A PROBABILISTIC APPROACH TOWARD CONJUGACY CLASSES IN THE FINITE GENERAL LINEAR AND UNITARY GROUPS

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Proposed Running Head: A Probabilistic Approach

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Abstract

The conjugacy classes of the finite general linear and unitary groups are used to define probability measures on the set of all partitions of all natural numbers. Probabilistic algorithms for growing random partitions according to these measures are obtained. These algorithms are applied to prove group theoretic results which are typically proved by techniques such as character theory and Moebius inversion. Among the theorems studied are Steinberg's count of unipotent elements, Rudvalis' and Shindoda's work on the fixed space of a random matrix, and Lusztig's count of nilpotent matrices of a given rank. Generalizations of these algorithms based on Macdonald's symmetric functions are given.

1 Introduction and Background

In thesis work done under the guidance of Persi Diaconis, the author [1] defined and studied measures $M_{(u,q)}$ on the set of all partitions of all natural numbers. The definition uses the following standard notation from [14]. One calls λ a partition of $n = |\lambda|$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and the λ_i are integers which sum to n. The λ_i are referred to as the parts of λ, and $m_i(\lambda)$ is defined as the number of parts of λ equal to i. Then for $q > 1$ and $0 < u < 1$ the following formula defines a probability measure:

$$M_{(u,q)}(\lambda) = \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right] \frac{u^{|\lambda|}}{q^{2[\sum_{h<i} hm_h(\lambda)m_i(\lambda)+\frac{1}{2}\sum_i (i-1)m_i(\lambda)^2]}\prod_i |GL(m_i(\lambda),q)|}.$$

For $0 < u < 1$ and q a prime power, the measures $M_{(u,q)}$ have a group theoretic description. For this recall (for instance from Chapter 6 of [8]) that the conjugacy classes of $GL(n,q)$ are parameterized by rational canonical form. Each such matrix corresponds to the following combinatorial data. To every monic non-constant irreducible polynomial ϕ over F_q, associate a partition (perhaps the trivial partition) λ_ϕ of some non-negative integer $|\lambda_\phi|$. The only restrictions necessary for this data to represent a conjugacy class are that $|\lambda_z| = 0$ and $\sum_\phi |\lambda_\phi| deg(\phi) = n$.

To be explicit, and for use in Section 3, a representative of the conjugacy class corresponding to the data λ_ϕ may be given as follows. Define the companion matrix $C(\phi)$ of a polynomial $\phi(z) = z^{deg(\phi)} + \alpha_{deg(\phi)-1} z^{deg(\phi)-1} + \cdots + \alpha_1 z + \alpha_0$ to be:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{deg(\phi)-1} \end{pmatrix}$$

Let $\phi_1, \cdots, \phi_k$ be the polynomials such that $|\lambda_{\phi_i}| > 0$. Denote the parts of λ_{ϕ_i} by $\lambda_{\phi_i,1} \geq \lambda_{\phi_i,2} \geq \cdots$. Then a matrix corresponding to the above conjugacy class data is:

$$\begin{pmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & R_k \end{pmatrix}$$

where R_i is the matrix:

$$\begin{pmatrix} C(\phi_i^{\lambda_{\phi_i,1}}) & 0 & 0 \\ 0 & C(\phi_i^{\lambda_{\phi_i,2}}) & 0 \\ 0 & 0 & \cdots \end{pmatrix}$$

Now consider the following procedure for putting a probability measure on the set of all partitions of all natural numbers. Fix u such that $0 < u < 1$. Pick a non-negative integer such that the chance of choosing n is equal to $(1-u)u^n$. Then pick α uniformly in $GL(n,q)$ and take λ to be the paritition corresponding to the polynomial $z-1$ in the rational canonical form of α (if $n=0$ take λ to be the trivial partition). It is proved in [1] that the random partition so defined obeys $M_{(u,q)}$ measure.

The measures $M_{(u,q)}$ have further remarkable properties. To state them, we use the notation that $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$, where $m_i(\lambda)$ is the number of parts of λ equal to i. Then the following three equations hold (for the third equation assume that $k \geq 2$):

$$\sum_{\lambda} x^{|\lambda|} M_{(u,q)}(\lambda) = \prod_{r=1}^{\infty} \frac{(1-\frac{u}{q^r})}{(1-\frac{ux}{q^r})}$$

$$\sum_{\lambda:\lambda_1'=k} x^{|\lambda|} M_{(u,q)}(\lambda) = \frac{(ux)^k}{|GL(k,q)|} \frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})}$$

$$\sum_{\lambda:\lambda_1<k} M_{(1,q)}(\lambda) = \prod_{\substack{r=1 \\ r=0,\pm k (mod\ 2k+1)}}^{\infty} (1-\frac{1}{q^r}).$$

These equations are strong evidence that the measures $M_{(u,q)}$ are worthy of study. The first two equations will be proved probabilistically and interpreted group theoretically in this paper. The third equation is related to the Rogers-Ramanujan identities and seems to be the first appearance of the Rogers-Ramanujan identities in finite group theory [2].

Section 2 develops probabilistic algorithms for growing partitions according to the measure $M_{(u,q)}$. These algorithms will be key tools of this paper. Section 3 applies the tools of Section 2 to several settings. First, Steinberg's count of unipotent elements is proved for $GL(n,q)$ and $U(n,q)$. Second, a deeper understanding is given to work of Rudvalis and Shinoda [15] on the fixed space of a random element of $GL(n,q)$ or $U(n,q)$. The factorization in their formulas is interpreted as a statement about certain random variables being conditionally independent. This suggests the possibility of generalizing the methods of this paper to the finite symplectic and orthogonal groups. Third, Lusztig's [13] results on nilpotent matrices of a given rank are derived probabilistically. In principle, the algorithms of Section 2 are relevant for analyzing any conjugacy class function on $GL(n,q)$ or $U(n,q)$. The results of Section 3 only scratch the surface.

Section 4 connects the algorithms of Section 2 with symmetric function theory. (In fact it was the structure of the Hall-Littlewood polynomials which led the author to the algorithms of Section 2). The Macdonald symmetric functions are used to define measures generalizing the measure $M_{(u,q)}$, and a probabilistic algorithm is given for growing partitions according to these measures. Specializing to the Schur functions gives a q-analog of Plancherel measure of the symmetric group. This q-analog is shown to have natural properties and is different from the q-analog considered in [10] and [4].

2 Probabilistic Algorithms

To begin we rewrite $M_{(u,q)}(\lambda)$ in a form more amenable to our analysis. Let $(\frac{1}{q})_r$ denote $(1-\frac{1}{q})\cdots(1-\frac{1}{q^r})$.

Lemma 1

$$M_{(u,q)}(\lambda) = \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right] \frac{u^{|\lambda|}}{q^{[\sum_i (\lambda_i')^2]}\prod_i (\frac{1}{q})_{m_i(\lambda)}}$$

PROOF: Write $(\frac{1}{q})_{m_i(\lambda)}$ as $\frac{|GL(m_i(\lambda),q)|}{q^{m_i(\lambda)^2}}$. Comparing powers of q reduces one to proving that

$$\sum_i (\lambda_i')^2 = \sum_i [i m_i(\lambda) + 2\sum_{h<i} h m_h(\lambda)] m_i(\lambda).$$

This last equation follows quickly after substituting $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. $\square$

We first develop the "Young Tableau Algorithm". To state it, recall that one defines the diagram associated to λ as the set of points $(i,j) \in Z^2$ such that $1 \leq j \leq \lambda_i$. We follow Macdonald's convention that the row index i increases as one goes downward and the column index j increases as one goes across. So the diagram of the partition (432) is

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & \\ . & . & & \end{array}$$

Recall that a standard Young tableau T of size n is a partition of n with each dot replaced by one of $\{1, \cdots, n\}$ such that each of $\{1, \cdots, n\}$ appears exactly once and the numbers increase in each row and column of T. For instance,

$$\begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 & \\ 8 & 9 & & \end{array}$$

is a standard Young tableau. The Young Tableau Algorithm is so named because numbering the dots in the order in which they are created gives a standard Young tableau.

The Young Tableau Algorithm

Step 0 Start with $N = 1$ and λ the empty partition. Also start with a collection of coins indexed by the natural numbers, such that coin i has probability $\frac{u}{q^i}$ of heads and probability $1 - \frac{u}{q^i}$ of tails.

Step 1 Flip coin N.

Step 2a If coin N comes up tails, leave λ unchanged, set $N = N + 1$ and go to Step 1.

Step 2b If coin N comes up heads, choose an integer $S > 0$ according to the following rule. Set $S = 1$ with probability $\frac{q^{N-\lambda_1'}-1}{q^N-1}$. Set $S = s > 1$ with probability $\frac{q^{N-\lambda_s'}-q^{N-\lambda_{s-1}'}}{q^N-1}$. Then increase the size of column s of λ by 1 and go to Step 1.

As an example of the Young Tableau Algorithm, suppose we are at Step 1 with λ equal to the following partition:

$$\begin{array}{cccc} . & . & . & . \\ . & . & & \\ . & & & \end{array}$$

Suppose also that $N = 4$ and that coin 4 had already come up heads once, at which time we added to column 1, giving λ. Now we flip coin 4 again and get heads, going to Step 2b. We add to column 1 with probability $\frac{q-1}{q^4-1}$, to column 2 with probability $\frac{q^2-q}{q^4-1}$, to column 3 with probability $\frac{q^3-q^2}{q^4-1}$, to column 4 with probability 0, and to column 5 with probability $\frac{q^4-q^3}{q^4-1}$. We then return to Step 1.

Theorem 1 *For $0 < u < 1$ and $q > 1$, the Young Tableau Algorithm always halts and the resulting partition obeys $M_{(u,q)}$ measure.*

PROOF: To see that the algorithm always halts, recall the Borel-Cantelli lemma of probability theory, which says that if A_N are events with probability $P(A_N)$ and $\sum_N P(A_N) < \infty$, then with probability 1 only finitely many A_N occur. Letting A_N be the event that coin N comes up heads at least once, one concludes that only a finite number of coins come up heads at least once. A second application of Borel-Cantelli shows that any coin coming up heads does so finitely many times. Thus the algorithm halts with probability 1.

Let $P^N(\lambda)$ be the probability that the algorithm outputs λ when coin N comes up tails. We will prove by induction the assertion that

$$P^N(\lambda) = \begin{cases} \frac{u^{|\lambda|}(\frac{u}{q})_N(\frac{1}{q})_N}{(\frac{1}{q})_{N-\lambda_1'}} \prod_{i\geq 1} \frac{1}{q^{(\lambda_i')^2}(\frac{1}{q})_{m_i(\lambda)}} & \text{if } \lambda_1' \leq N \\ 0 & \text{if } \lambda_1' > N. \end{cases}$$

Taking the $N \to \infty$ limit implies by Lemma 1 that the algorithm generates partitions according to $M_{(u,q)}$ measure.

The assertion is clear if $N < \lambda_1'$ for then $P^N(\lambda) = 0$, and Step 2b does not permit the number of parts of the partition to exceed the number of the coin being tossed at any stage in the algorithm.

For the case $N \geq \lambda_1'$, use induction on $|\lambda| + N$. The base case is that λ is the empty partition. This means that coins $1, 2, \cdots, N$ all came up tails on their first tosses, which occurs with probability $(\frac{u}{q})_N$. So the base case checks.

Let $s_1 \leq s_2 \leq \cdots \leq s_k$ be the columns of λ with the property that changing λ by decreasing the size of one of these columns by 1 gives a partition λ^{s_i}. It then suffices to check that the claimed formula for $P^N(\lambda)$ satisfies the equation

$$\begin{aligned} P^N(\lambda) &= (1 - \frac{u}{q^N})P^{N-1}(\lambda) + \frac{u}{q^N}\frac{q^{N-\lambda_1'} - 1}{q^N - 1}P^N(\lambda^1) \\ &\quad + \sum_{s_i > 1} \frac{u}{q^N}\frac{q^{N-\lambda_{s_i}'+1} - q^{N-\lambda_{s_i-1}'}}{q^N - 1}P^N(\lambda^{s_i}). \end{aligned}$$

This equation is based on the following logic. Suppose that when coin N came up tails, the algorithm gave the partition λ. If coin N came up tails on its first toss, then we must have had λ when coin $N-1$ came up tails. Otherwise, for each s_i we add the probability that "The algorithm gave the partition λ^{s_i} on the penultimate toss of coin N and the partition λ on the last toss of coin N". It is not hard to see that this probability is equal to the probability of getting λ^{s_i} on the final toss of coin N, multiplied by the chance of a heads on coin N which then gives the partition λ from λ^{s_i}.

We divide both sides of this equation by $P^N(\lambda)$ and show that the terms on the right-hand side sum to 1. First consider the terms with $s_i > 1$. Induction gives that

$$
\begin{aligned}
& \sum_{s_i>1} \frac{u}{q^N} \frac{q^{N-\lambda'_{s_i}+1} - q^{N-\lambda'_{s_{i-1}}}}{q^N-1} \frac{P^N(\lambda^{s_i})}{P^N(\lambda)} \\
= & \sum_{s_i>1} \frac{q^{-\lambda'_{s_i}+1} - q^{-\lambda'_{s_{i-1}}}}{q^N-1} \frac{q^{\binom{\lambda'_{s_i}}{2}} (\frac{1}{q})_{\lambda'_{s_{i-1}}-\lambda'_{s_i}} (\frac{1}{q})_{\lambda'_{s_i}-\lambda'_{s_{i+1}}}}{q^{\binom{\lambda'_{s_i}-1}{2}} (\frac{1}{q})_{\lambda'_{s_{i-1}}-\lambda'_{s_i}+1} (\frac{1}{q})_{\lambda'_{s_i}-\lambda'_{s_{i+1}}-1}} \\
= & \sum_{s_i>1} \frac{q^{-\lambda'_{s_i}+1} - q^{-\lambda'_{s_{i-1}}}}{q^N-1} q^{2\lambda'_{s_i}-1} \frac{(1-\frac{1}{q^{\lambda'_{s_i}-\lambda'_{s_{i+1}}}})}{(1-\frac{1}{q^{\lambda'_{s_{i-1}}-\lambda'_{s_i}+1}})} \\
= & \sum_{s_i>1} \frac{q^{\lambda'_{s_i}} - q^{\lambda'_{s_{i+1}}}}{q^N-1} \\
= & \frac{q^{\lambda'_2}-1}{q^N-1}.
\end{aligned}
$$

Next consider the term coming from $P^{N-1}(\lambda)$. If $N = \lambda'_1$, then $\lambda'_1 > N-1$, so by what we have proven $P^{N-1}(\lambda) = 0$. Otherwise,

$$
\begin{aligned}
(1-\frac{u}{q^N}) \frac{P^{N-1}(\lambda)}{P^N(\lambda)} &= (1-\frac{u}{q^N}) \frac{(\frac{u}{q})_{N-1} (\frac{1}{q})_{N-1} (\frac{1}{q})_{N-\lambda'_1}}{(\frac{u}{q})_N (\frac{1}{q})_N (\frac{1}{q})_{N-\lambda'_1-1}} \\
&= \frac{(1-\frac{1}{q^{N-\lambda'_1}})}{(1-\frac{1}{q^N})} \\
&= \frac{q^N - q^{\lambda'_1}}{q^N-1}.
\end{aligned}
$$

Thus this term always contributes $\frac{q^N-q^{\lambda'_1}}{q^N-1}$.

Finally, consider the term coming from $P^N(\lambda^1)$. This vanishes if $\lambda'_1 = \lambda'_2$ since then λ^1 is not a partition. Otherwise,

$$
\begin{aligned}
\frac{u}{q^N} \frac{q^{N-\lambda'_1+1}-1}{q^N-1} \frac{P^N(\lambda^1)}{P^N(\lambda)} &= \frac{q^{-\lambda'_1+1} - q^{-N}}{q^N-1} \frac{(\frac{1}{q})_{N-\lambda'_1}}{(\frac{1}{q})_{N-\lambda'_1+1}} \frac{q^{\binom{\lambda'_1}{2}} (\frac{1}{q})_{\lambda'_1-\lambda'_2}}{q^{\binom{\lambda'_1-1}{2}} (\frac{1}{q})_{\lambda'_1-\lambda'_2-1}} \\
&= \frac{q^{-\lambda'_1+1} - q^{-N}}{q^N-1} \frac{1}{(1-\frac{1}{q^{N-\lambda'_1+1}})} q^{2\lambda'_1-1} (1-\frac{1}{q^{\lambda'_1-\lambda'_2}}) \\
&= \frac{q^{\lambda'_1} - q^{\lambda'_2}}{q^N-1}.
\end{aligned}
$$

Thus in all cases this term contributes $\frac{q^{\lambda'_1}-q^{\lambda'_2}}{q^N-1}$.

Adding the three terms completes the proof. $\square$

We derive a second way to calculate the measures $M_{(u,q)}$. This uses the so-called Young lattice, which is important in combinatorics and representation theory. The elements of this lattice are all partitions of all natural numbers. A directed edge is drawn from partition λ to partition Λ if the diagram of λ is contained in the diagram of Λ and $|\Lambda| = |\lambda| + 1$.

Theorem 2 *Put weights $m_{\lambda,\Lambda}$ on the edges of Young lattice according to the rules:*

1. $m_{\lambda,\Lambda} = \frac{u}{q^{\lambda'_1}(q^{\lambda'_1+1}-1)}$ *if the diagram of Λ is obtained from that of λ by adding a dot to column 1.*

2. $m_{\lambda,\Lambda} = \frac{u(q^{-\lambda'_s}-q^{-\lambda'_{s-1}})}{q^{\lambda'_1}-1}$ *if the diagram of Λ is obtained from that of λ by adding a dot to column $s > 1$.*

Then the following formula holds:

$$M_{(u,q)}(\lambda) = \left[\prod_{r=1}^{\infty}(1 - \frac{u}{q^r})\right] \sum_{\gamma} \prod_{i=0}^{|\lambda|-1} m_{\gamma_i,\gamma_{i+1}}$$

where the sum is over all directed paths γ from the empty partition to λ, and the γ_i are the partitions along the path γ.

Remark: Note that in Theorem 2 the sum of the edge weights out of the empty partition is $\frac{u}{q-1}$ and that the sum of the edge weights out of any other partition λ is

$$\begin{aligned} \frac{u}{q^{\lambda'_1}(q^{\lambda'_1+1}-1)} + \sum_{i\geq 2} \frac{u(q^{-\lambda'_s} - q^{-\lambda'_{s-1}})}{q^{\lambda'_1}-1} &= \frac{u}{q^{\lambda'_1}(q^{\lambda'_1+1}-1)} + \frac{u}{q^{\lambda'_1}} \\ &= \frac{uq}{q^{\lambda'_1+1}-1} \\ &< 1. \end{aligned}$$

Since the sum of the weights out of a partition λ to a larger partition Λ is less than 1, the weights can also be viewed as transition probabilities, provided that one allows for halting.

To prove Theorem 2 some further notation is needed. T will denote a standard Young tableau and $\lambda(T)$ the partition corresponding to T. Let $|T|$ be the size of T. Recall that the Young Tableau Algorithm constructs a Young tableau, and thus defines a measure on the set of all Young tableaux. Let $P(T)$ be the chance that the Young Tableau Algorithm outputs T, and let $P^N(T)$ be the chance that it outputs T when coin N comes up tails. Let $T_{(i,j)}$ be the entry in the (i,j) position of T (recall that i is the row number and j the column number). For $j \geq 2$, let $A_{(i,j)}$ be the number of entries $(i', j-1)$ such that $T_{(i',j-1)} < T_{(i,j)}$. Let $B_{(i,j)}$ be the number of entries $(i',1)$ such that $T_{(i',1)} < T_{(i,j)}$. For instance the tableau

$$\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 & \\ 8 & 9 & & \end{matrix}$$

has $T_{(1,3)} = 5$. Also $A_{(1,3)} = 2$ because there are 2 entries in column $3-1=2$ which are less than 5. Finally, $B_{(1,3)} = 2$ because there are 2 entries in column 1 which are less than 5.

The proof of Theorem 2 will also use the equivalence between a standard Young tableau T of shape λ and a path in the Young lattice from the empty partition to λ. This equivalence is given by growing the partition λ by adding dots in the order $1, \cdots, n$ in the positions determined by T. For instance the tableau

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array}$$

corresponds to the path

$$\begin{array}{ccccccccc} & & \cdot & & \cdot & & \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ \emptyset & \rightarrow & & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot \end{array}$$

PROOF: We prove the assertion that

$$P^N(T) = \begin{cases} \frac{u^{|T|}}{|GL(\lambda_1'(T),q)|} \frac{\prod_{r=1}^{N}(1-\frac{u}{q^r})(1-\frac{1}{q^r})}{\prod_{r=1}^{N-\lambda_1'(T)}(1-\frac{1}{q^r})} \prod_{\substack{(i,j)\in\lambda(T) \\ j\geq 2}} \frac{q^{1-i}-q^{-A(i,j)}}{q^{B(i,j)}-1} & \text{if } \lambda_1'(T) \leq N \\ 0 & \text{if } \lambda_1'(T) > N. \end{cases}$$

Theorem 2 then follows by letting $N \rightarrow \infty$ and using the fact that T corresponds to a unique path in the Young lattice.

The case where T has more than N parts is proven as in Theorem 1.

The case $\lambda_1' \leq N$ is proven by induction on $|T|+N$. If $|T|+N=1$, then T is the empty tableau and $N=1$. This means that coin 1 in the Tableau algorithm came up tails on the first toss, which happens with probability $1-\frac{u}{q}$. So the base case checks.

For the induction step, there are two cases. The first case is that the largest entry in T occurs in column $s>1$. Removing the largest entry from T gives a tableau T^s. This yields the equation

$$P^N(T) = (1-\frac{u}{q^N})P^{N-1}(T) + \frac{u}{q^N}\frac{q^{N-\lambda_s'+1}-q^{N-\lambda_{s-1}'}}{q^N-1}P^N(T^s).$$

The two terms in this equation correspond to the whether or not T was completed at time N. We divide both sides of the equation by $P^N(T)$, substitute in the conjectured assertion, and show that it satisfies this recurrence. The two terms on the right hand side then give

$$\frac{q^N-q^{\lambda_1'}}{q^N-1} + \frac{q^{-\lambda_s'+1}-q^{-\lambda_{s-1}'}}{q^N-1}\frac{1}{\frac{q^{-\lambda_s'+1}-q^{-\lambda_{s-1}'}}{q^{\lambda_1'}-1}} = 1.$$

The other case is that the largest entry of T occurs in column 1. This yields the equation

$$P^N(T) = (1-\frac{u}{q^N})P^{N-1}(T) + \frac{u}{q^N}\frac{q^{N-\lambda_1'}-1}{q^N-1}P^N(T^1).$$

As in the previous case, we divide both sides of the equation by $P^N(T)$, substitute into the conjectured assertion, and show that it satisfies this recurrence. The two terms on the right hand side then give

$$\frac{q^N-q^{\lambda_1'}}{q^N-1} + \frac{1}{q^N}\frac{q^{N-\lambda_1'+1}-1}{q^N-1}\frac{1}{(1-\frac{1}{q^{N-\lambda_1'+1}})}\frac{|GL(\lambda_1',q)|}{|GL(\lambda_1'-1,q)|} = 1.$$

This completes the induction, and the proof of the theorem. $\square$

A third method for generating partitions according to the measure $M_{(u,q)}$ is given in Section 4.

3 Applications

This section uses the algorithms of Section 2 to obtain results about the general linear and unitary groups. The concept of a "cycle index" connects the measures $M_{(u,q)}$ with these groups. The cycle index of the general linear groups is due to Kung [12] and Stong [18], though without the idea of measures on partitions. The cycle index for the unitary groups was found by the author [1].

1. General Linear Group Cycle Index: Let α be an element of $GL(n,q)$ and ϕ a monic irreducible polynomial with coefficients in F_q, the field of q elements. Let $\lambda_\phi(\alpha)$ be the partition corresponding to the ϕ in the rational canonical form of α. Then,

$$(1-u)\left(1+\sum_{n=1}^{\infty}\frac{u^n}{|GL(n,q)|}\sum_{\alpha\in GL(n,q)}\prod_{\phi\neq z}x_{\phi,\lambda_\phi(\alpha)}\right)=\prod_{\phi\neq z}\sum_{\lambda}x_{\phi,\lambda}M_{(u,q^{deg(\phi)})}(\lambda).$$

2. Unitary Group Cycle Index: The conjugacy classes of $U(n,q)\subset GL(n,q^2)$ have a description analogous to rational canonical form for $GL(n,q)$. Given a polynomial ϕ with coefficients in F_{q^2} and non-vanishing constant term, define a polynomial $\tilde{\phi}$ by

$$\tilde{\phi}=\frac{z^{deg(\phi)}\phi^q(\frac{1}{z})}{[\phi(0)]^q}$$

where ϕ^q raises each coefficient of ϕ to the qth power. Writing this out, a polynomial $\phi(z)=z^{deg(\phi)}+\alpha_{deg(\phi)-1}z^{deg(\phi)-1}+\cdots+\alpha_1 z+\alpha_0$ with $\alpha_0\neq 0$ is sent to $\tilde{\phi}(z)=z^{deg(\phi)}+(\frac{\alpha_1}{\alpha_0})^q z^{deg(\phi)-1}+\cdots+(\frac{\alpha_{deg(\phi)-1}}{\alpha_0})^q z+(\frac{1}{\alpha_0})^q$. It is proved in [1] that all ϕ satisfying $\phi=\tilde{\phi}$ have odd degree.

Wall [21] proves that the conjugacy classes of the unitary group correspond to the following combinatorial data. As was the case with $GL(n,q^2)$, an element $\alpha\in U(n,q)$ associates to each monic, non-constant, irreducible polynomial ϕ over F_{q^2} a partition λ_ϕ of some non-negative integer $|\lambda_\phi|$ by means of a rational canonical form. The data λ_ϕ corresponds to a conjugacy class if and only if $|\lambda_z|=0$, $\lambda_\phi=\lambda_{\tilde{\phi}}$, and $\sum_\phi|\lambda_\phi|deg(\phi)=n$.

This leads to the cycle index:

$$\begin{aligned}(1-u)\left(1+\sum_{n=1}^{\infty}\frac{u^n}{|U(n,q)|}\sum_{\alpha\in U(n,q)}\prod_{\phi\neq z}x_{\phi,\lambda_\phi(\alpha)}\right) &= \prod_{\phi\neq z,\phi=\tilde{\phi}}\sum_{\lambda}x_{\phi,\lambda}M_{((-u)^{deg(\phi)},(-q)^{deg(\phi)})}(\lambda)\\ &\quad\prod_{\phi\neq\tilde{\phi}}\sum_{\lambda}x_{\phi,\lambda}M_{(u^{2deg(\phi)},q^{2deg(\phi)})}(\lambda)\end{aligned}$$

It is elementary to see that $M_{(-u,-q)}$ is also a measure. The Young Tableau Algorithm can not be applied to pick from it, however, because some of the "probabilities" involved would be negative. Nevertheless, the description in terms of weights on the Young lattice (Theorem 2) does extend by replacing u and q by their negatives.

The following three elementary lemmas will be of use in the applications to follow.

Lemma 2

$$1-u=\prod_{\phi\neq z}\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})$$

PROOF: For all ϕ, perform the following substitutions in the cycle index of the general linear groups. If $|\lambda|>0$, set $x_{\phi,\lambda}=0$. If $|\lambda|=0$, set $x_{\phi,\lambda}=1$. $\square$

Lemma 3

$$1-u=\left(\prod_{\phi\neq z,\phi=\tilde{\phi}}\prod_{r=1}^{\infty}(1+(-1)^r\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})\right)\left(\prod_{\phi\neq\tilde{\phi}}\prod_{r=1}^{\infty}(1-\frac{u^{2deg(\phi)}}{q^{2deg(\phi)r}})\right)$$

PROOF: Make the same substitutions as in Lemma 2, but for the unitary groups. $\square$

Lemma 4 is elementary and is taken from page 280 of Hardy and Wright [7].

Lemma 4 *For real a,y such that $|a|\leq 1,|y|<1$,*

$$\frac{1}{(1-ay)\cdots(1-ay^k)}=1+ay\frac{1-y^k}{1-y}+a^2y^2\frac{(1-y^k)(1-y^{k+1})}{(1-y)(1-y^2)}+\cdots$$

Application 1: Counting unipotent elements

The following theorem of Steinberg is normally proven using the Steinberg character, as on page 156 of [9]. Recall that $\alpha\in GL(n,q)$ is called unipotent if all of its eigenvalues are equal to one.

Theorem 3 *The number of unipotent elements in a finite group of Lie type G^F is the square of the order of a p-Sylow of G^F, where p is the prime used in the construction of G^F (in the case of the classical groups, p is the characteristic of F_q).*

The goal of this application is to give a probabilistic proof of Steinberg's result for $GL(n,q)$ and $U(n,q)$. To this end, we obtain a generating function for the size of a partition λ chosen from the measure $M_{(u,q)}$.

Theorem 4

$$\sum_{\lambda}x^{|\lambda|}M_{(u,q)}(\lambda)=\prod_{r=1}^{\infty}\frac{(1-\frac{u}{q^r})}{(1-\frac{ux}{q^r})}$$

PROOF: Observe from the Young Tableau Algorithm that the size of the partition is equal to the total number of coins which come up heads. The $r=i$ term of the product on the right hand side corresponds to the tosses of coin i, and these terms are multiplied because the coin tosses of different coins are independent. $\square$

In Corollaries 1 and 2 we use the notation that $[u^n]f(u)$ is the coefficient of u^n in the power series $f(u)$.

Corollary 1 *The number of unipotent elements of $GL(n,q)$ is $q^{n(n-1)}$.*

PROOF: Setting $x_{z-1,\lambda} = x^{|\lambda|}$ and $x_{\phi,\lambda} = 0$ for $\phi \neq z-1$ in the cycle index for $GL(n,q)$ shows that the number of unipotent elements of $GL(n,q)$ is

$$\begin{aligned}
& |GL(n,q)|[u^n x^n]\frac{1}{1-u}\left(\sum_\lambda x^{|\lambda|} M_{(u,q)}(\lambda)\right)\left(\prod_{\phi \neq z-1}\prod_{r=1}^\infty (1-\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})\right) \\
= & |GL(n,q)|[u^n x^n]\frac{1}{1-u}\left(\prod_{r=1}^\infty \frac{(1-\frac{u}{q^r})}{(1-\frac{ux}{q^r})}\right)\left(\prod_{\phi \neq z-1}\prod_{r=1}^\infty (1-\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})\right) \\
= & |GL(n,q)|[u^n x^n]\prod_{r=1}^\infty (\frac{1}{1-\frac{ux}{q^r}}) \\
= & q^{n(n-1)}.
\end{aligned}$$

The first equality comes from Theorem 4. The second equality is Lemma 2, and the third equality is Lemma 4 with $a = ux, y = \frac{1}{q}$. □

A similar argument works for the unitary groups. It is well known that the order of $U(n,q)$ is $q^{\binom{n}{2}}\prod_{i=1}^n (q^i - (-1)^i)$.

Corollary 2 *The number of unipotent elements of $U(n,q)$ is $q^{n(n-1)}$.*

PROOF: Setting $x_{z-1,\lambda} = x^{|\lambda|}$ and $x_{\phi,\lambda} = 0$ for $\phi \neq z-1$ in the cycle index for $U(n,q)$ shows that the number of unipotent elements of $U(n,q)$ is

$$\begin{aligned}
& |U(n,q)|[u^n x^n]\frac{1}{1-u}\left(\sum_\lambda x^{|\lambda|} M_{(-u,-q)}(\lambda)\right)\left(\prod_{\phi \neq z-1, \phi = \tilde{\phi}}\prod_{r=1}^\infty (1+(-1)^r\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})\right) \\
& \left(\prod_{\phi \neq \tilde{\phi}}\prod_{r=1}^\infty (1-\frac{u^{2deg(\phi)}}{q^{2deg(\phi)r}})\right) \\
= & |U(n,q)|[u^n x^n]\frac{1}{1-u}\left(\prod_{r=1}^\infty \frac{(1+(-1)^r\frac{u}{q^r})}{(1+(-1)^r\frac{ux}{q^r})}\right)\left(\prod_{\phi \neq z-1, \phi = \tilde{\phi}}\prod_{r=1}^\infty (1+(-1)^r\frac{u^{deg(\phi)}}{q^{deg(\phi)r}})\right) \\
& \left(\prod_{\phi \neq \tilde{\phi}}\prod_{r=1}^\infty (1-\frac{u^{2deg(\phi)}}{q^{2deg(\phi)r}})\right) \\
= & |U(n,q)|[u^n x^n]\prod_{r=1}^\infty (\frac{1}{(1+(-1)^r\frac{ux}{q^r})}) \\
= & q^{n(n-1)}.
\end{aligned}$$

The first equality comes from the corresponding equality in Corollary 1 with u and q replaced by their negatives. The third equality uses Lemma 4 with $a = -ux, y = -\frac{1}{q}$. □

Remarks:

1. The proof technique of Corollaries 1 and 2 can be used to give formulas for the number of elements of $GL(n,q)$ and $U(n,q)$ with a given characteristic polynomial [3].

2. At least for $GL(n,q)$, there should be a proof of Steinberg's count of unipotents which is bijective (i.e. which maps a pair of elements in a p-Sylow to a unipotent element).

Application 2: Work of Rudvalis and Shinoda

Rudvalis and Shinoda [15] studied the distribution of fixed vectors for the classical groups over finite fields. Let $G = G(n)$ be a classical group (i.e. one of GL,U,Sp, or O) acting on an n dimensional vector space V over a finite field F_q (in the unitary case F_{q^2}) in its natural way. Let $P_{G,n}(k,q)$ be the chance that the fixed-space of an element of G is k dimensional, and let $P_{G,\infty}(k,q)$ be the $n \rightarrow \infty$ limit of $P_{G,n}(k,q)$. The following results emerged for the general linear and unitary cases.

1. $P_{GL,n}(k,q) = \frac{1}{|GL(k,q)|} \sum_{i=0}^{n-k} \frac{(-1)^i q^{\binom{i}{2}}}{q^{ki}|GL(i,q)|}$.

2. $P_{GL,\infty}(k,q) = \left[\prod_{r=1}^{\infty}(1-\frac{1}{q^r})\right] \frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q})^2 \cdots (1-\frac{1}{q^k})^2}$.

3. $P_{U,n}(k,q) = \frac{1}{|U(k,q)|} \sum_{i=0}^{n-k} \frac{(-1)^i (-q)^{\binom{i}{2}}}{(-q)^{ki}|U(i,q)|}$.

4. $P_{U,\infty}(k,q) = \left[\prod_{r=0}^{\infty}(1+\frac{1}{(-q)^r})\right] \frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q^2}) \cdots (1-\frac{1}{q^{2k}})}$.

At first glance it is not even clear that $P_{GL,\infty}(k,q)$ and $P_{U,\infty}(k,q)$ define probability distributions in k, but as noted in [15], this follows from identities of Euler. Proofs of the above results of Rudvalis and Shinoda used Moebius inversion on the lattice of subspaces of a vector space and a detailed knowledge of geometry over finite fields.

Theorem 2 of the previous section will lead to probabilistic proofs of the above four equations. Furthermore, a probabilistic interpretation will be given to the products in the formulas for $P_{GL,\infty}(k,q)$ and $P_{U,\infty}(k,q)$. The first step is to connect the theorems of Rudvalis and Shinoda with the partitions in the rational canonical form of α.

Lemma 5 *The dimension of the fixed space of an element α of $GL(n,q)$ is equal to $\lambda_{z-1}(\alpha)'_1$ (i.e. the number of parts of the partition corresponding to the polynomial $z-1$ in the rational canonical form of α).*

PROOF: It must be shown that the kernel of $\alpha - I$, where I is the identity map, has dimension $\lambda_{z-1}(\alpha)'_1$. By the explicit description of the rational canonical form of a matrix in Section 1, it is enough to prove that the kernel of the linear map with matrix $M = C((z-1)^i) - I$ is 1 dimensional for all i (as in Section 1, $C(\phi)$ is the companion matrix of a polynomial ϕ).

Each of the first $i-1$ rows of M sums to 0, and they are linearly independent. Thus it needs to be shown that the last row of M has sum 0. This follows from the fact that the coefficients of $(z-1)^i$ sum to 0. $\square$

To proceed further, some notation is necessary. Let T be a standard Young tableau with k parts. Let $T_{(i,j)}$ be the entry in row i and column j of T. We define numbers $h_1(T), \cdots, h_k(T)$ associated with T. Let $h_m(T) = T_{(m+1,1)} - T_{(m,1)} - 1$ for $1 \leq m \leq k-1$ and let $h_k(T) = |T| - T_{(k,1)}$. So if $k = 3$ and T is the tableau

$$\begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 & \\ 8 & 9 & & \end{array}$$

then $h_1(T) = 2-1-1 = 0$, $h_2(T) = 8-2-1 = 5$, and $h_3(T) = 9-8 = 1$. View T as being created by the Young Tableau Algorithm. Then for $1 \leq m \leq k-1$, $h_m(T)$ is the number of dots added to T after it becomes a tableau with m parts and before it becomes a tableau with $m+1$ parts. $h_k(T)$ is the number of dots added to T after it becomes a tableau with k parts. The proof of Theorem 5 will show that if one conditions T chosen from the measure $M_{(u,q)}$ on having k parts, then the random variables $h_1(T), \cdots, h_k(T)$ are independent geometrics with parameters $\frac{u}{q}, \cdots, \frac{u}{q^k}$. This will explain the factorization on the right-hand side of the formula in Theorem 5.

Theorem 5

$$\sum_{\lambda:\lambda_1'=k} x^{|\lambda|} M_{(u,q)}(\lambda) = \frac{(ux)^k}{|GL(k,q)|} \frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})}$$

Proof: We sum over all Young tableaux T with k parts "$x^{|T|}$ multiplied by the chance that the Tableau algorithm outputs T". The point is that one can easily compute the probability that the Tableau algorithm produces a tableau T with given values $h_1, \cdots, h_k$.

Suppose that one takes a step along the Young lattice from a partition with m parts. Theorem 2 implies that the weight for adding to column 1 is $\frac{u}{q^m(q^{m+1}-1)}$, and that the sum of the weights for adding to any other column is $\frac{u}{q^m}$. Thus $x^{|T|}$ multiplied by the chance that the Tableau algorithm yields a tableau with given values $h_1, \cdots, h_k$ is

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\frac{(xu)^k}{|GL(k,q)|}\prod_{m=1}^{k}(\frac{ux}{q^m})^{h_m}.$$

Summing over all possible values of $h_m \geq 0$ gives:

$$\begin{aligned}
\sum_{\lambda:\sum\lambda_1'=k} x^{|\lambda|} M_{(u,q)}(\lambda) &= \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\frac{(ux)^k}{|GL(k,q)|}\prod_{m=1}^{k}\sum_{h_m=0}^{\infty}(\frac{ux}{q^m})^{h_m} \\
&= \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\frac{(ux)^k}{|GL(k,q)|}\prod_{m=1}^{k}\frac{1}{(1-\frac{ux}{q^m})} \\
&= \frac{(ux)^k}{|GL(k,q)|}\frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})}.
\end{aligned}$$

□

To deduce the Rudvalis/Shinoda formulas for the general linear and unitary groups, two further easy lemmas will be used. We adhere to the notation that $[u^n]f(u)$ is the coefficient of u^n in the power series $f(u)$.

Lemma 6 *If $f(1) < \infty$ and the Taylor series of $f(u)$ around 0 converges at $u = 1$, then*

$$lim_{n\to\infty}[u^n]\frac{f(u)}{1-u} = f(1).$$

Proof: Write the Taylor expansion $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Then observe that $[u^n]\frac{f(u)}{1-u} = \sum_{i=0}^{n} a_i$. □

Lemma 7 *([5])*

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r}) = \sum_{i=0}^{\infty}\frac{(-u)^i}{(q^i-1)\cdots(q-1)}$$

The goal of our second application of the algorithms of Section 2 can now be attained.

Theorem 6 *([15])*

1. $P_{GL,n}(k,q) = \frac{1}{|GL(k,q)|}\sum_{i=0}^{n-k}\frac{(-1)^i q^{\binom{i}{2}}}{q^{ki}|GL(i,q)|}$
2. $P_{GL,\infty}(k,q) = \left[\prod_{r=1}^{\infty}(1-\frac{1}{q^r})\right]\frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q})^2\cdots(1-\frac{1}{q^k})^2}$
3. $P_{U,n}(k,q) = \frac{1}{|U(k,q)|}\sum_{i=0}^{n-k}\frac{(-1)^i(-q)^{\binom{i}{2}}}{(-q)^{ki}|U(i,q)|}$
4. $P_{U,\infty}(k,q) = \left[\prod_{r=0}^{\infty}(1+\frac{1}{(-q)^r})\right]\frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q^2})\cdots(1-\frac{1}{q^{2k}})}$

PROOF: In the cycle index for the general linear groups, set $x_{z-1,\lambda}=1$ if λ has k parts and $x_{\phi,\lambda}=0$ otherwise. By Lemma 5, Theorem 5 with $x=1$, and Lemma 7,

$$\begin{aligned}
P_{GL,n}(k,q) &= [u^n]\frac{1}{1-u}\sum_{\lambda:\lambda_1'=k}M_{(u,q)}(\lambda)\\
&= [u^n]\frac{u^k\prod_{r=1}^{\infty}(1-\frac{u}{q^{k+r}})}{(1-u)|GL(k,q)|}\\
&= \frac{1}{|GL(k,q)|}[u^{n-k}]\frac{1}{1-u}\sum_{i=0}^{\infty}\frac{(-1)^i(uq^{-k})^i}{(q^i-1)\cdots(q-1)}\\
&= \frac{1}{|GL(k,q)|}\sum_{i=0}^{n-k}\frac{(-1)^i q^{-ki}}{(q^i-1)\cdots(q-1)}.
\end{aligned}$$

For the second part of the theorem use Lemma 6 and Theorem 5 with $x=1, u=1$ to conclude that

$$\begin{aligned}
P_{GL,\infty}(k,q) &= lim_{n\to\infty}[u^n]\frac{1}{1-u}\sum_{\lambda:\lambda_1'=k}M_{(u,q)}(\lambda)\\
&= \sum_{\lambda:\lambda_1'=k}M_{(1,q)}(\lambda)\\
&= \frac{\prod_{r=k+1}^{\infty}(1-\frac{1}{q^r})}{|GL(k,q)|}\\
&= \left[\prod_{r=1}^{\infty}(1-\frac{1}{q^r})\right]\frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q})^2\cdots(1-\frac{1}{q^k})^2}.
\end{aligned}$$

For the third statement, set $x_{z-1,\lambda}=1$ if λ has k parts and $x_{\phi,\lambda}=0$ otherwise in the cycle index of the unitary groups. As for the general linear groups,

$$
\begin{aligned}
P_{U,n}(k,q) &= [u^n]\frac{1}{1-u}\sum_{\lambda:\lambda_1'=k} M_{(-u,-q)}(\lambda)\\
&= [u^n]\frac{(-u)^k\prod_{r=1}^{\infty}(1-\frac{-u}{(-q)^{k+r}})}{(1-u)|GL(k,-q)|}\\
&= \frac{1}{|U(k,q)|}[u^{n-k}]\frac{1}{1-u}\sum_{i=0}^{\infty}\frac{(-1)^i(-u(-q)^{-k})^i}{((-q)^i-1)\cdots(-q-1)}\\
&= \frac{1}{|U(k,q)|}\sum_{i=0}^{n-k}\frac{(-1)^i(-q)^{\binom{i}{2}}}{(-q)^{ki}|U(i,q)|}.
\end{aligned}
$$

For the fourth statement argue as for the general linear groups to conclude that

$$
\begin{aligned}
P_{U,\infty}(k,q) &= lim_{n\to\infty}[u^n]\frac{1}{1-u}\sum_{\lambda:\lambda_1'=k} M_{(-u,-q)}(\lambda)\\
&= \frac{\prod_{r=k+1}^{\infty}(1+\frac{1}{(-q)^r})}{|U(k,q)|}\\
&= \left[\prod_{r=1}^{\infty}(1-\frac{1}{(-q)^r})\right]\frac{(\frac{1}{q})^{k^2}}{\prod_{s=1}^{k}(1+\frac{1}{(-q)^s})(1-\frac{1}{(-q)^s})}\\
&= \left[\prod_{r=1}^{\infty}(1+\frac{1}{(-q)^r})\right]\frac{(\frac{1}{q})^{k^2}}{(1-\frac{1}{q^2})\cdots(1-\frac{1}{q^{2k}})}.
\end{aligned}
$$

□

Remark: Rudvalis and Shinoda obtained, after many pages of labor, the following analogous factorizations for the symplectic and orthogonal groups:

$$
P_{Sp,\infty}(k,q) = \left[\prod_{r=1}^{\infty}\frac{1}{1+\frac{1}{q^r}}\right]\frac{(\frac{1}{q})^{\frac{k^2+k}{2}}}{(1-\frac{1}{q})\cdots(1-\frac{1}{q^k})}
$$

$$
P_{O,\infty}(k,q) = \left[\prod_{r=0}^{\infty}\frac{1}{1+\frac{1}{q^r}}\right]\frac{(\frac{1}{q})^{\frac{k^2-k}{2}}}{(1-\frac{1}{q})\cdots(1-\frac{1}{q^k})}.
$$

It would be marvellous if there are analogs of the Young Tableau Algorithm for the unipotent conjugacy classes of the symplectic and orthogonal groups. This should lead to a probabilistic interpretation of the products in the Rudvalis/Shinoda formulas. Chapters 5 and 6 of [1] give some preliminary results in this direction.

Application 3: Work of Lusztig on nilpotent matrices of a given rank

This application uses Theorem 5 (which was established probabilistically) to prove results of Lusztig [13]. The two theorems which follow were substantive enough for Lusztig to devote a note toward them. We discovered these results by the arguments below. As in the previous applications, $[u^n]f(u)$ is the coefficient of u^n in the power series $f(u)$.

Theorem 7 *([13]) The number of rank $n-k$ nilpotent $n*n$ matrices is*

$$\frac{|GL(n,q)|}{|GL(k,q)|}\frac{(1-\frac{1}{q^k})\cdots(1-\frac{1}{q^{n-1}})}{q^{n-k}(1-\frac{1}{q})\cdots(1-\frac{1}{q^{n-k}})}.$$

PROOF: Adding the identity shows that it suffices to count unipotent matrices in $GL(n,q)$ with a k dimensional fixed space. In the cycle index for the general linear groups, set $x_{z-1,\lambda}=x^{|\lambda|}$ if λ has k parts and $x_{\phi,\lambda}=0$ otherwise. By Lemma 5 and Theorem 5, the sought number is

$$\begin{aligned}
& |GL(n,q)|[(ux)^n]\frac{1}{1-u}\frac{(ux)^k}{|GL(k,q)|}\frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})} \\
= & \frac{|GL(n,q)|}{|GL(k,q)|}[(ux)^{n-k}]\frac{1}{1-u}\frac{\sum_{i=0}^{\infty}\frac{(-u)^i}{(q^i-1)\cdots(q-1)}}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})} \\
= & \frac{|GL(n,q)|}{|GL(k,q)|}[(ux)^{n-k}]\frac{1}{\prod_{r=1}^{k}(1-\frac{ux}{q^r})} \\
= & \frac{|GL(n,q)|}{|GL(k,q)|}\frac{1}{q^{n-k}}\frac{(1-\frac{1}{q^k})\cdots(1-\frac{1}{q^{n-1}})}{(1-\frac{1}{q})\cdots(1-\frac{1}{q^{n-k}})}.
\end{aligned}$$

The first equality used Lemma 7 and the third equality used Lemma 4. □

Theorem 8 is the corresponding result for the unitary groups. Lusztig actually counted nilpotent matrices, but the statement below is equivalent. This can be seen using the Cayley Transform $M\rightarrow(1-M)(1+M)^{-1}$ between unipotent and nilpotent matrices (page 117 of [9]).

Theorem 8 *([13]) The number of unipotent elements of $U(n,q)$ with a k dimensional fixed space is*

$$\frac{|U(n,q)|}{|U(k,q)|}\frac{(1-\frac{1}{(-q)^k})\cdots(1-\frac{1}{(-q)^{n-1}})}{q^{n-k}(1-\frac{1}{(-q)})\cdots(1-\frac{1}{(-q)^{n-k}})}.$$

PROOF: Arguing as in Theorem 7, the sought number is

$$\begin{aligned}
& |U(n,q)|[(ux)^n]\frac{1}{1-u}\frac{(-ux)^k}{|GL(k,q)|}\frac{\prod_{r=1}^{\infty}(1-\frac{-u}{(-q)^r})}{\prod_{r=1}^{k}(1-\frac{-ux}{(-q)^r})} \\
= & \frac{|U(n,q)|}{|U(k,q)|}[(ux)^{n-k}]\frac{1}{1-u}\frac{\sum_{i=0}^{\infty}\frac{(u)^i}{((-q)^i-1)\cdots(-q-1)}}{\prod_{r=1}^{k}(1-\frac{-ux}{(-q)^r})} \\
= & \frac{|U(n,q)|}{|U(k,q)|}[(ux)^{n-k}]\frac{1}{\prod_{r=1}^{k}(1-\frac{-ux}{(-q)^r})} \\
= & \frac{|U(n,q)|}{|U(k,q)|}\frac{1}{q^{n-k}}\frac{(1-\frac{1}{(-q)^k})\cdots(1-\frac{1}{(-q)^{n-1}})}{(1-\frac{1}{-q})\cdots(1-\frac{1}{(-q)^{n-k}})}.
\end{aligned}$$

□

4 Symmetric Functions

This section places the results of Section 2 in the context of symmetric functions. The Macdonald symmetric functions will be used to define probability measures $P_{x_1,x_2,\cdots,y_1,y_2,\cdots,q,t}$ on the set of all partitions of all natural nubmers. For brevity these measures will be denoted as $P_{x,y,q,t}$. Each specialization of the x, y, q, t variables will give a distinct measure. Algebraic properties of the Macdonald symmetric functions will lead to a probabilistic algorithm for growing partitions according to the measures $P_{x,y,q,t}$.

Admittedly the case of greatest interest seems to be the Hall-Littlewood polynomials, which are a specialization of the Macdonald polynomials. As will emerge this is the case relevant to the measure $M_{(u,q)}$. Nevertheless, we feel that the extension to the Macdonald polynomial case is of interest. First, it is remarkable that the Macdonald polynomials should have any probabilistic structure at all. Second, we obtain algorithms different from the Young Tableau Algorithm for growing partitions according to the measure $M_{(u,q)}$. Third, specializing to the Schur functions leads to a natural q-analog of Plancherel measure.

Background on Macdonald symmetric functions

To begin it is helpful to introduce some additional notation. Given a dot s in the diagram of a partition λ, let $l'_\lambda(s), l_\lambda(s), a_\lambda(s), a'_\lambda(s)$ be the number of dots in the diagram of λ to the north, south, east, and west of s respectively. The subscript λ will sometimes be omitted if the partition λ is clear from context. For instance the diagram

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & s & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & & \end{array}$$

satisfies $l'(s) = l(s) = a'(s) = 1$ and $a(s) = 2$. Let $n(\lambda)$ be the quantity $\sum_{i\geq 1}(i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}$.

A skew-diagram is the set theoretic difference $\lambda - \mu$ of two diagrams λ and μ, where the diagram of λ contains the diagram of μ. A horizontal strip is a skew-diagram with at most one dot in each column. For instance the following diagram is a horizontal strip:

$$\begin{array}{ccccc} & & & & \cdot \\ & & & & \\ & & \cdot & \cdot & \\ \cdot & & & & \end{array}$$

Let $p_n(x) = \sum_i x_i^n$ be the nth power sum symmetric function.

The following notation is less widely known, and is taken from Chapter 6 of Macdonald [14].

1. Given a partition λ and a dot s, set $b_\lambda(s) = 1$ if $s \notin \lambda$. Otherwise set

$$b_\lambda(s) = \frac{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}{1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}}.$$

   Let $b_\lambda(q,t) = \prod_{s\in\lambda} b_\lambda(s)$.

2. Define

$$\phi_{\lambda/\mu}(q,t) = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}$$

where $C_{\lambda/\mu}$ is the union of the columns intersecting $\lambda - \mu$.

3. The skew Macdonald polynomials (in one variable) are defined as

$$P_{\lambda/\mu}(x;q,t) = \frac{b_\mu(q,t)}{b_\lambda(q,t)} \phi_{\lambda/\mu}(q,t) x^{|\lambda|-|\mu|}$$

if $\lambda - \mu$ is a horizontal strip, and 0 otherwise.

4. Let $(x,q)_\infty$ denote $\prod_{i=1}^{\infty}(1 - xq^{i-1})$. Then define $\prod(x,y;q,t)$ by

$$\prod(x,y;q,t) = \prod_{i,j=1}^{\infty} \frac{(tx_i y_j, q)_\infty}{(x_i y_j, q)_\infty}.$$

Let $g_n(y;q,t)$ be the coefficient of x^n in $\prod_j \frac{(txy_j,q)_\infty}{(xy_j,q)_\infty}$.

The Macdonald symmetric functions $P_\lambda(x_i;q,t)$ are a two-parameter family of symmetric functions in the x variables. A precise definition appears in Chapter 6 of [14]. For the present purposes only five properites of the Macdonald functions are needed. It is convenient to name them (the Pieri Formula and Principal Specialization Formula being already named).

1. Measure Identity [14], page 324:

$$\sum_\lambda P_\lambda(x;q,t) P_\lambda(y;q,t) b_\lambda(q,t) = \prod(x,y;q,t).$$

2. Factorization Theorem [14], page 310:

$$\prod(x,y;q,t) = \prod_{n \geq 1} e^{\frac{1}{n}\frac{1-t^n}{1-q^n} p_n(x) p_n(y)}.$$

3. Principal Specialization Formula [14], page 337:

$$P_\lambda(1,t,\cdots,t^{N-1};q,t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{N-l'(s)}}{1 - q^{a(s)} t^{l(s)+1}}.$$

4. Skew Expansion [14], pages 343-7:

$$P_\lambda(x_1,\cdots,x_N;q,t) = \sum_\mu P_\mu(x_1,\cdots,x_{N-1};q,t) P_{\lambda/\mu}(x_N;q,t).$$

5. Pieri Formula [14], page 340:

$$P_\mu(y;q,t)g_r(y;q,t) = \sum_{\substack{\lambda:|\lambda-\mu|=r \\ \lambda-\mu \ horiz.\ strip}} \phi_{\lambda/\mu}(q,t)P_\lambda(y;q,t).$$

The Pieri Formula has its history in algebraic geometry, as a rule for multiplying classes of Schubert varieties in the cohomology ring of Grassmanians.

Defining Measures $P_{x,y,q,t}$ from the Macdonald Symmetric Functions

For the remainder of this paper it will be assumed that x, y, q, t satisfy the following conditions:

1. $0 \le t, q < 1$
2. $x_i, y_i \ge 0$
3. $\sum_{i,j} \frac{x_i y_j}{1-x_i y_j} < \infty$.

Then the following formula defines a probability measure $P_{x,y,q,t}$ on the set of all partitions of all numbers:

$$P_{x,y,q,t}(\lambda) = \frac{P_\lambda(x;q,t)P_\lambda(y;q,t)b_\lambda(q,t)}{\prod(x,y;q,t)}.$$

Lemma 8 *$P_{x,y,q,t}$ is a probability measure.*

PROOF: By the Measure Identity and the fact that there are countably many partitions, it suffices to check that $0 \le P_{x,y,q,t}(\lambda) < \infty$ for all λ. For this it is sufficient to show (again by the Measure Identity) that $P_\lambda(x;q,t), b_\lambda(q,t) \ge 0$ for all λ and that $0 \le \prod(x,y;q,t) < \infty$.

Condition 1 implies that $b_\lambda(q,t) \ge 0$ for all λ. We claim that $x_i \ge 0$ implies that $P_\lambda(x;q,t) \ge 0$. To see this, note that when $P_\lambda(x;q,t)$ is expanded in monomials in the x variables, all coefficients are non-negative. For any particular monomial, this follows by repeated use of the Skew Expansion.

By the Factorization Theorem, showing that $0 \le \prod(x,y;q,t) < \infty$ is equivalent to showing that

$$0 \le \sum_{n \ge 1} \frac{1}{n}\frac{1-t^n}{1-q^n} p_n(x)p_n(y) < \infty.$$

Conditions 1 and 2 imply that this expression is non-negative. To see that it is finite, use Condition 3 as follows:

$$\begin{aligned} \sum_{n\ge1} \frac{1}{n}\frac{1-t^n}{1-q^n} p_n(x)p_n(y) &\le \frac{1}{1-q}\sum_{n\ge1} p_n(x)p_n(y) \\ &= \frac{1}{1-q}\sum_{i,j\ge1}\frac{x_i y_j}{1-x_i y_j} \\ &< \infty. \end{aligned}$$

□

Define truncated measures $P^N_{x,y,q,t}(\lambda)$ to be 0 if λ has more than N parts, and otherwise

$$P^N_{x,y,q,t}(\lambda) = \frac{P_\lambda(x_1,\cdots,x_N,0,\cdots;q,t)P_\lambda(y;q,t)b_\lambda(q,t)}{\prod(x_1,\cdots,x_N,0,\cdots,y;q,t)}.$$

Let $P^0(x,y,q,t)$ be 1 on the empty partition and 0 elsewhere. Arguing as in Lemma 8 shows that the $P^N_{x,y,q,t}$ are probability measures. It is also clear that $lim_{N\to\infty}P^N_{x,y,q,t} = P_{x,y,q,t}$. We remark that there are other possible definitions of $P^N_{x,y,q,t}$ which converge to $P_{x,y,q,t}$ in the $N\to\infty$ limit (for instance one can truncate both the x and y variables).

A Probabilistic Algorithm for Picking From $P_{x,y,q,t}$

Step 0 Start with λ the empty partition and N (which we call the interval number) equal to 1.

Step 1 Pick an integer n_N so that $n_N = k$ with probability $\prod_j \frac{(x_N y_j;q)_\infty}{(tx_N y_j;q)_\infty} g_k(y;q,t)x_N^k$. (These probabilities sum to 1 by the definition of g_k).

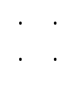
Step 2 Let Λ be a partition containing λ such that the difference $\Lambda - \lambda$ is a horizontal strip of size n_N. There are at most a finite number of such Λ. Change λ to Λ with probability:

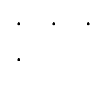
$$\frac{\phi_{\Lambda/\lambda}(q,t)}{g_{n_N}(y;q,t)}\frac{P_\Lambda(y;q,t)}{P_\lambda(y;q,t)}$$

(These probabilities sum to 1 by the Pieri Formula). Then set $N = N+1$ and go to Step 1.

As an example of the algorithm, suppose one is at Step 1 with $N = 3$ and the partition λ:

One then picks n_3 according to the rule in Step 1. Suppose that $n_3 = 2$. One thus adds a horizontal strip of size 2 to λ, giving Λ equal to one the following four partitions with probability given by the rule in Step 2:





One then sets $N = 4$ and returns to Step 1.

Lemma 9 *The algorithm terminates with probability 1.*

PROOF: Recall the Borel-Cantelli lemma, which says that if A_N are events with probability $P(A_N)$ and $\sum_N P(A_N) < \infty$, then with probability 1 only finitely many A_N occur. Let A_N be the event that at least one box is added to the partition during interval N. To prove the lemma it is sufficient to show that only finitely many A_N occur.

The Factorization Theorem implies that $g_0 = 1$. Again using the Factorization Theorem and the fact that $1 - e^{-x} \leq x$ for $x \geq 0$ shows that

$$\begin{aligned}
\sum_{N\geq 1} P(A_N) &= \sum_{N\geq 1}[1 - (\prod_j \frac{(x_N y_j; q)_\infty}{(tx_N y_j; q)_\infty}) g_0] \\
&= \sum_{N\geq 1}[1 - e^{-\sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n}(x_N)^n p_n(y)}] \\
&\leq \sum_{N\geq 1}\sum_{n\geq 1}[\frac{1}{n}\frac{1-t^n}{1-q^n}(x_N)^n p_n(y)] \\
&= \sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n} p_n(x) p_n(y) \\
&\leq \frac{1}{1-q}\sum_{n\geq 1} p_n(x) p_n(y) \\
&= \frac{1}{1-q}\sum_{i,j\geq 1}\frac{x_i y_j}{1 - x_i y_j} \\
&< \infty.
\end{aligned}$$

□

Theorem 9 proves that the algorithm picks from the measure $P_{x,y,q,t}$. Since q and t are fixed, the notation in the proof of Theorem 9 will be abbreviated by omitting the explicit dependence on these variables.

Theorem 9 *The chance that the algorithm yields the partition λ at the end of interval N is $P^N_{x,y,q,t}(\lambda)$. Consequently, the algorithm picks from $P_{x,y,q,t}$.*

PROOF: Since the algorithm proceeds by adding horizontal strips, the partition produced at the end of interval N has at most N parts. The base case $N = 0$ is clear since the algorithm starts with the empty partition and $P^0_{x,y,q,t}$ is 1 on the empty partition and 0 elsewhere. For the induction step, the Skew Expansion gives:

$$\begin{aligned}
P^N_{x,y,q,t}(\Lambda) &= [\prod_{i=1}^N \prod_j \frac{(x_i y_j, q)_\infty}{(tx_i y_j, q)_\infty}] P_\Lambda(x_1, \cdots, x_N) P_\Lambda(y) b_\Lambda \\
&= [\prod_{i=1}^N \prod_j \frac{(x_i y_j, q)_\infty}{(tx_i y_j, q)_\infty}] P_\Lambda(y) b_\Lambda \sum_{\lambda\subset\Lambda} P_\lambda(x_1, \cdots, x_{N-1}) P_{\Lambda/\lambda}(x_N) \\
&= [\prod_{i=1}^N \prod_j \frac{(x_i y_j, q)_\infty}{(tx_i y_j, q)_\infty}] P_\Lambda(y) b_\Lambda \sum_{\substack{\lambda\subset\Lambda \\ \Lambda-\lambda \text{ horiz. strip}}} P_\lambda(x_1, \cdots, x_{N-1}) x_N^{|\Lambda|-|\lambda|} \frac{b_\lambda}{b_\Lambda} \phi_{\Lambda/\lambda}
\end{aligned}$$

$$= \sum_{\substack{\lambda \subset \Lambda \\ \Lambda-\lambda \ horiz.\ strip}} [(\prod_{i=1}^{N-1} \prod_j \frac{(x_i y_j, q)_\infty}{(tx_i y_j, q)_\infty}) P_\lambda(x_1, \cdots, x_{N-1}) P_\lambda(y) b_\lambda]$$
$$[\prod_j \frac{(x_N y_j, q)_\infty}{(tx_N y_j, q)_\infty} g_{|\Lambda|-|\lambda|}(y) x_N^{|\Lambda|-|\lambda|}][\frac{\phi_{\Lambda/\lambda}}{g_{|\Lambda|-|\lambda|}(y)} \frac{P_\Lambda(y)}{P_\lambda(y)}]$$
$$= \sum_{\substack{\lambda \subset \Lambda \\ \Lambda-\lambda \ horiz.\ strip}} [P_{x,y,q,t}^{N-1}(\lambda)][\prod_j \frac{(x_N y_j, q)_\infty}{(tx_N y_j, q)_\infty} g_{|\Lambda|-|\lambda|}(y) x_N^{|\Lambda|-|\lambda|}]$$
$$[\frac{\phi_{\Lambda/\lambda}}{g_{|\Lambda|-|\lambda|}(y)} \frac{P_\Lambda(y)}{P_\lambda(y)}].$$

Probabilistically, this equality says that the chance that the algorithm gives Λ at the end of interval N is equal to the sum over all λ such that Λ/λ is a horizontal strip of the chance that the algorithm gives λ at the end of interval $N-1$ and that λ then grows to Λ in interval N. $\Box$

Example 1: Hall-Littlewood Polynomials

In this example the measure $P_{x,y,q,t}$ is specialized to $y^i = t^{i-1}, q = 0$. Then we set $t = \frac{1}{q}$, where this q is the size of a finite field. Theorem 10 shows that this case corresponds to the measures $M_{(u,q)}$.

Theorem 10 $M_{(u,q)} = P_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}$.

PROOF: We prove that stronger assertion that $P^N(\lambda)$ as defined in the proof of Theorem 1 of Section 2 is equal to $P^N_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}(\lambda)$ for all λ. This assertion will follow by showing that $P^N_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}(\lambda) = 0$ if λ has more than N parts, and that otherwise

$$P^N_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}(\lambda) = \frac{u^{|\lambda|} (\frac{u}{q})_N (\frac{1}{q})_N}{(\frac{1}{q})_{N-\lambda_1'}} \prod_{i \geq 1} \frac{1}{q^{(\lambda_i')^2} (\frac{1}{q})_{m_i(\lambda)}}.$$

The vanishing of $P^N_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}$ when λ has more than N parts is clear by defintion. If λ has at most N parts, then the Principal Specialization Formula yields that:

$$\begin{aligned}
P^N_{\frac{u}{q^i}, \frac{1}{q^{i-1}}, 0, \frac{1}{q}}(\lambda) &= \frac{P_\lambda(\frac{u}{q}, \cdots, \frac{u}{q^N}, 0, \cdots; 0, \frac{1}{q}) P_\lambda(\frac{1}{q^{i-1}}; 0, \frac{1}{q}) b_\lambda(0,t)}{\prod(\frac{u}{q}, \cdots, \frac{u}{q^N}, 0, \cdots, \frac{1}{q^{i-1}}; 0, \frac{1}{q})} \\
&= [\prod_{i=1}^N (1 - \frac{u}{q^i})] P_\lambda(\frac{u}{q}, \cdots, \frac{u}{q^N}, 0, \cdots; 0, \frac{1}{q}) P_\lambda(\frac{1}{q^{i-1}}; 0, \frac{1}{q}) b_\lambda(0,t) \\
&= \frac{\prod_{i=1}^N (1 - \frac{u}{q^i})(1 - \frac{1}{q^i})}{\prod_{i=1}^{N-\lambda'_1} (1 - \frac{1}{q^i})} \frac{P_\lambda(\frac{u}{q}, \cdots, \frac{u}{q^N}, 0, \cdots; 0, \frac{1}{q})}{q^{n(\lambda)}} \\
&= \frac{u^{|\lambda|} (\frac{u}{q})_N (\frac{1}{q})_N}{(\frac{1}{q})_{N-\lambda_1'}} \frac{P_\lambda(\frac{1}{q}, \cdots, \frac{1}{q^N}, 0, \cdots; 0, \frac{1}{q})}{q^{n(\lambda)}} \\
&= \frac{u^{|\lambda|} (\frac{u}{q})_N (\frac{1}{q})_N}{(\frac{1}{q})_{N-\lambda_1'}} \frac{1}{q^{|\lambda|+2n(\lambda)}} \prod_{s \in \lambda : a(s)=0} \frac{1}{1 - \frac{1}{q^{l(s)+1}}}
\end{aligned}$$

$$= \quad \frac{u^{|\lambda|}(\frac{u}{q})_N (\frac{1}{q})_N}{(\frac{1}{q})_{N-\lambda'_1}} \prod_{i \geq 1} \frac{1}{q^{(\lambda'_i)^2} (\frac{1}{q})_{m_i(\lambda)}}.$$

The last equality used the fact that $n(\lambda) = \sum_i \binom{\lambda'_i}{2}$. $\square$

Supposing further that $\sum_i x_i < 1$, there is a simplified algorithm (different from the Young Tableau Algorithm) which allows one to grow the partition λ by adding one dot at a time. Using the Borel-Cantelli lemmas it is straightforward to check that this algorithm always halts.

Simplified Algorithm for Picking from $P_{x,t^{i-1},0,t}$

Step 0 Start with λ the empty partition and $N = 1$. Also start with a collection of coins indexed by the natural numbers such that coin i has probability x_i of heads and probability $1 - x_i$ of tails.

Step 1 Flip coin N.

Step 2a If coin N comes up tails, leave λ unchanged, set $N = N + 1$ and go to Step 1.

Step 2b If coin N comes up heads, let j be the number of the last column of λ whose size was increased during a toss of coin N (on the first toss of coin N which comes up heads, set $j = 0$). Pick an integer $S > j$ according to the rule that $S = j + 1$ with probability $t^{\lambda'_{j+1}}$ and $S = s > j + 1$ with probability $t^{\lambda'_s} - t^{\lambda'_{s-1}}$ otherwise. Then increase the size of column S of λ by 1 and go to Step 1.

For example, suppose one is at Step 1 with λ equal to the following partition:

$$\begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

Suppose also that $N = 4$ and that coin 4 had already come up heads once, at which time one added to column 1, giving λ. Now one flips coin 4 again and get heads, going to Step 2b. One has that $j = 1$. Thus one adds a dot to column 1 with probability 0, to column 2 with probability t^2, to column 3 with probability $t - t^2$, to column 4 with probability 0, and to column 5 with probability $1 - t$. One then returns to Step 1. Note that the dots added during the tosses of a given coin form a horizontal strip.

Theorem 11 *The simplified algorithm for picking from $P_{x,t^{i-1},0,t}$ refines the general algorithm.*

PROOF: Let interval N denote the period between the first and last tosses of coin N. To prove the theorem, it will be shown that the two algorithms add horizontal strips in the same way during interval N.

For this observe that the size of the strips added in interval N is the same for the two algorithms. Since $q = 0$, the integer n_N in Step 1 of the general algorithm is equal to k with probability $(1 - x_N)x_N^k$. This is equal to the chance of k heads of coin N in the simplified algorithm.

Given that a strip of size k is added during interval N, the general algorithm then increases λ to Λ with probability

$$\frac{\phi_{\Lambda/\lambda}(0,t)}{g_k(t^{i-1};0,t)} \frac{P_\Lambda(1,t,t^2,\cdots;0,t)}{P_\lambda(1,t,t^2,\cdots;0,t)}.$$

This probability can be simplified. By Lemma 7 of Section 3, $g_k(t^{i-1};0,t) = 1$. The definition of $\phi_{\Lambda/\lambda}(0,t)$ and the Principal Specialization Formula show that the probability can be rewritten as

$$\left(\prod_{s \in C_{\Lambda/\lambda}} \frac{b_\Lambda(s)}{b_\lambda(s)}\right) \frac{t^{n(\Lambda)} \prod_{s \in \Lambda} \frac{1}{1-0^{a_\Lambda(s)} t^{l_\Lambda(s)+1}}}{t^{n(\lambda)} \prod_{s \in \lambda} \frac{1}{1-0^{a_\lambda(s)} t^{l_\lambda(s)+1}}}$$

where $0^0 = 1$. Let A be the set of column numbers $a > 1$ such that $\Lambda - \lambda$ intersects column a but not column $a-1$. Let A' be the set of column numbers a such that either $a = 1$ or $a > 1$ and $\Lambda - \lambda$ intersects both columns a and $a-1$. Most of the terms in the above expression cancel, giving

$$\frac{t^{n(\Lambda)}}{t^{n(\lambda)}} \prod_{a \in A} (1 - t^{\lambda'_{a-1} - \lambda'_a}) = \prod_{a \in A'} t^{\lambda'_a} \prod_{a \in A} (t^{\lambda'_a} - t^{\lambda'_{a-1}}).$$

It is easily seen that the simplified algorithm can go from λ to Λ in exactly 1 way, and that this also happens with probability equal to

$$\prod_{a \in A'} t^{\lambda'_a} \prod_{a \in A} (t^{\lambda'_a} - t^{\lambda'_{a-1}}).$$

□

Example 2: A q-analog of the Plancherel Measure of the Symmetric Group

To begin, let us recall Plancherel measure, a measure on the partitions λ of size n. Letting $h(s) = a(s) + l(s) + 1$ be the hook-length of $s \in \lambda$, Plancherel measure assigns to λ the probability $\frac{n!}{\prod_{s \in \lambda} h(s)^2}$. Plancherel measure has been the object of serious study, and algorithms have been given for growing partitions according to it [6], [11] [19], [20]. Particularly interesting is [11] which relates Plancherel measure with the Markov moment problem and the asymptotics of the separation of roots of the Hermite polynomials.

One property of Plancherel measure is its connection with the representation theory of the symmetric group: the irreducible representation of S_n parameterized by λ has dimension $\frac{n!}{\prod_{s \in \lambda} h(s)}$. A second property of Plancherel measure is that it can be defined by means of the Robinson-Schensted correspondence. Recall that Robinson and Schensted found a bijection from the symmetric group to the set of pairs (P,Q) of standard Young tableau of the same shape [16]. Let $\lambda(\pi)$ be the shape associated to π under the Robinson-Schensted correspondence. Then $\lambda(\pi)$ has Plancherel measure if π is chosen uniformly from the symmetric group. This follows from the fact that the dimension of the irreducible representation of S_n corresponding to the partition λ is the number of standard tableaux of shape λ.

A q-analog of Plancherel measure can be defined by renormalizing a certain specialization of $P_{x,y,q,t}$ to live on partitions of size n. More precisely, we define

$$P_{q,n}(\lambda) = \begin{cases} \frac{P_{\frac{1}{q^i}, \frac{1}{q^{i-1}}, \frac{1}{q}, \frac{1}{q}}(\lambda)}{\sum_{\lambda : |\lambda| = n} P_{\frac{1}{q^i}, \frac{1}{q^{i-1}}, \frac{1}{q}, \frac{1}{q}}(\lambda)} & \text{if } |\lambda| = n \\ 0 & \text{if } |\lambda| \neq n. \end{cases}$$

Propositions 1 and 2 show that this q-analog has natural properties, in direct analogy with the corresponding properties for the Plancherel measure of the symmetric group. We use the notation that $[i] = q^{i-1} + \cdots + q + 1$, the q-analog of the number i.

Proposition 1 *$P_{q,n}(\lambda)$ is proportional to the square of the degree of the unipotent representation of $GL(n,q)$ parameterized by λ'.*

PROOF: ¿From the Principal Specialization Formula one concludes that

$$P_{\frac{1}{q^i},\frac{1}{q^{i-1}},\frac{1}{q},\frac{1}{q}}(\lambda) = [\prod_{r=1}^{\infty}\prod_{t=0}^{\infty}(1-\frac{1}{q^{r+t}})]\frac{\frac{q^{|\lambda|^2-|\lambda|-2n(\lambda)}[(\frac{1}{q})_{|\lambda|}]^2}{\prod_{s\in\lambda}(1-\frac{1}{q^{h(s)}})^2}}{q^{|\lambda|^2}(1-\frac{1}{q})^2\cdots(1-\frac{1}{q^{|\lambda|}})^2}.$$

Thus $P_{q,n}(\lambda)$ is proportional to

$$\begin{aligned}
\frac{q^{|\lambda|^2-|\lambda|-2n(\lambda)}[(\frac{1}{q})_{|\lambda|}]^2}{\prod_{s\in\lambda}(1-\frac{1}{q^{h(s)}})^2} &= q^{|\lambda|^2-|\lambda|-2n(\lambda)}\frac{q^{2\sum_{s\in\lambda}h(s)}}{\prod_{s\in\lambda}(q^{h(s)}-1)^2}\frac{\prod_{i=1}^{|\lambda|}(q^i-1)^2}{q^{|\lambda|^2+|\lambda|}}\\
&= q^{2\sum_{s\in\lambda}h(s)-2|\lambda|-2n(\lambda)}(\frac{[|\lambda|]!}{\prod_{s\in\lambda}[h(s)]})^2\\
&= q^{2n(\lambda')}(\frac{[|\lambda'|]!}{\prod_{s\in\lambda'}[h(s)]})^2.
\end{aligned}$$

The final equality used the elementary fact that $\sum_{s\in\lambda}h(s) = n(\lambda)+n(\lambda')+|\lambda|$.

The proposition follows from the fact from Chapter 4 of [14] that the degree of the unipotent representation of $GL(n,q)$ corresponding to λ' is

$$q^{n(\lambda')}\frac{[|\lambda'|]!}{\prod_{s\in\lambda'}[h(s)]}.$$

□

Recall that the major index of a permutation $\pi\in S_n$ is defined by

$$maj(\pi) = \sum_{\substack{i:1\leq i\leq n-1\\ \pi(i)>\pi(i+1)}} i.$$

Proposition 2 *Choose $\pi\in S_n$ with probability proportional to $q^{maj(\pi)+maj(\pi^{-1})}$. Then $\lambda(\pi)'$, the transpose of the partition associated to π through the Robinson-Schensted correspondence, obeys our q-analog of Plancherel measure.*

PROOF: Define the major index of a standard Young tableau as the sum of the entries i such that $i+1$ is in a row below that of i. Reasoning similar to that of page 243 of [14] shows that

$$q^{n(\lambda')}\frac{[|\lambda'|]!}{\prod_{s\in\lambda'}[h(s)]} = \sum_{T\in SYT(\lambda')} q^{maj(T)},$$

where the sum is over all standard Young tableaux of shape λ'.

¿From the definition of the Robinson-Schensted correspondence (pages 97-101 of [16]), one sees that if π corresponds to the pair (P,Q), then $maj(\pi) = maj(Q)$. It is also known (Theorem 3.86 of [16]) that if π corresponds to the pair (P,Q), then π^{-1} corresponds to the pair (Q,P).

The proposition follows because $P_{q,n}(\lambda)$ is proportional to:

$$\begin{aligned}
(q^{n(\lambda')}\frac{[|\lambda'|]!}{\prod_{s\in\lambda'}[h(s)]})^2 &= [\sum_{T\in SYT(\lambda')} q^{maj(T)}]^2 \\
&= \sum_{(P,Q)\in\{SYT(\lambda')\times SYT(\lambda')\}} q^{maj(P)}q^{maj(Q)} \\
&= \sum_{\pi\in S_n:\lambda(\pi)=\lambda'} q^{maj(\pi)+maj(\pi^{-1})}.
\end{aligned}$$

□

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