\varepsilon := H_\varepsilon^0 \wedge \varepsilon^5$ ,  $c_\varepsilon := \frac{1}{3} c_0 \varepsilon^5$ . Note that  $H_\varepsilon \leq H$  for  $\varepsilon$  small enough, and by Ekren *et al.* (2012a) (2.8),

$$\sup_{\mathbb{P} \in \mathcal{P}_L} \mathbb{P}(H_\varepsilon \neq \varepsilon^5) = \sup_{\mathbb{P} \in \mathcal{P}_L} \mathbb{P}(H_\varepsilon^0 < \varepsilon^5) \leq CL^4 \varepsilon^{-4} \varepsilon^{10} \leq C\varepsilon c_\varepsilon. \quad (27)$$

Then

$$\bar{\mathcal{E}}[\varepsilon^5 - H_\varepsilon] \leq \bar{\mathcal{E}}[\varepsilon^5 \mathbf{1}_{\{H_\varepsilon \neq \varepsilon^5\}}] \leq C\varepsilon c_\varepsilon.$$

Thus, for  $\varepsilon$  small, it follows from  $\varphi^0 \in \bar{\mathcal{A}}^L \underline{u}(0, \mathbf{0})$  that

$$\begin{aligned} X_0^0 - \bar{\mathcal{E}}[X_{H_\varepsilon}^0] &= [\varphi^0 - \underline{u}]_0 - \bar{\mathcal{E}}\left[(\varphi^0 - \underline{u})_{H_\varepsilon} - \frac{c_0}{2} H_\varepsilon\right] \\ &\geq \bar{\mathcal{E}}[(\varphi^0 - \underline{u})_{H_\varepsilon}] - \bar{\mathcal{E}}\left[(\varphi^0 - \underline{u})_{H_\varepsilon} - \frac{c_0}{2} H_\varepsilon\right] \\ &\geq \underline{\mathcal{E}}\left[\frac{c_0}{2} H_\varepsilon\right] = \frac{c_0 \varepsilon^5}{2} - \frac{c_0}{2} \bar{\mathcal{E}}[\varepsilon^5 - H_\varepsilon] \geq \frac{3c_\varepsilon}{2} - C\varepsilon c_\varepsilon \geq c_\varepsilon > 0. \end{aligned} \quad (28)$$

Let  $h_k \downarrow 0$  be a sequence such that

$$\lim_{k \rightarrow \infty} u_0^{h_k} = \underline{u}_0, \quad (29)$$

and simplify the notations:  $u^k := u^{h_k}, X^k := X^{h_k}$ . Then (28) leads to

$$\begin{aligned} c_\varepsilon &\leq \left[ \varphi_0 - \liminf_{h \rightarrow 0} u_0^h \right] - \bar{\mathcal{E}} \left[ \varphi_{H_\varepsilon} - \liminf_{h \rightarrow 0} u_{H_\varepsilon}^h \right] \\ &\leq \left[ \varphi_0 - \lim_{k \rightarrow \infty} u_0^k \right] - \bar{\mathcal{E}} \left[ \varphi_{H_\varepsilon} - \liminf_{k \rightarrow \infty} u_{H_\varepsilon}^k \right]. \end{aligned}$$

Note that  $X^k$  is uniformly bounded. Then by (27) we have

$$\bar{\mathcal{E}}[|X_{H_\varepsilon}^k - X_{\varepsilon^5}^k|] \leq C\varepsilon c_\varepsilon.$$

Since  $u^h$  is uniformly continuous, applying the monotone convergence theorem under nonlinear expectation  $\bar{\mathcal{E}}$ , see e.g. (Ekren *et al.*, 2014b) Proposition 2.5, we have

$$\begin{aligned} c_\varepsilon &\leq \lim_{k \rightarrow \infty} [\varphi_0 - u_0^k] - \bar{\mathcal{E}} \left[ \limsup_{k \rightarrow \infty} [\varphi_{H_\varepsilon} - u_{H_\varepsilon}^k] \right] \\ &\leq \lim_{k \rightarrow \infty} X_0^k - \bar{\mathcal{E}} \left[ \limsup_{k \rightarrow \infty} X_{\varepsilon^5}^k \right] + C\varepsilon c_\varepsilon = \lim_{k \rightarrow \infty} X_0^k - \bar{\mathcal{E}} \left[ \lim_{m \rightarrow \infty} \sup_{k \geq m} X_{\varepsilon^5}^k \right] + C\varepsilon c_\varepsilon \\ &= \lim_{k \rightarrow \infty} X_0^k - \lim_{m \rightarrow \infty} \bar{\mathcal{E}} \left[ \sup_{k \geq m} X_{\varepsilon^5}^k \right] + C\varepsilon c_\varepsilon \leq \lim_{k \rightarrow \infty} X_0^k - \lim_{k \rightarrow \infty} \sup_{k \rightarrow \infty} \bar{\mathcal{E}}[X_{\varepsilon^5}^k] + C\varepsilon c_\varepsilon \\ &\leq \lim_{k \rightarrow \infty} X_0^k - \limsup_{k \rightarrow \infty} \bar{\mathcal{E}}[X_{H_\varepsilon}^k] + C\varepsilon c_\varepsilon = \liminf_{k \rightarrow \infty} [X_0^k - \bar{\mathcal{E}}[X_{H_\varepsilon}^k]] + C\varepsilon c_\varepsilon. \end{aligned}$$

Then, for all  $\varepsilon$  small enough and  $k$  large enough,

$$X_0^k - \bar{\mathcal{E}}[X_{H_\varepsilon}^k] \geq \frac{c_\varepsilon}{2}. \tag{30}$$

Now for each  $k$ , define

$$Y_t^k(\omega) := \sup_{\tau \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L[(X^k)_{\tau \wedge H_\varepsilon}^{t, \omega}], \quad t \leq H_\varepsilon(\omega), \quad \text{and} \quad \tau_k := \inf\{t \geq 0 : Y_t^k = X_t^k\}.$$

We remark that here  $Y^k, \tau_k$  depend on  $\varepsilon$  as well, but we omit the superscript  $\varepsilon$  for notational simplicity. Applying (Ekren *et al.*, 2014b) Theorem 3.6, we know  $\tau_k \leq H_\varepsilon$  is an optimal stopping time for  $Y_0^k$  and thus

$$0 < \frac{c_\varepsilon}{2} \leq X_0^k - \bar{\mathcal{E}}[X_{H_\varepsilon}^k] \leq Y_0^k - \bar{\mathcal{E}}[X_{H_\varepsilon}^k] = \bar{\mathcal{E}}[X_{\tau_k}^k] - \bar{\mathcal{E}}[X_{H_\varepsilon}^k].$$

By Lemma 3.5, for  $k$  large enough so that  $h_k \leq \delta_0(\frac{c_\varepsilon}{2}, L, d, \rho)$ , there exists  $\omega^k \in \Omega$  such that

$$t_*^k := \tau_k(\omega^k) < H_\varepsilon(\omega^k) \quad \text{and} \quad \sup_{\mathbb{P} \in \mathcal{P}_L^{t_*^k}} \mathbb{P}(H_\varepsilon^k - t_*^k \leq \delta) \leq C\delta^2 \quad \text{for all } \delta \leq h_k, \tag{31}$$

where  $H_\varepsilon^k := H_\varepsilon^{t^k, \omega^k}$ . Let  $\{t_i^k, i = 0, \dots, n_k\}$  denote the time partition corresponding to  $h_k$ , and assume  $t_{i-1}^k \leq t_*^k < t_i^k$ . Note that

$$X_{t_*^k}^k(\omega^k) = Y_{t_*^k}^k(\omega^k) \geq \bar{\mathcal{E}}_{t_*^k}^L[(X^k)_{\tau \wedge H_\varepsilon^k}^{t_*^k, \omega^k}], \quad \forall \tau \in \mathcal{T}^{t_*^k}.$$

Set  $\delta_k := t_i^k - t_*^k \leq h_k$  and  $\tau := t_i^k$ . Combine the inequality above and (31) we have

$$[\varphi - u^k](t_*^k, \omega^k) \geq \bar{\mathcal{E}}_{t_*^k}^L[(\varphi - u^k)_{t_i^k \wedge H_\varepsilon^k}^{t_*^k, \omega^k}] \geq \bar{\mathcal{E}}_{t_*^k}^L[(\varphi - u^k)_{t_i^k}^{t_*^k, \omega^k}] - C\delta_k^2.$$

This implies

$$\bar{\mathcal{E}}_{t_*^k}^L[(\varphi)_{t_i^k}^{t_*^k, \omega^k} - [\varphi - u^k](t_*^k, \omega^k) - C\delta_k^2] - (u^k)_{t_i^k}^{t_*^k, \omega^k} \leq 0.$$

By the monotonicity condition (15) we have

$$u^k(t_*^k, \omega^k) = \mathbb{T}_{\delta_k}^{t_*^k, \omega^k}[u_{t_i^k}^k] \leq \mathbb{T}_{\delta_k}^{t_*^k, \omega^k}[\varphi_{t_i^k} - [\varphi - u^k](t_*^k, \omega^k) - C\delta_k^2]. \quad (32)$$

We next use the consistency condition (14). For  $(t, \omega) = (0, \mathbf{0})$ , set

$$t' := t_*^k, \quad \omega' := \omega^k, \quad h := \delta_k, \quad c := -[\varphi - u^k](t_*^k, \omega^k) - C\delta_k^2.$$

By first sending  $k \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ , we see that

$$\mathbf{d}((t_*^k, \omega^k), (0, \mathbf{0})) \leq H_\varepsilon + \sup_{0 \leq t \leq H_\varepsilon} |\omega_t^k| \leq 2\varepsilon \rightarrow 0, \quad h \leq h_k \rightarrow 0,$$

which, together with (25), (29), and the uniform continuity of  $\varphi$  and  $u^k$ , implies

$$|c| \leq |[\varphi - u^k](t_*^k, \omega^k) - [\varphi - u^k](0, \mathbf{0})| + |u_0^k - \underline{u}_0| + C\delta_k^2 \rightarrow 0.$$

Then, by the consistency condition (14) we obtain from (32) that

$$\begin{aligned} 0 &\leq \frac{u^k(t_*^k, \omega^k) - \mathbb{T}_{\delta_k}^{t_*^k, \omega^k}[\varphi_{t_i^k} - [\varphi - u^k](t_*^k, \omega^k) - C\delta_k^2]}{\delta_k} \\ &= \frac{[c + \varphi](t_*^k, \omega^k) - \mathbb{T}_{\delta_k}^{t_*^k, \omega^k}[[c + \varphi]_{t_i^k}]}{\delta_k} + C\delta_k \rightarrow \mathcal{L}\varphi(0, \mathbf{0}). \end{aligned}$$

This contradicts with (26). □

#### 4. An Illustrative Monotone Scheme

We first remark that the monotonicity condition (15) is solely due to our definition of viscosity solution of PPDEs. It is sufficient but not necessary for the convergence of the scheme. In Markovian case, the PPDE (1) is reduced back to PDE (2). The schemes proposed in Fahim *et al.* (2011) and Guo *et al.* (2013) satisfy the

traditional monotonicity condition in Theorem 2.7, but violates our new monotonicity condition (15). However, as proved in (Fahim et al., 2011; Soner et al., 2012), we know those schemes do converge.

The goal of this section is to propose a scheme which satisfies all the conditions in Theorem 3.2 and thus converges. However, to ensure the monotonicity condition (15), we will need certain conditions which are purely technical. Finding monotone schemes for general parabolic PPDEs is a challenging problem and we shall leave it for future research. We also remark that efficient implementation of such schemes, especially in high dimensions, is also a very challenging problem and will also be left for future research.

Our scheme will involve some parameters:

$$\mu_i > 0, \quad \sigma_i > 0, \quad i = 1, \dots, d. \tag{33}$$

Let  $e_i \in \mathbb{R}^d$  be the vector whose  $i$ th component is 1 and all other components are 0, and  $e_{ij} \in \mathbb{R}^{d \times d}$  be the matrix whose  $(i, j)$ th component is 1 and all other components are 0. Given  $(t, \omega) \in [0, T) \times \Omega$ , recall (5) and introduce the following probability measures on  $\Omega^t$ : for  $i, j = 1, \dots, d$ ,

$$\begin{aligned} \mathbb{P}^0 : \alpha^{\mathbb{P}^0} = \mathbf{0}, \beta^{\mathbb{P}^0} = \mathbf{0}; \quad \mathbb{P}^i : \alpha^{\mathbb{P}^i} = \mu_i e_i, \beta^{\mathbb{P}^i} = \mathbf{0}; \\ \mathbb{P}^{ii} : \alpha^{\mathbb{P}^{ii}} = \mathbf{0}, \beta^{\mathbb{P}^{ii}} = \sigma_i e_{ii}; \quad \mathbb{P}^{ij} : \alpha^{\mathbb{P}^{ij}} = \mathbf{0}, \beta^{\mathbb{P}^{ij}} = \sigma_j e_{ij} + \sigma_i e_{ji}, \quad i \neq j. \end{aligned} \tag{34}$$

Now for  $h \in (0, T - t)$  and  $\varphi \in \mathbb{L}^0(\mathcal{F}_{t+h}^t)$ , define

$$\mathbb{T}_h^{t, \omega}[\varphi] := \mathcal{D}^{(0)}\varphi + hG(t, \omega, \mathcal{D}^{(0)}\varphi, \mathcal{D}^{(1)}\varphi, \mathcal{D}^{(2)}\varphi), \tag{35}$$

where  $\mathcal{D}^{(0)}\varphi, \mathcal{D}^{(1)}\varphi, \mathcal{D}^{(2)}\varphi$  take values in  $\mathbb{R}, \mathbb{R}^d, \mathbb{S}^d$ , respectively, with each component defined by

$$\begin{aligned} \mathcal{D}^{(0)}\varphi &:= \mathbb{E}^{\mathbb{P}^0}[\varphi], \quad \mathcal{D}_i^{(1)}\varphi := \frac{\mathbb{E}^{\mathbb{P}^i}[\varphi] - \mathbb{E}^{\mathbb{P}^0}[\varphi]}{\mu_i h}, \quad \mathcal{D}_{i,i}^{(2)}\varphi := \frac{\mathbb{E}^{\mathbb{P}^{ii}}[\varphi] - \mathbb{E}^{\mathbb{P}^0}[\varphi]}{\sigma_i^2 h/2}, \\ \mathcal{D}_{i,j}^{(2)}\varphi &:= \frac{\mathbb{E}^{\mathbb{P}^{ij}}[\varphi] - \mathbb{E}^{\mathbb{P}^{ii}}[\varphi] - \mathbb{E}^{\mathbb{P}^{jj}}[\varphi] + \mathbb{E}^{\mathbb{P}^0}[\varphi]}{\sigma_i \sigma_j h}, \quad i \neq j. \end{aligned} \tag{36}$$

We now verify the conditions in Theorem 3.2.

**Lemma 4.1 (Consistency).** *Under Assumption 3.1,  $\mathbb{T}_h^{t, \omega}$  satisfies the consistency condition (14).*

**Proof.** Without loss of generality, we assume  $(t, \omega) = (0, \mathbf{0})$ . Let  $(t', \omega', h, c)$  be as in (14), and for notational simplicity, at below we write  $(t', \omega')$  as  $(t, \omega)$ . Now for  $\varphi \in C^{1,2}(\Lambda)$ , denote  $\psi := c + \varphi^{t, \omega}(t + h, \cdot) \in \mathbb{L}^0(\mathcal{F}_{t+h}^t)$ , and  $\varphi|_t^s := \varphi^{t, \omega}(s, B^t) - \varphi(t, \omega)$ . Send  $(t, \omega, h, c) \rightarrow (0, \mathbf{0}, 0, 0)$ , by the functional Itô formula and the

smoothness of  $\varphi$ , one can easily check that

$$\mathcal{D}^{(0)}\psi = c + \varphi(t + h, \omega_{\cdot \wedge t}) \rightarrow \varphi(0, \mathbf{0});$$

$$\frac{\mathcal{D}^{(0)}\psi - [c + \varphi](t, \omega)}{h} = \frac{1}{h} \mathbb{E}^{\mathbb{P}^0}[\varphi|_t^{t+h}] = \frac{1}{h} \int_t^{t+h} \partial_t \varphi(s, \omega_{\cdot \wedge t}) ds \rightarrow \partial_t \varphi(0, \mathbf{0});$$

$$\begin{aligned} \mathcal{D}_i^{(1)}\psi &= \frac{1}{\mu_i h} \mathbb{E}^{\mathbb{P}^i}[\varphi|_t^{t+h}] - \frac{1}{\mu_i h} \mathbb{E}^{\mathbb{P}^0}[\varphi|_t^{t+h}] \\ &= \frac{1}{\mu_i h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^i}[(\partial_t + \mu_i \partial_{\omega^i})\varphi(s, \omega \otimes_t B^t) - \partial_t \varphi(s, \omega_{\cdot \wedge t})] ds \rightarrow \partial_{\omega^i} \varphi(0, \mathbf{0}); \end{aligned}$$

$$\begin{aligned} \mathcal{D}_{i,i}^{(2)}\psi &= \frac{2}{\sigma_i^2 h} \mathbb{E}^{\mathbb{P}^{ii}}[\varphi|_t^{t+h}] - \frac{2}{\sigma_i^2 h} \mathbb{E}^{\mathbb{P}^0}[\varphi|_t^{t+h}] \\ &= \frac{2}{\sigma_i^2 h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^{ii}} \left[ \left( \partial_t + \frac{\sigma_i^2}{2} \partial_{\omega^i \omega^i}^2 \right) \varphi(s, \omega \otimes_t B^t) \right. \\ &\quad \left. - \partial_t \varphi(s, \omega_{\cdot \wedge t}) \right] ds \rightarrow \partial_{\omega^i \omega^i}^2 \varphi(0, \mathbf{0}); \end{aligned}$$

$$\begin{aligned} \mathcal{D}_{i,j}^{(2)}\psi &= \frac{1}{\sigma_i \sigma_j h} \mathbb{E}^{\mathbb{P}^{ij}}[\varphi|_t^{t+h}] - \frac{1}{\sigma_i \sigma_j h} \mathbb{E}^{\mathbb{P}^{ii}}[\varphi|_t^{t+h}] - \frac{1}{\sigma_i \sigma_j h} \mathbb{E}^{\mathbb{P}^{jj}}[\varphi|_t^{t+h}] \\ &\quad + \frac{1}{\sigma_i \sigma_j h} \mathbb{E}^{\mathbb{P}^0}[\varphi|_t^{t+h}] \\ &= \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^{ij}} \left[ \left( \partial_t + \frac{1}{2} \sigma_i^2 \partial_{\omega^i \omega^i}^2 + \frac{1}{2} \sigma_j^2 \partial_{\omega^j \omega^j}^2 \right. \right. \\ &\quad \left. \left. + \sigma_i \sigma_j \partial_{\omega^i \omega^j}^2 \right) \varphi(s, \omega \otimes_t B^t) \right] ds \\ &\quad - \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^{ii}} \left[ \left( \partial_t + \frac{1}{2} \sigma_i^2 \partial_{\omega^i \omega^i}^2 \right) \varphi(s, \omega \otimes_t B^t) \right] ds \\ &\quad - \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^{jj}} \left[ \left( \partial_t + \frac{1}{2} \sigma_j^2 \partial_{\omega^j \omega^j}^2 \right) \varphi(s, \omega \otimes_t B^t) \right] ds \\ &\quad + \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \partial_t \varphi(s, \omega_{\cdot \wedge t}) ds \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \mathbb{E}^{\mathbb{P}^{ij}} \left[ \left( \partial_t + \frac{1}{2} \sigma_i^2 \partial_{\omega^i \omega^i}^2 + \frac{1}{2} \sigma_j^2 \partial_{\omega^j \omega^j}^2 + \sigma_i \sigma_j \partial_{\omega^i \omega^j}^2 \right) \varphi |_t^s \right] ds \\
 &\quad - \frac{1}{\sigma_i \sigma_j h} \int_t^{t+h} \left( \mathbb{E}^{\mathbb{P}^{ii}} \left[ \left( \partial_t + \frac{1}{2} \sigma_i^2 \partial_{\omega^i \omega^i}^2 \right) \varphi |_t^s \right] \right. \\
 &\quad \left. + \mathbb{E}^{\mathbb{P}^{jj}} \left[ \left( \partial_t + \frac{1}{2} \sigma_j^2 \partial_{\omega^j \omega^j}^2 \right) \varphi |_t^s \right] \right) ds + \partial_{\omega^i \omega^j}^2 \varphi(t, \omega) \rightarrow \partial_{\omega^i \omega^j}^2 \varphi(0, \mathbf{0}).
 \end{aligned}$$

Plug these into (35) and (36), we obtain (14) immediately. □

To ensure the monotonicity condition (15), we need some additional conditions.

**Assumption 4.2.** Assume  $G$  is differentiable in  $(z, \gamma)$  and one may choose  $\mu_i, \sigma_i$  so that

$$\begin{aligned}
 &\partial_{z_i} G \geq 0, \quad \partial_{\gamma_{ij}} G \geq 0, \quad 2\partial_{\gamma_{ii}} G / \sigma_i \geq \sum_{j \neq i} [\partial_{\gamma_{ij}} G + \partial_{\gamma_{ji}} G] / \sigma_j, \\
 &\sum_{i=1}^d \frac{\partial_{z_i} G}{\mu_i} + \sum_{i=1}^d \frac{2\partial_{\gamma_{ii}} G}{\sigma_i^2} - \sum_{i \neq j} \frac{\partial_{\gamma_{ij}} G}{\sigma_i \sigma_j} \leq 1 - \varepsilon_0 \quad \text{for some } \varepsilon_0 \in (0, 1).
 \end{aligned} \tag{37}$$

**Remark 4.3.** (i) The differentiability of  $G$  is just for convenience. For notational simplicity, at below we shall assume  $G$  is differentiable in  $y$  as well.

(ii) By setting  $\sigma_i$  all equal, a sufficient condition for the third inequality in (37) is the following diagonal dominance condition:

$$2\partial_{\gamma_{ii}} G \geq \sum_{j \neq i} [\partial_{\gamma_{ij}} G + \partial_{\gamma_{ji}} G]. \tag{38}$$

(iii) Since the derivatives of  $G$  are uniformly bounded, thanks to Assumption 3.1, then the last inequality in (37) always holds true when  $\mu_i, \sigma_i$  are large enough.

**Lemma 4.4 (Monotonicity).** Under Assumptions 3.1 and 4.2,  $\mathbb{T}_h^{t, \omega}$  satisfies the monotonicity condition (15) for  $L \geq L_0$  large enough and  $h$  small enough.

**Proof.** Without loss of generality, we assume  $(t, \omega) = (0, \mathbf{0})$  and denote  $\mathbb{T}_h := \mathbb{T}_h^{t, \omega}$ . Assume  $L \geq L_0$  is large enough so that the  $\mathbb{P}^i$  and  $\mathbb{P}^{ij}$  in (34) are in  $\mathcal{P}_L$ . Let  $\varphi_1, \varphi_2 \in UC_b(\mathcal{F}_h)$  satisfy

$$\bar{\mathcal{E}}^L[\psi] \leq 0, \quad \text{where } \psi := \varphi_1 - \varphi_2. \tag{39}$$

Then, recalling (35),

$$\mathbb{T}_h \varphi_1 - \mathbb{T}_h \varphi_2 = \mathcal{D}^{(0)} \psi + h[\partial_y G \mathcal{D}^{(0)} \psi + \partial_z G \cdot \mathcal{D}^{(1)} \psi + \partial_\gamma G : \mathcal{D}^{(2)} \psi].$$



Note that here  $\partial_y G$  etc. are deterministic. By (36), we have

$$\mathbb{T}_h \varphi_1 - \mathbb{T}_h \varphi_2 = a_0 \mathbb{E}^{\mathbb{P}^0}[\psi] + \sum_{i=1}^d a_i \mathbb{E}^{\mathbb{P}^i}[\psi] + \sum_{i=1}^d a_{ii} \mathbb{E}^{\mathbb{P}^{ii}}[\psi] + \sum_{i \neq j} a_{ij} \mathbb{E}^{\mathbb{P}^{ij}}[\psi], \tag{40}$$

where

$$a_0 := 1 + h \partial_y G - \sum_{i=1}^d \frac{\partial_{z_i} G}{\mu_i} - \sum_{i=1}^d \frac{\partial_{\gamma_{ii}} G}{\sigma_i^2/2} + \sum_{i \neq j} \frac{\partial_{\gamma_{ij}} G}{\sigma_i \sigma_j} \tag{41}$$

$$a_i := \frac{\partial_{z_i} G}{\mu_i}, \quad a_{ii} := \frac{2 \partial_{\gamma_{ii}} G}{\sigma_i^2} - \sum_{j \neq i} \frac{\partial_{\gamma_{ij}} G + \partial_{\gamma_{ji}} G}{\sigma_i \sigma_j}, \quad a_{ij} := \frac{\partial_{\gamma_{ij}} G}{\sigma_i \sigma_j}.$$

By (37), we see that  $a_0, a_i, a_{ij} \geq 0$ , provided  $h$  is small enough. Note that

$$a_0 + \sum_{i=1}^d a_i + \sum_{i,j=1}^d a_{ij} = 1 + h \partial_y G.$$

Then one may define the following probability measure:

$$\hat{\mathbb{P}} := \frac{1}{1 + h \partial_y G} \left[ a_0 \mathbb{P}^0 + \sum_{i=1}^d a_i \mathbb{P}^i + \sum_{i,j=1}^d a_{ij} \mathbb{P}^{ij} \right], \tag{42}$$

and rewrite (40) as

$$\mathbb{T}_h \varphi_1 - \mathbb{T}_h \varphi_2 = (1 + h \partial_y G) \mathbb{E}^{\hat{\mathbb{P}}}[\psi]. \tag{43}$$

Since  $\mathbb{P}^0, \mathbb{P}^i, \mathbb{P}^{ij} \in \mathcal{P}_L$ , (39) implies  $\mathbb{E}^{\mathbb{P}^0}[\psi], \mathbb{E}^{\mathbb{P}^i}[\psi], \mathbb{E}^{\mathbb{P}^{ij}}[\psi] \leq 0$  and thus  $\mathbb{E}^{\hat{\mathbb{P}}}[\psi] \leq 0$ . This leads to (15) immediately.  $\square$

**Remark 4.5.** In general  $\hat{\mathbb{P}}$  may not be in  $\mathcal{P}_L$ . However, we still have  $\underline{\mathcal{E}}^L \leq \mathbb{E}^{\hat{\mathbb{P}}} \leq \bar{\mathcal{E}}^L$ .

We now verify the stability condition.

**Lemma 4.6 (Stability).** *Let Assumptions 3.1 and 4.2 hold, and assume further that  $G$  and  $g$  are uniformly Lipschitz continuous in  $\omega$ . Then  $\mathbb{T}_h^{t,\omega}$  satisfies the stability condition in Theorem 3.2 for  $L$  large enough and  $h$  small enough.*

**Proof.** We assume  $L$  and  $h$  are chosen so that  $\mathbb{T}_h^{t,\omega}$  satisfies the monotonicity condition (15).

(i) We first show that  $u^h$  is uniformly bounded. Denote  $C_i := C_i^h := \sup_{\omega \in \Omega} |u^h(t_i, \omega)|$ ,  $\varphi := [u^h(t_{i+1}, \cdot)]^{t_i, \omega}$ , and recall (13). By (35) we have

$$u^h(t_i, \omega) = \mathcal{D}^{(0)}\varphi + hG(t_i, \omega, \mathcal{D}^{(0)}\varphi, \mathcal{D}^{(1)}\varphi, \mathcal{D}^{(2)}\varphi) - hG(t_i, \omega, 0, \mathbf{0}, \mathbf{0}) + hG(t_i, \omega, 0, \mathbf{0}, \mathbf{0}).$$

Following the arguments for (43), for some  $\hat{\mathbb{P}}$  defined in the spirit of (41) and (42), we have

$$u^h(t_i, \omega) = (1 + h\partial_y G)\mathbb{E}^{\hat{\mathbb{P}}}[\varphi] + hG(t_i, \omega, 0, \mathbf{0}, \mathbf{0}). \tag{44}$$

Then

$$|u^h(t_i, \omega)| \leq (1 + h\partial_y G)\|\mathbb{E}^{\hat{\mathbb{P}}}[\varphi]\| + h|G(t_i, \omega, 0, \mathbf{0}, \mathbf{0})| \leq (1 + Ch)C_{i+1} + Ch.$$

That is,

$$C_i \leq [1 + Ch]C_{i+1} + Ch.$$

Note that  $C_n = \|g\|_\infty$ . Then by the discrete Gronwall inequality we see that  $\max_{0 \leq i \leq n} C_i \leq C$ , where the constant  $C$  is independent of  $h$ .

Finally, for  $t \in (t_i, t_{i+1})$ , following similar arguments we can easily show that  $|u^h(t, \omega)| \leq [1 + Ch]C_{i+1} + Ch \leq C$ . Therefore,  $u^h$  is uniformly bounded.

(ii) We next show that  $u^h$  is uniformly Lipschitz continuous in  $\omega$ . Let  $L_i := L_h^i$  denote the Lipschitz constant of  $u^h(t_i, \cdot)$ . Given  $\omega^1, \omega^2 \in \Omega$ , denote  $\psi := [u^h(t_{i+1}, \cdot)]^{t_i, \omega^1} - [u^h(t_{i+1}, \cdot)]^{t_i, \omega^2}$ , then

$$|\psi| \leq L_{i+1} \|\omega^1 \otimes_{t_i} B^{t_i} - \omega^2 \otimes_{t_i} B^{t_i}\|_{t_{i+1}} = L_{i+1} \|\omega^1 - \omega^2\|_{t_i}.$$

Note that  $G$  is uniform Lipschitz continuous in  $\omega$  with certain Lipschitz constant  $L_G$ . Then similar to (i) above, we have

$$\begin{aligned} |u^h(t_i, \omega^1) - u^h(t_i, \omega^2)| &\leq (1 + h\partial_y G)\mathbb{E}^{\hat{\mathbb{P}}}[\|\psi\|] + L_G h \|\omega^1 - \omega^2\|_{t_i} \\ &\leq L_{i+1} \|\omega^1 - \omega^2\|_{t_i} [1 + Ch] + L_G h \|\omega^1 - \omega^2\|_{t_i}. \end{aligned}$$

Then

$$L_i \leq L_{i+1}[1 + Ch] + L_G h.$$

Since  $L_n = L_g$  is the Lipschitz constant of  $g$  which is independent of  $h$ , we see that  $\max_{0 \leq i \leq n} L_i$  is independent of  $h$ . Finally, as in the end of (i) above we see that  $u^h(t, \cdot)$  is uniformly Lipschitz continuous in  $\omega$ , uniformly in  $t$  and  $h$ .

(iii) We now prove the following time regularity in two steps:

$$|u^h(t, \omega) - u^h(t', \omega_{\wedge t})| \leq C\sqrt{t' - t + h}, \quad \text{for all } 0 \leq t < t' \leq T. \tag{45}$$

**Step 1.** We first assume  $t' = T$  and  $t = t_i$ . For  $j = i + 1, \dots, n$ , in the spirit of (44), we may define  $\hat{\mathbb{P}}_j$  such that  $\hat{\mathbb{P}}_{j+1} = \hat{\mathbb{P}}_j$  on  $\mathcal{F}_{t_j}^{t_i}$  and

$$u^h(t_j, \omega \otimes_{t_i} B^{t_i}) = [1 + hb_j] \mathbb{E}^{\hat{\mathbb{P}}_{j+1}} [u^h(t_{j+1}, \omega \otimes_{t_i} B^{t_i}) | \mathcal{F}_{t_j}^{t_i}] + hc_j,$$

where  $b_j := \partial_y G(t_j)$  and  $c_j := G(t_j, \omega \otimes_{t_i} B^{t_i}, \mathbf{0}, \mathbf{0}, \mathbf{0})$  are in  $\mathbb{L}^\infty(\mathcal{F}_{t_j}^{t_i})$ . Denote  $\Gamma_i := 1, \Gamma_{j+1} := \prod_{k=i}^j [1 + hb_k]$ . By induction we have

$$\begin{aligned} u^h(t_i, \omega) &= \mathbb{E}^{\hat{\mathbb{P}}_n} \left[ \Gamma_n u^h(t_n, \omega \otimes_{t_i} B^{t_i}) + h \sum_{j=i}^{n-1} \Gamma_j c_j \right] \\ &= \mathbb{E}^{\hat{\mathbb{P}}_n} \left[ \Gamma_n g(\omega \otimes_{t_i} B^{t_i}) + h \sum_{j=i}^{n-1} \Gamma_j c_j \right]. \end{aligned}$$

One may easily check that

$$|\Gamma_j| \leq C, \quad |\Gamma_j - 1| \leq C(j - i)h \leq C(n - i)h = C(T - t_i).$$

Thus

$$\begin{aligned} |u^h(t_i, \omega) - u^h(t_n, \omega_{\wedge t_i})| &= \left| \mathbb{E}^{\hat{\mathbb{P}}_n} \left[ \Gamma_n g(\omega \otimes_{t_i} B^{t_i}) + h \sum_{j=i}^{n-1} \Gamma_j c_j \right] - g(\omega_{\wedge t_i}) \right| \\ &\leq \mathbb{E}^{\hat{\mathbb{P}}_n} [|\Gamma_n - 1| |g(\omega \otimes_{t_i} B^{t_i})| + |g(\omega \otimes_{t_i} B^{t_i}) \\ &\quad - g(\omega_{\wedge t_i})| + C(n - i)h] \\ &\leq C(T - t_i) + C \mathbb{E}^{\hat{\mathbb{P}}_n} [\|B^{t_i}\|_T]. \end{aligned}$$

One can easily show that  $\mathbb{E}^{\hat{\mathbb{P}}_n} [\|B^{t_i}\|_T] \leq C\sqrt{T - t_i}$ . Then (45) holds in this case.

**Step 2.** We now verify the general case. Assume  $t_{i-1} \leq t < t_i$  and  $t_{j-1} \leq t' < t_j$ , then clearly  $i \leq j$ . Since  $u^h(t_i, \cdot)$  and  $u^h(t_j, \cdot)$  are Lipschitz continuous in  $\omega$ , by (44) and following the arguments in Step 1, one can similarly show that

$$\begin{aligned} |u^h(t, \omega) - u^h(t_i, \omega_{\wedge t})| &\leq C\sqrt{t_i - t} \leq C\sqrt{h}, \\ |u^h(t', \omega_{\wedge t}) - u^h(t_j, \omega_{\wedge t})| &\leq C\sqrt{t_j - t'} \leq C\sqrt{h}, \\ |u^h(t_i, \omega_{\wedge t}) - u^h(t_j, \omega_{\wedge t})| &\leq C\sqrt{t_j - t_i} \leq C\sqrt{t' - t + h}. \end{aligned}$$

These lead to (45) immediately. □

Combine Lemmas 4.1, 4.4 and (38), it follows from Theorem 3.2 that

**Theorem 4.7.** Assume all the conditions in Lemma 4.6 hold. Then  $u^h$  converges locally uniformly to the unique viscosity solution  $u$  of PPDE (1). Moreover,

$$|u(t, \omega) - u(t', \omega')| \leq Cd((t, \omega), (t', \omega')), \quad \text{for all } (t, \omega), (t', \omega') \in \Lambda. \quad (46)$$

### 5. The Case with Classical Solution

In this section, we obtain the rate of convergence of our scheme, provided that the PPDE has smooth enough solution. Denote

$$C_b^{2,4} := \{u \in C_b^{1,2} : \partial_t u, \partial_\omega u, \partial_{\omega\omega}^2 u \in C_b^{1,2}(\Lambda)\}. \tag{47}$$

We shall remark though, as we see in [Buckdahn et al. \(2013\)](#), in general  $\partial_t, \partial_{\omega_i}, \partial_{\omega_j}$  do not commute, and  $\partial_{\omega_i\omega_j}^2 u = \frac{1}{2}[\partial_{\omega_i}(\partial_{\omega_j} u) + \partial_{\omega_j}(\partial_{\omega_i} u)]$ .

We first have the following general result, in the spirit of [Theorem 3.2](#).

**Theorem 5.1.** *Let Assumption 3.1 hold and the PPDE (1) has a classical solution  $u \in C_b^{2,4}(\Lambda)$ . Assume a discretization scheme  $\mathbb{T}_h^{t,\omega}$  satisfies:*

(i) *For any  $(t, \omega) \in [0, T) \times \Omega$  and  $\varphi \in C_b^{2,4}(\Lambda^t)$ ,*

$$\left| \frac{\varphi(t, \omega) - \mathbb{T}_h^{t,\omega}[\varphi(t+h, \cdot)]}{h} - \mathcal{L}\varphi(t, \omega) \right| \leq Ch, \quad \forall h \in (0, T-t). \tag{48}$$

(ii) *There exists  $L \geq L_0$  such that, for any  $(t, \omega) \in [0, T) \times \Omega$  and  $\varphi, \psi \in UC_b(\Lambda^t)$  uniformly Lipschitz continuous,*

$$|\mathbb{T}_h^{t,\omega}[\varphi] - \mathbb{T}_h^{t,\omega}[\psi]| \leq (1 + Ch)\bar{\mathcal{E}}_t^L[|\varphi - \psi|]. \tag{49}$$

Then we have

$$\max_{0 \leq i \leq n} |u^h(t_i, \omega) - u(t_i, \omega)| \leq Ch. \tag{50}$$

**Proof.** Denote  $\varepsilon_i := \sup_{\omega \in \Omega} |u^h(t_i, \omega) - u(t_i, \omega)|$ . Since  $\mathcal{L}u = 0$ , then (48) implies

$$|\mathbb{T}_h^{t_i,\omega}[u(t_{i+1}, \cdot)] - u(t_i, \omega)| \leq Ch^2.$$

Now by (13) and (49) we have

$$\begin{aligned} |[u^h - u](t_i, \omega)| &\leq |\mathbb{T}_h^{t_i,\omega}[u^h(t_{i+1}, \cdot)] - \mathbb{T}_h^{t_i,\omega}[u(t_{i+1}, \cdot)]| + Ch^2 \\ &\leq (1 + Ch)\varepsilon_{i+1} + Ch^2. \end{aligned}$$

This implies

$$\varepsilon_i \leq (1 + Ch)\varepsilon_{i+1} + Ch^2. \tag{51}$$

Since  $\varepsilon_n = 0$ , then by discrete Gronwall inequality we obtain (50) immediately. □

We next apply the above result to the scheme proposed in [Sec. 4](#).

**Theorem 5.2.** *Assume all the conditions in [Lemma 4.6](#) hold true, and the PPDE (1) has a classical solution  $u \in C_b^{2,4}(\Lambda)$ . Then for the scheme introduced in [Sec. 4](#), it holds that  $|u^h(t_i, \omega) - u(t_i, \omega)| \leq Ch$ .*

**Proof.** First, (49) follows directly from (43) and Remark 4.5. By Theorem 5.1 it suffices to check (48). Without loss of generality, we assume  $(t, \omega) = (0, \mathbf{0})$ .

Fix  $\varphi \in C_b^{2,4}(\Lambda)$ , and set  $\psi := \varphi(h, \cdot)$ . Recall the computation in Lemma 4.1 with  $(t, \omega) = (0, \mathbf{0})$  and  $c = 0$ , we have

$$\begin{aligned} \mathcal{D}^{(0)}\psi &= \varphi(t+h, \mathbf{0}) = \varphi(0, \mathbf{0}) + \int_0^h \partial_t \varphi(s, \mathbf{0}) ds \\ &= \varphi(0, \mathbf{0}) + \partial_t \varphi(0, \mathbf{0})h + \int_0^h \int_0^s \partial_t \partial_t \varphi(r, \mathbf{0}) dr ds \\ &= \varphi(0, \mathbf{0}) + \partial_t \varphi(0, \mathbf{0})h + O(h^2); \end{aligned}$$

$$\frac{\mathcal{D}^{(0)}\psi - \varphi(0, \mathbf{0})}{h} = \partial_t \varphi(0, \mathbf{0}) + O(h);$$

$$\begin{aligned} \mathcal{D}_i^{(1)}\psi &= \frac{1}{\mu_i h} \int_0^h \mathbb{E}^{\mathbb{P}^i} [(\partial_t + \mu_i \partial_{\omega^i})\varphi(s, B) - \partial_t \varphi(s, \mathbf{0})] ds \\ &= \partial_{\omega^i} \varphi(0, \mathbf{0}) + \frac{1}{\mu_i h} \int_0^h \mathbb{E}^{\mathbb{P}^i} [(\partial_t + \mu_i \partial_{\omega^i})\varphi(s, B) \\ &\quad - (\partial_t + \mu_i \partial_{\omega^i})\varphi(0, \mathbf{0}) - [\partial_t \varphi(s, \mathbf{0}) - \partial_t \varphi(0, \mathbf{0})]] ds \\ &= \partial_{\omega^i} \varphi(0, \mathbf{0}) + \frac{1}{\mu_i h} \int_0^h \int_0^s \mathbb{E}^{\mathbb{P}^i} [(\partial_t + \mu_i \partial_{\omega^i})(\partial_t + \mu_i \partial_{\omega^i})\varphi(r, B) \\ &\quad - \partial_t \partial_t \varphi(r, \mathbf{0})] dr ds \\ &= \partial_{\omega^i} \varphi(0, \mathbf{0}) + O(h). \end{aligned}$$

Similarly, we can show that

$$\mathcal{D}_{i,j}^{(2)}\psi = \partial_{\omega^i \omega^j}^2 \varphi(0, \mathbf{0}) + O(h), \quad i, j = 1, \dots, d.$$

Plug all these into (35) and recall that  $G$  is uniformly Lipschitz continuous in  $(y, z, \gamma)$ , we obtain (48), and hence prove the theorem.  $\square$

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