A Probabilistic Proof of the Rogers-Ramanujan Identities

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Abstract

The asymptotic probability theory of conjugacy classes of the finite general groups leads to a probability measure on the set of all partitions of natural numbers. A simple method of understanding this measure in terms of Markov chains is given, leading to an elementary probabilistic proof of the Rogers-Ramanujan identities. This is compared with work on the uniform measure. The main case of Bailey's lemma is interpreted as finding eigenvectors of the transition matrix of a Markov chain. It is shown that the viewpoint of Markov chains extends to quivers.

Key words: Rogers-Ramanujan, Markov chain, quiver, conjugacy class

1 Introduction

The Rogers-Ramanujan identities [34], [35], [37]

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

$$1+\sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

are among the most interesting partition identities in number theory and combinatorics. Combinatorial aspects of these identities are discussed by Andrews [2],[5]. Lepowsky and Wilson [28], [29], [30] connect the Rogers-Ramanujan identities with affine Lie algebras and conformal field theory. Feigen and Frenkel [15] interpret them as a character formula for the Virasoro algebra. The author [18] applies the product form to computational group theory, obtaining a simple formula for the $n \to \infty$ limit of the probability that an element of $GL(n,q)$ is semisimple. Stembridge [38] proves them by adapting a method of Macdonald for calculating partial fraction expansions of symmetric formal power series. Garsia and Milne [22] offer a lengthy bijective proof, giving birth to the "involution principle" in combinatorics. Garrett, Ismail, and Stanton [21] recast Roger's original proof using orthogonal polynomials. Relations with statistical mechanics appear in [8], [4], and [40]. The paper [9] gives an account of their appearance in physics.

One of the main results of this note is a first probabilistic proof of the Rogers-Ramanujan identities. In fact the $i=1$ and $i=k$ cases of Theorem 1 will be proved. Theorem 1 is due to Andrews [1]. Bressoud [13] then connected it with an earlier combinatorial result of Gordon [23].

Theorem 1 *([1]) Let $(x)_n$ denote $(1-x)\cdots(1-x^n)$. For $1 \le i \le k, k \ge 2$,*

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{x^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(x)_{n_1}\cdots(x)_{n_{k-1}}} = \prod_{\substack{r=1 \\ r\neq 0,\pm i (mod \ \ 2k+1)}}^{\infty} \frac{1}{1-x^r}$$

where $N_j = n_j + \cdots n_{k-1}$.

There are other elementary proofs of the Rogers-Ramanujan identities available (e.g. [11]), but the one offered here is probably the most natural and gives insight into Bailey's Lemma.

The basic object of study in this paper is a certain one-parameter family of probability measures M_u on the set of all partitions of all natural numbers, studied in the prior article [17]. Section 2 recalls these measures and gives their group theoretic motivation. (For now we remark that they arise in the study of conjugacy classes of finite classical groups. As conjugacy classes of compact Lie groups are essentially eigenvalues up to the action of the Weyl group, their probabilistic study can be regarded as philosophically analogous to work of Dyson [14], who studied the eigenvalues of random matrices of compact Lie groups).

Section 3 shows how to construct these measures using a non-upward-moving Markov chain on the integers. The Markov chain is diagonalizable with eigenvalues $1, \frac{u}{q}, \frac{u^2}{q^4}, \cdots$, and a basis of eigenvectors is given. Analogous computations are done for a measure related to the uniform measure on partitions, which by work of Fristedt [16] has a Markov chain approach. It would be interesting to make a connection with the articles [36], in which a fascinating continuous space Markov chain arises in the asymptotic probability theory of the symmetric group.

With these preliminaries in place, Section 4 gives a proof of the Rogers-Ramanujan identities. The idea of the proof is simple. We compute in two ways the $L \to \infty$ probability that the Markov chain started at L is absorbed at the point 0 after k steps. (Since the Markov chain is absorbed at 0 with probability 1 and the measure M_u corresponds to the $L \to \infty$ limit, the time to absorption really is the most natural quantity one could consider). The sum side of the Rogers-Ramanujan identities follows from the definition of the probability measures. For the product side, the fact that the transition matrix is explicitly diagonalizable gives a sum expression. One then applies the Jacobi triple product identity (which as explained on page 21 of [2] follows easily from the q-binomial theorem) and the proof is complete.

Section 4 continues by discussing Bailey's Lemma, which is the only non-trivial step in many of the simplest proofs of the Rogers-Ramanujan identities. The most useful case of Bailey's Lemma follows immediately once a basis of eigenvectors of the transition matrix has been found. The only non-trivial step in our proof of the Rogers-Ramanujan identities is finding a basis of eigenvectors; however either by hand or with Mathematica this is easy. By contrast it is unclear how one would guess at Bailey's Lemma.

Section 5 reviews the theory of quivers and shows that the Markov chain method extends to quivers. Although we have not invested serious effort into finding analogs of Bailey's Lemma for quivers other than the one point quiver (which corresponds to conjugacy classes of the finite general linear groups), it is not hard to see that the resulting Bailey Lemmas differ from those of [32] and [6]. The follow-up paper [19] shows that the viewpoint of Markov chains extends to the finite symplectic and orthogonal groups.

It is tempting to speculate that there is a direct relationship between conjugacy classes of the finite general linear groups and modular forms. Aside from this paper, there are two good reasons to suspect this. One reason is that the conjugacy classes are related to Hall-Littlewood polynomials, which in turn are related to vertex operators [26]. Other evidence is work of Bloch and Okounkov [10], who relate a version of the uniform measure on partitions to quasi-modular forms.

2 Measures on Partitions and Group Theory

We begin by reviewing some standard notation about partitions, as on pages 2-5 of Macdonald [31]. Let λ be a partition of some non-negative integer $|\lambda|$ into parts $\lambda_1 \geq \lambda_2 \geq \cdots$. Let $m_i(\lambda)$ be the number of parts of λ of size i, and let λ' be the partition dual to λ in the sense that $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Let $n(\lambda)$ be the quantity $\sum_{i \geq 1}(i-1)\lambda_i$. It is also useful to define the diagram associated to λ as the set of points $(i,j) \in Z^2$ such that $1 \leq j \leq \lambda_i$. We use the convention that the row index i increases as one goes downward and the column index j increases as one goes across. So the diagram of the partition (5441) is:

$$\begin{array}{ccccc} . & . & . & . & . \\ . & . & . & . & \\ . & . & . & . & \\ . & & & & \end{array}$$

The rest of this section follows the paper [17]. Let q be the size of a finite field. To begin we recall a way of defining a one parameter family of probability measures $M_u(\lambda)$ on the set of all partitions of all natural numbers. If one simply wants a formula, then all of the following definitions are equivalent. In the third expression, P_λ denotes a Hall Littlewood polynomial as in [31].

$$
\begin{aligned}
M_u(\lambda) &= \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right] \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} \\
&= \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right] \frac{u^{|\lambda|}}{q^{2[\sum_{h<i} h m_h(\lambda) m_i(\lambda) + \frac{1}{2}\sum_i (i-1) m_i(\lambda)^2]} \prod_i |GL(m_i(\lambda),q)|} \\
&= \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right] \frac{u^{|\lambda|} P_\lambda(\frac{1}{q},\frac{1}{q^2},\cdots;0,\frac{1}{q})}{q^{n(\lambda)}}
\end{aligned}
$$

The only fact from this section which is required for the proof of the Rogers-Ramanujan identities is the fact that M_u defines a probability measure for $0 < u < 1$. This can be seen using an identity from either [31] or else following Stong [39], who uses the fact that there are q^{n^2-n} unipotent elements in $GL(n,q)$. The first proof of Theorem 2 will use the fact that M_u is a probability measure without further comment. As there has been interest in simplifying the proofs of the Rogers-Ramanujan identities as much as possible, a second completely elementary proof of Theorem 2 will be given. From this second proof it will follow that $M_u(\lambda)$ is a probability measure.

Although not logically necessary for this paper, we mention that for $0 < u < 1$ and q a prime power, the measures M_u have a group theoretic description, referring the reader to the survey [20] for further discussion. Recall that the conjugacy classes of $GL(n,q)$ are parameterized by rational canonical form. Each such matrix corresponds to the following combinatorial data. To every monic non-constant irreducible polynomial ϕ over F_q, associate a partition (perhaps the trivial partition) λ_ϕ of some non-negative integer $|\lambda_\phi|$. The only restrictions necessary for this data to represent a conjugacy class are that $|\lambda_z| = 0$ and $\sum_\phi |\lambda_\phi| deg(\phi) = n$. To be explicit, a representative of the conjugacy class corresponding to the data λ_ϕ may be given as follows. Define the companion matrix $C(\phi)$ of a polynomial $\phi(z) = z^{deg(\phi)} + \alpha_{deg(\phi)-1} z^{deg(\phi)-1} + \cdots + \alpha_1 z + \alpha_0$ to be:

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
-\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{deg(\phi)-1}
\end{pmatrix}
$$

Let $\phi_1, \cdots, \phi_k$ be the polynomials such that $|\lambda_{\phi_i}| > 0$. Denote the parts of λ_{ϕ_i} by $\lambda_{\phi_i,1} \geq \lambda_{\phi_i,2} \geq \cdots$. Then a matrix corresponding to the above conjugacy class data is:

$$\begin{pmatrix} C(\phi_i^{\lambda_{\phi_i,1}}) & 0 & 0 \\ 0 & C(\phi_i^{\lambda_{\phi_i,2}}) & 0 \\ 0 & 0 & \cdots \end{pmatrix}$$

Now consider the following procedure for putting a measure on the set of all partitions of all natural numbers. Fix u such that $0 < u < 1$. Pick a non-negative integer such that the chance of choosing n is equal to $(1-u)u^n$. Then pick α uniformly in $GL(n,q)$ and take λ to be the paritition corresponding to the polynomial $z-1$ in the rational canonical form of α. If $n=0$ take λ to be the trivial partition. The random partition so defined obeys M_u measure. (The polynomial $z-1$ is considered without loss of generality. Partitions corresponding to other irreducible polynomials are probabilistically independent, and in all formulas one just replaces q and u by raising them to the degree of the polynomial). In the limit $u \to 1$, one is simply studying random elements in $GL(n,q)$ with $n \to \infty$. The substitutions $u \to -u$ and $q \to -q$ correspond to the finite unitary groups. The idea of auxilliary randomization of the dimension n is analogous to the idea of canonical ensembles in statistical mechanics.

3 Markov chains

3.1 Group theoretical measures

The first result of this paper describes the measure M_u in terms of Markov chains. Two proofs will be given. The first proof is given in the interest of clarity and assumes that M_u is a probability measure. The second proof is more elementary. It is surprising that M_u has such a simple probabilistic description.

It is convenient to set λ_0' (the height of an imaginary zeroth column) equal to ∞. For the entirety of this subsection, $(x)_n$ will denote $(1-x)(1-x/q)\cdots(1-x/q^{n-1})$. Thus $(x)_0 = 1$ and $(x)_n = 0$ for $n < 0$. For convenience of notation, let $P(a)$ be the M_u probability that $\lambda_1' = a$. $Prob(E)$ will denote the probability of an event E under the measure M_u.

Theorem 2 *Starting with $\lambda_0' = \infty$, define in succession $\lambda_1', \lambda_2', \cdots$ according to the rule that if $\lambda_i' = a$, then $\lambda_{i+1}' = b$ with probability*

$$K(a,b) = \frac{u^b (\frac{1}{q})_a (\frac{u}{q})_a}{q^{b^2} (\frac{1}{q})_{a-b} (\frac{1}{q})_b (\frac{u}{q})_b}.$$

Then the resulting partition is distributed according to M_u.

PROOF: The M_u probability of choosing a partition with $\lambda_i' = r_i$ for all i is

$$Prob.(\lambda_0' = \infty) \frac{Prob.(\lambda_0' = \infty, \lambda_1' = r_1)}{Prob.(\lambda_0' = \infty)} \prod_{i=1}^{\infty} \frac{Prob.(\lambda_0' = \infty, \lambda_1' = r_1, \cdots, \lambda_{i+1}' = r_{i+1})}{Prob.(\lambda_0' = \infty, \lambda_1' = r_1, \cdots, \lambda_i' = r_i)}.$$

Thus it is enough to prove that

$$\frac{Prob.(\lambda_0' = \infty, \lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1}, \lambda_i' = a, \lambda_{i+1}' = b)}{Prob.(\lambda_0' = \infty, \lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1}, \lambda_i' = a)} = \frac{u^b (\frac{1}{q})_a (\frac{u}{q})_a}{q^{b^2} (\frac{1}{q})_{a-b} (\frac{1}{q})_b (\frac{u}{q})_b},$$

for all $i, a, b, r_1, \cdots, r_{i-1} \geq 0$.

For the case $i = 0$, the equation

$$P(a) = \frac{u^a (\frac{u}{q})_\infty}{q^{a^2} (\frac{1}{q})_a (\frac{u}{q})_a}$$

is given a probabilistic proof in [17]. For an elementary proof of this identity, see the second proof of this theorem. For $i > 0$ one calculates that

$$\sum_{\substack{\lambda:\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1} \\ \lambda_i' = a}} M_u(\lambda) = \frac{u^{r_1 + \cdots + r_{i-1}}}{q^{r_1^2 + \cdots + r_{i-1}^2} (\frac{1}{q})_{r_1 - r_2} \cdots (\frac{1}{q})_{r_{i-2} - r_{i-1}} (\frac{1}{q})_{r_{i-1} - a}} P(a).$$

Similarly, observe that

$$\sum_{\substack{\lambda:\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1} \\ \lambda_i' = a, \lambda_{i+1}' = b}} M_u(\lambda) = \frac{u^{r_1 + \cdots + r_{i-1} + a}}{q^{r_1^2 + \cdots + r_{i-1}^2 + a^2} (\frac{1}{q})_{r_1 - r_2} \cdots (\frac{1}{q})_{r_{i-2} - r_{i-1}} (\frac{1}{q})_{r_{i-1} - a} (\frac{1}{q})_{a-b}} P(b).$$

Thus the ratio of these two expressions is

$$\frac{u^b (\frac{1}{q})_a (\frac{u}{q})_a}{q^{b^2} (\frac{1}{q})_{a-b} (\frac{1}{q})_b (\frac{u}{q})_b},$$

as desired. Note that the transition probabilities automatically sum to 1 because

$$\sum_{b \leq a} \frac{\sum_{\substack{\lambda:\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1} \\ \lambda_i' = a, \lambda_{i+1}' = b}} M_u(\lambda)}{\sum_{\substack{\lambda:\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1} \\ \lambda_i' = a}} M_u(\lambda)} = 1$$

for any measure on partitions. $\square$

PROOF: (Second Proof) This proof needs only that M_u is a measure; it will emerge that M_u is a probability measure. As before, $P(a)$ denotes the M_u mass that $\lambda_1' = a$.

One calculates that

$$\sum_{\substack{\lambda:\lambda_1'=r_1,\cdots,\lambda_{i-1}'=r_{i-1}\\ \lambda_i'=a}} M_u(\lambda) = \frac{u^{r_1+\cdots+r_{i-1}}}{q^{r_1^2+\cdots+r_{i-1}^2}(\frac{1}{q})_{r_1-r_2}\cdots(\frac{1}{q})_{r_{i-2}-r_{i-1}}(\frac{1}{q})_{r_{i-1}-a}}P(a).$$

Similarly, observe that

$$\sum_{\substack{\lambda:\lambda_1'=r_1,\cdots,\lambda_{i-1}'=r_{i-1}\\ \lambda_i'=a,\lambda_{i+1}'=b}} M_u(\lambda) = \frac{u^{r_1+\cdots+r_{i-1}+a}}{q^{r_1^2+\cdots+r_{i-1}^2+a^2}(\frac{1}{q})_{r_1-r_2}\cdots(\frac{1}{q})_{r_{i-2}-r_{i-1}}(\frac{1}{q})_{r_{i-1}-a}(\frac{1}{q})_{a-b}}P(b).$$

Thus the ratio of these two expressions is

$$\frac{P(b)u^a}{P(a)q^{a^2}(\frac{1}{q})_{a-b}}.$$

Since M_u is a measure, it follows that

$$\sum_{b\leq a}\frac{P(b)u^a}{P(a)q^{a^2}(\frac{1}{q})_{a-b}} = 1.$$

From this recursion and the fact that $P(0) = (\frac{u}{q})_\infty$, one solves for $P(a)$ inductively, finding that

$$P(a) = \frac{u^a(\frac{u}{q})_\infty}{q^{a^2}(\frac{1}{q})_a(\frac{u}{q})_a}.$$

Identity 2.2.8 on page 20 of [2] now implies that $\sum_a P(a) = 1$, so that M_u is a probability measure. $\square$

The algorithm of Theorem 2 can be implemented on a computer.

Theorem 3 explicitly diagonalizes the transition matrix K, which is fundamental for understanding the Markov chain. Note that if the current distribution of the Markov chain is given by a row vector, the distribution at the next step is obtained by multiplying the row vector on the right by K.

Theorem 3

1. *Let C be the diagonal matrix with (i,i) entry $(\frac{1}{q})_i(\frac{u}{q})_i$. Let M be the matrix $\left(\frac{u^j}{q^{j^2}(\frac{1}{q})_{i-j}}\right)$. Then $K = CMC^{-1}$, which reduces the problem of diagonalizing K to that of diagonalizing M.*

2. *Let A be the matrix $\left(\frac{1}{(\frac{1}{q})_{i-j}(\frac{u}{q})_{i+j}}\right)$. Then the columns of A are eigenvectors of M for right multiplication, the jth column having eigenvalue $\frac{u^j}{q^{j^2}}$.*

3. *The inverse matrix A^{-1} is $\left(\frac{(1-u/q^{2i})(-1)^{i-j}(\frac{u}{q})_{i+j-1}}{q^{\binom{i-j}{2}}(\frac{1}{q})_{i-j}}\right)$.*

PROOF: The first part is obvious. The second part is a special case of Lemma 1 of [11]. The third part is a lemma of [3]. □

The point of the proof of Theorem 3 is that once one knows (either from Mathematica or by implementing algorithms from linear algebra) what the eigenvectors are, it is a simple matter to verify the computation.

The following corollary will be useful for the proof of the Rogers-Ramanujan identities.

Corollary 1 *Let E be the diagonal matrix with (i,i) entry $\frac{u^i}{q^{i^2}}$. Then $K^r = CAE^rA^{-1}C^{-1}$. More explicitly,*

$$K^r(L,j) = \frac{(\frac{1}{q})_L(\frac{u}{q})_L}{(\frac{1}{q})_j(\frac{u}{q})_j} \sum_{n=0}^{\infty} \frac{u^{rn}(1-u/q^{2n})(-1)^{n-j}(\frac{u}{q})_{n+j-1}}{q^{rn^2}(\frac{1}{q})_{L-n}(\frac{u}{q})_{L+n} q^{\binom{n-j}{2}}(\frac{1}{q})_{n-j}}.$$

PROOF: This is immediate from Theorem 3. □

3.2 Mixture of uniform measures

For this subsection $q < 1$. The measure assigns probability $q^{|\lambda|}\prod_{i=1}^{\infty}(1-q^i)$ to the partition λ. Conditioning this measure to live on partitions of a given size gives the uniform measure, an observation exploited by Fristedt [16]. As is clear from [33], this measure is very natural from the viewpoint of representation theory.

Fristedt (loc. cit.) proved that this measure has a Markov chain description. His chain affects row lengths rather than column lengths (though the algorithm would work on columns too as the measure is invariant under transposing diagrams). Nevertheless, we adhere to his notation. We use the notation that $(x)_n = (1-x)\cdots(1-x^n)$.

Theorem 4 *[16] Starting with $\lambda_0 = \infty$, define in succession $\lambda_1, \lambda_2, \cdots$ according to the rule that if $\lambda_i = a$, then $\lambda_{i+1} = b$ with probability*

$$K(a,b) = \frac{q^b(q)_a}{(q)_b}.$$

Then the resulting partition is distributed according to the measure of this subsection.

Theorem 5 diagonalizes this Markov chain, giving a basis of eigenvectors. The proof is analogous to that of Theorem 3, the second part being proved by induction.

Theorem 5 *1. Let C be the diagonal matrix with (i,i) entry equal to $\frac{(q)_i}{q^i}$. Let M be the matrix with (i,j) entry q^i if $i \geq j$ and 0 otherwise. Then $K = CMC^{-1}$.*

2. Let D be the diagonal matrix with (i,i) entry equal to q^i. Let A be the matrix $\left(\frac{(-1)^{i-j}}{q^{\binom{i-j}{2}}(\frac{1}{q})_{i-j}}\right)$, so that its inverse is $\left(\frac{1}{(\frac{1}{q})_{i-j}}\right)$ by part b of Theorem 3. Then the eigenvectors of M are the columns of A, the jth column having eigenvalue q^j.

As a corollary, one obtains a simple expression for the chance that under the measure of this subsection, the rth row has size j.

Corollary 2

$$K^r(L,j) = \frac{q^j q^{L(r-1)}(q)_L(\frac{1}{q})_{L-j+r-1}}{(q)_j(\frac{1}{q})_{L-j}(\frac{1}{q})_{r-1}}.$$

Letting $L \mapsto \infty$, the chance that the rth row has size j becomes

$$\frac{(q)_\infty q^{rj}}{(q)_j(q)_{r-1}}.$$

PROOF: To obtain the first expression, one multiplies out $K^r = CAD^rA^{-1}C^{-1}$ and uses the q-binomial theorem

$$\sum_{m=0}^{\infty} y^m q^{(m^2+m)/2} \frac{(q)_n}{(q)_m(q)_{m-n}} = (1+yq)(1+yq^2)\cdots(1+yq^n).$$

□

The second part of the corollary can be proved directly without recourse to Markov chain theory; one simply attaches to an $r * j$ square two partitions: one with at most $r-1$ rows and another with at most j columns.

4 Rogers-Ramanujan Identities and Bailey's Lemma

The first result of this section proves the following identity of Andrews, which contains the Rogers-Ramanujan identities. In this section $(x)_n$ denotes $(1-x)(1-x/q)\cdots(1-x/q^{n-1})$.

Theorem 6 *[1] For $k \geq 2$,*

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{1}{q^{N_1^2+\cdots+N_{k-1}^2}(1/q)_{n_1}\cdots(1/q)_{n_{k-1}}} = \prod_{\substack{r=1\\ r\neq 0,\pm k(mod\ \ 2k+1)}}^{\infty} \frac{1}{1-(1/q)^r}$$

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{1}{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}(1/q)_{n_1}\cdots(1/q)_{n_{k-1}}} = \prod_{\substack{r=1 \\ r\neq 0,\pm 1 (mod \;\; 2k+1)}}^{\infty} \frac{1}{1-(1/q)^r}$$

where $N_j = n_j + \cdots n_{k-1}$.

PROOF: For the first identity, we compute in two ways

$$\sum_{\lambda:\lambda_1<k} M_1(\lambda).$$

One obtains the sum side by using the first definition of M_1 in Section 2. For the product side, let $u=1$, $j=0$, $r=k$, and $L\to\infty$ in Corollary 1. The rest is now a standard argument.

$$\begin{aligned} 1+\sum_{n=1}^{\infty}\frac{(1+1/q^n)(-1)^n}{q^{rn^2}q^{\binom{n}{2}}} &= 1+\sum_{n=1}^{\infty}(-1)^n(\frac{1}{q})^{(k+1/2)n^2-n/2}+(\frac{1}{q})^{(k+1/2)n^2+n/2} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n(\frac{1}{q})^{(k+1/2)n^2}(\frac{1}{q})^{n/2}. \end{aligned}$$

Now invoke Jacobi's triple product identity (see e.g. [2] for an elementary proof)

$$\sum_{n=-\infty}^{\infty}(-1)^n v^n w^{n^2} = \prod_{n=1}^{\infty}(1-vw^{2n-1})(1-w^{2n-1}/v)(1-w^{2n}).$$

The proof of the second identity is the same except that one takes $u=\frac{1}{q}$ instead of $u=1$. $\square$

Next we discuss the most important case of Bailey's Lemma, which was alluded to in [7] and stated explicitly in [3]. A pair of sequences $\{\alpha_L\}$ and $\{\beta_L\}$ is called a Bailey pair if

$$\beta_L = \sum_{r=0}^{L}\frac{\alpha_r}{(1/q)_{L-r}(u/q)_{L+r}}.$$

Bailey's Lemma states that if $\alpha_L' = \frac{u^L}{q^{L^2}}\alpha_L$ and $\beta_L' = \sum_{r=0}^{L}\frac{u^r}{q^{r^2}(1/q)_{L-r}}\beta_r$, then $\{\alpha_L'\}$ and $\{\beta_L'\}$ are a Bailey pair.

From our viewpoint, this case of Bailey's Lemma is clear. Namely let A, D, M be as in Theorem 3 (recall that $M = ADA^{-1}$). Viewing $\alpha = \vec{\alpha_L}$ and $\beta = \vec{\beta_L}$ as column vectors, the notion of a Bailey pair means that $\beta = A\alpha$. This case of Bailey's Lemma follows because

$$\beta' = M\beta = ADA^{-1}\beta = AD\alpha = A\alpha'.$$

It would be interesting to obtain all of Bailey's Lemma (and Theorem 1) by probabilistic arguments, and also to understand the Bailey lattice probabilistically. Note that the idea of iterating Bailey's Lemma (e.g. Section 3.4 of [5]) is really taking steps according to the Markov chain K. The reader may enjoy the survey by Bressoud [12].

5 Quivers

This section uses the notion of a quiver, as surveyed in Kac [27], to which the reader is referred for more detail. The basic set-up is as follows. Let Γ be a connected graph with n vertices labelled as $1, \cdots, n$ (where we allow loops). Let N, Z denote the natural numbers and integers respectively. Let f_{ij} be the number of edges between i, j. Associated to Γ is a natural bilinear form on Z^n and a root system $\Delta \subset Z^n$. Choose an arbitrary orientation of Γ so that Γ is a quiver. For a given dimension $\alpha \in N^n - \{0\}$, let $A_\Gamma(\alpha, q)$ be the number of classes of absolutely indecomposable representations of Γ over the algebraic closure of a field of q elements. It is proved in [27] that $A_\Gamma(\alpha, q)$ is a polynomial in q with integer coefficients, and that this polynomial is independent of the orientation of the graph. Kac (loc. cit.) formulated many conjectures about this polynomial. One such conjecture, which is still open, is that the constant term in $A_\Gamma(\alpha, q)$ is the multiplicity of α in the root system.

In recent work, Hua [24, 25] has given a completely combinatorial reformulation of this conjecture. To explain, let $(1/q)_n$ denote $(1-\frac{1}{q})\cdots(1-\frac{1}{q^n})$, and for any two partitions λ, μ define $<\lambda, \mu> = \sum_{i \geq 1} \lambda_i' \mu_i'$. Let t_j^α be the coefficient of q^j in the polynomial $A_\Gamma(\alpha, q)$. Let $b_\lambda = \prod_{i \geq 1} (\frac{1}{q})_{m_i(\lambda)}$, where m_i is the number of parts of λ of size i. Let $\lambda(1), \cdots, \lambda(n)$ be an n-tuple of partitions. Set $U_\alpha = U_1^{\alpha_1} \cdots U_n^{\alpha_n}$. Hua's result, which reduces Kac's conjecture to a combinatorial assertion, is that

$$\sum_{\lambda(1),\cdots,\lambda(n)} \frac{\prod_{1 \leq i \leq j \leq n} q^{f_{ij} <\lambda(i),\lambda(j)>} U_1^{|\lambda(1)|} \cdots U_n^{|\lambda(n)|}}{\prod_{1 \leq i \leq n} q^{<\lambda(i),\lambda(i)>} b_{\lambda(i)}} = \prod_{\alpha \in \Delta^+} \prod_{j=0}^{deg(A_{\Gamma(\alpha,q)})} \prod_{i=1}^{\infty} \left(\frac{1}{1 - U_\alpha q^{j-i}}\right)^{t_j(\alpha)}.$$

A few points are in order. First, the right hand side of this equation is different from the statements in [24, 25], due to the fact that Lemma 4.8 of [24] and many of its consequences are false (fortunately the slip is minor). Second, observe that the expression converges in the ring of formal power series in the variables $U_1, \cdots, U_n$. This leads one to define a "probability" measure on n-tuples of partitions $M_{\Gamma,\vec{U}}$ by assigning mass

$$\prod_{\alpha \in \Delta^+} \prod_{j=0}^{deg(A_{\Gamma(\alpha,q)})} \prod_{i=1}^{\infty} (1 - U_\alpha q^{j-i})^{t_j(\alpha)} \frac{\prod_{1 \leq i \leq j \leq n} q^{f_{ij} <\lambda(i),\lambda(j)>} U_1^{|\lambda(1)|} \cdots U_n^{|\lambda(n)|}}{\prod_{1 \leq i \leq n} q^{<\lambda(i),\lambda(i)>} b_{\lambda(i)}}$$

to the n-tuple $\lambda(1), \cdots, \lambda(n)$. For quivers of finite type this is a true probability measure for values of $U_1, \cdots, U_n$ sufficiently small, but in general we abuse notation by using terms from probability theory when $U_1, \cdots, U_n$ are variables.

Note that when the graph consists of a single point, this measure is simply the measure M_u from Section 2. Theorem 7 shows that the structure of a Markov chain is still present. As the idea of the proof is the same as the second step of Theorem 2, the algebra is omitted. Let $P(a_1,\cdots,a_n)$ denote the $M_{\Gamma,\vec{U}}$ probability that $\lambda(1),\cdots,\lambda(n)$ have $a_1,\cdots,a_n$ parts respectively.

Theorem 7 *Let $\lambda(1)_1',\cdots,\lambda(n)_1'$ be distributed as $P(a_1,\cdots,a_n)$. Define $(\lambda(1)_2',\cdots,\lambda(n)_2')$ then $(\lambda(1)_3',\cdots,\lambda(n)_3')$, etc. successively according to the rule that if $(\lambda(1)_i',\cdots,\lambda(n)_i')$ is equal to $(a_1,\cdots,a_n)$, then $(\lambda(1)_{i+1}',\cdots,\lambda(n)_{i+1}')$ is equal to $(b_1,\cdots,b_n)$ with probability*

$$K(\vec{a},\vec{b}) = \prod_{1\leq i\leq j\leq n} q^{f_{ij}a_ia_j} \prod_{i=1}^{n} \frac{U_i^{a_i}}{q^{a_i^2}(\frac{1}{q})_{a_i-b_i}} \frac{P(b_1,\cdots,b_n)}{P(a_1,\cdots,a_n)}.$$

The resulting n-tuple of partitions is distributed according to $M_{\Gamma,\vec{U}}$.

As a final remark, observe that letting C be diagonal with entries $\frac{1}{P(a_1,\cdots,a_n)}$, one obtains a factorization $K = CMC^{-1}$, where M is defined by

$$M(\vec{a},\vec{b}) = \prod_{1\leq i\leq j\leq n} q^{f_{ij}a_ia_j} \prod_{i=1}^{n} \frac{U_i^{a_i}}{q^{a_i^2}(\frac{1}{q})_{a_i-b_i}}.$$

A very natural problem is to investigate the eigenvector matrix E of M, and E^{-1}, in order to obtain new Bailey Lemmas.

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