STEIN'S METHOD AND NARAYANA NUMBERS

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Abstract. Narayana numbers appear in many places in combinatorics and probability, and it is known that they are asymptotically normal. Using Stein's method of exchangeable pairs, we provide an error of approximation in total variation to a symmetric binomial distribution of order n^{-1}, which also implies a Kolmogorov bound of order $n^{-1/2}$ for the normal approximation. Our exchangeable pair is based on a birth-death chain and has remarkable properties, which allow us to perform some otherwise tricky moment computations.

1. Introduction

We use the convention that the Narayana numbers $N(n,k)$ are defined as

$$N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}, \; 1 \leqslant k \leqslant n$$

(some authors define them as $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, where $0 \leqslant k \leqslant n-1$). The Narayana numbers refine the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, since

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

The Catalan numbers are ubiquitous; see Stanley's book [8] for 214 objects enumerated by Catalan numbers. The Narayana numbers also appear in interesting places, and a good discussion of them is Chapter 2 of Petersen's book [2]. Some places (there are many others!) in combinatorics and probability where the Narayana numbers appear are: enumerating Dyck paths by peaks [2], enumerating 231-avoiding permutations by descents [2], enumerating non-crossing set partitions by the number of blocks [2], in the stationary distribution for the partially symmetric exclusion process [1], and in the enumeration of totally positive Grassmann cells [9].

Given these appearances of the Narayana numbers, it is natural to study their limiting distribution. We define a probability distribution π on the set $\{1, \cdots, n\}$ by

$$\pi(k) = \frac{N(n,k)}{C_n} = \frac{(n+1)\binom{n}{k}\binom{n}{k-1}}{n\binom{2n}{n}}$$

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We let K be a random variable which is equal to k with probability $\pi(k)$, and we define a random variable W by

$$W = \frac{K - \mu_n}{\sigma_n}$$

where

$$\mu_n = \frac{n+1}{2}, \quad \sigma_n^2 = \frac{(n-1)(n+1)}{4(2n-1)}.$$

It is observed in online lecture notes of Xi Chen ("Asymptotic normality in combinatorics", 2017) that W has mean 0 and variance 1, and is asymptotically normal. This follows from Harper's method for proving central limit theorems for numbers with real rooted generating functions [3] by representing them as a sum of independent random Bernoulli random variables, together with the fact that the generating function for Narayana polynomials has real roots (see Section 4.6 of [2]). This approach combined with, for example, the classical Berry-Esseen bound or even a refined translated Poisson approximation of independent indicators [4, Example 3.3], will yield a rate of convergence of σ_n^{-1}, which is order $n^{-1/2}$. However, using a direct exchangeable pairs approach, we show in this article that one can obtain better rates.

First, we define the *translated (almost) symmetric binomial distribution*, which will serve as the approximating distribution. While, ideally, we would want to approximate K by the symmetric binomial distribution $\mathrm{Bi}(n, 1/2)$ with n chosen to match the variance of K and shifted appropriately to match the mean of K, the restriction that both n and the shift have to be integer-valued requires some care in the exact definition.

For any real number x, let $\lceil x \rceil$ be the smallest integer that is larger or equal to x, let $\lfloor x \rfloor$ be the largest integer that is smaller than or equal to x, and let $\langle x \rangle = x - \lfloor x \rfloor$. Note that $x = \lfloor x \rfloor + \langle x \rangle$ and $x = \lceil x \rceil - \langle -x \rangle$. Assume $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are given. Let $\delta = \langle -4\sigma^2 \rangle$, so that $\lceil 4\sigma^2 \rceil = 4\sigma^2 + \delta$. Moreover, let $t = \langle -\mu + 2\sigma^2 + \delta/2 \rangle / \lceil 4\sigma^2 \rceil$. Denote by $\hat{\mathrm{Bi}}(\mu, \sigma^2)$ the binomial distribution $\mathrm{Bi}(\lceil 4\sigma^2 \rceil, 1/2 - t)$ shifted by $\mu - \lceil 4\sigma^2 \rceil (1/2 - t)$.

It is not difficult to check that if $X \sim \hat{\mathrm{Bi}}(\mu, \sigma^2)$, then X is integer-valued, that $\mathbb{E}X = \mu$ and that $\sigma^2 - 1/(4\sigma^2) \leqslant \operatorname{Var} X \leqslant \sigma^2 + 1/4$; in the context of distributional approximation, we like to think of $\hat{\mathrm{Bi}}(\mu, \sigma^2)$ as a discrete analogue to the normal distribution with mean μ and variance (almost) σ^2. The fact that we cannot match the variance exactly introduces only a very small error in the setting we are concerned with.

Finally, for probability measures P and Q on $\mathbb{Z}$, define the total variation metric

$$d_{\mathrm{TV}}(P, Q) = \sup_{A \subset \mathbb{Z}} |P[A] - Q[A]|$$

The purpose of this article is to prove the following explicit result, where $\mathscr{L}(K)$ denotes the law of K.

Theorem 1.1. *For K defined as above, we have*

$$d_{\mathrm{TV}}\big(\mathscr{L}(K), \hat{\mathrm{Bi}}(\mu,\sigma^2)\big) \leqslant \frac{12}{n}.$$

It is possible to deduce a Berry-Esseen-type bound from Theorem 1.1, but the rates of convergence for an integer-valued random variable to the normal distribution in Kolmogorov distance can never be better than the scaling factor.

Corollary 1.2. *There is a universal constant C such that*

$$\sup_{x\in\mathbb{R}} \lvert \mathbb{P}[W \leqslant x] - \Phi(x)\rvert \leqslant \frac{C}{n^{1/2}}, \tag{1.1}$$

where $\Phi(x)$ is the standard normal distribution.

Remark 1.3. (1) The bound (1.1) also follows from the classical Berry-Esseen bound for sums $S_n = \sigma^{-1}(X_1+\dots+X_n)$ of centered and independent, but non-identically distributed random variables; we have

$$\sup_{x\in\mathbb{R}} \lvert \mathbb{P}[S_n \leqslant x] - \Phi(x)\rvert \leqslant \frac{C_0\sum_{i=1}^n \mathbb{E}\lvert X_i\rvert^3}{\sigma^3}, \tag{1.2}$$

where C_0 is no bigger than 0.5606 [7]. Since the distribution of K can be represented as a sum of independent indicator random variables, and since the third central moment of an indicator random variable can always be upper-bounded by its variance, we obtain from (1.2) that the left hand side of (1.1) is bounded by $C_0\sigma_n^{-1}$, which yields that, on the right hand side of (1.1), we can take $C = 1.59$, since $\sigma_n \geqslant n^{1/2}/(2\sqrt{2})$ for $n \geqslant 2$, as is easy to prove.

(2) Applying a result of Shao and Su [6] to the exchangeable pair in this paper, one can show that one can take $C = 10$ in Corollary 1.2. The calculations involved are very similar to those in the present paper, but we omit the details as the bound in the previous remark is sharper.

We can also deduce a local limit theorem from Theorem 1.1, since the difference of the point probabilities are upper bounded by the total variation distance and since the total variation rates are better than σ_n^{-1}. It should also be fairly easy to prove this result directly using Stirling's approximation.

Corollary 1.4. *There is a universal constant C such that*

$$\sigma_n^{1/2} \sup_{k\in\mathbb{Z}} \left\lvert \mathbb{P}[K=k] - \frac{1}{\sigma_n}\varphi\Big(\frac{k-\mu_n}{\sigma_n}\Big)\right\rvert \leqslant \frac{C}{n^{1/2}},$$

where $\varphi(x)$ is the standard normal density.

We will apply the following result, which we will prove later using similar ideas as in [4, 5]. For its statement (and for the rest of the paper), recall that a pair of random variables (X, X') on a state space is called exchangeable if for all x_1 and x_2, we have $\mathbb{P}[X = x_1, X' = x_2] = \mathbb{P}[X = x_2, X' = x_1]$. Also, as is typical in probability theory, let $\mathbb{E}(X|Y)$ denote the expected value of X given Y.

Theorem 1.5. *Assume (X, X') is an exchangeable pair of integer-valued random variables with $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \sigma^2$, such that $X' - X \in \{-1, 0, 1\}$ almost surely and such that*

$$\mathbb{E}[X' - \mu | X] = (1 - \lambda)(X - \mu). \tag{1.3}$$

Then, with $S = S(X) = \mathbb{E}\{\mathrm{I}[X' \neq X] | X\}$,

$$d_{\mathrm{TV}}\big(\mathscr{L}(X), \hat{\mathrm{Bi}}(\mu, \sigma^2)\big) \leqslant \frac{\sqrt{\operatorname{Var} S}}{2\lambda\sigma^2} + \frac{1.4}{\sigma^2}. \tag{1.4}$$

Here $\mathscr{L}(X)$ denotes the law of X.

One of our contributions is to show how to apply Theorem 1.5 to prove Theorem 1.1. This is not straightforward for two reasons. First, it is not at all obvious how to construct an exchangeable pair (K, K') satisfying the linearity condition (1.3). As we show in Section 2, we do this using a birth-death chain on the state space $\{1, \dots, n\}$. We discovered this birth-death chain through experimentation. Second, in order to compute $Var(S)$, it turns out to be necessary to know the first four moments of the random variable K. The generating function for Narayana numbers is complicated. Indeed, from page 25 of [2], one has that

$$\sum_{n \geqslant 0} z^n \sum_{k=1}^{n} N(n, k) t^{k-1}$$

is equal to

$$\frac{1 + z(t-1) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2tz}. \tag{1.5}$$

It is not at all clear how to extract the fourth moment of K from (1.5). We show how to use properties of the exchangeable pair (W, W') to compute the first four moments of W (of course the first and third moments are zero by symmetry).

2. Proof of Theorem 1.1

Throughout we assume that $n \geqslant 2$ to avoid division by zero. In order to study the asymptotic behaviour of K by Stein's method, we will construct an exchangeable pair (K, K'). To do this, we first define a birth-death chain on the set $\{1, \dots, n\}$ by

$$p(k, k+1) = \frac{(n-k)(n-k+1)}{n(n-1)},$$
$$p(k, k-1) = \frac{k(k-1)}{n(n-1)},$$
$$p(k, k) = \frac{2(k-1)(n-k)}{n(n-1)}.$$

It is easy to see that $\pi(i)p(i, j) = \pi(j)p(j, i)$ for all i and j. Thus this birth death chain is reversible with respect to π. This allows us to construct an exchangeable pair (K, K') as follows: choose $K \in \{1, \cdots, n\}$ from π and then obtain K' by taking one step according to the birth-death chain.

The next result shows that the exchangeable pair (K, K') satisfies the linearity condition (1.3) of Theorem 1.5.

Lemma 2.1.
$$\mathbb{E}[K' - \mu | K] = \left(1 - \frac{2}{n-1}\right)(K - \mu).$$

Proof. We have
$$\begin{aligned}\mathbb{E}[K' - K|K] &= p(K, K+1) - p(K, K-1) \\ &= \frac{(n-K)(n-K+1) - K(K-1)}{n(n-1)} \\ &= \frac{n - 2K + 1}{n-1} = -\frac{2}{n-1}\big(K - \tfrac{n+1}{2}\big).\end{aligned}$$ □

As a corollary, we obtain the following result, which is also immediate from the symmetry of the distribution π, but it is interesting to deduce it using the pair (K, K').

Corollary 2.2. *We have*
$$\mathbb{E}K = \frac{n+1}{2}.$$

Proof. From the proof of Lemma 2.1 and the exchangeability of K and K', one has that
$$0 = \mathbb{E}[K' - K] = \mathbb{E}[\mathbb{E}[K' - K|K]] = -\frac{2}{n-1}\,\mathbb{E}\left[K - \frac{n+1}{2}\right].$$
□

Next we use the exchangeable pair to calculate $\mathbb{E}K^2$. The value of $\mathbb{E}K^2$ was stated in the online notes lecture notes of Xi Chen ("Asymptotic normality in combinatorics", 2017). Our derivation uses the exchangeable pair (K, K').

Lemma 2.3. *We have*
$$\mathbb{E}K^2 = \frac{n^3 + 2n^2 - 1}{4n - 2}.$$

Proof. Consider the quantity
$$\mathbb{E}[(K')^2 - K^2 | K].$$
On one hand, its expected value is equal to $\mathbb{E}[(K')^2 - K^2] = 0$. On the other hand, the construction of (K, K') gives that its expected value is equal to the expected value of
$$p(K, K+1)\big((K+1)^2 - K^2\big) + p(K, K-1)\big((K-1)^2 - K^2\big).$$
Using the definition of $p(K, K+1)$ and $p(K, K-1)$, it follows that
$$0 = \mathbb{E}[(n-K)(n-K+1)(2K+1) - K(K-1)(2K-1)].$$

Expanding this gives that

$$\mathbb{E}[K^2(2-4n)+n(1+n)+2K(n^2-1)]=0.$$

So we can solve for $\mathbb{E}K^2$ in terms of $\mathbb{E}K$, which we computed in Corollary 2.2. □

Lemma 2.4. *We have*

$$\mathbb{E}K^3 = \frac{(n^2+2n-2)(n+1)^2}{8n-4}.$$

Proof. By symmetry of the distribution π, we have $\mathbb{E}(K-\mu)^3=0$; that is,

$$\mathbb{E}\left[K^3 - 3K^2\left(\frac{n+1}{2}\right) + 3K\left(\frac{n+1}{2}\right)^2 - \left(\frac{n+1}{2}\right)^3\right] = 0.$$

Thus

$$\mathbb{E}K^3 = 3\left(\frac{n+1}{2}\right)\mathbb{E}K^2 - 3\left(\frac{n+1}{2}\right)^2\mathbb{E}K + \left(\frac{n+1}{2}\right)^3.$$

The lemma now follows from Corollary 2.2 and Lemma 2.3. □

Lemma 2.5. *We have*

$$\mathbb{E}K^4 = \frac{(n^5+4n^4-3n^3-12n^2+2n+6)(n+1)}{4(2n-1)(2n-3)}$$

Proof. Consider the quantity

$$\mathbb{E}[(K')^4 - K^4|K].$$

On one hand, its expected value is equal to $\mathbb{E}[(K')^4-K^4]=0$. On the other hand, the construction of (K',K) gives that its expected value is equal to the expected value of

$$\begin{aligned} &p(K,K+1)\big((K+1)^4-K^4\big)+p(K,K-1)\big((K-1)^4-K^4\big)\\ &\quad=\frac{(n-K)(n-K+1)}{n(n-1)}\big((K+1)^4-K^4\big)+\frac{K(K-1)}{n(n-1)}\big((K-1)^4-K^4\big)\end{aligned}$$

Since the K^5 and K^6 terms cancel out, this equality lets us solve for $\mathbb{E}K^4$ in terms of $\mathbb{E}K^3$, $\mathbb{E}K^2$ and $\mathbb{E}K$ (which were computed in Lemmas 2.4 and 2.3 and Corollary 2.2), yielding the final expression. □

Next, we give an exact formula for $\operatorname{Var} S$, where

$$S = S(K) = \mathbb{E}[(K'-K)^2|K].$$

Lemma 2.6. *For $n \geqslant 2$,*

$$\operatorname{Var} S = \frac{(n+1)(n-2)}{(2n-1)^2(2n-3)(n-1)}.$$

Proof. Clearly

$$
\begin{aligned}
S(K) &= p(K, K+1) + p(K, K-1) \\
&= 1 - p(K, K) = 1 - \frac{2(K-1)(n-K)}{n(n-1)}.
\end{aligned}
$$

Thus

$$
\operatorname{Var} S = \frac{4}{n^2(n-1)^2} \operatorname{Var}[(K-1)(n-K)]. \tag{2.1}
$$

To compute the variance of $(K-1)(n-K)$, we compute $\mathbb{E}[(K-1)(n-K)]$ and $\mathbb{E}[(K-1)^2(n-K)^2]$.

It follows from Corollary 2.2 and Lemma 2.3 that

$$
\mathbb{E}[(K-1)(n-K)] = -\mathbb{E}K^2 + (n+1)\,\mathbb{E}K - n = \frac{n(n-1)(n-2)}{(4n-2)}.
$$

Similarly, since $\mathbb{E}[(K-1)^2(n-K)^2]$ is equal to

$$
\mathbb{E}K^4 - (2n+2)\,\mathbb{E}K^3 + (n^2+4n+1)\,\mathbb{E}K^2 - (2n^2+2n)\,\mathbb{E}K + n^2.
$$

we can apply Corollary 2.2 and Lemmas 2.3, 2.4, and 2.5, to conclude that

$$
\mathbb{E}\big[(K-1)^2(n-K)^2\big] = \frac{n^2(n^4 - 7n^3 + 19n^2 - 23n + 10)}{4(4n^2 - 8n + 3)}.
$$

Thus

$$
\operatorname{Var}[(K-1)(n-K)] = \frac{(n+1)n^2(n-1)(n-2)}{4(2n-1)^2(2n-3)}.
$$

The lemma now follows from (2.1). □

Putting the pieces together, we now prove the main result of this paper.

Proof of of Theorem 1.1. The theorem clearly holds for $n \leqslant 2$. For $n > 2$, we apply Theorem 1.5. By Lemma 2.1, $\mathbb{E}[K' - \mu | K] = (1-\lambda)(K-\mu)$ with $\lambda = \frac{2}{n-1}$. It follows easily from Lemma 2.6 that

$$
d_{\mathrm{TV}}\big(\mathscr{L}(K), \hat{\mathrm{Bi}}(\mu, \sigma^2)\big) \leqslant \frac{1}{n}\sqrt{\frac{n^2(n-2)}{(2n-3)(n-1)(n+1)}} + \frac{5.6(2n-1)}{(n-1)(n+1)}
$$

It is not difficult to see that the fraction inside the square root is less than $1/2$ and that $5.6n(2n-1)(n-1)^{-1}(n+1)^{-1} \leqslant 11.2$ for $n \geqslant 2$. □

3. Proof of Theorem 1.5

We only give a very compact proof, since much of the material is explained in detail in [4, 5].

Proof of Theorem 1.5. Recall the definition of $\hat{\mathrm{Bi}}(\mu, \sigma^2)$ with the choices of n, δ and t. It can be shown [5, (2.8)] that Z has distribution $\hat{\mathrm{Bi}}(\mu, \sigma^2)$ if and only if

$$
\mathbb{E}\big\{(Z-\mu)\Theta g(Z) - \sigma^2 \Delta g(Z) + a(Z)\Delta g(Z)\big\} = 0
$$

for all, say, bounded functions $g : \mathbb{Z} \to \mathbb{R}$, where $\Theta g(k) = (g(k+1)+g(k))/2$, where $\Delta g(k) = g(k+1) - g(k)$, and where $a(k) = nt^2 - (k-\mu)t - \delta/4$. This motivates the definition of the Stein operator

$$(\mathcal{B}g)(k) := (k-\mu)\Theta g(k) - \sigma^2 \Delta g(k) + a(k)\Delta g(k)$$

and setting up the Stein equation

$$(\mathcal{B}g)(k) = \mathrm{I}[k \in A] - \mathbb{P}[Z \in A] \tag{3.1}$$

for $A \subset \mathcal{T} := \{0, \dots, n\} + \mu - n(1/2 - t)$ and $Z \sim \hat{\mathrm{Bi}}(\mu, \sigma^2)$.

Now, it was shown [5, (2.4) and (2.8)] that there is a solution g_A to (3.1) that satisfies

$$\|\Delta g_A\|_\infty \leqslant 1 \wedge \frac{1}{\sigma^2}. \tag{3.2}$$

Using identity (3.1), the triangle inequality and (3.2), we obtain

$$\begin{aligned} d_{\mathrm{TV}}\bigl(\mathscr{L}(X), \hat{\mathrm{Bi}}(\mu,\sigma^2)\bigr) &= \sup_{A\subset\mathbb{Z}} |\mathbb{P}[X \in A] - \mathbb{P}[Z \in A]| \\ &= \sup_{A\subset\mathcal{T}} |(\mathcal{B}g_A)(X)| + \mathbb{P}[X \notin \mathcal{T}] \\ &\leqslant \sup_{A\subset\mathcal{T}} \bigl|\mathbb{E}\{(X-\mu)\Theta g(X) - \sigma^2\Delta g(X)\}\bigr| + \frac{\mathbb{E}|a(X)|}{\sigma^2} + \mathbb{P}[X \notin \mathcal{T}]. \end{aligned} \tag{3.3}$$

We can use Chebychev's inequality to bound

$$\mathbb{P}[X \notin \mathcal{T}] \leqslant \mathbb{P}[|X-\mu| \geqslant 2\sigma^2 - 1] \leqslant \frac{\sigma^2}{(2\sigma^2-1)^2} \leqslant \frac{0.61}{\sigma^2}, \tag{3.4}$$

where the last inequality holds as long as $\sigma^2 \geqslant 1.4$, which we may assume without loss of generality since otherwise (1.4) is trivial. Moreover,

$$\mathbb{E}|a(X)| \leqslant nt^2 + \sigma t + \delta/4 \leqslant 3/4 \tag{3.5}$$

(see [5, after (2.17)] for the second inequality).

It remains to bound the first expression on the right hand side of (3.3); we follow the line of argument in [4]. Using exchangeability and anti-symmetry,

$$\begin{aligned} 0 &= \mathbb{E}\bigl\{(X'-X)\bigl(g(X') + g(X)\bigr)\bigr\} \\ &= \mathbb{E}\bigl\{(X'-X)\bigl(g(X') - g(X)\bigr)\bigr\} + 2\,\mathbb{E}\{(X'-X)g(X)\} \end{aligned} \tag{3.6}$$

Using (1.3), the second term equals

$$2\,\mathbb{E}\{(X'-X)g(X)\} = -2\lambda\,\mathbb{E}\{(X-\mu)g(X)\}.$$

so that (3.6) can be written as

$$\mathbb{E}\{(X-\mu)g(X)\} = \frac{1}{2\lambda}\,\mathbb{E}\bigl\{(X'-X)\bigl(g(X')-g(X)\bigr)\bigr\}. \tag{3.7}$$

To simplify the right hand side of (3.7), let $I_i := \mathrm{I}[X' - X = i]$ for $i \in \{-1, +1\}$, and making the case distinction whether $X' - X = +1$ or -1, write

$$\mathbb{E}\bigl\{(X'-X)\bigl(g(X')-g(X)\bigr)\bigr\} = \mathbb{E}\{I_{+1}\Delta g(X)\} + \mathbb{E}\{I_{-1}\Delta g(X-1)\}.$$

Using exchangeability,

$$\mathbb{E}\{I_{-1}\Delta g(X-1)\} = \mathbb{E}\{I_{+1}\Delta g(X)\},$$

so that (3.7) yields

$$\mathbb{E}\{(X-\mu)g(X)\} = \frac{1}{\lambda}\,\mathbb{E}\{I_{+1}\Delta g(X)\} \tag{3.8}$$

Replacing $g(X)$ by $g(X+1)$ and using exchangeability again,

$$\mathbb{E}\{(X-\mu)g(X+1)\} = \frac{1}{\lambda}\,\mathbb{E}\{I_{+1}\Delta g(X+1)\} = \frac{1}{\lambda}\,\mathbb{E}\{I_{-1}\Delta g(X)\} \tag{3.9}$$

Adding (3.8) and (3.9) and dividing by two,

$$\mathbb{E}\{(X-\mu)\Theta g(X)\} = \frac{1}{2\lambda}\,\mathbb{E}\{(I_{+1}+I_{-1})\Delta g(X)\} = \frac{1}{2\lambda}\,\mathbb{E}\{S\Delta g(X)\},$$

and it follows that

$$\mathbb{E}\{(X-\mu)\Theta g(X) - \sigma^2\Delta g(X)\} = \mathbb{E}\Big\{\Big(\frac{S}{2\lambda} - \sigma^2\Big)\Delta g(X)\Big\}. \tag{3.10}$$

Now, noticing that $\mathbb{E}S = 2\lambda\sigma^2$, the right hand side of (3.10) can be bounded by

$$\frac{\sqrt{\operatorname{Var} S}}{2\lambda}\|\Delta g\|_\infty \leqslant \frac{\sqrt{\operatorname{Var} S}}{2\lambda\sigma^2}. \tag{3.11}$$

Combining bounds (3.4), (3.5), and (3.11) with (3.3) yields the claim. □

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