Descent Identities, Hessenberg Varieties, and the Weil Conjectures

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Abstract

We apply the Weil conjectures to the Hessenberg Varieties to obtain information about the combinatorics of descents in the symmetric group. Combining this with elementary linear algebra leads to elegant proofs of some identities from the theory of descents.

1 Introduction

The purpose of this introduction is to give background on the following three topics: permutation statistics, Hessenberg varieties, and the Weil conjectures. The topics will be described in this order, and the emphasis will be on their relationship to each other as is relevant to this note. Permutation statistics are functions from the symmetric group S_n to the non-negative integers. Many permutations statistics have interesting combinatorial properties (pages 17-31 of Stanley [11]) and give rise to metrics which are important in the statistical theory of ranking (Chapter 6 of Diaconis [5]). Volume 3 of Knuth [9] connects permutation statistics with the theory of sorting.

One important statistic is the number of inversions of a permutation. This is denoted $Inv(\pi)$ and is equal to the number of pairs (i,j) such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. The number of inversions of π is also equal to the length of π in terms of the generating reflections $\{(i,i+1) : 1 \leq i \leq n-1\}$. Inversions have the following well-known generating function (e.g. page 21 of Stanley [11])

$$\sum_{\pi \in S_n} q^{Inv(\pi)} = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1}$$

which can be used to prove that the distribution of inversions is asymptotically normal when n goes to infinity (e.g. Chapter 6 of [5]). The number of complete flags (i.e. $V_0 = id \subset V_1 \subset \cdots \subset V_n = V$ with $dim(V_i) = i$) in an n-dimensional vector space V over a finite field of size q is $\prod_{i=1}^{n} \frac{q^i-1}{q-1}$. This suggests a connection with algebraic geometry.

A second permutation statistic of interest is the number of descents of a permutation. The number of descents is denoted $d(\pi)$ and is defined to be the number of pairs $(i, i+1)$ with $1 \leq i \leq n-1$ and $\pi(i) > \pi(i+1)$. The generating function for descents gives rise to the Eulerian polynomials

$$A_n(q) = \sum_{\pi \in S_n} q^{d(\pi)+1}.$$

Pages 243-246 of Comtet [3] describe some properties of the Eulerian polynomials. Two of the most explicit generating functions involving Eulerian polynomials are

$$\frac{A_n(q)}{(1-q)^{n+1}} = \sum_{l \geq 0} l^n q^l$$

$$\sum_{n \geq 0} \frac{A_n(q) t^n}{n!} = \frac{1-q}{1 - q e^{t(1-q)}}.$$

Some recurrences for the Eulerian polynomials will be found in Section 2.

Next, we recall the Hessenberg varieties defined by DeMari, Procesi, and Shayman [4]. Let G be a complex, semisimple algebraic group, B a Borel subgroup, and T a maximal torus in B. Let $\mathbf{g}, \mathbf{b}, \mathbf{t}$ be the lie algebras of G, B, T respectively. Let $\mathbf{h}$ be a subspace of $\mathbf{g}$ which contains $\mathbf{b}$ and is a $\mathbf{b}$ submodule. Let $s \in \mathbf{t}$ be a regular, semi-simple element. Define the corresponding Hessenberg variety (which turns out to depend on G and $\mathbf{h}$ but not on s) by

$$X_H(s) = \{g \in G | Ad(g^{-1})[s] \in \mathbf{h}\}$$

where Ad is the Adjoint action of Lie theory (conjugation in the case of matrix groups).

The main example to be considered in this note is the following, which we will call $Hess(n,p)$. Let p be an integer such that $1 \leq p \leq n-1$. Let $G = SL(n,C)$ and $\mathbf{h}$ be the subspace of $sl(n,C)$ consisting of those matrices (h_{ij}) for which $h_{ij} = 0$ if $i - j > p$. Let s be any diagonalizable element of G with distinct eigenvalues. Then the corresponding Hessenberg variety $X_H(s)$ can be more simply described as all complete flags $V_0 \subset V_1 \subset \cdots \subset V_n$ satisfying the condition that $s(V_i) \subset V_{i+p}$. For example, $Hess(n, n-1)$ is the flag variety of $SL(n,C)$.

DeMari, Procesi, and Shayman [4] study the varieties $X_H(s)$, proving that they are smooth toric varieties and computing their Betti numbers. We will require only the following special case. Our interest is the fact that the Betti numbers of the varieties $Hess(n,p)$ are permutation statistics. For a different application of the Hessenberg varieties to combinatorics, see [13].

Theorem 1 *(DeMari, Shayman [6])*

1. *The varieties $Hess(n,p)$ are smooth.*
2. *The odd Betti numbers $\beta_{2k-1}(Hess(n,p))$ vanish. The even Betti numbers $\beta_{2k}(Hess(n,p))$ are equal to the number of π in the symmetric groups S_n such that $|\{(i,j) : 1 \leq i < j \leq n, j - i \leq p, \pi(i) > \pi(j)\}| = k$.*
3. *For q sufficiently large, the equations defining $Hess(n,p)$, reduced to a field of q elements, define a smooth variety.*

The third part of Theorem 1 was not stated explicitly in DeMari and Shayman [6], but follows by the same arguments as in the smooth case, given on pages 224-5 of their article. Note that if $q \leq n$ then there does not exist an invertible n by n diagonal matrix with distinct eigenvalues all of which lie in a field of q elements. As two examples of Theorem 1, $\beta_{2k}(Hess(n, n-1))$ is the number of permutations in S_n with k inversions and $\beta_{2k}(Hess(n,1))$ is the number of permutations in S_n with k descents.

Next let us review the Weil conjectures. One use of them is to compute Betti numbers of continuous varieties by counting points in varieties defined over finite fields. This will be done in Section 2, thereby proving identities about the Eulerian polynomials. The version of the Weil conjectures considered here can be found in Appendix C of Hartshorne [8]. These conjectures are now, of course, theorems.

Theorem 2 *(Weil Conjecture) Given a smooth variety V, its Betti numbers can be computed as follows. Reduce the equations defining V to equations over a field of q^s elements where q is a prime power, and let $N(q^s)$ be the number of solutions to these reduced equations. If the reduced variety is smooth for all such reductions then*

$$exp\left(\sum_{s=1}^{\infty} \frac{N(q^s)x^s}{s}\right) = \frac{P_1(x)P_3(x)...P_{2\delta-1}(x)}{P_0(x)P_2(x)...P_{2\delta}(x)}$$

where δ is the dimension of V and $P_k(x) = \prod_{j=1}^{\beta_k}(1 - \alpha_{k,l}x)$, with $|\alpha_{k,l}| = q^{\frac{k}{2}}$.

Stanley [12] has written Theorem 2 in a form which is somewhat more useful for our purposes.

Proposition 1 *(Stanley [12]) Suppose in addition to the assumptions of Theorem 2 that $N(q^s)$ is a polynomial $\sum_k \gamma_k q^{ks}$ in q^s. Then $\beta_{2k} = \gamma_k$.*

PROOF:

$$\begin{aligned} exp\left(\sum_{s=1}^{\infty}\frac{N(q^s)x^s}{s}\right) &= exp\left(\sum_{s=1}^{\infty}(\sum_k \gamma_k q^{ks})\frac{x^s}{s}\right) \\ &= exp\left(\sum_k \gamma_k \sum_{s=1}^{\infty}\frac{(q^k x)^s}{s}\right) \\ &= exp\left(\sum_k -\gamma_k ln(1-q^k x)\right) \\ &= \prod_k (1-q^k x)^{-\gamma_k} \end{aligned}$$

□

2 Descent Identities

As an example of the concepts in the introduction, let us use the Weil conjectures to find the generating function for permutations in S_n by inversions (also known as the Poincaré series of S_n).

Theorem 3

$$\sum_{\pi \in S_n} q^{Inv(\pi)} = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1}$$

PROOF: Theorem 1 and Proposition 1 applied to $Hess(n, n-1)$ show that

$$\sum_{\pi \in S_n} q^{Inv(\pi)} = N(q),$$

is the number of complete flags $V_0 \subset V_1 \subset \cdots \subset V_n$ over a field of q elements. V_1 can be chosen in $\frac{q^n-1}{q-1}$ ways. Given this choice of V_1, quotienting out the flag by V_1 shows that V_2 can be chosen in $\frac{q^{n-1}-1}{q-1}$ ways. Continuing in this way and multiplying proves that

$$N(q) = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1}$$

for infinitely many q, hence for all q since both sides are polynomials. □

Bott [1] and Chevalley [2] used the topology of compact Lie groups to prove the factorization of the Poincaré series for Weyl groups. The argument in Theorem 3 extends to the other Weyl groups, but this would be somewhat circular because one must know the size of the flag variety G/B where G is a finite algebraic group with Weyl group W, and historically $|G|$ was computed using the Bruhat decomposition and the factorization of the Poincaré series for Weyl groups.

The linear algebra involved in using the Weil conjectures to study the Eulerian polynomials $A_n(q)$ is slightly more involved. We thus establish two straightforward lemmas.

Lemma 1 *Let $M \in GL(n, K)$ act on an n-dimensional vector space V over a field K such that M has distinct eigenvalues which are all contained in K. Then there are exactly $\binom{n}{m}$ subspaces of dimension m which are invariant under M.*

PROOF: Let W be a subspace of dimension m which is invariant under M. The characteristic polynomial of M restricted to W divides the characteristic polynomial of M on V, since W is invariant. Since M has distinct eigenvalues on V, its characteristic polynomial consists of distinct linear factors, so the same is true for the characteristic polynomial of M on W. Thus W is spanned by some m of the n 1-dimensional eigenspaces for the action of M on V. □

Given a linear transformation M on a n-dimensional vector space V, call a vector $\vec{v}$ primitive if the set $\{\vec{v}, M\vec{v}, M^2\vec{v}, \ldots, M^{n-1}\vec{v}\}$ forms a basis of V.

Lemma 2 *Let $M \in GL(n,K)$ act on an n-dimensional vector space V over a field K such that M has distinct eigenvalues which are all contained in K. Then a vector $\vec{v}$ is primitive if and only if its components with respect to a basis of eigenvectors of M are all non-zero. Thus when the field K is finite of cardinality q, there are $(q-1)^n$ primitive vectors.*

PROOF: Pick a basis of eigenvectors $\vec{e_1}, ..., \vec{e_n}$ of M with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\vec{v}$ have components $(v_1, \cdots, v_n)$ with respect to this basis. Then $M^i\vec{v}$ has components $(\lambda_1^i v_1, \ldots, \lambda_n^i v_n)$. Clearly $\vec{v}$ is primitive if and only if the determinant of the matrix with rows $\vec{v}, M\vec{v}, \ldots, M^{n-1}\vec{v}$ written with respect to the basis of eigenvectors, is non-vanishing. The value of this determinant is

$$\prod_{i=1}^{n} v_i \cdot det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} = \prod_{i=1}^{n} v_i \cdot \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i),$$

which is non-vanishing precisely when all v_i are non-vanishing because the eigenvalues λ_i of M are distinct. □

Recall that $A_n(q)$ denotes the nth Eulerian polynomial $\sum_{\pi \in S_n} q^{d(\pi)+1}$. For convenience set $A_0(q) = q$. Theorem 4 is likely known, though we have not seen it in the literature before.

Theorem 4 $A_n(q) = \sum_{i=1}^{n} \binom{n}{i} (q-1)^{i-1} A_{n-i}(q)$.

PROOF: Let $N(q,n)$ be the number of flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ over the field of q elements such that $MV_j \subset V_{j+1}$ for all j, where $M \in GL(n,q)$ is a diagonal matrix with distinct eigenvalues. Let i be the smallest number between 1 and n such that $MV_i = V_i$. By Lemma 1, there are $\binom{n}{i}$ ways of choosing V_i. The part of the flag between V_1 and V_i is determined by V_1, which is spanned by a primitive vector in the i dimensional space V_i. There are, by Lemma 2, $(q-1)^i$ primitive vectors for V_i, and hence $(q-1)^{i-1}$ choices for V_1. Quotienting out the part of the flag between V_i and $V_n = V$ by V_i shows that there are $N(q, n-i)$ possibilities for this part of the flag. We thus have the recurrence

$$N(q,n) = \sum_{i=1}^{n} \binom{n}{i} (q-1)^{i-1} N(q, n-i).$$

By Proposition 1, $A_n(q) = qN(q,n)$, proving the theorem. □

The recurrence in Theorem 4 was proved by splitting the flag at the first subspace invariant under M and summing over such splittings. Theorem 5 will come from splitting the flag at all the W_i invariant under M, and then summing over all such splittings. The identity in Theorem 5 is known and goes back to [7], though the proof is completely different. The Stirling number of the second kind $S(n,r)$ is defined to be the number of partitions of the set $\{1, \cdots, n\}$ into r blocks.

Theorem 5 *(Frobenius [7])* $A_n(q) = q\sum_{r=1}^{n} r!S(n,r)(q-1)^{n-r}$.

PROOF: Proposition 1 shows that $A_n(q) = qN(q,n)$, where $N(q,n)$ is the number of flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $MV_j \subset V_{j+1}$ for $1 \leq j \leq n-1$ and $M \in GL(n,q)$ is a diagonal matrix with distinct eigenvalues. We count these flags by the set I of $i > 0$ such that V_i is invariant under M. For each subset $I = \{i_1, i_2, \cdots, i_r = n\}$ of $\{1, \cdots, n\}$, there are, by Lemma 1, $\binom{n}{i_{r-1}}\binom{i_{r-1}}{i_{r-2}}...\binom{i_2}{i_1}$ ways of picking the invariant subspaces $V_{i_1}, V_{i_2}, \cdots, V_{i_r=n}$ of dimensions $\{i_1, i_2, \cdots, i_r = n\}$ so as to respect the inclusion relations. Consider the portion of the flag between two consecutive invariant subspaces, say $V_{i_1} \subset V_{i_1+1} \subset ... \subset V_{i_2}$. Quotienting out this whole sequence by V_{i_1} shows that V_{i_1+1}/V_{i_1} must be spanned by a primitive vector for the action of M on V_{i_2}/V_{i_1}. The dimension of the quotient is $i_2 - i_1$ so by Lemma 2 there are $(q-1)^{i_2-i_1}$ primitive vectors. Since we are only interested in the subspace spanned by the vector, we divide out by $q-1$. Multiplying out these choices of primitive vectors, one sees that there are $(q-1)^{n-r}$ such choices. Therefore,

$$\begin{aligned} A_n(q) &= qN(q,n) \\ &= q\sum_{r=1}^{n}(q-1)^{n-r}\sum_{\substack{I\subseteq\{1,\cdots,n\}\\ n\in I, |I|=r}}\binom{n}{i_{r-1}}\binom{i_{r-1}}{i_{r-2}}...\binom{i_2}{i_1} \\ &= q\sum_{r=1}^{n} r!S(n,r)(q-1)^{n-r}. \end{aligned}$$

The last equality follows because

$$\sum_{\substack{I\subseteq\{1,\cdots,n\}\\ n\in I, |I|=r}}\binom{n}{n-i_{r-1}}\binom{i_{r-1}}{i_{r-1}-i_{r-2}}...\binom{i_2}{i_2-i_1}$$

is the number of ways of choosing a set partition of $\{1, \cdots, n\}$ into r blocks with an ordering on the blocks (the first block has size $n - i_{r-1}$ and can be chosen in $\binom{n}{n-i_{r-1}}$ ways, the second block has size $i_{r-1} - i_{r-2}$ and can be chosen in $\binom{i_{r-1}}{i_{r-1}-i_{r-2}}$ ways, etc.) However by the definition of the Stirling numbers of the second kind, the number of set partitions of $\{1, \cdots, n\}$ into r blocks with an ordering on the blocks is equal to $r!S(n,r)$. □

3 Concluding Remarks

The proofs of the descent identities given here admittedly use a lot of machinery. Nevertheless, given this machinery, the method of counting flags employed in Theorems 3, 4, and 5 is natural and gives one a feel for where the recurrences come from. However, suppose one wants a recurrence for the Eulerian numbers $A(n,k)$ which are the number of permutations on n symbols with $k+1$ descents. It is easy to see combinatorially that

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k).$$

Thus direct combinatorics seems superior for finding recursions satisfied by the coefficients of the Eulerian polynomials, but the flag counting methods seem well-adapted toward finding recurrences satisfied by the polynomials themselves.

It would be interesting to use the Weil conjectures to find recurrences for the generating functions for the permutation statistics defined by $|\{(i,j) : 1 \leq i < j \leq n, j-i \leq p, \pi(i) > \pi(j)\}|$ at the permutation π. Descents and inversions are the limiting cases $p = 1, n-1$ and are the Betti numbers of $Hess(n,1)$ and $Hess(n,n-1)$ respectively. We have not made much progress for other values of p. Direct combinatorial arguments have not been successfully applied to this problem either.

Finally, it is natural to extend these results to other finite Coxeter groups W. The definition of a descent of an element $w \in W$ is a simple positive root which w maps to a negative root. Reiner [10] has studied the distribution of descents in other Coxeter groups.

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