

SEPARATION CUTOFFS FOR RANDOM WALK ON IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. Random walk on the irreducible representations of the symmetric and general linear groups is studied. A separation distance cutoff is proved and the exact separation distance asymptotics are determined. A key tool is a method for writing the multiplicities in the Kronecker tensor powers of a fixed representation as a sum of non-negative terms. Connections are made with the Lagrange-Sylvester interpolation approach to Markov chains.

1. INTRODUCTION

The study of convergence rates of random walk on a finite group is a rich subject; three excellent surveys are [Al], [Sal] and [D1]. In recent papers ([F1],[F3],[F6]), the author posed and studied a dual question, namely what can be said about the convergence rate of random walk on $Irr(G)$, the set of irreducible representations of a finite group G .

To define a random walk on $Irr(G)$, let η be a (not necessarily irreducible) representation of G . In order that the Markov chain we define be reversible (a concept explained in Section 2), it is assumed that the character of η is real valued. From an irreducible representation λ , one transitions to the irreducible representation ρ with probability

$$K(\lambda, \rho) := \frac{d_\rho m_\rho(\lambda \otimes \eta)}{d_\lambda d_\eta}.$$

Here d_λ denotes the dimension of λ and $m_\rho(\lambda \otimes \eta)$ denotes the multiplicity of ρ in the tensor product (also called the Kronecker product) of λ and η . Whereas random walk on G has the uniform measure as a stationary distribution, random walk on $Irr(G)$ has the Plancherel measure as a stationary distribution. The Plancherel measure assigns a representation λ probability $\frac{d_\lambda^2}{|G|}$.

There are many motivations for the study of these random walks: six motivations (with literature references and discussion) appear in the introduction of the recent paper [F6]. There is no need to repeat the discussion here, but let us just mention that similar processes have been studied for

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compact Lie groups and Lie algebras ([ER],[BBO]), and that the decomposition of iterated tensor products of finite groups has been studied by combinatorialists ([F3],[GC],[GK]). Moreover, these random walks arise in quantum computing ([MR],[F6]) and have been used to derive the first error bounds in limit theorems for the distribution of random character ratios ([F1], [F4]). As is clear from [F6] and this paper, these random walks are also a tractable testing ground for results in Markov chain theory.

To illustrate a result in this paper, we discuss the case of random walk on $\text{Irr}(S_n)$. To do this, recall that two commonly used distances between probability distributions P, Q on a finite set X are total variation distance

$$\|P - Q\| := \frac{1}{2} \sum_{x \in X} |P(x) - Q(x)|$$

and separation distance

$$s(P, Q) := \max_{x \in X} \left[1 - \frac{P(x)}{Q(x)} \right].$$

(In applications, $Q(x)$ is the stationary distribution of a Markov chain and satisfies $Q(x) \neq 0$ for all x). The following recent result gave a sharp total variation distance convergence rate estimate.

Theorem 1.1. ([F6]) *Let G be the symmetric group S_n and let π be the Plancherel measure of G . Let η be the n -dimensional defining representation of S_n . Let K^r denote the distribution of random walk on $\text{Irr}(G)$ after r steps, started from the trivial representation.*

(1) *If $r = \frac{1}{2}n \log(n) + cn$ with $c \geq 1$ then*

$$\|K^r - \pi\| \leq \frac{e^{-2c}}{2}.$$

(2) *If $r = \frac{1}{2}n \log(n) - cn$ with $0 \leq c \leq \frac{1}{6} \log(n)$, then there is a universal constant a (independent of c, n) so that*

$$\|K^r - \pi\| \geq 1 - ae^{-4c}.$$

This paper gives precise separation distance asymptotics. Letting $s(r)$ denote the separation distance after r steps, it will be shown that for the walk in Theorem 1.1 and c fixed in \mathbb{R} ,

$$s(n \log(n) + cn) = 1 - e^{-e^{-c}}(1 + e^{-c}) + O\left(\frac{\log(n)}{n}\right).$$

This expression goes to 0 as $c \rightarrow \infty$ and to 1 as $c \rightarrow -\infty$ and a cutoff (defined precisely in Section 2) occurs since $cn = o(n \log(n))$. Note that whereas the total variation cutoff occurs at time $\frac{1}{2}n \log(n)$, the separation distance cutoff occurs at time $n \log(n)$; for a heuristic discussion of why separation distance often takes twice as long as total variation distance to become small, see Section 10.2 of [W1] or Section 7 of [AD2].

The proof of the above separation distance asymptotics (and also the corresponding result for $GL(n, q)$) consists of two steps:

- (1) One must determine at which element λ of $Irr(G)$ the separation distance is attained. This is equivalent to finding the λ which minimizes $\frac{m_\lambda(\eta^r)}{d_\lambda}$. This is tricky since the usual formula for multiplicities in tensor products involves character values and so both positive and negative terms. Our solution is to give a subtle rewriting of this expression as a sum of non-negative terms.
- (2) Once one knows at which representation the separation distance is attained, one needs a formula for the separation distance. For the cases in this paper we indicate how to do this using combinatorial arguments and the diagonalization (i.e. eigenvalues and eigenvectors) of the random walk on $Irr(G)$.

A rather remarkable fact is that for the cases studied in this paper, one can use Lagrange-Sylvester interpolation to carry out Step 2 knowing only the eigenvalues (and not the eigenvectors) of random walk on $Irr(G)$. A similar trick had been usefully applied in the one-dimensional setting of birth-death chains ([Br], [DSa]), but the state spaces $Irr(S_n)$ and $Irr(GL(n, q))$ are high-dimensional so it is interesting that the trick can be extended to this context.

The organization of this paper is as follows. Section 2 gives background from Markov chain theory and recalls the diagonalization of the Markov chain on $Irr(G)$. Section 3 derives separation distance asymptotics for random walk on $Irr(S_n)$ when η is the defining representation (whose character on a permutation is the number of fixed points). Section 4 obtains separation distance asymptotics for random walk on $Irr(GL(n, q))$ when η is the representation whose character is $q^{d(g)}$, where $d(g)$ is the dimension of the fixed space of g . Section 5 discusses Lagrange-Sylvester interpolation, giving eigenvector-free proofs of some results of Sections 3 and 4.

To close the introduction, we mention the follow-up paper [F7]. This revisits the current paper using the combinatorics of commutation relations, and gives some further examples (including one of biological interest) to which the methodology applies.

2. PRELIMINARIES

This section collects some background on finite Markov chains, using random walk on $Irr(G)$ as a running example. Let X be a finite set and K a matrix indexed by $X \times X$ whose rows sum to 1. Let π be a distribution such that K is reversible with respect to π ; this means that $\pi(x)K(x, y) = \pi(y)K(y, x)$ for all x, y and implies that π is a stationary distribution for the Markov chain corresponding to K .

As an example, the Markov chain on $Irr(G)$ defined in the introduction is reversible with respect to the Plancherel measure π . To see this let χ

denote the character of a representation, and recall the formula

$$m_\rho(\lambda \otimes \eta) = \frac{1}{|G|} \sum_{g \in G} \chi^\lambda(g) \chi^\eta(g) \overline{\chi^\rho(g)}.$$

The equation $\pi(\lambda)K(\lambda, \rho) = \pi(\rho)K(\rho, \lambda)$ follows because η was assumed to be real valued.

Define $\langle f, g \rangle = \sum_{x \in X} f(x)g(x)\pi(x)$ for real valued functions f, g on X , and let $L^2(\pi)$ denote the space of such functions. Then when K is considered as an operator on $L^2(\pi)$ by

$$Kf(x) := \sum_y K(x, y)f(y),$$

it is self adjoint. Hence K has an orthonormal basis of eigenvectors $f_i(x)$ with $Kf_i(x) = \beta_i f_i(x)$, where both f_i and β_i are real. It is easily shown that the eigenvalues satisfy $-1 \leq \beta_{|X|-1} \leq \dots \leq \beta_1 \leq \beta_0 = 1$. One calls K ergodic if $|\beta_{|X|-1}|, |\beta_1| < 1$.

As an example, Lemma 2.1 determines an orthonormal basis of eigenvectors for the Markov chains on $\text{Irr}(G)$.

Lemma 2.1. ([F2], Proposition 2.3) *Let K be the Markov chain on $\text{Irr}(G)$ defined using a representation η whose character is real valued. The eigenvalues of K are indexed by conjugacy classes C of G :*

- (1) *The eigenvalue parameterized by C is $\frac{\chi^\eta(C)}{d_\eta}$.*
- (2) *An orthonormal basis of eigenfunctions f_C in $L^2(\pi)$ is defined by $f_C(\rho) = \frac{|C|^{1/2} \chi^\rho(C)}{d_\rho}$.*

For instance when $G = S_n$ and η is the n -dimensional defining representation, the eigenvalues are $\frac{i}{n}$ where $0 \leq i \leq n-2$ or $i = n$, with multiplicity equal to the number of conjugacy classes of permutations with i fixed points. Similarly, suppose that $G = GL(n, q)$ and that η is the representation whose character is the number of fixed vectors of g in its natural action on the n -dimensional vector space V . Then the eigenvalues are q^{-i} for $i = 0, \dots, n$, with multiplicity equal to the number of conjugacy classes of elements of $GL(n, q)$ with an $n-i$ dimensional fixed space.

A common way to quantify convergence rates of Markov chains is using total variation distance. This was defined in the introduction and it is not hard to see that

$$\|P - Q\| = \max_{A \subseteq X} |P(A) - Q(A)|.$$

Let K_x^r be the probability measure given by taking r steps from the starting state x . Researchers in Markov chains are interested in the behavior of $\|K_x^r - \pi\|$ as a function of r .

Lemma 2.2 relates total variation distance to the spectrum of K . Part 1 is the usual method for computing the power of a diagonalizable matrix. Part 2 is classical (see [DH] for a proof) and upper bounds $\|K_x^r - \pi\|$ in

terms of eigenvalues and eigenvectors and is effective in many examples; it was crucial in the proof of Theorem 1.1 stated in the introduction.

Lemma 2.2. (1) $K^r(x, y) = \sum_{i=0}^{|X|-1} \beta_i^r f_i(x) f_i(y) \pi(y)$ for any $x, y \in X$.
 (2)

$$4\|K_x^r - \pi\|^2 \leq \sum_y \frac{|K^r(x, y) - \pi(y)|^2}{\pi(y)} = \sum_{i=1}^{|X|-1} \beta_i^{2r} |f_i(x)|^2.$$

Note that the final sum does not include $i = 0$.

Another frequently used method to quantify convergence rates of Markov chains is to use separation distance, introduced in [AD1],[AD2] and defined in the introduction of this paper. Note that

$$\|P - Q\| = \sum_{x:Q(x) \geq P(x)} [Q(x) - P(x)] \leq s(P, Q).$$

Specializing to random walk on $Irr(G)$ started at the trivial representation $\hat{1}$, one has that

$$s(K_{\hat{1}}^r, \pi) = \max_{\lambda} \left[1 - \frac{|G|K^r(\hat{1}, \lambda)}{d_{\lambda}^2} \right].$$

Lemma 3.2 of [F6] gives that $K^r(\hat{1}, \lambda) = \frac{d_{\lambda}}{(d_{\eta})^r} m_{\lambda}(\eta^r)$. Thus the separation distance is attained at the λ which minimizes $\frac{m_{\lambda}(\eta^r)}{d_{\lambda}}$.

Finally, let us give a precise definition of the cutoff phenomenon. A nice survey of the subject is [D2]; we use the definition from [Sal]. Consider a family of finite sets X_n , each equipped with a stationary distribution π_n . Let $p_n^{r_n}$ denote the distribution of the random walk on X_n after r_n steps. One says that there is a total variation cutoff for the family (X_n, π_n) if there exists a sequence (t_n) of positive reals such that

- (1) $\lim_{n \rightarrow \infty} t_n = \infty$;
- (2) For any $\epsilon \in (0, 1)$ and $r_n = \lfloor (1 + \epsilon)t_n \rfloor$, $\lim_{n \rightarrow \infty} \|p_n^{r_n} - \pi_n\| = 0$;
- (3) For any $\epsilon \in (0, 1)$ and $r_n = \lfloor (1 - \epsilon)t_n \rfloor$, $\lim_{n \rightarrow \infty} \|p_n^{r_n} - \pi_n\| = 1$.

For the definition of a separation cutoff, one replaces $\|p_n^{r_n} - \pi_n\|$ by $s(p_n^{r_n}, \pi_n)$.

3. THE SYMMETRIC GROUP

This section studies the random walk K on $Irr(S_n)$ defined from the representation η whose character is the number of fixed points. Although not needed for the results in this section, it should be mentioned that when $Irr(S_n)$ is viewed as the partitions of n , the random walk K has a description in terms of removing and then reattaching a corner box at each step (see [F6] for a proof).

The primary purpose of this section is to determine the asymptotic behavior of the separation distance

$$s(r) = \max_{\lambda} \left[1 - \frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} \right].$$

The first step in studying $s(r)$ is determine for which λ the maximum is attained. Part 1 of Lemma 2.2 and Lemma 2.1 imply that

$$\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} = \sum_{i=0}^n \left(\frac{i}{n} \right)^r \sum_{g \in S_n: fp(g)=i} \frac{\chi^\lambda(g)}{d_\lambda},$$

where $fp(g)$ is the number of fixed points of g . However since characters can take both positive and negative values, it is not clear which λ minimizes this expression.

Theorem 3.3 will circumvent this difficulty by giving an expression for $\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)}$ as a sum of non-negative terms. This result was first derived in our earlier paper [F5] using the Robinson-Schensted-Knuth correspondence. The proof presented here is different and uses instead inclusion-exclusion and the branching rules for the irreducible representations of S_n . As will be seen in Section 4, it generalizes perfectly to the group $GL(n, q)$.

As a first step, the following lemma is useful. In its statement, and throughout this section, we assume familiarity with the concept of standard tableaux as in Chapters 2 and 3 of [Sag].

Lemma 3.1. *Let $d_{\lambda/\mu}$ denote the number of standard tableaux of shape λ/μ . Then*

$$\sum_{g \in S_n: fp(g)=i} \chi^\lambda(g) = \frac{n!}{i!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j!} d_{\lambda/(n-i-j)}.$$

Proof. Let $Fix(g)$ denote the set of fixed points of a permutation g . Then

$$\begin{aligned}
& \sum_{g \in S_n: fp(g)=i} \chi^\lambda(g) \\
&= \binom{n}{i} \sum_{g: Fix(g)=\{n-i+1, \dots, n\}} \chi^\lambda(g) \\
&= \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \sum_{\substack{A \subseteq \{1, \dots, n-i\} \\ |A|=j}} \sum_{g: Fix(g) \supseteq A \cup \{n-i+1, \dots, n\}} \chi^\lambda(g) \\
&= \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{g: Fix(g) \supseteq \{n-i-j+1, \dots, n\}} \chi^\lambda(g) \\
&= \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (n-i-j)! \langle Res_{S_{n-i-j}}^{S_n} [\chi^\lambda], \hat{1} \rangle \\
&= \frac{n!}{i!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j!} d_{\lambda/(n-i-j)}.
\end{aligned}$$

The first and third equalities are since character values are invariant under conjugacy. The second equality is the inclusion-exclusion principle (Chapter 10 of [VW]). In the fourth equality, $Res_{S_{n-i-j}}^{S_n} [\chi^\lambda]$ denotes the restriction of χ^λ from S_n to S_{n-i-j} , and the inner product is the usual inner product on complex valued functions on G (i.e. $\langle \psi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\phi(g)}$). The final equality follows from the branching rules for irreducible representations of symmetric groups [Sag]. Note also that when $i = 0$, the set $\{n-i+1, \dots, n\}$ should be interpreted as the empty set. \square

In what follows $P(a, r, n)$ will denote the probability that when r balls are dropped at random into n boxes, there are exactly a occupied boxes. Lemma 3.2 gives an explicit expression for $P(a, r, n)$. This expression is an exercise on page 103 of [Fe], but since the proof is simple and motivates an analogous result in Section 4, we include it.

Lemma 3.2. ([Fe])

$$P(a, r, n) = \binom{n}{a} \sum_{b=n-a}^n (-1)^{b-(n-a)} \binom{a}{n-b} \left(1 - \frac{b}{n}\right)^r.$$

Proof. Clearly $P(a, r, n)$ is $\binom{n}{a}$ multiplied by the probability that the occupied boxes are the first a boxes. By the principle of inclusion and exclusion, this is

$$\binom{n}{a} \sum_{s=0}^a (-1)^{a-s} \binom{a}{s} P_{\leq}(s)$$

where $P_{\leq}(s)$ is the probability that the set of occupied boxes is contained in $\{1, \dots, s\}$. Noting that $P_{\leq}(s) = \left(\frac{s}{n}\right)^r$, the result follows from the change of variables $b = n - s$. \square

Theorem 3.3 gives the needed expression for $\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)}$ as a sum of non-negative quantities.

Theorem 3.3. *Let $d_{\lambda/\mu}$ denote the number of standard tableaux of shape λ/μ . Then*

$$\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} = \sum_{a=0}^n P(a, r, n) (n-a)! \frac{d_{\lambda/(n-a)}}{d_{\lambda}}.$$

Proof. As noted earlier, part 1 of Lemma 2.2 and Lemma 2.1 imply that

$$\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} = \sum_{i=0}^n \left(\frac{i}{n}\right)^r \sum_{g \in S_n: fp(g)=i} \frac{\chi^{\lambda}(g)}{d_{\lambda}}.$$

By Lemma 3.1 this is

$$\sum_{i=0}^n \left(\frac{i}{n}\right)^r \frac{n!}{i!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j!} \frac{d_{\lambda/(n-i-j)}}{d_{\lambda}}.$$

Letting $a = i + j$, this becomes

$$\begin{aligned} & \sum_{i=0}^n \sum_{a=i}^n (-1)^{a-i} \frac{n!}{i!} \frac{1}{(a-i)!} \left(\frac{i}{n}\right)^r \frac{d_{\lambda/(n-a)}}{d_{\lambda}} \\ &= \sum_{a=0}^n \sum_{i=0}^a (-1)^{a-i} \frac{n!}{i!} \frac{1}{(a-i)!} \left(\frac{i}{n}\right)^r \frac{d_{\lambda/(n-a)}}{d_{\lambda}}. \end{aligned}$$

Letting $b = n - i$, this becomes

$$\sum_{a=0}^n \binom{n}{a} \sum_{b=n-a}^n (-1)^{b-(n-a)} \binom{a}{n-b} \left(1 - \frac{b}{n}\right)^r (n-a)! \frac{d_{\lambda/(n-a)}}{d_{\lambda}}.$$

The result follows from Lemma 3.2. \square

Corollary 3.4. *The quantity $\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)}$ is minimized for $\lambda = (1^n)$, corresponding to the sign representation.*

Proof. By Theorem 3.3 one wants to find λ minimizing

$$\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} = \sum_{a=0}^n P(a, r, n) (n-a)! \frac{d_{\lambda/(n-a)}}{d_{\lambda}}.$$

Note that the $a = n - 1$ and $a = n$ terms in this expression are independent of λ . Moreover, all other terms are non-negative, and vanish when λ is the sign representation (which corresponds to the partition all of whose parts have size 1). The result follows. \square

Next we use Theorem 3.3 and Corollary 3.4 to derive both a formula and a precise asymptotic expression for the separation distance of the Markov chain K .

Theorem 3.5. *Let $s(r)$ be the separation distance between K^r started at the trivial representation and the Plancherel measure π .*

(1)

$$s(r) = \sum_{i=0}^{n-2} (-1)^{n-i} \binom{n}{i} (n-i-1) \left(\frac{i}{n}\right)^r.$$

(2) For c fixed in \mathbb{R} and $n \rightarrow \infty$,

$$s(n \log(n) + cn) = 1 - e^{-e^{-c}} (1 + e^{-c}) + O\left(\frac{\log(n)}{n}\right).$$

Proof. (First proof) Theorem 3.3 and Corollary 3.4 imply that

$$\begin{aligned} s(r) &= 1 - \sum_{a=0}^n P(a, r, n) (n-a)! \frac{d_{(1^n)/(n-a)}}{d_{(1^n)}} \\ &= 1 - P(n, r, n) - P(n-1, r, n). \end{aligned}$$

By Lemma 3.2 this is equal to

$$\begin{aligned} &1 - \sum_{b=0}^n (-1)^b \binom{n}{b} \left(1 - \frac{b}{n}\right)^r - n \sum_{b=1}^n (-1)^{b-1} \binom{n-1}{n-b} \left(1 - \frac{b}{n}\right)^r \\ &= \sum_{b=1}^n (b-1) \binom{n}{b} (-1)^b \left(1 - \frac{b}{n}\right)^r. \end{aligned}$$

Letting $i = n - b$ proves the first assertion.

For the second assertion, we use asymptotics of the coupon collector's problem: it follows from Section 6 of [CDM] that when $n \log(n) + cn$ balls are dropped into n boxes, the number of unoccupied boxes converges to a Poisson distribution with mean e^{-c} , and that the error term in total variation distance is $O\left(\frac{\log(n)}{n}\right)$. The chance that a Poisson random variable with mean e^{-c} takes value not equal to 0 or 1 is $1 - e^{-e^{-c}}(1 + e^{-c})$, which completes the proof. \square

There is a second proof of Theorem 3.5, which uses a connection with the top to random shuffle of the symmetric group. We prefer the first proof as the ideas are more elementary (one doesn't need the RSK correspondence) and generalize to $GL(n, q)$ (see Section 4).

Proof. (Second proof) By Corollary 3.4, $s(r) = 1 - n! K^r(\hat{1}, (1^n))$. Theorem 3.1 of [F3] gives that for any shape λ , one has that $K^r(\hat{1}, \lambda)$ is equal to the chance of obtaining a permutation with Robinson-Schensted-Knuth (RSK) shape λ after r top to random shuffles started from the identity. The only permutation with RSK shape (1^n) is the "longest" permutation π , defined by $\pi(i) = n - i + 1$ for all i . It follows from Corollary 2.1 of [DFP] that

the separation distance for the top to random shuffle is attained at this π . Thus the chain K and the top to random shuffle have the same separation distance $s(r)$, so the result follows from page 142 of [DFP]. \square

From the second proof the reader might think that the theory of the chain K can be entirely understood in terms of the top to random shuffle. This is untrue. For example if one measures convergence to the stationary distribution using total variation distance, the top to random shuffle takes $n \log(n) + cn$ steps to be close to random [AD1], but the chain K requires only $\frac{1}{2}n \log(n) + cn$ steps [F6].

As a final result, we use Corollary 3.4 and the cycle index of the symmetric group to give a third proof of part 1 of Theorem 3.5.

Proof. By Corollary 3.4, $s(r) = 1 - \frac{K^r(\hat{1}, (1^n))}{\pi(1^n)}$. Part 1 of Lemma 2.2 and Lemma 2.1 imply that

$$\frac{K^r(\hat{1}, \lambda)}{\pi(\lambda)} = \sum_{i=0}^n \left(\frac{i}{n}\right)^r \sum_{g \in S_n: fp(g)=i} \frac{\chi^\lambda(g)}{d_\lambda}.$$

Specializing to $\lambda = (1^n)$ implies that

$$s(r) = - \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^r \sum_{g \in S_n: fp(g)=i} \text{sign}(g).$$

Here $\text{sign}(g) = (-1)^{n-c(g)}$, where $c(g)$ is the number of cycles of g .

A classic result in combinatorics (see [W]) is the ‘‘cycle index’’ of the symmetric group, which states that

$$1 + \sum_{n \geq 1} \frac{u^n}{n!} \sum_{g \in S_n} \prod_{j \geq 1} x_j^{n_j(g)} = \exp \left(\sum_{m \geq 1} \frac{x_m u^m}{m} \right).$$

Here $n_j(g)$ is the number of cycles of length j of g . Making the substitutions $x_1 = -x$, $x_i = -1$ for $i \geq 2$ and replacing u by $-u$, the cycle index implies that

$$\begin{aligned} 1 + \sum_{n \geq 1} \frac{u^n}{n!} \sum_{g \in S_n} \text{sign}(g) \cdot x^{fp(g)} &= \exp \left(xu - \sum_{m \geq 2} \frac{(-u)^m}{m} \right) \\ &= \exp(xu - u) \exp(\log(1 + u)) \\ &= \frac{e^{xu}(1 + u)}{e^u}. \end{aligned}$$

Taking the coefficient of $\frac{u^n x^i}{n!}$ on both sides shows that if $0 \leq i \leq n-1$, then

$$\begin{aligned} \sum_{g \in S_n: fp(g)=i} \text{sign}(g) &= \frac{n!}{i!} \left[\frac{(-1)^{n-i}}{(n-i)!} + \frac{(-1)^{n-i-1}}{(n-i-1)!} \right] \\ &= (-1)^{n-i+1} \binom{n}{i} (n-i-1). \end{aligned}$$

The result now follows from the previous paragraph. \square

4. THE GENERAL LINEAR GROUP

This section studies random walk on $\text{Irr}(GL(n, q))$ in the case that η is the representation of $GL(n, q)$ whose character is $q^{d(g)}$, where $d(g)$ is the dimension of the fixed space of g . As in Section 3, we aim to determine the asymptotic behavior of the separation distance

$$s(r) = \max_{\Lambda} \left[1 - \frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)} \right].$$

The first step is to find the irreducible representation Λ for which the maximum is attained. Part 1 of Lemma 2.2 and Lemma 2.1 imply that

$$\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)} = \sum_{i=0}^n q^{-r(n-i)} \sum_{g \in GL(n, q): d(g)=i} \frac{\chi^\Lambda(g)}{d_\Lambda}.$$

Since characters can take both positive and negative values, it is not at all clear which Λ minimizes this expression. As in the symmetric group case, the key is to find a way to write $\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)}$ as a sum of non-negative terms.

To begin we recall some facts about the representation theory of $GL(n, q)$. A full treatment of the subject with proofs appears in [Ma], [Z]. As usual a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ is identified with its geometric image $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ and $|\lambda| = \lambda_1 + \dots + \lambda_m$ is the total number of boxes. Let \mathbb{Y} denote the set of all partitions, including the empty partition of size 0.

Given an integer $1 \leq k < n$ and two characters χ_1, χ_2 of the groups $GL(k, q)$ and $GL(n-k, q)$, their parabolic induction $\chi_1 \circ \chi_2$ is the character of $GL(n, q)$ induced from the parabolic subgroup of elements of the form

$$P = \left\{ \begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} : g_1 \in GL(k, q), g_2 \in GL(n-k, q) \right\}$$

by the function $\chi_1(g_1)\chi_2(g_2)$.

A character is called cuspidal if it is not a component of any parabolic induction. Let C_m denote the set of cuspidal characters of $GL(m, q)$ and let $C = \bigcup_{m \geq 1} C_m$; it is known that $|C_m| = \frac{1}{m} \sum_{d|m} \mu(d)(q^{m/d} - 1)$ where μ is the Moebius function. The unit character of $GL(1, q)$ plays an important role and will be denoted e ; it is one of the $q-1$ elements of C_1 . Given a family $\Lambda : C \mapsto \mathbb{Y}$ with finitely many non-empty partitions $\Lambda(c)$, its

degree $|\Lambda|$ is defined as $\sum_{m \geq 1} \sum_{c \in C_m} m \cdot |\Lambda(c)|$. A fundamental result is that the irreducible representations of $GL(n, q)$ are in bijection with the families of partitions of degree n , so we also let Λ denote the corresponding representation.

Let $\vec{e}_1, \dots, \vec{e}_n$ be the standard basis of the vector space V on which $GL(n, q)$ acts (so the k th component of \vec{e}_j is $\delta_{j,k}$). Define $H(k, q)$ as the subgroup of $GL(n, q)$ consisting of g which fix all of $\vec{e}_1, \dots, \vec{e}_k$. Equivalently, the elements of $H(k, q)$ are block matrices of the form

$$\begin{pmatrix} I_k & X \\ 0_{n-k} & Y \end{pmatrix}$$

where I_k is a k by k identity matrix, X is any k by $n - k$ matrix with entries in \mathbb{F}_q , and Y is any element of $GL(n - k, q)$. Thus $|H(k, q)| = q^{k(n-k)} |GL(n - k, q)|$. For Λ an element of $Irr(GL(n, q))$, it will be helpful to let

$$c_k(\Lambda) = \sum_{g \in H(k, q)} \chi^\Lambda(g).$$

Then $c_k(\Lambda)$ is non-negative, since it is the product of $|H(k, q)|$ and the multiplicity of the trivial representation of $H(k, q)$ in the restricted representation $Res_{H(k, q)}^{GL(n, q)}[\Lambda]$.

It will also be convenient to let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the q -binomial coefficient $\frac{(q^n - 1) \cdots (q - 1)}{(q^k - 1) \cdots (q - 1)(q^{n-k} - 1) \cdots (q - 1)}$, which is equal to the number of k dimensional subspaces of an n dimensional vector space over a finite field \mathbb{F}_q .

Lemma 4.1. *Let Λ be an irreducible representation of $GL(n, q)$. Then*

$$\sum_{\substack{g \in GL(n, q) \\ d(g) = i}} \chi^\Lambda(g) = \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} c_{i+j}(\Lambda).$$

Proof. Let $Fix(g)$ denote the fixed space of g . Also if A is a set of vectors, $\langle A \rangle$ will denote their span. Then

$$\begin{aligned} \sum_{\substack{g \in GL(n, q) \\ d(g) = i}} \chi^\Lambda(g) &= \begin{bmatrix} n \\ i \end{bmatrix} \sum_{\substack{g \in GL(n, q) \\ Fix(g) = \langle \vec{e}_1, \dots, \vec{e}_i \rangle}} \chi^\Lambda(g) \\ &= \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \sum_{\substack{W \supseteq \langle \vec{e}_1, \dots, \vec{e}_i \rangle \\ \dim(W) = i+j}} (-1)^j q^{\binom{j}{2}} \sum_{\substack{g \in GL(n, q) \\ Fix(g) \supseteq W}} \chi^\Lambda(g) \\ &= \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} \sum_{\substack{g \in GL(n, q) \\ Fix(g) \supseteq \langle \vec{e}_1, \dots, \vec{e}_{i+j} \rangle}} \chi^\Lambda(g) \\ &= \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} c_{i+j}(\Lambda). \end{aligned}$$

The first and third equalities used the fact that if W_1, W_2 are subspaces of V of equal dimension, then $\{g : \text{Fix}(g) = W_1\}$ and $\{g : \text{Fix}(g) = W_2\}$ are conjugate in $GL(n, q)$. The second equality used Moebius inversion on the lattice of subspaces of a vector space (Chapter 25 of the text [VW]). The third equality also used the fact that the number of $i + j$ dimensional subspaces of V containing $\langle \vec{e}_1, \dots, \vec{e}_i \rangle$ is $\begin{bmatrix} n-i \\ j \end{bmatrix}$. \square

In what follows we let $P_q(a, r, n)$ denote the probability that the span of $\vec{v}_1, \dots, \vec{v}_r$ is a dimensional, where the r vectors are chosen uniformly at random from an n -dimensional vector space over \mathbb{F}_q . Lemma 4.2 gives a formula for $P_q(a, r, n)$.

Lemma 4.2.

$$P_q(a, r, n) = \begin{bmatrix} n \\ a \end{bmatrix} \sum_{b=n-a}^n (-1)^{b-(n-a)} q^{\binom{b-(n-a)}{2}} \begin{bmatrix} a \\ n-b \end{bmatrix} q^{-rb}.$$

Proof. Clearly $P_q(a, r, n)$ is $\begin{bmatrix} n \\ a \end{bmatrix}$ multiplied by the chance that the span of $\vec{v}_1, \dots, \vec{v}_r$ is exactly the a dimensional subspace consisting of vectors whose last $n - a$ coordinates are 0. One applies Moebius inversion on the lattice of subspaces of an n -dimensional vector space over \mathbb{F}_q (Chapter 25 of [VW]) to conclude that

$$P_q(a, r, n) = \begin{bmatrix} n \\ a \end{bmatrix} \sum_{s=0}^a \sum_{\substack{W \subseteq \langle \vec{e}_1, \dots, \vec{e}_a \rangle \\ \dim(W)=s}} (-1)^{a-s} q^{\binom{a-s}{2}} P_{\leq}(W).$$

Here $\vec{e}_1, \dots, \vec{e}_n$ is the standard basis of V and $P_{\leq}(W)$ is the probability that the span of $\vec{v}_1, \dots, \vec{v}_r$ is contained in W . Clearly $P_{\leq}(W) = q^{-r(n-\dim(W))}$. Thus

$$P_q(a, r, n) = \begin{bmatrix} n \\ a \end{bmatrix} \sum_{s=0}^a \begin{bmatrix} a \\ s \end{bmatrix} (-1)^{a-s} q^{\binom{a-s}{2}} q^{-r(n-s)},$$

and the result follows by the change of variables $b = n - s$. \square

Theorem 4.3 is a key result of this section; it expresses $\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)}$ as a sum of non-negative terms.

Theorem 4.3.

$$\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)} = \sum_{a=0}^n P_q(a, r, n) \frac{c_a(\Lambda)}{d_\Lambda}.$$

Proof. Part 1 of Lemma 2.2 and Lemma 2.1 imply that

$$\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)} = \sum_{i=0}^n q^{-r(n-i)} \sum_{g \in GL(n, q): d(g)=i} \frac{\chi^\Lambda(g)}{d_\Lambda}.$$

By Lemma 4.1, this is

$$\sum_{i=0}^n q^{-r(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} \frac{c_{i+j}(\Lambda)}{d_\Lambda}.$$

Letting $a = i + j$, this becomes

$$\begin{aligned} & \sum_{i=0}^n q^{-r(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} \sum_{a=i}^n \begin{bmatrix} n-i \\ a-i \end{bmatrix} (-1)^{a-i} q^{\binom{a-i}{2}} \frac{c_a(\Lambda)}{d_\Lambda} \\ &= \sum_{a=0}^n \frac{c_a(\Lambda)}{d_\Lambda} \sum_{i=0}^a q^{-r(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ a-i \end{bmatrix} (-1)^{a-i} q^{\binom{a-i}{2}}. \end{aligned}$$

Setting $b = n - i$ this becomes

$$\begin{aligned} & \sum_{a=0}^n \frac{c_a(\Lambda)}{d_\Lambda} \sum_{b=n-a}^n q^{-rb} \begin{bmatrix} n \\ b \end{bmatrix} \begin{bmatrix} b \\ a-(n-b) \end{bmatrix} (-1)^{a-(n-b)} q^{\binom{a-(n-b)}{2}} \\ &= \sum_{a=0}^n \frac{c_a(\Lambda)}{d_\Lambda} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{b=n-a}^n (-1)^{b-(n-a)} q^{\binom{b-(n-a)}{2}} \begin{bmatrix} a \\ n-b \end{bmatrix} q^{-rb}. \end{aligned}$$

The result now follows from Lemma 4.2. \square

Corollary 4.4. *Suppose that $GL(n, q) \neq GL(1, 2)$.*

- (1) *The quantity $\frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)}$ is minimized for any Λ which satisfies $\Lambda(e) = \emptyset$, and such Λ exist.*
- (2) *The separation distance $s(r)$ of K started at the trivial representation is*

$$1 - P_q(n, r, n) = \sum_{b=1}^n (-1)^{b+1} q^{\binom{b}{2}} \begin{bmatrix} n \\ b \end{bmatrix} q^{-rb}.$$

Proof. The formula for $|C_m|$ stated earlier in this section implies the existence of Λ with $\Lambda(e) = \emptyset$. Proposition 5.4 of [F6] states that for any Λ , if $K^r(\hat{1}, \Lambda) > 0$, then the largest part of $\Lambda(e)$ is at least $n - r$. It follows that if $\Lambda(e) = \emptyset$ then $K^{n-1}(\hat{1}, \Lambda) = 0$. By Theorem 4.3 and the fact that $P_q(a, n-1, n) > 0$ for $0 \leq a \leq n-1$, it follows that if $\Lambda(e) = \emptyset$ then $c_a(\Lambda) = 0$ for $0 \leq a \leq n-1$. Thus if $\Lambda(e) = \emptyset$, only the $a = n$ term in Theorem 4.3 can be non-vanishing, but this term is independent of Λ , since $\frac{c_n(\Lambda)}{d_\Lambda} = 1$ for all Λ . This implies the first part of the lemma since all terms in Theorem 4.3 are nonnegative. It also implies that $s(r) = 1 - P_q(n, r, n)$, and the equality in the second assertion follows from Lemma 4.2. \square

Theorem 4.5 bounds the separation distance $s(r)$ and determines its exact asymptotic behavior.

Theorem 4.5. *Suppose that $GL(n, q) \neq GL(1, 2)$.*

(1) If $r < n$, then $s(r) = 1$. If $c \geq 0$, then

$$\frac{1}{q^{c+1}} - \frac{4}{q^{2c+3}} \leq s(n+c) \leq \frac{2}{q^{c+1}}.$$

(2) Let $c \geq 0$ be fixed. Then

$$\lim_{n \rightarrow \infty} s(n+c) = 1 - \prod_{m=1}^{\infty} (1 - q^{-(c+m)}).$$

Proof. The first two sentences of the proof of Corollary 4.4 imply that if $r < n$, then $s(r) = 1$. To upper bound $s(n+c)$, one checks that since $q \geq 2$, the sum in the second part of Corollary 4.4 is alternating with terms of decreasing magnitude. Thus the sum is upper bounded by its first term, $\begin{bmatrix} n \\ 1 \end{bmatrix} q^{-(n+c)}$, which is easily seen to be less than $2q^{-(c+1)}$, since $q \geq 2$. For the lower bound, note that the first term in the second part of Corollary 4.4 is at least $q^{-(c+1)}$ and that the second term $-\frac{\begin{bmatrix} n \\ 2 \end{bmatrix}}{q^{2n+2c-1}}$ is at least $-4q^{-(2c+3)}$ since $q \geq 2$. This proves the first part of the theorem.

To prove the second part, rewrite the expression for $s(n+c)$ in part 2 of Corollary 4.4 as

$$-\sum_{b=1}^n \frac{(-1)^b (1 - 1/q^n)(1/q - 1/q^n) \cdots (1/q^{b-1} - 1/q^n)}{q^{(c+1)b} (1 - 1/q) \cdots (1 - 1/q^b)}.$$

It is straightforward to see that as $n \rightarrow \infty$ this converges to

$$-\sum_{b=1}^{\infty} \frac{(-1)^b}{q^{(c+1)b} q^{\binom{b}{2}} (1 - 1/q) \cdots (1 - 1/q^b)}.$$

An identity of Euler (Corollary 2.2 in [An]) states that

$$1 + \sum_{b=1}^{\infty} \frac{t^b}{q^{\binom{b}{2}} (1 - 1/q) \cdots (1 - 1/q^b)} = \prod_{m=0}^{\infty} (1 + tq^{-m})$$

for $|t| < 1, |q| > 1$. The result follows by applying this identity with $t = -1/q^{(c+1)}$. \square

5. LAGRANGE-SYLVESTER INTERPOLATION

For certain one dimensional random walks, namely stochastically monotone birth-death chains, Lagrange-Sylvester interpolation allows one to study separation distance knowing only the eigenvalues (and not the eigenvectors) of the Markov chain; see [Br], [DSa] and the remarks following Proposition 5.1. The purpose of this section is to give examples of higher dimensional state spaces (namely $Irr(S_n)$ and $Irr(GL(n, q))$) where the methodology is useful. Further examples appear in the follow-up paper [F7].

To begin we review the Lagrange-Sylvester interpolation approach to diagonalizable matrices, and hence to reversible Markov chains. A textbook discussion in the matrix setting appears in [Ga], and the paper [Br] uses the language of Markov chains.

As usual, K is a Markov chain on a finite set X and is reversible with respect to a distribution π . If $\pi(x) > 0$ for all x , then letting A be a diagonal matrix whose (x, x) entry is $\pi(x)$, it follows that $A^{1/2}KA^{-1/2}$ is symmetric. Hence K is conjugate to a diagonal matrix D , whose entries d_1, \dots, d_n are the eigenvalues of K . Thus if f, g are polynomials with $f(d_i) = g(d_i)$ for $i = 1, \dots, n$ then $f(K) = g(K)$.

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of K (so $m \leq |X|$). Define $g_r(s) = s^r$ and

$$f_r(s) = \sum_{i=1}^m \lambda_i^r \left[\prod_{j \neq i} \frac{s - \lambda_j}{\lambda_i - \lambda_j} \right].$$

Since $f_r(\lambda_i) = g_r(\lambda_i)$ for $i = 1, \dots, m$, it follows that $f_r(D) = g_r(D)$. Thus $f_r(K) = g_r(K)$ which gives that

$$K^r = \sum_{i=1}^m \lambda_i^r \left[\prod_{j \neq i} \frac{K - \lambda_j I}{\lambda_i - \lambda_j} \right].$$

As noted in [Br], expanding this expresses K^r in terms of I, K, \dots, K^{m-1} as follows:

$$K^r = \sum_{a=1}^m K^{a-1} (-1)^{m-a} \sum_{i=1}^m \lambda_i^r \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1} \sum_{\alpha \in c(m, i, m-a)} \left(\prod_{s \in \alpha} \lambda_s \right).$$

Here $c(m, i, m-a)$ consists of the $\binom{m-1}{a-1}$ subsets of size $m-a$ from $\{j : 1 \leq j \leq m, j \neq i\}$.

For the next proposition it is useful to define the distance $\text{dist}(x, y)$ between $x, y \in X$ as the smallest r such that $K^r(x, y) > 0$. For the special case of birth-death chains on the set $\{0, 1, \dots, d\}$, Proposition 5.1 appears in [DF] and [Br].

Proposition 5.1. *Let K be a reversible ergodic Markov chain on a finite set X . Let $1, \lambda_1, \dots, \lambda_d$ be the distinct eigenvalues of K (so $d+1 \leq |X|$). Suppose that x, y are elements of X with $\text{dist}(x, y) = d$. Then for all $r \geq 0$,*

$$1 - \frac{K^r(x, y)}{\pi(y)} = \sum_{i=1}^d \lambda_i^r \left[\prod_{j \neq i} \frac{1 - \lambda_j}{\lambda_i - \lambda_j} \right].$$

Proof. Since $\text{dist}(x, y) = d$, the Lagrange-Sylvester expansion of K^r in terms of I, K, \dots, K^d gives that

$$K^r(x, y) = K^d(x, y) \left(\prod_j (1 - \lambda_j)^{-1} - \sum_{i=1}^d \lambda_i^r (1 - \lambda_i)^{-1} \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1} \right).$$

By Lemma 2.2, ergodicity of K implies that $\pi(y) = \lim_{r \rightarrow \infty} K^r(x, y)$. Thus

$$\pi(y) = K^d(x, y) \prod_j (1 - \lambda_j)^{-1},$$

which implies that

$$\frac{K^r(x, y)}{\pi(y)} = 1 - \sum_{i=1}^d \lambda_i^r \left[\prod_{j \neq i} \frac{1 - \lambda_j}{\lambda_i - \lambda_j} \right].$$

□

Remarks:

- (1) Let K be a reversible connected Markov chain on a finite set X . As in Proposition 5.1, let $1, \lambda_1, \dots, \lambda_d$ be the distinct eigenvalues of K (so $d+1 \leq |X|$). From the expansion of K^r in terms of I, K, \dots, K^d it follows that $\text{dist}(x, y) \leq d$ for any states of the chain. Thus the hypothesis of Proposition 5.1 is an extremal case.
- (2) Let K be a birth-death chain on the set $\{0, \dots, d\}$, with transition probabilities

$$\begin{aligned} a_x &= K(x, x-1), \quad x = 1, \dots, d \\ b_x &= K(x, x), \quad x = 0, \dots, d \\ c_x &= K(x, x+1), \quad x = 0, \dots, d-1. \end{aligned}$$

Suppose that $a_x > 0$ for $0 < x \leq d$ and that $c_x > 0$ for $0 \leq x < d$. Such chains are reversible with respect to the stationary distribution

$$\pi(x) = Z \prod_{i=1}^x \frac{c_{i-1}}{a_i},$$

where Z is a normalizing constant. Supposing further that $c_x + a_{x+1} \leq 1$ for $0 \leq x < d$ (such chains are called monotone chains), then [DF] showed that the separation distance for the chain started at 0 is equal to

$$s(r) = 1 - \frac{K^r(0, d)}{\pi(d)}.$$

Applying Proposition 5.1 with $x = 0, y = d$ one recovers the lovely result of Diaconis and Fill [DF] expressing the separation distance entirely in terms of the eigenvalues of K :

$$s(r) = \sum_{i=1}^d \lambda_i^r \left[\prod_{j \neq i} \frac{1 - \lambda_j}{\lambda_i - \lambda_j} \right].$$

This fact was used in [DSa] to give a necessary and sufficient spectral condition for the existence of a separation cutoff for monotone birth death chains.

Next we use Proposition 5.1 to give eigenvector-free proofs of the formulas for separation distance for the random walks on $\text{Irr}(S_n)$ and $\text{Irr}(GL(n, q))$ analyzed in Sections 3 and 4. It should be emphasized that as in the proofs of Sections 3 and 4, one still needs to know at what representation the separation distance is attained.

Proof. (Fourth proof of part 1 of Theorem 3.5) By Corollary 3.4, the separation distance is equal to

$$s(r) = 1 - \frac{K^r(\hat{1}, (1^n))}{\pi((1^n))},$$

where (1^n) is the partition corresponding to the sign representation. As was mentioned at the beginning of Section 3 and proved in [F6], the Markov chain K has a description as a random walk on partitions in which one removes and adds a box at each step. From that description it is clear that the trivial representation (corresponding to the partition (n)) and the sign representation (1^n) are distance $n - 1$ apart. By part 1 of Lemma 2.1, the chain K has n distinct eigenvalues, namely $\frac{i}{n}$ where $0 \leq i \leq n - 2$ or $i = n$. Thus Proposition 5.1 implies that

$$\begin{aligned} s(r) &= \sum_{i=0}^{n-2} \left(\frac{i}{n}\right)^r \prod_{\substack{j \neq i \\ 0 \leq j \leq n-2}} \frac{\left(1 - \frac{j}{n}\right)}{\left(\frac{i}{n} - \frac{j}{n}\right)} \\ &= \sum_{i=0}^{n-2} \left(\frac{i}{n}\right)^r \prod_{\substack{j \neq i \\ 0 \leq j \leq n-2}} \frac{n-j}{i-j} \\ &= \sum_{i=0}^{n-2} \left(\frac{i}{n}\right)^r \frac{n!}{n-i} \prod_{\substack{j \neq i \\ 0 \leq j \leq n-2}} \frac{1}{i-j} \\ &= \sum_{i=0}^{n-2} (-1)^{n-i} \binom{n}{i} (n-i-1) \left(\frac{i}{n}\right)^r. \end{aligned}$$

□

A similar argument works for the general linear case.

Proof. (Second proof of part 2 of Corollary 4.4) By part 1 of Corollary 4.4, the separation distance is equal to

$$s(r) = 1 - \frac{K^r(\hat{1}, \Lambda)}{\pi(\Lambda)},$$

where Λ is any representation satisfying $\Lambda(e) = \emptyset$. By part 1 of Lemma 2.1, the chain K has $n + 1$ distinct eigenvalues, namely q^{-i} for $0 \leq i \leq n$. By the first remark after Proposition 5.1, this implies that $\text{dist}(\hat{1}, \Lambda) \leq n$. Proposition 5.4 of [F6] states that for any Λ , if $K^r(\hat{1}, \Lambda) > 0$ then the largest part of $\Lambda(e)$ is at least $n - r$; it follows that $\text{dist}(\hat{1}, \Lambda) \geq n$. Thus $\text{dist}(\hat{1}, \Lambda) = n$, and so Proposition 5.1 can be applied with $x = \hat{1}, y = \Lambda$.

Using the notation $(1/q)_k = (1 - 1/q) \cdots (1 - 1/q^k)$, one obtains that

$$\begin{aligned}
 s(r) &= \sum_{i=1}^n q^{-ir} \prod_{\substack{j \neq i \\ 1 \leq j \leq n}} \frac{1 - q^{-j}}{q^{-i} - q^{-j}} \\
 &= \sum_{i=1}^n q^{-ir} \frac{(1/q)_n}{(1 - q^{-i})} \prod_{j=1}^{i-1} \frac{1}{q^{-i} - q^{-j}} \prod_{j=i+1}^n \frac{1}{q^{-i} - q^{-j}} \\
 &= \sum_{i=1}^n q^{-ir} \frac{(1/q)_n}{(1 - q^{-i})} \frac{(-1)^{i-1} q^{\binom{i}{2}}}{(1/q)_{i-1}} \frac{q^{i(n-i)}}{(1/q)_{n-i}} \\
 &= \sum_{i=1}^n (-1)^{i+1} q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} q^{-ir}.
 \end{aligned}$$

□

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