Orbifold Euler characteristics and the number of commuting m-tuples in the symmetric groups

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Abstract

Generating functions for the number of commuting m-tuples in the symmetric groups are obtained. We define a natural sequence of "orbifold Euler characteristics" for a finite group G acting on a manifold X. Our definition generalizes the ordinary Euler characteristic of X/G and the string-theoretic orbifold Euler characteristic. Our formulae for commuting m-tuples underlie formulas that generalize the results of Macdonald and Hirzebruch-Höfer concerning the ordinary and string-theoretic Euler characteristics of symmetric products.

1 Introduction

Let X be a manifold with the action of a finite group G. The Euler characteristic of the quotient space X/G can be computed by the Lefschetz fixed point formula:

$$\chi(X/G) = \frac{1}{|G|}\sum_{g\in G}\chi(X^g)$$

where X^g is the fixed point set of g. Motivated by string theory, physicists have defined an "orbifold characteristic" by

$$\chi(X,G) = \frac{1}{|G|}\sum_{gh=hg}\chi(X^{(g,h)})$$

where the sum runs over commuting pairs and $X^{(g,h)}$ denotes the common fixed point set of g and h.

We introduce a natural sequence of orbifold Euler characteristics $\chi_m(X,G)$ for $m = 1, 2, \ldots$ so that $\chi(X/G)$ and $\chi(X,G)$ appear as the first two terms. Namely, if we denote by $Com(G,m)$ the set of mutually commuting m-tuples $(g_1, \ldots, g_m)$ and by $X^{(g_1,\ldots,g_m)}$ the simultaneous fixed point set, then we define the *m-th orbifold characteristic* to be

$$\chi_m(X,G) = \frac{1}{|G|}\sum_{Com(G,m)}\chi(X^{(g_1,\ldots,g_m)}). \tag{1}$$

In the case of a symmetric product, *i.e.* X is the n-fold product M^n and G is the symmetric group S_n, there are combinatorial formulas for χ_1 and χ_2 due to Macdonald [5] and Hirzebruch-Höfer [3] respectively. The main result of this note (Theorem 1) is a generalization of those formulas to χ_m for arbitrary m. In the case where M has (ordinary) Euler characteristic 1, our formulas specialize to generating functions for $|Com(S_n,m)|$, the number of commuting m-tuples in S_n.

Finally, we remark that the first two terms in our sequence $\chi_m(X,G)$ of orbifold Euler characteristics are the Euler characteristics of the cohomology theories $H^*_G(X;\mathbf{Q})$ and $K^*_G(X;\mathbf{Q})$ respectively. This was observed by Segal, [1] who was led to speculate that the heirarchy of generalized cohomology theories investigated by Hopkins and Kuhn [4] may have something to do with the sequence of Euler characteristics defined in this paper (our definition is implicitly suggested in [1]). We hope that our combinatorial formulas will provide clues to the nature of these theories.

2 Formulae

In this section we specialize to the case of symmetric products so that $X = M^n$ and $G = S_n$. For $(\pi_1, \cdots, \pi_m) \in Com(S_n, m)$, let $\#(\pi_1, \cdots, \pi_m)$ be the number of connected components in the graph on vertex set $\{1, \cdots, n\}$ defined by connecting the vertices according to the permutations $\pi_1, \cdots, \pi_m$. For instance, $\#(\pi_1)$ is the number of cycles of π. The main result of this note is the following theorem.

Theorem 1 *Let χ denote the (ordinary) Euler characteristic of M. The generating function for the orbifold Euler characteristic $\chi_m(M^n, S_n)$ satisfies the following formulas:*

$$\sum_{n=0}^{\infty} \chi_m(M^n, S_n) u^n = \sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\pi_1,\ldots,\pi_m \in Com(S_n,m)} \chi^{\#(\pi_1,\ldots,\pi_m)} \tag{2}$$

$$= \left(\sum_{n=0}^{\infty} |Com(S_n, m)| \frac{u^n}{n!} \right)^{\chi} \tag{3}$$

$$= \prod_{i_1,\cdots,i_{m-1}=1}^{\infty} (1 - u^{i_1 \cdots i_{m-1}})^{-\chi i_1^{m-2} i_2^{m-3} \cdots i_{m-2}}. \tag{4}$$

Remarks: We will show that Equation 2 follows directly from the definitions and a straightforward geometric argument. Equation 3 is proved in Lemma 1 and shows that it suffices to prove Equation 4 in the case $\chi = 1$. Our main result then should be regarded as Equation 4 which in light of Equation 3 gives a generating function for the number of commuting m-tuples in S_n. Note also that for $m = 1$ Equation 4 is Macdonald's formula $(1-u)^{-\chi}$ for the Euler characteristic of a symmetric product and for $m = 2$ Equation 4 is Hirzebruch and Höfer's formula for the string-theoretic orbifold Euler characteristic of a symmetric product.

To prove Equation 2 it suffices to see that

$$\chi(M^{(\pi_1,\ldots,\pi_m)}) = (\chi(M))^{\#(\pi_1,\ldots,\pi_m)}.$$

Partition $\{1, \ldots, n\}$ into disjoint subsets $I_1, \ldots, I_{\#(\pi_1,\ldots,\pi_m)}$ according to the components of the graph associated to $(\pi_1, \ldots, \pi_m)$. Then the small diagonal in the product $\prod_{i \in I_j} M_i$ is fixed by $(\pi_1, \ldots, \pi_m)$ and is homeomorphic to M. The full fixed set of $(\pi_1, \ldots, \pi_m)$ is then the product of all the small diagonals in the subproducts associated to the I_j's. By the multiplicative properties of Euler characteristic we see that $\chi(M^{(\pi_1,\ldots,\pi_m)}) = (\chi(M))^{\#(\pi_1,\ldots,\pi_m)}$.

Lemma 1 *For χ a natural number,*

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\pi_1,\cdots,\pi_m \in Com(S_n,m)} \chi^{\#(\pi_1,\cdots,\pi_m)} = \left(\sum_{n=0}^{\infty} \frac{u^n |Com(S_n, m)|}{n!} \right)^{\chi}$$

Proof: It suffices to show that an ordered m-tuple $(\pi_1, \cdots, \pi_m)$ of mutually commuting elements of S_n contributes equally to the coefficient of $\frac{u^n}{n!}$ on both sides of the equation. The contribution to this coefficient on the left-hand side is $\chi^{\#(\pi_1,\cdots,\pi_m)}$.

The right hand side can be rewritten as

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{n_1,\cdots,n_\chi : \sum n_i = n} \binom{n}{n_1, \cdots, n_\chi} |Com(S_{n_1}, m)| \cdots |Com(S_{n_\chi}, m)|.$$

Observe that $\binom{n}{n_1,\cdots,n_\chi} |Com(S_{n_1}, m)| \cdots |Com(S_{n_\chi}, m)|$ is the number of ways of decomposing the vertex set $\{1, \cdots, n\}$ into χ ordered subsets $S_1, \cdots, S_\chi$ of sizes $n_1, \cdots, n_\chi$ and defining an ordered m-tuple of mutually commuting elements of S_{n_i} on each subset. Gluing these together defines an ordered m-tuple of mutually commuting elements of S_n. Note that the m-tuple $(\pi_1, \cdots, \pi_m)$ arises in $\chi^{\#(\pi_1,\cdots,\pi_m)}$ ways, because each of the $\#(\pi_1, \cdots, \pi_m)$ connected components of the graph corresponding to $(\pi_1, \cdots, \pi_m)$ could have come from any of the χ subsets $S_1, \cdots, S_\chi$. $\square$

Let us now recall some facts about wreath products of groups. All of this can be found in Sections 4.1 and 4.2 of James and Kerber [2]. Given a group G, the wreath product $GWrS_n$ is defined as a set by $(g_1, \cdots, g_n; \pi)$ where $g_i \in G$ and $\pi \in S_n$. Letting permutations act on the right, the group multiplication is defined by:

$$(g_1, \cdots, g_n; \pi)(h_1, \cdots, h_n; \tau) = (g_1 h_{(1)\pi^{-1}}, \cdots, g_n h_{(n)\pi^{-1}}; \pi\tau)$$

Furthermore, the conjugacy classes of $GWrS_n$ are parameterized as follows. Let $Cl_1, \cdots, Cl_i$ be the conjugacy classes of G. Then the conjugacy classes of $GWrS_n$ correspond to arrays $(M_{j,k})$ satisfying the properties:

1. $M_{j,k} = 0$ if $j > i$
2. $\sum_{j,k} k M_{j,k} = n$

The correspondence can be made explicit. For $(g_1, \cdots, g_n; \pi) \in GWrS_n$, let $M_{j,k}$ be the number of k-cycles of π such that multiplying the k g_i whose subscripts lies in the k-cycle gives an element of G belonging to the conjugacy class Cl_j of G. The matrix so-defined clearly satisfies the above two conditions.

Lemma 2 is a key ingredient of this paper. It says that centralizers of elements of wreath products can be expressed in terms of wreath products; this will lead to an inductive proof of Theorem 1.

Lemma 2 *Let C_i denote a cyclic group of order i. Then the centralizer in $C_i W r S_n$ of an element in the conjugacy class corresponding to the data $M_{j,k}$ is isomorphic to the direct product*

$$\prod_{j,k} C_{ik} W r S_{M_{j,k}}$$

PROOF: To start, let us construct an element $(g_1, \cdots, g_n; \pi)$ of $C_i W r S_n$ with conjugacy class data $M_{j,k}$. This can be done as follows:

1. Pick π to be any permutation with $\sum_j M_{j,k}$ k-cycles
2. For each j choose $M_{j,k}$ of the k-cycles of π and think of them as k-cycles of type j
3. Assign (in any order) to the g_i whose subscripts are contained in a k-cycle of type j of π the values $(c_j, 1, \cdots, 1)$ where c_j is an element in the jth conjugacy class of the group C_i

To describe the centralizer of this element $(g_1, \cdots, g_n; \pi)$, note that conjugation in $GWrS_n$ works as

$$\begin{aligned} & (h_1, \cdots, h_n; \tau)(g_1, \cdots, g_n; \pi)(h_{(1)\tau}^{-1}, \cdots, h_{(n)\tau}^{-1}; \tau^{-1}) \\ = \; & (h_1 g_{(1)\tau^{-1}} h_{(1)\tau\pi^{-1}\tau^{-1}}^{-1}, \cdots; \tau\pi\tau^{-1}) \end{aligned}$$

It is easy to see that if $(h_1, \cdots, h_n; \tau)$ commutes with $(g_1, \cdots, g_n; \pi)$, then τ operates on the $M_{j,k}$ k-cycles of π of type j by first permuting these cycles amongst themselves and then performing some power of a cyclic shift within each cycle. Further, among the h_i whose subscripts lie in a k-cycle of π of type j exactly one can be chosen arbitrarily in C_i–the other h's with subscripts in that k-cycle then have determined values.

The direct product assertion of the theorem is then easily checked; the only non-trivial part is to see the copy of $C_{ik} WrS_{M_{j,k}}$. Here the $S_{M_{j,k}}$ permutes the $M_{j,k}$ k-cycles of type j, and the generator of the C_{ik} corresponds to having τ cyclically permuting within the k cycle and having the h's with subscripts in that k-cycle equal to $\{c_j, 1, \cdots, 1\}$, where c_j is a generator of C_i. □

With these preliminaries in hand, induction can be used to prove the following result. Note that by Lemma 1, only the $i = 1$ case of Theorem 2 is needed to prove the main result of this paper, Theorem 1. However, the stronger statement (general i) in Theorem 2 makes the induction work by making the induction hypothesis stronger.

Theorem 2 *For $m \geq 2$,*

$$\sum_{n=0}^{\infty} \frac{u^n |Com(C_i WrS_n, m)|}{|C_i WrS_n|} = \prod_{i_1,\cdots,i_{m-1}=1}^{\infty} (\frac{1}{1-u^{i_1 \cdots i_{m-1}}})^{i^{m-1} i_1^{m-2} i_2^{m-3} \cdots i_{m-2}}$$

PROOF: The proof proceeds by induction on m. We use the notation that if λ denotes a conjugacy class of a group G, then $C_G(\lambda)$ is the centralizer in G of some element of λ (hence $C_G(\lambda)$ is well defined up to isomorphism). For the base case $m = 2$ observe that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{u^n |Com(C_i WrS_n, 2)|}{|C_i WrS_n|} \\
= & \sum_{n=0}^{\infty} \frac{u^n}{|C_i WrS_n|} \sum_{\substack{(M_{j,k}):1 \leq j \leq i \\ \sum_{j,k} kM_{j,k}=n}} \frac{|C_i WrS_n|}{|C_{C_i WrS_n}(M_{j,k})|} |C_{C_i WrS_n}(M_{j,k})| \\
= & \sum_{n=0}^{\infty} u^n \sum_{\substack{(M_{j,k}):1 \leq j \leq i \\ \sum_{j,k} kM_{j,k}=n}} 1 \\
= & \prod_{i_1=1}^{\infty} (\frac{1}{1-u^{i_1}})^i
\end{aligned}$$

For the induction step, the parameterization of conjugacy classes of wreath products and Lemma 2 imply that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{u^n |Com(C_i WrS_n, m)|}{|C_i WrS_n|} \\
= & \sum_{n=0}^{\infty} \frac{u^n}{|C_i WrS_n|} \sum_{\substack{(M_{j,k}):1 \leq j \leq i \\ \sum_{j,k} kM_{j,k}=n}} \frac{|C_i WrS_n|}{|C_{C_i WrS_n}(M_{j,k})|} |Com(C_{C_i WrS_n}(M_{j,k}), m-1)| \\
= & [\prod_{k=1}^{\infty} \sum_{a=0}^{\infty} \frac{u^{ka} |Com(C_{ik} WrS_a, m-1)|}{|C_{ik} WrS_a|}]^i \\
= & [\prod_{k=1}^{\infty} \prod_{i_2,\cdots,i_{m-1}=1}^{\infty} (\frac{1}{1-u^{ki_2 \cdots i_{m-1}}})^{(ik)^{m-2} i_2^{m-3} \cdots i_{m-2}}]^i \\
= & \prod_{i_1,\cdots,i_{m-1}=1}^{\infty} (\frac{1}{1-u^{i_1 \cdots i_{m-1}}})^{i^{m-1} i_1^{m-2} i_2^{m-3} \cdots i_{m-2}}
\end{aligned}$$

□

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