Finite Affine Groups: Cycle Indices, Hall-Littlewood Polynomials, and Probabilistic Algorithms

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Abstract

The study of asymptotic properties of the conjugacy class of a random element of the finite affine group leads one to define a probability measure on the set of all partitions of all positive integers. Four different probabilistic understandings of this measure are given–three using symmetric function theory and one using Markov chains. This leads to non-trivial enumerative results. Cycle index generating functions are derived and are used to compute the large dimension limiting probabilities that an element of the affine group is separable, cyclic, or semisimple and to study the convergence to these limits. The semisimple limit involves both Rogers-Ramanujan identities. This yields the first examples of such computations for a maximal parabolic subgroup of a finite classical group.

Key words: Conjugacy class, classical group, affine group, Hall-Littlewood polynomial, symmetric function, random matrix.

1 Introduction

The conjugacy classes of the unitary groups with complex entries are simply the set of its eigenvalues. Thus the enormous body of recent work on eigenvalues of random matrices is to a large extent a study of conjugacy classes. Given the power of random matrix models (see for instance [KS1] and [KS2] which show the predictive power of random matrices for the study of local properties of the zeros of the Riemann zeta function), it is natural to study conjugacy in random matrices over finite fields as well.

The papers [F1], [F2], [B], [F3] give a purely probabilistic understanding of conjugacy classes in the finite classical groups and in the group of upper triangular matrices over a finite field. One outcome was a simple and motivated proof of the Rogers-Ramanujan identities [F4].

The current paper considers the affine group $A(n,q)$ (all matrices in $GL(n+1,q)$ with $x_{11}=1$

and $x_{j1} = 0$ for $j \geq 2$) and the maximal parabolic subgroup $P(n,q)$ (all matrices in $GL(n+1,q)$ with $x_{11} \neq 0$ and $x_{j1} = 0$ for $j \geq 2$).

Section 2 derives an expression for the chance that an element of $A(n,q)$ or $P(n,q)$ has a given rational canonical form in $GL(n+1,q)$, giving rise to cycle index generating functions for these groups. Section 2 also makes useful contact with the Hall-Littlewood symmetric functions.

Section 3 opens by defining a new and natural probability measure $N_{u,q}$ on the set of all partitions of all positive integers and by relating it to an analogous measure for the finite general linear groups. Then it gives four purely probabilistic algorithms for "growing" random partitions according to $N_{u,q}$ and also an algorithm which arises when considering conjugacy classes only of unipotent elements in $A(n,q)$ or $P(n,q)$.

Section 4 gives non-trivial enumerative applications of the algorithms of Section 3. For instance the $n \rightarrow \infty$ chance that an element of $A(n,q)$ or $P(n,q)$ has a fixed space of a given dimension is shown to have a simple product formula, where each factor in the product has a clear probabilistic meaning. The number of unipotent elements in $A(n,q)$ or $P(n,q)$ of a given rank is computed.

Section 5 of this paper uses the generating functions of Section 2 to calculate the large n limiting probabilities that an element of $A(n,q)$ or $P(n,q)$ is separable, cyclic, or semisimple. Analogous results are known for the finite classical groups [F5],[W],[FNP] and are important for computational group theory [NP],[NP4] and for estimating sizes of images of maps between curves [FG]. The motivation for Section 5 is the fact that probabilistic estimates in maximal subgroups of the finite classical groups are also useful for group theory. To state a typical result, recall that a matrix is called cyclic if its minimal polynomial is equal to its characteristic polynomial. Whereas the $n \rightarrow \infty$ limiting probability that an element of $GL(n,q)$ is cyclic is equal to $\frac{1-1/q^5}{1+1/q^3}$, Section 5 shows that the corresponding probability for $A(n,q)$ or $P(n,q)$ is $\frac{1-1/q}{1-1/q+1/q^2} \frac{1-1/q^5}{1+1/q^3}$, which is asymptotically $1-1/q^2+O(1/q^3)$ rather than $1-1/q^3+O(1/q^4)$. It is worth emphasizing that at present even for

$GL(n,q)$ the only method for performing such exact calculations is through the use of generating functions.

2 Cycle Indices, Partitions and Hall-Littlewood Polynomials

To begin we recall some standard notation about partitions which will be used throughout the paper. Let λ be a partition of some non-negative integer $|\lambda|$ into parts $\lambda_1 \geq \lambda_2 \geq \cdots$. Let $m_i(\lambda)$ be the number of parts of λ of size i, and let λ' be the partition dual to λ in the sense that $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Let $n(\lambda)$ be the quantity $\sum_{i \geq 1}(i-1)\lambda_i$. It is also useful to define the diagram associated to λ as having the jth row consist of λ_j boxes. For example the diagram of the partition (5441) is:

□ □ □ □ □

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□

The notation $P_\lambda(x_1, x_2, \cdots; t)$ denotes the Hall-Littlewood symmetric function. The reader is referred to Chapter 3 of [Mac] for a thorough discussion of its properties. The symbol $(\frac{1}{q})_i$ will denote $(1-1/q)\cdots(1-1/q^i)$.

Next recall (e.g. Chapter 6 of [H]) that the conjugacy classes of $GL(n,q)$ are parameterized by rational canonical form. This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial ϕ over the field of q elements F_q, associate a partition (perhaps the trivial partition) λ_ϕ of some non-negative integer $|\lambda_\phi|$. Let $deg(\phi)$ denote the degree of ϕ. The only restrictions necessary for this data to represent a conjugacy class are that the partition corresponding to the polynomial $\phi(z) = z$ is empty and that $\sum_\phi |\lambda_\phi| deg(\phi) = n$.

This conjugacy data arises explicitly as follows. Given an element $\alpha \in GL(n,q)$ operating on the vector space V, there is a unique direct sum decomposition $V = \bigoplus V_\phi$ where the characteristic polynomial of α on V_ϕ is a power of ϕ and the characteristic polynomials on any two summands are relatively prime. Furthermore α decomposes each V_ϕ into a sum of cyclic subspaces. This decomposition need not be unique but the dimensions of the cyclic subspaces in the decomposition of V_ϕ are unique and dividing them by $deg(\phi)$ gives the row lengths of the partition λ_ϕ. For example, the identity matrix has λ_{z-1} equal to (1^n) and an elementary transvection with $a \neq 0$ in the $(1,2)$ position, ones on the diagonal and zeros elsewhere has λ_{z-1} equal to $(2, 1^{n-2})$. As another example, the characteristic polynomial of an element with conjugacy class data $\{\lambda_\phi\}$ is $\prod_{\phi \neq z} \phi^{|\lambda_\phi|}$.

The first aim of this section is to find expressions for the chance that an element of $A(n,q)$ or $P(n,q)$ has given rational canonical form data. This is a cruder invariant than conjugacy in $A(n,q)$ or $P(n,q)$. However most conjugacy class functions on $A(n,q)$ or $P(n,q)$ of interest depend only on the Jordan form in $GL(n+1,q)$. (A similar reduction proved useful [B],[F3] for upper triangular matrices over a finite field, for which the theory of wild quivers implies that there is no finite parameterization of conjugacy classes).

To find the chance that an element of $A(n,q)$ or $P(n,q)$ has a given rational canonical form we use a result of Nakada and Shinoda [NaS] (and subsequently in [Mu]) on the parameterization and sizes of conjugacy classes in $A(n,q)$ or $P(n,q)$. Then a combinatorial reformulation followed by a summation gives the result we seek.

For the statement of Theorem 1, we use the notation that $|Z_G(c)|$ is the size of the centralizer in the group G of any element in the conjugacy class c. The symbol F_q^* denotes the non-zero elements in F_q. We use the convention that $GL(0,q)$ consists of one conjugacy class of size one.

Theorem 1 *([NaS])*

1. *The conjugacy classes of $A(n,q)$ are parameterized by pairs (c_{n+1-k}, k) where $0 < k \leq n+1$ and c_{n+1-k} is a conjugacy class of $GL(n+1-k,q)$. Letting $\{\overline{\lambda_\phi}\}$ be the conjugacy class*

data corresponding to c_{n+1-k}*, the* $GL(n+1,q)$ *rational canonical form of the corresponding class in* $A(n,q)$ *is given by* $\lambda_\phi = \overline{\lambda_\phi}$ *for* $\phi \neq z-1$ *and by letting* $\lambda_{z-1} = (k) \cup \overline{\lambda_{z-1}}$ *be the partition formed by adding a row of length* k *to* $\overline{\lambda_{z-1}}$*. The centralizer size of the corresponding conjugacy class is*

$$|Z_{GL(n+1-k,q)}(c_{n+1-k})|q^{k-1+2\sum_{i=1}^{k-1} im_i(\overline{\lambda_{z-1}})+(2k-1)\sum_{i=k}^{n-k} m_i(\overline{\lambda_{z-1}})}.$$

2. *The conjugacy classes of* $P(n,q)$ *are parameterized by triples* (c_{n+1-k}, k, a) *where* $0 < k \leq n+1$*,* $a \in F_q^*$ *and* c_{n+1-k} *is a conjugacy class of* $GL(n+1-k,q)$*. Letting* $\{\overline{\lambda_\phi}\}$ *be the conjugacy class data corresponding to* c_{n+1-k}*, the* $GL(n+1,q)$ *rational canonical form of the corresponding class in* $A(n,q)$ *is given by* $\lambda_\phi = \overline{\lambda_\phi}$ *for* $\phi \neq z-a$ *and by letting* $\lambda_{z-a} = (k) \cup \overline{\lambda_{z-a}}$ *be the partition formed by adding a row of length* k *to* $\overline{\lambda_{z-a}}$*. The centralizer size of the corresponding conjugacy class is*

$$|Z_{GL(n+1-k,q)}(c_{n+1-k})|(q-1)q^{k-1+2\sum_{i=1}^{k-1} im_i(\overline{\lambda_{z-1}})+(2k-1)\sum_{i=k}^{n-k} m_i(\overline{\lambda_{z-1}})}.$$

Lemma 1 below was Theorem 9 in the thesis [F0]. Of course the formula for the conjugacy class sizes in $GL(n,q)$ is not due to the author (see for instance page 219 of [Mac] for the third expression), but the combinatorial rewritings of it in Lemma 1 are very useful.

Lemma 1 *([F0]) The conjugacy class of* $GL(n,q)$ *corresponding to the data* $\{\lambda_\phi\}$ *has size*

$$\frac{|GL(n,q)|}{\prod_{\phi \neq z} c_{GL,\phi,q}(\lambda_\phi)}$$

where

$$\begin{aligned} c_{GL,\phi,q}(\lambda) &= q^{2deg(\phi)[\sum_{h<i} hm_h(\lambda)m_i(\lambda)+\frac{1}{2}\sum_i (i-1)m_i(\lambda)^2]} \prod_i |GL(m_i(\lambda), q^{deg(\phi)})| \\ &= q^{deg(\phi)[\sum_i (\lambda_i')^2]} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)} \\ &= \frac{q^{deg(\phi)n(\lambda)}}{P_\lambda(\frac{1}{q^{deg(\phi)}}, \frac{1}{q^{2deg(\phi)}}, \cdots; \frac{1}{q^{deg(\phi)}})} \end{aligned}$$

Theorem 2 is our first main result. The notation used in the proof is the same as in the statement of Theorem 1. Recall that $\lambda'_{\phi,1}$ is the size of the first column of λ_ϕ.

Theorem 2 *1. The number of elements of $A(n,q)$ with $GL(n+1,q)$ rational canonical form data $\{\lambda_\phi\}$ is equal to*

$$\frac{|A(n,q)|(q^{\lambda'_{z-1,1}}-1)}{\prod_{\phi \neq z} q^{deg(\phi)\cdot \sum_i (\lambda'_{\phi,i})^2} (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda_\phi)}}.$$

2. The number of elements of $P(n,q)$ with $GL(n+1,q)$ rational canonical form data $\{\lambda_\phi\}$ is equal to

$$\sum_{a \in F_q^*} \frac{|P(n,q)|(q^{\lambda'_{z-a,1}}-1)}{q-1} \frac{1}{\prod_{\phi \neq z} q^{deg(\phi)\cdot \sum_i (\lambda'_{\phi,i})^2} (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda_\phi)}}.$$

Proof: First we consider $A(n,q)$. From part 1 of Theorem 1 the number of elements of $A(n,q)$ with $GL(n+1,q)$ rational canonical form data $\{\lambda_\phi\}$ is

$$\sum_{\substack{(c_{n+1-k},k) \\ k \geq 1, (k) \cup \overline{\lambda_{z-1}} = \lambda_{z-1}}} \frac{|A(n,q)|}{|Z_{GL(n+1-k,q)}(c_{n+1-k})| q^{k-1+2\sum_{i=1}^{k-1} i m_i(\overline{\lambda_{z-1}}) + (2k-1)\sum_{i=k}^{n-k} m_i(\overline{\lambda_{z-1}})}}.$$

From Lemma 1, $|Z_{GL(n+1-k,q)}(c_{n+1-k})|$ can be written as

$$\prod_{\phi \neq z} \prod_i q^{deg(\phi)\cdot(\overline{\lambda'_{\phi,i}})^2} (\frac{1}{q^{deg(\phi)}})_{m_i(\overline{\lambda_\phi})}.$$

Thus the number of elements of $A(n,q)$ with $GL(n+1,q)$ rational canonical form data $\{\lambda_\phi\}$ is

$$|A(n,q)| \sum_{\substack{(k,\overline{\lambda_{z-1}}) \\ k \geq 1, (k) \cup \overline{\lambda_{z-1}} = \lambda_{z-1}}} \frac{1}{q^{k-1+2\sum_{i=1}^{k-1} i m_i(\overline{\lambda_{z-1}}) + (2k-1)\sum_{i=k}^{n-k} m_i(\overline{\lambda_{z-1}})}} \frac{1}{\prod_i q^{(\overline{\lambda'_{z-1,i}})^2} (\frac{1}{q})_{m_i(\overline{\lambda_{z-1}})}}$$

$$\cdot \prod_{\phi \neq z, z-1} \prod_i \frac{1}{q^{deg(\phi)\cdot(\overline{\lambda'_{\phi,i}})^2} (\frac{1}{q^{deg(\phi)}})_{m_i(\overline{\lambda_\phi})}}$$

Consequently it is sufficient to prove that

$$\begin{aligned}
& \sum_{\substack{(k,\overline{\lambda_{z-1}}) \\ k \geq 1, (k) \cup \overline{\lambda_{z-1}} = \lambda_{z-1}}} \frac{1}{q^{k-1+2\sum_{i=1}^{k-1} im_i(\overline{\lambda_{z-1}})+(2k-1)\sum_{i=k}^{n-k} m_i(\overline{\lambda_{z-1}})} \prod_i q^{(\overline{\lambda'_{z-1,i}})^2} (\frac{1}{q})_{m_i(\overline{\lambda_{z-1}})}} \\
= \quad & \frac{(q^{\lambda'_{z-1,1}}-1)}{\prod_i q^{(\lambda'_{z-1,i})^2}(\frac{1}{q})_{m_i(\lambda_{z-1})}}.
\end{aligned}$$

Since only partitions corresponding to the polynomial $z-1$ are involved in this last equation, we simplify notation by suppressing the dependence on $z-1$ and by further replacing $m_i(\lambda)$ and $m_i(\overline{\lambda})$ by m_i and $\overline{m_i}$.

From Lemma 1 one sees that

$$\begin{aligned}
& \sum_{\substack{(k,\overline{\lambda}) \\ k \geq 1, (k) \cup \overline{\lambda} = \lambda}} \frac{1}{q^{k-1+2\sum_{i=1}^{k-1} i\overline{m_i}+(2k-1)\sum_{i=k}^{n-k} \overline{m_i}} \prod_i q^{(\overline{\lambda}'_i)^2}(\frac{1}{q})_{\overline{m_i}}} \\
= \quad & \sum_{\substack{(k,\overline{\lambda}) \\ k \geq 1, (k) \cup \overline{\lambda} = \lambda}} \frac{1}{q^{k-1+2\sum_{i=1}^{k-1} i\overline{m_i}+(2k-1)\sum_{i=k}^{n-k} \overline{m_i}} q^{2\sum_{i<j} i\overline{m_i m_j}+\sum_i i\overline{m_i}^2} \prod_i (\frac{1}{q})_{\overline{m_i}}}
\end{aligned}$$

Clearly if $(k) \cup \overline{\lambda} = \lambda$ then $m_i = \overline{m_i}$ for $i \neq k$ and $m_k = \overline{m_k} + 1$. Elementary combinatorics then shows that this last expression is equal to

$$\begin{aligned}
& \sum_{\substack{(k,\overline{\lambda}) \\ k \geq 1, (k) \cup \overline{\lambda} = \lambda}} \frac{1}{q^{2\sum_{i<j} im_im_j+\sum_i im_i^2-\sum_{i\geq k} m_i} \prod_i (\frac{1}{q})_{\overline{m_i}}} \\
= \quad & \frac{1}{q^{2\sum_{i<j} im_im_j+\sum_i im_i^2} \prod_i (\frac{1}{q})_{m_i}} \sum_{\substack{(k,\overline{\lambda}) \\ k \geq 1, (k) \cup \overline{\lambda} = \lambda}} (1-1/q^{m_k}) q^{\lambda'_k} \\
= \quad & \frac{1}{q^{2\sum_{i<j} im_im_j+\sum_i im_i^2} \prod_i (\frac{1}{q})_{m_i}} \sum_{k \geq 1} (1-1/q^{m_k}) q^{\lambda'_k} \\
= \quad & \frac{1}{q^{2\sum_{i<j} im_im_j+\sum_i im_i^2} \prod_i (\frac{1}{q})_{m_i}} \sum_{k \geq 1} (q^{\lambda'_k} - q^{\lambda'_{k+1}}) \\
= \quad & (q^{\lambda'_1}-1) \frac{1}{q^{2\sum_{i<j} im_im_j+\sum_i im_i^2} \prod_i (\frac{1}{q})_{m_i}} \\
= \quad & (q^{\lambda'_1}-1) \frac{1}{\prod_i q^{(\lambda'_i)^2}(\frac{1}{q})_{m_i}},
\end{aligned}$$

where the final equality is Lemma 1. This completes the proof of part 1 of the theorem. Part 2 follows from part 1 and Theorem 1. □

Given Theorem 2 it is now straightforward to write down the corresponding cycle index generating functions. These will be applied in Section 5.

Corollary 1 *Let $x_{\phi,\lambda}$ be a collection of variables. Let $\{\lambda_\phi(\alpha)\}$ be the rational canonical form data of any $\alpha \in GL(n+1,q)$. Then*

1.

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{u^n}{|A(n,q)|} \sum_{\alpha \in A(n,q)} \prod_{\phi \neq z} x_{\phi,\lambda_\phi(\alpha)} \\ = & \left(\sum_{\lambda:|\lambda|>0} \frac{x_{z-1,\lambda} u^{|\lambda|-1}(q^{\lambda_1'}-1)}{\prod_i q^{(\lambda_i')^2}(\frac{1}{q})_{m_i(\lambda)}} \right) \prod_{\phi \neq z,z-1} \left(\sum_\lambda \frac{x_{\phi,\lambda} u^{|\lambda| \cdot deg(\phi)}}{\prod_i q^{deg(\phi)\cdot(\lambda_i')^2}(\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right). \end{aligned}$$

2.

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{u^n}{|P(n,q)|} \sum_{\alpha \in P(n,q)} \prod_{\phi \neq z} x_{\phi,\lambda_\phi(\alpha)} \\ = & \sum_{a \in F_q^*} \left(\sum_{\lambda:|\lambda|>0} \frac{x_{z-a,\lambda} u^{|\lambda|-1}(q^{\lambda_1'}-1)}{(q-1)\prod_i q^{(\lambda_i')^2}(\frac{1}{q})_{m_i(\lambda)}} \right) \prod_{\phi \neq z,z-a} \left(\sum_\lambda \frac{x_{\phi,\lambda} u^{|\lambda| \cdot deg(\phi)}}{\prod_i q^{deg(\phi)\cdot(\lambda_i')^2}(\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right). \end{aligned}$$

3 A Measure on Partitions and Probabilistic Algorithms

To begin we define a probability measure which will be the object of study in this section. Recall that $n(\lambda)$ is the quantity $\sum_{i \geq 1}(i-1)\lambda_i$.

Definition: For $0 < u < 1$ and $q > 1$ the measure $N_{u,q}$ on the set of all partitions of all positive integers is defined by

$$\begin{aligned} N_{u,q}(\lambda) &= \prod_{r=1}^{\infty}(1-\frac{u}{q^r}) \frac{u^{|\lambda|-1}(q^{\lambda_1'}-1)}{\prod_i q^{(\lambda_i')^2}(\frac{1}{q})_{m_i(\lambda)}} \\ &= \prod_{r=1}^{\infty}(1-\frac{u}{q^r}) \frac{u^{|\lambda|-1}(q^{\lambda_1'}-1)P_\lambda(\frac{1}{q},\frac{1}{q^2},\cdots;\frac{1}{q})}{q^{n(\lambda)}}. \end{aligned}$$

The equivalence between the two definitions of $N_{u,q}(\lambda)$ follows from Lemma 1.

Before proving that $N_{u,q}$ is indeed a probability measure, we recall the analogous measure $M_{u,q}$ on the set of all partitions of all natural numbers. The survey [F3] summarizes what is known about $M_{u,q}$.

$$\begin{aligned} M_{u,q}(\lambda) &= \prod_{r=1}^{\infty}(1-\frac{u}{q^r})\frac{u^{|\lambda|}}{\prod_i q^{(\lambda_i')^2}(\frac{1}{q})_{m_i(\lambda)}} \\ &= \prod_{r=1}^{\infty}(1-\frac{u}{q^r})\frac{u^{|\lambda|}P_\lambda(\frac{1}{q},\frac{1}{q^2},\cdots;\frac{1}{q})}{q^{n(\lambda)}}. \end{aligned}$$

Lemma 2 proves that $N_{u,q}$ is a probability measure. We include two proofs–one using an identity of Macdonald from symmetric function theory and another using the cycle index of $A(n,q)$ and the fact that $M_{u,q}$ is a probability measure for $q>1$ and $0<u<1$. (Two other proof methods are to use either Steinberg's result that there are q^{n^2} unipotent elements in $A(n,q)$ together with the cycle index or else to use the Markov chain construction later in this section together with an identity of Cauchy).

Lemma 2 *If $q>1$ and $0<u<1$ then $N_{u,q}$ defines a probability measure.*

PROOF: (First proof) Equation 3 on page 219 of [Mac] states that

$$\prod_{i\geq 1}(1+x_iy)/(1-x_i) = \sum_{\lambda} t^{n(\lambda)} \prod_{j=1}^{\lambda_1'}(1+t^{1-j}y)P_\lambda(x;t).$$

The result now follows by first setting $x_i = u/q^i, t=1/q$ and taking coefficients of y on both sides, and then using the fact that $P_\lambda(u/q,u/q^2,\cdots;1/q) = u^{|\lambda|}P_\lambda(1/q,1/q^2,\cdots;1/q)$. □

PROOF: (Second proof) Setting all $x_{\phi,\lambda}$ in the cycle index of $A(n,q)$ equal to 1 gives the identity

$$\frac{1}{1-u} = (\prod_{r=1}^{\infty}\frac{1}{1-u/q^r}\sum_{\lambda:|\lambda|>0}N_{u,q}(\lambda))\prod_{\phi\neq z,z-1}(\prod_{r=1}^{\infty}\frac{1}{1-u^{deg(\phi)}/q^{r\cdot deg(\phi)}})\sum_{\lambda}M_{u^{deg(\phi)},q^{deg(\phi)}}(\lambda).$$

Since $M_{u,q}$ is a probability measure it follows that

$$\frac{1}{1-u} = (\prod_{\phi \neq z} \prod_{r=1}^{\infty} \frac{1}{1-u^{deg(\phi)}/q^{r \cdot deg(\phi)}}) \sum_{\lambda:|\lambda|>0} N_{u,q}(\lambda).$$

Setting all variables in the cycle index for $GL(n,q)$ equal to 1 and using the fact that $M_{u,q}$ is a probability measure implies that

$$\frac{1}{1-u} = \prod_{\phi \neq z} \prod_{r=1}^{\infty} \frac{1}{1-u^{deg(\phi)}/q^{r \cdot deg(\phi)}}$$

(of course this can be proved directly). The lemma follows. □

We require an elementary lemma about Taylor series. For its statement $[u^n]f(u)$ denotes the coefficient of u^n in a polynomial $f(u)$.

Lemma 3 *If $f(1) < \infty$ and the Taylor series of f around 0 converges at $u = 1$, then*

$$lim_{n \to \infty} [u^n] \frac{f(u)}{1-u} = f(1).$$

PROOF: Write the Taylor expansion $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Then observe that $[u^n]\frac{f(u)}{1-u} = \sum_{i=0}^{n} a_i$. □

Theorem 3 shows that the measure $N_{u,q}$ is a fundamental object for understanding the probability theory of $A(n,q)$. We remark that the idea behind it–auxiliary randomization–is a mainstay of statistical mechanics.

Theorem 3 *1. Fix u with $0 < u < 1$. Then choose a random number N with probability of getting n equal to $(1-u)u^n$. Choose α uniformly in $A(N,q)$. Then as ϕ varies any finite number of the random partitions $\lambda_\phi(\alpha)$ are independent random variables, with λ_{z-1} distributed according to the measure $N_{u,q}$ and all other λ_ϕ distributed according to the measure $M_{u^{deg(\phi)},q^{deg(\phi)}}$.*

2. *Choose α uniformly in $A(n,q)$. Then as $n \to \infty$, any finite number of the random partitions $\lambda_{\phi}(\alpha)$ are independent random variables, with λ_{z-1} distributed according to the measure $N_{1,q}$ and all other λ_{ϕ} distributed according to the measure $M_{1,q^{deg(\phi)}}$.*

3. *Fix u with $0 < u < 1$. Then choose a random number N with probability of getting n equal to $(1-u)u^n$. Choose α uniformly in $P(N,q)$. Then as ϕ varies over polynomials of degree at least two, any finite number of the random partitions $\lambda_{\phi}(\alpha)$ are independent random variables distributed according to the measure $M_{u^{deg(\phi)},q^{deg(\phi)}}$. The partitions λ_{z-a} are independent of λ_{ϕ} for $deg(\phi) \geq 2$ but depend on each other; any particular λ_{z-a} has as its distribution the mixture $\frac{N_{u,q}}{q-1} + \frac{(q-2)M_{u,q}}{q-1}$.*

4. *Choose α uniformly in $P(n,q)$. Then as $n \to \infty$ and ϕ varies over polynomials of degree at least two, any finite number of the random partitions $\lambda_{\phi}(\alpha)$ are independent random variables distributed according to the measure $M_{1,q^{deg(\phi)}}$. The partitions λ_{z-a} are independent of λ_{ϕ} for $deg(\phi) \geq 2$ but depend on each other; any particular λ_{z-a} has as its distribution the mixture $\frac{N_{1,q}}{q-1} + \frac{(q-2)M_{1,q}}{q-1}$.*

PROOF: As explained in the second proof of Lemma 2, we know that

$$\frac{1}{1-u} = \prod_{\phi \neq z} \prod_{r=1}^{\infty} \frac{1}{1 - u^{deg(\phi)}/q^{r \cdot deg(\phi)}}.$$

Multiplying this by the cycle index of $A(n,q)$ gives that

$$\sum_{n=0}^{\infty} \frac{(1-u)u^n}{|A(n,q)|} \sum_{\alpha \in A(n,q)} \prod_{\phi \neq z} x_{\phi,\lambda_{\phi}(\alpha)} = (\sum_{\lambda:|\lambda|>0} x_{z-1,\lambda} N_{u,q}(\lambda)) \prod_{\phi \neq z, z-1} (\sum_{\lambda} x_{\phi,\lambda} M_{u^{deg(\phi)},q^{deg(\phi)}}(\lambda)).$$

This proves the first assertion of the theorem. For the second assertion use Lemma 3. The proofs of the third and fourth assertions are almost identical so are omitted. □

The remainder of this section considers probabilistic methods for growing random partitions according to the measure $N_{u,q}$, analogous to those for $M_{u,q}$ in [F0],[F1]. For this recall that a standard Young tableau T of size n is a partition of n with each box containing one of $\{1, \cdots, n\}$ such that each of $\{1, \cdots, n\}$ appears exactly once and the numbers increase in each row and column of T. For instance,

$$\begin{array}{cccc} \boxed{1} & \boxed{3} & \boxed{5} & \boxed{6} \\ \boxed{2} & \boxed{4} & \boxed{7} & \\ \boxed{8} & \boxed{9} & & \end{array}$$

is a standard Young tableau.

Our first method for growing partitions according to the measure $N_{u,q}$ is an algorithm we call the Affine Young Tableau Algorithm, so as to distinguish it from the Young Tableau Algorithm for growing partitions according to the measure $M_{u,q}$. For its statement, we need the Young Tableau Algorithm [F0],[F1].

The Young Tableau Algorithm

Step 0 Start with $N = 1$ and λ the empty partition. Also start with a collection of coins indexed by the natural numbers, such that coin i has probability $\frac{u}{q^i}$ of heads and probability $1 - \frac{u}{q^i}$ of tails.

Step 1 Flip coin N.

Step 2a If coin N comes up tails, leave λ unchanged, set $N = N + 1$ and go to Step 1.

Step 2b If coin N comes up heads, choose an integer $S > 0$ according to the following rule. Set $S = 1$ with probability $\frac{q^{N-\lambda'_1}-1}{q^N-1}$. Set $S = s > 1$ with probability $\frac{q^{N-\lambda'_s}-q^{N-\lambda'_{s-1}}}{q^N-1}$. Then increase the size of column s of λ by 1 and go to Step 1.

As an example of the Young Tableau Algorithm, suppose we are at Step 1 with λ equal to the following partition:

□ □ □ □

□ □

□

Suppose also that $N = 4$ and that coin 4 had already come up heads once, at which time we added to column 1, giving λ. Now we flip coin 4 again and get heads, going to Step 2b. We add to column 1 with probability $\frac{q-1}{q^4-1}$, to column 2 with probability $\frac{q^2-q}{q^4-1}$, to column 3 with probability $\frac{q^3-q^2}{q^4-1}$, to column 4 with probability 0, and to column 5 with probability $\frac{q^4-q^3}{q^4-1}$. We then return to Step 1.

Theorem 4 *([F0],[F1]) For $0 < u < 1$ and $q > 1$, the Young Tableau Algorithm generates partitions which are distributed according to the measure $M_{u,q}$.*

Next we point out that the Young Tableau Algorithm can easily be made to terminate. This idea was developed in joint work with Mark Huber surveyed in [F3] and goes as follows. Let a_N be the number of times that coin N comes up heads; the idea is simply to first determine the random vector $(a_1, a_2, \cdots)$ and then grow the partitions as in Step 2b of the Young Tableau Algorithm. So let us explain how to determine $(a_1, a_2, \cdots)$. For $N \geq 1$ let $t^{(N)}$ be the probability that all tosses of all coins numbered N or greater are tails. For $N \geq 1$ and $j \geq 0$ let $t_j^{(N)}$ be the probability that some toss of a coin numbered N or greater is a head and that coin N comes up heads j times. It is simple to write down expressions for $t^{(N)}, t_0^{(N)}, t_1^{(N)}, \cdots$ and clearly $t^{(N)} + \sum_{j \geq 0} t_j^{(N)} = 1$. The basic operation a computer can perform is to produce a random variable U distributed uniformly in the interval $[0, 1]$. By dividing $[0, 1]$ into intervals of length $t^{(1)}, t_0^{(1)}, t_1^{(1)}, \cdots$ and seeing where U is located, one arrives at the value of a_1. Furthermore, if U landed in the interval of length $t^{(1)}$ then all coins come up tails and the vector $(a_1, a_2, \cdots)$ is determined. Otherwise, move on to coin 2, dividing $[0, 1]$ into intervals of length $t^{(2)}, t_0^{(2)}, t_1^{(2)}, \cdots$ and so on.

Now we describe the Affine Young Tableau Algorithm.

The Affine Young Tableau Algorithm

Step 1 Run the Young Tableau Algorithm so as to generate a partition λ distributed as $M_{u,q}$.

Step 2 Set $S = 1$ with probability $\frac{1}{q^{\lambda'_1}}$ and $S = s > 1$ with probability $\frac{1}{q^{\lambda'_s}} - \frac{1}{q^{\lambda'_{s-1}}}$. Then increase the size of column s of λ by 1.

Theorem 6 shows that the Affine Young Tableau Algorithm grows partitions distributed as $N_{u,q}$. For its proof, we use a different method for generating $M_{u,q}$. Recall that the Young lattice is the set of all partitions of all natural numbers, with a directed edge drawn from the partition λ to partition Λ if the diagram of λ is contained in the diagram of Λ and $|\Lambda| = |\lambda| + 1$.

Theorem 5 *([F1]) Put weights $m_{\lambda,\Lambda}$ on the edges of Young lattice according to the rules:*

1. *$m_{\lambda,\Lambda} = \frac{u}{q^{\lambda'_1}(q^{\lambda'_1+1}-1)}$ if the diagram of Λ is obtained from that of λ by adding a box to column 1.*

2. *$m_{\lambda,\Lambda} = \frac{u(q^{-\lambda'_s}-q^{-\lambda'_{s-1}})}{q^{\lambda'_1}-1}$ if the diagram of Λ is obtained from that of λ by adding a box to column $s > 1$.*

Then the following formula holds:

$$M_{u,q}(\lambda) = \left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\sum_{\gamma}\prod_{i=0}^{|\lambda|-1} m_{\gamma_i,\gamma_{i+1}}$$

where the sum is over all directed paths γ from the empty partition to λ, and the γ_i are the partitions along the path γ. Furthermore, if the Young tableau T corresponds to the path γ then the probability that the Young Tableau Algorithm outputs T is equal to

$$\left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\prod_{i=0}^{|\lambda|-1} m_{\gamma_i,\gamma_{i+1}}.$$

Now we can show that the Affine Young Tableau Algorithm works.

Theorem 6 *The Affine Young Tableau Algorithm grows partitions distributed as $N_{u,q}$.*

PROOF: Let $\lambda^{(s)}$ denote the the shape obtained by decreasing the size of column s of λ by one and let $M_{u,q}(\lambda^{(s)}) = 0$ if $\lambda^{(s)}$ is not a partition. To prove the theorem it is sufficient to show that for $|\lambda| > 0$,

$$\frac{1}{q^{\lambda_1'-1}}M_{u,q}(\lambda^{(1)}) + \sum_{s>1}(\frac{1}{q^{\lambda_s'-1}} - \frac{1}{q^{\lambda_{s-1}'}})M_{u,q}(\lambda^{(s)}) = N_{u,q}(\lambda).$$

Theorem 5 shows that

$$\frac{1}{q^{\lambda_1'-1}}M_{u,q}(\lambda^{(1)}) + \sum_{s>1}(\frac{1}{q^{\lambda_s'-1}} - \frac{1}{q^{\lambda_{s-1}'}})M_{u,q}(\lambda^{(s)})$$

$$= \prod_{r=1}^{\infty}(1-\frac{u}{q^r})[\sum_{s\geq 1}\sum_{\gamma:\emptyset\rightarrow\lambda^{(s)}\rightarrow\lambda}(\prod_{i=0}^{|\lambda|-2} m_{\gamma_i,\gamma_{i+1}})(\frac{(q^{\lambda_1'}-1)m_{\lambda^{(s)},\lambda}}{u})]$$

where the inner sum is over all paths in the Young lattice from the empty partition to $\lambda^{(s)}$ to λ and all other notation is as in Theorem 5. Simplifying further and using Theorem 5 again gives that

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r})[\frac{q^{\lambda_1'}-1}{u}\sum_{\gamma:\emptyset\rightarrow\lambda}\prod_{i=0}^{|\lambda|-1} m_{\gamma_i,\gamma_{i+1}}] = \frac{q^{\lambda_1'}-1}{u}M_{u,q}(\lambda) = N_{u,q}(\lambda).$$

□

Remarks:

1. Theorem 6 gives another proof that $N_{u,q}$ is a probability measure.

2. From Theorem 5 one has an explicit formula for the chance that the Young Tableau Algorithm outputs a given Young tableau (page 565 of [F1]). Since the Affine Young Tableau Algorithm randomly adds on one additional box, one can write down a formula for the chance that it generates a given Young Tableau.

3. Two other probabilistic algorithms are given for growing partitions according to $M_{u,q}$ on pages 581 and 585 of [F1]. These yield algorithms for sampling from $N_{u,q}$ simply by adding an additional box as in Step 2 of the Affine Young Tableau Algorithm.

Next we describe a rather surprising method for sampling from $N_{u,q}$ using Markov chains. We use the notation that $(\frac{u}{q})_i$ is equal to $(1-u/q)\cdots(1-u/q^i)$. $Prob.(E)$ will denote the probability of an event E under the measure $N_{u,q}$.

Theorem 7 *Starting with $\lambda_1' = a \geq 1$ with probability $Q(a) = \frac{(\prod_{r=1}^{\infty}(1-u/q^r))u^{a-1}}{q^{a^2-a}(\frac{u}{q})_a(\frac{1}{q})_{a-1}}$, define in succession $\lambda_2', \lambda_3', \cdots$ according to the rule that if $\lambda_i' = a$, then $\lambda_{i+1}' = b$ with probability*

$$K(a,b) = \frac{u^b(\frac{1}{q})_a(\frac{u}{q})_a}{q^{b^2}(\frac{1}{q})_{a-b}(\frac{1}{q})_b(\frac{u}{q})_b}.$$

Then the resulting partition is distributed according to $N_{u,q}$.

Proof: The $N_{u,q}$ probability of choosing a partition with $\lambda_i' = r_i$ for all i is

$$Prob.(\lambda_1' = r_1)\prod_{i=1}^{\infty}\frac{Prob.(\lambda_1' = r_1, \cdots, \lambda_{i+1}' = r_{i+1})}{Prob.(\lambda_1' = r_1, \cdots, \lambda_i' = r_i)}.$$

Theorem 10 of Section 4 shows that $Prob.(\lambda_1' = r_1) = Q(r_1)$. Thus it is enough to prove that

$$\frac{Prob.(\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1}, \lambda_i' = a, \lambda_{i+1}' = b)}{Prob.(\lambda_1' = r_1, \cdots, \lambda_{i-1}' = r_{i-1}, \lambda_i' = a)} = \frac{u^b(\frac{1}{q})_a(\frac{u}{q})_a}{q^{b^2}(\frac{1}{q})_{a-b}(\frac{1}{q})_b(\frac{u}{q})_b},$$

for all $i \geq 1, a, b, r_1, \cdots, r_{i-1}$.

Letting $T(a)$ be the $M_{u,q}$ probability that $\lambda_1' = a$ it is proved in [F1] that $T(a) = \frac{u^a(\frac{u}{q})_\infty}{q^{a^2}(\frac{1}{q})_a(\frac{u}{q})_a}$. Next one calculates that for $i \geq 2$

$$\sum_{\substack{\lambda:\lambda_1'=r_1,\cdots,\lambda_{i-1}'=r_{i-1}\\ \lambda_i'=a}} N_{u,q}(\lambda) = \frac{(q^{r_1}-1)u^{r_1+\cdots+r_{i-1}-1}}{q^{r_1^2+\cdots+r_{i-1}^2}(\frac{1}{q})_{r_1-r_2}\cdots(\frac{1}{q})_{r_{i-2}-r_{i-1}}(\frac{1}{q})_{r_{i-1}-a}}T(a).$$

Similarly, observe that

$$\sum_{\substack{\lambda:\lambda_1'=r_1,\cdots,\lambda_{i-1}'=r_{i-1}\\ \lambda_i'=a,\lambda_{i+1}'=b}} N_{u,q}(\lambda) = \frac{(q^{r_1}-1)u^{r_1+\cdots+r_{i-1}+a-1}}{q^{r_1^2+\cdots+r_{i-1}^2+a^2}(\frac{1}{q})_{r_1-r_2}\cdots(\frac{1}{q})_{r_{i-2}-r_{i-1}}(\frac{1}{q})_{r_{i-1}-a}(\frac{1}{q})_{a-b}}T(b).$$

Thus the ratio of these two expressions is

$$\frac{u^b(\frac{1}{q})_a(\frac{u}{q})_a}{q^{b^2}(\frac{1}{q})_{a-b}(\frac{1}{q})_b(\frac{u}{q})_b},$$

as desired. The case $i = 1$ must be checked separately but the same ratio results. □

The algorithm of Theorem 7 runs on a computer because of the well known fact that to sample from a discrete distribution one divides $[0, 1]$ into intervals of the appropriate lengths and generates a uniform variable in $[0, 1]$.

To conclude the section we explain how to sample from $N_{u,q}$ given that the size of the partition is $n+1$, which is clearly the same as sampling the partition λ_{z-1} for a unipotent element of $A(u, q)$. One naive method is to keep picking from $N_{u,q}$ until one obtains a partition of size n; however this is very slow and not theoretically useful.

Algorithm for Sampling from $N_{u,q}$ given that $|\lambda| = n + 1$

Step 0 Start with $N = 1$ and λ the empty partition.

Step 1 If $n = 0$ then go to step 3. Otherwise set $h = 1 - \frac{1}{q^n}$.

Step 2 Flip a coin with probability of heads h.

Step 2a If the toss of Step 2 came up tails, increase the value of N by 1 and go to Step 2.

Step 2b If the toss of Step 2 comes up heads, decrease the value of n by 1, increase λ according to the rule of Step 2b of the Young Tableau Algorithm (which depends on N), and then go to Step 1.

Step 3 Perform Step 2 of the Affine Young Tableau Algorithm.

Before proving Theorem 8, we remark that it is an adaptian of an analogous result for $GL(n, q)$ which was obtained in joint work with Mark Huber, surveyed in [F3].

Theorem 8 *The algorithm for sampling from $N_{u,q}$ given that $|\lambda| = n+1$ is valid.*

Proof: The survey [F3] proved that if one replaced Step 3 in the above algorithm by stopping then one would sample from $M_{u,q}$ given that $|\lambda| = n$. Let $\lambda^{(s)}$ denote λ with the size of column s decreased by one and let $E(s)$ be the probability that Step 2 of the Affine Young Tableau Algorithm adds to column s of $\lambda^{(s)}$. The result follows because

$$\begin{aligned}
\frac{N_{u,q}(\lambda)}{\sum_{\lambda:|\lambda|=n+1} N_{u,q}(\lambda)} &= \frac{N_{u,q}(\lambda)}{\sum_{\lambda:|\lambda|=n} M_{u,q}(\lambda)}\\
&= \frac{\sum_s M_{u,q}(\lambda^{(s)})E(s)}{\sum_{\lambda:|\lambda|=n} M_{u,q}(\lambda)}\\
&= \sum_s E(s)\frac{M_{u,q}(\lambda^{(s)})}{\sum_{\lambda:|\lambda|=n} M_{u,q}(\lambda)}.
\end{aligned}$$

□

4 Applications of Probabilistic Algorithms

This section consider applications of the probabilistic algorithms of Section 3. The results here parallel those of [F1] for $GL(n,q)$ but are genuinely different. We do omit the formula for the chance that an element of $A(n,q)$ has a given characteristic polynomial, as this follows from the $GL(n,q)$ result and offers no new insights. Furthermore we only state the results for $A(n,q)$ as the extensions to $P(n,q)$ are trivial. This section uses the notation that $[u^n]f(u)$ denotes the coefficient of u^n in a polynomial $f(u)$.

Let $P_{A,n}(k,q)$ be the chance that an element of $A(n,q)$ has a k dimensional fixed space and let $P_{A,\infty}(k,q)$ be the $n \rightarrow \infty$ limit of $P_{A,n}(k,q)$. Theorem 10 will show that

$$P_{A,n}(k,q) = \frac{1}{|A(k-1,q)|} \sum_{i=0}^{n-k+1} \frac{(-1)^i}{q^{ki}(q^i-1)\cdots(q-1)}$$

and

$$P_{A,\infty}(k,q) = [\prod_{r=1}^{\infty}(1-\frac{1}{q^r})]\frac{(1/q)^{k^2-k}}{(1-1/q)^2\cdots(1-1/q^{k-1})^2(1-1/q^k)},$$

where we use the notation that $|A(-1,q)| = 0$. Furthermore, a probabilistic interpretation will be given to the products in $P_{A,\infty}(k,q)$. The first step is to connect the chance that an element α of $A(n,q)$ has a k dimensional fixed space with the partitions in the rational canonical form of α.

Lemma 4 *([F1]) The dimension of the fixed space of an element α of $GL(n,q)$ is equal to $\lambda'_{z-1,1}(\alpha)$ (i.e. the number of parts of the partition corresponding to the polynomial $z-1$ in the rational canonical form of α).*

To proceed further, some notation is necessary. Let T be a standard Young tableau with k parts. Let $T_{(i,j)}$ be the entry in row i and column j of T. We define numbers $h_1(T), \cdots, h_k(T)$ associated with T. Let $h_m(T) = T_{(m+1,1)} - T_{(m,1)} - 1$ for $1 \leq m \leq k-1$ and let $h_k(T) = |T| - T_{(k,1)}$. So if $k = 3$ and T is the tableau

$$\begin{array}{cccc} \boxed{1} & \boxed{3} & \boxed{5} & \boxed{6} \\ \boxed{2} & \boxed{4} & \boxed{7} & \\ \boxed{8} & \boxed{9} & & \end{array}$$

then $h_1(T) = 2-1-1 = 0$, $h_2(T) = 8-2-1 = 5$, and $h_3(T) = 9-8 = 1$. View T as being created by the Affine Young Tableau Algorithm. Then for $1 \leq m \leq k-1$, $h_m(T)$ is the number of boxes added to T after it becomes a tableau with m parts and before it becomes a tableau with $m+1$ parts. $h_k(T)$ is the number of boxes added to T after it becomes a tableau with k parts. The proof of Theorem 9 will show that if one conditions T chosen from the measure $N_{u,q}$ on having k parts, then the random variables $h_1(T), \cdots, h_k(T)$ are independent, where $h_1(T), \cdots, h_k(T)$ are geometric with parameters $\frac{u}{q}, \cdots, \frac{u}{q^k}$.

Theorem 9

$$\sum_{\lambda:\lambda_1'=k} N_{u,q}(\lambda) = \frac{u^{k-1}}{|A(k-1,q)|}\frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{u}{q^r})}$$

PROOF: We sum over all Young tableaux T with k parts the chance that the Affine Young Tableau Algorithm outputs T. Recall from the second remark after Theorem 5 that one can compute the probability that the Affine Young Tableau Algorithm outputs any given Young tableau. The point is that one can in fact compute the probability that the Affine Young Tableau Algorithm produces a tableau T with given values of the h's.

To do this, consider what happens during Step 1 of the Affine Young Tableau Algorithm. Suppose that one moves along the Young lattice from a partition with m parts. Theorem 5 implies that the weight for adding to column 1 is $\frac{u}{q^m(q^{m+1}-1)}$, and that the sum of the weights for adding to some other column is $\frac{u}{q^m}$. Thus the chance that Step 1 of the Affine Young Tableau Algorithm yields a tableau with given values $h_1, \cdots, h_{k-1}$ and all other h_i=0 is

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\frac{u^{k-1}}{|GL(k-1,q)|}\prod_{m=1}^{k-1}(\frac{u}{q^m})^{h_m}.$$

In order for the Affine Young Tableau Algorithm to generate a partition with m parts, either Step 1 generates a partition with $m-1$ parts and Step 2 increases the size of column 1 or else Step 1 generates a partition with m parts and Step 2 increases the size of column $s>1$. Thus the probability that the Affine Young Tableau Algorithm generates a partition with k parts is

$$\begin{aligned}
&\sum_{\lambda:\sum\lambda_1'=k} N_{u,q}(\lambda)\\
= &\left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\frac{u^{k-1}}{|GL(k-1,q)|}\prod_{m=1}^{k-1}\sum_{h_m=0}^{\infty}(\frac{u}{q^m})^{h_m}[\frac{1}{q^{k-1}}+\frac{u}{q^{k-1}(q^k-1)}\sum_{h_k\geq 0}(\frac{u}{q^k})^{h_k}(1-\frac{1}{q^k})]\\
= &\left[\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right]\frac{u^{k-1}}{q^{k-1}|GL(k-1,q)|}\prod_{m=1}^{k-1}\sum_{h_m=0}^{\infty}(\frac{u}{q^m})^{h_m}[\sum_{h_k\geq 0}(\frac{u}{q^k})^{h_k}]
\end{aligned}$$

$$= \frac{u^{k-1}}{|A(k-1,q)|} \frac{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})}{\prod_{r=1}^{k}(1-\frac{u}{q^r})}.$$

□

One further lemma will be used.

Lemma 5 *([GoR])*

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r}) = \sum_{i=0}^{\infty} \frac{(-u)^i}{(q^i-1)\cdots(q-1)}$$

Now the formulas for $P_{A,n}(k,q)$ and $P_{A,\infty}(k,q)$ can be proved. Recall that we use the notation that $|A(-1,q)| = 0$.

Theorem 10 *1.*

$$P_{A,n}(k,q) = \frac{1}{|A(k-1,q)|} \sum_{i=0}^{n-k+1} \frac{(-1)^i}{q^{ki}(q^i-1)\cdots(q-1)}.$$

2.

$$P_{A,\infty}(k,q) = [\prod_{r=1}^{\infty}(1-\frac{1}{q^r})] \frac{(1/q)^{k^2-k}}{(1-1/q)^2\cdots(1-1/q^{k-1})^2(1-1/q^k)}.$$

Proof: In the equation of part one of Corollary 1 set $x_{z-1,\lambda} = 1$ if λ has k parts and $x_{z-1,\lambda} = 0$ otherwise. Also set $x_{\phi,\lambda} = 1$ for $\phi \neq z-1$. It follows from Theorem 9 and Lemma 5 that

$$\begin{aligned}
P_{A,n}(k,q) &= [u^n] \prod_{\phi \neq z} \prod_{r=1}^{\infty} (\frac{1}{1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}}) \sum_{\lambda:\lambda_1'=k} N_{u,q}(\lambda) \\
&= [u^n] \frac{1}{1-u} \sum_{\lambda:\lambda_1'=k} N_{u,q}(\lambda) \\
&= [u^n] \frac{u^{k-1}\prod_{r=1}^{\infty}(1-\frac{u}{q^{k+r}})}{(1-u)|A(k-1,q)|} \\
&= \frac{1}{|A(k-1,q)|}[u^{n-k+1}] \frac{1}{1-u} \sum_{i=0}^{\infty} \frac{(-1)^i(uq^{-k})^i}{(q^i-1)\cdots(q-1)} \\
&= \frac{1}{|A(k-1,q)|} \sum_{i=0}^{n-k+1} \frac{(-1)^i}{q^{ki}(q^i-1)\cdots(q-1)}.
\end{aligned}$$

For the second assertion of the theorem use Lemma 3 and Theorem 9 to conclude that

$$\begin{aligned} P_{A,\infty}(k,q) &= lim_{n\rightarrow\infty}[u^n]\frac{1}{1-u}\sum_{\lambda:\lambda_1'=k} N_{u,q}(\lambda) \\ &= \sum_{\lambda:\lambda_1'=k} N_{1,q}(\lambda) \\ &= \frac{1}{|A(k-1,q)|}\frac{\prod_{r=1}^{\infty}(1-\frac{1}{q^r})}{\prod_{r=1}^{k}(1-\frac{1}{q^r})} \end{aligned}$$

as desired. □

We remark that the analog of Theorem 10 for $GL(n,q)$ was first proved by Rudvalis and Shinoda [RS], using Moebius inversion on the lattice of subspaces of a vector space. Theorem 10 can be proved along the same lines but we omit the details as the argument is less insightful and incongruous with the theme of this paper.

The final result in this section is an enumeration of unipotent elements in $A(n,q)$ of a given rank.

Theorem 11 *The number of unipotent elements of rank k in $A(n,q)$ is equal to*

$$\frac{|A(n,q)|}{|A(k-1,q)|}\frac{(1-1/q^k)\cdots(1-1/q^n)}{q^{n-k+1}(1-1/q)\cdots(1-1/q^{n-k+1})}.$$

Proof: In the equation of part 2 of Corollary 1 set $x_{z-1,\lambda}=1$ if λ has k parts and $x_{\phi,\lambda}=0$ otherwise. Using Theorem 9 one sees that the sought number is

$$\begin{aligned} & |A(n,q)|[u^n]\frac{u^{k-1}}{|A(k-1,q)|}\frac{1}{\prod_{r=1}^{k}(1-\frac{u}{q^r})} \\ = & \frac{|A(n,q)|}{|A(k-1,q)|}[u^{n-k+1}]\prod_{r=1}^{k}\frac{1}{1-\frac{u}{q^r}} \\ = & \frac{|A(n,q)|}{|A(k-1,q)|}\frac{(1-1/q^k)\cdots(1-1/q^n)}{q^{n-k+1}(1-1/q)\cdots(1-1/q^{n-k+1})}. \end{aligned}$$

□

5 Applications of Cycle Indices

The purpose of this section is to apply the cycle index generating functions of Section 2 to obtain precise estimates for the probabilities that an element of $A(n,q)$ is separable, cyclic, or semisimple (these terms will be defined as needed). As these probabilities are identical for $P(n,q)$ results will only be stated for $A(n,q)$. The section is organized by discussing separable, cyclic, and semisimple probabilities, and in that order.

The use of generating function methods to provide similar estimates for the finite classical groups has appeared in earlier papers (the paper [F5] computes all three limits for GL, [W] computes the separable and cyclic limits and bounds the convergence rates, and [FNP] obtains convergence rates for the semisimple case and extensions to the finite classical groups). The reader unfamiliar with the cycle index of $GL(n,q)$ may wish to consult [St] or perhaps [F3] or [F5] which give worked examples in our notation.

The contribution of this section is the not obvious fact that cycle index methods can be applied to $P(n,q)$, a maximal parabolic subgroup of $GL(n,q)$. It is also interesting that the $n \rightarrow \infty$ limiting separable, cyclic, and semisimple probabilities are all different from the corresponding limits in $GL(n,q)$.

The following notation will be used throughout this section. $N(d,q)$ will denote the number of monic degree d irreducible polynomials over F_q. $N^{'}(d,q)$ is defined by $N^{'}(d,q) = N(d,q)$ for $d > 1$ and $N'(1,q) = N(1,q) - 2$. The following elementary lemmas will be useful.

Lemma 6

$$\prod_{\phi} (1 - \frac{u^{deg(\phi)}}{q^{deg(\phi)}}) = 1 - u$$

Proof: Expanding $\frac{1}{1-\frac{u^{deg(\phi)}}{q^{deg(\phi)}}}$ as a geometric series and using unique factorization in $F_q[x]$, one sees that the coefficient of u^d in the reciprocal of the left hand side is $\frac{1}{q^d}$ times the number of monic polynomials of degree d, hence 1. Comparing with the reciprocal of the right hand side completes the proof. □

Lemma 7 *(Darboux [O]) Suppose that $f(u)$ is analytic for $|u| < r, r > 0$ and has only simple poles on $|u| = r$. Letting w_j denote the poles, and $g_j(u)$ be such that $f(u) = \frac{g_j(u)}{1-u/w_j}$ and $g_j(u)$ is analytic near w_j, one has that as $n \rightarrow \infty$, the coefficient of u^n in $f(u)$ is*

$$\sum_j \frac{g_j(w_j)}{w_j^n} + o(1/r^n).$$

5.1 Separable Matrices

An element α in $A(n,q)$ is called separable if its characteristic polynomial is square free. We remark that the results of this subsection (but not those of the cyclic and semisimple subsections) could also by obtained by using the cycle index for $GL(n,q)$.

Let $s_A(n,q)$ be the proportion of separable elements in $A(n,q)$ and define the generating function $S_A(u,q) = \sum_{n=0}^{\infty} u^n s_A(n,q)$ and let $S_{GL}(u,q)$ be the corresponding generating function for GL.

Theorem 12

$$\begin{aligned} S_A(u,q) &= \prod_{d \geq 1} (1 + \frac{u^d}{q^d - 1})^{N'(d,q)} \\ &= \frac{S_{GL}(u,q)}{(1 + \frac{u}{q-1})} \end{aligned}$$

PROOF: The first equality follows from Corollary 1 together with the fact that an element α of $GL(n,q)$ is separable if and only if all $\lambda_\phi(\alpha)$ have size at most 1. The second equality follows from the cycle index for $GL(n,q)$. □

Wall [W] shows that $S_{GL}(u,q)$ is analytic in $|u| < q$ except for a simple pole at $u = 1$ which has residue $1 - 1/q$, implying that $s_{GL}(\infty, q) = 1 - \frac{1}{q}$. To compute the corresponding limit for the affine case note that for $q > 2$, $S_A(u,q)$ is analytic in $|u| < q - 1$ except for a simple pole at $u = 1$

which has residue $\frac{1-\frac{1}{q}}{1+\frac{1}{q-1}}$. For $q = 2$, since $N'(1, q) = 0$ the function $S_A(u, q)$ is analytic in $|u| < q$ except for a simple pole at $u = 1$ which has residue $\frac{1-\frac{1}{q}}{1+\frac{1}{q-1}}$. Thus $s_A(\infty, q) = \frac{1-\frac{1}{q}}{1+\frac{1}{q-1}}$.

Theorem 13 bounds the convergence rate of $s_A(n, q)$ to its limit. The statement of the theorem is not fully simplified so as to make the proof easier to follow.

Theorem 13 *Let $c = 3/2$ and $K^+ = \frac{kc}{c-1}(1 + \frac{q-2}{c^2})(\frac{c}{q-c})$. Then*

1. $|s_A(n, q) - s_A(\infty, q)| \leq \frac{2K^+c(c/(q-1))^n}{(c-1)(1-c/(q-1))} + \frac{1}{(1-1/(q-1))(q-1)^{n+1}}$ *for $q > 2$.*

2. $|s_A(n, q) - s_A(\infty, q)| \leq \frac{2K^+(c/q)^{n+1}}{1-(c/q)^2}$ *for $q = 2$.*

Proof: Theorem 12 implies that

$$(1 - u)S_A(u, q) = \frac{(1 - u)}{(1 + \frac{u}{q-1})} S_{GL}(u, q).$$

Taking coefficients of u^{m+1} on both sides and using the fact that $s_{GL}(1, q) = s_{GL}(0, q)$ one obtains that

$$|s_A(m + 1, q) - s_A(m, q)| \leq (\frac{1}{q-1})^m [\sum_{i=1}^{m} (q - 1)^i |s_{GL}(i + 1, q) - s_{GL}(i, q)|] + \frac{1}{(q-1)^{m+1}}.$$

For any c, K^+ as above, page 273 of [W] gives the bound

$$|s_{GL}(i, q) - s_{GL}(\infty, q)| \leq K^+(c/q)^i.$$

Thus by the triangle inequality,

$$|s_{GL}(i + 1, q) - s_{GL}(i, q)| \leq 2K^+(c/q)^i.$$

Now summing over $m \geq n$ and using the triangle inequality gives

$$|s_A(n, q) - s_A(\infty, q)| \quad \leq \quad 2 \sum_{m \geq n} (\frac{1}{q-1})^m [\sum_{i=1}^{m} (\frac{q-1}{q})^i K^+ c^i] + \sum_{m \geq n} \frac{1}{(q-1)^{m+1}}$$

$$\begin{aligned}
&\leq 2\sum_{m\geq n}(\frac{1}{q-1})^m[\sum_{i=1}^{m}K^+c^i]+\frac{1}{(1-1/(q-1))(q-1)^{n+1}}\\
&\leq \frac{2K^+c}{c-1}\sum_{m\geq n}(\frac{c}{q-1})^m+\frac{1}{(1-1/(q-1))(q-1)^{n+1}}\\
&= \frac{2K^+c(c/(q-1))^n}{(c-1)(1-c/(q-1))}+\frac{1}{(1-1/(q-1))(q-1)^{n+1}}.
\end{aligned}$$

For $q=2$ observe that taking coefficients of u^{m+2} on both sides of

$$(1-u^2)S_A(u,q)=(1-u)S_{GL}(u,q)$$

gives that

$$|s_A(m+2,q)-s_A(m,q)|\leq|s_{GL}(m+2,q)-s_{GL}(m+1,q)|.$$

Now use Wall's bound for $|s_{GL}(i,q)-s_{GL}(\infty,q)|$ to conclude that

$$\begin{aligned}
|s_A(n,q)-s_A(\infty,q)| &\leq \sum_{\substack{m\geq n\\ m-n\ even}}|s_A(m+2,q)-s_A(m,q)|\\
&\leq \sum_{\substack{m\geq n\\ m-n\ even}}|s_{GL}(m+2,q)-s_{GL}(m+1,q)|\\
&\leq 2\sum_{\substack{m\geq n\\ m-n\ even}}K^+(c/q)^{m+1}\\
&\leq \frac{2K^+(c/q)^{n+1}}{1-(c/q)^2}.
\end{aligned}$$

□

5.2 Cyclic matrices

An element α of $GL(n,q)$ is called cyclic if its characteristic polynomial is equal to its minimal polynomial. Let $c_A(n,q)$ be the proportion of cyclic elements in $A(n,q)$ and let $c_A(\infty,q)$ be the $n\to\infty$ limit of $c_A(n,q)$. Let $C_A(u,q)=\sum_{n=0}^{\infty}u^nc_A(n,q)$ and let $C_{GL}(u,q)$ be the corresponding generating function for GL.

Theorem 14

$$\begin{aligned} C_A(u,q) &= \frac{1}{1-u/q}\prod_{d\geq 1}(1+\frac{u^d}{(q^d-1)(1-u^d/q^d)})^{N'(d,q)} \\ &= \frac{1}{1-u/q}\frac{1}{1+\frac{u}{(q-1)(1-u/q)}}C_{GL}(u,q). \end{aligned}$$

PROOF: The first equality follows from Corollary 1 together with the fact that an element α of $GL(n,q)$ is cyclic if and only if all $\lambda_\phi(\alpha)$ have at most 1 part. The second equality follows from the cycle index for $GL(n,q)$. □

Corollary 2

$$c_A(\infty,q) = \frac{1-\frac{1}{q}}{1-\frac{1}{q}+\frac{1}{q^2}}\frac{1-1/q^5}{1+1/q^3}.$$

PROOF: Wall [W] shows that $C_{GL}(u,q)$ is analytic in $|u| < q^2$ except for a simple pole at $u=1$ which has residue $\frac{1-1/q^5}{1+1/q^3}$. Theorem 14 implies that $C_A(u,q)$ is analytic in $|u| < q^2-q$ except for a simple pole at $u=1$ which has residue $\frac{1-\frac{1}{q}}{1-\frac{1}{q}+\frac{1}{q^2}}\frac{1-1/q^5}{1+1/q^3}$. Thus $c_A(\infty,q)$ is equal to $\frac{1-\frac{1}{q}}{1-\frac{1}{q}+\frac{1}{q^2}}\frac{1-1/q^5}{1+1/q^3}$. □

For large q the limit in Theorem 2 is of the form $1-1/q^2+O(1/q^3)$. It would be interesting to understand this in terms of algebraic geometry; the $1-1/q^3+O(q^4)$ behavior in the case of $GL(n,q)$ is a finite analog of Steinberg's result that the variety of regular semisimple elements has codimension three [St]. See [NP3] for further discussion.

Adapting a trick of Wall [W] from the GL case gives instant bounds on the convergence rate of $c_A(n,q)$ to its limit.

Lemma 8

$$(1-u)C_A(u,q) = (1-\frac{u}{q})S_A(\frac{u}{q},q).$$

PROOF: Theorems 14, 12 and Lemma 6 imply that

$$(1-u)C_A(u,q) = \frac{1}{1-u/q}\prod_{d\geq 1}(1+\frac{u^d}{q^d(1-u^d/q^d)(1-1/q^d)})^{N'(q,d)}\prod_{d\geq 1}(1-u^d/q^d)^{N(d,q)}$$

$$\begin{aligned} &= (1-u/q)\prod_{d\geq 1}(1+\frac{u^d}{q^d(q^d-1)})^{N'(q,d)} \\ &= (1-u/q)S_A(\frac{u}{q},q). \end{aligned}$$

□

Corollary 3

$$|c_A(\infty,q)-c_A(n,q)| \leq \frac{1}{q^{n+1}(1-1/q)}$$

PROOF: Taking coefficients of u^{m+1} on both sides of the equation in Lemma 8 gives that

$$c_A(m+1,q)-c_A(m,q) = \frac{1}{q^{m+1}}[s_A(m+1,q)-s_A(m,q)].$$

Using the triangle inequality and the fact that $0 \leq s_A(m+1,q), s_A(m,q) \leq 1$, it follows that

$$\begin{aligned} |c_A(\infty,q)-c_A(n,q)| &\leq \sum_{m=n}^{\infty} |c_A(m+1,q)-c_A(m,q)| \\ &= \sum_{m=n}^{\infty} \frac{1}{q^{m+1}}|s_A(m+1,q)-s_A(m,q)| \\ &\leq \sum_{m=n}^{\infty} \frac{1}{q^{m+1}} \\ &= \frac{1}{q^{n+1}(1-1/q)}. \end{aligned}$$

□

5.3 Semisimple matrices

An element α of $GL(n,q)$ is called semisimple if it diagonalizable over the algebraic closure of F_q. Treatments of semisimple probabilities without generating functions appear in [IsKanSp] and [GuL], which proves the lovely result that if G is a simple Chevalley group, then the probability of not being semisimple is at most $3/(q-1)+2/(q-1)^2$.

A crude asymptotic understanding of the behavior of the proportion of semisimple elements in $GL(n,q)$ is in [St], who used generating functions. The paper [F5] uses one of the Rogers-Ramanujan identities to show that $n \to \infty$ probability that an element of $GL(n,q)$ is semisimple is

$$\prod_{\substack{r=1\\ r=0,\pm 2 (mod\ 5)}}^{\infty} \frac{(1-\frac{1}{q^{r-1}})}{(1-\frac{1}{q^r})}.$$

The paper [FNP] gives bounds for finite n.

Let $ss_A(n,q)$ be the probability that an element of $A(n,q)$ is semisimple and let $ss_A(\infty,q)$ be the $n \to \infty$ limit of $ss_A(n,q)$. Let $SS_A(u,q) = \sum_{n=0}^{\infty} u^n ss_A(n,q)$ and let SS_{GL} be the corresponding generating function for GL.

Theorem 15

$$\begin{aligned} SS_A(u,q) &= (\sum_{k \geq 0} \frac{u^k}{q^{k^2+k}(\frac{1}{q})_k}) \prod_{d \geq 1} (\sum_{k \geq 0} \frac{u^{kd}}{q^{kd}(\frac{1}{q^d})_k})^{N'(d,q)} \\ &= \frac{(\sum_{k \geq 0} \frac{u^k}{q^{k^2+k}(\frac{1}{q})_k})}{(\sum_{k \geq 0} \frac{u^k}{q^{k^2}(\frac{1}{q})_k})} SS_{GL}(u,q). \end{aligned}$$

Proof: An element α of $GL(n,q)$ is semisimple if and only if all $\lambda_{\phi}(\alpha)$ have at most one column. Now use Corollary 1. □

To calculate $ss_A(\infty,q)$ we will use (with q replaced by $1/q$) the well known Rogers-Ramanujan identities (see [A] for discussion)

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}.$$

Corollary 4

$$ss_A(\infty,q) = \frac{\prod_{\substack{r=1\\ r=0,\pm 1\ mod\ 5}}^{\infty} (1-1/q^r) \prod_{\substack{r=1\\ r=0,\pm 2\ mod\ 5}}^{\infty} (1-1/q^{r-1})}{\prod_{\substack{r=1\\ r=0,\pm 2\ mod\ 5}}^{\infty} (1-1/q^r)^2}$$

PROOF: The discussion in [FNP] shows that $SS_{GL}(u,q)$ is analytic within a circle of radius greater than 1, except for a simple pole at $u=1$. Since

$$\frac{(\sum_{k \geq 0} \frac{u^k}{q^{k^2+k}(\frac{1}{q})_k})}{(\sum_{k \geq 0} \frac{u^k}{q^{k^2}(\frac{1}{q})_k})}$$

is also analytic within a circle of radius greater than 1, it follows that

$$ss_A(\infty,q) = \frac{(\sum_{k \geq 0} \frac{1}{q^{k^2+k}(\frac{1}{q})_k})}{(\sum_{k \geq 0} \frac{1}{q^{k^2}(\frac{1}{q})_k})} ss_{GL}(\infty,q).$$

Now simply use both Rogers-Ramanujan identities and the formula for $ss_{GL}(\infty,q)$ in [F5] stated at the beginning of this subsection. □

Bounding the convergence rate of $ss_A(n,q)$ through generating functions is an involved analytic excursion which we omit. The following elementary bound is sufficient for practical purposes, given the results of the previous subsections and the fact that $ss_A(n,q)$ can be computed explicitly for small n from the generating function.

Theorem 16

$$s_A(n,q) \leq ss_A(n,q) \leq s_A(n,q) + (1 - c_A(n,q)).$$

PROOF: The first inequality follows because semisimple matrices are separable. The second inequality follows because a matrix which is not separable is either not cyclic or not semisimple. □

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