The B.E. Journal of Theoretical Economics

Contributions

Volume 8, *Issue* 1	2008	*Article* 23

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Recommended Citation
Jaksa Cvitanic, Xuhu Wan, and Jianfeng Zhang (2008) "Principal-Agent Problems with Exit Options," *The B.E. Journal of Theoretical Economics*: Vol. 8: Iss. 1 (Contributions), Article 23.
Available at: http://www.bepress.com/bejte/vol8/iss1/art23

Principal-Agent Problems with Exit Options*

Jaksa Cvitanic, Xuhu Wan, and Jianfeng Zhang

Abstract

We consider the problem of when to deliver the contract payoff, in a continuous-time principal-agent setting, in which the agent's effort is unobservable. The principal can design contracts of a simple form that induce the agent to ask for the payoff at the time of the principal's choosing. The optimal time of payment depends on the agent's and the principal's outside options. We develop a theory for general utility functions, while with CARA utilities we are able to specify conditions under which the optimal payment time is not random. However, in general, the optimal payment time is typically random. One illustrative application is the case when the agent can be fired, after having been paid a severance payment, and then replaced by another agent. The methodology we use is the stochastic maximum principle and its link to Forward-Backward Stochastic Differential Equations.

KEYWORDS: principal-agent problems, real options, exit decisions, forward backward stochastic differential equations

*An earlier version of this paper was titled "Optimal Contracting with Random Time of Payment and Outside Options." We are very grateful to the editor and the anonymous referees for helpful suggestions that significantly improved the exposition of the paper. We are solely responsible for any remaining errors, and the opinions, findings and conclusions or suggestions in this article do not necessarily reflect anyone's opinions but the authors'. Research supported in part by NSF grants DMS 04-03575, DMS 06-31298, DMS 06-31366, the grant DAG 05/06.BM28 from HKUST, and through the Programme "GUEST" of the National Foundation For Science, Higher Education and Technological Development of the Republic of Croatia.

1 Introduction

Standard exit problems are of the type

$$\sup_{\tau} E[U(\tau, X_\tau - C_\tau)] \tag{1.1}$$

where X_t is the time t value of an output process, C_t is the cost of liquidating, and τ is the exit time. Alternatively, τ can be thought of as the entry time, X_t as the present value at time t of a project, and C_τ as the cost incurred when entering the investment project. Classical references include McDonald and Siegel (1986) and the book Dixit and Pindyck (1994). For a very general model see, for example, Johnson and Zervos (2006), who also show how to reduce mixed entry and exit problems with intertemporal profit/loss rate to the standard optimal stopping problem of the type (1.1). We consider exit problems in the case when the output process X_t can be influenced by actions of an agent, and C_τ is interpreted as the payment from a principal to the agent. In other words, we combine some of the classical real options problem of optimal timing of investment/disinvestment decisions, with a contract theory framework in which the value obtained from a project depends on the agent's effort. Our setting is mostly suited for exit problems, while we leave entry problems for future research.

Some motivating examples for our work are the following. Company executives are often given options which they are free to exercise at any time during a given time period; the possibility of exercising early (being paid early) is definitely beneficial for executives, but is it beneficial for the company? Another application that we analyze in our framework is the question for a company of when to fire the executive while paying her the severance payment, and replace her with a new one.

In order to address questions like these, we develop a general principal-agent theory with flexible time of payment, in a standard, stylized continuous-time principal-agent models, in which the agent can influence the drift of the process by her unobservable effort, while suffering a certain cost. The agent is paid only once, at a random time τ. In our model, the timing of the payment depends crucially on the "outside options" of the agent and of the principal.

By outside options we mean the benefits and the costs the agent and the principal will be exposed to, after the payment has occurred. In our general framework, we model these as stochastic processes which are flexible enough to include a possibility of the agent leaving the project, maybe being replaced by another agent maybe not, or the agent staying with the project and applying substandard effort, or the agent being retired with a severance package or regular annuity payments, or any other modeling of the events taking place after the payment time.

We allow for two different kinds of outside options: a benefit/cost which is not separable from the principal/agent utility, which is suitable for modeling cash payments the principal/agent receive from or have to pay to a third party at or after the payment time; we also allow the outside option to be separable from the principal/agent utility, which is suitable for modeling non-monetary utility/cost they expect to incur after the payment time. Our contributions are mostly methodological, providing tools and models for solving general problems. On the other hand, we do illustrate the methods with some examples.

The paper that started the continuous-time principal-agent literature is Holmström and Milgrom (1987). That paper considers a model with moral hazard, lump-sum payment at the end of the time horizon, and exponential utilities. Because of the latter, the optimal contract is linear. Their framework was extended by Schättler and Sung (1993, 1997), Sung (1995, 1997), Detemple, Govindaraj, and Loewenstein (2001). See also Dybvig, Farnsworth and Carpenter (2001), Hugonnier, J. and R. Kaniel (2001), Müller (1998, 2000), and Hellwig and Schmidt (2003). The papers Williams (2004) and Cvitanić, Wan and Zhang (2008) (henceforth CWZ 2008), use the stochastic maximum principle and Forward-Backward Stochastic Differential Equations (FBSDEs) to characterize the optimal compensation for more general utility functions, under moral hazard. Cvitanić and Zhang (2007) (henceforth CZ 2007) consider adverse selection in the special case of separable and quadratic cost function on the agent's action. Another paper with adverse selection in continuous time is Sung (2005), in the special case of exponential utility functions and only the initial and the final value of the output being observable. A continuous-time paper which considers a random time of retiring the agent is Sannikov (2007). Moreover, He (2007) has extended Sannikov's work to the case of the agent controlling the size of the company. The paper Mason and Välimäki (2007) considers a continuous-time model in which the risk-neutral agent is paid only when the project succeeds, by a risk-neutral principal, and the agent's actions influence only the probability of success. While their time of payment is random, it is not a part of the contract, as in our case. The optimal contract payment in their model is a linear function of the agent's remaining utility and the marginal cost of effort, while we work with general utility functions, typically leading to nonlinear contracts. Another recent work in this spirit is Philippon and Sannikov (2007). In their framework, the compensation payment to the agent is continuous, while the investment occurs at an optimal random time.

We discuss now the main contributions and results of our paper, and, in particular, the main differences with CWZ (2008). First, as already mentioned above, we find a convenient and very general way to model outside options for the principal and the agent. In the previous literature this is usually either not modeled at all (CWZ 2008), or it is modeled in a very simple way, as a constant payment at the time of exiting the contract, or as a constant level of promised utility (Sannikov 2007). Second, we show that when τ is interpreted as the exercise time of payment to be decided by the agent, the principal can "force" the agent to exercise at a time of the principal's choosing, by an appropriate payoff design. We show that this design can be accomplished in a natural way, and leads to simple looking contracts in which the agent is paid a low contract value unless she waits until the output hits a certain level. The previous literature does not consider the possibility for the agent to choose the optimal time of exiting the contract.

Next, we find general necessary conditions for the optimality of hidden actions of the agent, with arbitrary utility functions for the principal and the agent, and a separable cost function for the agent. This part is an extension of CWZ (2008) to the case of exit option, and is technically similar to that paper, but we state all the results and prove or sketch their proofs, for the convenience of the reader. In particular, as usual in dynamic stochastic control problems of this type, the solution to the agent's problem depends on her "value function", that is, on her remaining expected utility process * (what Sannikov 2007 calls "promised value"). However, in the current paper this process is no longer a solution to a standard Backward Stochastic Differential Equation (BSDE), but a reflected BSDE, because of the optimal stopping component. The solution to the principal's problem depends, in general, not only on his and the agent's remaining expected utilities, but also on the remaining expected ratio of marginal utilities (which is constant in the first-best case, with no moral hazard).

We obtain new results in the variation on the classical Holmström-Milgrom (1987) set-up, with exponential utilities and quadratic cost. That is, we describe more precisely how to find the optimal exit time, something which was not modeled in the previous literature. It turns out that under a wide range of "stationarity conditions", it is either optimal to have the agent be paid right away (to be interpreted as the end of the vesting period), or not be paid early, but wait until the end. In other words, it is often not optimal for the principal that the agent be given an option to exercise the payment at a random time. For example, if the risk aversions and the cost of effort are small, and the "total output process", which is the sum of the output plus the certainty equivalents of the outside options, is a submartingale (has positive drift), then it is optimal not to have early payment. In general, the optimal exit time problem reduces to an optimal stopping problem involving the total output process. If the agent is risk-neutral, in analogy with the classical models, the principal "sells the whole firm" to the agent, in exchange for a payment at the optimal stopping time in the future. Moreover, the agent would choose the same optimal payment time as the principal, even if she was not forced to do so.

In case of non-exponential utilities, we are able to provide semi-explicit results, assuming that the cost function of the agent is quadratic and separable. This is possible because with the quadratic cost function the agent's optimal utility and the principal's problem can both be represented in a simple form which involves explicitly the contracted payoff only, and not the agent's effort

*In continuous-time stochastic control literature this method is known at least since Davis and Varaiya (1973). In dynamic principal-agent problems in discrete-time, it is used, among others, in Abreu, Pearce and Stacchetti (1986), (1990), and Phelan and Townsend (1991).

process. The ratio of the marginal utilities of the principal and the agent depends now also on the principal's utility. The optimal payoff depends in a nonlinear way on the value of the output at the time of payment. The results just described parallel those of CWZ (2008). Again, the new and different aspect is finding the optimal payment time. We show that it is determined as a solution to an optimal stopping problem of a standard type. The presence of the option to exit makes the problem much more difficult than in CWZ (2008): while, as just mentioned above, the payoff part of the contract is determined in a direct way as a function of the output, the optimal payment time is determined as a solution to a potentially hard optimal stopping problem. In an example with a risk-neutral principal and a log agent, the optimal payoff is a simple linear function, but the optimal payment time is much more complex than in the exponential utilities case. The associated optimal stopping problem involves a nonlinear function of the value of the output, the value of the principal's outside option and it also depends on the agent's outside option process. These type of problems can, in general, be solved only numerically, using PDE methods.

The paper is organized as follows: In Section 2 we consider a general model with hidden action, while the case of exponential utilities is studied in Section 3. The quadratic cost case with general utilities is analyzed in Section 4. We conclude in Section 5, and delegate most proofs to Appendix.

2 General moral hazard model

2.1 Optimization problems

We first describe the optimization problems of the agent and the principal, before giving the details of the model. We assume first that the principal has the right to choose the exercise time. However, we will show below that this is equivalent to the case when the agent has that right. The agent's problem is, given an exercise time τ and a random payment C_τ at time τ,

$$V^A(\tau, C_\tau) := \sup_u E^u \Big\{ U_1(\tau, C_\tau, A(\tau, T)) - \int_0^\tau g(u_t)dt \Big\}. \tag{2.1}$$

Here u is the effort of the agent, $A(\tau, T)$ is the value of the outside option, discussed in more details below, function U_1 is a utility function, and g is a cost function. Note here that we assume that the cost is separable from the utility due to the payoff and the outside options, which will not be the case later below when we consider the Holmström-Milgrom (1987) framework with exponential utilities.

Introduce the agent's cumulative cost corresponding to not exercising early:

$$G_t := \int_0^t g(u_s)ds; \tag{2.2}$$

Also introduce a possibly random function $\tilde{U}_1(t, c)$, expected remaining utility for the agent if she is paid c at time t:

$$\tilde{U}_1(t, c) := \tilde{E}_t[U_1(t, c, A(t, T))]. \tag{2.3}$$

Then, we can write

$$V^A(\tau, C_\tau) = \sup_u E^u\Big\{\tilde{U}_1(\tau, C_\tau) - G_\tau\Big\}. \tag{2.4}$$

If we consider only such contracts (τ, C_τ) for which the agent's problem has a unique solution $\hat{u} = \hat{u}^{\tau, C_\tau}$, then, the principal's problem is

$$V^P := \sup_{\tau, C_\tau} V^P(\tau, C_\tau) := \sup_{\tau, C_\tau} E^{\hat{u}}\Big\{U_2(\tau, X_\tau, C_\tau, P(\tau, T))\Big\}; \tag{2.5}$$

where U_2 is a function representing the principal's utility, X_τ is the underlying output process and $P(\tau, T)$ is the value of the outside option of the principal. The above problem has to be solved under the standard *individual rationality* (IR) constraint, or *participation constraint*:

$$V^A(\tau, C_\tau) \geq R_0 \tag{2.6}$$

In other words, the agent would not work for the principal for less than a given constant R_0, in terms of expected utility.

2.2 Model details

We now present the model from CWZ (2008), which, in turn, is a variation on the classical model from Holmström and Milgrom (1987) and Schattler and Sung (1993). Let B be a standard Brownian motion under some probability space with probability measure P, and $\mathbf{F}^B = \{\mathcal{F}_t\}_{0\le t\le T}$ be the information filtration generated by B up to time $T > 0$. For a given $\mathbf{F}^B$-adapted process $v > 0$ such that $E\int_0^T v_t^2 dt < \infty$, we introduce the value process of the output

$$X_t := x + \int_0^t v_s dB_s. \tag{2.7}$$

Note that $\mathbf{F}^X = \mathbf{F}^B$.

As is standard for hidden action models, we will assume that the agent changes the distribution of the output process X, by making the underlying probability measure P^u depend on agent's action u. More precisely, for any $\mathbf{F}^B$-adapted process u, to be interpreted as the agent's action, and for a fixed time horizon T, we let

$$B_t^u := B_t - \int_0^t u_s ds; \quad M_t^u := \exp\Big(\int_0^t u_s dB_s - \frac{1}{2}\int_0^t |u_s|^2 ds\Big); \quad P^u(A) := E[M_T^u \mathbf{1}_A]. \tag{2.8}$$

We assume here that u satisfies the conditions required by the Girsanov Theorem (e.g. Novikov condition). In other words, we assume that u is such that P^u is a probability measure and M_t^u is a P^u-martingale on $[0, T]$. Moreover, B^u is a P^u-Brownian motion and

$$dX_t = v_t dB_t = u_t v_t dt + v_t dB_t^u. \tag{2.9}$$

Thus, the fact that the agent controls the distribution P^u by her effort will be interpreted as the agent controlling the drift process u_t.

We suppose that the principal specifies a stopping time $\tau \le T$ and a random payoff $C_\tau \in \mathcal{F}_\tau$ at time 0. We call τ the *exercise time*, in accordance with the option pricing terminology. As we will see in Section 2.3.1, under certain technical conditions, this is equivalent to the model that the principal offers a family of contracts $\{C_t\}_{0\le t\le T}$ and the agent chooses a stopping time

τ, at which the payoff C_τ is paid to the agent. For some applications, we should interpret time $t = 0$ as the end of the vesting period before which the agent cannot exercise the payment.

- 1. Dynamics for $t \le \tau$: For $t < \tau$, the agent applies effort u_t and the dynamics is as in (2.9).

- 2. Profit/Loss after exercise, if $\tau < T$: We need to model what happens if the contract is exercised early. We denote by $\tilde{P}, \tilde{E}, \tilde{B}$ the probability measure, the corresponding expectation operator, and the corresponding Brownian Motion for the probability model after exercise time, and we introduce the following notation:

- $A(\tau, T) =$ the agent's benefit/cost due to the early exercise of the contract.

- $P(\tau, T) =$ the principal's benefit/cost due to the early exercise of the contract.

- $A_t = \tilde{E}_t[A(t, T)] =$ the agent's remaining expected benefit/cost due to the early exercise of the contract.

- $P_t = \tilde{E}_t[P(t, T)] =$ the principal's remaining expected benefit/cost due to the early exercise of the contract.

Here, $\tilde{E}_t$ denotes conditional expectation under $\tilde{P}$ with respect to $\mathcal{F}_t$. Random variables $A(t, T)$ and $P(t, T)$ don't have to be adapted to $\mathcal{F}_T$, they may depend on some outside random factors, too. Note that $A(t, T), P(t, T)$ do not depend on u or τ. Also note that if $A(t, T)$ is deterministic then $A_t = A(t, T)$, and similarly for P_t.

For example, we can have

$$A(\tau, T) = -\int_\tau^T c_t^A dt \tag{2.10}$$

and it may represent the cost the agent is facing after exercise, or, perhaps more realistically, $(-c^A)$ determines the value of an outside option the agent has of going to work for another principal, or simply a benefit for not applying active effort. Similarly, we could have

$$P(\tau, T) = \int_\tau^T [\tilde{u}_t v_t - c_t^P] dt + \int_\tau^T v_t d\tilde{B}_t \tag{2.11}$$

where $\tilde{u}$ has the interpretation of the drift after the exercise, and it may have several components: some fixed effort by the agent if she has not left the company, an "inertia" drift present without any effort, and/or an effort applied by whoever is in charge after the agent has left. On the other hand, c^P may measure the cost faced by the principal after exercise, maybe for hiring a new agent. The term $\int_\tau^T v_t d\tilde{B}_t$ is due to the noise term in the output, in analogy to the same type of noise term before exercise.

In general, A_t, P_t are flexible enough to include a possibility of the agent leaving the company, being replaced by another agent, the agent staying with the company and applying substandard effort, firing of the the agent after paying her a severance package or regular annuity payments, and many other possibilities for taking into account the events occurring after the exercise time.

Remark 2.1 Our formulation is suited for exit problems. If we wanted to model entry problems, we would have to allow for a possibility that the entry never happens, while we assume in this paper that the payment will definitely be paid, at time T if not sooner. Moreover, with entry problems, it might be more realistic to assume that the contract may be renegotiated at the entry time.

2.3 Solving agent's problem

Recall the agent's problem (2.4), in which the admissible set for the effort processes u will be specified in Definition 2.1 below.

It is by now standard in the continuous-time principal-agent literature to consider the agent's remaining utility process W^A, and represent it using the so-called Backward Stochastic Differential Equation (BSDE) form. More precisely, in our model we can write W^A in terms of its "volatility" process w^A for $t < \tau$ in the backward form as follows:[†]

[†]We note that in general $\mathbf{F}^{B^u}$ is smaller than $\mathbf{F}^B$, so one cannot apply directly the standard Martingale Representation Theorem to guarantee the existence of an adapted process $w^{A,u}$ in (2.12). Nevertheless, we can obtain $w^{A,u}$ by using a modified martingale representation theorem (see, CWZ (2008) Lemma 3.1).

$$W_t^{A,u} = E_t^u[\tilde{U}_1(\tau, C_\tau) - \int_t^\tau g(u_s)ds] = \tilde{U}_1(\tau, C_\tau) - \int_t^\tau g(u_s)ds - \int_t^\tau w_s^{A,u} dB_s^u. \tag{2.12}$$

We now specify some technical conditions. [‡]

Assumption 2.1 *(i) Cost function g is continuously twice differentiable with $g'' > 0$;*

(ii) Utility function $U_1(t,c,a)$ is continuously differentiable in c with $U_1' > 0, U_1'' \leq 0$. Here U_1', U_1'' denote the partial derivatives of U_1 with respect to c.

Definition 2.1 *The set $\mathcal{A}_1$ of admissible effort processes u is the space of $\mathbf{F}^B$-adapted processes u such that*

(i) $P(\int_0^T |u_t|^2 dt < \infty) = 1$;

(ii) $E\{|M_T^u|^4\} < \infty$;

(iii) $E\{(\int_0^T |g(u_t)|dt)^{\frac{8}{3}} + (\int_0^T |u_t g'(u_t)|dt)^{\frac{8}{3}} + (\int_0^T |g'(u_t)|^2 dt)^{\frac{4}{3}}\} < \infty$.

Condition (i) and (ii) are needed for the Novikov condition

$$E\left\{e^{2\int_0^T |u_t|^2 dt}\right\} < \infty; \tag{2.13}$$

which implies that Girsanov Theorem holds for (B^u, P^u) (see CZ 2007). It is seen in the proof of the proposition below that condition (iii) is sufficient to guarantee that $E^u\{|\tilde{U}_1(\tau, C_\tau)|^2\} < \infty$, $E^u\{|\int_0^\tau g(u_s)ds|^2 ds\} < \infty$ and $E^u\{|\int_0^\tau g'(u_s)ds|^2 ds\} < \infty$, which are standard conditions needed when studying BSDEs.

The following result has been known in one form or another from previous work, with fixed $\tau = T$; see Schattler and Sung (1993), Sannikov (2007), Williams (2003) and CWZ (2008). The result characterizes the agent's optimal expected utility process W_t^A as a solution to a BSDE with terminal condition determined by the given contract, and it characterizes the optimal control of the agent in terms of the associated volatility process w_t^A:

[‡]We mention that, in general, in this paper we do not aim to find the minimum set of sufficient conditions.

Proposition 2.1 *Given a contract (τ, C_τ), assume the following BSDE has a unique solution (W^A, w^A):*

$$W_t^A = \tilde{U}_1(\tau, C_\tau) - \int_t^\tau [g(I_1(w_s^A)) - w_s^A I_1(w_s^A)]ds - \int_t^\tau w_s^A dB_s, \qquad (2.14)$$

such that $I_1(w^A) \in \mathcal{A}_1$, where

$$I_1 := (g')^{-1}$$

and $w_t^A := 0$ for $t > \tau$. Then the agent's unique optimal action is

$$u_t^A = I_1(w_t^A)$$

and the agent's optimal utility process is $W_t^A = W_t^{A,u^A}$ for $t \leq \tau$. In particular, the optimal agent's expected utility satisfies $V^A(\tau, C_\tau) = W_0^A$. ■

We see that at the optimum

$$g'(u_t^A) = w_t^A$$

which means that *the marginal cost of effort is equal to the sensitivity (or volatility) of the agent's remaining utility with respect to the underlying uncertainty*, described by Brownian Motion B.

2.3.1 Implementability of the exercise time

In this subsection we assume that the agent has the right to choose the exercise time and show that this is in fact equivalent to the model we discussed above. To be precise, given a contract process $\{C_t\}_{0 \leq t \leq T}$, the agent's problem is:

$$V^A(C) := \sup_\tau \sup_{u \in \mathcal{A}_1} E^u[\tilde{U}_1(\tau, C_\tau) - \int_0^\tau g(u_s)ds]. \qquad (2.15)$$

Then we have the following result.

Proposition 2.2 *Assume that a pair $(\tau_0, C^0_{\tau_0})$ satisfies the condition of Proposition 2.1, and let (W^A, u^A) be the solution to the corresponding BSDE. Then, there exists a process C such that $C_{\tau_0} = C^0_{\tau_0}$ and $W_0^A = V^A(C)$.*

Proof. Note that

$$W_t^A = W_0^A + \int_0^t g(u_s^A)ds + \int_0^t g'(u_s^A)dB_s^{u^A}. \qquad (2.16)$$

For $t \in [0,T]$, let $C_t := J(t, W_t^A)$ where

$$J(t,\cdot) := \tilde{U}_1(t,\cdot)^{-1} \qquad (2.17)$$

Then obviously $C_{\tau_0} = C_{\tau_0}^0$. Moreover, for any τ, by the proof of Proposition 2.1 in Appendix, we know that (τ, C_τ) satisfies the condition of Proposition 2.1 and by the proposition, since $\tilde{U}_1(\tau, C_\tau) = W_\tau^A$, we get from (2.16),

$$\sup_{u\in\mathcal{A}_1} E^u[\tilde{U}_1(\tau, C_\tau) - \int_0^\tau g(u_s)ds] = W_0^A.$$

This ends the proof. ■

In fact, the proof shows that *given the contract* $C_t = J(t, W_t^A)$*, the agent is indifferent with respect to the exercise time.* This is because with this contract, for any t the principal is offering C_t which is the certainty equivalent of the remaining expected utility. When indifferent, we assume that the agent will choose the exercise time which is the best for the principal.

Remark 2.2 In this remark we further discuss how to construct a contract process C_t in order to implement a desired contract $(\tau_0, C_{\tau_0}^0)$. Assume W^A is given as in Proposition 2.2.

(i) The principal can induce the agent to exercise the contract at τ_0 by offering C_t such that $C_{\tau_0} = J(\tau_0, W_{\tau_0}^A)$ and $C_t < J(t, W_t^A)$ for $t \neq \tau_0$. In particular, if we assume that C_t has a lowest possible value L (maybe $-\infty$) and that $J(t, W_t^A) > L$, then the contract $C_t := J(\tau_0, W_{\tau_0}^A)1_{\{t=\tau_0\}} + L1_{\{t\neq\tau_0\}}$ will "force" the agent to choose the exercise time τ_0.

(ii) When the model is Markovian, as in Remark 4.3 (ii) below, we have $\tau_0 = \inf\{t : f_1(t, X_t) = 0\}$ for some deterministic function $f_1(t,x) \le 0$. By the Markovian structure, one can show further that $J(t, W_t^A) = f_2(t, X_t)$ for some deterministic function $f_2(t,x)$ when $t < \tau_0$. We may choose some function f_3 such that $f_3(t,x) < f_2(t,x)$ when $f_1(t,x) < 0$ and $f_3(t,x) = f_2(t,x)$ when $f_1(t,x) = 0$ (e.g. set $f_3(t,x) := f_1(t,x) + f_2(t,x)$). Then $C_t := f_3(t, X_t)$

will induce the agent to choose exercise time τ_0. In practice, in recent years companies have started to modify usual executive compensation packages due to related scandals, and one of the suggestions has been to allow payment exercise only if the performance has been good enough, which is a contract reminiscent of the above type.

Remark 2.3 Assume $v_t = \sigma(t, X_t)$ and $\tilde{U}_1(t, C_t) = l(t, X_t)$ for some deterministic functions σ and l (e.g., if $A(t,T)$ is deterministic and C_t is a deterministic function of (t, X_t)). Then, under certain technical conditions, the agent's problem is associated with the following PDE obstacle problem:

$$\begin{cases} \max(\varphi_t + \frac{1}{2}\varphi_{xx}\sigma^2 - g(I_1(\varphi_x\sigma)) + \varphi\sigma I_1(\varphi\sigma), l - \varphi) = 0; \\ \varphi(T, x) = l(T, x); \end{cases}$$

in the sense that $W_t^A = \varphi(t, X_t)$. Moreover, the first optimal exercise time of the agent is $\tau := \inf\{t : \varphi(t, X_t) = l(t, X_t)\}$, and before τ we always have $\varphi(t, X_t) > l(t, X_t)$.

2.4 Solving principal's problem

We now fix the agent's utility value to be R_0, so that the IR constraint is satisfied:

$$V^A(\tau, C_\tau) = W_0^A = R_0. \tag{2.18}$$

In most cases this is without loss of generality, as we explain in Remark 2.4 below.

From now on we always assume (2.18) and that the pair (τ, C_τ) satisfies the conditions in Proposition 2.1. Recalling the principal's problem (2.5), we can write it as

$$V^P = \sup_{\tau, C_\tau} V^P(\tau, C_\tau) := \sup_{\tau, C_\tau} E^{I_1(w^A)}\Big\{U_2(\tau, X_\tau, C_\tau, P(\tau, T))\Big\};$$

where w^A corresponding to (τ, C_τ) is determined by Proposition 2.1 .

For $u := I_1(w^A)$, we have

$$W_t^A = R_0 + \int_0^t g(u_s)ds + \int_0^t g'(u_s)dB_s^u, \quad t \le \tau, \tag{2.19}$$

By Proposition 2.1 we can rewrite the principal's problem as

$$\begin{aligned} V^P &:= \sup_\tau V^P(\tau) := \sup_\tau \sup_{u\in\mathcal{A}_2(\tau)} V^P(\tau;u) \\ &:= \sup_\tau \sup_{u\in\mathcal{A}_2(\tau)} E^u\Big\{U_2(\tau, X_\tau, J(\tau, W^A_\tau), P(\tau,T))\Big\}; \end{aligned} \tag{2.20}$$

where $\mathcal{A}_2(\tau) \subset \mathcal{A}_1$ will be specified later in Definition 2.2. From now on, we consider τ and u (instead of (τ, C_τ)) as the principal's control, and we call u an incentive compatible effort process.

Introduce a possibly random function $\tilde{U}_2(t,x,c)$, expected remaining utility for the principal if the agent is paid c at time t:

$$\tilde{U}_2(t,x,c) := \tilde{E}_t\Big\{U_2(t,x,c,P(t,T))\Big\}. \tag{2.21}$$

Then we have, for the principal's utility $V^P(\tau;u)$ introduced in (2.20),

$$V^P(\tau;u) = E^u\Big\{\tilde{U}_2(\tau, X_\tau, J(\tau, W^A_\tau))\Big\}. \tag{2.22}$$

We are now ready to describe a general system of necessary conditions for the principal's problem in terms of four variables: the output X, the agent's remaining utility W^A, the principal's remaining utility W^P, and the remaining "ratio of marginal utilities" Y, where the latter two are defined by

$$W^P_t := E^u_t\left[\tilde{U}_2\left(\tau, X_\tau, J(\tau, W^A_\tau)\right)\right]; \quad Y_t := E^u_t\left[\frac{\tilde{U}_2'\left(\tau, X_\tau, J(\tau, W^A_\tau)\right)}{\tilde{U}_1'(\tau, J(\tau, W^A_\tau))}\right],$$

where $\tilde{U}_2'(t,x,c)$ denotes the partial derivative of $\tilde{U}_2$ with respect to c. Fix τ and u. Consider the following system of Forward-Backward SDEs, for $t \in$

$[0,\tau]$, which has to be solved for processes $X, W^A, (W^P, w^P), (Y, Z)$:

$$\begin{cases} X_t = x + \int_0^t v_s dB_s; \\ W_t^A = R_0 + \int_0^t \Big[g(u_s) - g'(u_s)u_s\Big]ds + \int_0^t g'(u_s)dB_s; \\ W_t^P = \tilde{U}_2(\tau, X_\tau, J(\tau, W_\tau^A)) + \int_t^\tau w_s^P u_s ds - \int_t^\tau w_s^P dB_s; \\ Y_t = \frac{\tilde{U}_2'(\tau, X_\tau, J(\tau, W_\tau^A))}{\tilde{U}_1'(\tau, J(\tau, W_\tau^A))} + \int_t^\tau Z_s u_s ds - \int_t^\tau Z_s dB_s; \end{cases} \tag{2.23}$$

When there is a need to emphasize the dependence on the parameters τ, u, we may use $W^{P,\tau,u}$ instead of W^P. The other notations can be defined similarly.

We now specify the technical conditions needed to derive the necessary conditions of optimality.

Assumption 2.2 *Function $U_2(t,x,c,p)$ is continuously differentiable in c with $U_2' < 0, U_2'' \leq 0$, where U_2', U_2'' denote the partial derivatives of U_2 with respect to c; and for almost all ω, $\tilde{U}_2(t,x,c,\omega)$ is uniformly continuous in t, uniformly in (x,c).*

Definition 2.2 *For any stopping time τ, the set $\mathcal{A}_2(\tau)$ of admissible incentive compatible effort processes u the principal can choose from is the space of $\mathbf{F}^B$-adapted processes u over $[0,\tau]$ such that*

(i) $u \in \mathcal{A}_1$, where we take the convention that $u_t := 0$ for $t \in (\tau, T]$;

(ii) $E^u\{|\tilde{U}_2(\tau, X_\tau, J(\tau, W_\tau^{A,u}))|^2 + |\tilde{U}_2'(\tau, X_\tau, J(\tau, W_\tau^{A,u}))/\tilde{U}_1'(\tau, J(\tau, W_\tau^{A,u}))|^2\} < \infty$;

(iii) For any bounded $\Delta u \in \mathbf{F}^B$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $u^\varepsilon := u + \varepsilon \Delta u$ satisfies (i) and (ii) above and

$$M_\tau^{u^\varepsilon}\Big[|\tilde{U}_2(\tau, X_\tau, J(\tau, W_\tau^{A,u^\varepsilon}))|^2 + |\tilde{U}_2'(\tau, X_\tau, J(\tau, W_\tau^{A,u^\varepsilon}))/\tilde{U}_1'(\tau, J(\tau, W_\tau^{A,u^\varepsilon}))|^2\Big]$$

are uniformly integrable under E, uniformly in ε.

Condition (ii) is a standard L^2-condition needed for solving BSDEs in (2.23). Condition (iii) is needed in order to be able to take the limit when $\varepsilon \to 0$.

Remark 2.4 Regarding the assumption (2.18), we could always work with general $R \geq R_0$ instead, and then maximize over R as the final step. In other words, we would need to solve the problem $V^P = \sup_{R \geq R_0} V^P(R)$. However, recall that $U_1' > 0$ and $U_2' < 0$. Then, for any fixed τ and u, the expectation in the right side of (2.20) is decreasing in R and therefore, ignoring a possible impact of R on the admissibility set $\mathcal{A}_2(\tau)$, $V^P(R)$ would be decreasing in R. So the principal's optimal choice would typically be $R = R_0$.

We have the following necessary condition for optimality:

Proposition 2.3 *Let $u^\tau \in \mathcal{A}_2(\tau)$ be the optimal incentive compatible effort for the problem (2.20), but with τ fixed. Then the FBSDE (2.23) is satisfied by a multiple of adapted processes $(X^\tau, W^{A,\tau}, W^{P,\tau}, Y^\tau, w^{P,\tau}, Z^\tau)$, such that*

$$w_t^{P,\tau} + g''(u_t^\tau)Z_t^\tau = 0, \quad \forall t \leq \tau. \tag{2.24}$$

Moreover, the principal's problem becomes

$$V^P = \sup_\tau W_0^{P,\tau}, \tag{2.25}$$

and the optimal payoff satisfies

$$C_\tau = J(\tau, W_\tau^A).$$

We remark that in CWZ (2008) it was shown that if the condition (2.24) uniquely determines u as a function of $w^{P,\tau}, Z^\tau$, and if the corresponding FBSDE (2.23) is well-posed in the sense of having a unique solution and satisfies a stability condition, then sufficiency also holds true, that is, u^τ is optimal. However, in general it is difficult to check the well-posedness of such a coupled FBSDE. On the other hand, when g is quadratic, we argue below directly that the necessary conditions we obtain are also sufficient. Note also that the quadrati case is special, in the sense that the necessary condition gives that the sensitivity to the underlying uncertainty of the principal's remaining utility is proportional to that sensitivity of the expected ratio of the final marginal utilities.

3 Exponential utilities

In this section we study the classical set-up of Holmström-Milgrom (1987), with exponential utilities, but with the time of payment chosen optimally, and possibly random. The main economic conclusions will be the following:

- The optimal payment time depends on the nature of the "total output process", equal to the output process plus certainty equivalents of the outside options. In particular, the payment time will depend on the relationship between the trends and the volatilities of such processes.

- Under specific stationarity conditions on the trends and volatilities, it is optimal to either pay right away (to be interpreted as the end of the vesting period) or wait until the end.

- If, in addition, the outside options are independent of the randomness driving the output process, the optimal contract is linear and of the same form as in Holmström-Milgrom (1987).

- A risk neutral agent will be given the whole output value in exchange for cash, as is usual in these types of problems. She will agree with the principal on what is the optimal payment time.

We assume

$$U_1(t,x) = -\frac{1}{\gamma_A}\exp(-\gamma_A x); \quad U_2(t,x) = -\frac{1}{\gamma_P}\exp(-\gamma_P x). \tag{3.1}$$

Introduce the certainty equivalents of the expected benefits/costs after exercise:

$$\tilde{A}_t = -\frac{1}{\gamma_A}\log \tilde{E}_t[e^{-\gamma_A A(t,T)}]; \qquad \tilde{P}_t = -\frac{1}{\gamma_P}\log \tilde{E}_t[e^{-\gamma_P P(t,T)}].$$

Moreover, assume utilities non-separable in the contract payoff, outside options $A(t,T), P(t,T)$ and the cost, that is, assume the agent and the principal are maximizing

$$E^u[U_1(C_\tau - G_\tau + \tilde{A}_\tau)], \quad E^u[U_2(X_\tau - C_\tau + \tilde{P}_\tau)].$$

The state process X is still given by (2.9).

We note that this model is not covered in Section 2 because the cost is non-separable here. In general, it is difficult to extend the previous results to the case with a general utility and non-separable cost. However, due to the special structure here, we are still able to solve the problem following similar arguments as in Section 2, as we do next. For simplicity, in this section we omit the technical conditions and assume all the terms involved have good integrability properties so that all the calculations are valid.

Given a contract (τ, C_τ), the agent's remaining utility W_t^A can be represented as, for some adapted process Z^A,

$$\begin{aligned} W_t^A &:= E_t^u[U_1(C_\tau - G_\tau + \tilde{A}_\tau)] \\ &= -\frac{1}{\gamma_A}\exp\left[-\gamma_A\left(C_\tau - \int_0^\tau g(u_s)ds + \tilde{A}_\tau\right)\right] + \int_t^\tau W_s^A Z_s^A dB_s^u. \end{aligned}$$

Introduce the "certainty equivalent" process

$$\tilde{W}_t^A := -\frac{1}{\gamma_A}\log(-\gamma_A W_t^A) + \int_0^t g(u_s)ds$$

We have the following result for the agent's problem.

Proposition 3.1 *Given a contract (τ, C_τ), the optimal effort u of the agent satisfies the necessary and sufficient condition*

$$\gamma_A g'(u_t) = Z_t^A. \tag{3.2}$$

Moreover, for the optimal u we have

$$C_\tau = \tilde{W}_0^A - \tilde{A}_\tau + \int_0^\tau \left[\frac{1}{2}\gamma_A(g'(u_s))^2 + g(u_s) - u_s g'(u_s)\right] ds + \int_0^\tau g'(u_s)dB_s \tag{3.3}$$

Condition (3.2) is the usual condition that the marginal cost of effort is proportional to the sensitivity to the underlying uncertainty of the agent's remaining utility. Expression (3.3) shows how the optimal contract depends on the cost of effort and the marginal cost of effort.

Introduce now the certainty equivalent $\tilde{R}_0$ of the agent's reservation wage R_0:

$$\tilde{R}_0 = -\frac{1}{\gamma_A}\log(-\gamma_A R_0)$$

We now assume $\tilde{W}_0^A = \tilde{R}_0$ and consider (τ, u) as the principal's control. Let C_τ be determined by (3.3) with $\tilde{W}_0^A = \tilde{R}_0$. Define the principal's remaining utility

$$\begin{aligned} W_t^P &:= E_t^u[U_2(X_\tau - C_\tau + \tilde{P}_\tau)] \\ &= -\frac{1}{\gamma_P} \exp\left[-\gamma_P \left(X_\tau - C_\tau + \tilde{P}_\tau\right)\right] + \int_t^\tau W_s^P Z_s^P dB_s^u, \end{aligned}$$

We will also need in the proofs the principal's "certainty equivalent" process

$$\tilde{W}_t^P := -\frac{1}{\gamma_P} \log(-\gamma_P W_t^P). \tag{3.4}$$

We have the following characterization of the optimal effort:

Proposition 3.2 *Given τ, the optimal incentive compatible effort u for the principal has to satisfy the necessary condition*

$$Z_t^P = \frac{\gamma_A \gamma_P g'(u_t) g''(u_t)}{1 + \gamma_P g''(u_t)}. \tag{3.5}$$

Recall that the volatility of the agent's remaining utility $Z_t^A = \gamma_A g'(u_t)$ depends only on the agent's risk aversion and marginal cost of effort, while we see that the volatility of the principal's remaining utility depends, in addition, on the principal's risk aversion and the rate of change of marginal utility g''. However, if the cost function g is quadratic, then also the volatility of the principal's remaining utility is proportional to the marginal cost of effort.

3.1 Quadratic cost and exponential utilities

The above analysis does not tell us how to determine the optimal payoff time τ. In order to get some results in that direction, we assume now

$$g(u) = ku^2/2.$$

Denote

$$\alpha := \frac{1 + \gamma_P k}{(\gamma_A + \gamma_P)k^2 + k}; \qquad \beta := \frac{1 + \gamma_P k - \gamma_A \gamma_P k^2}{(\gamma_A + \gamma_P)\, k^2 + k};$$

and introduce the expected "total output" process S_t, the sum of the current output and the certainty equivalents of the after-exercise benefits/costs for the agent and the principal:

$$S_t := X_t + \not{A}_t + \not{P}_t.$$

We have the following

Proposition 3.3 *(i) Given (τ, C_τ), u is optimal for the agent if and only if $u_t = \frac{1}{k\gamma_A} Z_t^A$.*

(ii) An incentive-compatible u is optimal for the principal if and only if (introducing new notation $\tilde{Z}$)

$$u_t = \alpha \tilde{Z}_t := \alpha \left[\frac{Z_t^P}{\gamma_P} + g'(u_t) \right]. \tag{3.6}$$

(iii) If $\beta = 0$, the optimal stopping problem is equivalent to $\sup_\tau E(S_\tau)$.

(iv) If $\beta > 0$, the optimal stopping problem is equivalent to $\sup_\tau E\left\{e^{\beta S_\tau}\right\}$.

(v) If $\beta < 0$, the optimal stopping problem is equivalent to $\inf_\tau E\{e^{\beta S_\tau}\}$.

Results (i) and (ii) are the optimality conditions specialized to the quadratic cost. Results (iii)-(v) give a complete characterization of the optimal stopping problem faced by the principal.

The following results specify the optimal stopping time more explicitly. Whenever not specified, we assume that sub-,super-, or regular martingale property refers to the probability P. The first proposition is a direct consequence of Proposition 3.3 (iii)-(v).

Proposition 3.4 *(i) In the following cases it is optimal to exercise right away (i.e. $\tau = 0$):*

- $\beta \le 0$ and S_t is a super-martingale;

- $\beta > 0$ and $e^{\beta S_t}$ is a super-martingale;

- $\beta < 0$ and $e^{\beta S_t}$ is a sub-martingale.

(ii) In the following cases it is optimal to wait until time T:

- $\beta \ge 0$ and S_t is a sub-martingale;

- $\beta > 0$ and $e^{\beta S_t}$ is a sub-martingale;

- $\beta < 0$ and $e^{\beta S_t}$ is a super-martingale.

Remark 3.1 Note that $\beta > 0$ if the risk aversion parameters γ_A, γ_P and/or the cost parameter k are small enough, and $\beta < 0$ if the risk aversions or the cost are large enough. Thus, the proposition tells us: (i) in case the risk aversions and cost are small enough: if the drift of $e^{\beta S_t}$ is negative then it is optimal not to wait at all, while if that drift, or the drift of S_t is positive, it is optimal to wait until the end; (ii) in case the risk-aversions or the cost are large enough, if the drift of S_t or the drift of $-e^{\beta S_t}$ is negative then it is optimal not to wait at all, while if the drift of $-e^{\beta S_t}$ is positive, it is optimal to wait until the end.

Intuitively, if there is a tendency of (a monotone increasing transformation of) the expected total output to move in the positive direction, it is better to postpone the payment, while if the tendency is in the negative direction, it is better to pay right away. However, if the expected total output can go both up and down in expected value, the optimal time of payment is likely to be a random time between zero and T.

We now provide more specific results if the total output S is a Gaussian process:

Proposition 3.5 *Assume that the total output process satisfies*

$$dS_t = \mu_t dt + \rho_t dB_t,$$

for some deterministic μ and ρ. Then the optimal stopping time τ is deterministic. Moreover, we have

(i) If $\beta = 0$, then the problem is equivalent to $\max_\tau \int_0^\tau \mu_t dt$. If particular, if $\mu \le 0$, then $\tau = 0$; and if $\mu \ge 0$, then $\tau = T$.

(ii) If $\beta > 0$, then the problem is equivalent to $\max_\tau \int_0^\tau [\frac{1}{2}\beta^2\rho_t^2 + \beta\mu_t]dt$. In particular, if $\frac{1}{2}\beta^2\rho^2 + \beta\mu \le 0$, then $\tau = 0$; and if $\frac{1}{2}\beta^2\rho^2 + \beta\mu \ge 0$, then $\tau = T$.

(iii) If $\beta < 0$, then the problem is equivalent to $\min_\tau \int_0^\tau [\frac{1}{2}\beta^2\rho_t^2 + \beta\mu_t]dt$. In particular, if $\frac{1}{2}\beta^2\rho^2 + \beta\mu \ge 0$, then $\tau = 0$; and if $\frac{1}{2}\beta^2\rho^2 + \beta\mu \le 0$, then $\tau = T$.

(iv) If we assume furthermore that $\tilde{A}_t$ and $\tilde{P}_t$ are deterministic (that is, after exercise benefits/costs $A(t,T)$ and $P(t,T)$ are independent of $\mathcal{F}_t$), and

v is deterministic, then

$$\tilde{Z}_t = \rho_t = v_t, \quad \forall t \leq \tau; \tag{3.7}$$

and

$$C_\tau = f(\tau) + k\alpha X_\tau, \tag{3.8}$$

for some deterministic function f. That is, the optimal contract is of the same linear form as in the case of the fixed exercise time of Holmström and Milgrom (1987).

Remark 3.2 Intuitively, (ii) and (iii) tell us that if the drift μ of the total output sufficiently overwhelms its uncertainty ρ^2, then it is optimal to pay right away or at the end, depending on the sign of μ. Actually, and more precisely, higher uncertainty leads to postponing the payment if the risk aversions or the cost are low enough ($\beta > 0$), and it leads to speeding up the payment if the risk aversions and the cost are high enough.

(b) On one hand, we see that Holmström and Milgrom (1987) result is robust assuming enough stationarity in the underlying model. On the other hand, without such assumptions, their simple linear contract would not be optimal, in general.

Remark 3.3 *Risk-neutral agent.* Assume $U_1(t, x) = x$. We can formally get the results by modifying the agent's utility in (3.1) to $U_1(t, x) = -\frac{1}{\gamma_A}[e^{-\gamma_A x} - 1]$, and sending $\gamma_A \to 0$. We note that, however, all the results here can be proved rigorously. First, by (3.5) and $\gamma_A = 0$ we get $Z^p = 0$. This, together with (6.23) in Appendix, implies a simple expression for the optimal contract, with $\tilde{W}^P$ defined in (3.4):

$$C_\tau = X_\tau + \tilde{P}_\tau - \tilde{W}_0^P \tag{3.9}$$

That is, as usual with a risk-neutral agent, the principal "sells the whole business" $X_\tau + \tilde{P}_\tau$ to the agent in exchange for a cash payment, equal to the principal's initial certainty equivalent $\tilde{W}_0^P$. It can also readily be shown that the agent's optimal stopping problem is equivalent to the principal's, hence the agent will implement the exercise time τ which is optimal for the principal, without being told when to exercise.

4 Moral hazard with general utilities and quadratic cost function

We now specialize the general model of Section 2 with separable cost to the case of quadratic cost. As analyzed in CWZ (2008) and CZ (2007), the case when the agent's cost is quadratic makes the cases of non-exponential utilities more tractable. We follow those papers and in this section we assume

$$g(u) = \frac{1}{2}u^2. \tag{4.1}$$

We could use the theory developed in Section 2, but we choose to provide here an alternative direct approach, which requires weaker conditions[§].

Recall that the agent's problem is to maximize $E^u[\tilde{U}_1(\tau, C_\tau) - G_\tau]$. As in Section 2.1 we consider $u \in \mathcal{A}_1$ as the agent's control. But unlike in Section 2.2 where we consider (τ, u) as the principal's control, in this case we consider (τ, C_τ) as the control. We first note that, by (2.13), Definition 2.1 (iii) is the consequence of (i) and (ii). We next specify the technical conditions (τ, C_τ) should satisfy, which is in general not equivalent to $\mathcal{A}_2$ in Section 2.2.

Definition 4.1 *The admissible set $\mathcal{A}_3$ of the principal is the space of (τ, C_τ) satisfying*

(i) $E\{|\tilde{U}_1(\tau, C_\tau)|^4 + e^{4\tilde{U}_1(\tau, C_\tau)}\} < \infty$.

(ii) $E\{|\tilde{U}_2(\tau, X_\tau, C_\tau)|^2 + e^{\tilde{U}_1(\tau, C_\tau)}|\tilde{U}_2(\tau, X_\tau, C_\tau)|\} < \infty$.

Moreover, in this section Assumptions 2.1 and 2.2 are always in force.

We have the following result, analogous to CWZ (2008), but extended to our framework of the random time of exercise and random benefits/costs after exercise. Again, without loss of generality we will always assume $W_0^A = R_0$.

Proposition 4.1 *For any (τ, C_τ), the optimal effort $\hat{u}$ for the agent is obtained by solving the BSDE*

$$\bar{W}_t = E_t[e^{\tilde{U}_1(\tau, C_\tau)}] = e^{\tilde{U}_1(\tau, C_\tau)} - \int_t^\tau \hat{u}_s \bar{W}_s dB_s \tag{4.2}$$

[§]It should be mentioned, though, that we originally used the general theory to solve problems like this, and only then realized that there was a different direct approach.

Moreover, the agent's remaining expected utility is determined by

$$W_t^A = \log(\bar{W}_t).$$

In particular, the agent's expected utility is

$$R_0 = W_0^A = \log \bar{W}_0 = \log E[e^{\tilde{U}_1(\tau, C_\tau)}]. \tag{4.3}$$

In addition, with the change of probability measure density M^u defined in (2.8)*, we have, for $t \leq \tau$,*

$$M_t^{\hat{u}} = \exp(W_t^A - R_0) \ , \quad \textit{hence} \quad M_\tau^{\hat{u}} = e^{-R_0} e^{\tilde{U}_1(\tau, C_\tau)}. \tag{4.4}$$

Proof: First by Definition 4.1 (i) and the arguments in CWZ (2008), we know (4.2) is well-posed and $\hat{u} \in \mathcal{A}_1$. Denote $W_t^A := \log(\bar{W}_t), w_t^A := \hat{u}_t$. By Ito's formula one can check straightforwardly that (W^A, w^A) satisfy (2.14), and thus, by Proposition 2.1, $\hat{u}$ is the agent's optimal action. Moreover, by (4.2) we have $\bar{W}_t = \bar{W}_0 M_t^{\hat{u}}$. Since we assume $W_0^A = R_0$, the other claims are obvious now. ∎

Remark 4.1 (i) Simple relationships (4.3) and (4.4) between the agent's optimal utility, the "optimal change of probability" M_τ^u and the given contract C_τ are possible because of the assumption of quadratic cost. These expressions make the problem tractable. In particular, at the optimum the agent's remaining expected utility is obtained simply by exponentiating her utility, taking conditional expectation and then inverting the exponentiation by taking the logarithm. And the optimal effort is simply the sensitivity of the remaining expected utility with respect to the underlying uncertainty.

(ii) In the language of option pricing theory finding optimal u by solving (4.2) is equivalent to finding a replicating portfolio for the option with payoff $e^{\tilde{U}_1(\tau, C_\tau)}$. Various methods have been developed for this purpose, including PDE methods.

We now investigate the principal's problem. Denote by λe^{-R_0} the Lagrange multiplier for the IR constraint (4.3). By (2.22), Proposition 4.1 and recalling that $E^u[X_\tau] = E[M_\tau^u X_\tau]$ for an $\mathcal{F}_\tau-$measurable random variable X_τ, we can rewrite the constrained principal's problem as

$$\sup_{\tau, C_\tau} E\Big\{ M_\tau^u \tilde{U}_2(\tau, X_\tau, C_\tau) + \lambda e^{-R_0} e^{\tilde{U}_1(\tau, C_\tau)} \Big\}$$
$$= \sup_{\tau, C_\tau} e^{-R_0} E\Big\{ e^{\tilde{U}_1(\tau, C_\tau)} [\tilde{U}_2(\tau, X_\tau, C_\tau) + \lambda] \Big\}. \tag{4.5}$$

The principal wants to maximize this expression over C_τ. We have the following result, extending an analogous result from CWZ (2008) to our framework.

Proposition 4.2 *Assume that the contract C_t is required to satisfy*

$$L_t \leq C_t \leq H_t$$

for some $\mathcal{F}_t-$measurable random variables L_t, H_t, which may take infinite values. Suppose that, with probability one, there exists a finite value $\hat{C}_\tau^\lambda(\omega) \in [L_\tau(\omega), H_\tau(\omega)]$ that maximizes

$$e^{\tilde{U}_1(\tau, C_\tau)} [\tilde{U}_2(\tau, X_\tau, C_\tau) + \lambda], \tag{4.6}$$

that there exists an optimal exercise time $\tau(\lambda)$ that solves

$$\sup_\tau E\Big\{ e^{\tilde{U}_1(\tau, \hat{C}_\tau^\lambda)} [\tilde{U}_2(\tau, X_\tau, \hat{C}_\tau^\lambda) + \lambda] \Big\} \tag{4.7}$$

and that λ can be found so that

$$E[e^{\tilde{U}_1(\tau(\lambda), \hat{C}_{\tau(\lambda)}^\lambda)}] = e^{R_0}.$$

Then, $\hat{C}_{\tau(\lambda)}^\lambda$ is the optimal contract, and $\tau(\lambda)$ is the optimal exercise time.

Note that the problem of maximizing (4.6) over C_τ is a one-variable deterministic optimization problem (for any given ω), thus much easier than the original problem.

Remark 4.2 In parts (i) and (ii) of this remark we consider the case when there is an interior solution for the problem of maximizing (4.6) over C_τ.

(i) The first order condition for that problem is given by

$$-\frac{\tilde{U}_2'(\tau, X_\tau, C_\tau)}{\tilde{U}_1'(\tau, C_\tau)} = \lambda + \tilde{U}_2(\tau, X_\tau, C_\tau). \tag{4.8}$$

This extends the standard Borch rule for risk-sharing in the first-best (full information) case, with fixed $\tau = T$:

$$-\frac{U_2'(X_T, C_T)}{U_1'(C_T)} = \lambda. \tag{4.9}$$

We conclude that the second-best contract is "more nonlinear" than the first-best. For example, if both utility functions are exponential, $U_2(x, c) = U_2(x - c)$, and we require $C_t \geq L > -\infty$, the first-best contract C_T is linear in X_T for $C_T > L$. The second-best contract is nonlinear. In addition, in our framework the contract also needs to take into account the future uncertainty about the benefit/cost after exercise, which is why U_i is replaced by $\tilde{U}_i$.

(ii) Here is an explanation of the difference between the first-best and the second-best first order condition. Assume for simplicity that $A_t \equiv P_t \equiv 0$. In the first best case, what is maximized is the expected utility of $U_2(X_T, C_T) + \lambda U_1(C_T)$, which leads directly to the Borch condition of marginal utilities being proportional. However, in the second best case, according to Proposition 4.1, the agent chooses the action u which corresponds to the sensitivity to the underlying uncertainty of $\log E_t[e^{U_1(C_T)}]$ and not of $\log E_t[e^{U_2(X_T, C_T) + \lambda U_1(C_T)}]$. Moreover, this action is such that the probability measure is changed to $P^u(A) = e^{-R_0} E[\mathbf{1}_A e^{U_1(C_T)}]$. That is, *the agent chooses the distribution which puts more weight on the outcomes in which the contract payoff has a high value.* The principal needs to maximize her expected utility under this particular choice of distribution, and under the IR constraint. Because this distribution favors the states in which the agent is paid more, the principal's marginal utility, marginal with respect to the contract payoff, has to go down relative to the first best case. It is optimal for the principal to reduce it by the amount of $U_1' U_2$. In other words, *in the states the principal has more utility, he sacrifices more of his marginal utility relative to the agent's.*

(iii) In our model with quadratic cost and the separable utility for the agent, the optimal contract still has a relatively simple form, as it is a (possibly random) function of τ and the value of the output X_τ at the time of payment. It was noted in CWZ (2008) in case of fixed $\tau = T$, and it's also true here, that the sensitivity of the contract with respect to X_τ is higher in the second-best case than in the first-best, as expected. Moreover, it was observed that higher marginal utility for either party causes the slope of the contract to increase relative to the first-best case, but more so for higher marginal utility of the agent.

(iv) With exponential utilities, under a wide range of conditions provided in Proposition 3.4, the optimal stopping time is either $\tau = 0$ or $\tau = T$. However, here, the optimal stopping time in (4.7) would be equal to 0 or T only under much more restrictive conditions.

Remark 4.3 We discuss here how to solve the optimal stopping problem (4.7).

(i) Denote

$$\Theta_t := e^{\tilde{U}_1(t,\hat{C}_t^\lambda)}[\tilde{U}_2(t, X_t, \hat{C}_t^\lambda) + \lambda].$$

Assume Θ is a continuous process and the following Reflected BSDE has a unique solution (W^P, w^P, K^P):

$$\begin{cases} W_t^P = \Theta_T - \int_t^T w_s^P dB_s + K_T^P - K_t^P; \\ W_t^P \ge \Theta_t; \quad \int_0^T [W_t^P - \Theta_t] dK_t^P = 0. \end{cases} \tag{4.10}$$

Then the principal's optimal utility is W_0^P, and the optimal exercise time is $\tau(\lambda) := \inf\{t : W_t^P = \Theta_t\}$.

(ii) Assume the following Markovian structure: 1) $X_t = x + \int_0^t \sigma(s, X_s) dB_s$ where σ is a deterministic function; 2) X is Markovian under $\tilde{P}$ (e.g., $\tilde{u}$ is a deterministic function of (t, X_t)); 3) $A(t,T)$ and $P(t,T)$ are conditionally independent of $\mathbf{F}_t^B$ under $\tilde{P}$, given X_t (for example, if $A(t,T)$ and $P(t,T)$ are deterministic); and 4) $L_t = \bar{L}(t, X_t)$ and $H_t = \bar{H}(t, X_t)$ for some deterministic functions $\bar{L}$ and $\bar{H}$ (which may take values ∞ and $-\infty$). Then $\tilde{U}_1(t,c) = \bar{U}_1(t, c, X_t)$ and $\tilde{U}_2(t,x,c) = \bar{U}_2(t, x, X_t, c)$ for some deterministic functions $\bar{U}_1, \bar{U}_2$. Therefore, when maximizing (4.6) we have $\hat{C}_t^\lambda = \bar{C}(t, X_t)$ and thus $\Theta_t = \bar{\Theta}(t, X_t)$ for some deterministic functions $\bar{C}(t,x)$ and $\bar{\Theta}(t,x)$. In this case the Reflected BSDE (4.10) is associated to the following PDE obstacle problem:

$$\begin{cases} \max\left(\varphi_t(t,x) + \frac{1}{2}\varphi_{xx}(t,x)\sigma^2(t,x), \bar{\Theta}(t,x) - \varphi(t,x)\right) = 0; \\ \varphi(T,x) = \bar{\Theta}(T,x); \end{cases} \tag{4.11}$$

in the sense that $W_t^P = \varphi(t, X_t)$. Moreover, the optimal exercise time is $\tau := \inf\{t : \varphi(t, X_t) = \bar{\Theta}(t, X_t)\}$.

We now show that with no outside options for the agent, a risk-neutral principal typically would not want to pay early in case the drift of his after exercise benefits/costs process is positive.

Proposition 4.3 *Assume $U_2(t,x,c,p) = x - c + p, U_1(t,c,a) = U_1(c)$ and*

$$\lim_{c\to-\infty} ce^{U_1(c)} = 0; \quad L = -\infty; \quad H = \infty. \tag{4.12}$$

If the principal's after exercise benefits/costs process P_t is a $P-$submartingale, then the optimal exercise time is $\tau = T$.

4.1 Example: Risk neutral principal and log utility for the agent; hiring a new agent

Assume now that $U_1(t,c,A) = \gamma[\log(c) + A]$, $U_2(t,x,c,p) = x - c + p$ and the model is

$$dX_t = \sigma X_t \left(u_t dt + dB_t^u\right) \tag{4.13}$$

where σ is a known constant. Thus, $X_t > 0$ for all t. From (4.5) and the IR constraint, the principal's problem can be shown to reduce to

$$\sup_{\tau, C_\tau} E^\theta \left\{ e^{\gamma[A_\tau + \log(C_\tau)]} \left[X_\tau - C_\tau + P_\tau + \lambda\right]\right\} \tag{4.14}$$

We get, assuming the following value C_τ is positive, that

$$C_\tau = \frac{\gamma}{1+\gamma}[X_\tau + P_\tau + \lambda] \tag{4.15}$$

where λ will be obtained from the IR constraint

$$e^{R_0} = E[C_\tau^\gamma e^{\gamma A_\tau}]. \tag{4.16}$$

We assume that the model is such that $C_\tau > 0$ (see the example below). Then, substituting C_τ from (4.15) into (4.14), we get that the principal has to solve

$$\sup_\tau E\left\{e^{\gamma A_\tau}\left(X_\tau + P_\tau + \lambda\right)^{1+\gamma}\right\} \tag{4.17}$$

Let's summarize the previous in the following

Proposition 4.4 *Assume a risk-neutral principal and a log agent, and model (4.13). Consider the stopping time $\hat\tau = \hat\tau(\lambda)$ which solves the problem (4.17) and the contract $C_{\hat\tau}^\lambda$ from (4.15). Assume that there exists a unique $\hat\lambda$ which solves (4.16) with $C_\tau = C_{\hat\tau}^\lambda$, and that $C_{\hat\tau}^{\hat\lambda}$ is a strictly positive random variable. Then, $(\hat\tau, C_{\hat\tau}^{\hat\lambda})$ is the optimal contract.*

Remark 4.4 If $A_t = 0$, and P_t is a non-negative P-submartingale, then the process $(X_t + P_t + \lambda)^{1+\gamma}$ is a $P-$submartingale, and it is optimal to wait until maturity, $\tau = T$. In general, the optimal time depends on the properties of the process $e^{\gamma A_\tau}\left(X_\tau + P_\tau + \lambda\right)^{1+\gamma}$. If this process is a $P-$submartingale, then it is not optimal to exercise early, and if it is a supermartingale, then it is optimal to exercise right away. However, there seem to be no general natural conditions for this to happen when process A is not zero, unlike the conditions of Proposition 3.4 in the CARA case. Thus, it is more likely in this framework that the optimal time of payment will, indeed, be random. We work out a specific example in this spirit next.

4.1.1 Paying off the agent and hiring a new one

In this subsection we assume that the principal can pay off the agent, then hire another agent, or not hire anyone. If no agent is hired after τ, we assume that after τ the effort u is fixed, and normalized to zero. We also assume that if the new agent is hired, she will stay until time T (for simplicity). Thus, the principal makes two decisions, when to pay (fire) the first agent, and whether to hire another one.¶

As above, the principal is risk-neutral. We can then model the option to fire/hire as an outside option for the principal as follows:

$$P_\tau = \tilde{E}_\tau[P(\tau, T)] = \max\left(P^h_\tau, P^n_\tau\right)$$

where P^n_τ is the (optimal) conditional expected utility if the principal doesn't hire the agent at time τ,

$$P^n_\tau := E^0_\tau \left\{X_T - X_\tau\right\}$$

and P^h_τ is the (optimal) conditional expected utility of the principal if he hires a new agent at time τ,

$$P^h_\tau = \sup_{C^{\text{new}}_T, u} E^u_\tau \left\{X_T - C^{\text{new}}_T - X_\tau\right\}$$

under the IR constraint

$$E^u_\tau \left\{U^{\text{new}}_1 \left(C^{\text{new}}_T\right) - \int_\tau^T g^{\text{new}}(u_s)ds\right\} \geq R(\tau),$$

where $R(t)$ is the reservation wage of the new agent, prevailing at time t. There is no cost of searching for another agent, who can be hired immediately, and at no extra cost.‖ The principal's problem at time zero is then

$$\sup_{\tau, C_\tau, u} E^u \left\{X_\tau - C_\tau + P_\tau\right\}$$

under the IR constraint

$$E^u \left\{U_1\left(\tau, C_\tau, A(\tau, T)\right) - \int_0^\tau g(u_s)ds\right\} \geq R_0$$

We now show that, with log agents, if the new agent is sufficiently expensive, and if the time to maturity T is small relative to the variance of the

¶A similar problem is considered in Wang (2005), but with a fixed time of firing, in a different, much simpler model.

‖However, we could easily add a one time fixed cost of hiring the new agent as an additional term in P^h_τ.

output σ^2, the principal will not fire/pay the first agent before the terminal time T. However, if either the new agent is not very expensive, or the time to maturity T is large relative to the variance σ^2, the time of payment will be random. Moreover, the principal will never fire the first agent right away, at $\tau = 0$. The reader not interested in the technical details of the example, can skip the rest of the section.

Assume

$$U_1^{\text{new}}(x) = \log(x), \quad dX_t = \sigma X_t dB_t, \quad g(u) = g^{\text{new}}(u) = u^2/2$$

We have

$$P_\tau^n = E_\tau^0 \{X_T - X_\tau\} = 0$$

Similarly as in (4.15) (with $\gamma = 1$, $\tau = T$ and $P_T = 0$)), we get

$$C_T^{\text{new}} = \frac{1}{2}(X_T + \lambda_\tau)$$

where λ_τ is chosen so that $E_\tau \left[e^{\log\left(C_T^{\text{new}}\right)} \right] = e^{R(\tau)}$, that is

$$\lambda_\tau = -2e^{R(\tau)} + X_\tau$$

so that

$$C_T^{\text{new}} = \frac{1}{2}(X_T - X_\tau) + e^{R(\tau)}$$

We assume that the reservation wage $R(\tau)$ is sufficiently large to make $C_T^{\text{new}} > 0$, that is, we assume

$$e^{R(t)} > \frac{1}{2} X_t$$

We then have, noting that $E_\tau[X_T^2] = X_\tau^2 e^{\sigma^2(T-\tau)}$,

$$\begin{aligned} P_\tau^h + X_\tau &= E_\tau \{C_T^{\text{new}}(X_T - C_T^{\text{new}})\} \\ &= E_\tau \left\{ \left(\frac{1}{2}(X_T - X_\tau) + e^{R(\tau)} \right) \left(\frac{1}{2}(X_T + X_\tau) - e^{R(\tau)} \right) \right\} \\ &= -e^{2R(\tau)} + e^{R(\tau)} X_\tau + \frac{1}{4} X_\tau^2 [e^{\sigma^2(T-\tau)} - 1] \end{aligned}$$

Consider now the case when the first agent also has log utility: $U_1(t, x, A) =$

$\log(x) + A$. As in (4.17) (with $\gamma = 1$), the principal's problem at time zero is now

$$\sup_{\tau} E\left\{e^{A_\tau}\left(X_\tau + P_\tau + \lambda\right)^2\right\} \tag{4.18}$$

where $P_\tau = \max\left(P^h_\tau, 0\right)$. Assume, moreover,

$$A_t \equiv 0\ , \quad e^{R(t)} = kX(t)\ , k > \frac{1}{2}$$

The first condition means that the first agent's expected cost/benefit after the payment is zero, which would be the case, if, for example the after-exercise benefit/cost satisfies $A(t,T) = cX_t$ for some constant c; the second condition means that the new agent's reservation utility is more than log of half of the output. We can now compute that

$$P_\tau = \max\left(0, X^2_\tau[k - k^2 + \frac{1}{4}(e^{\sigma^2(T-\tau)} - 1)] - X_\tau\right)$$

In particular, if k is large enough, meaning the new agent is sufficiently expensive, and if the time to maturity T is sufficiently small relative to the variance σ^2, we will have $P_\tau \equiv 0$ always, and, since $(X_t + \lambda)^2$ is a submartingale, the principal will not fire/pay the first agent before the terminal time T. However, if either the new agent is not very expensive, or the time to maturity T is not small relative to the variance σ^2, P_τ will oscillate between zero and positive values, $(X_t + P_t + \lambda)^2$ will not be a submartingale (nor a supermartingale), and the optimal time of payment will be random. It would have to be computed numerically, solving problem (4.18). Note also that the principal will never fire the first agent right away, at $\tau = 0$.

5 Conclusions

We have developed a methodology for studying continuous-time principal-agent problems with hidden action in case the agent is paid once, at an optimal random time. We have identified conditions under which it is optimal to pay the agent as soon as possible, and conditions under which it is optimal to pay her as late as possible. Our framework can be a basis for many possible natural extensions and applications, such as: (i) introduce an additional random time of auditing, after which the return of the output may change, due to the new information on whether the agent has manipulated the output; (ii) give the agent more bargaining power, and, in particular, let the agent dictate the timing of the (possibly multiple) payoffs; in the same spirit, allow the agent to quit at or after the time she is paid; (iii) in general, model more precisely the uncertainty about the future outside options; (iv) allow renegotiation to take place and consider reputation effects; (v) add intermediate consumption and possibility of paying the agent at a continuous rate, as in Sannikov (2007) and Williams (2004), but in our setup; (vi) adapt the methods developed here to the case of entry problems, such as the case when τ is the time when a big pharmaceutical company enters a project with a small biotech firm, or it is the time when a venture capitalist decides to fund a project.

It is also possible to study hidden type/adverse selection problems with random time of payment, extending CZ (2007). In this context, it would be of interest to consider the case in which the agent is also uncertain about her type; for example, if the type influences the return of the output, then even without existence of outside options, the principal and the agent might want the payment to be paid early, as they update their information on the true return.

A different direction would be to allow the agent to also control the volatility of the output, as is the case in delegated portfolio management problems. However, this will require studying a combined problem of stochastically controlling the volatility of a random process together with an optimal stopping problem. There is very little theory for these problems, and no general conditions under which the solution can be found; see Karatzas and Wang (2001) and Henderson and Hobson (2008) for some special cases.

6 Appendix

Proof of Proposition 2.1: It suffices to prove $W_t^A \geq W_t^{A,u}$ for any $u \in \mathcal{A}_1$. Without loss of generality, we assume $t = 0$. Our proof here follows the arguments of CWZ (2008).

First, note that

$$\tilde{U}_1(\tau, C_\tau) = W_0^A + \int_0^\tau [g(u_s^A) - u_s^A g'(u_s^A)]ds + \int_0^\tau g'(u_s^A)dB_s.$$

Let Γ denote a constant which may vary from line to line. Then by Definition 2.1 (iii) we have

$$E\{|\tilde{U}_1(\tau, C_\tau)|^{\frac{8}{3}}\} \le \Gamma E\Big\{1+(\int_0^T |g(u_s^A)|ds)^{\frac{8}{3}}+(\int_0^T |u_s^A g'(u_s^A)|ds)^{\frac{8}{3}}+(\int_0^T |g'(u_s^A)|^2 ds)^{\frac{4}{3}}\Big\} < \infty.$$

Thus

$$E^u\{|\tilde{U}_1(\tau, C_\tau)|^2\} = E\{M_T^u|\tilde{U}_1(\tau, C_\tau)|^2\} \le E\{|M_T^u|^4\}^{\frac{1}{4}} E\{|\tilde{U}_1(\tau, C_\tau)|^{\frac{8}{3}}\}^{\frac{3}{4}} < \infty,$$

which, together with

$$E^u\{|\int_0^\tau g(u_s)ds|^2\} \le E\{M_T^u|\int_0^\tau |g(u_s)|ds|^2\} \le E\{|M_T^u|^4\}^{\frac{1}{4}} E\{|\int_0^\tau |g(u_s)|ds|^{\frac{8}{3}}\}^{\frac{3}{4}} < \infty,$$

implies that (2.12) is well-posed and

$$E^u\{\int_0^T |w_t^{A,u}|^2 dt\} < \infty.$$

Moreover,

$$E^u\{\int_0^T |w_t^A|^2 dt\} = E\{M_T^u \int_0^T |g'(u_t^A)|^2 dt\} < \infty.$$

Thus

$$E^u\{\int_0^T |w_t^A - w_t^{A,u}|^2 dt\} < \infty. \tag{6.19}$$

Now recalling (2.12) and (2.14), we have

$$W_0^A - W_0^{A,u} = \int_0^\tau \Big[[g(u_s) - u_s w_s^{A,u}] - [g(I_1(w_s^A)) - w_s^A I_1(W_s^A)]\Big]ds + \int_0^\tau [w_s^{A,u} - w_s^A]dB_s.$$

Since g is convex, we have

$$g(u_s) - g(I_1(w_s^A)) \geq g'(I_1(w_s^A))[u_s - I_1(w_s^A)] = w_s^A[u_s - I_1(w_s^A)]$$

with the equality holding true if and only if $u = I_1(w^A)$. Then

$$W_0^A - W_0^{A,u} \geq \int_0^\tau u_s[w_s^A - w_s^{A,u}]ds + \int_0^\tau [w_s^{A,u} - w_s^A]dB_s = \int_0^\tau [w_s^{A,u} - w_s^A]dB_s^u. \tag{6.20}$$

By (6.19), taking expected values we prove $W_0^A \geq W_0^{A,u}$. ■

Proof of Proposition 2.3: First, by Definition 2.2 (ii), (2.23) is well-posed. If $u = u^\tau$ is optimal, along Δu we can show, using arguments similar to those in CWZ (2008), that

$$\begin{aligned}\nabla V^P(\tau; u) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[W_0^{P,\tau,u^\varepsilon} - W_0^{P,\tau,u}] = E^u\Big\{\tilde{U}_2(\tau, X_\tau, J(\tau, W_\tau^{1,u}))\int_0^\tau \Delta u_t dB_t^u \\ + \tilde{U}_2'(\tau, X_\tau, J(\tau, W_\tau^{1,u}))/\tilde{U}_1'(\tau, J(\tau, W_\tau^A))\int_0^\tau g''(u_t)\Delta u_t dB_t^u\Big\}.\end{aligned}$$

and the condition (2.24) is a consequence of maximum principle arguments, again as in CWZ (2008). ■

Proof of Proposition 3.1: Note that $\tilde{W}_0^A = -\frac{1}{\gamma_A}\exp\left[-\gamma_A W_0^A\right]$, so the optimization of the agent's utility W_0^A is equivalent to the optimization of $\tilde{W}_0^A$. By Ito's rule, we get

$$\begin{aligned}\tilde{W}_t^A &= C_\tau + \tilde{A}_\tau - \int_t^\tau \left[\frac{1}{2\gamma_A}(Z_s^A)^2 + g(u_s)\right]ds - \int_t^\tau \frac{Z_s^A}{\gamma_A}dB_s^u \\ &= C_\tau + \tilde{A}_\tau - \int_t^\tau \left[\frac{1}{2\gamma_A}(Z_s^A)^2 + g(u_s) - \frac{Z_s^A}{\gamma_A}u_s\right]ds - \int_t^\tau \frac{Z_s^A}{\gamma_A}dB_s \end{aligned} \tag{6.21}$$

By the Comparison Theorem for BSDEs **, the optimal u is obtained by minimizing the integrand in the first integral in the previous expression, so

**By the comparison theorem we mean the result of the type as in Proposition 2.1. In the standard BSDE literature it is proved under Lipschitz conditions, while in Proposition 2.1 we prove it under weaker conditions. Here, we omit all the technical conditions needed for the comparison theorem.

that the optimal u is determined from (3.2). This gives us, for the optimal u,

$$\tilde{W}_t^A = C_\tau + \tilde{A}_\tau - \int_t^\tau \left[\frac{1}{2}\gamma_A(g'(u_s))^2 + g(u_s) - u_s g'(u_s)\right] ds - \int_t^\tau g'(u_s)dB_s,$$

which obviously implies (3.3). ■

Proof of Proposition 3.2 . Define

$$\tilde{W}_t := \tilde{W}_t^P + \tilde{W}_t^A - \tilde{R}_0. \tag{6.22}$$

Note that $\tilde{W}_0 = \tilde{W}_0^P = -\frac{1}{\gamma_P}\log(-\gamma_P W_0^P)$. Thus, the principal's problem is equivalent to maximizing $\tilde{W}_0$. Applying Ito's formula we have

$$\tilde{W}_t^P = X_\tau - C_\tau + \tilde{P}_\tau - \int_t^\tau \left[\frac{1}{2\gamma_P}(Z_s^P)^2 - \frac{Z_s^P u_s}{\gamma_P}\right] ds - \int_t^\tau \left(\frac{Z_s^P}{\gamma_P}\right) dB_s \tag{6.23}$$

Denote

$$\tilde{Z}_t := \frac{Z_t^P}{\gamma_P} + g'(u_t), \tag{6.24}$$

Recalling (6.21) and (3.2), by straightforward calculation we have

$$\begin{aligned}\tilde{W}_t =& X_\tau + \tilde{A}_\tau + \tilde{P}_\tau - \tilde{R}_0 - \int_t^\tau \tilde{Z}_s dB_s \\ & - \int_t^\tau \left[\frac{\gamma_P}{2}\tilde{Z}_s^2 - (u_s + \gamma_P g'(u_s))\,\tilde{Z}_s + \frac{\gamma_A+\gamma_P}{2}(g'(u_s))^2 + g(u_s)\right] ds.\end{aligned} \tag{6.25}$$

We now mimic the proof of Proposition 2.3. For any Δu, denote $u^\varepsilon := u + \varepsilon\Delta u$, let $\tilde{W}^\varepsilon, \tilde{Z}^\varepsilon$ be the corresponding processes, and

$$\nabla\tilde{W} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\tilde{W}^\varepsilon - \tilde{W}]; \quad \nabla\tilde{Z} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\tilde{Z}^\varepsilon - \tilde{Z}].$$

Then, it can be shown that

$$\begin{aligned}\nabla\tilde{W}_0 \;=\; & -\int_0^\tau \nabla\tilde{Z}_s dB_s - \int_0^\tau \left[\gamma_P\tilde{Z}_s\nabla\tilde{Z}_s - (u_s + \gamma_P g'(u_s))\,\nabla\tilde{Z}_s\right] ds \\ & - \int_0^\tau \left[-(1+\gamma_P g''(u_s))\,\tilde{Z}_s + (\gamma_A+\gamma_P)g'(u_s)g''(u_s) + g'(u_s)\right]\Delta u_s ds.\end{aligned}$$

If u is optimal, then $\nabla \tilde{W}_0 \leq 0$ for any Δu. Thus

$$(1 + \gamma_P g''(u_t))\tilde{Z}_t = (\gamma_A + \gamma_P) g'(u_t) g''(u_t) + g'(u_t)$$

which obviously implies (3.5). ■

Proof of Proposition 3.3 : Note that $g'(u) = ku, g''(u) = k$. Then (i) is a direct consequence of Proposition 3.1.

To prove (ii), first note that by Proposition 3.2 and (6.24), (3.6) is necessary. On the other hand, for any z,

$$g(u) + \frac{\gamma_A + \gamma_P}{2}(g'(u))^2 + z[\gamma_P g'(u) + u] = [k + \frac{\gamma_A + \gamma_P}{2}]u^2 + z[\gamma_P k + 1]u$$

is a convex function of u. Then by (6.25) and the comparison theorem for BSDE's we know (3.6) is also sufficient.

It remains to prove (iii)-(v). By (6.24) and (3.6), (6.25) leads to

$$\tilde{W}_t = S_\tau - \tilde{R}_0 + \int_t^\tau \left[\frac{\beta}{2}\tilde{Z}_s^2\right] ds - \int_t^\tau \tilde{Z}_s dB_s. \tag{6.26}$$

If $\beta = 0$, we get $\tilde{W}_0 = E\{S_\tau\} - \tilde{R}_0$, which obviously implies (iii).

If $\beta \neq 0$, denote $W_t := \exp(\beta \tilde{W}_t)$. Then

$$dW_t = \beta W_t \tilde{Z}_t dB_t, \tag{6.27}$$

and thus

$$W_0 = E\{W_\tau\} = e^{-\beta \tilde{R}_0} E\{e^{\beta S_\tau}\}.$$

If $\beta > 0$, the optimal stopping problem is equivalent to maximizing W_0, which is further equivalent to maximizing $E\{e^{\beta S_\tau}\}$. This proves (iv). Finally, (v) can be proved analogously. ■

Proof of Proposition 3.5. (i) Note that for any stopping time τ,

$$E\{S_\tau\} = S_0 + E\{\int_0^\tau \mu_s ds\} \leq S_0 + \max_t \int_0^t \mu_s ds.$$

If $\beta = 0$, by Proposition 3.3 (iii) we prove the result immediately.

(ii) Assume $\beta > 0$. Define a new probability measure Q by

$$\frac{dQ}{dP} = \exp\left\{\int_0^T \beta\rho_t dB_t - \frac{1}{2}\int_0^T \beta^2\rho_t^2 dt\right\}.$$

Then

$$E\left\{e^{\beta S_\tau}\right\} = E^Q\left\{e^{\beta S_0 + \int_0^\tau [\frac{1}{2}\beta^2\rho_s^2 + \beta\mu_s]ds}\right\} \le \exp\left(\beta S_0 + \max_t \int_0^t [\frac{1}{2}\beta^2\rho_s^2 + \beta\mu_s]ds\right).$$

This proves (ii). One can prove (iii) similarly.

(iv) Recall (3.5). Since $\tilde{A}_t$ and $\tilde{P}_t$ are deterministic, it is obvious that $\rho_t = v_t$. Moreover, as above, τ is deterministic. Then for $t \le \tau$, by (6.27) we have

$$W_t = E_t\{W_\tau\} = E_t\{e^{\beta(S_\tau - \tilde{R}_0)}\} = \exp\left(\beta[S_0 + \int_0^\tau \mu_s ds - \tilde{R}_0] + \beta\int_0^t v_s dB_s + \frac{1}{2}\beta^2\int_t^\tau v_s^2 ds\right).$$

This implies that, for $t < \tau$,

$$dW_t = \beta W_t v_t dB_t,$$

which, combined with (6.27), implies that $\tilde{Z}_t = v_t$.

Finally, by (3.3) and (3.6), we can compute C_τ as in (3.8). ■

Proof of Proposition 4.3: The principal wants to maximize, over C_τ,

$$e^{U_1(C_\tau)}[X_\tau - C_\tau + P_\tau + \lambda]$$

We change the variables as $Y_\tau := e^{U_1(C_\tau)} > 0$. Then $C_\tau = J_1(\log(Y_\tau))$ where $J_1 := U_1^{-1}$. Denote

$$f(y;x) := y[x - J_1(\log(y))]; \quad \hat{f}(x) := \sup_{y>0} f(y;x).$$

Then the principal wants to maximize

$$\sup_{Y_\tau > 0} f(Y_\tau; X_\tau + P_\tau + \lambda) = \hat{f}(X_\tau + P_\tau + \lambda).$$

It is easily shown that $yJ_1(\log(y))$ is a convex function. By (4.12) the conjugate $\hat{f}(x)$ is well defined and is an increasing convex function. If P_t is a submartingale, then so is $X_t + P_t + \lambda$, and therefore $\hat{f}(X_t + P_t + \lambda)$ is also a submartingale. So the solution to the principal's optimal stopping problem $\sup_\tau E[\hat{f}(X_\tau + P_\tau + \lambda)]$ is $\tau = T$. ■

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