# Counting Semisimple Orbits of Finite Lie Algebras by Genus

By Jason Fulman

Dartmouth College

Department of Mathematics

Running Head: Counting Semisimple Orbits

Mailing Address:

Jason Fulman

Dartmouth College

Department of Mathematics

6188 Bradley Hall

Hanover, NH 03755

email:fulman@dartmouth.edu

**Abstract**

The adjoint action of a finite group of Lie type on its Lie algebra is studied. A simple formula is conjectured for the number of semisimple orbits of a given split genus. This conjecture is proved for type $A$, and partial results are obtained for other types. For type $A$ a probabilistic interpretation is given in terms of card shuffling.

# 1 Introduction

Let $G$ be a reductive, connected, simply connected group of Lie type defined over an algebraically closed field of characteristic $p$. Let $F$ denote a Frobenius map and $G^F$ the corresponding finite group of Lie type. Suppose also that $G^F$ is $F$-split. Two semisimple elements $x, y \in G^F$ are said to be of the same genus if their centralizers $C_G(x), C_G(y)$ are conjugate by an element of $G^F$. It is well known in the theory of finite groups of Lie type that character values on semisimple conjugacy classes of the same genus behave in a unified way.

Deriziotis [5] showed that a genus of semisimple elements of $G^F$ corresponds to a pair $(J, [w])$, where $\emptyset \subseteq J \subset \tilde{\Delta}, J \neq \tilde{\Delta}$ is a proper subset of the vertex set $\tilde{\Delta}$ of the extended Dynkin diagram up to equivalence under the action of $W$, and $[w]$ is a conjugacy class representative of the normalizer quotient $N_W(W_J)/W_J$. A centralizer corresponding to the data $(J, [w])$ can be obtained by twisting by $w$ the group generated by a maximal torus $T$ and the root groups $U_{\pm\alpha}$ for $\alpha \in J$.

Many authors ([4], [8], [9], [10], [16]) have considered the problem of counting semisimple conjugacy classes of $G^F$ according to genus. As emerges from their work, the number of semisimple classes belonging to the genus $(J, [w])$ is equal to $f(J, [w])/|C_{N_W(W_J)/W_J}(w)|$ where $f(J, [w])$ is the number of $t$ in a maximal torus $T$ of $G$ such that $w \cdot F(t) = t$ and the subgroup of $W$ fixing $t$ is $W_J$. Determining $f(J, [w])$ explicitly is an elaborate computation involving Moebius inversion on a collection of closed subsystems of the root system.

Let $g$ be the Lie algebra of $G$. Much less seems to be known about the semisimple orbits of the adjoint action of $G^F$ on $g^F$. Letting $r$ denote the rank of $G$, it is known from [12] that the number of such orbits is equal to $q^r$. By a result of Steinberg [18], the number of semisimple conjugacy classes of $G^F$ is also equal to $q^r$, though no correspondence between these sets is known.

Two semisimple elements $x, y \in g^F$ are said to be in the same genus if $C_G(x), C_G(y)$ are conjugate by an element of $G^F$. Experimentation with small examples such as $SL(3,5)$ suggests that there is not an obvious relation between this decomposition of semisimple orbits according to genus and the decomposition of semisimple conjugacy classes of $G^F$ according to genus.

To parametrize the genera, a semisimple element $x \in g^F$ is said to be in the genus $(J, [w])$ where $\emptyset \subseteq J \subset \tilde{\Delta}, J \neq \tilde{\Delta}$ if $C_G(x)$ is conjugate to the group obtained by twisting by $w$ the group generated by a maximal torus $T$ and the root groups $U_{\pm\alpha}$ for $\alpha \in J$.

Lehrer [12] obtained some results concerning this parametrization. In the case where $p$ is a prime which is good and regular (these notions are defined in Section 2) he obtained formulae for the total number of split orbits (i.e. $[w] = [id]$) and the total number of regular orbits (i.e. $J = \emptyset$).

The main conjecture of this paper is a formula for the number of orbits in the genus $(J, [id])$ for any $J$. This formula has a different flavor from Lehrer's formulae and counts solutions to equations which arose in a geometric setting in Sommers' work on representations of the affine Weyl group on sets of affine flags [17]. Section 2 states our conjecture, proves it for special cases such as type $A$, and shows that it is consistent with Lehrer's count of split orbits. Section 3 gives a probabilistic interpretation for type $A$ involving the theory of card shuffling. This connection is likely not as ad-hoc as it seems, given the companion papers [6],[7] defining card shuffling for all finite Coxeter groups and relating it to the semisimple orbits of $G^F$ on $g^F$.

# 2 Main Results

To state the main conjecture of this paper, some further notation is necessary. Let $\Phi$ be an irreducible root system of rank $r$ which spans the inner product space $V$. The coroots $\check{\Phi}$ are the elements of $V$ defined as $2\alpha/ < \alpha, \alpha >$ where $\alpha \in \Phi$. Let $L$ be the lattice in $V$ generated by $\check{\Phi}$ and set

$$\hat{L} = \{v \in V| < v, \alpha > \in Z \ for \ all \ \alpha \in \Phi\}.$$

Let $f = [\hat{L} : L]$ be the index $L$ in $\hat{L}$. Let $\Pi = \{\alpha_i\} \subset \Phi^+$ be a set of simple roots contained in a set of positive roots and let $\theta$ be the highest root in $\Phi^+$. For convenience set $\alpha_0 = -\theta$. Let $\tilde{\Pi} = \Pi \cup \{\alpha_0\}$. Define coefficients $c_\alpha$ of $\theta$ with respect to $\tilde{\Pi}$ by the equations $\sum_{\alpha \in \tilde{\Pi}} c_\alpha \alpha = 0$ and $c_{\alpha_0} = 1$.

As is standard in the theory of finite groups of Lie type, define a prime $p$ to be bad if it divides the coefficient of some root $\alpha$ when expressed as a combination of simple roots. Following [12], define a prime $p$ to be regular if the lattice of hyperplane intersections corresponding to $\Phi$ remains the same upon reduction mod $p$. For example in type $A$ a prime $p$ dividing $n$ is not regular.

For subsets $S_1, S_2$ of $\tilde{\Pi}$, we write $S_1 \sim S_2$ if there is an element $w \in W$ such that $w(S_1) = S_2$. For $S \subset \tilde{\Pi}, S \neq \tilde{\Pi}$ and $J \subset \Pi$, the notation $W_S \sim W_J$ means that $W_S$ is conjugate to the parabolic subgroup $W_J$. (As Eric Sommers explained to us, $W_S$ is conjugate to such a subgroup if and only if all the coefficients $c_\alpha$ are relatively prime where one considers those $\alpha$ not in $S$).

For $\emptyset \subseteq S \subset \tilde{\Pi}, S \neq \tilde{\Pi}$, define as in Sommers [17] $p(S,t)$ to be the number of solutions $\mathbf{y}$ in strictly positive integers to the equation

$$\sum_{\alpha \in \tilde{\Pi} - S} c_\alpha y_\alpha = t.$$

With these preliminaries in hand, the main conjecture of this paper can be stated.

**Conjecture 1:** Let $G$ be a reductive, connected, simply connected group of Lie type which is $F$-split where $F$ denotes a Frobenius automorphism of $G$. Suppose that the corresponding prime $p$ is good and regular. Then the number of semisimple orbits of $G^F$ on $g^F$ of genus $(J, [id])$ is equal to

$$\frac{\sum_{S \sim J} p(S,q)}{f}.$$

Remark: Sommers [17] studies the quantity $\sum_{S \sim J} p(S,t)$. He shows that it can be reexpressed in either of the following two ways, both of which will be of use to this paper.

1. Let $\hat{U}_t$ be the permutation representation of $W$ on the set $\hat{L}/tL$. Let $P_1, \cdots, P_m$ be representatives of the conjugacy classes of parabolic subgroups of $W$. Then

$$\hat{U}_t = \oplus_{i=1}^m [\sum_{S : W_S \sim P_i} p(S,t)] Ind_{P_i}^W (1).$$

2. Let $A$ be a set of hyperplanes in $V = R^n$ such that $\cap_{H \in A} H = 0$. Let $L = L(A)$ be the set of intersections of these hyperplanes, where we consider $V \in L$. Partially order $L$ by reverse inclusion and define a Moebius function $\mu$ on $L$ by: $\mu(X,X) = 1$ and $\sum_{X \leq Z \leq Y} \mu(Z,Y) = 0$ if $X < Y$ and $\mu(X,Y) = 0$ otherwise. The characteristic polynomial of $L$ is then

$$\chi(L,t) = \sum_{X \in L} \mu(V,X) t^{dim(X)}.$$

For $P$ a parabolic subgroup of $W$, let $X \in L(A)$ be the fixed point set of $P$ on $V$. Define the lattice $L^X$ to be the sublattice of $L$ whose elements are $\{X \cap H | H \in A \ and \ (A-H) \cap X \neq \emptyset\}$. Let $N_W(P)$ be the normalizer in $W$ of $P$. Then

$$\sum_{S:W_S\sim W_J} p(S,t) = \frac{f}{[N_W(W_J):W_J]}\chi(L^X,t).$$

The first piece of evidence for Conjecture 1 is Theorem 1, which shows consistency with Lehrer's result [12] that under the hypotheses of Conjecture 1, the total number of split, semisimple orbits of $G^F$ on $g^F$ is equal to $\frac{\prod_i (q+m_i)}{|W|}$, where the $m_i$ are the exponents of $W$.

**Theorem 1**

$$\frac{1}{f}\sum_{\substack{S\subset\tilde{\Pi}\\ S\neq\tilde{\Pi}}} p(S,q) = \frac{\prod_i (q+m_i)}{|W|}$$

Proof: The left hand side is equal to $\frac{1}{f}$ times the number of solutions in non-negative integers of the equation

$$\sum_{\alpha\in\tilde{\Pi}} c_\alpha y_\alpha = q.$$

Sommers [17] shows that each such solution corresponds to an orbit of $W$ on $\hat{L}/qL$. By Proposition 3.9 of [17], the number of fixed points of $w$ on $\hat{L}/qL$ is equal to $fq^{dim(fix(w))}$, where $dim(fix(w))$ is the dimension of the fixed space of $w$ in its natural action on $V$. Burnside's Lemma states that the number of orbits of a finite group $G$ on a finite set $S$ is equal to

$$\frac{1}{|G|}\sum_{g\in G} Fix(g),$$

where $Fix(g)$ is the number of fixed points of $g$ on $S$. Thus

$$\frac{1}{f}\sum_{\substack{S\subset\tilde{\Pi}\\ S\neq\tilde{\Pi}}} p(S,q) = \frac{1}{|W|}\sum_{w\in W} q^{dim(fix(w))} = \frac{\prod_i (q+m_i)}{|W|}.$$

The second equality is a theorem of Shephard and Todd [15]. □

A second piece of evidence in support of Conjecture 1 is its truth for regular split semisimple orbits (i.e. genus $(\emptyset,[id])$) for $G$ of classical type.

**Theorem 2** *Conjecture 1 predicts that for $q=p^a$ with $p$ regular and good, the number of regular split semisimple orbits of $G^F$ on $\mathrm{g}^{\mathrm{F}}$ is equal to*

$$\frac{\prod_i q-m_i}{|W|}.$$

*This checks for types $A, B, C$ and $D$.*

Proof: Let $P_i$ be a parabolic subgroup of $W$ and $sgn$ be the alternating character of $W$. Note by Frobenius reciprocity that $< Ind_{P_i}^W(1), sgn >_W = 0$ unless $P_i$ is the trivial subgroup, in which case $< Ind_1^W(1), sgn >_W = 1$. Therefore, recalling the first formula in the remark after Conjecture 1,

$$< \hat{U}_q, sgn >_W = \sum_{i=1}^{m} < [\sum_{S:W_S \sim P_i} p(S,q)] Ind_{P_i}^{W}(1), sgn >_W = p(\emptyset, q).$$

However $< \hat{U}_q, sgn >_W$ can be computed directly from its definition. From [17], if the characteristic is good then the number of fixed points of $w$ on $\hat{L}/qL$ is equal to $fq^{dim(fix(w))}$, where $dim(fix(w))$ is the dimension of the fixed space of $w$ in its natural action on $V$. Thus,

$$\begin{aligned} < \hat{U}_q, sgn >_W &= \frac{f}{|W|} \sum_{w \in W} (-1)^{sgn(w)} q^{dim(fix(w))} \\ &= \frac{f(-1)^r}{|W|} \sum_{w \in W} (-q)^{dim(fix(w))} \\ &= \frac{f \prod_i (q - m_i)}{|W|} \end{aligned}$$

where the final equality is a theorem from [15]. Comparing these expressions for $< \hat{U}_q, sgn >_W$ shows that Conjecture 1 predicts that the number of regular split semisimple orbits of $G^F$ on $g^F$ is equal to $\frac{\prod_i q - m_i}{|W|}$.

Let us now check this for the classical types. For type $A$, the split semisimple orbits correspond to monic degree $n$ polynomials which factor into distinct linear factors and have vanishing coefficient of $x^{n-1}$. Since $p$ does not divide $n$ ($p$ is regular), by the argument in Theorem 3 this is $\frac{1}{q}$ times the number of monic degree $n$ polynomials which factor into distinct linear factors, with no constraint on the coefficient of $x^{n-1}$. As elementary counting shows the number of such polynomials to be $\frac{q(q-1)\cdots(q-n+1)}{n!}$, the result follows.

For types $B_n$ and $C_n$, split semisimple orbits correspond to orbits of the hyperoctahedral group of size $2^n n!$ on the maximal toral subalgebra $diag(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$, where $x_i \in F_q$ and an element $w$ of the hyperoctahedral group acts by permuting the $x_i$, possibly with sign changes. The regular orbits are simply those not stabilized by any non-identity $w$. To count these orbits of the hyperoctahedral group, note first that the hypotheses of Conjecture 1 imply that the characteristic is odd, since 2 is a bad prime for types $B$ and $C$. In odd characteristic the only element of $F_q$ equal to its negative is 0. Thus $x_1$ may be any of the $q-1$ non-0 elements of $F_q$, $x_2$ may be any of the $q-3$ elements of $F_q$ such that $x_2 \neq 0, \pm x_1$, and so on. As each such hyperoctahedral orbit has size $|B_n| = |C_n|$, and the exponents for types $B_n, C_n$ are $1, 3, \cdots, 2n-1$, Conjecture 1 checks for these cases.

For type $D_n$, split semisimple orbits correspond to orbits of $D_n$ on the maximal toral subalgebra $diag(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$ where $x_i \in F_q$ and an element $w$ of $D_n$ acts by permuting the $x_i$, possibly with an even number of sign changes. The regular orbits are simply those not stabilized by any non-identity $w$. Here also one may assume odd characteristic, as 2 is a bad prime for type $D$. Let us consider the possible values of $x_1, \cdots, x_n$. The first possibility is that all $x_i \neq 0$. This can happen in $(q-1)(q-3)\cdots(q-(2n-1))$ ways, as $x_2 \neq \pm x_1$, $x_3 \neq \pm x_1, x_2$, and so on. The second possibility is that exactly one $x_i$ is equal to 0. As this $i$ can be chosen in $n$ ways, the second possibility can arise in a total of $n(q-1)(q-3)\cdots(q-(2n-3))$ ways. Thus the total number of possible values of $x_1, \cdots, x_n$ is equal to $(q-1)(q-3)\cdots(q-(2n-3))(q-(n-1))$. As each such orbit of $D_n$ has size $|D_n|$ and the exponents for $D_n$ are $1, 3, \cdots, 2n-3, n-1$, the result follows for type $D_n$. $\square$

As a third piece of evidence for Conjecture 1, we prove it for $W$ of type $A$ (i.e. $SL(n,q)$). Let us make some preliminary remarks about this case. All $p$ are good for type $A$, and it is easy to see that if $p$ divides $n$ then $p$ is not regular for $SL(n,q)$. The split semisimple orbits of $SL(n,q)$ on $sl(n,q)$ correspond to monic degree $n$ polynomials $f(x)$ which factor into linear polynomials and have vanishing coefficient of $x^{n-1}$. The genera are parameterized by partitions $\lambda = (i^{r_i})$, where $r_i$ is the number of irreducible factors of $f(x)$ which occur with multiplicity $i$.

**Theorem 3** *Conjecture 1 holds for type A. Furthermore, the number of split, semisimple orbits of $SL(n,q)$ on $sl(n,q)$ of genus $\lambda = (i^{r_i})$ is equal to*

$$\frac{(q-1)\cdots(q+1-\sum r_i)}{r_1!\cdots r_n!}.$$

PROOF: Note that because $p$ does not divide $n$, for any $c_1, c_2$ there is a bijection between the set of split, monic polynomials with coefficient of $x^{n-1}$ equal to $c_1$ and factorization $\lambda$, and the set of split, monic polynomials with coefficient of $x^{n-1}$ equal to $c_2$ and factorization $\lambda$. This bijection is given by sending $x \rightarrow x+a$ for suitable $a$. An easy combinatorial argument shows that the number of split, monic degree $n$ polynomials (with no restriction on the coefficient of $x^{n-1}$) of factorization $\lambda$ is equal to

$$\frac{q(q-1)\cdots(q+1-\sum r_i)}{r_1!\cdots r_n!}.$$

Dividing by $q$ establishes the desired count.

To show that Conjecture 1 holds for type $A$, take $J$ of type $\lambda$ (i.e. $W \simeq \prod S_i^{r_i}$) in the second formula in the remark after Conjecture 1. One obtains that

$$\frac{\sum_{S\sim J} p(S,t)}{f} = \frac{\chi(L^X,q)}{[N_W(W_J):W_J]} = \frac{(q-1)\cdots(q-\sum r_i+1)}{r_1!\cdots r_n!}$$

where the formula for $\chi(L^X,q)$ used in the second equality is Proposition 2.1 of [14].□

## 3 A Connection with Card Shuffling

The purpose of this section is to give a probabilistic proof using card shuffling of the following identity of Lehrer [12] (which also follows from Theorem 3):

$$\sum_{\lambda=(i^{r_i})\vdash n} \frac{q(q-1)\cdots(q-\sum r_i+1)}{r_1!\cdots r_n!} = \frac{q(q+1)\cdots(q+n-1)}{n!}.$$

Persi Diaconis suggested that a probabilistic interpretation might exist.

Before doing so, we indicate the importance of card shuffling in Lie theory and give some necessary background. For any finite Coxeter group $W$ and $x \neq 0$, the author [6] defined signed probability measures $H_{W,x}$ on $W$ as follows. For $w \in W$, let $D(w)$ be the set of simple positive roots mapped to negative roots by $w$ (also called the descent set of $w$). Let $\lambda$ be an equivalence class of subsets of $\Pi$ under $W$-conjugacy and let $\lambda(K)$ be the equivalence class of $K$. Then define

$$H_{W,x}(w) = \sum_{K\subseteq \Pi - Des(w)} \frac{|W_K|\chi(L^{Fix(W_K)},x)}{x^n|N_W(W_K)||\lambda(K)|}.$$

The measure $H_{S_n,x}$ arises from the theory of card shuffling, as will be described below. It is also (expressed differently) related to the Poincare-Birkhoff-Witt theorem and splittings of Hochschild and cyclic homology [13]. There is an alternate definition of the measures $H_{W,x}$ using the theory of hyperplane arrangements [7]. This definition leads to a concept of riffle shuffling for any real hyperplane arrangement or oriented matroid. The measures $H_{W,x}$ have interesting properties [7], the most remarkable of which are:

1. (Convolution) $\left(\sum_{w \in W} H_{W,x}(w)w\right)\left(\sum_{w \in W} H_{W,y}(w)w\right) = \sum_{w \in W} H_{W,xy}(w)w$..
   This holds at least for $W$ of types $A, B, C, H_3$ and rank 2 groups.

2. (Non-negativity) $H_{W,p}(w) \geq 0$ for all $w \in W$, provided that $W$ is crystallographic and $p$ is a good prime for $W$.

There is one more aspect of $H_{W,x}$ which should be mentioned in the context of this paper. Lehrer [12] defined a map from semisimple orbits of $G^F$ on $g^F$ to conjugacy classes of $W$ as follows. Let $\alpha \in g^F$ be semisimple. $G'$ simply connected implies that $C_G(\alpha)$ is connected. Take $T$ to be an $F$-stable maximal torus in $C_G(\alpha)$ such that $T^F$ is maximally split. All such $T$ are conjugate in $G^F$. As there is a bijection between $G^F$ conjugacy classes of $F$-stable maximal tori in $G$ and conjugacy classes of the Weyl group $W$, one can associate a conjugacy class of $W$ to a semisimple orbit of $G^F$ on $g^F$. It is shown in [6] (for types $A$,$B$) that for $p$ good and regular, if one of the $q^r$ semisimple orbits of $G^F$ on $g^F$ is chosen uniformly at random, then the probability that the associated conjugacy class in $W$ is a given conjugacy class $C$ is equal to the chance that an element of $W$ chosen according to the measure $H_{W,q}$ belongs to $C$.

The measure $H_{S_n,x}$ has an explicit "physical" description when $x$ is a positive integer. This is called inverse $x$-shuffling by Bayer and Diaconis [1]. Start with a deck of $n$ cards held face down. Cards are turned face up and dealt into one of $q$ piles uniformly and independently. Then, after all cards have been dealt, the piles are assembled from left to right and the deck of cards is turned face down. The chance that an inverse $q$-shuffle leads to the permutation $\pi^{-1}$ is equal to the mass the measure $H_{S_n,x}$ places on $\pi$.

**Theorem 4**

$$\sum_{\lambda=(i^{r_i})\vdash n} \frac{q(q-1)\cdots(q-\sum r_i+1)}{r_1!\cdots r_n!} = \frac{q(q+1)\cdots(q+n-1)}{n!}$$

Proof: The right hand side is equal to the number of ways that an inverse $q$-shuffle results in the identity. To see this, note that an inverse $q$-shuffle gives the identity if and only if for all $r < s$, all cards in pile $r$ have lower numbers than all cards in pile $s$. Thus, letting $x_j$ be the number of cards which end up in the $j$th pile, inverse $q$-shuffles resulting in the identity are in $1-1$ correspondence with solutions of the equations $x_1 + \cdots + x_q = n, x_j \geq 0$. Elementary combinatorics shows there to be $\binom{q+n-1}{q-1}$ solutions.

The left hand side also counts the number of ways in which an inverse $q$-shuffle can yield the identity. As before let $x_j$ be the number of cards which end up in the $j$th pile. The term corresponding to $\lambda = (i^{r_i})$ counts the number of solutions to the equation $x_1 + \cdots + x_q = n, x_j \geq 0$ and exactly $r_i$ of the $x$'s equal to $i$. This is because such solutions are counted by the multinomial coefficient $\binom{q}{q-\sum r_i, r_1, \cdots, r_n}$. $\Box$

A natural problem suggested by Theorem 4 is to find a probabilistic interpretation of the concept of genus.

## References


[1] Bayer, D. and Diaconis, P., *Trailing the dovetail shuffle to its lair*, Annals of Appl. Probab., **2** (1992), 294-313.

[2] Bergeron, F., Bergeron, N., Howlett R.B., and Taylor D.E., *A decomposition of the descent algebra of a finite Coxeter group*, J. Algebraic Combinatorics **1** (1992), 23-44.

[3] Carter, R., *Finite groups of Lie type*, John Wiley and Sons, 1985.

[4] Chang, B., *The conjugate classes of Chevalley groups of type ($G_2$)*, J. Algebra **9** (1968), 190-211.

[5] Deriziotis, D.I., *Centralizers of semisimple elements in a Chevalley group*, Comm. Algebra **9** (1981), 1997-2014.

[6] Fulman, J., *Semisimple orbits of Lie algebras and card-shuffling measures on Coxeter groups*, Preprint.

[7] Fulman, J., *Descent algebras, hyperplane arrangements, and shuffling cards*, Preprint.

[8] Fleischmann, P., and Janiszczak, I., *The number of regular semisimple elements for Chevalley groups of classical type*, J. Algebra **155** (1993), 482-528.

[9] Fleischmann, P., and Janiszczak, I., *The semisimple conjugacy classes of finite groups of Lie type $E_6$ and $E_7$*, Comm. Algebra **21** (1993), 93-161.

[10] Fleischmann, P., and Janiszczak, I., *The semisimple conjugacy classes and the generic class number of the finite simple groups of Lie type $E_8$*, Comm. Algebra **22** (1994), 2221-2303.

[11] Humphreys, J., *Conjugacy classes in semisimple algebraic groups*, AMS Mathematical Surverys and Monographs **43** (1995).

[12] Lehrer, G., *Rational tori, semisimple orbits and the topology of hyperplane complements*, Comment. Math. Helvetici **67** (1992), 226-251.

[13] Loday, J.-L., *Cyclic homology*, Springer-Verlag, 1998.

[14] Orlik, P., and Solomon, L., *Coxeter arrangements*, Proc. of Symposia in Pure Math., Vol. 40 (1983) Part 2, 269-291.

[15] Shephard, G.C., and Todd, J.A., *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274-304.

[16] Shoji, T., *The conjugacy classes of Chevalley groups of type* ($F_4$) *over finite fields of characteristic* $p \neq 2$, J. Fac. Sci. Univ. Tokyo **21** (1974), 1-17.

[17] Sommers, E., *A family of affine Weyl group representations*, Transform. Groups **2** (1997), 375-390.

[18] Steinberg, R., *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80** (1968), 1-108.