STEIN'S METHOD AND CHARACTERS OF COMPACT LIE GROUPS

JASON FULMAN

Abstract. Stein's method is used to study the trace of a random element from a compact Lie group or symmetric space. Central limit theorems are proved using very little information: character values on a single element and the decomposition of the square of the trace into irreducible components. This is illustrated for Lie groups of classical type and Dyson's circular ensembles. The approach in this paper will be useful for the study of higher dimensional characters, where normal approximations need not hold.

1. Introduction

There is a large literature on the traces of random elements of compact Lie groups. One of the earliest results is due to Diaconis and Shahshahani [DS]. Using the method of moments, they show that if g is random from the Haar measure of the unitary group $U(n,\mathbb{C})$, and $Z = X + iY$ is a standard complex normal with X and Y independent, mean 0 and variance $\frac{1}{2}$ normal variables, then for $j = 1, 2, \cdots$, $Tr(g^j)$ are independent and distributed as $\sqrt{j}Z$ asymptotically as $n \to \infty$. They give similar results for the orthogonal group $O(n,\mathbb{R})$ and the group of unitary symplectic matrices $USp(2n,\mathbb{C})$. The moment computations of [DS] use representation theory. It is worth noting that there are other approaches to their moment computations: [PV] uses a version of integration by parts (and also treats $SO(n,\mathbb{R})$), and [CoSz] uses an "extended Wick calculus" (and also treats symmetric spaces).

Concerning the error in the normal approximation in the [DS] results, Diaconis conjectured that for fixed j, it decreases exponentially or even superexponentially in n. Stein [St2] uses "Stein's method" to show that $Tr(g^k)$ on $O(n,\mathbb{R})$ is asymptotically normal with error $O(n^{-r})$ for any fixed r. Johansson [J] proved Diaconis' conjecture for classical compact Lie groups using Toeplitz determinants and a very detailed analysis of characteristic functions.

Date: Submitted 6/13/08; Revised for Comm. Math. Phys. on 7/10/08.

Key words and phrases. Stein's method, central limit theorem, compact Lie group, circular ensembles, symmetric space, random matrix.

The author received funding from NSF grant DMS-0503901.

Address: Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA; fulman@usc.edu.

One direction in which the [DS] results have been extended is the study of linear statistics of eigenvalues: see [J], [DE], [So] and the numerous references therein. There is also work by D'Aristotile, Diaconis, and Newman [DDN] on central limit theorems for linear functions such as $Tr(Ag)$ where A is a fixed $n \times n$ real matrix and g is from the Haar measure of $O(n, \mathbb{R})$. In recent work, Meckes [Me2] refined Stein's technique from [St2] to establish a sharp total variation distance error term (order n^{-1}) for the [DDN] result.

A natural goal is to prove limit theorems (with error terms) for the distribution of traces in other irreducible representations: i.e. $\chi^{\tau}(g)$, where g is a random element of a compact Lie group and χ^{τ} is the character of an irreducible representation τ. This would have direct implications for Katz's work [Ka] on exponential sums; see Section 4.7 of [KLR] for details. We do not attain this goal, but make a useful contribution to it.

More precisely, the current paper presents a formulation of Stein's method designed for the study of $\chi^{\tau}(g)$. In the case of normal approximation, we obtain $O(n^{-1})$ bounds for the error term using only two pieces of information:

- The value of the "character ratios" $\frac{\chi^{\phi}(\alpha)}{dim(\phi)}$ where ϕ may be arbitrary but α is a single element of G (typically chosen to be close to the identity)
- The decomposition of τ^2 into irreducible representations

In contrast, the method of moments approach requires knowing the multiplicity of the trivial representation in τ^k for all $k \geq 1$ (which could be tricky to compute) and does not give an immediate bound on the error. Johansson's paper [J] gives sharper bounds when χ^{τ} is the trace of an element from a classical compact Lie group, but required knowledge of high order moments and deep analytical tools which might not extend to arbitrary representations τ. Even the Stein's method approaches of Stein [St2] and Meckes [Me2] use information about the distribution of matrix entries; very little is known about this for arbitrary τ, whereas the main ingredient for our approach (character theory) is well-developed.

Let us explain our statement in the abstract that the methods of this paper will prove useful for approximation other than normal approximation. We use Stein's method of exchangeable pairs which involves the construction of a pair (W, W') of exchangeable random variables. Our pair (which is somewhat different from those of Stein [St2] and Meckes [Me2]) satisfies the linearity condition that $\mathbb{E}(W'|W)$ is proportional to W, and we find representation theoretic formulas for quantities such as $\mathbb{E}(W' - W)^k$. These computations are completely general and apply to arbitrary distributional approximation. Stein's method of exchangeable pairs is still quite undeveloped for continuous distributions other than the normal, but that is temporary and there are some results: see [Mn], [Re] for the chi-squared distribution, [Lu] for the Gamma distribution, and [GoT] for the semicircle law. Closest to the current paper is [CF], which develops error terms for exponential approximation using quantities like $\mathbb{E}(W'-W)^k$ with k small.

We remark that the bounds in our paper are all given in the Kolmogorov metric. Similar results can be proved in the slightly stronger total variation metric (see the remarks after Theorem 2.1). However we prefer to work in the Kolmogorov metric as it underscores the similarity with discrete settings such as [Fu], where total variation convergence does not occur. We also mention that all bounds obtained in this paper are given with explicit constants.

The organization of this paper is as follows. Section 2 gives background on Stein's method and normal approximation. Section 3 develops general theory for the case that G is a compact Lie group and χ^τ an irreducible character. It treats the trace of random elements of $O(n,\mathbb{R})$, $USp(2n,\mathbb{C})$, and $U(n,\mathbb{C})$ as examples. Section 4 extends the methods of Section 3 to study spherical functions of compact symmetric spaces. The symmetric space setting is natural from the viewpoint of random matrix theory [Dn], [KaS]. After illustrating the technique on the sphere, we treat Dyson's circular ensembles as examples, obtaining an error term.

2. Stein's Method for Normal Approximation

In this section we briefly review Stein's method for normal approximation, using the method of exchangeable pairs [St1]. For more details, one can consult the survey [RR] and the references therein.

Two random variables W, W' are called an exchangeable pair if (W, W') has the same distribution as (W', W). As is typical in probability theory, let $\mathbb{E}(A|B)$ denote the expected value of A given B. The following result of Stein uses an exchangeable pair (W, W') to prove a central limit theorem for W.

Theorem 2.1. *(*[St1]*) Let (W, W') be an exchangeable pair of real random variables such that $\mathbb{E}(W^2) = 1$ and $\mathbb{E}(W'|W) = (1-a)W$ with $0 < a < 1$. Then for all real x_0,*

$$\begin{aligned} &\left| \mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx \right| \\ \leq \quad & \frac{\sqrt{Var(\mathbb{E}[(W'-W)^2|W])}}{a} + (2\pi)^{-\frac{1}{4}} \sqrt{\frac{1}{a}\mathbb{E}|W'-W|^3}. \end{aligned}$$

Remarks:

(1) There are variations of Theorem 2.1 (for instance Theorem 6 of [Me1]) which can be combined with our calculations to prove normal approximation in the total variation metric. However Theorem 2.1 is quite convenient for our purposes.
(2) In recent work, Röellin [Rl] has given a version of Theorem 2.1 in which the exchangeability condition can be replaced by the slightly

weaker condition that W and W' have the same law. Since exchangeability holds in our examples and may be useful for other applications involving Stein's method, we adhere to using Theorem 2.1.

To apply Theorem 2.1, one needs bounds on $Var(\mathbb{E}[(W'-W)^2|W])$ and $\mathbb{E}|W-W'|^3$. The following lemmas are helpful for this purpose.

Lemma 2.2. *Let (W,W') be an exchangeable pair of random variables such that $\mathbb{E}(W'|W) = (1-a)W$ and $\mathbb{E}(W^2) = 1$. Then $\mathbb{E}(W'-W)^2 = 2a$.*

Proof. Since W and W' have the same distribution,

$$\begin{aligned}
\mathbb{E}(W'-W)^2 &= \mathbb{E}(\mathbb{E}(W'-W)^2|W) \\
&= \mathbb{E}((W')^2) + \mathbb{E}(W^2) - 2\mathbb{E}(W\mathbb{E}(W'|W)) \\
&= 2\mathbb{E}(W^2) - 2\mathbb{E}(W\mathbb{E}(W'|W)) \\
&= 2\mathbb{E}(W^2) - 2(1-a)\mathbb{E}(W^2) \\
&= 2a.
\end{aligned}$$

□

Lemma 2.3 is a well known inequality (already used in the monograph [St1]) and useful because often the right hand side is easier to compute or bound than the left hand side. To make this paper as self-contained as possible, we include a proof. Here x is an element of the state space X.

Lemma 2.3.

$$Var(\mathbb{E}[(W'-W)^2|W]) \leq Var(\mathbb{E}[(W'-W)^2|x]).$$

Proof. Jensen's inequality states that if g is a convex function, and Z a random variable, then $g(\mathbb{E}(Z)) \leq \mathbb{E}(g(Z))$. There is also a conditional version of Jensen's inequality (Section 4.1 of [Du]) which states that for any σ subalgebra F of the σ-algebra of all subsets of X,

$$\mathbb{E}(g(\mathbb{E}(Z|F))) \leq \mathbb{E}(g(Z)).$$

The lemma follows by setting $g(t) = t^2$, $Z = \mathbb{E}((W'-W)^2|x)$, and letting F be the σ-algebra generated by the level sets of W. □

3. Compact Lie groups

This section uses Stein's method to study the distribution of a fixed irreducible character χ^τ of a compact Lie group G. Subsection 3.1 develops general theory for the case that χ^τ is real valued. This is applied to study the trace of a random element of $USp(2n,\mathbb{C})$ in Subsection 3.2 and the trace of a random orthogonal matrix in Subsections 3.3 and 3.4. Subsection 3.5 indicates the relevant amendments for the complex setting and Subsection 3.6 illustrates the theory for $U(n,\mathbb{C})$.

3.1. **General theory (real case).** Let G be a compact Lie group and χ^τ a non-trivial real-valued irreducible character of G. The random variable of interest to us is $W = \chi^\tau(g)$, where g is chosen from the Haar measure of G. It follows from the orthogonality relations for irreducible characters of G that $\mathbb{E}(W) = 0$ and $\mathbb{E}(W^2) = 1$.

The following functional equation will be useful.

Lemma 3.1. *([He2], p. 392) Let G be a compact Lie group and χ^ϕ an irreducible character of G. Then*

$$\int_G \chi^\phi(h\alpha h^{-1}g)dh = \frac{\chi^\phi(\alpha)}{dim(\phi)}\chi^\phi(g)$$

for all $\alpha, g \in G$.

We now define a pair (W, W') by letting $W = \chi^\tau(g)$ where g is chosen from Haar measure and $W' = W(\alpha g)$ where α is chosen uniformly at random from a fixed self-inverse conjugacy class of G. Exchangeability of (W, W') follows since the conjugacy class of α is self-inverse. Moreover, since $\chi^\phi(\alpha^{-1}) = \overline{\chi^\phi(\alpha)}$, one has that $\chi^\phi(\alpha)$ is real for all irreducible representations ϕ, a fact which will be used freely throughout this subsection.

Remark: Non-identity self-inverse conjugacy classes always exist. Indeed, if G has rank r then any maximal torus has $2^r - 1$ elements of order 2, and these are naturally self-inverse. Moreover if G has all characters real valued (as is the case for symplectic and orthogonal groups), then all conjugacy classes are self inverse, since class functions can be uniformly approximated by sums of characters and $\chi^\phi(\alpha^{-1}) = \overline{\chi^\phi(\alpha)}$.

The remaining results in this subsection show that the exchangeable pair (W, W') has desirable properties.

Lemma 3.2.

$$\mathbb{E}(W'|W) = \left(\frac{\chi^\tau(\alpha)}{dim(\tau)}\right)W.$$

Proof. Applying Lemma 3.1 with $\phi = \tau$, one has that

$$\mathbb{E}(W'|g) = \int_{h\in G} \chi^\tau(h\alpha h^{-1}g)dh = \left(\frac{\chi^\tau(\alpha)}{dim(\tau)}\right)\chi^\tau(g).$$

The result follows since this depends on g only through W. □

Lemma 3.3.

$$\mathbb{E}(W' - W)^2 = 2\left(1 - \frac{\chi^\tau(\alpha)}{dim(\tau)}\right).$$

Proof. This is immediate from Lemmas 2.2 and 3.2. □

For the remainder of this subsection, if ϕ is an irreducible representation of G, we let $m_\phi(\tau^r)$ denote the multiplicity of ϕ in the r-fold tensor product of τ (which has character $(\chi^\tau)^r$).

Lemma 3.4.
$$\mathbb{E}[(W')^2|g] = \sum_{\phi} m_\phi(\tau^2) \frac{\chi^\phi(\alpha)}{dim(\phi)} \chi^\phi(g),$$
where the sum is over all irreducible representations of G.

Proof. Write $(W')^2 = \sum_\phi m_\phi(\tau^2)\chi^\phi(g')$. Lemma 3.1 gives that
$$\mathbb{E}[\chi^\phi(g')|g] = \int_G \chi^\phi(h\alpha h^{-1}g)dh = \frac{\chi^\phi(\alpha)}{dim(\phi)}\chi^\phi(g),$$
and the result follows. □

Lemma 3.5 writes $Var([\mathbb{E}(W'-W)^2|g])$ as a sum of positive quantities.

Lemma 3.5.
$$Var([\mathbb{E}(W'-W)^2|g]) = \sum_{\phi}^{*} m_\phi(\tau^2)^2 \left(1 + \frac{\chi^\phi(\alpha)}{dim(\phi)} - \frac{2\chi^\tau(\alpha)}{dim(\tau)}\right)^2,$$
where the star signifies that the sum is over all nontrivial irreducible representations of G.

Proof. By Lemmas 3.2 and 3.4,
$$\begin{aligned}
\mathbb{E}((W'-W)^2|g) &= \mathbb{E}[(W')^2|g] - 2W\mathbb{E}(W'|g) + W^2 \\
&= \mathbb{E}[(W')^2|g] + \left(1 - \frac{2\chi^\tau(\alpha)}{dim(\tau)}\right) W^2 \\
&= \sum_\phi m_\phi(\tau^2)\left(1 + \frac{\chi^\phi(\alpha)}{dim(\phi)} - \frac{2\chi^\tau(\alpha)}{dim(\tau)}\right)\chi^\phi(g).
\end{aligned}$$
The orthogonality relation for irreducible characters of G gives that
$$\mathbb{E}[\mathbb{E}((W'-W)^2|g)^2] = \sum_\phi m_\phi(\tau^2)^2 \left(1 + \frac{\chi^\phi(\alpha)}{dim(\phi)} - \frac{2\chi^\tau(\alpha)}{dim(\tau)}\right)^2.$$
Finally, note that
$$Var([\mathbb{E}(W'-W)^2|g]) = \mathbb{E}[\mathbb{E}((W'-W)^2|g)^2] - (\mathbb{E}(W'-W)^2)^2,$$
and since the multiplicity of the trivial representation in τ^2 is 1, the result follows from Lemma 3.3. □

Lemma 3.6. *Let k be a positive integer.*

(1) $\mathbb{E}(W'-W)^k = \sum_{r=0}^k (-1)^{k-r}\binom{k}{r}\sum_\phi m_\phi(\tau^r)m_\phi(\tau^{k-r})\frac{\chi^\phi(\alpha)}{dim(\phi)}$.
(2) $\mathbb{E}(W'-W)^4 = \sum_\phi m_\phi(\tau^2)^2\left[8\left(1 - \frac{\chi^\tau(\alpha)}{dim(\alpha)}\right) - 6\left(1 - \frac{\chi^\phi(\alpha)}{dim(\phi)}\right)\right]$.

Proof. For the first assertion, note that
$$\mathbb{E}[(W'-W)^k|g] = \sum_{r=0}^k (-1)^{k-r}\binom{k}{r}\chi^\tau(g)^{k-r}\mathbb{E}[(W')^r|g].$$

Arguing as in Lemma 3.4 gives that this is equal to

$$\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\chi^{\tau}(g)^{k-r}\sum_{\phi} m_{\phi}(\tau^{r})\frac{\chi^{\phi}(\alpha)}{dim(\phi)}\chi^{\phi}(g).$$

Thus $\mathbb{E}(W'-W)^k$ is equal to

$$\begin{aligned} & \mathbb{E}(\mathbb{E}[(W'-W)^k|g]) \\ = & \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{\phi} m_{\phi}(\tau^{r})\frac{\chi^{\phi}(\alpha)}{dim(\phi)}\int_{g\in G}\chi^{\tau}(g)^{k-r}\chi^{\phi}(g) \\ = & \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{\phi} m_{\phi}(\tau^{r})m_{\phi}(\tau^{k-r})\frac{\chi^{\phi}(\alpha)}{dim(\phi)}. \end{aligned}$$

For the second assertion, note by the first assertion that

$$\mathbb{E}(W'-W)^4 = \sum_{r=0}^{4}(-1)^{r}\binom{4}{r}\sum_{\phi} m_{\phi}(\tau^{r})m_{\phi}(\tau^{4-r})\frac{\chi^{\phi}(\alpha)}{dim(\phi)}.$$

If α is the identity element of G, then $W'=W$ which implies that

$$0 = \sum_{r=0}^{4}(-1)^{r}\binom{4}{r}\sum_{\phi} m_{\phi}(\tau^{r})m_{\phi}(\tau^{4-r}).$$

Thus for general α,

$$\mathbb{E}(W'-W)^4 = -\sum_{r=0}^{4}(-1)^{r}\binom{4}{r}\sum_{\phi} m_{\phi}(\tau^{r})m_{\phi}(\tau^{4-r})\left(1-\frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right).$$

Observe that the $r=0,4$ terms in this sum vanish, since the only contribution could come from the trivial representation, which contributes 0. The $r=2$ term is

$$-6\sum_{\phi}\left(1-\frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)m_{\phi}(\tau^2)^2.$$

The $r=1,3$ terms are equal and together contribute

$$\begin{aligned} 8\left(1-\frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right)m_{\tau}(\tau^3) &= 8\left(1-\frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right)\int_{g\in G}\chi^{\tau}(g)^4 \\ &= 8\left(1-\frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right)\int_{g\in G}\left|\sum_{\phi} m_{\phi}(\tau^2)\chi^{\phi}(g)\right|^2 \\ &= 8\left(1-\frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right)\sum_{\phi} m_{\phi}(\tau^2)^2. \end{aligned}$$

This completes the proof. □

The above lemmas are completely general. Specializing to normal approximation, one obtains the following result.

Theorem 3.7. *Let G be a compact Lie group and let τ be a non-trivial irreducible representation of G whose character is real valued. Fix a non-identity element α with the property that α and α^{-1} are conjugate. Let $W = \chi^{\tau}(g)$ where g is chosen from the Haar measure of G. Then for all real x_0,*

$$\left|\mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx\right|$$
$$\leq \sqrt{\sum_{\phi}^{*} m_{\phi}(\tau^2)^2 \left[2 - \frac{1}{a}\left(1 - \frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)\right]^2}$$
$$+ \left[\frac{1}{\pi} \sum_{\phi} m_{\phi}(\tau^2)^2 \left(8 - \frac{6}{a}\left(1 - \frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)\right)\right]^{1/4}.$$

Here $a = 1 - \frac{\chi^{\tau}(\alpha)}{dim(\tau)}$, the first sum is over all non-trivial irreducible representations of G, and the second sum is over all irreducible representations of G.

Proof. One applies Theorem 2.1 to the exchangeable pair (W, W') of this subsection. By Lemmas 2.3 and 3.5, the first term in Theorem 2.1 gives the first term in the theorem. To upper bound the second term in Theorem 2.1, note by the Cauchy-Schwartz inequality that

$$\mathbb{E}|W' - W|^3 \leq \sqrt{\mathbb{E}(W' - W)^2 \mathbb{E}(W' - W)^4}.$$

Now use Lemma 3.3 and part 2 of Lemma 3.6. □

3.2. **Example:** $USp(2n, \mathbb{C})$**.** This subsection studies the distribution of $\chi^{\tau}(g)$, where τ is the $2n$ dimensional defining representation of $USp(2n, \mathbb{C})$. The only representation theoretic fact needed is Lemma 3.8, which is the $k = 2$ case of a formula from page 200 of [Su] giving the decomposition of τ^k into irreducible representations. In its statement, we let $x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}$ denote the eigenvalues of an element of $USp(2n, \mathbb{C})$.

Lemma 3.8. *For $n \geq 2$, the square of the defining representation of the group $USp(2n, \mathbb{C})$ decomposes in a multiplicity free way as the sum of the following three irreducible representations:*

- *The trivial representation, with character* 1
- *The representation with character* $\frac{1}{2}(\sum_i x_i + x_i^{-1})^2 + \frac{1}{2}\sum_i (x_i^2 + x_i^{-2})$
- *The representation with character* $\frac{1}{2}(\sum_i x_i + x_i^{-1})^2 - \frac{1}{2}\sum_i (x_i^2 + x_i^{-2}) - 1$

Remark: Lemma 3.8 could also be easily guessed (and proved) by looking at the character formulas for $USp(2n, \mathbb{C})$ on page 219 of [W].

Theorem 3.9. *Let g be chosen from the Haar measure of $USp(2n,\mathbb{C})$, where $n \geq 2$. Let $W(g)$ be the trace of g. Then for all real x_0,*

$$\left| \mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx \right| \leq \frac{\sqrt{2}}{n}.$$

Proof. One applies Theorem 3.7, with τ the defining representation, and α an element of type $\{x_1^{\pm 1}, \cdots, x_n^{\pm n}\}$ where $x_1 = \cdots = x_{n-1} = 1$ and $x_n = e^{i\theta}$. Then α is conjugate to α^{-1} and one computes that $a = \frac{1-cos(\theta)}{n}$. Using Lemma 3.8, one calculates that the first error term in Theorem 3.7 is equal to $\frac{2\sqrt{4cos(\theta)^2-4cos(\theta)+2}}{2n+1}$. One computes that the second error term is equal to $\left[\frac{24(1-cos(\theta))}{\pi(2n+1)}\right]^{1/4}$. Since these bounds hold for all θ and are continuous in θ, the bounds hold in the limit that $\theta \to 0$. This gives an upper bound of $\frac{2\sqrt{2}}{2n+1} \leq \frac{\sqrt{2}}{n}$, as claimed. □

3.3. **Example:** $SO(2n+1,\mathbb{R})$**.** We investigate the distribution of $\chi^\tau(g)$, where τ is the $2n+1$-dimensional defining representation of $SO(2n+1,\mathbb{R})$. The only ingredient from representation theory needed is Lemma 3.10, which is the $k=2$ case of a formula from page 204 of [Su] giving the decomposition of τ^k into irreducible representations (it is also easily obtained by inspecting the character formulas on page 228 of [W]). In its statement, we let $x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}, 1$ be the eigenvalues of an element of $SO(2n+1,\mathbb{R})$.

Lemma 3.10. *For $n \geq 2$, the square of the defining representation of $SO(2n+1,\mathbb{R})$ decomposes in a multiplicity free way as the sum of the following three irreducible representations:*

- *The trivial representation, with character* 1
- *The representation with character* $\frac{1}{2}(\sum_i x_i + x_i^{-1})^2 + \frac{1}{2}\sum_i (x_i^2 + x_i^{-2}) + \sum_i (x_i + x_i^{-1})$
- *The representation with character* $\frac{1}{2}(\sum_i x_i + x_i^{-1})^2 - \frac{1}{2}\sum_i (x_i^2 + x_i^{-2}) + \sum_i (x_i + x_i^{-1})$

This leads to the following theorem.

Theorem 3.11. *Let g be chosen from the Haar measure of $SO(2n+1,\mathbb{R})$, where $n \geq 2$. Let $W(g)$ be the trace of g. Then for all real x_0,*

$$\left| \mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx \right| \leq \frac{\sqrt{2}}{n}.$$

Proof. One applies Theorem 3.7, taking τ to be the defining representation, and α to be an element of type $\{x_1^{\pm 1}, \cdots, x_n^{\pm n}, 1\}$ where $x_1 = \cdots = x_{n-1} = 1$ and $x_n = e^{i\theta}$ (i.e. α is a rotation by θ). Then α is conjugate to α^{-1} and $a = \frac{2(1-cos(\theta))}{2n+1}$. Using Lemma 3.10 one computes that in the $\theta \to 0$ limit the first error term in Theorem 3.7 is equal to $\frac{\sqrt{2}}{n}$. One calculates that the second error term is equal to $\left[\frac{12(2n+1)(1-cos(\theta))}{\pi n(2n+3)}\right]^{1/4}$. The proof of the theorem is completed by noting that this goes to 0 as $\theta \to 0$. $\square$

3.4. **Example:** $O(2n, \mathbb{R})$**.** We consider the distribution of $\chi^\tau(g)$, where τ is the $2n$-dimensional defining representation of $O(2n, \mathbb{R})$. The only representation theoretic information needed is Lemma 3.12, which is the $k = 2$ case of a result of Proctor [Pr] (and also not difficult to obtain from the character formulas on page 228 of [W]). In its statement, we let $x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}$ be the eigenvalues of an element of $O(2n, \mathbb{R})$.

Lemma 3.12. *For $n \geq 2$, the square of the defining representation of $O(2n, \mathbb{R})$ decomposes in a multiplicity free way as the sum of the following three irreducible representations:*

- *The trivial representation, with character* 1
- *The representation with character* $\frac{1}{2}\left(\sum_i x_i + \overline{x_i}\right)^2 - \frac{1}{2}\sum_i \left(x_i^2 + \overline{x_i}^2\right)$
- *The representation with character* $\frac{1}{2}\left(\sum_i x_i + \overline{x_i}\right)^2 + \frac{1}{2}\sum_i \left(x_i^2 + \overline{x_i}^2\right) - 1$

This leads to the following result.

Theorem 3.13. *Let g be chosen from the Haar measure of $O(2n, \mathbb{R})$, where $n \geq 2$. Let $W(g)$ be the trace of g. Then for all real x_0,*

$$\left|\mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx\right| \leq \frac{\sqrt{2}}{n-1}.$$

Proof. Apply Theorem 3.7, with τ the defining representation of $O(2n, \mathbb{R})$. We take α to be an element of type $\{x_1^{\pm 1}, \cdots, x_n^{\pm n}\}$ where $x_1 = \cdots = x_{n-1} = 1$ and $x_n = e^{i\theta}$ (i.e. α is a rotation by θ). Then α is conjugate to α^{-1} and $a = \frac{1-cos(\theta)}{n}$. Lemma 3.12 gives the decomposition of τ^2 into irreducibles, and from this one calculates the first error term in Theorem 3.7, and sees that in the $\theta \to 0$ limit it is equal to $\frac{\sqrt{8}}{2n-1}$. One computes that the second error term is equal to $\left[\frac{24n(1-cos(\theta))}{\pi(n+1)(2n-1)}\right]^{1/4}$. The proof of the theorem is completed by noting that this goes to 0 as $\theta \to 0$. $\square$

3.5. **General theory (complex case).** Let G be a compact Lie group and τ be an irreducible representation of G whose character is not real valued. The random variable of interest to us is $W = \frac{1}{\sqrt{2}}\left(\chi^\tau(g) + \overline{\chi^\tau(g)}\right)$, where g is chosen from the Haar measure of G. It follows from the orthogonality relations for irreducible characters of G that $\mathbb{E}(W) = 0$ and $\mathbb{E}(W^2) = 1$.

We now define a pair (W, W') by letting W be as above and $W' = W(\alpha g)$ where α is chosen uniformly at random from a fixed self-inverse conjugacy class of G. As in Subsection 3.1, the pair (W, W') is exchangeable and all $\chi^\phi(\alpha)$ are real.

The remaining results in this subsection are proved by minor modifications of the arguments in Subsection 3.1.

Lemma 3.14.

$$\mathbb{E}(W'|W) = \left(\frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right) W.$$

Lemma 3.15.

$$\mathbb{E}(W' - W)^2 = 2\left(1 - \frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right).$$

Lemma 3.16.

$$\mathbb{E}[(W')^2|g] = \frac{1}{2}\sum_{\phi} m_{\phi}[(\tau+\overline{\tau})^2]\frac{\chi^{\phi}(\alpha)}{dim(\phi)}\chi^{\phi}(g),$$

where the sum is over all irreducible representations of G.

Lemma 3.17.

$$Var([\mathbb{E}(W' - W)^2|g]) = \frac{1}{4}\sum_{\phi}^{*} m_{\phi}[(\tau+\overline{\tau})^2]^2\left(1 + \frac{\chi^{\phi}(\alpha)}{dim(\phi)} - \frac{2\chi^{\tau}(\alpha)}{dim(\tau)}\right)^2,$$

where the star signifies that the sum is over all nontrivial irreducible representations of G.

Lemma 3.18. *Let k be a positive integer.*

(1) $\mathbb{E}(W' - W)^k$ *is equal to*

$$\frac{1}{2^{k/2}}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{\phi} m_{\phi}[(\tau+\overline{\tau})^r]m_{\phi}[(\tau+\overline{\tau})^{k-r}]\frac{\chi^{\phi}(\alpha)}{dim(\phi)}.$$

(2) $\mathbb{E}(W' - W)^4 = \sum_{\phi} m_{\phi}[(\tau+\overline{\tau})^2]^2\left[2\left(1 - \frac{\chi^{\tau}(\alpha)}{dim(\tau)}\right) - \frac{3}{2}\left(1 - \frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)\right].$

Finally, one obtains the following central limit theorem.

Theorem 3.19. *Let G be a compact Lie group and τ an irreducible representation of G whose character is not real valued. Let $\alpha \neq 1$ be such that α and α^{-1} are conjugate. Let $W = \frac{1}{\sqrt{2}}\left(\chi^{\tau}(g) + \overline{\chi^{\tau}(g)}\right)$ where g is chosen from the Haar measure of G. Then for all real x_0,*

$$\left|\mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}dx\right|$$

$$\leq \frac{1}{2}\sqrt{\sum_{\phi}^{*} m_{\phi}[(\tau+\overline{\tau})^2]^2\left[2 - \frac{1}{a}\left(1 - \frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)\right]^2}$$

$$+ \left[\frac{1}{\pi}\sum_{\phi} m_{\phi}[(\tau+\overline{\tau})^2]^2\left(2 - \frac{3}{2a}\left(1 - \frac{\chi^{\phi}(\alpha)}{dim(\phi)}\right)\right)\right]^{1/4}.$$

Here $a = 1 - \frac{\chi^{\tau}(\alpha)}{dim(\tau)}$, the first sum is over all non-trivial irreducible representations of G, and the second sum is over all irreducible representations of G.

3.6. **Example:** $U(n,\mathbb{C})$**.** Every element of $U(n,\mathbb{C})$ is conjugate to a diagonal matrix with entries $(x_1,\cdots,x_n)$ and the representation theory of $U(n,\mathbb{C})$ is well understood (see for instance [Bu]). The irreducible representations of $U(n,\mathbb{C})$ are parameterized by integer sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The corresponding character value on an element of type $\{x_1,\cdots,x_n\}$ is given by the Schur function $s_\lambda(x_1,\cdots,x_n)$. (The usual definition of Schur functions requires that $\lambda_n \geq 0$, so if $\lambda_n = -k < 0$, this should be interpreted as $(x_1\cdots x_n)^{-k} s_{\lambda+(k)^n}$, where $\lambda + (k)^n$ is given by adding k to each of $\lambda_1,\cdots,\lambda_n$). The complex conjugate of a character with data $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ has data $-\lambda_n \geq -\lambda_{n-1} \geq \cdots \geq -\lambda_1$.

Combining the above information with Theorem 3.19, one obtains the following result.

Theorem 3.20. *Let g be chosen from the Haar measure of $U(n,\mathbb{C})$, where $n \geq 2$. Let $W(g) = \frac{1}{\sqrt{2}}[Tr(g) + \overline{Tr(g)}]$ where Tr denotes trace. Then for all real x_0,*

$$\left|\mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}dx\right| \leq \frac{2}{n-1}.$$

Proof. One applies Theorem 3.19 with τ the n-dimensional defining representation. We take α to be an element of type $\{x_1,\cdots,x_n\}$ with $x_1 = \cdots = x_{n-2} = 1$, $x_{n-1} = e^{i\theta}$, and $x_n = e^{-i\theta}$. Then α and α^{-1} are conjugate and $a = \frac{2(1-cos(\theta))}{n}$. By the Pieri rule for multiplying Schur functions (page 73 of [Mac]), the decomposition of the character of $(\tau+\overline{\tau})^2$ in terms of Schur functions is given by

$$\begin{aligned}
(s_{(1)} + \overline{s_{(1)}})^2 &= s_{(1)}s_{(1)} + 2\frac{s_{(1)}s_{(1^{n-1})}}{x_1\cdots x_n} + \overline{s_{(1)}s_{(1)}} \\
&= s_{(2)} + s_{(1,1)} + 2\left[\frac{s_{(1^n)} + s_{(2,1^{n-2})}}{x_1\cdots x_n}\right] + s_{(-2)} + s_{(-1,-1)} \\
&= 2 + s_{(2)} + s_{(1,1)} + s_{(-2)} + s_{(-1,-1)} + 2s_{(1,0^{n-2},-1)}.
\end{aligned}$$

One then computes that in the $\theta \to 0$ limit, the first error term in Theorem 3.19 is equal to $\frac{2\sqrt{n^2+2}}{n^2-1} \leq \frac{2}{n-1}$. One computes that the second error term is equal to $\left[\frac{12(2n-1)(1-cos(\theta))}{\pi(n^2-1)}\right]^{1/4}$. The proof is completed by noting that this approaches 0 as $\theta \to 0$. □

4. Compact Symmetric Spaces

This section extends the methods of Section 3 to study the distribution of a fixed spherical function ω_τ on a random element of a compact symmetric space G/K. Subsection 4.1 gives general theory for the case that ω_τ is real valued. This is illustrated for the sphere in Subsection 4.2, giving a different perspective on a result of [DF] and [Me1]. We note that since compact Lie groups can be viewed as symmetric spaces (see Section 4.1), the examples in Subsections 3.2, 3.3, and 3.4 give further examples. Our theorems should also prove useful for Jacobi-type ensembles arising from other root systems (see for instance [Vr]).

Subsection 4.3 indicates the changes needed to treat spherical functions ω_τ which are not real valued, and Subsections 4.4 and 4.5 study the trace of elements from Dyson's circular orthogonal and circular symplectic ensembles as special cases (the circular unitary ensemble is equivalent to $U(n,\mathbb{C})$, so was already treated in Subsection 3.6). Central limit theorems are known for the trace of an element from the circular ensembles (see pages 48-9 of [Ra], [BF], [CoSz]), but our approach gives an error term.

4.1. **General theory (real case).** To begin we recall some concepts about spherical functions of symmetric spaces. Standard references which contain more details are [He1], [He2], [Te], and [V]. Chapter 7 of [Mac] is also very helpful.

A Riemannian manifold X is said to be a symmetric space if the geodesic symmetry $\sigma : X \mapsto X$ with center at any point x_0 is an isometry. Then X can be identified with G/K, where G is a connected transitive Lie group of isometries of X, and K is a compact group which up to finite index is given by $K = \{g \in G : gx_0 = x_0\}$.

A function $\omega_\phi \in L^2(G/K)$ is called spherical if $\omega_\phi(1) = 1$ and the following functional equation is satisfied:

$$\int_K \omega_\phi(xky)dk = \omega_\phi(x)\omega_\phi(y) \ \forall x, y \in G.$$

This equation implies that ω_ϕ is K bi-invariant (i.e. $\omega_\phi(k_1gk_2) = \omega_\phi(g)$ for all k_1, k_2 in K), which justifies our writing $\omega_\phi(g)$ instead of $\omega_\phi(gK)$.

One reason that spherical functions are important is that if G/K is compact and H_ϕ is the G-invariant subspace of $L^2(G/K)$ generated by ω_ϕ, then H_ϕ is a finite dimensional irreducible representation of G and $L^2(G/K)$ is a direct sum of all such H_ϕ. We let $dim(\phi)$ denote the dimension of H_ϕ.

In reading this section it is useful to keep in mind that a compact Lie group U can be viewed as a compact symmetric space. Indeed, one can take $G = U \times U$ and K the diagonal subgroup of U; then G/K is identified with U under the mapping $(u_1, u_2)K \mapsto u_1 u_2^{-1}$. The spherical functions ω_ϕ of G/K are indexed by irreducible representations ϕ of U and are precisely the character ratios $\frac{\chi^\phi(u)}{\chi^\phi(1)}$; moreover, $dim(H_\phi) = \chi^\phi(1)^2$.

Let ω_τ be a non-trivial real valued spherical function of G/K. We are interested in the distribution of $\omega_\tau(g)$ (normalized to have variance 1). Here gK is chosen from the "Haar measure" μ on G/K. This is induced from the Haar measure of G using the projection map $G \mapsto G/K$, and is invariant under the action of G.

The following orthogonality relation will be used; see for instance page 45 of [Kl] for a proof.

Lemma 4.1.
$$\int_{G/K} \omega_\phi(g)\overline{\omega_\eta(g)} = \frac{\delta_{\phi,\eta}}{dim(\phi)}.$$

In particular, Lemma 4.1 implies that $W := [dim(\tau)]^{1/2}\omega_\tau$ has mean 0 and variance 1.

The following lemma is immediate from the functional equation for ω_ϕ and K bi-invariance of ω_ϕ.

Lemma 4.2. *Let G/K be a compact symmetric space, and ω_ϕ a spherical function of G/K. Then*
$$\int_{K \times K} \omega_\phi(k_1 \alpha k_2 g) dk_1 dk_2 = \omega_\phi(\alpha)\omega_\phi(g)$$
for all $\alpha, g \in G$.

We define the pair (W, W') by letting $W = [dim(\tau)]^{1/2}\omega_\tau(g)$ where gK is from the "Haar measure" of G/K, and $W' = W(\alpha g)$ where α is chosen uniformly at random from a fixed double coset $K\alpha K \neq K$ which satisfies the property that $K\alpha K = K\alpha^{-1}K$. Since $K\alpha^{-1}K = (K\alpha K)^{-1}$, it follows that (W, W') is exchangeable. Moreover the integral formula for spherical functions (page 417 of [He2]) implies that all $\omega_\phi(\alpha)$ are real.

The analysis of the exchangeable pair (W, W') can be carried out exactly as in Subsection 3.1, using Lemmas 4.1 and 4.2 instead of the orthogonality relations for compact Lie groups and Lemma 3.1. Hence we simply record the results.

Lemma 4.3.
$$\mathbb{E}(W'|W) = \omega_\tau(\alpha)W.$$

Lemma 4.4.
$$\mathbb{E}(W' - W)^2 = 2(1 - \omega_\tau(\alpha)).$$

In the statements of the remaining results, we define the "multiplicity" $m_\phi(\tau^r)$ by the expansion
$$[\omega_\tau(g)]^r = \sum_\phi \left[\frac{dim(\phi)}{dim(\tau)^r}\right]^{1/2} m_\phi(\tau^r)\omega_\phi(g).$$
The numbers $m_\phi(\tau^r)$ are real and non-negative (argue as on page 396 of [Mac] with sums replaced by integrals) but need not be integers. Note that by Lemma 4.1,
$$m_\phi(\tau^r) = [dim(\phi)dim(\tau)^r]^{1/2} \int_{G/K} \omega_\tau(g)^r \overline{\omega_\phi(g)}.$$

Lemma 4.5.
$$\mathbb{E}[(W')^2|gK] = \sum_\phi m_\phi(\tau)^2 [dim(\phi)]^{1/2} \omega_\phi(\alpha)\omega_\phi(g),$$
where the sum is over all spherical functions of G/K.

Lemma 4.6.

$$Var([\mathbb{E}(W'-W)^2|gK]) = \sum_{\phi}^{*} m_\phi(\tau^2)^2\left(1+\omega_\phi(\alpha)-2\omega_\tau(\alpha)\right)^2,$$

where the star signifies that the sum is over all nontrivial spherical functions of G/K.

Lemma 4.7. *Let k be a positive integer.*

(1) $\mathbb{E}(W'-W)^k = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_\phi m_\phi(\tau^r)m_\phi(\tau^{k-r})\omega_\phi(\alpha)$.
(2) $\mathbb{E}(W'-W)^4 = \sum_\phi m_\phi(\tau^2)^2\left[8(1-\omega_\tau(\alpha))-6(1-\omega_\phi(\alpha))\right]$.

Finally, one has the following central limit theorem.

Theorem 4.8. *Let G/K be a compact symmetric space and let ω_τ be a non-trivial real-valued spherical function of G/K. Fix an element $\alpha \notin K$ such that $K\alpha K = K\alpha^{-1}K$. Let $W = [dim(\tau)]^{1/2}\omega_\tau(g)$ where gK is chosen from the "Haar measure" of G/K. Then for all real x_0,*

$$\left|\mathbb{P}(W\le x_0)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0}e^{-\frac{x^2}{2}}dx\right|$$
$$\le \sqrt{\sum_{\phi}^{*} m_\phi(\tau^2)^2\left[2-\frac{1}{a}\left(1-\omega_\phi(\alpha)\right)\right]^2}$$
$$+\left[\frac{1}{\pi}\sum_{\phi} m_\phi(\tau^2)^2\left(8-\frac{6}{a}\left(1-\omega_\phi(\alpha)\right)\right)\right]^{1/4}.$$

Here $a = 1-\omega_\tau(\alpha)$, the first sum is over all non-trivial spherical functions of G/K, and the second sum is over all spherical functions of G/K.

4.2. **Example: the sphere.** This subsection studies the unit sphere in $\mathbb{R}^n$, viewed as the symmetric space $SO(n,\mathbb{R})/SO(n-1,\mathbb{R})$. Chapter 9 of [V] is a good reference for the spherical functions of this symmetric space, and Chapter 4 of [DyM] is a very clear textbook treatment for the special case $n=3$. Letting $e_1,\cdots,e_n$ be the standard basis of $\mathbb{R}^n$ and embedding $SO(n-1,\mathbb{R})$ inside $SO(n,\mathbb{R})$ as the subgroup fixing e_n, then $Kg_1K = Kg_2K$ if and only if $g_1(e_n)$ and $g_2(e_n)$ have the same last coordinate. From this it is not difficult to check that $KgK = Kg^{-1}K$ for all g, and that the double cosets of $SO(n-1,\mathbb{R})$ in $SO(n,\mathbb{R})$ are parameterized by x_n, the final coordinate of a point $(x_1,\cdots,x_n)$ on the sphere. In what follows we let x denote x_n.

From page 461 of [V], the spherical functions ω_l are parameterized by integers $l \ge 0$ and satisfy

$$\omega_l(x) = \frac{l!(n-3)!}{(l+n-3)!}C_l^{\frac{n-2}{2}}(x).$$

Here C_l^ρ is the Gegenbauer polynomial, defined by the generating function

$$\sum_{l\geq 0} C_l^\rho(x)t^l = (1-2xt+t^2)^{-\rho}.$$

For instance,

$$C_0^\rho(x)=1,\ C_1^\rho(x)=2\rho x,\ C_2^\rho(x)=-\rho+2\rho(1+\rho)x^2$$

and

$$\omega_0(x)=1,\ \omega_1(x)=x,\ \omega_2(x)=\frac{nx^2-1}{n-1}.$$

By page 462 of [V], $dim(l)=\frac{(2l+n-2)(n+l-3)!}{(n-2)!l!}$.

We study the random variable $W(x)=[dim(1)]^{1/2}\omega_1=\sqrt{n}x$. In fact sharp (up to constants) normal approximations for W are known: see Diaconis and Freedman [DF] and also Meckes [Me1], who uses Stein's method to obtain an error term of $\frac{2\sqrt{3}}{n-1}$ in total variation distance. Our viewpoint leads to the following result.

Theorem 4.9. *Let $W=\sqrt{n}x$, where x is the last coordinate of a random point on the unit sphere in $\mathbb{R}^n$. Then for all real x_0,*

$$\left|\mathbb{P}(W\leq x_0)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0}e^{-\frac{x^2}{2}}dx\right|\leq\frac{2\sqrt{2}}{n-1}.$$

Proof. We apply Theorem 4.8 with $\tau=\omega_1$. Writing $\omega_1(x)^2$ as a linear combination of $\omega_0(x)$ and $\omega_2(x)$, one computes that $m_{(0)}(\tau^2)=1, m_{(2)}(\tau^2)=\sqrt{\frac{2(n-1)}{n+2}}$, and that all other multiplicities in τ^2 vanish. Letting α be less than one but close to 1, one computes that the first error term in Theorem 4.8 is exactly $\frac{2\sqrt{2}}{n-1}$. Letting α tend to 1 from below, the second error term in Theorem 4.8 vanishes and the result follows. □

4.3. **General theory (complex case).** In this subsection G/K is a compact symmetric space and ω_τ is a spherical function which is not real valued. The random variable we study is $W=\left[\frac{dim(\tau)}{2}\right]^{1/2}\left(\omega_\tau(g)+\overline{\omega_\tau(g)}\right)$. As in Subsection 4.1, let $W'=W(\alpha g)$ where α is chosen uniformly at random from a fixed double coset $K\alpha K\neq K$ which satisfies the property that $K\alpha K=K\alpha^{-1}K$. Then (W,W') is exchangeable and all $\omega_\phi(\alpha)$ are real.

The remaining results in this subsection extend those of Subsection 3.5, and are proved by nearly identical arguments, using Lemmas 4.1 and 4.2.

Lemma 4.10. $\mathbb{E}(W'|W)=\omega_\tau(\alpha)W$.

Lemma 4.11. $\mathbb{E}(W'-W)^2=2(1-\omega_\tau(\alpha))$.

For the remaining results in this subsection, we define the "multiplicity" $m_\phi[(\tau+\overline{\tau})^r]$ by the expansion

$$(\omega_\tau + \overline{\omega_\tau})^r = \sum_\phi \left[\frac{dim(\phi)}{dim(\tau)^r}\right]^{1/2} m_\phi[(\tau+\overline{\tau})^r]\omega_\phi.$$

Arguing as on page 396 of [Mac], one has that the numbers $m_\phi[(\tau+\overline{\tau})^r]$ are real and non-negative (though not necessarily integers). Note that by Lemma 4.1,

$$m_\phi[(\tau+\overline{\tau})^r] = [dim(\phi)dim(\tau)^r]^{1/2}\int_{G/K}\left(\omega_\tau(g)+\overline{\omega_\tau(g)}\right)^r\overline{\omega_\phi(g)}.$$

Lemma 4.12.

$$\mathbb{E}[(W')^2|gK] = \frac{1}{2}\sum_\phi m_\phi[(\tau+\overline{\tau})^2]dim(\phi)^{1/2}\omega_\phi(\alpha)\omega_\phi(g),$$

where the sum is over all spherical functions of G/K.

Lemma 4.13.

$$Var([\mathbb{E}(W'-W)^2|gK]) = \frac{1}{4}\sum_\phi^* m_\phi[(\tau+\overline{\tau})^2]^2(1+\omega_\phi(\alpha)-2\omega_\tau(\alpha))^2,$$

where the star signifies that the sum is over all nontrivial spherical functions of G/K.

Lemma 4.14. *Let k be a positive integer.*

(1) $\mathbb{E}(W'-W)^k$ *is equal to*

$$\frac{1}{2^{k/2}}\sum_{r=0}^k(-1)^{k-r}\binom{k}{r}\sum_\phi m_\phi[(\tau+\overline{\tau})^r]m_\phi[(\tau+\overline{\tau})^{k-r}]\omega_\phi(\alpha).$$

(2) $\mathbb{E}(W'-W)^4$ *is equal to*

$$\sum_\phi m_\phi[(\tau+\overline{\tau})^2]^2\left[2(1-\omega_\tau(\alpha))-\frac{3}{2}(1-\omega_\phi(\alpha))\right].$$

Putting the pieces together, one has the following central limit theorem.

Theorem 4.15. *Let G/K be a compact symmetric space and let ω_τ be a spherical function of G/K which is not real valued. Fix an element $\alpha \notin K$ such that $K\alpha K = K\alpha^{-1}K$. Let $W = \sqrt{\frac{dim(\tau)}{2}}\left(\omega_\tau(g)+\overline{\omega_\tau(g)}\right)$, where gK*

is from the "Haar measure" of G/K. Then for all real x_0,

$$\left|\mathbb{P}(W \le x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx\right|$$
$$\le \frac{1}{2}\sqrt{\sum_{\phi}^{*} m_{\phi}[(\tau+\overline{\tau})^2]^2 \left[2 - \frac{1}{a}(1-\omega_{\phi}(\alpha))\right]^2}$$
$$+ \left[\frac{1}{\pi}\sum_{\phi} m_{\phi}[(\tau+\overline{\tau})^2]^2 \left(2 - \frac{3}{2a}(1-\omega_{\phi}(\alpha))\right)\right]^{1/4}.$$

Here $a = 1-\omega_{\tau}(\alpha)$, the first sum is over all non-trivial spherical functions of G/K, and the second sum is over all spherical functions of G/K.

4.4. **Example:** $U(n,\mathbb{C})/O(n,\mathbb{R})$**.** This symmetric space can be identified with the set of symmetric unitary matrices by the map $g \mapsto gg^T$ (see [Dn] or [Fo] for details), and the resulting matrix ensemble is known as Dyson's circular orthogonal ensemble. For a thorough discussion of this ensemble, see the texts [Fo] or [Mt]. In particular, it is known that if a function f depends only on the eigenvalues $x_1, \cdots, x_n$ of a matrix from Dyson's ensemble, then

$$\int_{G/K} f = \frac{[\Gamma(3/2)]^n}{\Gamma(\frac{n}{2}+1)} \int_{\mathbb{T}^n} f(x_1,\cdots,x_n) \prod_{1\le i<j\le n} |x_i - x_j| \prod_{k=1}^{n} \frac{dx_k}{2\pi}.$$

In this integral, $\mathbb{T}^n$ is the n-dimensional torus with coordinates

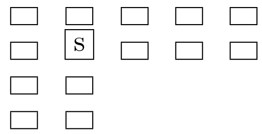
$$x_1,\cdots,x_n,\ x_i \in \mathbb{C},\ |x_i| = 1.$$

It is convenient to let the inner product $\langle f, g\rangle$ of two functions be defined by

$$\langle f,g\rangle = \frac{[\Gamma(3/2)]^n}{\Gamma(\frac{n}{2}+1)} \int_{\mathbb{T}^n} f(x_1,\cdots,x_n)\overline{g(x_1,\cdots,x_n)} \prod_{1\le i<j\le n} |x_i - x_j| \prod_{k=1}^{n} \frac{dx_k}{2\pi}$$

The spherical functions for this symmetric space are parameterized by integer sequences $\lambda_1 \ge \lambda_2 \cdots \ge \lambda_n$ and are $\omega_\lambda := \frac{P_\lambda(x_1,\cdots,x_n;2)}{P_\lambda(1,\cdots,1;2)}$, the normalized Jack polynomials with parameter 2. An excellent reference for Jack polynomials is Section 6.10 of [Mac]. There one assumes that $\lambda_n \ge 0$, so if $\lambda_n = -k < 0$, P_λ should be interpreted as $(x_1 \cdots x_n)^{-k} P_{\lambda+(k)^n}$, where $\lambda + (k)^n$ is given by adding k to each of $\lambda_1, \cdots, \lambda_n$.

To describe some useful combinatorial properties of Jack polynomials, we use the notation that if λ is a partition and s is a box of λ, then $l'(s), l(s), a(s), a'(s)$ are respectively the number of squares in the diagram of λ to the north of s (in the same column), south of s (in the same column), east of s (in the same row), and west of s (in the same row). For example the box marked s in the partition below



would have $l'(s) = 1$, $l(s) = 2$, $a'(s) = 1$, and $a(s) = 3$.

Letting λ be a partition of n, and using this notation, two useful formulas are the "principal specialization formula" (page 381 of [Mac])

$$P_\lambda(1, \cdots, 1; 2) = \prod_{s \in \lambda} \left[\frac{n + 2a'(s) - l'(s)}{2a(s) + l(s) + 1} \right].$$

and the formula

$$dim(\lambda) = \prod_{s \in \lambda} \frac{(n + 1 + 2a'(s) - l'(s))\,(n + 2a'(s) - l'(s))}{(2a(s) + l(s) + 2)\,(2a(s) + l(s) + 1)},$$

which follows from the formula for $\langle P_\lambda, P_\lambda \rangle$ on page 383 of [Mac] and the fact (Lemma 4.1) that

$$dim(\lambda) = \frac{P_\lambda(1, \cdots, 1; 2)^2}{\langle P_\lambda, P_\lambda \rangle}.$$

Theorem 4.16. *Let* $W = \frac{1}{2}\sqrt{1 + \frac{1}{n}} \left(Tr(g) + \overline{Tr(g)} \right)$, *where* g *is random from Dyson's circular orthogonal ensemble,* Tr *denotes trace, and* $n \geq 2$. *Then for all real* x_0,

$$\left| \mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx \right| \leq \frac{4}{n}.$$

Proof. Apply Theorem 4.15 to the spherical function $\tau = \omega_{(1)}(g) = \frac{Tr(g)}{n}$. To compute $m_\phi[(\tau + \overline{\tau})^2]$ for all ϕ, one has to decompose $(\omega_{(1)} + \overline{\omega_{(1)}})^2$ into spherical functions, which is equivalent to decomposing $(P_{(1)} + \overline{P_{(1)}})^2$ in terms of Jack polynomials. From the Pieri rule for Jack polynomials ([Mac], page 340), one calculates that

$$\begin{aligned} (P_{(1)} + \overline{P_{(1)}})^2 &= P_{(1)}P_{(1)} + 2\frac{P_{(1)}P_{(1^{n-1})}}{x_1 \cdots x_n} + \overline{P_{(1)}P_{(1)}} \\ &= P_{(2)} + \frac{4}{3}P_{(1^2)} + 2\left[\frac{\frac{2n}{n+1}P_{(1^n)} + P_{(2,1^{n-2})}}{x_1 \cdots x_n} \right] + \overline{P_{(2)}} + \frac{4}{3}\overline{P_{(1^2)}} \\ &= \frac{4n}{n+1} + P_{(2)} + \frac{4}{3}P_{(1^2)} + 2P_{(1,0^{n-2},-1)} + \overline{P_{(2)}} + \frac{4}{3}\overline{P_{(1^2)}}. \end{aligned}$$

We choose α to be an element of type $(1, \cdots, 1, e^{i\theta}, e^{-i\theta})$. Then $K\alpha K = K\alpha^{-1}K$ and $a = \frac{2(1-cos(\theta))}{n}$. One computes that in the $\theta \to 0$ limit, the first error term of Theorem 4.15 is $\frac{1}{n}\sqrt{\frac{8(n^3+2n^2+5n+6)}{n^3+4n^2+n-6}} \leq \frac{4}{n}$. The proof is completed by computing that the second error term is $\left[\frac{24(n+1)^2(1-cos(\theta))}{\pi n^2(n+3)}\right]^{1/4}$ which goes to 0 as $\theta \to 0$. □

4.5. **Example:** $U(2n,\mathbb{C})/USp(2n,\mathbb{C})$**.** This symmetric space corresponds to Dyson's circular symplectic ensemble ([Dn]); see [Fo] or [Mt] for background on this ensemble. In particular, it is known that if f depends only on the eigenvalues $x_1, \cdots, x_n$ of a matrix from this ensemble, then

$$\int_{G/K} f = \frac{2^n}{(2n)!} \int_{\mathbb{T}^n} f(x_1, \cdots, x_n) \prod_{1 \leq i < j \leq n} |x_i - x_j|^4 \prod_{k=1}^n \frac{dx_k}{2\pi},$$

where $\mathbb{T}^n$ is as in the previous example. We let the inner product $\langle f, g \rangle$ of two functions be defined by

$$\langle f, g \rangle = \frac{2^n}{(2n)!} \int_{\mathbb{T}^n} f(x_1, \cdots, x_n) \overline{g(x_1, \cdots, x_n)} \prod_{1 \leq i < j \leq n} |x_i - x_j|^4 \prod_{k=1}^n \frac{dx_k}{2\pi}.$$

The spherical functions for this symmetric space are parameterized by integer sequences $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ and are $\omega_\lambda := \frac{P_\lambda(x_1,\cdots,x_n;\frac{1}{2})}{P_\lambda(1,\cdots,1;\frac{1}{2})}$, the normalized Jack polynomials with parameter 1/2. As mentioned in the previous example, Jack polynomials are usually defined assuming that $\lambda_n \geq 0$, so if $\lambda_n = -k < 0$, P_λ should be interpreted as $(x_1 \cdots x_n)^{-k} P_{\lambda+(k)^n}$, where $\lambda + (k)^n$ is given by adding k to each of $\lambda_1, \cdots, \lambda_n$.

Letting λ be a partition of n and using the notation of the previous example, two useful formulas are the "principal specialization formula" (page 381 of [Mac])

$$P_\lambda(1, \cdots, 1; \frac{1}{2}) = \prod_{s \in \lambda} \left[\frac{n + \frac{a'(s)}{2} - l'(s)}{\frac{a(s)}{2} + l(s) + 1} \right].$$

and the formula

$$dim(H_\lambda) = \prod_{s \in \lambda} \frac{\left(n + \frac{a'(s)}{2} - l'(s)\right)(2n - 1 + a'(s) - 2l'(s))}{\left(\frac{a(s)}{2} + l(s) + 1\right)(a(s) + 2l(s) + 1)},$$

which follows from the formula for $\langle P_\lambda, P_\lambda \rangle$ on page 383 of [Mac] and the fact (Lemma 4.1) that

$$dim(\lambda) = \frac{P_\lambda(1, \cdots, 1; \frac{1}{2})^2}{\langle P_\lambda, P_\lambda \rangle}.$$

Theorem 4.17. *Let* $W = \sqrt{1 - \frac{1}{2n}} \left(Tr(g) + \overline{Tr(g)}\right)$*, where* g *is random from Dyson's circular symplectic ensemble and* $n \geq 2$*. Then for all real* x_0*,*

$$\left| \mathbb{P}(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx \right| \leq \frac{4}{n}.$$

Proof. Apply Theorem 4.15 to the spherical function $\tau = \omega_{(1)}(g) = \frac{Tr(g)}{n}$. To compute $m_\phi[(\tau + \overline{\tau})^2]$ for all ϕ, one has to decompose $(\omega_{(1)} + \overline{\omega_{(1)}})^2$ into spherical functions, which is equivalent to decomposing $(P_{(1)} + \overline{P_{(1)}})^2$ in terms of Jack polynomials. From the Pieri rule for Jack polynomials ([Mac], page 340), one calculates that $(P_{(1)} + \overline{P_{(1)}})^2$ is equal to

$$\begin{aligned} & P_{(1)}P_{(1)} + 2\frac{P_{(1)}P_{(1^{n-1})}}{x_1 \cdots x_n} + \overline{P_{(1)}P_{(1)}} \\ = \quad & P_{(2)} + \frac{2}{3}P_{(1^2)} + 2\left[\frac{\frac{n}{2n-1}P_{(1^n)} + P_{(2,1^{n-2})}}{x_1 \cdots x_n}\right] + \frac{2}{3}\overline{P_{(1^2)}} + \overline{P_{(2)}} \\ = \quad & \frac{2n}{2n-1} + P_{(2)} + \frac{2}{3}P_{(1^2)} + 2P_{(1,0^{n-2},-1)} + \frac{2}{3}\overline{P_{(1^2)}} + \overline{P_{(2)}}. \end{aligned}$$

Now take α to be an element of type $(1, \cdots, 1, e^{i\theta}, e^{-i\theta})$; then $K\alpha K = K\alpha^{-1}K$ and $a = \frac{2(1-cos(\theta))}{n}$. One computes that in the $\theta \to 0$ limit, the first error term of Theorem 4.15 is $\frac{1}{2n}\sqrt{\frac{8(4n^3-4n^2+5n-3)}{4n^3-8n^2+n+3}} \leq \frac{4}{n}$. The second error term is computed to be $\left[\frac{6(2n-1)(4n-5)(1-cos(\theta))}{\pi n^2(2n-3)}\right]^{1/4}$ which goes to 0 as $\theta \to 0$. □

References

[BF] Baker, T. and Forrester, P., Finite-n fluctuation formulas for random matrices, *J. Stat. Phys.* **88** (1997), 1371-1386.

[Bu] Bump, D., *Lie Groups*, Graduate Texts in Mathematics, 225. Springer-Verlag, New York, 2004.

[CF] Chatterjee, S. and Fulman, J., Exponential approximation by exchangeable pairs and spectral graph theory, arXiv:math.PR/0605552 (2006).

[CoSz] Collins, B. and Stolz, M., Borel theorems for random matrices from the classical compact symmetric spaces, *Ann. Probab.* **36** (2008), 876-895.

[DDN] D'Aristotile, A., Diaconis, P., and Newman, C., Brownian motion and the classical groups, in *Probability, statistics and their applications: papers in honor of Rabi Bhattacharya*, 97-116, IMS Lecture Notes Ser. 41, 2003.

[DE] Diaconis, P. and Evans, S., Linear functionals of eigenvalues of random matrices, *Transac. Amer. Math. Soc.* **353** (2001), 2615-2633.

[DF] Diaconis, P. and Freedman, D., A dozen de Finetti-style results in search of a theory, *Ann. Inst. H. Poincaré Probab. Statist.* **23** (1987), 397-423.

[DS] Diaconis, P. and Shahshahani, M., On the eigenvalues of random matrices, *J. Appl. Probab.* **31A** (1994), 49-62.

[Dn] Dueñez, E., Random matrix ensembles associated to compact symmetric spaces, *Comm. Math. Phys.* **244** (2004), 29-61.

[Du] Durrett, R., *Probability: Theory and examples, Second edition*, Duxbury Press, Belmont, CA, 1996.

[DyM] Dym, H. and McKean, H., *Fourier series and integrals*, Academic Press, New York and London, 1972.

[Fo] Forrester, P., *Log-gases and random matrices*, book in preparation. Available at http://www.ms.unimelb.edu.au/∼matpjf/matpjf.html

[Fu] Fulman, J., Stein's method and random character ratios, *Transac. Amer. Math. Soc.* **360** (2008), 3687-3730.

[GoT] Götze, F. and Tikhomirov, A.N., Limit theorems for spectra of random matrices with martingale structure, *Theory Probab. Appl.* **51** (2007), 42-64.

[He1] Helgason, S., *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, San Diego, CA, 1978.

[He2] Helgason, S., *Groups and geometric analysis*, Corrected reprint of the 1984 original, American Mathematical Society, Providence, RI, 2000.

[J] Johansson, K., On random matrices from the compact classical groups, *Ann. of Math.* **145** (1997), 519-545.

[Ka] Katz, N., *Exponential sums and differential equations*, Annals of Mathematical Studies, 124. Princeton University Press, Princeton, NJ 1990.

[KaS] Katz, N. and Sarnak, P., Zeroes of zeta functions and symmetry, *Bull. Amer. Math. Soc.* **36** (1999), 1-26.

[KLR] Keating, J.P., Linden, N., and Rudnick, Z., Random matrix theory, the exceptional Lie groups and L functions, *J. Phys. A* **36** (2003), 2933-2944.

[Kl] Klyachko, A., Random walks on symmetric spaces and inequalities for matrix spectra, *Lin. Alg. Applic.* **319** (2000), 37-59.

[Lu] Luk, H.M., Stein's method for the gamma distribution and related statistical applications, University of Southern California Ph.D. thesis, 1994.

[Mac] Macdonald, I., *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, New York, 1995.

[Mn] Mann, B., Stein's method for χ^2 of a multinomial, unpublished manuscript, 1997.

[Me1] Meckes, E., On the approximate normality of eigenfunctions of the Laplacian, arXiv:0705.1342 (2007), to appear in *Transac. Amer. Math. Soc.*

[Me2] Meckes, E., Linear functions on the classical matrix groups, arXiv:math.PR/0509441 (2005), to appear in *Transac. Amer. Math. Soc.*

[Mt] Mehta, M., Random matrices, Third edition, Elsevier/Academic Press, Amsterdam, 2004.

[PV] Pastur, L. and Vasilchuk, V., On the moments of traces of matrices of classical groups, *Comm. Math. Phys.* **252** (2004), 149-166.

[Pr] Proctor, R., A Schensted algorithm which models tensor representations of the orthogonal group, *Canad. J. Math.* **42** (1990), 28-49.

[Ra] Rains, E., Topics in probability on compact Lie groups, Harvard University Ph.D. thesis, 1997.

[Re] Reinert, G., Three general approaches to Stein's method, in *An introduction to Stein's method*, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., Volume 4, 2005.

[RR] Rinott, Y. and Rotar, V., Normal approximations by Stein's method, *Decis. Econ. Finance* **23** (2000), 15-29.

[Rl] Röellin, A., A note on the exchangeability condition in Stein's method, arXiv:math.PR/0611050 (2006).

[So] Soshnikov, A., The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities, *Ann. Probab.* **28** (2000), 1353-1370.

[St1] Stein, C., *Approximate computation of expectations*, Institute of Mathematical Statistics Lecture Notes, Volume 7, 1986.

[St2] Stein, C., The accuracy of the normal approximation to the distribution of the traces of powers of random orthogonal matrices, Stanford University Statistics Department technical report no. 470, March 1995.

[Su] Sundaram, S., Tableaux in the representation theory of compact Lie groups, in *Invariant theory and tableaux*, IMA Volumes in Mathematics, Volume 19 (1990), 191-225.

[Te] Terras, A., *Harmonic analysis on symmetric spaces and applications*, Volumes I, II, Springer Verlag, N.Y., 1985, 1988.

[V] Vilenkin, N.J., *Special functions and the theory of group representations*, Translations of Mathematics Monographs, Volume 2, American Math Society, 1968.

[Vr] Vretare, L., Formulas for elementary spherical functions and generalized Jacobi polynomials, *SIAM J. Math. Anal.* **15** (1984), 805-833.

[W] Weyl, H., *The classical groups*, Fifteenth printing, Princeton University Press, Princeton, NJ, 1997.