

RANDOM PARTITIONS AND COHEN-LENSTRA HEURISTICS

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ABSTRACT. We investigate combinatorial properties of a family of probability distributions on finite abelian p -groups. This family includes several well-known distributions as specializations. These specializations have been studied in the context of Cohen-Lenstra heuristics and cokernels of families of random p -adic matrices.

1. INTRODUCTION

Friedman and Washington study a distribution on finite abelian p -groups G of rank at most d in [12]. In particular, a finite abelian p -group G of rank $r \leq d$, is chosen with probability

$$(1) \quad P_d(G) = \frac{1}{|\mathrm{Aut}(G)|} \left(\prod_{i=1}^d (1 - 1/p^i) \right) \left(\prod_{i=d-r+1}^d (1 - 1/p^i) \right).$$

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ be a partition. A finite abelian p -group G has *type* λ if

$$G \cong \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{\lambda_r}\mathbb{Z}.$$

Note that r is equal to the rank of G .

There is a correspondence between measures on the set of integer partitions and on isomorphism classes of finite abelian p -groups. Let \mathcal{L} denote the set of isomorphism classes of finite abelian p -groups. Given a measure ν on partitions, we get a corresponding measure ν' on \mathcal{L} by setting $\nu'(G) = \nu(\lambda)$ where $G \in \mathcal{L}$ is the isomorphism class of finite abelian p -groups of type λ . We analogously define a measure on partitions given a measure on \mathcal{L} . When G is a finite abelian group of type λ , we write $|\mathrm{Aut}(\lambda)|$ for $|\mathrm{Aut}(G)|$, and

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from page 181 of [19],

$$(2) \quad |\text{Aut}(\lambda)| = p^{\sum(\lambda'_i)^2} \prod_i (1/p)_{m_i(\lambda)}.$$

The notation used in (2) is standard, and we review it in Section 1.2.

We introduce and study a more general distribution on integer partitions and on finite abelian p -groups G of rank at most d . We choose a partition λ with $r \leq d$ parts with probability

$$(3) \quad P_{d,u}(\lambda) = \frac{u^{|\lambda|}}{p^{\sum(\lambda'_i)^2} \prod_i (1/p)_{m_i(\lambda)}} \prod_{i=1}^d (1 - u/p^i) \prod_{i=d-r+1}^d (1 - 1/p^i).$$

This gives a distribution on partitions for all real $p > 1$ and $0 < u < p$. We can include p as an additional parameter and write $P_{d,u}^p(\lambda)$. For clarity, we will suppress this additional notation except in Section 3. When p is prime, we can interpret (3) as a distribution on \mathcal{L} . When p is not prime it does not make sense to talk about automorphisms of a finite abelian p -group, but in this case we can take (2) as the definition of $|\text{Aut}(\lambda)|$.

The main goal of this paper is to investigate combinatorial properties of the family of distributions of (3). We begin by noting six interesting specializations of this measure.

- Setting $u = 1$ in $P_{d,u}$ recovers P_d .
- We define a distribution $P_{\infty,u}$ by

$$\lim_{d \rightarrow \infty} P_{d,u}(\lambda) = P_{\infty,u}(\lambda) = \frac{u^{|\lambda|}}{|\text{Aut}(\lambda)|} \prod_{i \geq 1} (1 - u/p^i).$$

It is not immediately clear that this limit defines a distribution on partitions, but this follows from the sentence after Proposition 2.1, from Theorem 2.2, or from Theorem 5.3, taking μ to be the trivial partition.

For $0 < u < 1$, this probability measure arises by choosing a random non-negative integer N with probability $P(N = n) = (1 - u)u^n$, and then looking at the $z - 1$ piece of a random element of the finite group $\text{GL}(N, p)$. See [13] for details.

- Note that

$$P_{\infty,1}(\lambda) = \frac{1}{|\text{Aut}(\lambda)|} \prod_{i \geq 1} (1 - 1/p^i).$$

This is the measure on partitions corresponding to the usual Cohen-Lenstra measure on finite abelian p -groups [5]. It also arises from studying the $z - 1$ piece of a random element of the finite group $\text{GL}(d, p)$ in the $d \rightarrow \infty$ limit [13], or from studying the cokernel of a random $d \times d$ p -adic matrix in the $d \rightarrow \infty$ limit [12].

- Let w be a positive integer and λ a partition. The w -probability of λ , denoted by $P_w(\lambda)$, is the probability that a finite abelian p -group of type λ is obtained by the following three step random process:
 - Choose randomly a p -group H of type μ with respect to the measure $P_{\infty,1}(\mu)$.
 - Then choose w elements g_1, \dots, g_w of H uniformly at random.
 - Finally, output $H/\langle g_1, \dots, g_w \rangle$, where $\langle g_1, \dots, g_w \rangle$ denotes the group generated by g_1, \dots, g_w .
 From Example 5.9 of Cohen and Lenstra [5], it follows that $P_w(\lambda)$ is a special case of (3):

$$(4) \quad P_w(\lambda) = P_{\infty,1/p^w}(\lambda).$$

- We now mention two analogues of Proposition 1 of [12] for rectangular matrices. Let w be a non-negative integer. Friedman and Washington do not discuss this explicitly, but using the same methods as in [12] one can show that taking the limit as $d \rightarrow \infty$ of the probability that a randomly chosen $d \times (d+w)$ matrix over \mathbb{Z}_p has cokernel isomorphic to a finite abelian p -group of type λ is given by $P_{\infty,1/p^w}(\lambda)$. See the discussion above Proposition 2.3 of [26].
 Similarly, Tse considers rectangular matrices with more rows than columns and shows that $P_{\infty,1/p^w}(\lambda)$ is equal to the $d \rightarrow \infty$ probability that a randomly chosen $(d+w) \times d$ matrix over \mathbb{Z}_p has cokernel isomorphic to $\mathbb{Z}_p^w \oplus G$, where G is a finite abelian p -group of type λ [23].
- In Section 3 we see that the measure on partitions studied by Bhargava, Kane, Lenstra, Poonen and Rains [1], arising from taking the cokernel of a random alternating p -adic matrix is also a special case of $P_{d,u}$. Taking a limit as the size of the matrix goes to infinity gives a distribution consistent with heuristics of Delaunay for Tate-Shafarevich groups of elliptic curves defined over \mathbb{Q} [8].

A few of these specializations have received extensive attention in prior work:

- When p is an odd prime, Cohen and Lenstra conjecture that $P_{\infty,1}$ models the distribution of p -parts of class groups of imaginary quadratic fields and $P_{\infty,1/p}$ models the distribution of p -parts of class groups of real quadratic fields [5]. Theorem 6.3 in [5] gives the probability that a group chosen from $P_{\infty,1/p^w}$ has given p -rank. For any n odd, they show that the average number of elements of order exactly n of a group drawn from $P_{\infty,1}$ is 1, and that this average for a group drawn from $P_{\infty,1/p}$ is $1/n$ [5, Section 9]. Delaunay generalizes these results in Corollary 11 of [9], where he computes the probability that a group drawn from $P_{\infty,u}$ simultaneously has specified p^j -rank for several values of j . Delaunay and Jouhet compute averages of

even more complicated functions involving moments of the number of p^j -torsion points for varying j in [6].

The distribution of 2-parts of class groups of quadratic fields is not modeled by $P_{\infty,u}$ and several authors have worked to understand these issues. Motivated by work of Gerth [15, 16], Fouvry and Klüners study the conjectural distribution of p^j -ranks and moments for the number of torsion points of C_D^2 , the square of the ideal class group of a quadratic field [11].

- Delaunay [9], and Delaunay and Jouhet [6], prove analogues of the results described in the previous paragraphs for groups drawn from the $n \rightarrow \infty$ specialization of the distribution we study in Section 3. In [7], they prove analogues of the results of Fouvry and Klüners [11] for this distribution.

1.1. Outline of the Paper. In Section 2 we interpret $P_{d,u}$ in terms of Hall-Littlewood polynomials and use this interpretation to compute the probability that a partition chosen from $P_{d,u}$ has given size, given number of parts, or given size and number of parts. In Theorem 2.2 we give an algorithm for producing a partition according to the distribution $P_{d,u}$.

In Section 3 we show how a measure studied in [1] that arises from distributions of cokernels of random alternating p -adic matrices is given by a specialization of $P_{d,u}$. In Section 4 we briefly study a measure on partitions that arises from distributions of cokernels of random symmetric p -adic matrices that is studied in [4, 24]. We give an algorithm for producing a partition according to this distribution.

In Section 5 we combinatorially compute the moments of the distribution $P_{d,u}$ for all d and u . These moments were already known for the case $d = \infty, u = 1$, and our method is new even in that special case. We also show that in many cases these moments determine a unique distribution. This is a generalization of a result of Ellenberg, Venkatesh, and Westerland [10], that the moments of the Cohen-Lenstra distribution determine the distribution, and of Wood [26], that the moments of the distribution P_w determine the distribution.

1.2. Notation. Throughout this paper, when p is a prime number we write \mathbb{Z}_p for the ring of p -adic integers.

For a ring R , let $M_d(R)$ denote the set of all $d \times d$ matrices with entries in R and let $\text{Sym}_d(R)$ denote the set of all $d \times d$ symmetric matrices with entries in R . For an even integer d , let $\text{Alt}_d(R)$ denote the set of all $d \times d$ alternating matrices with entries in R (that is, matrices A with zeros on the diagonal satisfying that the transpose of A is equal to $-A$).

For groups G and H we write $\text{Hom}(G, H)$ for the set of homomorphisms from G to H , $\text{Sur}(G, H)$ for the set of surjective homomorphisms from G to H , and $\text{Aut}(G)$ for the set of automorphisms of G . If G is a finite abelian p -group of type λ and H is a finite abelian p -group of type μ , we sometimes write $|\text{Sur}(\lambda, \mu)|$ for $|\text{Sur}(G, H)|$.

For a partition λ , we let λ_i denote the size of the i^{th} part of λ and $m_i(\lambda)$ denote the number of parts of λ of size i . We let λ'_i denote the size of the i^{th} column in the diagram of λ (so $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \dots$). We also let $n(\lambda) = \sum_i \binom{\lambda'_i}{2}$. We generally use r or $r(\lambda)$ to denote the number of parts of λ . We use $|\lambda| = n$ to say that λ is a partition of n , or equivalently $\sum \lambda_i = n$.

We let $n_\lambda(\mu)$ denote the number of subgroups of type μ of a finite abelian p -group of type λ . For a finite abelian group G , the number of subgroups $H \subseteq G$ of type μ equals the number of subgroups for which G/H has type μ [19, Equation (1.5), page 181].

We also let $(x)_i = (1-x)(1-x/p)\cdots(1-x/p^{i-1})$. So $(1/p)_i = (1-1/p)\cdots(1-1/p^i)$. With this notation, (3) is equivalent to

$$P_{d,u}(\lambda) = \frac{u^{|\lambda|} (u/p)_d}{p^{\sum (\lambda'_i)^2} \prod_i (1/p)_{m_i(\lambda)}} \frac{(1/p)_d}{(1/p)_{d-r(\lambda)}}.$$

We use some notation related to q -binomial coefficients, namely:

$$\begin{aligned} [n]_q &= \frac{q^n - 1}{q - 1} = 1 + q + \dots + q^{n-1}; \\ [n]_q! &= [n]_q [n-1]_q \cdots [2]_q; \\ \binom{n}{j}_q &= \frac{[n]_q!}{[j]_q! [n-j]_q!}. \end{aligned}$$

Finally if $f(u)$ is a power series in u , we let $\text{Coef. } u^n$ in $f(u)$ denote the coefficient of u^n in $f(u)$.

2. PROPERTIES OF THE MEASURE $P_{d,u}$

To begin we give a formula for $P_{d,u}(\lambda)$ in terms of Hall-Littlewood polynomials. We let P_λ denote a Hall-Littlewood polynomial, defined for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of length at most n by

$$P_\lambda(x_1, \dots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right),$$

where

$$v_\lambda(t) = \prod_{i \geq 0} \prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t},$$

the permutation $w \in S_n$ permutes the x variables, and we note that some parts of λ may have size 0. For background on Hall-Littlewood polynomials, see Chapter 3 of [19].

Proposition 2.1. *For a partition λ with $r \leq d$ parts,*

$$P_{d,u}(\lambda) = \prod_{i=1}^d (1 - u/p^i) \cdot \frac{P_\lambda(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}}.$$

Proof. From page 213 of [19],

$$\prod_{i=1}^d (1 - u/p^i) \cdot \frac{P_\lambda(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}}$$

is equal to

$$\frac{u^{|\lambda|} \prod_{i=1}^d (1 - u/p^i)}{\prod_i (1/p)_{m_i(\lambda)}} \frac{(1/p)_d}{p^{|\lambda|+2n(\lambda)} (1/p)_{d-r}}.$$

Since $|\lambda| + 2n(\lambda) = \sum (\lambda'_i)^2$, this is equal to (3), and the proposition follows. \square

The fact that $\sum_\lambda P_{d,u}(\lambda) = 1$ follows from Proposition 2.1 and the identity of Example 1 on page 225 of [19]. It is also immediate from Theorem 2.2.

There are two ways to generate random partitions λ according to the distribution $P_{d,u}$. The first is to run the ‘‘Young tableau algorithm’’ of [13], stopped when coin d comes up tails. The second method is given by the following theorem.

Theorem 2.2. *Starting with $\lambda'_0 = d$, define in succession $d \geq \lambda'_1 \geq \lambda'_2 \geq \dots$ according to the rule that if $\lambda'_i = a$, then $\lambda'_{i+1} = b$ with probability*

$$K(a, b) = \frac{u^b (1/p)_a (u/p)_a}{p^{b^2} (1/p)_{a-b} (1/p)_b (u/p)_b}.$$

Then the resulting partition is distributed according to $P_{d,u}$.

Proof. One must compute

$$K(d, \lambda'_1) K(\lambda'_1, \lambda'_2) K(\lambda'_2, \lambda'_3) \cdots$$

There is a lot of cancellation, and (recalling that $\lambda'_1 = r$), what is left is:

$$\frac{(u/p)_d (1/p)_d u^{|\lambda|}}{(1/p)_{d-r} p^{\sum (\lambda'_i)^2} \prod_i (1/p)_{m_i(\lambda)}}.$$

This is equal to $P_{d,u}(\lambda)$, completing the proof. \square

The following corollary is immediate from Theorem 2.2.

Corollary 2.3. *Choose λ from $P_{d,u}$. Then the chance that λ has $r \leq d$ parts is equal to*

$$\frac{u^r (1/p)_d (u/p)_d}{p^{r^2} (1/p)_{d-r} (1/p)_r (u/p)_r}.$$

Proof. From Theorem 2.2, the sought probability is $K(d, r)$. \square

The $u = 1$ case of this result appears in another form in work of Stanley and Wang [22]. In Theorem 4.14 of [22], the authors compute the probability $Z_d(p, r)$ that the Smith normal form of a certain model of random integer matrix has at most r diagonal entries divisible by p . Setting $u = 1$ in Corollary 2.3 gives $Z_d(p, r) - Z_d(p, r - 1)$. This expression also appears in [3] where the authors study finite abelian groups arising as \mathbb{Z}^d/Λ for random

sublattices $\Lambda \subset \mathbb{Z}^d$; isolating the prime p and the $i = r$ term in Corollary 1.2 of [3] gives the $u = 1$ case of Corollary 2.3.

The next result computes the chance that λ chosen from $P_{d,u}$ has size n .

Theorem 2.4. *The chance that λ chosen from $P_{d,u}$ has size n is equal to*

$$\frac{u^n (u/p)_d (1/p)_{d+n-1}}{p^n (1/p)_{d-1} (1/p)_n}.$$

Proof. By Proposition 2.1, the sought probability is equal to

$$\begin{aligned} \sum_{|\lambda|=n} P_{d,u}(\lambda) &= (u/p)_d \sum_{|\lambda|=n} \frac{P_\lambda(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}} \\ &= (u/p)_d \sum_{|\lambda|=n} u^n \frac{P_\lambda(\frac{1}{p}, \frac{1}{p^2}, \dots, \frac{1}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}} \\ &= u^n (u/p)_d \text{Coef. } u^n \text{ in } \sum_{\lambda} \frac{P_\lambda(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}} \\ &= u^n (u/p)_d \text{Coef. } u^n \text{ in } \frac{1}{(u/p)_d} \\ &= \frac{u^n (u/p)_d (1/p)_{d+n-1}}{p^n (1/p)_{d-1} (1/p)_n}. \end{aligned}$$

The fourth equality used Proposition 2.1 and the fact that $P_{d,u}$ defines a probability distribution, and the final equality used Theorem 349 of [17]. \square

Theorem 2.5. *The probability that λ chosen from $P_{d,u}$ has size n and $r \leq \min\{d, n\}$ parts is equal to*

$$\frac{u^n (u/p)_d (1/p)_d (1/p)_{n-1}}{p^{r^2} (1/p)_{d-r} (1/p)_r p^{n-r} (1/p)_{r-1} (1/p)_{n-r}}.$$

Proof. From the definition of $P_{d,u}$, one has that

$$\begin{aligned}
\sum_{\substack{\lambda'_1=r \\ |\lambda|=n}} P_{d,u}(\lambda) &= \sum_{\substack{\lambda'_1=r \\ |\lambda|=n}} \frac{u^n (u/p)_d (1/p)_d}{|\text{Aut}(\lambda)|(1/p)_{d-r}} \\
&= u^n (u/p)_d \sum_{\substack{\lambda'_1=r \\ |\lambda|=n}} \frac{(1/p)_d}{|\text{Aut}(\lambda)|(1/p)_{d-r}} \\
&= u^n (u/p)_d \text{Coef. } u^n \text{ in } \sum_{\lambda'_1=r} \frac{u^{|\lambda|} (1/p)_d}{|\text{Aut}(\lambda)|(1/p)_{d-r}} \\
&= u^n (u/p)_d \text{Coef. } u^n \text{ in } \frac{1}{(u/p)_d} \sum_{\lambda'_1=r} P_{d,u}(\lambda) \\
&= u^n (u/p)_d \text{Coef. } u^n \text{ in } \frac{1}{(u/p)_d p^{r^2} (1/p)_{d-r} (1/p)_r (u/p)_r} u^r (1/p)_d (u/p)_d \\
&= \frac{u^n (u/p)_d (1/p)_d}{p^{r^2} (1/p)_{d-r} (1/p)_r} \text{Coef. } u^{n-r} \text{ in } \frac{1}{(u/p)_r} \\
&= \frac{u^n (u/p)_d (1/p)_d}{p^{r^2} (1/p)_{d-r} (1/p)_r} \frac{(1/p)_{n-1}}{p^{n-r} (1/p)_{r-1} (1/p)_{n-r}}.
\end{aligned}$$

The fifth equality used Corollary 2.3, and the final equality used Theorem 349 of [17]. \square

In the rest of this section we give another view of the distributions given by (1) and (3). When p is prime, equation (19) in [20] implies that

$$(5) \quad P_d(\lambda) = \frac{1}{p^{|\lambda|d}} \left(\prod_{i=1}^{\lambda_1} p^{\lambda_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i-\lambda'_{i+1}}_p \right) \prod_{i=1}^d (1-1/p^i).$$

Comparing this to the expression for $P_d(\lambda)$ given in (1) shows that

$$(6) \quad \frac{1}{p^{|\lambda|d}} \left(\prod_{i=1}^{\lambda_1} p^{\lambda_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i-\lambda'_{i+1}}_p \right) = \frac{1}{|\text{Aut}(\lambda)|} \left(\prod_{i=d-r+1}^d (1-1/p^i) \right).$$

A direct proof is given in Proposition 4.7 of [3]. Therefore, we get a second expression for $P_{d,u}(\lambda)$,

$$(7) \quad P_{d,u}(\lambda) = \frac{u^{|\lambda|}}{p^{|\lambda|d}} \left(\prod_{i=1}^{\lambda_1} p^{\lambda_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i-\lambda'_{i+1}}_p \right) \prod_{i=1}^d (1-u/p^i).$$

We give a combinatorial proof of (6) that applies for any real $p > 1$, so (7) applies for any $p > 1$ and $0 < u < p$.

Proof of Equation (6). It is sufficient to show that for a partition λ with $r \leq d$ parts,

$$(8) \quad |\text{Aut}(\lambda)| \left(\prod_{i=1}^{\lambda_1} p^{\lambda'_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i - \lambda'_{i+1}}_p \right) = p^{|\lambda|d} \prod_{j=0}^{r-1} (1 - p^{-d+j}).$$

Clearly

$$\begin{aligned} & \prod_{i=1}^{\lambda_1} p^{\lambda'_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i - \lambda'_{i+1}}_p \\ &= p^{d(|\lambda|-\lambda'_1) - \sum_i \lambda'_i \lambda'_{i+1}} \prod_i \binom{d-\lambda'_{i+1}}{\lambda'_i - \lambda'_{i+1}}_p \\ &= p^{d(|\lambda|-\lambda'_1) - \sum_i \lambda'_i \lambda'_{i+1}} \frac{[d]_p!}{[d-\lambda'_1]_p! [\lambda'_1 - \lambda'_2]_p! [\lambda'_2 - \lambda'_3]_p! \cdots} \\ &= p^{d(|\lambda|-\lambda'_1) - \sum_i \lambda'_i \lambda'_{i+1}} \frac{(p-1)^{\lambda'_1} [d]_p!}{[d-\lambda'_1]_p! p^{\sum_i (\lambda'_i - \lambda'_{i+1} + 1)}} \prod_i (1/p)_{m_i(\lambda)} \\ &= \frac{p^{d(|\lambda|-\lambda'_1)} (p-1)^{\lambda'_1} [d]_p!}{[d-\lambda'_1]_p! p^{\frac{1}{2}[\sum_i (\lambda'_i)^2 + (\lambda'_{i+1})^2 + \lambda'_i - \lambda'_{i+1}]} \prod_i (1/p)_{m_i(\lambda)}} \\ &= \frac{p^{d(|\lambda|-\lambda'_1)} p^{(\lambda'_1)^2/2} (p-1)^{\lambda'_1} [d]_p!}{[d-\lambda'_1]_p! p^{\lambda'_1/2}} \cdot \frac{1}{p^{\sum_i (\lambda'_i)^2} \prod_i (1/p)_{m_i(\lambda)}}. \end{aligned}$$

Since $\lambda'_1 = r$, equation (2) implies that the left-hand side of (8) is equal to

$$\begin{aligned} & \frac{p^{d|\lambda| - dr + r^2/2 - r/2} (p-1)^r [d]_p!}{[d-r]_p!} \\ &= p^{d|\lambda| - dr + r^2/2 - r/2} (p^d - 1) \cdots (p^{d-r+1} - 1), \end{aligned}$$

which simplifies to the right-hand side of (8). \square

We now use the alternate expression of (7) to give an additional proof of Theorem 2.4 in the case when p is prime. The zeta function of \mathbb{Z}^d is defined by

$$\zeta_{\mathbb{Z}^d}(s) = \sum_{H \leq \mathbb{Z}^d} [\mathbb{Z}^d : H]^{-s},$$

where the sum is taken over all finite index subgroups of \mathbb{Z}^d . It is known that

$$(9) \quad \begin{aligned} \zeta_{\mathbb{Z}^d}(s) &= \zeta(s) \zeta(s-1) \cdots \zeta(s-(d-1)) \\ &= \prod_p \left((1-p^{-s})^{-1} (1-p^{-(s-1)})^{-1} \cdots (1-p^{-(s-(d-1))})^{-1} \right), \end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta function, and the product is taken over all primes. See the book of Lubotzky and Segal for five proofs of this fact [18].

Second Proof of Theorem 2.4 for p prime. From (7), we need only prove

$$(10) \quad \sum_{|\lambda|=n} \frac{u^n}{p^{nd}} \left(\prod_{i=1}^{\lambda_1} p^{\lambda'_{i+1}(d-\lambda'_i)} \binom{d-\lambda'_{i+1}}{\lambda'_i-\lambda'_{i+1}}_p \right) = \frac{u^n}{p^n} \frac{(1/p)_{d+n-1}}{(1/p)_{d-1}(1/p)_n}.$$

Let $\lambda^* = (\lambda_1, \dots, \lambda_1)$, where there are d entries in the tuple. The discussion around equation (19) in [20] says that the term in parentheses of the left-hand side of (10) is equal to the number of subgroups of a finite abelian p -group of type λ^* that have type λ , $n_{\lambda^*}(\lambda)$, which is also equal to the number of subgroups $\Lambda \subset \mathbb{Z}^d$ such that \mathbb{Z}^d/Λ is a finite abelian p -group of type λ .

After some obvious cancelation, we need only show that

$$\sum_{|\lambda|=n} n_{\lambda^*}(\lambda) = \frac{p^{n(d-1)}(1/p)_{d+n-1}}{(1/p)_{d-1}(1/p)_n}.$$

The left-hand side is the number of subgroups $\Lambda \subset \mathbb{Z}^d$ such that \mathbb{Z}^d/Λ has order p^n . This is the p^{-sn} coefficient of $\zeta_{\mathbb{Z}^d}(s)$. Using (9), this is equal to

$$\begin{aligned} & \text{Coef. } p^{-sn} \text{ in } (1-p^{-s})^{-1}(1-p^{-(s-1)})^{-1} \dots (1-p^{-(s-(d-1))})^{-1} \\ &= \text{Coef. } x^n \text{ in } (1-x)^{-1}(1-px)^{-1}(1-p^2x)^{-1} \dots (1-p^{d-1}x)^{-1}. \end{aligned}$$

By Theorem 349 of [17], this is equal to

$$\frac{p^{n(d-1)}(1/p)_{d+n-1}}{(1/p)_{d-1}(1/p)_n},$$

and the proof is complete. \square

3. COKERNELS OF RANDOM ALTERNATING p -ADIC MATRICES

In this section we consider a distribution on finite abelian p -groups that arises in the study of cokernels of random p -adic alternating matrices. We show that this is a special case of the distributions $P_{d,u}^p$.

Let n be an even positive integer and let $A \in \text{Alt}_n(\mathbb{Z}_p)$ be a random matrix chosen with respect to additive Haar measure on $\text{Alt}_n(\mathbb{Z}_p)$. The cokernel of A is a finite abelian p -group of the form $G \cong H \times H$ for some H of type λ with at most $n/2$ parts, and is equipped with a nondegenerate alternating pairing $[\cdot, \cdot]: H \times H \mapsto \mathbb{Q}/\mathbb{Z}$. Let $\text{Sp}(G)$ be the group of automorphisms of H respecting $[\cdot, \cdot]$. Let r be the number of parts of λ , and $|\lambda|$, $n(\lambda)$, $m_i(\lambda)$ be as in Section 1.2.

Lemma 3.1. *Let n be an even positive integer and $A \in \text{Alt}_n(\mathbb{Z}_p)$ be a random matrix chosen with respect to additive Haar measure on $\text{Alt}_n(\mathbb{Z}_p)$. The probability that the cokernel of A is isomorphic to G is given by*

$$(11) \quad P_{n,p}^{\text{Alt}}(\lambda) = \frac{\prod_{i=n-2r+1}^n (1 - 1/p^i) \prod_{i=1}^{n/2-r} (1 - 1/p^{2i-1})}{p^{|\lambda|+4n(\lambda)} \prod_i \prod_{j=1}^{m_i(\lambda)} (1 - 1/p^{2j})}.$$

Proof. Formula (6) and Lemma 3.6 of [1] imply that the probability that the cokernel of A is isomorphic to G is given by

$$\frac{|\text{Sur}(\mathbb{Z}_p^n, G)|}{|\text{Sp}(G)|} \prod_{i=1}^{n/2-r} (1 - 1/p^{2i-1}) |G|^{1-n}.$$

We can rewrite this expression in terms of the partition λ . Clearly $|G| = p^{2|\lambda|}$. Proposition 3.1 of [5] implies that since G has rank $2r$,

$$|\text{Sur}(\mathbb{Z}_p^n, G)| = p^{2n|\lambda|} \prod_{i=n-2r+1}^n (1 - 1/p^i).$$

An identity on the bottom of page 538 of [9] says that,

$$\begin{aligned} |\text{Sp}(G)| &= p^{|\lambda|} p^{2\sum_i (\lambda_i')^2} \prod_i \prod_{j=1}^{m_i(\lambda)} (1 - 1/p^{2j}) \\ &= p^{4n(\lambda)+3|\lambda|} \prod_i \prod_{j=1}^{m_i(\lambda)} (1 - 1/p^{2j}). \end{aligned}$$

Putting these results together completes the proof. \square

The next theorem shows that (11) is a special case of (3).

Theorem 3.2. *Let n be an even positive integer. For any partition λ ,*

$$P_{n/2,p}^{p^2}(\lambda) = P_{n,p}^{\text{Alt}}(\lambda).$$

Proof. Rewrite (3) as

$$\frac{u^{|\lambda|} \prod_{i=1}^d (1 - u/p^i) \prod_{i=d-r+1}^d (1 - 1/p^i)}{p^{2n(\lambda)+|\lambda|} \prod_i \prod_{j=1}^{m_i(\lambda)} (1 - 1/p^j)}.$$

Replacing d by $n/2$, u by p , and p by p^2 gives

$$\frac{\prod_{i=1}^{n/2} (1 - 1/p^{2i-1}) \prod_{i=n/2-r+1}^{n/2} (1 - 1/p^{2i})}{p^{4n(\lambda)+|\lambda|} \prod_i \prod_{j=1}^{m_i(\lambda)} (1 - 1/p^{2j})}.$$

Comparing with (11), we see that it suffices to prove

$$\prod_{i=1}^{n/2} (1 - 1/p^{2i-1}) \prod_{i=n/2-r+1}^{n/2} (1 - 1/p^{2i}) = \prod_{i=n-2r+1}^n (1 - 1/p^i) \prod_{i=1}^{n/2-r} (1 - 1/p^{2i-1}).$$

To prove this equality, note that when each side is multiplied by

$$(1 - 1/p^2)(1 - 1/p^4) \cdots (1 - 1/p^{n-2r}),$$

each side becomes $(1/p)_n$. \square

4. COKERNELS OF RANDOM SYMMETRIC p -ADIC MATRICES

Let $A \in \text{Sym}_n(\mathbb{Z}_p)$ be a random matrix chosen with respect to additive Haar measure on $\text{Sym}_n(\mathbb{Z}_p)$. Let r be the number of parts of λ . Theorem 2 of [4] shows that the probability that the cokernel of A has type λ is equal to:

$$(12) \quad P_n^{\text{Sym}}(\lambda) = \frac{\prod_{j=n-r+1}^n (1 - 1/p^j) \prod_{i=1}^{\lfloor (n-r)/2 \rfloor} (1 - 1/p^{2i-1})}{p^{n(\lambda)+|\lambda|} \prod_{i \geq 1} \prod_{j=1}^{\lfloor m_i(\lambda)/2 \rfloor} (1 - 1/p^{2j})}.$$

Note that $P_n^{\text{Sym}}(\lambda) = 0$ if λ has more than n parts. As in earlier sections, when p is prime (12) has an interpretation in terms of finite abelian p -groups, but defines a distribution on partitions for any $p > 1$. This follows directly from Theorem 4.1 below.

Taking $n \rightarrow \infty$ gives a distribution on partitions where λ is chosen with probability

$$(13) \quad P_\infty^{\text{Sym}}(\lambda) = \frac{\prod_{i \text{ odd}} (1 - 1/p^i)}{p^{n(\lambda)+|\lambda|} \prod_{i \geq 1} \prod_{j=1}^{\lfloor m_i(\lambda)/2 \rfloor} (1 - 1/p^{2j})}.$$

The distribution of (13) is studied in [24], where Wood shows that it arises as the distribution of p -parts of sandpile groups of large Erdős-Rényi random graphs. Combinatorial properties of this distribution are considered in [14], where it is shown that this distribution is a specialization of a two parameter family of distributions. It is unclear whether the distribution of (12) also sits within a larger family.

The following theorem allows one to generate partitions from the measure (12), and is a minor variation on Theorem 3.1 of [14].

Theorem 4.1. *Starting with $\lambda'_0 = n$, define in succession $n \geq \lambda'_1 \geq \lambda'_2 \geq \cdots$ according to the rule that if $\lambda'_i = a$, then $\lambda'_{i+1} = b$ with probability*

$$K(a, b) = \frac{\prod_{i=1}^a (1 - 1/p^i)}{p^{\binom{b+1}{2}} \prod_{i=1}^b (1 - 1/p^i) \prod_{j=1}^{\lfloor (a-b)/2 \rfloor} (1 - 1/p^{2j})}.$$

Then the resulting partition with at most n parts is distributed according to (12).

Proof. It is necessary to compute

$$K(n, \lambda'_1) K(\lambda'_1, \lambda'_2) K(\lambda'_2, \lambda'_3) \cdots$$

There is a lot of cancelation, and (recalling that $\lambda'_1 = r$), what is left is:

$$\frac{\prod_{j=1}^n (1 - 1/p^j)}{\prod_{j=1}^{\lfloor (n-r)/2 \rfloor} (1 - 1/p^{2j})} \frac{1}{p^{n(\lambda)+|\lambda|} \prod_{i \geq 1} \prod_{j=1}^{\lfloor m_i(\lambda)/2 \rfloor} (1 - 1/p^{2j})}.$$

So to complete the proof, it is necessary to check that

$$\frac{\prod_{j=1}^n (1 - 1/p^j)}{\prod_{j=1}^{\lfloor (n-r)/2 \rfloor} (1 - 1/p^{2j})} = \prod_{j=n-r+1}^n (1 - 1/p^j) \prod_{i=1}^{\lfloor (n-r)/2 \rfloor} (1 - 1/p^{2i-1}).$$

This equation is easily verified by breaking it into cases based on whether $n - r$ is even or odd. \square

The following corollary is immediate.

Corollary 4.2. *Let λ be chosen from (12). Then the chance that λ has $r \leq n$ parts is equal to*

$$\frac{\prod_{j=r+1}^n (1 - 1/p^j)}{p^{\binom{r+1}{2}} \prod_{j=1}^{\lfloor (n-r)/2 \rfloor} (1 - 1/p^{2j})}.$$

Proof. By Theorem 4.1, the sought probability is equal to $K(n, r)$. \square

Taking $n \rightarrow \infty$ in this result recovers Theorem 2.2 of [14], which is also Corollary 9.4 of [24].

5. COMPUTATION OF H -MOMENTS

We recall that \mathcal{L} denotes the set of isomorphism classes of finite abelian p -groups and that a probability distribution ν on \mathcal{L} gives a probability distribution on the set of partitions in an obvious way. Similarly, a measure on partitions gives a measure on \mathcal{L} , setting $\nu(G) = \nu(\lambda)$ when G is a finite abelian p -group of type λ . When $G, H \in \mathcal{L}$ we write $|\text{Sur}(G, H)|$ for the number of surjections from any representative of the isomorphism class G to any representative of the isomorphism class H .

Let ν be a probability measure on \mathcal{L} . For $H \in \mathcal{L}$, the H -moment of ν is defined as

$$\sum_{G \in \mathcal{L}} \nu(G) |\text{Sur}(G, H)|.$$

When H is a finite abelian p -group of type μ this is

$$\sum_{\lambda} \nu(\lambda) |\text{Sur}(\lambda, \mu)|.$$

The distribution ν gives a measure on partitions and we refer to this quantity as the μ -moment of the measure. For an explanation of why these are called the moments of the distribution, see Section 3.3 of [4].

The Cohen-Lenstra distribution is the probability distribution on \mathcal{L} for which a finite abelian group G of type λ is chosen with probability $P_{\infty,1}(\lambda)$. One of the most striking properties of the Cohen-Lenstra distribution is that the H -moment of $P_{\infty,1}$ is 1 for every H , or equivalently, for any finite abelian p -group H of type μ ,

$$\sum_{\lambda} P_{\infty,1}(\lambda) |\text{Sur}(\lambda, \mu)| = 1.$$

There is a nice algebraic explanation of this fact using the interpretation of $P_{\infty,1}$ as a limit of the $P_{d,1}$ distributions given by (1) (see for example [21]).

Lemma 8.2 of [10] shows that the Cohen-Lenstra distribution is determined by its moments.

Lemma 5.1. *Let p be an odd prime. If ν is any probability measure on \mathcal{L} for which*

$$\sum_{G \in \mathcal{L}} \nu(G) |\text{Sur}(G, H)| = 1$$

for any $H \in \mathcal{L}$, then $\nu = P_{\infty,1}$.

Our next goal is to compute the moments for the measure $P_{d,u}$; see Theorem 5.3 below. Our method is new even in the case $P_{\infty,1}$.

There has been much recent interest in studying moments of distributions related to the Cohen-Lenstra distribution, and showing that these moments determine a unique distribution [2, 24, 26]. At the end of this section, we add to this discussion by proving a version of Lemma 5.1 for the distribution $P_{d,u}$.

The following lemma counts the number of surjections from G to H . Recall that $n_{\lambda}(\mu)$ is the number of subgroups of type μ of a finite abelian group of type λ .

Lemma 5.2. *Let G, H be finite abelian p -groups of types λ and μ respectively. Then*

$$|\text{Sur}(G, H)| = |\text{Sur}(\lambda, \mu)| = n_{\lambda}(\mu) |\text{Aut}(\mu)|.$$

For a proof, see page 28 of [27]. The main idea is that $|\text{Sur}(G, H)|$ is the number of *injective* homomorphisms from \widehat{H} to \widehat{G} , where these are the dual groups of H and G , respectively. The image of such a homomorphism is a subgroup of \widehat{G} of type μ .

The distributions $P_{d,u}$ are defined for all $p > 1$. It is not immediately clear what the μ -moment of this distribution should mean when p is not prime, since $|\text{Sur}(\lambda, \mu)|$ is defined in terms of surjective homomorphisms between finite abelian p -groups. In (2) we saw how to define $|\text{Aut}(\lambda)|$ in terms of the parts of the partition λ and the parameter p , even in the case where p is not prime. Similarly, Lemma 5.2 gives a way to define $|\text{Sur}(\lambda, \mu)|$ in terms of the parameter p and the partitions λ and μ even when p is not prime. We first define $|\text{Aut}(\mu)|$ using (2), and then note that $n_{\lambda}(\mu)$ is a polynomial in p that we can evaluate for any $p > 1$.

Theorem 5.3. *The μ -moment of the distribution $P_{d,u}$ is equal to*

$$\begin{cases} \frac{u^{|\mu|} (1/p)_d}{(1/p)_{d-r(\mu)}} & \text{if } r(\mu) \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Here, as above, $r(\mu)$ denotes the number of parts of μ .

Proof. Clearly we can suppose that $r(\mu) \leq d$. By Lemma 5.2, the μ -moment of the distribution $P_{d,u}$ is equal to

$$\sum_{\lambda} P_{d,u}(\lambda) |\text{Sur}(\lambda, \mu)| = |\text{Aut}(\mu)| \sum_{\lambda} P_{d,u}(\lambda) n_{\lambda}(\mu).$$

Let $n_{\lambda}(\mu, \nu)$ be the number of subgroups M of G so that M has type μ and G/M has type ν . This is a polynomial in p (see Chapter II Section 4 of [19]). Then by Proposition 2.1, the μ -moment becomes

$$|\text{Aut}(\mu)| \prod_{i=1}^d (1 - u/p^i) \cdot \sum_{\lambda} \frac{P_{\lambda}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}} \sum_{\nu} n_{\lambda}(\mu, \nu).$$

Reversing the order of summation, this becomes

$$|\text{Aut}(\mu)| \prod_{i=1}^d (1 - u/p^i) \cdot \sum_{\nu} \sum_{\lambda} \frac{P_{\lambda}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\lambda)}} n_{\lambda}(\mu, \nu).$$

From Section 3.3 of [19], it follows that for any values of the x variables,

$$\sum_{\lambda} n_{\lambda}(\mu, \nu) \frac{P_{\lambda}(x; \frac{1}{p})}{p^{n(\lambda)}} = \frac{P_{\mu}(x; \frac{1}{p})}{p^{n(\mu)}} \frac{P_{\nu}(x; \frac{1}{p})}{p^{n(\nu)}}.$$

Specializing $x_i = u/p^i$ for $i = 1, \dots, d$ and 0 otherwise, it follows that the μ -moment of $P_{d,u}$ is equal to

$$\begin{aligned} & |\text{Aut}(\mu)| \prod_{i=1}^d (1 - u/p^i) \cdot \sum_{\nu} \frac{P_{\mu}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\mu)}} \\ & \cdot \frac{P_{\nu}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\nu)}} \\ & = |\text{Aut}(\mu)| \frac{P_{\mu}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\mu)}} \\ & \cdot \sum_{\nu} \prod_{i=1}^d (1 - u/p^i) \cdot \frac{P_{\nu}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\nu)}}. \end{aligned}$$

By Proposition 2.1, this is equal to

$$|\text{Aut}(\mu)| \frac{P_{\mu}(\frac{u}{p}, \frac{u}{p^2}, \dots, \frac{u}{p^d}, 0, \dots; \frac{1}{p})}{p^{n(\mu)}}.$$

By pages 181 and 213 of [19], this simplifies to

$$\frac{u^{|\mu|} (1/p)_d}{(1/p)_{d-r(\mu)}}.$$

□

Remarks:

- The exact same argument proves the analogous result for the distribution $P_{\infty, u}$.
- Setting $d = \infty$ and $u = 1/p^w$ (with w a positive integer) gives the distribution (4), and in this case Theorem 5.3 recovers Lemma 3.2 of [25].
- The argument used in the proof of Theorem 5.3 does not require that p is prime.

We use Theorem 5.3 to determine the expected number of p^ℓ -torsion elements of a finite abelian group H drawn from $P_{d, u}$. Let T_ℓ be defined by

$$T_\ell(H) = |H[p^\ell]| = |\{x \in H : p^\ell \cdot x = 0\}|.$$

The number of elements of H of order exactly p^ℓ is $T_\ell(H) - T_{\ell-1}(H)$.

For a finite abelian p -group H , let $r_{p^k}(H)$ denote the p^k -rank of H , that is,

$$r_{p^k}(H) = \dim_{\mathbb{Z}/p\mathbb{Z}}(p^{k-1}H/p^kH).$$

If H is of type λ , then $r_{p^k}(H) = \lambda'_k$, the number of parts of λ of size at least k . The number of parts of λ of size exactly k is $\lambda'_k - \lambda'_{k+1}$. It is clear that

$$T_\ell(H) = p^{r_p(H) + r_{p^2}(H) + \dots + r_{p^\ell}(H)} = p^{\lambda'_1 + \lambda'_2 + \dots + \lambda'_\ell}.$$

Theorem 5.4. *Let p be a prime, ℓ be a positive integer, and $0 < u < p$. The expected value of $T_\ell(H)$ for a finite abelian p -group H drawn from $P_{d, u}$ is*

$$(u^\ell + u^{\ell-1} + \dots + u)(1 - p^{-d}) + 1.$$

The expected value of $T_\ell(H) - T_{\ell-1}(H)$ is $u^\ell(1 - p^{-d})$.

Remarks:

- The exact same argument proves the analogous result for the distribution $P_{\infty, u}$.
- Taking $d = \infty$, $u = p^{-w}$ recovers a result of Delaunay, the first part of Corollary 3 of [9]. Delaunay's result generalizes work of Cohen and Lenstra for $P_{\infty, 1}$ and $P_{\infty, 1/p}$ [5].
- Theorem 5.3 can likely be used to compute moments of more complicated functions involving $T_\ell(H)$ giving results similar to those of Delaunay and Jouhet in [6]. We do not pursue this further here.

Lemma 5.5. *Let H be a finite abelian p -group of type λ and let $\ell \geq 1$. Then*

$$\#\text{Hom}(H, \mathbb{Z}/p^\ell\mathbb{Z}) = p^{r_{p^\ell}(H) + r_{p^{\ell-1}}(H) + \dots + r_p(H)} = p^{\lambda'_1 + \lambda'_2 + \dots + \lambda'_\ell} = T_\ell(H).$$

Proof. Suppose

$$H \cong \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{\lambda_{r_p(H)}}\mathbb{Z},$$

and consider the particular generating set for H

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_{r_p(H)} = (0, \dots, 0, 1).$$

Note that e_i has order p^{λ_i} .

A homomorphism from H to $\mathbb{Z}/p^\ell\mathbb{Z}$ is uniquely determined by the images of $e_1, \dots, e_{r_p(H)}$. When $\lambda_i \geq \ell$ there are p^ℓ choices for the image of e_i . If $1 \leq \lambda_i \leq \ell$, there are p^{λ_i} choices for the image of e_i . Therefore, the total number of homomorphisms is

$$p^{\ell\lambda'_\ell + (\ell-1)(\lambda'_{\ell-1} - \lambda'_\ell) + \dots + 1 \cdot (\lambda'_1 - \lambda'_2)}.$$

□

Proof of Theorem 5.4. We compute the expected value of

$$\#\text{Hom}(H, \mathbb{Z}/p^\ell\mathbb{Z}) - \#\text{Hom}(H, \mathbb{Z}/p^{\ell-1}\mathbb{Z})$$

and apply Lemma 5.5 to complete the proof.

Let H be a finite abelian p -group drawn from $P_{d,u}$. Every element of $\text{Hom}(H, \mathbb{Z}/p^\ell\mathbb{Z})$ is either a surjection, or surjects onto a unique proper subgroup of $\mathbb{Z}/p^\ell\mathbb{Z}$. Every proper subgroup of $\mathbb{Z}/p^\ell\mathbb{Z}$ is contained in the unique proper subgroup of $\mathbb{Z}/p^\ell\mathbb{Z}$ that is isomorphic to $\mathbb{Z}/p^{\ell-1}\mathbb{Z}$. Therefore,

$$\#\text{Sur}(H, \mathbb{Z}/p^\ell\mathbb{Z}) = \#\text{Hom}(H, \mathbb{Z}/p^\ell\mathbb{Z}) - \#\text{Hom}(H, \mathbb{Z}/p^{\ell-1}\mathbb{Z}).$$

Lemma 5.5 implies $T_\ell(H) - T_{\ell-1}(H) = \#\text{Sur}(H, \mathbb{Z}/p^\ell\mathbb{Z})$. Applying Theorem 5.3, noting that $T_0(H) = 1$ for any H , completes the proof. □

We close this section by proving a version of Lemma 5.1 for the distribution $P_{d,u}$. The proof of Lemma 8.2 from [10] carries over almost exactly to this more general setting.

Theorem 5.6. *Suppose that $p > 1$ and $0 < u < p$ are such that*

$$(14) \quad \frac{1}{(u/p)_d} = \prod_{i=1}^d (1 - u/p^i)^{-1} < 2.$$

If ν is any probability measure on the set of partitions for which

$$(15) \quad \sum_{\lambda} \nu(\lambda) |\text{Sur}(\lambda, \mu)| = \begin{cases} \frac{u^{|\mu|} (1/p)_d}{(1/p)_{d-r(\mu)}} & \text{if } r(\mu) \leq d \\ 0 & \text{otherwise,} \end{cases}$$

then $\nu = P_{d,u}$.

Remarks:

- When p is prime this result has an interpretation in terms of probability measures on \mathcal{L} .
- The exact same argument proves the analogous result for the distribution $P_{\infty,u}$.
- The expression on the left-hand side of (14) is decreasing in p and in u . Setting $d = \infty$, $u = 1$ and noting that this inequality holds for all $p \geq 3$ gives Lemma 5.1.
- Similarly, setting $d = \infty$, $u = 1/p^w$ (with p prime and w a positive integer) gives Proposition 2.3 of [26].

- Theorem 5.6 only applies when $1/(u/p)_d < 2$. Results of Wood imply that the moments determine the distribution in additional cases where p is prime, for example when $p = 2$, $d = \infty$, and $u = 1$. See Theorem 3.1 in [25] and Theorem 8.3 in [24].

Proof. The assumption gives, for every μ

$$(16) \quad |\text{Aut}(\mu)|\nu(\mu) + \sum_{\lambda \neq \mu} |\text{Sur}(\lambda, \mu)|\nu(\lambda) = \begin{cases} \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}} & \text{if } r(\mu) \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Since the second term in the left-hand side of (16) is non-negative, for $r(\mu) > d$ we have $|\text{Aut}(\mu)|\nu(\mu) = 0$, so $\nu(\mu) = 0$.

Now suppose that $r(\mu) \leq d$. Our goal is to show that

$$\nu(\mu) = \frac{u^{|\mu|}(u/p)_d}{|\text{Aut}(\mu)|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}}.$$

By Theorem 5.3, in the particular case $\nu = P_{d,u}$, (16) is equal to

$$\frac{u^{|\mu|}(u/p)_d(1/p)_d}{(1/p)_{d-r(\mu)}} + \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} u^{|\lambda|}(u/p)_d \frac{|\text{Sur}(\lambda, \mu)|}{|\text{Aut}(\lambda)|} \frac{(1/p)_d}{(1/p)_{d-r(\lambda)}} = \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}}.$$

This gives

$$\sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} u^{|\lambda|} \frac{|\text{Sur}(\lambda, \mu)|}{|\text{Aut}(\lambda)|(1/p)_{d-r(\lambda)}} = \frac{u^{|\mu|}}{(1/p)_{d-r(\mu)}} \left(\frac{1}{(u/p)_d} - 1 \right).$$

Let

$$\beta = \frac{(1/p)_{d-r(\mu)}}{u^{|\mu|}} \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} u^{|\lambda|} \frac{|\text{Sur}(\lambda, \mu)|}{|\text{Aut}(\lambda)|(1/p)_{d-r(\lambda)}} = \frac{1}{(u/p)_d} - 1.$$

It is enough to show that

$$(17) \quad |\text{Aut}(\mu)|\nu(\mu) = u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} \frac{1}{\beta + 1}.$$

By assumption, $|\beta| < 1$, so we verify (17) by showing that $|\text{Aut}(\mu)|\nu(\mu)$ is bounded by the alternating partial sums of the series

$$u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} \frac{1}{\beta + 1} = u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} (1 - \beta + \beta^2 - \dots).$$

Equation (16) implies that

$$|\text{Aut}(\mu)|\nu(\mu) \leq \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}}.$$

For any λ with $r(\lambda) \leq d$ this gives

$$\nu(\lambda) \leq \frac{u^{|\lambda|}(1/p)_d}{|\text{Aut}(\lambda)|(1/p)_{d-r(\lambda)}}.$$

Using this bound in (16) gives

$$\begin{aligned} |\text{Aut}(\mu)|\nu(\mu) &= u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} - \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} |\text{Sur}(\lambda, \mu)|\nu(\lambda) \\ &\geq u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} - \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} u^{|\lambda|} \frac{|\text{Sur}(\lambda, \mu)|}{|\text{Aut}(\lambda)|} \frac{(1/p)_d}{(1/p)_{d-r(\lambda)}} \\ &= \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}} - \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}} \beta = \frac{u^{|\mu|}(1/p)_d}{(1/p)_{d-r(\mu)}} (1 - \beta). \end{aligned}$$

Similarly, for any λ with $r(\lambda) \leq d$, this gives

$$\nu(\lambda) \geq \frac{u^{|\lambda|}}{|\text{Aut}(\lambda)|} \frac{(1/p)_d}{(1/p)_{d-r(\lambda)}} (1 - \beta).$$

Using this bound in (16) gives

$$\begin{aligned} |\text{Aut}(\mu)|\nu(\mu) &= u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} - \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} |\text{Sur}(\lambda, \mu)|\nu(\lambda) \\ &\leq u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} - \sum_{\substack{\lambda \neq \mu \\ r(\lambda) \leq d}} u^{|\lambda|} \frac{|\text{Sur}(\lambda, \mu)|}{|\text{Aut}(\lambda)|} \frac{(1/p)_d}{(1/p)_{d-r(\lambda)}} (1 - \beta) \end{aligned}$$

which implies

$$\begin{aligned} |\text{Aut}(\mu)|\nu(\mu) &\leq u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} - u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} \beta (1 - \beta) \\ &= u^{|\mu|} \frac{(1/p)_d}{(1/p)_{d-r(\mu)}} (1 - \beta + \beta^2). \end{aligned}$$

Continuing in this way completes the proof. \square

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