The Eigenvalue Distribution of a Random Unipotent Matrix in its Representation on Lines

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Proposed Running Head: The Eigenvalue Distribution

Abstract

The eigenvalue distribution of a uniformly chosen random finite unipotent matrix in its permutation action on lines is studied. We obtain bounds for the mean number of eigenvalues lying in a fixed arc of the unit circle and offer an approach toward other asymptotics. For the case of all unipotent matrices, the proof gives a probabilistic interpretation to identities of Macdonald from symmetric function theory. For the case of upper triangular matrices over a finite field, connections between symmetric function theory and a probabilistic growth algorithm of Borodin and Kirillov emerge.

Key words: Random matrix, symmetric functions, Hall-Littlewood polynomial.

1 Introduction

The subject of eigenvalues of random matrices is very rich. The eigenvalue spacings of a complex unitary matrix chosen from Haar measure relate to the spacings between the zeros of the Riemann zeta function ([O], [RS1], [RS2]). For further recent work on random complex unitary matrices, see [DiS], [R], [So], [W]. The references [Dy] and [Me] contain much of interest concerning the eigenvalues of a random matrix chosen from Dyson's orthogonal, unitary, and symplectic circular ensembles, for instance connections with the statistics of nuclear energy levels. The papers [AD],[BaiDeJ],[Ok] and the references contained in them give exciting recent results relating eigenvalue distributions of matrices to statistics of random permutations such as the longest increasing subsequence.

Little work seems to have been done on the eigenvalue statistics of matrices chosen from finite groups. One recent step is Chapter 5 of Wieand's thesis [W]. She studies the permutation eigenvalues of a random element of the symmetric group in its representation on the set $\{1, \cdots, n\}$. This note gives two natural q-analogs of Wieand's work. For the first q-analog, let $\alpha \in GL(n,q)$ be a random unipotent matrix. Letting V be the vector space on which α acts, we consider the eigenvalues of α in the permutation representation of $GL(n,q)$ on the lines of V. Let $X^{\theta}(\alpha)$ be the number of eigenvalues of α lying in a fixed arc $(1, e^{i2\pi\theta}], 0 < \theta < 1$ of the unit circle. Bounds are obtained for the mean of X^{θ} (we suspect that as $n \to \infty$ with q fixed, a normal limit theorem holds). A second q-analog which we analyze is the case when α is a randomly chosen unipotent upper triangular matrix over a finite field. A third interesting q-analog would be taking α uniformly chosen in $GL(n,q)$; however this seems intractable. It would also be of interest to extend Wieand's work to more general representations of the symmetric group, using formulas of Stembridge [Ste].

The main method of this paper is to interpret identities of symmetric function theory in a probabilistic setting. Section 2 gives background and results in this direction. This interaction appears fruitful, and it is shown for instance that a probabilistic algorithm of Borodin and Kirillov describing the Jordan form of a random unipotent upper triangular matrix [Bo],[Ki1] follows from the combinatorics of symmetric functions. This ties in with work on analogous algorithms for the unipotent conjugacy classes of finite classical groups [F1]. The applications to the eigenvalue problems described above appear in Section 3. We remark that quite different computations in symmetric function theory plays the central role in work of Diaconis and Shahshahani [DiS] on the eigenvalues of random complex classical matrices.

2 Symmetric functions

To begin we describe some notation, as on pages 2-5 of [Ma]. Let λ be a partition of a non-negative integer $n = \sum_i \lambda_i$ into non-negative integral parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. The notation $|\lambda| = n$ will mean that λ is a partition of n. Let $m_i(\lambda)$ be the number of parts of λ of size i, and let λ' be the partition dual to λ in the sense that $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Let $n(\lambda)$ be the quantity $\sum_{i \geq 1}(i-1)\lambda_i$. It is also useful to define the diagram associated to λ as the set of points $(i,j) \in Z^2$ such that $1 \leq j \leq \lambda_i$. We use the convention that the row index i increases as one goes downward and the column index j increases as one goes across. So the diagram of the partition (5441) is:

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & & \end{array}$$

Let G_λ be an abelian p-group isomorphic to $\bigoplus_i Cyc(p^{\lambda_i})$. We write $G = \lambda$ if G is an abelian p-group isomorphic to G_λ. Finally, let $(\frac{1}{p})_r = (1-\frac{1}{p})\cdots(1-\frac{1}{p^r})$.

The rest of the paper will treat the case $GL(n,p)$ with p prime as opposed to $GL(n,q)$. This reduction is made only to make the paper more accessible at places, allowing us to use the language of abelian p-groups rather than modules over power series rings. From Chapter 2 of Macdonald [Ma] it is clear that everything works for prime powers.

2.1 Unipotent elements of $GL(n,p)$

It is well known that the unipotent conjugacy classes of $GL(n,p)$ are parametrized by partitions λ of n. A representative of the class λ is given by

$$\begin{pmatrix} M_{\lambda_1} & 0 & 0 & 0 \\ 0 & M_{\lambda_2} & 0 & 0 \\ 0 & 0 & M_{\lambda_3} & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix},$$

where M_i is the $i * i$ matrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Lemmas 1-3 recall elementary facts about unipotent elements in $GL(n,p)$.

Lemma 1 *([Ma] page 181,[SS]) The number of unipotent elements in $GL(n,p)$ with conjugacy class type λ is*

$$\frac{|GL(n,p)|}{p^{\sum(\lambda'_i)^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}}.$$

Chapter 3 of [Ma] defines Hall-Littlewood symmetric functions $P_\lambda(x_1,x_2,\cdots;t)$ which will be used extensively. There is an explicit formula for the Hall-Littlewood polynomials. Let the permutation w act on the x-variables by sending x_i to $x_{w(i)}$. There is also a coordinate-wise action of w on $\lambda = (\lambda_1,\cdots,\lambda_n)$ and S_n^λ is defined as the subgroup of S_n stabilizing λ in this action. For a partition $\lambda = (\lambda_1,\cdots,\lambda_n)$ of length $\leq n$, two formulas for the Hall-Littlewood polynomial restricted to n variables are:

$$
\begin{aligned}
P_\lambda(x_1,\cdots,x_n;t) &= \left[\frac{1}{\prod_{i\geq 0}\prod_{r=1}^{m_i(\lambda)}\frac{1-t^r}{1-t}}\right]\sum_{w\in S_n} w(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}) \\
&= \sum_{w\in S_n/S_n^\lambda} w(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{\lambda_i>\lambda_j}\frac{x_i-tx_j}{x_i-x_j})
\end{aligned}
$$

Lemma 2 *The probability that a unipotent element of $GL(n,p)$ has conjugacy class of type λ is equal to either of*

1. $\frac{p^n(\frac{1}{p})_n}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}}$
2. $\frac{p^n(\frac{1}{p})_n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}{p^{n(\lambda)}}$

PROOF: The first statement follows from Lemma 1 and Steinberg's theorem that $GL(n,p)$ has $p^{n(n-1)}$ unipotent elements. The second statement follows from the first and from elementary manipulations applied to Macdonald's principal specialization formula (page 337 of [Ma]). Full details appear in [F2]. □

One consequence of Lemma 2 is that in the $p\to\infty$ limit, all mass is placed on the partition $\lambda=(n)$. Thus the asymptotics in this paper will focus on the more interesting case of the fixed p, $n\to\infty$ limit.

Lemma 3

$$\sum_{\lambda\vdash n}\frac{1}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}} = \frac{1}{p^n(\frac{1}{p})_n}$$

PROOF: Immediate from Lemma 2. □

Lemmas 4 and 5 relate to the theory of Hall polynomials and Hall-Littlewood symmetric functions [Ma]. Lemma 4, for instance, is the duality property of Hall polynomials.

Lemma 4 *(Page 181 of [Ma]) For all partitions λ,μ,ν,*

$$|\{G_1\subseteq G_\lambda : G_\lambda/G_1=\mu, G_1=\nu\}| = |\{G_1\subseteq G_\lambda : G_\lambda/G_1=\nu, G_1=\mu\}|.$$

Lemma 5 *Let G_λ denote an abelian p-group of type λ, and G_1 a subgroup. Then for all types μ,*

$$\sum_{\lambda\vdash n}\frac{\{|G_1\subseteq G_\lambda : G_1=\mu|\}}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}} = \frac{1}{p^{\sum(\mu_i')^2}\prod_i(\frac{1}{p})_{m_i(\mu)}}\frac{1}{p^{n-|\mu|}(\frac{1}{p})_{n-|\mu|}}.$$

PROOF: Macdonald (page 220 of [Ma]), using Hall-Littlewood symmetric functions, establishes for any partitions μ,ν, the equation:

$$\sum_{\lambda:|\lambda|=|\mu|+|\nu|}\frac{|\{G_1\subseteq G_\lambda : G_\lambda/G_1=\mu, G_1=\nu\}|}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}} = \frac{1}{p^{\sum(\mu_i')^2}\prod_i(\frac{1}{p})_{m_i(\mu)}}\frac{1}{p^{\sum(\nu_i')^2}\prod_i(\frac{1}{p})_{m_i(\nu)}}.$$

Fixing μ, summing the left hand side over all ν of size $n-|\mu|$, and applying Lemma 4 yields

$$\sum_{\lambda}\sum_{\nu}\frac{|\{G_1\subseteq G_\lambda : G_\lambda/G_1=\mu, G_1=\nu\}|}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}} = \sum_{\lambda}\sum_{\nu}\frac{|\{G_1\subseteq G_\lambda : G_\lambda/G_1=\nu, G_1=\mu\}|}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}}$$
$$= \sum_{\lambda}\frac{|\{G_1\subseteq G_\lambda : G_1=\mu\}|}{p^{\sum(\lambda_i')^2}\prod_i(\frac{1}{p})_{m_i(\lambda)}}.$$

Fixing μ, summing the right hand side over all ν of size $n-|\mu|$, and applying Lemma 3 gives that

$$\frac{1}{p^{\sum(\mu_i')^2}\prod_i(\frac{1}{p})_{m_i(\mu)}}\sum_{\nu\vdash n-|\mu|}\frac{1}{p^{\sum(\nu_i')^2}\prod_i(\frac{1}{p})_{m_i(\nu)}} = \frac{1}{p^{\sum(\mu_i')^2}\prod_i(\frac{1}{p})_{m_i(\mu)}}\frac{1}{p^{n-|\mu|}(\frac{1}{p})_{n-|\mu|}},$$

proving the lemma. □

2.2 Upper triangular matrices over a finite field

Let $T(n,p)$ denote the set of upper triangular elements of $GL(n,p)$ with 1's along the main diagonal. From the theory of wild quivers there is a provable sense in which the conjugacy classes of $T(n,p)$ have no simple classification. Nevertheless, as emerges from work of Kirillov [Ki1, Ki2] and Borodin [Bo], it is interesting to study the Jordan form of elements of $T(n,p)$. As with the unipotent conjugacy classes of $GL(n,p)$, the possible Jordan forms correspond to partitions λ of n.

Theorem 1 gives five expressions for the probability that an element of $T(n,p)$ has Jordan form of type λ. As is evident from the proof, most of the hard work at the heart of these formulas has been carried out by others. Nevertheless, at least one of these expressions is useful, and to the best of our knowledge none of these formulas has appeared elsewhere. P_λ will denote the Hall-Littlewood polynomial of the previous subsection. By a standard Young tableau S of size $|S|=n$ is meant an assignment of $\{1,\cdots,n\}$ to the dots of the partition such that each of $\{1,\cdots,n\}$ appears exactly once, and the entries increase along the rows and columns. For instance,

$$\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 & \\ 8 & 9 & & \end{matrix}$$

is a standard Young tableau.

Theorem 1 *The probability that a uniformly chosen element of $T(n,p)$ has Jordan form of type λ is equal to each of the following:*

1. $\frac{(p-1)^n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) fix_\lambda(p)}{p^{n(\lambda)}}$, *where $fix_\lambda(p)$ is the number of complete flags of an n-dimensional vector space over a field of size p which are fixed by a unipotent element u of type λ.*

2. $\frac{(p-1)^n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) Q^\lambda_{(1)^n}(p)}{p^{n(\lambda)}}$, *where $Q^\lambda_{(1)^n}(p)$ is a Green's polynomial as defined on page 247 of [Ma].*

3. $(p-1)^n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})\sum_\mu dim(\chi^\mu)K_{\mu,\lambda}(\frac{1}{p})$, *where μ is a partition of n, $dim(\chi^\mu)$ is the dimension of the irreducible representation of S_n of type μ, and $K_{\mu,\lambda}$ is the Kostka-Foulkes polynomial as on page 239 of [Ma].*

4. $\frac{(p-1)^n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) chain_\lambda(p)}{p^{n(\lambda)}}$, *where* $chain_\lambda(p)$ *is the number of maximal length chains of subgroups in an abelian* p*-group of type* λ.

5. $P_\lambda(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})\sum_S \prod_{j=1}^n (1-\frac{1}{p^{m^*(\Lambda_j)}})$, *where the sum is over all standard Young tableaux of shape* λ, *and* $m^*(\Lambda_j)$ *is the number of parts in the subtableau formed by* $\{1,\cdots,j\}$ *which are equal to the column number of* j.

PROOF: For the first assertion, observe that complete flags correspond to cosets $GL(n,p)/B(n,p)$ where $B(n,p)$ is the subgroup of all invertible upper triangular matrices. Note that $u \in GL(n,p)$ fixes the flag $gB(n,p)$ exactly when $g^{-1}ug \in B(n,p)$. The unipotent elements of $B(n,p)$ are precisely $T(n,p)$. Thus the number of complete flags fixed by u is $\frac{1}{(p-1)^n|T(n,p)|}|\{g : g^{-1}ug \in T(n,p)\}|$. It follows that the sought probability is equal to $(p-1)^n fix_\lambda(p)$ multiplied by the probability that an element of $GL(n,p)$ is unipotent of type λ. The first assertion then follows from Lemma 1.

The second part follows from the first part since by page 187 of [Ma], $Q^\lambda_{(1)^n}(p)$ is the number of complete flags of an n-dimensional vector space over a field of size p which are fixed by a unipotent element of type λ. The third part follows from the second part and a formula for $Q^\lambda_{(1)^n}(p)$ on page 247 of [Ma]. The fourth part follows from the third part and a formula for $\sum_\mu dim(\chi^\mu)K_{\mu,\lambda}(\frac{1}{p})$ in [Kir]. For the fifth assertion, a result on page 197 of [Ma] gives that the number of maximal length chains of subgroups in an abelian p-group of type λ is equal to $\frac{p^{n(\lambda)}}{(1-\frac{1}{p})^n}\sum_S \prod_{j=1}^n(1-\frac{1}{p^{m^*(\Lambda_j)}})$. Observing that for a partition λ of n, $P_\lambda(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) = p^n P_\lambda(\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})$, the result follows. $\square$

As a corollary of Theorem 1, we recover the "Division Algorithm" of Borodin [Bo] and Kirillov [Kir], which gives a probabilistic way of growing partitions a dot at a time such that the chance of getting λ after n steps is equal to the chance that a uniformly chosen element of $T(n,p)$ has Jordan type λ. We include our proof as it uses symmetric functions, which aren't mentioned in the literature on probability in the upper triangular matrices.

We remark that a wonderful application of the division algorithm was found by Borodin [Bo], who proved asymptotic normality theorems for the lengths of the longest parts of the partition corresponding to a random element of $T(n,p)$, and even found the covariance matrix. We give another application in Section 3.2.

Corollary 1 *([Bo],[Kir]) Stopping the following procedure after* n *steps produces a partition distributed as the Jordan form of a random element of* $T(n,p)$*. Starting with the empty partition, at each step transition from a partition* λ *to a partition* Λ *by adding a dot to column* i *chosen according to the rules*

- $i=1$ *with probability* $\frac{1}{p^{\lambda'_1}}$
- $i=j>1$ *with probability* $\frac{1}{p^{\lambda'_j}} - \frac{1}{p^{\lambda'_{j-1}}}$

PROOF: For a standard Young tableau S, let $\Lambda_j(S)$ be the partition formed by the entries $\{1,\cdots,j\}$ of S. It suffices to prove that at step j the division algorithm goes from Λ_{j-1} to Λ_j with probability $\frac{P_{\Lambda_j}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}{P_{\Lambda_{j-1}}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}(1-\frac{1}{p^{m^*(\Lambda_j)}})$, because then the probability that the algorithm gives λ at step $n=|\lambda|$ is

$$\sum_{S:shape(S)=\lambda} \prod_{j=1}^{n} \frac{P_{\Lambda_j}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}{P_{\Lambda_{j-1}}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}(1-\frac{1}{p^{m^*(\Lambda_j)}}) = P_\lambda(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) \sum_S \prod_{j=1}^{n}(1-\frac{1}{p^{m^*(\Lambda_j)}}),$$

as desired from part 5 of Theorem 1. The fact that the division algorithm goes from Λ_{j-1} to Λ_j with probability $\frac{P_{\Lambda_j}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}{P_{\Lambda_{j-1}}(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p})}(1-\frac{1}{p^{m^*(\Lambda_j)}})$ follows, after algebraic manipulations, from Macdonald's principle specialization formula (page 337 of [Ma])

$$P_\lambda(1,\frac{1}{p},\frac{1}{p^2},\frac{1}{p^3},\cdots;\frac{1}{p}) = p^{n+n(\lambda)} \prod_i \frac{1}{p^{\lambda_i'^2}(\frac{1}{p})_{m_i(\lambda)}}.$$

□

The follow-up paper [F3] relates the probability theory of Jordan forms of elements of $T(n,p)$ to potential theory on Bratteli diagrmas, giving a more conceptual proof of Corollary 1. The author has been informed that the boundaries of the branchings in [F3] are homeomorphic to those introduced by [Ke2] and studied by [KOO] as referenced there, and that the branchings are multiplicative. Nevertheless, given the connection with $T(n,p)$, the formulation in [F3] should prove useful.

We remark that Corollary 1 ties in with the algorithms of [F1] for growing random paritistions distributed according to the $n \to \infty$ law of the partition corresponding to the polynomial $z-1$ in the Jordan form of a random element of $GL(n,p)$. The precise relationship is that if each coin in the algorithm on page 585 of [F1] is conditioned to come up heads exactly once, the resulting algorithm is that of Corollary 1.

3 Applications

In this section we return to the problem which motivated this paper: studying the eigenvalues of unipotent matrices in the permutation representation on lines. Lemma 6 describes the cycle structure of the permutation action of a unipotent element α of $GL(n,p)$ on lines in V in terms of the partition parametrizing the conjugacy class of α.

Lemma 6 *Let α be a unipotent element of $GL(n,p)$ with conjugacy class of type λ. Every orbit of the action of α on the set of lines in V has size p^r for some $r \geq 0$. The number of orbits of size p^r is*

$$\begin{array}{ll} \frac{p^{\lambda_1'+\cdots+\lambda_{p^r}'}-p^{\lambda_1'+\cdots+\lambda_{p^{r-1}}'}}{p-1} & \text{if } r \geq 1 \\ \frac{p^{\lambda_1'}-1}{p-1} & \text{if } r = 0. \end{array}$$

PROOF: As discussed at the beginning of Section 2, the matrix α may be assumed to be

$$\begin{pmatrix} M_{\lambda_1} & 0 & 0 & 0 \\ 0 & M_{\lambda_2} & 0 & 0 \\ 0 & 0 & M_{\lambda_3} & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix},$$

where M_i is the $i*i$ matrix with 1's along and right above the diagonal, and 0's elsewhere. Let $E_i = M_i - Id$, where Id is the identity matrix.

From this explicit form all eigenvalues of $\alpha^m, m \geq 0$ are 1. Thus if α^m fixes a line, it fixes it pointwise. Hence the number of lines fixed by α^m is one less than the number of points it fixes, all divided by $p-1$, and we are reduced to studying the action of α of non-zero vectors. It is easily proved that M_i has order p^a, where $p^{a-1} < i \leq p^a$. Hence if $\alpha^m(x_1, \cdots, x_n) = (x_1, \cdots, x_n)$ with some x_i non-zero, and m is the smallest non-negative integer with this property, then m is a power of p. Thus all orbits of α on the lines of V have size p^r for $r \geq 0$.

We next claim that $\alpha^{p^r}, r \geq 0$ fixes a vector

$$(x_1, \cdots, x_{\lambda_1}, x_{\lambda_1+1}, \cdots, x_{\lambda_1+\lambda_2}, x_{\lambda_1+\lambda_2+1}, \cdots, x_{\lambda_1+\lambda_2+\lambda_3}, \cdots, x_n)$$

if and only if

$$x_{\lambda_1+\cdots+\lambda_{i-1}+p^a+1} = x_{\lambda_1+\cdots+\lambda_{i-1}+p^a+2} = \cdots = x_{\lambda_1+\cdots+\lambda_i} = 0 \ for\ i : \lambda_i > p^r.$$

It suffices to prove this claim when λ has one part λ_1 of size n. Observe that the ith coordinate of $\alpha^{p^r}(x_1, \cdots, x_n)$ is $\sum_{j=i}^{n} \binom{p^r}{j-i} x_j$. Thus α^{p^r} fixes the x_i for $i > n - p^r$, but sends all other x_i to $x_i + x_{i+p^r}$. To summarize α^{p^r} fixes $(x_1, \cdots, x_n)$ if and only if $x_{p^r+1} = \cdots = x_n = 0$, as desired.

This explicit description of fixed vectors (hence of fixed lines) of α^{p^a} yields the formula of the lemma for $r \geq 1$, because the number of lines in an orbit of size p^r is the difference between the number of lines fixed by α^{p^r} and the number of lines fixed by $\alpha^{p^{r-1}}$. The formula for the number of lines in an orbit of size 1 follows because there are a total of $\frac{p^n-1}{p-1}$ lines.□

3.1 Unipotent elements of $GL(n,p)$

Let α be a uniformly chosen unipotent element of $GL(n,p)$. Each element of $GL(n,p)$ permutes the lines in V and thus defines a permutation matrix, which has complex eigenvalues. Each size p^r orbit of α on lines gives p^r eigenvalues, with one at each of the p^rth roots of unity. For $\theta \in (0,1)$, define a random variable X^θ by letting $X^\theta(\alpha)$ be the number of eigenvalues of α in the interval $(1, e^{2\pi i\theta}]$ on the unit circle. For $r \geq 1$, define random variables X_r on the unipotent elements of $GL(n,p)$ by

$$X_r(\alpha) = \frac{p^{\lambda_1'(\alpha)+\cdots+\lambda_r'(\alpha)} - p^{\lambda_1'(\alpha)+\cdots+\lambda_{r-1}'(\alpha)}}{p-1}.$$

Clearly $X_r(\alpha) = 0$ if $r > n$. Let $\lfloor y \rfloor$ denote the greatest integer less than y. Lemma 6 implies that

$$X^\theta = X_1 \lfloor \theta \rfloor + \sum_{r \geq 1} \frac{X_{p^{r-1}+1} + \cdots + X_{p^r}}{p^r} \lfloor p^r \theta \rfloor.$$

This relationship (analogous to one used in [W]) will reduce the computation of the mean of X^θ to similar computations for the random variables X_r, which will now be carried out.

Let E_n denote the expected value with respect to the uniform distribution on the unipotent elements of $GL(n,p)$.

Theorem 2 *For $1 \leq r \leq n$,*

$$E_n(X_r) = \frac{p^r(1 - \frac{1}{p^{n-r+1}}) \cdots (1 - \frac{1}{p^n})}{p-1}.$$

Proof: By Lemma 2,

$$E_n(X_r) = \sum_{\lambda \vdash n} \frac{p^n(\frac{1}{p})_n}{p^{\sum (\lambda_i')^2} \prod_i (\frac{1}{p})_{m_i(\lambda)}} \frac{p^{\lambda_1'(\alpha)+\cdots+\lambda_r'(\alpha)} - p^{\lambda_1'(\alpha)+\cdots+\lambda_{r-1}'(\alpha)}}{p-1}.$$

Observe that $\frac{p^{\lambda_1'+\cdots+\lambda_r'} - p^{\lambda_1'+\cdots+\lambda_{r-1}'}}{p^r - p^{r-1}}$ is the number of subgroups of G_λ of type $\nu = (r)$. This is because the total number of elements of order p^r in G_λ is $p^{\lambda_1'+\cdots+\lambda_r'} - p^{\lambda_1'+\cdots+\lambda_{r-1}'}$, and every subgroup of type $\nu = (r)$ has $p^r - p^{r-1}$ generators. Therefore, using Lemma 5,

$$\begin{aligned} E_n(X_r) &= p^n(\frac{1}{p})_n \frac{p^r - p^{r-1}}{p-1} \sum_{\lambda \vdash n} \frac{|\{G_1 \subseteq G_\lambda : G_1 = (r)\}|}{p^{\sum (\lambda_i')^2} \prod_i (\frac{1}{p})_{m_i(\lambda)}} \\ &= \left(p^n(\frac{1}{p})_n \frac{p^r - p^{r-1}}{p-1} \right) \left(\frac{1}{p^r(1-\frac{1}{p})} \frac{1}{p^{n-r}(\frac{1}{p})_{n-r}} \right) \\ &= \frac{p^r(1-\frac{1}{p^{n-r+1}}) \cdots (1-\frac{1}{p^n})}{p-1}. \end{aligned}$$

□

Corollary 2 uses Theorem 2 to bound the mean of X^θ.

Corollary 2 $E_n(X^\theta) = \theta \frac{p^n - 1}{p-1} - O(\frac{p^n}{n})$.

Proof: Let $\{y\} = y - \lfloor y \rfloor$ denote the fractional part of a positive number y. Theorem 2 and the writing of X^θ in terms of the X_r's imply that

$$\begin{aligned} E_n(X^\theta) &= \theta E_n(\sum_{i \geq 1} X_i) - \sum_{r \geq 1} \{p^r \theta\} E_n(\frac{X_{p^{r-1}+1} + \cdots + X_{p^r}}{p^r}) \\ &= \theta \frac{p^n - 1}{p-1} - \sum_{r \geq 1} \{p^r \theta\} E_n(\frac{X_{p^{r-1}+1} + \cdots + X_{p^r}}{p^r}) \\ &\geq \theta \frac{p^n - 1}{p-1} - \sum_{r \geq 1} E_n(\frac{X_{p^{r-1}+1} + \cdots + X_{p^r}}{p^r}) \\ &\geq \theta \frac{p^n - 1}{p-1} - (\sum_{r=1}^{\lfloor log_p(n) \rfloor)} \frac{p^{p^{r-1}+1} + \cdots + p^{p^r}}{(p-1)p^r}) - (\frac{p^{p^{\lfloor log_p(n) \rfloor}+1} + \cdots + p^n}{(p-1)p^{\lfloor log_p(n) \rfloor + 1}}). \end{aligned}$$

We suppose for simplicity that $n \neq p^{p^r} + 1$ for some r (the case $n = p^{p^r} + 1$ is similar). Continuing,

$$E_n(X^\theta) \geq \theta \frac{p^n - 1}{p-1} - (\sum_{r=1}^{\lfloor log_p(n) \rfloor} \frac{p^{p^r+1}}{(p-1)^2 \frac{n}{p^{\lfloor log_p(n) \rfloor - r + 1}}}) - \frac{p^{n+1}}{(p-1)^2 n} = \theta \frac{p^n - 1}{p-1} - O(\frac{p^n}{n}).$$

□

The approach here appears to extend to the computation of higher moments, but the computations are formidable. For example one can show that if $1 \leq r \leq s \leq n$, then

$$E_n(X_r X_s) = \frac{p^{r+s-1}}{p-1} [\frac{p}{p-1}(1 - \frac{1}{p^{n-s-r+1}}) \cdots (1 - \frac{1}{p^n}) + \sum_{a=0}^{r-1} (1 - \frac{1}{p^{n-a-s+1}}) \cdots (1 - \frac{1}{p^n})].$$

3.2 Upper triangular matrices over a finite field

Let α be a uniformly chosen element of $T(n,p)$. Recall that α is unipotent by the definition of $T(n,p)$. Each element of $T(n,p)$ permutes the lines in V and thus defines a permutation matrix, which has complex eigenvalues. Each size p^r orbit of α on lines gives p^r eigenvalues, with one at each of the p^rth roots of unity. For $\theta \in (0,1)$, define a random variable X^θ by letting $X^\theta(\alpha)$ be the number of eigenvalues of α in the interval $(1, e^{2\pi i\theta}]$ on the unit circle. For $r \geq 1$, define random variables X_r on the unipotent elements of $T(n,p)$ by

$$X_r(\alpha) = \frac{p^{\lambda_1'(\alpha)+\cdots+\lambda_r'(\alpha)} - p^{\lambda_1'(\alpha)+\cdots+\lambda_{r-1}'(\alpha)}}{p-1}.$$

Let $\lfloor y \rfloor$ denote the greatest integer less than y. Lemma 6 implies that

$$X^\theta = X_1 \lfloor \theta \rfloor + \sum_{r \geq 1} \frac{X_{p^{r-1}+1} + \cdots + X_{p^r}}{p^r} \lfloor p^r \theta \rfloor.$$

As for the case of $GL(n,p)$ this relationship reduces the computation of the mean of X^θ to similar computations for the random variables X_r.

Let E_n denote the expected value with respect to the uniform distribution on the unipotent elements of $T(n,p)$. Theorem 3 shows that the expected value of X_r is surprisingly simple. As one sees from the case $p=2$, the result is quite different from that of Theorem 2. Using the same technique one can compute higher moments.

Theorem 3 *For $1 \leq r \leq n$,*

$$E_n(X_r) = (p-1)^{r-1} \binom{n}{r}.$$

PROOF: We proceed by joint induction on n and r, the base case $n=r=1$ being clear. Let $Prob(S)$ denote the probability that the algorithm of Corollary 1 yields the standard Young tableau S after $|S|$ steps. Let $col(n)$ be the column number of n in S. With all sums being over standard Young tableaux, observe that

$$\begin{aligned}
E_n(p^{\lambda_1'+\cdots+\lambda_r'}) &= \sum_{S:|S|=n} p^{\lambda_1'(S)+\cdots+\lambda_r'(S)} Prob(S) \\
&= \sum_{S:|S|=n, col(n)=1} p^{\lambda_1'(S)+\cdots+\lambda_r'(S)} Prob(S) \\
&\quad + \sum_{S:|S|=n, 1<col(n)=j\leq r} p^{\lambda_1'(S)+\cdots+\lambda_r'(S)} Prob(S) \\
&\quad + \sum_{S:|S|=n, col(n)>r} p^{\lambda_1'(S)+\cdots+\lambda_r'(S)} Prob(S) \\
&= \sum_{S':|S'|=n-1} p^{\lambda_1'(S')+\cdots+\lambda_r'(S')+1} Prob(S') \frac{1}{p^{\lambda_1'(S')}} \\
&\quad + \sum_{j=2}^{r} \sum_{S':|S'|=n-1} p^{\lambda_1'(S')+\cdots+\lambda_r'(S')+1} Prob(S') \left(\frac{1}{p^{\lambda_j'(S')}} - \frac{1}{p^{\lambda_{j-1}'(S')}}\right) \\
&\quad + \sum_{j>r} \sum_{S':|S'|=n-1} p^{\lambda_1'(S')+\cdots+\lambda_r'(S')} Prob(S') \left(\frac{1}{p^{\lambda_j'(S')}} - \frac{1}{p^{\lambda_{j-1}'(S')}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= pE_{n-1}(p^{\lambda_2'+\cdots+\lambda_r'}) + pE_{n-1}(p^{\lambda_1'+\cdots+\lambda_{r-1}'} - p^{\lambda_2'+\cdots+\lambda_r'}) \\
&\quad +E_{n-1}(p^{\lambda_1'+\cdots+\lambda_r'} - p^{\lambda_1'+\cdots+\lambda_{r-1}'}) \\
&= (p-1)E_{n-1}(p^{\lambda_1'+\cdots+\lambda_{r-1}'}) + E_{n-1}(p^{\lambda_1'+\cdots+\lambda_r'}) \\
&= (p-1)^{r-1}\binom{n-1}{r-1} + (p-1)^{r-1}\binom{n-1}{r} \\
&= (p-1)^{r-1}\binom{n}{r}.
\end{aligned}
$$

□

Corollary 3 follows by using Theorem 3 and arguing along the lines of Corollary 2.

Corollary 3 $\theta\frac{p^n-1}{p-1} - p\sum_{r=1}^{n}\frac{(p-1)^{r-1}\binom{n}{r}}{r} \leq E_n(X^\theta) \leq \theta\frac{p^n-1}{p-1}$.

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