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Combinatorics of balanced carries

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ABSTRACT

We study the combinatorics of addition using balanced digits, deriving an analog of Holte's "amazing matrix" for carries in usual addition. The eigenvalues of this matrix for base b balanced addition of n numbers are found to be $1, 1/b, \cdots, 1/b^n$, and formulas are given for its left and right eigenvectors. It is shown that the left eigenvectors can be identified with hyperoctahedral Foulkes characters, and that the right eigenvectors can be identified with hyperoctahedral Eulerian idempotents. We also examine the carries that occur when a column of balanced digits is added, showing this process to be determinantal. The transfer matrix method and a serendipitous diagonalization are used to study this determinantal process.

1. Introduction

This paper studies the combinatorics of "carries" in basic arithmetic, using balanced digits. To begin we describe the motivation for using balanced digits. When ordinary integers are added, carries occur. Consider a carries table with rows and columns indexed

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by digits $0, 1, \cdots, b-1$ (working base b) and a carry at (i, j) if $i + j \geq b$. Thus when $b = 5$, labeling the rows and columns in the order $0, 1, 2, 3, 4$ the carries matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & b \\ 0 & 0 & b & b & b \\ 0 & b & b & b & b \end{pmatrix}.$$

In general there are $\binom{b}{2}$ carries (so 10 when $b = 5$). If digits are chosen uniformly at random, the chance of a carry is $\binom{b}{2}/b^2 = \frac{1}{2} - \frac{1}{2b}$.

The digits $0, 1, \cdots, b-1$ can be thought of as coset representatives for $b\mathbb{Z} \subseteq \mathbb{Z}$ and the carries are cocycles [14]. For b odd (as assumed throughout this paper), consider instead the balanced representatives $0, \pm 1, \cdots \pm (b-1)/2$. One motivation for using balanced representatives is that they lead to fewer carries. For example, when $b = 5$, writing $\bar{j}$ for $-j$, the digits are $\{0, 1, \bar{1}, 2, \bar{2}\}$. Labeling the rows and columns in the order $\bar{2}, \bar{1}, 0, 1, 2$, the carries matrix is

$$\begin{pmatrix} \bar{b} & \bar{b} & 0 & 0 & 0 \\ \bar{b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & b \end{pmatrix}.$$

(For example, $(-2) + (-2) = -4 = -5 + 1$.) Here there are 6 carries versus 10 for the classical choice. For general b, balanced carries lead to $(b^2 - 1)/4$ carries. This is the smallest number possible [2,12].

Balanced digits are elementary but unfamiliar: for example in base 5, 13 is equal to $1\bar{2}\bar{2}$, and -9 is equal to $\bar{2}1$. Negating numbers negates the digits and the sign of the number is the sign of its left-most digit. Balanced digits were introduced in 1726 by Colson [7]; see Cajori [6] or Chapter 4 of Knuth [16] for history and applications.

Of course, balanced digits may be used for addition with larger numbers. For example:

$$\begin{array}{c} \bar{1}011000 \\[1ex] 1\bar{2}122\bar{1}0 \\ \underline{1\bar{1}\bar{1}2221} \\[1ex] 1210\bar{1}11 \end{array}$$

Here the numbers along the top row are the carries. When two numbers are added the possible carries are $0, 1, \bar{1}$. If n numbers are added, the possible carries are $-\lfloor \frac{n}{2} \rfloor, \cdots, \lfloor \frac{n}{2} \rfloor$.

Suppose now that balanced digits base b are used, n numbers are added, and that the digits are chosen uniformly at random in $\{\overline{\frac{b-1}{2}}, \cdots \frac{b-1}{2}\}$. Consider the carries along the top, from right to left as $\kappa_1, \kappa_2, \cdots$. (Thus $\kappa_1 = 0$ always.) It is easy to see that the $\{\kappa_i\}$ form a Markov chain on the set $-\lfloor \frac{n}{2} \rfloor, \cdots, \lfloor \frac{n}{2} \rfloor$. For the classical choice of digits, this Markov chain was analyzed by Holte [13], with follow-up reviewed later in this introduction.

Let $K(i,j)$ denote the transition matrix of the balanced carries Markov chain. Of course $K(i,j)$ depends on b and n but this is suppressed. In the case that n is odd, we find that the matrix $K(i,j)$ is the same as Holte's amazing matrix for usual carries. However when n is even, new results emerge. In particular, the stationary distribution of the Markov chain is given by

$$\pi(j - n/2) = \frac{\overline{A(n,j)}}{2^n n!} \quad 0 \leq j \leq n.$$

Here the $\overline{A(n,j)}$ are "signed Eulerian numbers", defined carefully in Section 3. In particular, for $n = 2$, the chance of a carry $-1, 0, 1$ is $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. We find explicit formulae for the left and right eigenvectors of the balanced carries chain; we show that the left eigenvectors can be identified with Miller's hyperoctahedral Foulkes characters [17], and that the right eigenvectors can be identified with hyperoctahedral Eulerian idempotents of Bergeron and Bergeron [3]. Before we finished writing this paper, the preprint [18] appeared, and there is some overlap with our work; one can deduce our formula for the left eigenvectors of $K(i,j)$ from their paper.

Next we give a brief historical overview of the "type A" work that motivated this paper. The Markov chain of carries when n random integers are added mod b (with the usual choice of digits) was first studied by Holte [13], who dubbed the matrix "amazing". He found that the eigenvalues are $1, 1/b, 1/b^2, \cdots, 1/b^{n-1}$, and identified the stationary distribution as $A(n,k)/n!$, with $A(n,k)$ the kth Eulerian number—the number of permutations in S_n with k descents. This says what percent of carries are k $(0 \leq k \leq n-1)$. Holte further found closed formulae for the left and right eigenvectors of the transition matrix.

The connection between carries and shuffling was developed by the present authors. The first proofs used generating functions and symmetric function theory [8], and an "aha" bijective proof was later found in [9]. An algebraic combinatorics proof appeared in [20] and Pang [21] studied the entire descent pattern after shuffles using Hopf algebras. Further connections between carries and shuffling can be found in [19], which appeared after the current paper. We note that Holte's amazing matrix also appears in algebraic geometry, giving the Hilbert series of the Veronese embedding [5,8].

The connection of carries with Foulkes characters and Eulerian idempotents appears in [10], which identifies the left eigenvectors of Holte's matrix with the Foulkes characters of the symmetric groups, and the right eigenvectors with the Eulerian idempotents. The transition to other types of reflection groups is studied in Miller [17], and we identify the left eigenvectors of the balanced carries chain with Miller's hyperoctahedral

Foulkes characters. We give a new proof of Miller's recurrence for hyperoctahedral Foulkes characters. We also identify the right eigenvectors of the balanced carries chain with the Eulerian idempotents of the hyperoctahedral groups. For a different proof connecting the inverse of the Foulkes character table with Eulerian idempotents, one can see [17].

The above results describe "carries across the top", when several long numbers are added. It is also fruitful to study the "carries down a column" when a single column of random digits is added. For ordinary addition, this was studied in [4], which showed that the positions of the carries form a one-dependent determinantal point process, with explicitly computable correlation functions. We use the transfer matrix method and a serendipitous diagonalization to show that the same is true for "carries down a column" when a column of random balanced digits is added.

Carries and cocycles make sense for any subgroup H of any group G. Choosing coset representatives X for H in G and then picking elements x in X from some natural probability distribution leads to a carries process. This is developed in [4] and [12]. Developing a parallel theory involving a nested sequence of subgroups (as in the present paper) suggests a world of math to be done.

The organization of this paper is as follows. Section 2 begins by deriving an analog of Holte's amazing matrix for balanced carries. When an odd number of numbers is added, we show that this reduces to Holte's amazing matrix for ordinary carries, and when an even number of numbers is added, we show that it reduces to the type B carries chain of [8]. The argument is similar to Holte's, and the result can also be deduced from the paper [18]. Section 3 shows that the eigenvalues of the carries chain are $1, 1/b, \cdots, 1/b^n$ and studies its left eigenvectors, giving an explicit formula and identifying them with Miller's hyperoctahedral Foulkes characters. Section 4 gives a formula for its right eigenvectors, relating them to hyperoctahedral Eulerian idempotents. Section 5 studies "balanced carries down a column" as a determinantal point process.

2. Amazing matrix for balanced carries

We work in an odd base b, with digits $0, \pm 1, \pm 2, \cdots, \pm (b-1)/2$. For reasons which become clear in the remarks following Theorem 2.1, we add an even number n of numbers. Then the carries range from $-\frac{n}{2}$ to $\frac{n}{2}$, and form a Markov chain on the set $\{-\frac{n}{2}, -\frac{n}{2}+1, \cdots, \frac{n}{2}-1, \frac{n}{2}\}$. Theorem 2.1 works out the transition matrix for this Markov chain.

Theorem 2.1. *Let $K(i,j)$ be the transition probability of the balanced carries Markov chain on the set $\{-\frac{n}{2}, -\frac{n}{2}+1, \cdots, \frac{n}{2}-1, \frac{n}{2}\}$, corresponding to the addition of an even number n of numbers. Then $K(i,j)$ is equal to both:*

(1) *The coefficient of* $x^{jb+(n+1)(b-1)/2-i}$ *in* $(1+x+\cdots+x^{b-1})^{n+1}/b^n$.

(2)

$$\frac{1}{b^n}\sum_{l=0}^{\lfloor j+\frac{n+1}{b}\frac{b-1}{2}-\frac{i}{b}\rfloor}(-1)^l\binom{n+1}{l}\binom{n+jb+(n+1)\frac{b-1}{2}-i-lb}{n}.$$

As an example of Theorem 2.1, when $n=2$, with the rows indexed by $i=-1,0,1$ and the columns indexed by $j=-1,0,1$, the transition matrix is, for all odd b,

$$K(i,j)=\frac{1}{b^2}\begin{pmatrix}\frac{b^2+4b+3}{8} & \frac{3}{4}(b^2-1) & \frac{b^2-4b+3}{8}\\ \frac{b^2-1}{8} & \frac{3b^2+1}{4} & \frac{b^2-1}{8}\\ \frac{b^2-4b+3}{8} & \frac{3}{4}(b^2-1) & \frac{b^2+4b+3}{8}\end{pmatrix}.$$

Proof of Theorem 2.1. Suppose that the carry into a column is i, with $-\frac{n}{2}\le i\le\frac{n}{2}$. Let the n digits in the column be $X_1,\cdots,X_n$, with $-(b-1)/2\le X_i\le(b-1)/2$ for all i. Then the carry to the next column is j precisely if

$$jb-(b-1)/2\le i+X_1+\cdots+X_n\le jb+(b-1)/2.$$

Letting $X_i'=X_i+(b-1)/2$ for all i, one has that $0\le X_1',\cdots,X_n'\le b-1$, and that the carry to the next column is j exactly when

$$jb-(b-1)/2\le i+X_1'+\cdots+X_n'-n(b-1)/2\le jb+(b-1)/2,$$

which is equivalent to

$$jb+(n-1)(b-1)/2-i\le X_1'+\cdots+X_n'\le jb+(n+1)(b-1)/2-i.$$

Thus $K(i,j)$ is equal to $1/b^n$ multiplied by the number of solutions to

$$X_1'+\cdots+X_n'+Y=jb+(n+1)(b-1)/2-i,$$

where $0\le Y\le b-1$. This is equal to $1/b^n$ multiplied by the coefficient of $x^{jb+(n+1)(b-1)/2-i}$ in $(1+x+\cdots+x^{b-1})^{n+1}$, proving part 1.

For part 2, let $[x^a]f(x)$ denote the coefficient of x^a in $f(x)$. Then by part 1,

$$\begin{aligned}K(i,j)&=\frac{1}{b^n}\left[x^{jb+(n+1)(b-1)/2-i}\right]\left(\frac{1-x^b}{1-x}\right)^{n+1}\\&=\frac{1}{b^n}\sum_{l\ge0}(-1)^l\binom{n+1}{l}\left[x^{jb+(n+1)(b-1)/2-i-lb}\right](1-x)^{-(n+1)}\\&=\frac{1}{b^n}\sum_{l=0}^{\lfloor j+\frac{n+1}{b}\frac{b-1}{2}-\frac{i}{b}\rfloor}(-1)^l\binom{n+1}{l}\binom{n+jb+(n+1)\frac{b-1}{2}-i-lb}{n}.\end{aligned}$$ □

Remarks.

(1) The type B carries chain in the paper [8] has the same transition matrix as the balanced chain, when the number of numbers being added is even. More precisely the chain in the paper [8] has state space $\{0, 1, \cdots, n\}$, and the balanced chain has state space $\{-n/2, \cdots, 0, \cdots, n/2\}$. The chance that the type B chain in [8] (with b replaced by $(b-1)/2$) goes from i to j is the same as the chance that the balanced chain goes from $i - n/2$ to $j - n/2$. This follows immediately by comparing the formula in part 1 of Theorem 4.2 of [8] with the formula in Theorem 2.1.

(2) Consider the balanced carries chain when adding an odd number n of numbers. This is a Markov chain on the set $\{-(n-1)/2, \cdots, (n-1)/2\}$, with transition probabilities given by Theorem 2.1 above. This is the same as Holte's carries chain [13] on $\{0, 1, \cdots, n-1\}$. More precisely, for all $0 \leq i, j \leq n-1$, the chance that Holte's chain moves from i to j is equal to the chance that the balanced chain moves from $i - (n-1)/2$ to $j - (n-1)/2$. This follows by comparing the formula in Holte's paper with the formula in Theorem 2.1 above.

(3) Letting K_b denote the base b balanced carries transition matrix, one has that $K_a K_b = K_{ab}$. This follows from the fact (proved in Sections 3 and 4) that the eigenvalues of K_b are $1, 1/b, \cdots, 1/b^n$, and that the eigenvectors are independent of b.

(4) While it is not emphasized here, the papers [8,9] develop a card shuffling interpretation of the transition matrix and a host of card shuffling interpretations of the spectral properties developed here.

(5) Balanced arithmetic can be developed for even bases. For example, when $b = 10$ choose digits $0, \pm1, \pm2, \pm3, \pm4, 5$ (or replace 5 by -5). The results seem similar but we have not fully worked out the details.

3. Left eigenvectors of the amazing matrix for balanced carries

This section studies the left eigenvectors of the balanced carries matrix when an even number n of numbers is added. The eigenvalues turn out to be $1, 1/b, 1/b^2, \cdots, 1/b^n$, and we derive an explicit formula for the left eigenvectors, identifying them with hyperoctahedral Foulkes characters. In particular, these eigenvectors turn out to be independent of b. The left and right eigenvectors have myriad uses for quantifying rates of convergence and the behavior of features of the carries process. These are detailed in Section 2 of [11].

Theorem 3.1. *Let K be the transition matrix of the balanced carries chain of Section 2. The jth left eigenvector of K (where $0 \leq j \leq n$), corresponding to the eigenvalue $1/b^j$, evaluated at the state i (where $-n/2 \leq i \leq n/2$), is given by*

$$v_j^n[i] = \sum_{r=0}^{i+n/2} (-1)^r \binom{n+1}{r} (n + 2i - 2r + 1)^{n-j}.$$

For example, when $n = 2$, the matrix whose rows are the left eigenvectors of K with eigenvalue $1/b^j$ (with $0 \le j \le 2$) is given by

$$\begin{pmatrix} 1 & 6 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

When $n = 4$, the matrix of left eigenvectors is

$$\begin{pmatrix} 1 & 76 & 230 & 76 & 1 \\ 1 & 22 & 0 & -22 & -1 \\ 1 & 4 & -10 & 4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

The left eigenvector corresponding to the eigenvalue 1 is proportional to the stationary distribution of the balanced carries chain. This has an interpretation in terms of descents of signed permutations, analogous to Holte's interpretation of the stationary distribution of the usual carries chain in terms of descents in ordinary permutations. To describe this, we use the linear ordering

$$1 < 2 < \cdots < n < -n < \cdots < -2 < -1.$$

We say that

(1) σ has a descent at position i $(1 \le i \le n-1)$ if $\sigma(i) > \sigma(i+1)$.
(2) σ has a descent at position n if $\sigma(n) < 0$.

For example, $-1\ -2\ -3$ has three descents.

Let $\overline{A(n,k)}$ denote the number of signed permutations on n symbols with k descents. From Corollary 4.6 of [8], one has that

$$\overline{A(n,k)} = \sum_{r=0}^{k} (-1)^r \binom{n+1}{r} (2k-2r+1)^n.$$

Hence Theorem 3.1 implies that $v_0^n[i]$ is equal to the number of signed permutations with $i+n/2$ descents. For example when $n = 4$, the entries of the left eigenvector 1 76 230 76 1 are the number of signed permutations on 4 symbols with $0, 1, 2, 3, 4$ descents respectively.

Next we proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. First note that $\sum_{r=0}^{i+n/2} (-1)^r \binom{n+1}{r} (n+2i-2r+1)^{n-j}$ is the coefficient of x^{2i+n+1} in

$$\sum_{r\geq 0}(-1)^r\binom{n+1}{r}x^{2r}\cdot\sum_{k\geq 0}k^{n-j}x^k=\left(1-x^2\right)^{n+1}\sum_{k\geq 0}k^{n-j}x^k.$$

Using the well-known fact (easily proved by induction) that

$$\sum_{k\geq 0}k^n x^k=\left(x\frac{d}{dx}\right)^n(1-x)^{-1}, \tag{1}$$

it follows that $v_j^n[i]$ is the coefficient of x^{2i+n+1} in

$$\left(1-x^2\right)^{n+1}\left(x\frac{d}{dx}\right)^{n-j}(1-x)^{-1}.$$

Note that

$$\left(1-x^2\right)^{n+1}\left(x\frac{d}{dx}\right)^{n-j}(1-x)^{-1}$$

is a polynomial of degree $2n+1$ (being equal to $(1+x)^{n+1}(1-x)^j$ multiplied by the Eulerian polynomial $(1-x)^{n-j+1}(x\frac{d}{dx})^{n-j}(1-x)^{-1}$ of degree $n-j$).

Thus, using the notation for $[x^a]f(x)$ as the coefficient of x^a in a power series $f(x)$, it follows that

$$\begin{aligned}
&\sum_{i=-n/2}^{n/2}K(i,k)\cdot v_j^n[i]\\
&=\sum_{i=-\infty}^{\infty}K(i,k)\cdot v_j^n[i]\\
&=\frac{1}{b^n}\sum_{i=-\infty}^{\infty}\sum_{l\geq 0}(-1)^l\binom{n+1}{l}\left[x^{kb+(n+1)(b-1)/2-i-lb}\right](1-x)^{-(n+1)}\cdot v_j^n[i]\\
&=\frac{1}{b^n}\sum_{i=-\infty}^{\infty}\left[x^{2i+n+1}\right]\left(1-x^2\right)^{n+1}\left(x\frac{d}{dx}\right)^{n-j}(1-x)^{-1}\\
&\quad\cdot\sum_{l\geq 0}(-1)^l\binom{n+1}{l}\left[x^{kb+(n+1)(b-1)/2-i-lb}\right](1-x)^{-(n+1)}\\
&=\frac{1}{b^n}\sum_{i=-\infty}^{\infty}\left[x^{2i+n+1}\right]\left(1-x^2\right)^{n+1}\left(x\frac{d}{dx}\right)^{n-j}(1-x)^{-1}\\
&\quad\cdot\sum_{l\geq 0}(-1)^l\binom{n+1}{l}\left[x^{2kb+(n+1)(b-1)-2i-2lb}\right]\left(1-x^2\right)^{-(n+1)}\\
&=\frac{1}{b^n}\sum_{l\geq 0}(-1)^l\binom{n+1}{l}\left[x^{2kb+(n+1)b-2lb}\right]\left(x\frac{d}{dx}\right)^{n-j}(1-x)^{-1}
\end{aligned}$$

$$= \frac{1}{b^n} \sum_{l=0}^{k+n/2} (-1)^l \binom{n+1}{l} \big(2kb + (n+1)b - 2lb\big)^{n-j}$$

$$= \frac{1}{b^j} \sum_{l=0}^{k+n/2} (-1)^l \binom{n+1}{l} \big(2k + (n+1) - 2l\big)^{n-j}$$

$$= \frac{1}{b^j} v_j^n[k],$$

as needed. Note that the sixth equality used Eq. (1). □

Proposition 3.2 gives a recursive formula for the left eigenvectors of the balanced carries chain, which is very similar to that of the type A Foulkes characters on p. 306 of [15]. For $0 \le i, j \le n$, define

$$w_j^n[i] = \sum_{r=0}^{i} (-1)^r \binom{n+1}{r} (2i - 2r + 1)^{n-j}.$$

Note that for n even, $w_j^n[i] = v_j^n[i - n/2]$.

Proposition 3.2. *With notation as above,*

$$w_j^n[i] = w_{j-1}^{n-1}[i] - w_{j-1}^{n-1}[i-1]$$

for all $1 \le i, j \le n$. Moreover there are the boundary conditions $w_0^n[i] = \overline{A(n,i)}$ and $w_j^n[n] = (-1)^j$.

Proof. The recurrence $w_j^n[i] = w_{j-1}^{n-1}[i] - w_{j-1}^{n-1}[i-1]$ follows from the fact in the proof of Theorem 3.1 that $w_j^n[i]$ is the coefficient of x^{2i+1} in

$$\big(1 - x^2\big)^{n+1} \left(x \frac{d}{dx}\right)^{n-j} (1 - x)^{-1}.$$

The equation $w_0^n[i] = \overline{A(n,i)}$ is clear from the formula for $A(n,i)$ just preceding the proof of Theorem 3.1. To see that $w_j^n[n] = (-1)^j$, note from the proof of Theorem 3.1 that $w_j^n[n]$ is the coefficient of x^{2n+1} in the product of $(1+x)^{n+1}(1-x)^j$ with the $n-j$th Eulerian polynomial. Since the $n-j$th Eulerian polynomial is monic of degree $n-j$, it follows that $w_j^n[n] = (-1)^j$. □

Remark. Comparing Theorem 5 of Miller's paper [17] (in the case $r = 2$ of the hyperoctahedral group) with Theorem 3.1 shows that our left eigenvectors for the balanced carries chain are indeed equal to Miller's hyperoctahedral Foulkes characters. The recurrence in our Proposition 3.2 is equivalent to the recurrence in his Theorem 7, though the proof is completely different.

There is another derivation of the stationary distribution (left eigenvector corresponding to eigenvalue 1) of the balanced carries chain, when an even number of numbers is added.

Theorem 3.3. *For $0 \le j \le n$, with n fixed,*

$$\lim_{r\to\infty} K^r(0, j - n/2) = \frac{\overline{A(n,j)}}{2^n n!}.$$

To begin, recall the following lemma from [8].

Lemma 3.4. *Let $U_1, \cdots, U_n$ be independent, identically distributed continuous uniform random variables on $[0,1]$. Then*

$$P\left(j - \frac{1}{2} \le U_1 + \cdots + U_n \le j + \frac{1}{2}\right) = \frac{\overline{A(n,j)}}{2^n n!}.$$

Now we prove Theorem 3.3.

Proof of Theorem 3.3. From the proof of Theorem 2.1, $K^r(0, j - n/2)$ is equal to the probability that

$$(j - n/2)b^r + \frac{(n-1)(b^r - 1)}{2} \le X_1 + \cdots + X_n \le (j - n/2)b^r + \frac{(n+1)(b^r - 1)}{2},$$

where $X_1, \cdots, X_n$ are discrete uniforms on $\{0, \cdots, b^r - 1\}$. This is equal to the probability that

$$(j - n/2)b^r + \frac{(n-1)(b^r - 1)}{2} \le \sum_{i=1}^n Y_i - \sum_{i=1}^n \bigl(Y_i - \lfloor Y_i \rfloor\bigr) \le (j - n/2)b^r + \frac{(n+1)(b^r - 1)}{2},$$

where $Y_1, \cdots, Y_n$ are continuous iid uniforms on $[0, b^r]$. Letting $U_i = Y_i/b^r$ be iid uniforms on $[0,1]$, it follows that $K^r(0, j - n/2)$ is equal to the probability that

$$j - 1/2 - (n-1)/\bigl(2b^r\bigr) \le \sum_{i=1}^n U_i - E \le j + 1/2 - (n+1)/\bigl(2b^r\bigr),$$

where $E = \sum_{i=1}^n (Y_i - \lfloor Y_i \rfloor)/b^r$. Since E, $(n-1)/(2b^r)$, and $(n+1)/(2b^r)$ all tend to 0 with probability 1 as $r \to \infty$ and n, b are fixed, it follows from Slutsky's theorem that

$$\lim_{r\to\infty} K^r(0, j-n/2) = P\left(j - 1/2 \le \sum_{i=1}^{n} U_i \le j + 1/2\right).$$

Applying Lemma 3.4 finishes the proof of the theorem. □

Remark. There is a representation of the transition probabilities of the balanced carries chain which might be useful for bounding the total variation convergence rate of the balanced carries chain to its stationary distribution π (arguing as in Theorem 3.4 of [8]). Indeed, Theorem 3.3 and Lemma 3.4 give that for $0 \le j \le n$,

$$\pi(j - n/2) = P\big(\lfloor U_1 + \cdots + U_n + 1/2\rfloor = j\big).$$

We want to quantify the convergence of $K^r(0, j - n/2)$ to $\pi(j)$ as $r \to \infty$. By the proof of Theorem 2.1, it follows that $K^r(0, j - n/2)$ is equal to

$$P\left(j - (n-1)/\big(2b^r\big) \le \frac{X_1}{b^r} + \cdots + \frac{X_n}{b^r} + 1/2 \le j + 1 - (n+1)/\big(2b^r\big)\right),$$

where $X_1, \cdots, X_n$ are discrete uniforms on $\{0, 1, \cdots, b^r - 1\}$. For r large enough with respect to n, this *almost* says that

$$K^r(0, j-n/2) = P\left(\left\lfloor \frac{1}{b^r}\sum_{i=1}^{n} X_i + 1/2\right\rfloor = j\right).$$

4. Right eigenvectors of the amazing matrix for balanced carries

This section describes the right eigenvectors of the balanced carries matrix K. As usual, we assume that the base b is odd, and that an even number of numbers is being added.

Theorem 4.1. *Let K be the transition matrix of the balanced carries chain of Section 2. The jth right eigenvector of K (where $0 \le j \le n$) corresponding to the eigenvalue $1/b^j$, evaluated at the state i (where $-n/2 \le i \le n/2$), is given by*

$$u_j^n[i] = \big[x^{n-j}\big](x - n - 2i + 1)(x - n - 2i + 3)\cdots(x - n - 2i + 2n - 1),$$

where $[x^a]f(x)$ denotes the coefficient of x^a in $f(x)$.

For example, when $n = 2$, the matrix whose columns are the right eigenvectors of K with eigenvalue $1/b^j$ (with $0 \le j \le 2$) is given by

$$\begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & -1 \\ 1 & -4 & 3 \end{pmatrix}.$$

When $n = 4$, the matrix of right eigenvectors is

$$\begin{pmatrix} 1 & 16 & 86 & 176 & 105 \\ 1 & 8 & 14 & -8 & -15 \\ 1 & 0 & -10 & 0 & 9 \\ 1 & -8 & 14 & 8 & -15 \\ 1 & -16 & 86 & -176 & 105 \end{pmatrix}.$$

Remark. The right eigenvectors allow describing the distribution of functionals of the carries chain $\{\kappa_j\}$. For example, the second eigenvector (corresponding to eigenvalue $1/b$) forming the second column of the matrices above, is $-2ni$. By scaling, the function i is an eigenvector. Translating into probability language, for s less than t,

$$E(\kappa_t|\kappa_s = m) = m/b^{t-s}.$$

Similarly, the formulae of Theorem 4.1 show that explicit polynomials of degree a in κ are eigenvectors of the chain with eigenvalues $1/b^a$.

Now we prove Theorem 4.1.

Proof of Theorem 4.1. Let V denote the matrix of left eigenvectors of the balanced carries chain. Thus the jth row of V has ith entry $v_j^n[i]$. Let U denote the matrix whose columns are the right eigenvectors of the balanced carries chain; thus the jth column of U has ith entry $u_j^n[i]$. We will prove that $U = 2^n n! \cdot V^{-1}$, which implies the theorem since by Theorem 3.1, the eigenvalues of the carries matrix are distinct.

Letting $u_{ik} = u_k^n[i]$ be the (i,k) entry of U (where $-n/2 \le i \le n/2$ and $0 \le k \le n$) and $v_{kj} = v_k^n[j]$ be the (k,j) entry of V (where $-n/2 \le j \le n/2$ and $0 \le k \le n$), one computes that

$$\begin{aligned}
\sum_{k=0}^{n} u_{ik} v_{kj} &= \sum_{k=0}^{n} \left[x^{n-k}\right](x-n-2i+1)(x-n-2i+3)\cdots(x-n-2i+2n-1) v_{kj} \\
&= \sum_{k=0}^{n} \left[x^{n-k}\right](x-n-2i+1)(x-n-2i+3)\cdots(x-n-2i+2n-1) \\
&\quad \cdot \sum_{r=0}^{j+n/2} (-1)^r \binom{n+1}{r} (n+2j-2r+1)^{n-k} \\
&= \sum_{r=0}^{j+n/2} (-1)^r \binom{n+1}{r} \sum_{k=0}^{n} (n+2j-2r+1)^{n-k} \\
&\quad \cdot \left[x^{n-k}\right](x-n-2i+1)(x-n-2i+3)\cdots(x-n-2i+2n-1) \\
&= \sum_{r=0}^{j+n/2} (-1)^r \binom{n+1}{r} \bigl(2(j-i)-2r+2\bigr)\cdots\bigl(2(j-i)-2r+2n\bigr)
\end{aligned}$$

$$= 2^n \sum_{r=0}^{j+n/2} (-1)^r \binom{n+1}{r} \bigl(n + (j-i) - r\bigr) \cdots \bigl(1 + (j-i) - r\bigr)$$

$$= 2^n n! \sum_{r=0}^{j+n/2} (-1)^r \binom{n+1}{r} \binom{n+j-i-r}{n}$$

$$= 2^n n! \sum_{r \geq 0} (-1)^r \binom{n+1}{r} \binom{n+j-i-r}{n}$$

$$= 2^n n! \cdot \delta_{i,j},$$

where the final equality is explained on p. 147 of [13]. □

Remark. The right eigenvectors are related to the type B riffle shuffles studied in [3]. More precisely, for $1 \leq k \leq n$, one has the following generating function:

$$\sum_{k=1}^{n} E_{n,k} x^k = \frac{1}{2^n n!} \sum_{\pi \in B_n} \bigl(x - 2d(\pi) + 1\bigr)\bigl(x - 2d(\pi) + 3\bigr) \cdots \bigl(x - 2d(\pi) + 2n - 1\bigr)\pi,$$

where the $E_{n,k}$ are the Eulerian idempotents of the hyperoctahedral group B_n. Here $d(\pi)$ is what [3] calls the number of descents of π (the definition of descents in [3] is slightly different than the definition earlier in this section). Thus

$$\sum_{k=1}^{n} E_{n,k} (x-n)^k = \frac{1}{2^n n!} \sum_{\pi \in B_n} \bigl(x - n - 2d(\pi) + 1\bigr)\bigl(x - n - 2d(\pi) + 3\bigr) \cdots \bigl(x - n - 2d(\pi) + 2n - 1\bigr)\pi.$$

Letting $E_{n,k}[i]$ denote the value of $E_{n,k}$ on a permutation with i descents, it follows that $u_j^n[i]$ is equal to $2^n n!$ multiplied by the coefficient of x^{n-j} in $\sum_{k=1}^{n} E_{n,k}[i](x-n)^k$.

5. Balanced carries as a point process

This section works with an odd base b and digits $0, \pm 1, \pm 2, \ldots, \pm (b-1)/2$. Then successive carries when adding a column of digits are $0, \pm b$. As such, independent uniformly chosen digits generate a stationary, one-dependent marked point process. Call this $X_1, X_2, \ldots$. See [4] for background on one-dependent processes.

Example 5.1. Working mod 5, consider Table 1. The carries are shown in the central column as $X_1 = -5$, $X_2 = X_3 = 0$, $X_4 = 5, \ldots$ (the carries are $\pm b$ in general). On the right are remainders $-2, 1, 2, \ldots$. If the digits are independent and identically distributed in $\{0, \pm 1, \pm 2\}$, so are the remainders. If the remainders are R_i, there is a carry of -5 iff $R_i - R_{i+1} \leq -3$, a carry of 5 iff $R_i - R_{i+1} \geq 3$, and a zero carry otherwise. Replace "3" by $(b+1)/2$ for general bases.

Table 1
Carries down a column for $b = 5$ with signed digits. The right column shows the remainders. There is a $-$ or $+$ in the central column for a carry of $-b$ or b.

$\bar{2}$	$-$	$\bar{2}$
$\bar{2}$		1
1		2
0	$+$	2
2		$\bar{1}$
2		1
$\bar{2}$		$\bar{1}$
1		0
1		1
1	$+$	2
1		$\bar{2}$
$3 = 1\bar{2}$		

From this description, it is easy to see that for any base, two successive ++ or −− carries are impossible. For $b \geq 5$, all other patterns occur with positive probability. For $b = 3$, ++, −−, +0+, and −0− are impossible. The probability distribution of this balanced carries process can be expressed via determinantal formulae from [4]. These authors determine the joint distribution for the process that records 1 or 0 as there is a carry or not (e.g., replace all $\pm b$ symbols by 1). Let $a_i = P(i-1 \text{ consecutive ones})$ with $a_1 = 1$. Theorem 4.1 in [4] gives

Theorem 5.2. *For a stationary binary one-dependent process with $a_i = P(X_1 = X_2 = \ldots = X_{i-1} = 1)$, $a_1 = 1$ and a binary string $t_1, t_2, \ldots, t_{n-1}$ with k zeros at positions $S = \{s_1 < s_2 < \ldots < s_k\} \subseteq [n-1]$,*

$$P(t_1, \ldots, t_{n-1}) = \det(a_{s_{j+1}-s_i})_{i,j=0}^{k}. \tag{2}$$

The determinant is of a $(k+1) \times (k+1)$ matrix and $s_0 = 0$, $s_{k+1} = n$, $a_0 = 1$, and $a_i = 0$ for $i < 0$.

Example 5.3.

$$P(0,0,0) = \det \begin{pmatrix} 1 & a_2 & a_3 & a_4 \\ 1 & 1 & a_2 & a_3 \\ 0 & 1 & 1 & a_2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 1 - 3a_2 + a_2^2 + 2a_3 - a_4.$$

Theorem 4.2 and Corollary 4.3 of [4] give expressions for (2) as skew Schur functions and for the higher order correlations in terms of the a_i.

It thus remains to determine a_i. These can be determined by the transfer matrix method and a serendipitous diagonalization.

Proposition 5.4. *For odd $b \geq 3$ and the balanced coset representatives $0, \pm 1, \ldots, \pm(b-1)/2$, $a_1 = 1$ and the chance a_i of $i-1$ consecutive carries is*

$$a_i = \begin{cases} \frac{8}{b^{i+1}} \sum_{r=1}^{(b-1)/2} \lambda_r^{(i-1)/2} v_r^2 & \textit{if } i > 1 \textit{ is odd,} \\ \frac{8}{b^{i+1}} \sum_{r=1}^{(b-1)/2} \lambda_r^{(i-2)/2} v_r w_r & \textit{if } i > 0 \textit{ is even} \end{cases}$$

where

$$1/\lambda_r = 4\sin^2\bigl((2r-1)\pi/2b\bigr), \quad 1 \le r \le (b-1)/2,$$

$$v_r = \sum_{j=1}^{(b-1)/2} \sin\bigl((2r-1)j\pi/b\bigr), \quad 1 \le r \le (b-1)/2,$$

and

$$w_r = \sum_{j=1}^{(b-1)/2} j \cdot \sin\bigl((2r-1)j\pi/b\bigr), \quad 1 \le r \le (b-1)/2.$$

Proof. The chance of any digit sequence of length i in the remainder column is $1/b^i$. For the pattern $+ - + - \cdots$ of length $i-1$, sequences of digit choices $x_1, x_2, \ldots, x_i$ must be chosen so that x_1, x_2 yield a $+$ (so $x_1 - x_2 \ge (b+1)/2$), x_2, x_3 result in a $-$ (so $x_2 - x_3 \le -(b+1)/2$), and so on. Admissible sequences can be enumerated as paths in a graph. As an example, for $b = 7$, the digits are $0, \pm 1, \pm 2, \pm 3$. The corresponding graph has adjacency matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the rows of M are labeled by the vertices $-3, -2, -1, 1, 2, 3$ from top to bottom, and the columns of M are labeled by the vertices $-3, -2, -1, 1, 2, 3$ from left to right. Thus

$$M^2 = \begin{pmatrix} AA^T & 0 \\ 0 & A^TA \end{pmatrix}, \qquad M^3 = \begin{pmatrix} 0 & AA^TA \\ A^TAA^T & 0 \end{pmatrix},$$
$$M^4 = \begin{pmatrix} AA^TAA^T & 0 \\ 0 & A^TAA^TA \end{pmatrix}.$$

Suppose next that i is odd. The sum of the entries in $(A^TA)^{(i-1)/2}$ counts paths resulting in $+ - + - \cdots$ ($i-1$ terms). The matrix A^TA has (a, b) entry $\min(a, b)$. This is the correlation matrix for random walk $S_1, S_2, \ldots, S_{(b-1)/2}$ with $S_i = Y_1 + \ldots + Y_i$ and Y_i are independent with mean 0, variance 1. Its spectral decomposition is known [1]: the $(b-1)/2 \times (b-1)/2$ matrix A^TA with (i, j) entry $\min(i, j)$ has

(1) eigenvalues λ_r with $1/\lambda_r = 4\sin^2((2r-1)\pi/2b)$, $1 \le r \le (b-1)/2$;
(2) corresponding eigenvectors

$$\psi_r(k) = \sin\bigl((2r-1)k\pi/b\bigr), \quad 1 \le r \le (b-1)/2;$$

(3) the eigenvectors are orthogonal with

$$\langle \psi_i, \psi_j \rangle = \sum_k \psi_i(k)\psi_j(k) = \delta_{ij} b/4.$$

From this, for any ℓ,

$$\bigl(A^T A\bigr)^\ell_{i,j} = \frac{4}{b} \sum_{r=1}^{(b-1)/2} \lambda_r^\ell \psi_r(i)\psi_r(j).$$

Summing in i and j gives

$$\mathbf{1}^T\bigl(A^T A\bigr)^\ell \mathbf{1} = \frac{4}{b} \sum_{r=1}^{(b-1)/2} \lambda_r^\ell v_r^2, \quad v_r = \sum_j \psi_r(j) = \sum_{j=1}^{(b-1)/2} \sin\bigl((2r-1)j\pi/b\bigr).$$

This gives the result for an even number of steps (so for i odd), since the probability of any length i digit sequence in the remainder column is $1/b^i$, and one must multiply by 2 to account for the string $+-+-\cdots+-$ ($i-1$ terms) and $-+-+\cdots-+$ ($i-1$ terms).

Suppose next that i is even. By arguing as in the i odd case, to compute the chance of $-+\cdots-$ ($i-1$ terms), we need to study

$$\mathbf{1}^T A\bigl(A^T A\bigr)^\ell \mathbf{1}.$$

Since $\mathbf{1}^T A = (1, 2, \cdots, (b-1)/2)$ we get that

$$\begin{aligned}\mathbf{1}^T A\bigl(A^T A\bigr)^\ell \mathbf{1} &= \frac{4}{b} \sum_{r=1}^{(b-1)/2} \lambda_r^\ell \sum_{i=1}^{(b-1)/2} \sum_{j=1}^{(b-1)/2} i\psi_r(i)\psi_r(j)\\ &= \frac{4}{b} \sum_{r=1}^{(b-1)/2} \lambda_r^\ell v_r w_r,\end{aligned}$$

where

$$v_r = \sum_{j=1}^{(b-1)/2} \sin\bigl((2r-1)j\pi/b\bigr), \qquad w_r = \sum_{j=1}^{(b-1)/2} j \cdot \sin\bigl((2r-1)j\pi/b\bigr).$$

To obtain the theorem, we now set $\ell = (i-2)/2$, divide by b^i (the probability of any length i digit sequence in the remainder column), and multiply by 2 (to also account for the length $i-1$ sequence $+-\cdots+$). □

Example 5.5 *(Balanced ternary).* With $b = 3$ and coset representatives $0, \pm 1$, the matrix M is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus $M^2 = \text{Id}$, $M^3 = M$, and so on. It follows directly that the chance of $+ - + - \cdots$ with length $i - 1$ is $a_i = 1/3^i$ for all i. For the zero/one consolidation the chance of $i - 1$ ones is $2/3^i$.

Example 5.6 *(Base 5).* With $b = 5$ and coset representatives $0, \pm 1, \pm 2$, the matrix M is

$$M = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with

$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \left(A^T A\right)^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \qquad \left(A^T A\right)^3 = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}, \quad \cdots$$

We recognize $(A^T A)^h = \left(\begin{smallmatrix} F_{2h-1} & F_{2h} \\ F_{2h} & F_{2h+1} \end{smallmatrix}\right)$, where F_i is the ith Fibonacci number. The sum of these entries is $F_{2h-1} + 2F_{2h} + F_{2h+1} = F_{2h+3}$. So $a_{2i+1} = 2F_{2i+3}/5^{2i+1}$. Similarly $a_{2i} = 2F_{2i+2}/5^{2i}$. This gives $a_i = 2F_{i+2}/5^i$ for all i:

$$a_i = \frac{2F_{i+2}}{5^i} = 10\sqrt{5}\left[\left(\frac{1+\sqrt{5}}{10}\right)^{i+2} - \left(\frac{1-\sqrt{5}}{10}\right)^{i+2}\right].$$

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