Math 655 - Topics in Partial Differential Equations
Hamilton-Jacobi Equations. Theory and Applications

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This course is designed as an introduction to the celebrated Hamilton-Jacobi Equation

$$\frac{\partial S}{\partial t} = H(x, \nabla S), \qquad \textbf{(HJE)}$$

HJE takes its origin in the Hamiltonian formalism for Classical Mechanics (certain contact transformations preserving the symplectic structure are generated by a function satisfying the HJE). Moreover, it provides an alternative formulation of Mechanics where the motion of a particle is represented as a wave, a realization of Huygens' ideas. In this respect, it is a predecessor of Schroedinger's equation in quantum mechanics.

Yet another equivalent formulation of Classical Mechanics is Hamilton's Least Action Principle. HJE thus appears naturally in the Calculus of Variations, where solutions represent conserved quantities, i.e. first integrals of the system (Jacobi's Theorem). One particular application is the determination of geodesics on a Riemannian manifold. Finally, the HJE (rather, a generalization usually called Hamilton-Jacobi-Bellman equation) is fundamental in Control Theory both for deterministic and stochastic systems.

The emphasis of the course is on the big picture rather than on technical details. I will present the basics of Hamiltonian formalism and canonical transformations to show the original motivation of the equations, as well as the derivation in the Variational Calculus context, where we will prove the celebrated Jacobi's Theorem on integrability of Hamiltonian systems. On the purely mathematical context, we will tackle the questions of existence and uniqueness of weak and viscosity solutions. If time permits, we will consider applications to Control Theory (Dynamic Programming). The course is aimed at Math and Applied Math majors willing to familiarize with some of the fundamental ideas of Mathematical Physics and with the mathematical tools needed to build a satisfactory theory.

• Bibliography

I will closely follow chapters 3 and 10 from Evans' book for the mathematical theory and applications to Control Theory

- L. C. Evans, Partial Differential Equations: Second Edition (Graduate Studies in Mathematics)

For connections with Classical Mechanics, I will follow

- V. I. Arnold, Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics)
- V. I. Arnold, Lectures on Partial Differential Equations, Universitext.

For connections with Variational Calculus,

- I. M. Gelfand and S. V. Fomin, Calculus of Variations (Dover Books on Mathematics)