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# Martingale representation theorem for the $G$-expectation[☆]

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**Abstract**

This paper considers the nonlinear theory of $G$-martingales as introduced by Peng (2007) in [16,17]. A martingale representation theorem for this theory is proved by using the techniques and the results established in Soner et al. (2009) [20] for the second-order stochastic target problems and the second-order backward stochastic differential equations. In particular, this representation provides a hedging strategy in a market with an uncertain volatility.


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## 1. Introduction

The notion of a $G$-expectation as recently introduced by Peng [16,17] has several motivations and applications. One of them is the study of financial problems with uncertainty about

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the volatility. This important problem was also considered earlier by Denis and Martini [4]. Motivated by this application, Denis and Martini developed an almost pathwise theory of stochastic calculus. In this second approach, probabilistic statements are required to hold *quasi-surely*: namely $\mathbb{P}$-almost surely for all probability measures $\mathbb{P}$ from a large class of mutually singular measures $\mathcal{P}$. Denis and Martini employ functional analytic techniques while Peng's approach utilizes the theory of viscosity solutions of parabolic partial differential equations.

Indeed, the $G$-expectation is defined by Peng using the nonlinear heat equation,

$$-\partial_t u - G(D^2 u) = 0 \quad \text{on } [0, 1),$$

where the time maturity is taken to be $T = 1$ and for given $d \times d$ symmetric matrices $\overline{a} > 0$ and $0 \le \underline{a} \le \overline{a}$, the nonlinearity $G$ is defined by

$$G(\gamma) := \frac{1}{2} \sup\{\mathrm{tr}\,[\gamma a] \mid \underline{a} \le a \le \overline{a}\}, \quad \gamma \in \mathbb{R}^{d\times d}. \tag{1.1}$$

Then for "Markov-like" random variables, the $G$-expectation and conditional expectations are defined through the solution of the above equation with this random variable as its terminal condition at time $T = 1$. A $G$-martingale is then defined easily as a process which satisfies the martingale property through this conditional expectation. A brief introduction to this theory is provided in Section 2 below.

Denis and Martini [4] also construct a similar structure of quasi-sure stochastic analysis. However, they use a quite different approach which utilizes the set $\mathcal{P}$ of all probability measures $\mathbb{P}$ such that the canonical map in the Wiener space is a martingale under $\mathbb{P}$ and the quadratic variation of this martingale lies between $\underline{a} \le \overline{a}$. Although the constructions of the quasi-sure analysis and the $G$-expectations are substantially different, these theories are very closely related as proved recently by Denis et al. [3]. The paper [3] also provides a dual representation of the $G$-expectation as the supremum of expectations over $\mathcal{P}$. This duality and, more generally, the dynamic programming principle are generalized by Nutz [13] who considers lower and upper bounds $\underline{a}, \overline{a}$ that are random processes.

A probabilistic construction similar to quasi-sure stochastic analysis and $G$-expectations is the theory of second-order backward stochastic differential equations (2BSDE). This theory is developed in [1,2,18] as a generalization of BSDEs as initially introduced in [7,14]. In particular, 2BSDEs provide a stochastic representation for fully nonlinear partial differential equations. Since the $G$-expectation is defined through such a nonlinear equation, one expects the $G$-expectations to be naturally connected to the 2BSDEs. Equivalently, 2BSDEs can be viewed as the extension of $G$-expectations to more general nonlinearities. Indeed, recently the authors developed such a generalization and a duality theory for 2BSDEs using probabilistic constructions similar to those of quasi-sure analysis [19–21].

In this paper, we investigate the problem of representing an arbitrary $G$-martingale in terms of stochastic integrals and other processes. Specifically, we fix a finite horizon, say $T = 1$. Since all martingales can be seen as conditional expectations, we also fix the final value $\xi$. We then would like to construct stochastic processes $H$ and $K$ such that

$$Y_t := \mathbb{E}_t^G[\xi] = \xi - \int_t^1 H_s \mathrm{d}B_s + K_1 - K_t = \mathbb{E}^G[\xi] + \int_0^t H_s \mathrm{d}B_s - K_t,$$

where $\mathbb{E}_t^G$ is the $G$-conditional expectation and the process $M := -K$ is a non-increasing $G$-martingale. The stochastic integral that appears in the above is the regular Itô one. But it is also defined quasi-surely. More precisely, the above statement holds almost surely for all probability measures in $\mathcal{P}$. Equivalently, the above equation holds quasi-surely in the sense of Denis and Martini. In particular, all the above processes as well as the stochastic integral are defined on the support of all measures in the set $\mathcal{P}$. This is an important property of this martingale representation as $\mathcal{P}$ contains measures which are mutually singular. Moreover, there is no measure that dominates all measures in $\mathcal{P}$. Hence the above processes are defined on a large subset of our probability space.

A partial answer to this question was already provided by Xu and Zhang [23] for the class of *symmetric $G$-martingales*, i.e. a process $N$ where both itself and $-N$ are $G$-martingales. Since the $G$-expectation is not linear, the class of symmetric martingales is a strict subset of all $G$-martingales. In particular, the representations of symmetric martingales are obtained using only the stochastic integrals. We obtain the martingale representation in Theorem 5.1 for almost all square-integrable martingales. This result essentially provides a complete answer to the question of representation for the integrable classes defined in [17].

Our analysis utilizes the already mentioned duality result of Denis et al. [3]. Like [3], we also provide a dual characterization of $G$-martingales as an immediate consequence of the results in [3,17]. This observation is one of the key ingredients of our representation proof. Moreover, it can be used to extend the definition of $G$-martingales to a class larger than the integrability class $\mathcal{L}_G^1$ of Peng. Indeed, the above martingale representation result could also be proved for a larger class of random variables. But this development also requires the extension of $G$-expectations and conditional expectations to this larger class. These results are not pursued here. But in an example, Example A.3, we show that the integrability class $\mathcal{L}_G^1$ does not include all bounded random variables. Thus it is desirable to extend the theory to a larger class of random variables using the equivalent definitions that do not refer to partial differential equations. Indeed such a theory is developed by the authors in [19–21].

After the completion and the submission of this manuscript, we became aware of the manuscript of Song [22] which proves a decomposition result for random variables in $\mathcal{L}_G^p$ with $p > 1$. He obtained this result after a preliminary version of this manuscript, without Lemma 4.1, was circulated. In view of Lemma 4.1, our results hold for $\mathcal{L}_G^p$ with $p > 2$ and in contrast to that of [22], we also consider the possibly degenerate case $\underline{a} \geq 0$; see Assumption (2.1).

The paper is organized as follows. In Section 2, we review the theory of $G$-expectations and $G$-martingales. Section 3 defines the quasi-sure analysis of Denis and Martini and also provides the dual formulation. The main ingredients for our approach, such as the norms and spaces, are collected in Section 4. The main result is then stated and proved in Section 5. In the Appendix, we provide an approximation argument for the solutions of the partial differential equation. Then the connection between the integrability class of Peng and the spaces utilized in this paper is given in Appendix A.2.

### 1.1. Notation and spaces

We collect all the spaces and the notation used in the paper with a reference to their definitions. We always assume that $\overline{a} > 0$, $0 \leq \underline{a} \leq \overline{a}$.

- $\mathbb{F} = \{\mathcal{F}_t^B, t \geq 0\}$ is the filtration generated by the canonical process $B$.
- $\mathbb{E}^G$ is the $G$-expectation, defined in [17] and in Section 2.1.
- $\mathbb{E}_t^G$ is the conditional $G$-expectation.

- $\mathcal{L}_{ip}$ is the space of random variables of the form $\varphi(B_{t_1}, \ldots, B_{t_n})$ with a bounded, Lipschitz deterministic function $\varphi$ and time points $0 \le t_1 \le \cdots \le t_n \le 1$.
- $\mathcal{L}^p_G$ is the integrability class defined in Section 2.1 as the closure of $\mathcal{L}_{ip}$.
- $\mathcal{H}^{p,0}_G$ is the space of piecewise constant $G$-stochastic integrands; see Section 2.2.
- $\mathcal{H}^p_G$ is the integrability class defined in Section 2.2 as the closure of $\mathcal{H}^{p,0}_G$.
- $\mathcal{P} = \overline{\mathcal{P}}^W_{[\underline{a},\overline{a}]}$ are measures under which the canonical process is a martingale and satisfies (3.1).
- $\mathcal{P}(t, \mathbb{P})$ is defined in (3.3).
- $\mathbb{L}^p_{\mathcal{P}}$ is the set of all $p$-integrable random variables; see (4.1).
- $\mathcal{L}^p_{\mathcal{P}}$ is the closure of $\mathcal{L}_{ip}$ under the norm $\mathbb{L}^p_{\mathcal{P}}$; see (4.1).
- $\mathbb{H}^p_{\mathcal{P}}$ is the set of all $p$-integrable, $\mathbb{R}^d$-valued stochastic integrands; see (4.2).
- $\mathcal{H}^p_{\mathcal{P}}$ is the closure of $\mathcal{H}^{p,0}_G$ under the norm $\|\cdot\|_{\mathbb{H}^p_{\mathcal{P}}}$; see Definition 4.2.
- $\mathbb{S}^p_{\mathcal{P}}$ is the set of all $p$-integrable, *continuous* processes; see Definition 4.2.
- $\mathbb{I}^p_{\mathcal{P}}$ is the subset of $\mathbb{S}^p_{\mathcal{P}}$ that are *non-decreasing* with initial value 0; see Definition 4.2.
- $\mathbb{S}_d$ is the set of all $d \times d$ symmetric matrices with the usual ordering and identity $I_d$.
- For $\nu, \eta \in \mathbb{R}^d$, $A := \nu \otimes \eta \in \mathbb{S}_d$ is defined by $Ax = (\eta \cdot x)\nu$ for any $x \in \mathbb{R}^d$.
- For $A \in \mathbb{S}_d$, $\nu_k \in \mathbb{R}^d$ are its orthonormal eigenvectors and $\lambda_k$ are the corresponding eigenvalues such that
$$A = \sum_k \lambda_k [\nu_k \otimes \nu_k].$$
- For $A \in \mathbb{S}_d$, and a real number, $A \vee cI_d \in \mathbb{S}_d$ is defined by
$$A \vee cI_d := \sum_k (\lambda_k \vee c)\, [\nu_k \otimes \nu_k].$$

## 2. *G*-stochastic analysis of Peng [16,17]

We fix the time horizon $T = 1$. Let $\Omega := \{\omega \in C([0, 1], \mathbb{R}^d) : \omega(0) = 0\}$ be the canonical space, $B$ the canonical process, and $\mathbb{P}_0$ the Wiener measure. $\mathbb{F} = \{\mathcal{F}^B_t, t \in [0, 1]\}$ is the filtration generated by $B$. We note that $\mathcal{F}^B_{t-} = \mathcal{F}^B_t \ne \mathcal{F}^B_{t+}$.

In what follows, we always use the space $\Omega$ together with the filtration $\mathbb{F}$. We remark that we do not augment the filtration, as is usually done in standard stochastic analysis literature. In fact, for any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_1)$, denoting by $\bar{\mathbb{F}}^{\mathbb{P}} = \{\bar{\mathcal{F}}^{\mathbb{P}}_t, 0 \le t \le 1\}$ the augmented filtration of $\mathbb{F}$ under $\mathbb{P}$, we have the following straightforward result.

**Lemma 2.1.** *For any $\bar{\mathcal{F}}^{\mathbb{P}}_t$-measurable random variable $\xi$, there exists a unique $\mathbb{P}$-a.s. $\mathcal{F}_t$-measurable random variable $\tilde{\xi}$ such that $\tilde{\xi} = \xi$, $\mathbb{P}$-a.s.*

*Similarly, for every $\bar{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process $X$, there exists a unique $\mathbb{F}$-progressively measurable process $\tilde{X}$ such that $\tilde{X} = X$, $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-a.s. Moreover, if $X$ is $\mathbb{P}$-almost surely continuous, then one can choose $\tilde{X}$ to be $\mathbb{P}$-almost surely continuous.*

**Proof.** Lemma 2.4 in [19] proves the analogous result for the right continuous filtration $\mathbb{F}^+ := \{\mathcal{F}^B_{t+}, 0 \le t \le 1\}$ and its augmentation, instead of $\mathbb{F}$ and its augmentation. However, the proof does not change in this context and we prove the above result following the proof Lemma 2.4 in [19] line by line. □

In what follows, quite often we make use of the above result. Indeed, when a probability measure $\mathbb{P}$ is given, we will consider any process in its $\mathbb{F}$-progressively measurable version. However, we emphasize that these versions, in general, may depend on $\mathbb{P}$.

### 2.1. G-expectation and G-martingales

Following Peng [16], let $G$ be as in (1.1) with two given $d \times d$ symmetric matrices satisfying

$$0 \leq \underline{a} \leq \overline{a}, \quad \overline{a} > 0. \tag{2.1}$$

Notice that we allow degenerate diffusion matrices as the only positivity assumption is placed on the upper bound.

For a bounded Lipschitz continuous function $\varphi$ on $\mathbb{R}^d$, let $u$ be the unique, bounded, Lipschitz continuous viscosity solution of the following parabolic equation:

$$-\partial_t u - G(D^2 u) = 0 \quad \text{on } [0, 1), \quad \text{and} \quad u(1, x) = \varphi(x). \tag{2.2}$$

Here, $\partial_t$ and $D^2$ denote, respectively, the partial derivative with respect to $t$, and the partial Hessian with respect to the space variable $x$. Then, the conditional $G$-expectation of the random variable $\varphi(B_1)$ at time $t$ is defined by

$$\mathbb{E}_t^G[\varphi(B_1)] := u(t, B_t).$$

In particular, the $G$-expectation of $\varphi(B_1)$ is given by

$$\mathbb{E}^G[\varphi(B_1)] := \mathbb{E}_0^G[\varphi(B_1)] = u(0, 0).$$

Next consider the random variables of the form $\xi := \varphi(B_{t_1}, \ldots, B_{t_{n-1}}, B_{t_n})$ for some bounded Lipschitz continuous function $\varphi$ on $\mathbb{R}^{d \times n}$ and $0 \leq t_1 < \cdots < t_n = 1$. For $t_{i-1} \leq t < t_i$, let

$$\mathbb{E}_t^G[\xi] = \mathbb{E}_t^G[\varphi(B_{t_1}, \ldots, B_{t_n})] := v_i(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t),$$

where $\{v_i\}_{i=1,\ldots,n-1}$ is the unique, bounded, Lipschitz viscosity solution of the following equation:

$$\begin{aligned} &-\partial_t v_i - G(D^2 v_i) = 0, \quad t_{i-1} \leq t < t_i \quad \text{and} \\ &v_i(t_i, x_1, \ldots, x_{i-1}, x) = v_{i+1}(t_i, x_1, \ldots, x_{i-1}, x, x), \end{aligned} \tag{2.3}$$

and $v_n$ solves the above equation with final data $v_n(1, x_1, \ldots, x_{n-1}, x) = \varphi(x_1, \ldots, x_{n-1}, x)$. Here, for $v_i$, the variables $(x_1, \ldots, x_{i-1})$ are (fixed) parameters and the Hessian $D^2$ is the second-order derivative on $x$. Moreover, if we set $u_i(x_1, \ldots, x_i) = v_{i+1}(t_i, x_1, \ldots, x_i, x_i)$, then for $t_{i-1} \leq t < t_i$ we have the following additional identity:

$$\mathbb{E}_t^G[\varphi(B_{t_1}, \ldots, B_{t_n})] = v_i(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t) = \mathbb{E}_t^G[u_i(B_{t_1}, \ldots, B_{t_i})].$$

Let $\mathcal{L}_{ip}$ denote the space of all random variables of the form $\varphi(B_{t_1}, \ldots, B_{t_n})$ with a bounded and Lipschitz function $\varphi$. For $p \geq 1$, $\mathcal{L}_G^p$ is the closure of $\mathcal{L}_{ip}$ under the norm

$$\|\xi\|_{\mathcal{L}_G^p}^p := \mathbb{E}^G[|\xi|^p].$$

We may then extend the definitions of the $G$-expectation and the conditional $G$-expectation to all $\xi \in \mathcal{L}_G^1$. In particular, the important tower property of the conditional expectation still holds:

$$\mathbb{E}^G[\mathbb{E}_t^G[\xi]] = \mathbb{E}^G[\xi] \quad \text{for all } \xi \in \mathcal{L}_G^1. \tag{2.4}$$

A characterization of this space, in particular a Lusin type theorem, is obtained in [3]. However, since these integrability classes are defined through the closure of a rather smooth space $\mathcal{L}_{ip}$, they require substantial "smoothness". Indeed, in the Appendix, we construct a bounded random variable which is not in $\mathcal{L}_G^1$ (see Example A.3).

We can now define $G$-martingales.

**Definition 2.2.** An $\mathbb{F}$-progressively measurable $\mathcal{L}_G^1$-valued process $M$ is called a $G$-martingale if and only if for any $0 \le s < t$, $M_s = \mathbb{E}_s^G[M_t]$.

$M$ is called a symmetric $G$-martingale if both $M$ and $-M$ are $G$-martingales.

A $G$-stochastic integral (as will be defined in the next subsection) is an example of a symmetric $G$-martingale. In particular, the canonical process $B$ is a symmetric $G$-martingale. But not all $G$-martingales are stochastic integrals and not all of them are symmetric.

### 2.2. *The stochastic integral and quadratic variation*

For $p \in [1, \infty)$, we let $\mathcal{H}_G^{p,0}$ be the space of $\mathbb{F}$-progressively measurable, $\mathbb{R}^d$-valued piecewise constant processes $H = \sum_{i\ge 0} H_{t_i} \mathbf{1}_{[t_i,t_{i+1})}$ such that $H_{t_i} \in \mathcal{L}_G^p$. For $H \in \mathcal{H}_G^{p,0}$, the $G$-stochastic integral is easily defined by

$$\int_0^t H_s \mathrm{d}_G B_s := \sum_{i\ge 0} H_{t_i}[B_{t\wedge t_{i+1}} - B_{t\wedge t_i}].$$

Notice that this definition is completely universal in the sense that it is pointwise and independent of $G$. Let $\mathcal{H}_G^p$ be the closure of $\mathcal{H}_G^{p,0}$ under the norm

$$\|H\|_{\mathcal{H}_G^p}^p := \int_0^1 \mathbb{E}^G[|H_t|^p]\mathrm{d}t.$$

By a closure argument the stochastic integral is defined for all $H \in \mathcal{H}_G^p$.

It is clear that the set of $G$-martingales does not form a linear space (unless $\underline{a} = \overline{a}$). However, for any $H \in \mathcal{H}_G^{p,0}$, one may directly verify that the stochastic integral process $M := \int_0^\cdot H_s \mathrm{d}_G B_s$ is a $G$-martingale and so is $-M$. Hence, any $G$-stochastic integral is a symmetric $G$-martingale.

This notion of the stochastic integral can be used to define the quadratic variation process $\langle B\rangle_t^G$ as well. Indeed, the $\mathbb{S}_d$-valued process is defined by the identity

$$\langle B\rangle_t^G := \frac{1}{2} B_t \otimes B_t - \int_0^t B_s \otimes \mathrm{d}_G B_s, \quad \forall 0 \le t \le 1, \tag{2.5}$$

where the tensor product $\otimes$ is as in the Notation 1.1. We can directly check that the integrand $B_t$ is in the integration class $\mathcal{H}_G^p$. Therefore, $\langle B\rangle_t^G$ is well-defined.

## 3. The quasi-sure stochastic analysis of Denis and Martini [4]

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathbb{F})$ such that the canonical process $B$ is a martingale. Then, the quadratic variation process $\langle B\rangle_t$ of $B$ under $\mathbb{P}$ exists. We consider the subset $\mathcal{P} := \overline{\mathcal{P}}_{[\underline{a},\overline{a}]}^W$ of such measures $\mathbb{P}$ that $\langle B\rangle_t$ satisfies the following for some deterministic constant $c = c(\mathbb{P}) > 0$:

$$0 < \left[cI_d \vee \underline{a}\right] \le \frac{\mathrm{d}\langle B\rangle_t}{\mathrm{d}t} \le \overline{a}, \quad \forall t \in [0, 1],\ \mathbb{P}\text{-a.s.}, \tag{3.1}$$

where $I_d$ is the identity matrix in $\mathbb{S}_d$. Notice that when $\underline{a}$ is positive definite, as required in Denis and Martini [4], we do not need $cI_d$ in the lower bound. Also, the constant $c = c(\mathbb{P})$ may be different for each measure. Denis and Martini [4] give the following definition.

**Definition 3.1.** We say that a property holds $\mathcal{P}$-quasi-surely, abbreviated as q.s., if it holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$.

**Remark 3.2.** All the results in this paper will also hold true if we let $\mathcal{P} := \overline{\mathcal{P}}^S_{[\underline{a},\overline{a}]}$ be the set of all probability measures $\mathbb{P}^\alpha$ given by

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where } X^\alpha_t := \int_0^t \alpha_s^{1/2} \mathrm{d}B_s,\ t \in [0,1], \mathbb{P}_0 - \text{a.s.}$$

for some $\mathbb{F}$-progressively measurable process $\alpha$ taking values in $\mathbb{S}_d$ and satisfying

$$\left[c(\alpha) I_d \vee \underline{a}\right] \le \alpha_t \le \overline{a}, \quad \forall t \in [0,1],\ \mathbb{P}_0 - \text{a.s.},$$

where the constant $c(\alpha) > 0$ may depend on $\alpha$. We note that $\overline{\mathcal{P}}^S_{[\underline{a},\overline{a}]}$ is a strict subset of $\overline{\mathcal{P}}^W_{[\underline{a},\overline{a}]}$ and each $\mathbb{P} \in \overline{\mathcal{P}}^S_{[\underline{a},\overline{a}]}$ satisfies the Blumenthal zero–one law and the martingale representation property. We remark that Denis and Martini [4] uses the space $\overline{\mathcal{P}}^W_{[\underline{a},\overline{a}]}$. But Denis et al. [3] and our subsequent work [21] essentially use $\overline{\mathcal{P}}^S_{[\underline{a},\overline{a}]}$. □

The following are immediate consequences of the definition of $G$-expectations.

**Proposition 3.3.** *Let $H \in \mathcal{H}^2_G$. Then, $H$ is Itô-integrable for every $\mathbb{P} \in \mathcal{P}$. Moreover,*

$$\int H_s \mathrm{d}_G B_s = \int H_s \mathrm{d}B_s, \quad \mathbb{P}\text{-a.s. for every } \mathbb{P} \in \mathcal{P}, \tag{3.2}$$

*where the right hand side is the usual Itô integral. Consequently, the quadratic variation process $\langle B\rangle^G$ defined in (2.5) agrees with the usual quadratic variation process quasi-surely.*

**Proof.** The above statements clearly hold for the integrands $H \in \mathcal{H}^{2,0}_G$ (i.e. the piecewise constant processes). For $H \in \mathcal{H}^2_G$, there exist $H^n \in \mathcal{H}^{2,0}_G$ such that $\lim_{n\to\infty} \|H^n - H\|_{\mathcal{H}^2_G} = 0$. For any fixed $\mathbb{P} \in \mathcal{P}$, since $\mathbb{E}^{\mathbb{P}}[\int_0^1 |H^n_t - H_t|^2 \mathrm{d}t] \le \|H^n - H\|^2_{\mathcal{H}^2_G}$, the equality (3.2) holds. The statement about the quadratic variation follows from the general statement about the stochastic integrals and the formula (2.5). □

Next we recall a dual characterization of the $G$-expectation as proved in [3]. We will then generalize that characterization to the $G$-conditional expectations. Like the previous result, this generalization is also an immediate consequence of the previous results. We need the following notation, for $t \in [0,1]$ and $\mathbb{P} \in \mathcal{P}$:

$$\mathcal{P}(t,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{P} :\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\}. \tag{3.3}$$

Notice that for any $\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})$ and $\xi \in \mathcal{L}^1_G$, the random variable $\mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$ is defined both $\mathbb{P}$-almost surely and $\mathbb{P}'$-almost surely. Also recall that $\operatorname{ess\,sup} = \operatorname{ess\,sup}^{\mathbb{P}}$ is the essential supremum of a class of $\mathbb{P}$-almost surely defined random variables. Clearly, it is also defined $\mathbb{P}$-almost surely (see Definition A.1 on page 323 in [10]). In particular, for $t \in [0,1]$, we may define

$$\underset{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})}{\operatorname{ess\,sup}}\ \mathbb{E}^{\mathbb{P}'}[\xi \mid \mathcal{F}_t] \tag{3.4}$$

as a $\mathbb{P}$-almost sure random variable. We remark that, for given $\mathbb{P}$, the above random variable can be first defined as $\mathcal{F}_t^{\mathbb{P}}$-measurable. However, in view of Lemma 2.1, we will always consider its $\mathcal{F}_t$-measurable version.

We now have the following characterization of the $G$-conditional expectation.

**Proposition 3.4.** *For any $\xi \in \mathcal{L}_G^1$, $t \in [0, 1]$, and $\mathbb{P} \in \mathcal{P}$,*

$$\mathbb{E}_t^G[\xi] = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi \mid \mathcal{F}_t], \quad \mathbb{P}\text{-a.s.}$$

*Moreover, an $\mathbb{F}$-progressively measurable $\mathcal{L}_G^1$ valued process $M$ is a $G$-martingale if and only if it satisfies the following dynamic programming principle for all $0 \le s \le t \le 1$ and $\mathbb{P} \in \mathcal{P}$:*

$$M_s = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[M_t \mid \mathcal{F}_s], \quad \mathbb{P}\text{-a.s.} \tag{3.5}$$

**Proof.** The characterization of the conditional expectation follows from [3] for $\xi \in \mathcal{L}_{ip}$. Indeed, [3] proves this result when the set of probability measures is $\overline{\mathcal{P}}_{[\underline{a},\overline{a}]}^S$ as defined in Remark 3.2. Moreover when $\xi = g(B_1)$, we can use the dynamic programming equation (2.2) and classical verification arguments as in [8] to conclude the claimed representation in our formulation. Then, a simple induction argument extends the result to all $\xi \in \mathcal{L}_{ip}$.

For $\xi \in \mathcal{L}_G^1$, there exist $\xi_n \in \mathcal{L}_{ip}$ such that $\lim_{n\to\infty} \mathbb{E}^G[|\xi_n - \xi|] = 0$. Then, for every $t \in [0, 1]$, by the definition of $\mathbb{E}_t^G[\xi]$,

$$\lim_{n\to\infty} \mathbb{E}^G[|\mathbb{E}_t^G[\xi_n] - \mathbb{E}_t^G[\xi]|] = 0.$$

Moreover, for any $t \in [0, 1]$ and $\mathbb{P} \in \mathcal{P}$,

$$\mathbb{E}^{\mathbb{P}}[|\mathbb{E}_t^G[\xi_n] - \mathbb{E}_t^G[\xi]|] \le \mathbb{E}^G[|\mathbb{E}_t^G[\xi_n] - \mathbb{E}_t^G[\xi]|].$$

Using these and (2.4), we directly estimate that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\Big| \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi_n] - \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi]\Big|\Big] &\le \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[|\xi_n - \xi|]\Big] \\
&\le \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^G[|\xi_n - \xi|]\Big] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}_t^G[|\xi_n - \xi|]] \\
&\le \mathbb{E}^G[\mathbb{E}_t^G[|\xi_n - \xi|]] = \mathbb{E}^G[|\xi_n - \xi|].
\end{aligned}$$

Therefore,

$$\mathbb{E}_t^G[\xi] = \lim_{n\to\infty} \mathbb{E}_t^G[\xi_n] = \lim_{n\to\infty} \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi_n] = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi], \quad \mathbb{P}\text{-a.s.}$$

The martingale property is a direct consequence of the tower property of the $G$-conditional expectation as proved in [16] and the above formula for the conditional expectation. $\square$

**Remark 3.5.** In their classical paper [5], El Karoui and Jeanblanc consider a very general stochastic optimal control problem. Their results in our context imply that

$$M_s^{\mathbb{P}} := \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi \mid \mathcal{F}_s]$$

is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$. Moreover $\mathbb{P}^*$ is a maximizer if and only if $M^{\mathbb{P}^*}$ is a $\mathbb{P}^*$-martingale. While this result provides a characterization of the optimal measure $\mathbb{P}^*$, it does not provide a "universal" hedge. More precisely their approach provides an optimal control which is defined only for the optimal measure and on its support. Indeed, the supermartingale property of $M^{\mathbb{P}}$ implies that there are an increasing process $K^{\mathbb{P}}$ and an integrand $H^{\mathbb{P}}$ such that

$$M_t^{\mathbb{P}} = \int_0^t H_s^{\mathbb{P}} \mathrm{d}B_s - K_t^{\mathbb{P}}.$$

However, aggregation of these processes into one universally defined $K$ and one universally defined $H$ is not immediate. In the standard Markovian context, this problem can be solved directly. However, it is exactly the non-Markovian generalization that motivates this paper and [4,17,16]. This interesting question of aggregation is further discussed in Remark 4.3. □

## 4. Spaces and norms

The particular case of $t = 0$ in (3.5) gives the following dual characterization proved in [3]:

$$\mathbb{E}^G[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi].$$

The above results enable us to extend the definitions of $G$-expectation and $G$-martingales to a possibly larger class of random variables. In particular, this extension has the advantage of not referring to the partial differential equation (2.2). We will not develop this theory here. However, in view of the results and the norms used in the theory of BSDEs, we introduce the following function spaces.

For $p \geq 1$, and an $\mathcal{F}_1$-measurable, non-negative random variable $\xi$, we set

$$\|\xi\|^p_{\mathbb{L}^p_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\operatorname*{ess\,sup}_{t\in[0,1]}(M_t^{\mathbb{P}}(\xi))^p], \quad \text{where } M_t^{\mathbb{P}}(\xi) := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t].$$

In the above definition, *a priori* we do not have any information on the time regularity of $M_t^{\mathbb{P}}(\xi)$. That is the reason for defining the norm through the random variable $\operatorname{ess\,sup}_{t\in[0,1]}(M_t^{\mathbb{P}}(\xi))$, which is, in view of Lemma 2.1, $\mathcal{F}_1$-measurable. Alternatively, one may first prove that $M_t^{\mathbb{P}}(\xi)$ is a $\mathbb{P}$-supermartingale and that it admits a càdlàg version. Then, $\sup_{t\in[0,1]}(M_t^{\mathbb{P}}(\xi))^p$ would be measurable and we could use it in the definition. However, we believe that this issue is tangential to the main thrust of the paper and we prefer to give the above quicker definition.

We next define

$$\mathbb{L}^p_{\mathcal{P}} := \{\xi \ : \ \mathcal{F}_1\text{-measurable and } \|\xi\|_{\mathbb{L}^p_{\mathcal{P}}} := \||\xi|\|_{\mathbb{L}^p_{\mathcal{P}}} < \infty\}, \tag{4.1}$$
$$\mathcal{L}^p_{\mathcal{P}} := \text{closure of } \mathcal{L}_{ip} \text{ under the norm} \mathbb{L}^p_{\mathcal{P}}.$$

Notice that if $\xi \in \mathcal{L}^1_G$, then $M_t^{\mathbb{P}}(\xi) = \mathbb{E}_t^G[\xi]$ for every $\mathbb{P} \in \mathcal{P}$. Moreover, for every $\xi \in \mathcal{L}_{ip}$, $\|\xi\|_{\mathbb{L}^p_{\mathcal{P}}} = \|\xi^*\|_{\mathcal{L}^p_G}$, where $\xi^* := \sup_{t\in[0,1]} \mathbb{E}_t^G[|\xi|]$.

In the Appendix, we compare the integrability classes defined by Peng [17] and the above spaces. The connection is related to the Doob maximal inequalities in the setting of $G$-expectations. In particular, we prove the following.

**Lemma 4.1.** $\cup_{p>2} \mathcal{L}^p_G \subset \mathcal{L}^2_{\mathcal{P}} \subset \mathbb{L}^2_{\mathcal{P}} \cap \mathcal{L}^2_G \subset \mathbb{L}^2_{\mathcal{P}}$. *Moreover, the final inclusion is strict.*

We also define the following norms for the processes. As usual $1 \leq p < \infty$. For an $\mathbb{F}$-progressively measurable integrand $H$ and a stochastic process $Y$, we set

$$\|H\|^p_{\mathbb{H}^p_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 (\mathrm{d}\langle B\rangle_t H_t \cdot H_t)\right)^{\frac{p}{2}}\right], \tag{4.2}$$

$$\|Y\|^p_{\mathbb{S}^p_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\operatorname*{ess\,sup}_{0\leq t\leq 1} |Y_t|^p]. \tag{4.3}$$

If $Y_t = \mathbb{E}^G_t[|\xi|]$ for some $\xi \in \mathcal{L}^1_G$, then $\|Y\|^p_{\mathbb{S}^p_{\mathcal{P}}} = \|\xi\|^p_{\mathbb{L}^p_{\mathcal{P}}}$. This identity also motivates the definition of the norm $\mathbb{L}^p_{\mathcal{P}}$. Moreover, when the lower bound $\underline{a}$ in (3.1) is non-degenerate, then the $\mathbb{H}^p_{\mathcal{P}}$ norm is equivalent to the norm used in [3,16]:

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 |H_t|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right].$$

By analogy with the standard notation in stochastic calculus, we define the following spaces.

**Definition 4.2.** Let $p \in [1, \infty)$ and $\mathcal{P}$ be as in Section 3.

- $\mathbb{H}^p_{\mathcal{P}}$ is the set of all $\mathbb{F}$-progressively measurable integrands with a finite $\|\cdot\|_{\mathbb{H}^p_{\mathcal{P}}}$-norm.
- $\mathcal{H}^p_{\mathcal{P}}$ is the closure of $\mathcal{H}^{p,0}_G$ under the norm $\|\cdot\|_{\mathbb{H}^p_{\mathcal{P}}}$.
- $\mathbb{S}^p_{\mathcal{P}}$ is the set of all $\mathbb{F}$-progressively measurable processes with quasi-surely *continuous* paths and finite $\|\cdot\|_{\mathbb{S}^p_{\mathcal{P}}}$-norm.
- $\mathbb{I}^p_{\mathcal{P}}$ is the subset of $\mathbb{S}^p_{\mathcal{P}}$ of *non-decreasing* processes with $X_0 = 0$. $\square$

Clearly all of the above spaces are defined as quasi-sure equivalence classes. As such, they are complete and therefore Banach spaces. Also $\|H\|_{\mathbb{H}^p_{\mathcal{P}}} \leq \|H\|_{\mathcal{H}^p_G}$ for $H \in \mathcal{H}^{p,0}_G$; then it is clear that $\mathcal{H}^p_G \subset \mathcal{H}^p_{\mathcal{P}}$. Therefore $\mathcal{H}^p_{\mathcal{P}}$ is the closure of $\mathcal{H}^p_G$ under the norm $\|\cdot\|_{\mathbb{H}^p_{\mathcal{P}}}$.

**Remark 4.3.** Given $\xi \in \mathbb{L}^1_{\mathcal{P}}$ (but not necessarily in $\mathcal{L}^1_G$) and an $\mathbb{F}$-stopping time $\tau$, it is not straightforward to define the conditional $G_{\mathcal{P}}$-expectation $\mathbb{E}^{\mathcal{P}}_\tau[\xi]$ as in (3.4). Indeed, set

$$M^{\mathbb{P}}_\tau := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(\tau,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_\tau[\xi], \quad \mathbb{P}\text{-a.s.}$$

Then, to define the conditional expectation, we need to aggregate this family of random variables $\{M^{\mathbb{P}}_\tau, \mathbb{P} \in \mathcal{P}\}$ into one "universally" defined random variable. A similar problem arises in the definition of a stochastic integral for a given integrand $H \in \mathbb{H}^2_{\mathcal{P}}$. Again, for $\mathbb{P} \in \mathcal{P}$, we set $M^{\mathbb{P}}_t := \int_0^t H_s \mathrm{d}B_s$. Then, to define the $G$-stochastic integral of $H$ we need to aggregate this family of stochastic processes.

The issue of aggregation is an interesting technical question. Generally, a solution to this technical issue is given by imposing regularity on the random variables. Indeed, for all random variables which are in $\mathcal{L}^p_G$, one can define the universal version through a closure argument. However, there are other alternatives and a comprehensive study of this question is given in our accompanying paper [19].

Finally we recall that, when the integrand $H$ has the additional regularity that it is a càdlàg process, then Karandikar [9] defines the stochastic integral $M_t^{\mathbb{P}} := \int_0^t H_s \mathrm{d}B_s$ pointwise. This definition can then be used as the aggregating process. □

## 5. The martingale representation theorem

To motivate the main result of this paper, we first consider the case $\xi = \varphi(B_1)$ for some smooth, bounded function $\varphi$. In this case, as in [15,16], a formal construction can be derived by simply using the Itô's formula. Now suppose that the solution $u(t, x)$ of (2.2) is smooth. Indeed, we can approximate the eqaution (2.2) such that the approximating equation admits smooth solutions as proved by Krylov [11]. This is done in the Appendix. Then, we set $Y_t := u(t, B_t) = \mathbb{E}_t^G[\xi]$, $H_t := \nabla u(t, B_t)$ and

$$K_t := \int_0^t \left( G(D^2 u(s, B_s)) - \frac{1}{2} \mathrm{tr}\,[\hat{a}_s D^2 u(s, B_s)] \right) \mathrm{d}s, \qquad \hat{a}_t := \frac{\mathrm{d}\langle B \rangle_t}{\mathrm{d}t}, \quad \text{q.s.}$$

Using (2.2), (3.1) and the definition of the nonlinearity $G$, one may directly check that

$$Y_t = \xi - \int_t^1 H_s \mathrm{d}B_s + K_1 - K_t, \quad \text{and} \quad \mathrm{d}K_t \geq 0 \quad \text{q.s.}$$

Also, the characterization of $G$-martingales in Proposition 3.4 and the definition of the nonlinearity $G$ imply that $-K$ is a $G$-martingale. Hence for the random variable $\xi = \varphi(B_1)$, we have the martingale representation. More importantly, this example also shows that in general a non-decreasing process $K$ is always present in this representation. The above construction is also the basic step in our construction. Indeed essentially for almost all random variables in $\mathcal{L}_{ip}$ the above construction proves the result. We then prove that stochastic integrals and non-decreasing martingales are closed subsets under the appropriate norms as defined in the preceding section. Finally, these results allow us to prove the result by a closure argument.

### 5.1. Main results

We first state the main result. Recall that function spaces are defined in Definition 4.2.

**Theorem 5.1.** *Assume that $\underline{a}$ and $\overline{a}$ satisfy (2.1). Then, for every $\xi \in \mathcal{L}^2_{\mathcal{P}}$, the conditional $G$-expectation process $Y_t := \mathbb{E}_t^G[\xi]$ is in $\mathbb{S}^2_{\mathcal{P}}$, and there exist unique $H \in \mathcal{H}^2_{\mathcal{P}}$, $K \in \mathbb{I}^2_{\mathcal{P}}$ such that $N := -K$ is a $G$-martingale and for every $t \in [0, T]$,*

$$Y_t = \xi - \int_t^1 H_s \mathrm{d}B_s + K_1 - K_t = \mathbb{E}^G[\xi] + \int_0^t H_s \mathrm{d}B_s - K_t, \quad q.s. \tag{5.1}$$

*In particular, the stochastic integrals are defined both as $G$-stochastic integrals and also quasi-surely. Moreover the following estimate is also satisfied with a universal constant $C^*$:*

$$\|Y\|_{\mathbb{S}^2_{\mathcal{P}}} + \|H\|_{\mathbb{H}^2_{\mathcal{P}}} + \|K\|_{\mathbb{S}^2_{\mathcal{P}}} \leq C^* \|\xi\|_{\mathbb{L}^2_{\mathcal{P}}}. \tag{5.2}$$

The proof of the above theorem will be completed in several lemmas below.

In the above theorem the integrand $H$ is not only in the class $\mathbb{H}^2_{\mathcal{P}}$ but also in the closure of $\mathcal{H}_G^{2,0}$ under the norm $\|\cdot\|_{\mathbb{H}^2_{\mathcal{P}}}$. Indeed this fact implies that stochastic integral is well-defined quasi-surely as is shown in the next subsection.

The following is an immediate corollary of the above martingale representation.

**Corollary 5.2.** *A $G$-martingale $M$ with $M_1 \in \mathcal{L}^2_{\mathcal{P}}$ is symmetric if and only if the process $K$ in the representation (5.1) is identically equal to zero.*

In addition to the estimate (5.2), an estimate of the differences of the solutions is known to be an important tool. Let $\xi_1, \xi_2 \in \mathcal{L}^2_{\mathcal{P}}$ and $(Y^i, H^i, K^i)$ be the processes in the martingale representation. We set $\delta\xi := \xi^1 - \xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$ and $\delta K := K^1 - K^2$.

**Theorem 5.3.** *There exists a universal constant $C^*$ such that*

$$\|\delta Y\|_{\mathbb{S}^2_{\mathcal{P}}} \le \|\delta\xi\|_{\mathbb{L}^2_{\mathcal{P}}},$$

$$\|\delta H\|_{\mathbb{H}^2_{\mathcal{P}}} + \|\delta K\|_{\mathbb{S}^2_{\mathcal{P}}} \le C^*[\|\delta\xi\|_{\mathbb{L}^2_{\mathcal{P}}} + (\|\xi^1\|^{\frac{1}{2}}_{\mathbb{L}^2_{\mathcal{P}}} + \|\xi^2\|^{\frac{1}{2}}_{\mathbb{L}^2_{\mathcal{P}}})\, \|\delta\xi\|^{\frac{1}{2}}_{\mathbb{L}^2_{\mathcal{P}}}].$$

## *5.2. The stochastic integral and symmetric $G$-martingales*

As discussed in Remark 4.3, for an integrand $H \in \mathbb{H}^2_{\mathcal{P}}$ it is not immediate to define the stochastic integral $\int_0^\cdot H_s \mathrm{d}B_s$ quasi-surely. However, the stochastic integral is defined in [17] for integrands $H \in \mathcal{H}^{2,0}_G$. Then, for integrands in $\mathcal{H}^2_{\mathcal{P}}$ a closure argument can be used to construct the stochastic integral quasi-surely. (Recall that $\mathcal{H}^2_{\mathcal{P}}$ is the closure of $\mathcal{H}^{2,0}_G$ under the norm $\|\cdot\|_{\mathbb{H}^2_{\mathcal{P}}}$.)

**Theorem 5.4.** *For any $H \in \mathcal{H}^2_{\mathcal{P}}$, the stochastic integral $\int_0^\cdot H_s \mathrm{d}B_s$ exists quasi-surely. Moreover, the stochastic integral satisfies the Burkholder–Davis–Gundy inequality*

$$\|H\|_{\mathbb{H}^2_{\mathcal{P}}} \le \left\| \int_0^\cdot H_s \mathrm{d}B_s \right\|_{\mathbb{S}^2_{\mathcal{P}}} \le 2\|H\|_{\mathbb{H}^2_{\mathcal{P}}}. \tag{5.3}$$

**Proof.** Let $H \in \mathcal{H}^2_{\mathcal{P}}$. Then, there is a sequence $\{H^n\}_n \subset \mathcal{H}^{2,0}_G$ such that $\|H^n - H\|_{\mathbb{H}^2_{\mathcal{P}}}$ converges to zero as $n$ tends to infinity. By relabeling the sequence we may assume that $\|H^n - H\|_{\mathbb{H}^2_{\mathcal{P}}} \le 2^{-n}$ for every $n$. Moreover, since $H \in \mathbb{H}^2_{\mathcal{P}}$, for every $\mathbb{P} \in \mathcal{P}$,

$$M^{\mathbb{P}}_t := \int_0^t H_s \mathrm{d}B_s, \quad t \in [0, 1],$$

is $\mathbb{P}$-almost surely well-defined. Since $H^n \in \mathcal{H}^{2,0}_G$, the $G$-stochastic integral

$$M^n_t := \int_0^t H^n_s \mathrm{d}B_s, \quad t \in [0, 1],$$

is also defined pointwise.

We now have to prove that the family $\{M^{\mathbb{P}}, P \in \mathcal{P}\}$ can be aggregated into a universal $\mathbb{F}$-progressively measurable process. For this, we define

$$\overline{M}_t := \overline{\lim_{n\to\infty}} M^n_t, \quad t \in [0, 1].$$

Notice that $\overline{M}$ is pointwise defined and $\mathbb{F}$-progressively measurable. We continue by showing that $\overline{M} = M^{\mathbb{P}}$, $\mathbb{P}$-almost surely, for every $\mathbb{P} \in \mathcal{P}$. Indeed for any $\mathbb{P} \in \mathcal{P}$, we use the

Burkholder–Davis–Gundy inequality to obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1}\left|M_t^n - M_t^{\mathbb{P}}\right|^2\right] &= \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1}\left|\int_0^t (H_s^n - H_s)\mathrm{d}B_s\right|^2\right] \\ &\le 4\mathbb{E}^{\mathbb{P}}\left[\left|\int_0^1 (H_s^n - H_s)\mathrm{d}B_s\right|^2\right] \\ &= 4\mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{a}_s^{1/2}(H_s^n - H_s)|^2\mathrm{d}s\right] \\ &\le 4\|H^n - H\|^2_{\mathbb{H}^2_{\mathcal{P}}} \le 2^{2-2n}.\end{aligned}$$

We then directly estimate that

$$\sum_{n=1}^{\infty}\mathbb{P}[\sup_{0\le t\le 1}|M_t^n - M_t^{\mathbb{P}}| \ge n^{-2}] \le \sum_{n=1}^{\infty} n^2\mathbb{E}^{\mathbb{P}}[\sup_{0\le t\le 1}|M_t^n - M_t^{\mathbb{P}}|^2]^{\frac{1}{2}} < \infty.$$

By the Borel–Cantelli Lemma,

$$\lim_{n\to\infty}\sup_{0\le t\le T}|M_t^n - M_t^{\mathbb{P}}| = 0, \quad \mathbb{P}\text{-a.s.}$$

This implies that $M^{\mathbb{P}} = \overline{M}$, $\mathbb{P}$-almost surely. Since this holds for every $\mathbb{P} \in \mathcal{P}$, we conclude that the process $\overline{M}$ is an aggregating process. Hence the stochastic integral is defined.

The Burkholder–Davis–Gundy inequalities follow directly from the definitions. □

We close this subsection by stating the following result for symmetric $G$-martingales, which is an immediate consequence of the main results.

**Theorem 5.5.** *Let $M$ be a $G$-martingale with $M_1 \in \mathcal{L}^2_{\mathcal{P}}$. The following are equivalent:*

(i) *$M$ is a $\mathbb{P}$-martingale for every $\mathbb{P} \in \mathcal{P}$.*
(ii) *$M$ is a symmetric $G$-martingale.*
(iii) *For any $G$-martingale $N$, both $N + M$ and $N - M$ are also $G$-martingales.*
(iv) *$\mathbb{E}^G\{-M_t\} = -\mathbb{E}^G\{M_t\}$ for any $t \ge 0$.*
(v) *There exists $H \in \mathcal{H}^2_{\mathcal{P}}$ such that $M_t := M_0 + \int_0^t H_s\mathrm{d}B_s$.*

**Remark 5.6.** The main reason for the requirements $\xi \in \mathcal{L}^2_{\mathcal{P}}$ and $H \in \mathcal{H}^2_{\mathcal{P}}$ is to ensure the existence of the universal version of the conditional $G$-expectation $E_t^G[\xi]$ and the stochastic integral $\int_0^t H_s\mathrm{d}B_s$. However, if we were given a $G$-martingale $M$ with $M_1 \in \mathbb{L}^2_{\mathcal{P}}$, then there would be no aggregation issue. Then, following the same arguments, one can easily show that Theorem 5.5 still holds true under the weaker assumption $M_1 \in \mathbb{L}^2_{\mathcal{P}}$. Moreover, (v) requires only $H \in \mathbb{H}^2_{\mathcal{P}}$.

Recall that $\mathbb{I}^2_{\mathcal{P}}$ is defined in Definition 4.2 as the set of all $\mathbb{F}$-progressively measurable, non-decreasing, continuous processes with finite $\|\cdot\|_{\mathbb{S}^p_{\mathcal{P}}}$. For $(H, K) \in \mathcal{H}^2_{\mathcal{P}} \times \mathbb{I}^2_{\mathcal{P}}$, define a process by

$$M_t := M_0 + \int_0^t H_s\mathrm{d}B_s - K_t. \tag{5.4}$$

An immediate corollary of the above result is the following.

**Corollary 5.7.** *The process $M$ defined in (5.4) is a $G$-martingale if and only if the non-increasing process $-K$ is a $G$-martingale.*

### 5.3. Increasing $G$-martingales

In this section we show that the set of non-decreasing $G$-martingales is a closed set. Indeed, let $MI^2_{\mathcal{P}}$ be the set of all processes $K \in \mathbb{I}^2_{\mathcal{P}}$ such that $-K$ is a $G$-martingale. Then we have the following closure result which is similar to Theorem 5.4.

**Theorem 5.8.** *The space $MI^2_{\mathcal{P}}$ is closed in $\mathbb{S}^2_{\mathcal{P}}$ under norm $\|\cdot\|_{\mathbb{S}^2_{\mathcal{P}}}$.*

**Proof.** Consider a sequence $K^n \in MI^2_{\mathcal{P}}$ converging to a process $K \in \mathbb{I}^2_{\mathcal{P}}$ in the norm $\|\cdot\|_{\mathbb{S}^2_{\mathcal{P}}}$. We claim that the limit $-K$ is also a $G$-martingale and therefore $K \in MI^2_{\mathcal{P}}$. Indeed, for every $0 \le s \le t \le 1$, set $A_t := K_t - K_s$ and $A^n_t := K^n_t - K^n_s$. Then, by the martingale property of the sequence, for every $n$ and $\mathbb{P} \in \mathcal{P}$, we have

$$\operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s [A^n_t] = 0, \quad \mathbb{P}\text{-a.s.}$$

Moreover, $\mathbb{P}$-a.s.,

$$\begin{aligned} \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s [A_t] &\le \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s |A_t - A^n_t| + \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s [A^n_t] \\ &= \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s |A_t - A^n_t|. \end{aligned}$$

The following can be shown directly from the definitions:

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s |A_t - A^n_t|\Big] \le \|A - A^n\|_{\mathcal{S}^2_{\mathcal{P}}}.$$

Hence by the convergence of $\|A - A^n\|_{\mathcal{S}^2_{\mathcal{P}}}$ to zero as $n$ tends to infinity, we conclude that

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(s,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_s |A_t - A^n_t| = 0, \quad \mathbb{P}\text{-a.s.}$$

Since $0 \le s \le t \le 1$ and $\mathbb{P} \in \mathcal{P}$ are arbitrary, the limit process $-K$ is also a $G$-martingale. □

### 5.4. Estimates

For $(H, K) \in \mathbb{H}^2_{\mathcal{P}} \times \mathbb{I}^2_{\mathcal{P}}$, let $M$ be defined as in (5.4). In this subsection, we prove certain estimates for $H$ and $K$ in terms of the process $M$. These estimates are similar to those obtained for reflected backward stochastic differential equations in [6].

**Proposition 5.9.** *Let $H, K, M$ be as in (5.4). There exists a constant $C$ depending only on the dimension such that*

$$\|H\|_{\mathbb{H}^2_{\mathcal{P}}} + \|K\|_{\mathbb{S}^2_{\mathcal{P}}} \le C\|M\|_{\mathbb{S}^2_{\mathcal{P}}}.$$

**Proof.** We directly calculate that

$$\mathrm{d}|M_t|^2 = 2M_t H_t \mathrm{d}B_t - 2M_t \mathrm{d}K_t + \mathrm{d}\langle B\rangle_t H_t \cdot H_t.$$

We integrate over $[t, 1]$ to obtain

$$|M_t|^2 + \int_t^1 \mathrm{d}\langle B\rangle_s H_s \cdot H_s = |M_1|^2 + 2\int_t^1 M_s \mathrm{d}K_s - 2\int_t^1 M_s H_s \mathrm{d}B_s.$$

We then take the expected value under an arbitrary $\mathbb{P} \in \mathcal{P}$ to arrive at

$$\mathbb{E}^{\mathbb{P}}\left[|M_t|^2 + \int_0^1 \mathrm{d}\langle B\rangle_t H_t \cdot H_t\right] \le \mathbb{E}^{\mathbb{P}}\left[|M_1|^2 + 2\int_0^1 |M_t|\mathrm{d}K_t\right].$$

Since $\mathrm{d}K_t \ge 0$, for any $\varepsilon > 0$, we have the following estimate:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[|M_t|^2 + \int_0^1 \mathrm{d}\langle B\rangle_t H_t \cdot H_t\right] &\le \mathbb{E}^{\mathbb{P}}[|M_1|^2 + 2(\sup_{t\in[0,1]} |M_t|)K_1] \\ &\le (1+\varepsilon^{-1})\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,1]} |M_t|^2] + \varepsilon \mathbb{E}^{\mathbb{P}}[K_1^2]. \qquad (5.5)\end{aligned}$$

Next we estimate $K$. Recall that $0 = K_0 \le K_t$. By the definition of $M_t$,

$$\begin{aligned}K_1^2 &= \left(M_1 - M_0 - \int_0^1 H_s \mathrm{d}B_s\right)^2 \\ &\le 3|M_1|^2 + 3|M_0|^2 + 3\left(\int_0^1 H_s \mathrm{d}B_s\right)^2.\end{aligned}$$

We now use (5.5) with $\varepsilon = \frac{1}{6}$. The result is

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[K_1^2] &\le \mathbb{E}^{\mathbb{P}}\left[3|M_1|^2 + 3|M_0|^2 + 3\int_0^1 \mathrm{d}\langle B\rangle_t H_t \cdot H_t\right] \\ &\le 27\,\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,1]} |M_t|^2] + \frac{1}{2}\mathbb{E}^{\mathbb{P}}[K_1^2].\end{aligned}$$

Hence,

$$\mathbb{E}^{\mathbb{P}}[K_1^2] \le 54\,\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,1]} |M_t|^2].$$

This, together with (5.5) and the definitions of the norms, implies the result. □

Next we prove an estimate for differences. So for any $(H^i, K^i) \in \mathbb{H}^2_{\mathcal{P}} \times \mathbb{I}^2_{\mathcal{P}}$, $i = 1, 2$, let $M^i$ be defined as in (5.4). As before, let $\delta M := M^1 - M^2$, $\delta H := H^1 - H^2$, $\delta K := K^1 - K^2$.

**Proposition 5.10.** *There exists a constant $C$ depending only on the dimension such that*

$$\|\delta H\|^2_{\mathbb{H}^2_{\mathcal{P}}} + \|\delta K\|^2_{\mathbb{S}^2_{\mathcal{P}}} \le C[\|\delta M\|^2_{\mathbb{S}^2_{\mathcal{P}}} + \|\delta M\|_{\mathbb{S}^2_{\mathcal{P}}}(\|K^1\|_{\mathbb{S}^2_{\mathcal{P}}} + \|K^2\|_{\mathbb{S}^2_{\mathcal{P}}})]. \qquad (5.6)$$

The terms $\|K^i\|_{\mathbb{S}^2_{\mathcal{P}}}$ in the above inequality can be estimated using Proposition 5.9.

**Proof.** The arguments are very similar to those in the proof of Proposition 5.9. The only difference is the fact that $\delta K$ is no longer a monotone function. We directly compute that

$$\delta M_t = \delta M_0 + \int_0^t \delta H_s \mathrm{d}B_s - \delta K_t.$$

Then we proceed as in the proof of the previous proposition to arrive at

$$\mathbb{E}^{\mathbb{P}}\left[|\delta M_t|^2 + \int_0^1 \mathrm{d}\langle B\rangle_t \delta H_t \cdot \delta H_t\right] \le \mathbb{E}^{\mathbb{P}}[|\delta M_1|^2] + \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\delta M_s|\mathrm{d}|\delta K|_s\right].$$

The last integral term is directly estimated as follows:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\delta M_s|\mathrm{d}|\delta K|_s\right] &\le \mathbb{E}^{\mathbb{P}}[(\sup_{t\in[0,1]} |\delta M_t|)(\sup_{t\in[0,1]}[|K_t^1| + |K_t^2|])] \\ &\le 2[\mathbb{E}^{\mathbb{P}} \sup_{t\in[0,1]} |\delta M_t|^2]^{1/2}\left(\sum_{i=1}^2 [\mathbb{E}^{\mathbb{P}} \sup_{t\in[0,1]} |K_t^i|^2]^{1/2}\right) \\ &\le 2\|\delta M\|_{\mathbb{S}^2_{\mathcal{P}}}(\|K^1\|_{\mathbb{S}^2_{\mathcal{P}}} + \|K^2\|_{\mathbb{S}^2_{\mathcal{P}}}).\end{aligned}$$

The estimate of $\|\delta K\|_{\mathbb{S}^2_{\mathcal{P}}}$ is obtained exactly as in the proof of Proposition 5.9. □

### 5.5. *Proof of Theorem 5.1*

We prove uniqueness first. Suppose that there are two pairs $(H^i, K^i)$ satisfying (5.1). Then, we can use Proposition 5.10 with $M_t^i = Y_t = E_t^G[\xi]$. In particular, $\delta M \equiv 0$. By (5.6), we conclude that $\|\delta H\|_{\mathbb{H}^2_{\mathcal{P}}} = \|\delta K\|_{\mathbb{S}^2_{\mathcal{P}}} = 0$.

For the existence, let $\mathcal{M}$ be the subset of $\mathbb{L}^2_{\mathcal{P}}$ such that the martingale representation (5.1) holds for all $\xi \in \mathcal{M}$. We will prove the result by showing that $\mathcal{M}$ is closed in $\mathbb{L}^2_{\mathcal{P}}$ and that $\mathcal{L}_{ip} \subset \mathcal{M}$. The second statement is proved in the Appendix, by an approximation argument. This is Proposition A.1. Then for $\xi \in \mathcal{L}^2_{\mathcal{P}}$ these two statements imply the existence of $(H, K)$ as $\mathcal{L}^2_{\mathcal{P}}$ is in the closure of $\mathcal{L}_{ip}$ under the norm $\mathbb{L}^2_{\mathcal{P}}$.

To show that $\mathcal{M}$ is closed, consider a sequence $\xi^n \in \mathcal{M}$ converging to $\xi \in \mathbb{L}^2_{\mathcal{P}}$. Since $\xi^n \in \mathcal{M}$, there are $H^n \in \mathcal{H}^2_{\mathcal{P}}$ and $K^n \in \mathbb{I}^2_{\mathcal{P}}$ such that (5.1) holds for each $n$ and $N^n := -K^n$ is a continuous, non-increasing $G$-martingale. We now use the estimate (5.6) with $M^1 = Y^n$ and $M^2 = Y^m$ for arbitrary $n$ and $m$. The identity $Y_t^n = E_t^G[\xi^n]$, together with the definition of the conditional expectation $\mathbb{E}_t^G$, implies that for every $t \in [0, 1]$,

$$|Y_t^n - Y_t^m|^2 \le \mathbb{E}_t^G[|\xi^n - \xi^m|^2].$$

Hence the definition of the norm $\|\cdot\|_{\mathbb{L}^2_{\mathcal{P}}}$ yields

$$\|Y^n - Y^m\|_{\mathbb{S}^2_{\mathcal{P}}} \le \|\xi^n - \xi^m\|_{\mathbb{L}^2_{\mathcal{P}}}.$$

We now use the results of Propositions 5.9 and 5.10 with $M^1 = Y^n$ and $M^2 = Y^m$. Proposition 5.9 yields, for each $n$,

$$\|K^n\|_{\mathbb{S}^2_{\mathcal{P}}} \le \|\xi^n\|_{\mathbb{L}^2_{\mathcal{P}}} \le c_0 := \sup_m \|\xi^m\|_{\mathbb{L}^2_{\mathcal{P}}} < \infty.$$

We use this in (5.6). The result is

$$\|H^n - H^m\|^2_{\mathbb{H}^2_{\mathcal{P}}} + \|K^n - K^m\|^2_{\mathbb{S}^2_{\mathcal{P}}} \le C^*[\|\xi^n - \xi^m\|^2_{\mathbb{L}^2_{\mathcal{P}}} + 2c_0\|\xi^n - \xi^m\|_{\mathbb{L}^2_{\mathcal{P}}}].$$

Hence $\{H^n\}_n$ is a Cauchy sequence in $\mathcal{H}^2_{\mathcal{P}}$. Therefore by the definition of $\mathcal{H}^2_{\mathcal{P}}$, we know that there is a limit $H \in \mathcal{H}^2_{\mathcal{P}}$. Moreover, by (5.3) the corresponding stochastic integrals converge in $\mathbb{S}^2_{\mathcal{P}}$. Also $\{K^n\}_n$ is a Cauchy sequence in $\mathbb{S}^2_{\mathcal{P}}$. By Theorem 5.8, we conclude that there is a limit $K \in \mathbb{I}^2_{\mathcal{P}}$ such that $N := -K$ is a $G$-martingale. Since $(Y^n, H^n, K^n)$ satisfies (5.1) with final data $Y^n_1 = \xi^n$, we conclude that the limit process $(Y, H, K)$ also satisfies (5.1) with final data $Y_1 = \xi$. Hence $\mathcal{M}$ is closed under the norm $\mathbb{L}^2_{\mathcal{P}}$. $\square$

### 5.6. Proof of Theorem 5.3

Since $Y^i_t = E^G_t[\xi^i]$, the dual representation of the $G$-conditional expectation yields that for each $t \in [0, 1]$,

$$|\delta Y_t| = |E^G_t[\xi^1] - E^G_t[\xi^2]| \le E^G_t[|\xi^1 - \xi^2|].$$

Hence,

$$\|\delta Y\|_{\mathbb{S}^2_{\mathcal{P}}} \le \|\delta\xi\|_{\mathbb{L}^2_{\mathcal{P}}}.$$

We now use Proposition 5.10. The result is

$$\|\delta H\|_{\mathbb{H}^2_{\mathcal{P}}} + \|\delta K\|_{\mathbb{S}^2_{\mathcal{P}}} \le C^*[\|\delta Y\|_{\mathbb{S}^2_{\mathcal{P}}} + \|\delta Y\|^{\frac{1}{2}}_{\mathbb{S}^2_{\mathcal{P}}}(\|K^1\|^{\frac{1}{2}}_{\mathbb{S}^2_{\mathcal{P}}} + \|K^2\|^{\frac{1}{2}}_{\mathbb{S}^2_{\mathcal{P}}})].$$

We now use the estimate (5.2) in the above inequality, together with the fact that $|\|\xi^2\|_{\mathbb{S}^2_{\mathcal{P}}} - \|\xi^1\|_{\mathbb{S}^2_{\mathcal{P}}}| \le \|\delta\xi\|_{\mathbb{S}^2_{\mathcal{P}}}$, to complete the proof of the Theorem. $\square$

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## Appendix

In this Appendix, we construct smooth approximations of the partial differential equations (2.2), (2.3) and study the properties of the integrability class $\mathbb{L}^2_{\mathcal{P}}$.

### A.1. Approximation

The main goal of this subsection is to construct a smooth approximation of solutions of (2.3). We require smoothness of these solutions in order to be able to apply the Itô rule. The first obstacle to regularity is the possible degeneracy of the nonlinearity $G$ or equivalently the possible degeneracy of the lower bound $\underline{a}$. Therefore, we do not expect the equation to regularize the final data. However, even in this case the solution remains twice differentiable provided the final data have this regularity. But the second difficulty in proving smoothness emanates from the fact that Eq. (2.3) is solved in several time intervals and in each interval $(t_i, t_{i+1})$ and the value $B_{t_i}$ enters into the equation as a parameter. Differentiability with respect to parameters of these types is harder to prove. Given these difficulties, we approximate the equation as follows.

For $\epsilon \in (0, 1]$, set $\underline{a}^\epsilon := \underline{a} \vee \epsilon I$ so that we have

$$\bar{G}^\epsilon(\gamma) := \sup\left\{\frac{1}{2}\mathrm{tr}\,[a\gamma] \mid \underline{a}^\epsilon \le a \le \overline{a}\right\}.$$

We then mollify $\bar{G}^\epsilon$. Indeed, let $\eta : \mathbb{S}^d \to [0, 1]$ be a regular bump function, i.e., the support of $\eta$ is the unitary ball $O_1$ and $\int_{O_1} \eta(\gamma)\mathrm{d}\gamma = 1$. We then define

$$G^\epsilon(\gamma) := \int_{O_1} \bar{G}^\epsilon(\gamma + \epsilon\gamma')\,\eta(\gamma')\,\mathrm{d}\gamma'.$$

It can be shown that

$$\frac{1}{2}\mathrm{tr}\,[\underline{a}^\epsilon\gamma'] \le G^\epsilon(\gamma + \gamma') - G^\epsilon(\gamma) \le \frac{1}{2}\mathrm{tr}\,[\overline{a}\gamma'],$$

and that there is a constant $C^*$ satisfying

$$0 \le G^\epsilon(\gamma) - \bar{G}^\epsilon(\gamma) \le C^*\epsilon,$$

where the left inequality is due to the fact that $\bar{G}^\varepsilon$ is convex. Moreover $G^\epsilon$ is smooth and convex. Thus, we can define the Legendre transform of $G^\epsilon$ by

$$L^\epsilon(a) := \sup_{\gamma\in\mathbb{S}_d}\left\{\frac{1}{2}\mathrm{tr}\,[a\gamma] - G^\epsilon(\gamma)\right\}.$$

Then $L^\epsilon(a)$ is finite only if $\underline{a}^\epsilon \le a \le \overline{a}$. Also, $-C^*\epsilon \le L^\varepsilon(a) \le 0$ for all $\underline{a}^\epsilon \le a \le \overline{a}$ and

$$G^\epsilon(\gamma) := \sup_{\underline{a}^\epsilon \le a \le \overline{a}}\left\{\frac{1}{2}\mathrm{tr}\,[a\gamma] - L^\epsilon(a)\right\}.$$

We are now ready to prove the approximation result. Recall that $\mathcal{M} \subset \mathbb{L}^2_{\mathcal{P}}$ is the subset for which the representation (5.1) holds.

**Proposition A.1.** *Assume that $\underline{a}$ and $\overline{a}$ satisfy* (2.1). *Then, $\mathcal{L}_{ip} \subset \mathcal{M}$.*

**Proof.** Let $\xi \in \mathcal{L}_{ip}$. Then $\xi = \varphi(B_{t_1}, \ldots, B_{t_n})$ for some bounded Lipschitz function $\varphi$ and $0 \le t_1 \le \cdots \le t_n = 1$. Let $\{v_i\}_{i=1}^n$ be the solutions of (2.3). Then, the $v_i$'s are bounded and Lipschitz continuous. Moreover, by the definition of the $G$-expectations

$$E_t^G[\xi] = v_i(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t), \quad t \in [t_{i-1}, t_i).$$

We approximate $v_i$ as follows. Let $\varphi^\epsilon$ be a smooth, bounded approximation of $\varphi$ such that $\|\varphi^\epsilon - \varphi\|_\infty$ tends to zero and $\|\nabla\varphi^\epsilon\|_\infty \le \|\nabla\varphi\|_\infty$. Define $v_i^\epsilon(t, x_1, \ldots, x_i, x)$ recursively as in the definition of $G$-expectations in Section 2 with data $\varphi^\epsilon(B_{t_1}, \ldots, B_{t_n})$ and the nonlinearity $G^\epsilon$. Indeed, $v_i^\epsilon$ is the solution of

$$-\frac{\partial}{\partial t}v_i^\epsilon(t, x_1, \ldots, x_{i-1}, x) - G^\epsilon(D_x^2 v_i^\epsilon(t, x_1, \ldots, x_{i-1}, x)) = 0, \tag{A.1}$$

on the interval $[t_{i-1}, t_i)$ with final data $v_i^\epsilon(t_i, x_1, \ldots, x_{i-1}, x) = v_{i+1}^\epsilon(t_i, x_1, \ldots, x_{i-1}, x, x)$. In the interval $[t_{n-1}, 1)$, $v_n^\epsilon(t, x_1, \ldots, x_{n-1}, x)$ solves (A.1) with data $v_n^\epsilon(1, x_1, \ldots, x_{n-1}, x) = \varphi^\epsilon(x_1, \ldots, x_{n-1}, x)$.

We claim that the celebrated regularity result of [11] (Theorem 1, Section 6.3, page 292) applies and that $v_i^\epsilon(t, x_1, \ldots, x_{i-1}, x)$ is a smooth function of $(t, x) \in (t_i, t_{i+1}) \times \mathbb{R}^d$. Indeed, the nonlinearity $G^\epsilon$ depends only on the Hessian variable. Moreover, it is constructed in such a way that all of its derivatives with respect to $\gamma$ are bounded on all of the space. Hence this nonlinearity $G^\epsilon$ can be directly shown to belong to the class of functions considered in the Definition 5.5.1 of [11]. Moreover, in the notation of Theorem 1 of Section 6.3 in [11] (page 292), the domain $Q = (0, 1) \times \mathbb{R}^d$. Therefore, this theorem applies, to yield existence and interior regularity. To obtain regularity up to the terminal condition, we use Theorem 2(b) in [11] (Section 6.3, page 295). We may then use the stochastic control representation of this smooth and classical solution to obtain bounds. Indeed, the boundedness and the Lipschitz estimate are immediate consequences of the fact that the equation is translation invariant (or equivalently, the nonlinearity $G^\epsilon$ depends only on the Hessian). Hence the solution is bounded and Lipschitz in all variables. Moreover the uniform Lipschitz constant of $\varphi$ is preserved and for each $i$, we have

$$\lim_{\varepsilon \to 0} \|v_i^\epsilon - v_i\|_\infty = 0, \qquad \sup_{0<\epsilon\le 1} \|\nabla v_i^\epsilon\|_\infty \le \|\nabla \varphi\|_\infty. \tag{A.2}$$

For $t \in (t_i, t_{i+1})$, we set

$$\begin{aligned} M_t^\epsilon &:= v_i^\epsilon(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t), \\ H_t^\epsilon &:= \nabla_x v_i^\epsilon(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t), \\ K_t^\epsilon &:= G^\epsilon(D_x^2 v_i^\epsilon(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t)) - \frac{1}{2}\mathrm{tr}\,[\hat{a}_t D_x^2 v_i^\epsilon(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t)], \end{aligned}$$

and so

$$\mathrm{d}M_t^\epsilon = H^\epsilon \cdot \mathrm{d}B_t - \mathrm{d}K_t^\epsilon.$$

Let $\mathcal{P}_\epsilon$ be defined exactly as $\mathcal{P}$ but with lower bound $\underline{a}_\epsilon$ in (3.1). Then, by the definition of $G^\epsilon$ and $\mathcal{P}_\epsilon$, we have that $K^\epsilon$ is non-decreasing $\mathbb{P}$-almost surely for every $\mathbb{P} \in \mathcal{P}_\epsilon$. But also since $L^\epsilon \ge -C^*\epsilon$, we have

$$-C^*\epsilon \le \sup_{\mathbb{P}\in\mathcal{P}_\epsilon} \mathbb{E}^{\mathbb{P}}[-\mathbb{K}_1^\epsilon] \le 0. \tag{A.3}$$

It follows from (A.2) that $M_t^\epsilon$ converges to $M_t := E_t^G[\xi]$. Also, $|H_t^\epsilon|$ is uniformly bounded in $\epsilon$ due to the Lipschitz estimate on $v_i^\epsilon$. Hence $H^\epsilon \in \mathcal{H}_G^2$. Also Proposition 5.9 (applied with $\mathcal{P}_\epsilon$ instead of $\mathcal{P}$) yields

$$\|K^\epsilon\|_{\mathbb{S}^2_{\mathcal{P}_\epsilon}} \le C\|M^\epsilon\|_{\mathbb{S}^2_{\mathcal{P}_\epsilon}} \le C\|\xi\|_\infty.$$

Moreover, noting that $\mathcal{P}_\varepsilon$ is decreasing as $\varepsilon$ increases, by Proposition 5.10 we obtain the following estimate:

$$\|H^\epsilon - H^{\epsilon'}\|_{\mathbb{H}^2_{\mathcal{P}_{\epsilon_0}}} + \|K^\epsilon - K^{\epsilon'}\|_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} \le C(\epsilon_0), \quad 0 < \epsilon, \epsilon' \le \epsilon_0,$$

where

$$\begin{aligned} C(\epsilon_0) &:= \sup_{0<\epsilon,\epsilon'\le\epsilon_0} \left(\|M^\epsilon - M^{\epsilon'}\|_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} + \|M^\epsilon - M^{\epsilon'}\|^{1/2}_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} \left(\|K^\epsilon\|_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} + \|K^{\epsilon'}\|_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}}\right)\right) \\ &\le \sup_{0<\epsilon,\epsilon'\le\epsilon_0} \left(\|M^\epsilon - M^{\epsilon'}\|_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} + \|M^\epsilon - M^{\epsilon'}\|^{1/2}_{\mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}} (2\|\xi\|_\infty)\right). \end{aligned}$$

Since $M^\epsilon$ converges uniformly to $M_t$, $C(\epsilon_0)$ tends to zero with $\epsilon_0$. Therefore $\{(H^\epsilon, K^\epsilon)\}_\epsilon$ is a Cauchy sequence in $\mathbb{H}^2_{\mathcal{P}_{\epsilon_0}} \times \mathbb{S}^2_{\mathcal{P}_{\epsilon_0}}$ for every $\epsilon_0$.

By the closure results, Theorems 5.4 and 5.8, we conclude that there are $H \in \mathbb{H}^2_{\mathcal{P}_\epsilon}$ and $K \in \mathbb{I}^2_{\mathcal{P}_\epsilon}$ for every $\epsilon > 0$ and that $(M, H, K)$ satisfies (5.4) and

$$\|H\|_{\mathbb{S}^2_{\mathcal{P}_\varepsilon}} + \|K\|_{\mathbb{S}^2_{\mathcal{P}_\varepsilon}} \le C\|\xi\|_\infty. \tag{A.4}$$

Clearly $H$ and $K$ are independent of $\varepsilon$. Since by definition and by (3.1)

$$\mathcal{P} = \bigcup_{\epsilon>0} \mathcal{P}_\epsilon,$$

we conclude from the uniform estimates (A.4) that $H \in \mathbb{H}^2_{\mathcal{P}}$, $K \in \mathbb{I}^2_{\mathcal{P}}$. Moreover, this yields that $H \in \mathcal{H}^2_{\mathcal{P}}$ and also $-K$ is a $G$-martingale by (A.3). Since $M_t = E^G_t[\xi]$, we have shown that there is a martingale representation for the arbitrary random variable $\xi \in \mathcal{L}_{ip}$. Hence $\xi \in \mathcal{M}$. $\square$

### A.2. $\mathbb{L}^p_{\mathcal{P}}$-spaces

In this section we study the properties of the $\mathbb{L}^2_{\mathcal{P}}$ space. The following result, together with the example that follows it, implies Lemma 4.1.

**Lemma A.2.** *For every $p > 2$, there exists $C_p$ such that for $\xi \in \mathcal{L}_{ip}$,*

$$\|\xi\|_{\mathbb{L}^2_{\mathcal{P}}} \le C_p \|\xi\|_{\mathcal{L}^p_G}.$$

**Proof.** Since $\xi \in \mathcal{L}_{ip}$, by its definition in Section 2.1, $M_t := E^G_t[\xi]$ is continuous. Moreover, for each $\mathbb{P} \in \mathcal{P}$, by Proposition 3.4 we have $M_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[\xi]$, $\mathbb{P}$-a.s. Set $M^*_t := \sup_{0\le s\le t} M_t$. It suffices to show that

$$\mathbb{E}^{\mathbb{P}}[|M^*_1|^2] \le C_p \|\xi\|^2_{\mathbb{L}^p_G} \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

Now fix $\mathbb{P} \in \mathcal{P}$. Without loss of generality we may assume $\xi \ge 0$.

For any $\lambda > 0$, set $\hat\tau := \hat\tau_\lambda := \inf\{t : M_t \ge \lambda\}$. Since $M$ is continuous, $\hat\tau$ is an $\mathbb{F}$-stopping time and

$$\mathbb{P}(M^*_1 \ge \lambda) = \mathbb{P}(\hat\tau \le 1) \le \frac{1}{\lambda}\mathbb{E}^{\mathbb{P}}[M_{\hat\tau}\mathbf{1}_{\{\hat\tau\le 1\}}].$$

By Neveu [12, Proposition VI-1-1], there exists a sequence $\{\mathbb{P}_j, j \ge 1\} \subset \mathcal{P}(\hat\tau, \mathbb{P})$ defined in (3.3) such that

$$M_{\hat\tau} = \sup_{j\ge1} \mathbb{E}^{\mathbb{P}_j}_{\hat\tau}[\xi], \ \mathbb{P}\text{-a.s.}$$

For each $n \ge 1$, define

$$M^n_{\hat\tau} := \sup_{1\le j\le n} \mathbb{E}^{\mathbb{P}_j}_{\hat\tau}[\xi].$$

Then $M^n_{\hat\tau} \uparrow M_{\hat\tau}$, $\mathbb{P}$-a.s. Fix $n$. Set $A_j := \{M^n_{\hat\tau} = \mathbb{E}^{\mathbb{P}_j}_{\hat\tau}[\xi]\}$, $1 \le j \le n$, and $\tilde A_1 := A_1$, $\tilde A_j := A_j \setminus \cup_{1\le i<j} A_i$, $j = 2, \ldots, n$. Then $\{\tilde A_j, 1 \le j \le n\} \subset \mathcal{F}^B_{\hat\tau}$ form a partition of $\Omega$.

Define $\hat{\mathbb{P}}^n$ by

$$\hat{\mathbb{P}}^n(E) := \sum_{j=1}^{n} \mathbb{P}_j(E \cap \tilde{A}_j).$$

We claim that

$$\hat{\mathbb{P}}^n \in \mathcal{P}(\hat{\tau}, \mathbb{P}) \quad \text{and} \quad M^n_{\hat{\tau}} = \mathbb{E}^{\hat{\mathbb{P}}^n}_{\hat{\tau}}[\xi], \quad \hat{\mathbb{P}}^n \text{a.s.} \tag{A.5}$$

In fact, $\hat{\mathbb{P}}^n$ is obviously a probability measure and, since $\mathbb{P}_j \in \mathcal{P}(\hat{\tau}, \mathbb{P})$, $\hat{\mathbb{P}}^n = \mathbb{P}$ on $\mathcal{F}^B_{\hat{\tau}}$. Then $B$ is a $\hat{\mathbb{P}}$-martingale on $[0, \hat{\tau}]$. Moreover, for any stopping time $\tau \geq \hat{\tau}$ and any bounded $\mathcal{F}^B_\tau$-measurable random variable $\eta$, since $B$ is a $\mathbb{P}_j$-martingale and $\tilde{A}_j \in \mathcal{F}^B_{\hat{\tau}} \subset \mathcal{F}^B_\tau$, we have

$$\begin{aligned}\mathbb{E}^{\hat{\mathbb{P}}^n}[B_1\eta] &= \sum_{j=1}^{n} \mathbb{E}^{\hat{\mathbb{P}}^n}[B_1\eta\mathbf{1}_{\tilde{A}_j}] = \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}_j}[B_1\eta\mathbf{1}_{\tilde{A}_j}]\\ &= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}_j}[B_1\eta\mathbf{1}_{\tilde{A}_j}] = \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}_j}[B_\tau\eta\mathbf{1}_{\tilde{A}_j}] = \mathbb{E}^{\hat{\mathbb{P}}^n}[B_\tau\eta].\end{aligned}$$

Therefore, $\mathbb{E}^{\hat{\mathbb{P}}^n}[B_1|\mathcal{F}^B_\tau] = B_\tau$, $\hat{\mathbb{P}}^n$-a.s. Hence $B$ is a $\hat{\mathbb{P}}^n$-martingale on $[\hat{\tau}, 1]$. So $\hat{\mathbb{P}}^n$ is a martingale measure. By (3.1), for each $j$ there exists a constant $c_j > 0$ such that $BB^T - \overline{a}$ and $BB^T - (c_j I_d \vee \underline{a})$ are a $\mathbb{P}_j$-supermartingale and a $\mathbb{P}_j$-submartingale, respectively. Set $c := \min_{1\leq j\leq n} c_j > 0$. Similarly one can show that $BB^T - \overline{a}$ and $BB^T - (cI_d \vee \underline{a})$ are a $\hat{\mathbb{P}}^n$-supermartingale and a $\hat{\mathbb{P}}^n$-submartingale, respectively. This implies that $\hat{\mathbb{P}}^n$ satisfies (3.1) and therefore $\hat{\mathbb{P}}^n \in \mathcal{P}(\hat{\tau}, \mathbb{P})$. Finally, for any bounded $\mathcal{F}^B_{\hat{\tau}}$-measurable random variable $\eta$, since $\tilde{A}_j \subset A_j$, we have

$$\begin{aligned}\mathbb{E}^{\hat{\mathbb{P}}^n}[\xi\eta] &= \sum_{j=1}^{n} \mathbb{E}^{\hat{\mathbb{P}}^n}[\xi\eta\mathbf{1}_{\tilde{A}_j}] = \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}_j}[\mathbb{E}^{\mathbb{P}_j}_{\hat{\tau}}[\xi]\eta\mathbf{1}_{\tilde{A}_j}]\\ &= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}_j}[M_{\hat{\tau}}\eta\mathbf{1}_{\tilde{A}_j}] = \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}}[M_{\hat{\tau}}\eta\mathbf{1}_{\tilde{A}_j}] = \mathbb{E}^{\mathbb{P}}[M_{\hat{\tau}}\eta] = \mathbb{E}^{\hat{\mathbb{P}}^n}[M_{\hat{\tau}}\eta].\end{aligned}$$

Hence $M^n_{\hat{\tau}} = \mathbb{E}^{\hat{\mathbb{P}}^n}_{\hat{\tau}}[\xi]$, $\hat{\mathbb{P}}^n$-a.s. and this proves the claim (A.5).

Now let $q := p/(p-1)$ be the conjugate of $p$. We directly estimate that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[M^n_{\hat{\tau}}\mathbf{1}_{\{\hat{\tau}\leq 1\}}] &= \mathbb{E}^{\hat{\mathbb{P}}^n}[M^n_{\hat{\tau}}\mathbf{1}_{\{\hat{\tau}\leq 1\}}] = \mathbb{E}^{\hat{\mathbb{P}}^n}[\mathbb{E}^{\hat{\mathbb{P}}^n}_{\hat{\tau}}[\xi]\mathbf{1}_{\{\hat{\tau}\leq 1\}}] = \mathbb{E}^{\hat{\mathbb{P}}^n}[\xi\mathbf{1}_{\{\hat{\tau}\leq 1\}}]\\ &\leq [\mathbb{E}^{\hat{\mathbb{P}}^n}(|\xi|^p)]^{\frac{1}{p}}[\hat{\mathbb{P}}^n(\hat{\tau}\leq 1)]^{\frac{1}{q}} = [\mathbb{E}^{\hat{\mathbb{P}}^n}(|\xi|^p)]^{\frac{1}{p}}[\mathbb{P}(M^*_1\geq\lambda)]^{\frac{1}{q}}\\ &\leq \|\xi\|_{\mathbb{L}^p_G}[\mathbb{P}(M^*_1\geq\lambda)]^{\frac{1}{q}}.\end{aligned}$$

We let $n \to \infty$ to arrive at

$$\mathbb{P}(M^*_1 \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}^{\mathbb{P}}[M^*_{\hat{\tau}}\mathbf{1}_{\{\hat{\tau}\leq 1\}}] \leq \lim_{n\to\infty} \frac{1}{\lambda}\mathbb{E}^{\mathbb{P}}[M^n_{\hat{\tau}}\mathbf{1}_{\{\hat{\tau}\leq 1\}}] \leq \frac{1}{\lambda}\|\xi\|_{\mathbb{L}^p_G}[\mathbb{P}(M^*_1\geq\lambda)]^{\frac{1}{q}}.$$

Therefore,

$$\mathbb{P}(M^*_1 \geq \lambda) \leq \frac{1}{\lambda^p}\|\xi\|^p_{\mathbb{L}^p_G},$$

and so for any fixed $\lambda_0$,

$$\mathbb{E}^{\mathbb{P}}[|M_1^*|^2] = 2\int_0^\infty \lambda \mathbb{P}(M_1^* \geq \lambda)\mathrm{d}\lambda \leq 2\int_0^{\lambda_0} \lambda \mathrm{d}\lambda + 2\int_{\lambda_0}^\infty \lambda \mathbb{P}(M_T^* \geq \lambda)\mathrm{d}\lambda$$
$$\leq \lambda_0^2 + 2\|\xi\|_{\mathbb{L}_G^p}^p \int_{\lambda_0}^\infty \frac{\mathrm{d}\lambda}{\lambda^{p-1}} = \lambda_0^2 + \frac{2}{p-2}\|\xi\|_{\mathbb{L}_G^p}^p \lambda_0^{2-p}.$$

We choose $\lambda_0 := \|\xi\|_{\mathbb{L}_G^p}$ to conclude that

$$\mathbb{E}^{\mathbb{P}}[|M_1^*|^2] \leq C_p \|\xi\|_{\mathbb{L}_G^p}^2. \quad \square$$

We next construct a bounded random variable which is not in $\mathcal{L}_G^1$.

**Example A.3.** Let $d = 1$, $\underline{a} = 1$, $\overline{a} = 2$, $E := \{\overline{\lim}_{t\downarrow 0} B_t/\sqrt{2t \ln\ln\frac{1}{t}} = 1\}$. We claim that $\mathbf{1}_E \notin \mathcal{L}_G^1$. Indeed, assume that $\mathbf{1}_E \in \mathcal{L}_G^1$. Then there exists $\xi_n = \varphi(B_{t_1}, \ldots, B_{t_n}) \in \mathcal{L}_{ip}$ such that $\mathbb{E}^G[|\xi_n - \mathbf{1}_E|] < \frac{1}{3}$. For $\theta \in [0, 1]$, define $a_t^\theta := 1 + \theta \mathbf{1}_{[0,t_1)}(t)$ and $\mathbb{P}^\theta := \mathbb{P}^{a^\theta}$. Define $\psi(x) := \mathbb{E}^{\mathbb{P}_0}[\varphi(x, x + B_{t_2-t_1}, \ldots, x + B_{t_n-t_1})]$. Since $E \in \mathcal{F}_{0+} \subset \mathcal{F}_{t_1}$, for any $\theta \in [0, 1]$, we have the following inequality:

$$\mathbb{E}^{\mathbb{P}^\theta}[|\psi(B_{t_1}) - \mathbf{1}_E|] = \mathbb{E}^{\mathbb{P}^\theta}[|\mathbb{E}_{t_1}^{\mathbb{P}^\theta}[\varphi(B_{t_1}, \ldots, B_{t_n})] - \mathbf{1}_E|]$$
$$\leq \mathbb{E}^{\mathbb{P}^\theta}[|\varphi(B_{t_1}, \ldots, B_{t_n}) - \mathbf{1}_E|] < \frac{1}{3}.$$

Note that $\mathbb{P}^0(E) = 1$ and $\mathbb{P}^\theta(E) = 0$ for all $\theta > 0$. Then

$$\mathbb{E}^{\mathbb{P}_0}[|\psi(B_{t_1}) - 1|] < \frac{1}{3} \quad \text{and} \quad \mathbb{E}^{\mathbb{P}^\theta}[|\psi(B_{t_1})|] < \frac{1}{3} \quad \text{for all } \theta > 0.$$

The latter implies that

$$\mathbb{E}^{\mathbb{P}_0}[|\psi(B_{t_1})|] = \lim_{\theta\downarrow 0} \mathbb{E}^{\mathbb{P}_0}[|\psi((1+\theta)^{\frac{1}{2}} B_{t_1})|] = \lim_{\theta\downarrow 0} \mathbb{E}^{\mathbb{P}^\theta}[|\psi(B_{t_1})|] \leq \frac{1}{3}.$$

Thus

$$1 \leq \mathbb{E}^{\mathbb{P}_0}[|\psi(B_{t_1}) - 1|] + \mathbb{E}^{\mathbb{P}_0}[|\psi(B_{t_1})|] \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$

yielding a contradiction. Hence $1_E \notin \mathcal{L}_G^1$. $\square$

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