THE HYPERBOLIC REVOLUTION: FROM TOPOLOGY TO GEOMETRY, AND BACK

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The late nineteen seventies and early eighties saw a surprising convergence between topology and rigid geometry. This followed the groundbreaking work of Bill Thurston on the geometrization of three-dimensional manifolds, but this was also part of a larger trend that resulted in a period of intense cross-fertilization between topology, geometry, dynamical systems, combinatorial group theory, and complex analysis.

First, we should begin with the traditional difference between topology and geometry. Both fields consider geometric objects, but topologists allow themselves to deform these objects and stretch distances, whereas geometers tend to focus on the fine properties of these distances. As an illustration, it is well-known that topologists like to turn doughnuts into coffee mugs, whereas a typical result in geometry would be the Polyhedron Rigidity Theorem of Cauchy, which says that it is impossible to deform a convex polyhedron in euclidean space without changing the shape of any of its faces.

1. The hyperbolic space

Among the geometries that can occur in dimension 3, the more fundamental one is hyperbolic geometry.

The *n–dimensional hyperbolic space* is the half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times [0, \infty)$ in $\mathbb{R}^n$, endowed with the hyperbolic metric defined as follows. First, for every differentiable curve $\gamma\colon [a, b] \to \mathbb{H}^n$ with $\gamma(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big)$, we define its *hyperbolic arc length* $\ell_{\mathrm{hyp}}(\gamma) = \int_a^b \frac{1}{x_n(t)} \sqrt{\sum_{i=1}^n x_i'(t)^2}\, dt$ (differing from the usual arc length only by the $\frac{1}{x_n(t)}$ factor). The *hyperbolic distance* $d_{\mathrm{hyp}}(P, Q)$ between two points P, $Q \in \mathbb{H}^n$ is then defined as the infimum of the hyperbolic arc lengths $\ell_{\mathrm{hyp}}(\gamma)$ over all curves γ joining $P = \gamma(a)$ to $Q = \gamma(b)$.

What is not obvious from the above description is that the hyperbolic space $\mathbb{H}^n$ is highly symmetric. In fact $\mathbb{H}^n$ is *homogeneous* in the sense that, for every P, $Q \in \mathbb{H}^n$, there is an isometry φ of the metric space $(\mathbb{H}^n, d_{\mathrm{hyp}})$ that sends P to Q. It is even *isotropic* in the sense that we can require the isometry φ to send an arbitrary direction at P to an arbitrary direction at Q. In this regard, it is as symmetric as the usual euclidean space $\mathbb{R}^n$.

Hyperbolic geometry made its first appearance in the context of Euclid's Fifth Postulate, in the early nineteenth century. Henri Poincaré [34, 35] was the first to connect it to another major branch of mathematics, namely complex analysis and

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the theory of linear differential equations. About a century later, Thurston placed hyperbolic geometry at the center of three-dimensional topology.

2. Knots in space

As an introduction to Thurston's geometrization results for three-dimensional manifolds, let us focus on their applications to knot theory. The results are then easier to state, and give a good illustration of the more general ideas.

The author likes to say that knot theory is to the topologist what the fruit fly is to the biologist: a small laboratory example where big theories can be tested, which is easier to handle and to visualize than the long-term problems motivating these theories, and which nevertheless is sufficiently complex to offer a challenging testing ground.

A *knot* is a closed curve K in the euclidean space $\mathbb{R}^3$ that is smooth, namely has a well-defined tangent (with no switchback) at each point, and that has no self-intersection. The main problem in knot theory is to decide when two knots K and K' can be deformed to each other, namely whether there exists a continuous family of homeomorphisms $(\varphi_t)_{t\in[0,1]}$ of $\mathbb{R}^3$ such that φ_0 is the identity and $\varphi_1(K) = K'$.

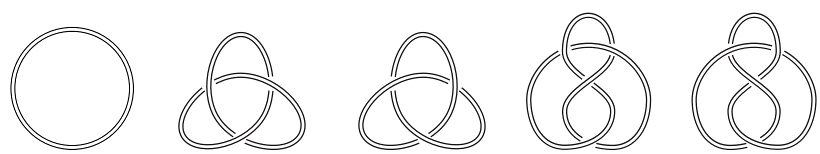


Figure 1. A few knots

Figure 1 offers a few examples. It is not immediately obvious that two of these knots can be deformed to each other, and deep mathematics is required to prove that no two of the remaining four knots can be deformed to each other.

This situation is fairly typical. To tackle the challenge of rigorously proving that two knots that appear different cannot be deformed one to the other, mathematicians have traditionally used techniques of algebraic topology. One of the early successes of such an approach was due to James W. Alexander and Garland B. Briggs [2] who showed in 1927 that the knots of up to nine crossings listed in the XIX-th century knot tables by Tait, Kirkman and Little [45, 16, 17, 20, 21] were actually different.[1] They did so by comparing the homology groups of certain branched covers of these knot. The following decades saw the development of ever more sophisticated methods of algebraic topology to attack problems in knot theory.

A less common approach to knot theory involved the cut-and-paste analysis of special surfaces in the complement of the knot, as in the innovative work of Horst Schubert [41, 42, 43].

Thurston's Hyperbolization Theorem for knot complements provided a completely different type of knot invariants. To state this result, we need to mention a couple of classical constructions of knots.

[1]To be completely accurate, there were a very small number of exceptions that Alexander and Briggs could not settle.

The first one is that of torus knots. These are the knots that can be drawn on the surface of a standard torus in $\mathbb{R}^3$. More precisely, for coprime integers p and q, the $\{p,q\}$*–torus knot* is represented by the curve parametrized by

$$t \longmapsto \big((R + r\cos qt)\cos pt, (R + r\cos pqt)\sin pt, r\sin qt\big)$$

for arbitrary $0 < r < R$. For instance, Figure 2 represents the $\{5,-4\}$–torus knot, and the first three knots of Figure 1 are the $\{1,0\}$, $\{2,3\}$ and $\{2,-3\}$–torus knots, respectively. Torus knots are very well understood. In particular, when p and q are different from ± 1, the $\{p,q\}$–torus knot can be deformed to the $\{p',q'\}$–torus knot if and only if the set $\{p,q\}$ is equal to $\{p',q'\}$ or to $\{-p',-q'\}$; when p or q are equal to ± 1, the $\{p,q\}$–torus knot can be deformed to the *unknot*, namely the first knot of Figure 1.

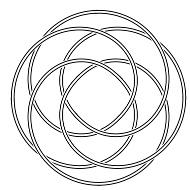
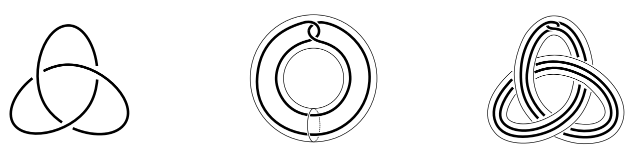


FIGURE 2. A torus knot

The second construction that we need is that of satellite knots. Suppose that we are given a first knot $K \subset \mathbb{R}^3$ that cannot be deformed to the unknot, as well as another knot L contained in the standard solid torus

$$V = \Big\{\big((R+\rho\cos v)\cos u, (R+\rho\cos v)\sin u, \rho\sin v\big); u, v, \rho \in \mathbb{R}, 0 \leq \rho \leq r\Big\}$$

consisting of those points which are at distance at most r from the horizontal circle C of radius R centered at the origin, for arbitrary r, R with $0 < r < R$. We assume in addition that L is *non-trivial* in the solid torus V, in the sense that it cannot be deformed in V to a knot L' which is disjoint from one of the cross-section disks where the coordinate u is constant, or to the central circle C of V.



The knot $K \subset \mathbb{R}^3$ The knot $L \subset V$ The satellite $K' \subset \mathbb{R}^3$

FIGURE 3. A satellite knot

We can then tie V as a tube around the knot K, and consider the image of L. More precisely, choose an injective continuous map $\phi\colon V \to \mathbb{R}^3$ which sends the central circle C to the knot K. Assume in addition that φ is differentiable, and that its jacobian is everywhere different from 0, so that the image $K' = \varphi(L)$ is now a new knot in $\mathbb{R}^3$. Any knot K' obtained in this way is said to be a *satellite* of the knot K.

Theorem 1 (Hyperbolization Theorem for knot complements)**.** *Let K be a knot in $\mathbb{R}^3$, and let $\widehat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ be obtained by adding to $\mathbb{R}^3$ a point ∞ at infinity. Then, exactly one of the following holds:*

(1) *K is a torus knot;*
(2) *K is a satellite of a non-trivial knot;*
(3) *the complement $\widehat{\mathbb{R}}^3 - K$ admits a complete metric d which induces the same topology as the euclidean metric of $\mathbb{R}^3$ and which is locally isometric to the hyperbolic metric of the hyperbolic space $\mathbb{H}^3$.*

The first alternative is somewhat trivial, since torus knots are very well understood (and very rare). The second alternative can be essentially reduced to the other two, through a unique factorization process of satellite knots into non-satellite links [42, 14, 13, 5]. In practice, almost all knots satisfy the third alternative, and therefore admit a hyperbolic metric, namely a metric d as in this third alternative.

The Hyperbolization Theorem is greatly enhanced by the following earlier result of George Mostow [27, 28].

Theorem 2 (Mostow's Rigidity Theorem)**.** *When the third case of Theorem* 1 *holds, the metric d is unique up to isometry. Namely, for any two such metrics d and d', there exists a map $\phi\colon \widehat{\mathbb{R}}^3 - K \to \widehat{\mathbb{R}}^3 - K$ such that $d'\big(\phi(x), \phi(y)\big) = d(x, y)$ for every x, $y \in \widehat{\mathbb{R}}^3 - K$.*

The incredible power of the combination of Theorems 1 and 2 is that they turn the topological problem of deciding when two knots can be deformed to each other into the rigid geometric problem of deciding when their associated hyperbolic metrics are isometric.

These metrics carry a lot of information. For instance they have a well-defined volume. Theorem 2 then shows that, if two knots satisfy the third conclusion of Theorem 1 and can be deformed to each other, then they must have the same volume. This simple test is remarkably efficient to show that two knots cannot be deformed to each other.

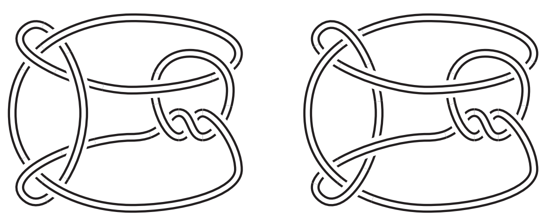


FIGURE 4. Two very similar knots

A more powerful invariant of the hyperbolic metric of a knot complement is its *Ford domain.* This object was introduced in a 2–dimensional setting [10] by Lester Ford,[2] and generalized to knot complements by Bob Riley [38, 39, 40]. They provide a tessellation of the euclidean plane by polygons, which is invariant under two linearly independent translations and which carries additional pairing information. See for instance [4, §12.4] for a more precise description. It then follows from Theorem 2 that two knots can be deformed to each other *if and only if* there is a similitude (namely a composition of an isometry with a homothety) of the Euclidean plane that carries the tessellation associated to the first knot to the tessellation

[2]Also famous for being the President of the Mathematical Association of America from 1947 to 1948

associated to the second one, and that preserves the pairing information. The "if and only if" part of this statement is particularly impressive and useful.

As an example taken from [4], consider the two knots of Figure 4. These are somewhat difficult to distinguish with the tools of algebraic topology that were available before hyperbolic geometry techniques became available, and their complements even have the same hyperbolic volume. However a quick inspection, for instance counting the number of edges emanating from each vertex, shows that there is no similitude of the euclidean plane that exchanges their respective Ford tessellations, represented in Figure 5. It immediately follows that these two knots cannot be deformed to each other.

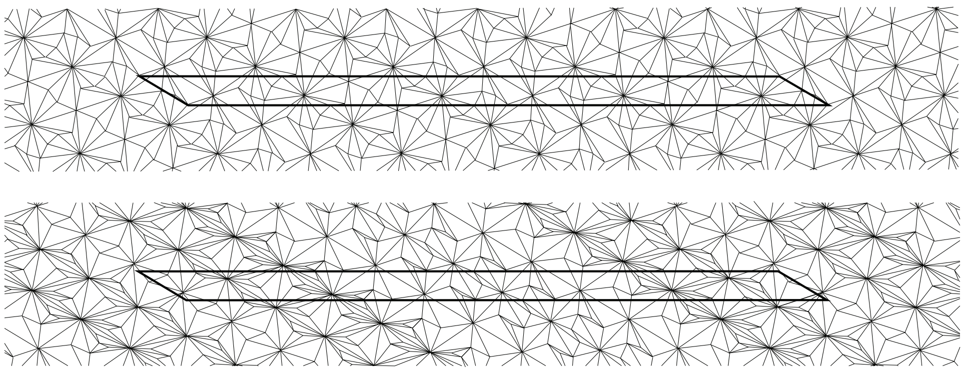


FIGURE 5. The Ford tessellations of the knots of Figure 4

Another important property of these results is that they can be explicitly implemented on a computer. Following up on the early pioneering work of Bob Riley [38, 39], the software *SnapPea* developed by Jeff Weeks [53] has been particularly influential among researchers in knot theory; see [9] for a current incarnation, *Python*-based and called *SnapPy*, of the same tool.

3. GEOMETRIZATION OF GENERAL THREE-DIMENSIONAL MANIFOLDS

Theorem 1 is a special case of a more general geometrization result for three-dimensional manifolds. An *n–dimensional (topological) manifold* is a topological space that is locally homeomorphic to the usual n–dimensional euclidean space $\mathbb{R}^n$. For instance, a surface is a two-dimensional manifold. The space $\widehat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ of Theorem 1 is a three-dimensional manifold, even near the point ∞ (and is homeomorphic to the three-dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$); a knot complement $\widehat{\mathbb{R}}^3 - K$ is also a three-dimensional manifold.

More generally, an *n–dimensional manifold-with-boundary* (in a single word) is a topological space M locally homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$, and its *boundary* ∂M consists of the points that go to $\mathbb{R}^{n-1} \times \{0\}$ under the corresponding local homeomorphisms. In particular, a manifold as defined in the previous paragraph is a manifold-with-boundary with empty boundary.

The general Geometrization Theorem for three-dimensional manifold is a little difficult to state precisely while remaining within the limited scope of this article, and we just want to give the flavor of this result.

First of all, the Geometrization Theorem involves more geometries than hyperbolic geometry. A geometric structure on a manifold can be interpreted as a metric that is locally homogeneous, in the sense that any two points have isometric neighborhoods. These geometries are locally modeled on the homogeneous spaces associated to Lie groups and, in dimension 3, the classification of Lie groups shows that there is a limited number of possible models. In fact, there are only eight geometries that are relevant for the Geometrization Theorem:

(1) the three isotropic geometries of the euclidean space $\mathbb{R}^3$, the three-dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, and the three-dimensional hyperbolic space $\mathbb{H}^3$;
(2) the two product geometries $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, product of the euclidean line $\mathbb{R}$ with the 2–dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and the hyperbolic plane $\mathbb{H}^2$, respectively;
(3) two suitable defined twisted product geometries $\mathbb{H}^2 \widetilde{\times} \mathbb{R}$ and $\mathbb{R}^2 \widetilde{\times} \mathbb{R}$;
(4) the Sol geometry, related to the unique three-dimensional solvable Lie group.

See [36, 3] for a precise description of these geometries.

Then, one needs a topological notion of triviality for a surface contained in a three-dimensional manifold. We cannot precisely describe this concept here, except by saying that the definition is consistent with the terminology: a trivial surface is obtained by a method that is too straightforward to be of much use. For instance, the boundary of a small ball in a three-dimensional manifold M is a trivial sphere in M; similarly, the boundary of a thin tube around a simple closed curve in M gives a trivial torus.

Theorem 3 (Geometrization Theorem for three-dimensional manifolds)**.** *Let M be a connected three-dimensional manifold that is topologically finite, in the sense that $M = \overline{M} - \partial\overline{M}$ is obtained by removing its (possibly empty) boundary from a compact manifold-with-boundary $\overline{M}$. Then, at least one of the following holds:*

(1) *M admits a complete metric d which is locally isometric to one of the eight geometric models listed above;*
(2) *M contains a non-trivial sphere, projective plane, torus or Klein bottle.*

In most cases, the geometry that occurs in the first case is that of the hyperbolic space $\mathbb{H}^3$, and the other geometries occur only for a limited array of three-dimensional manifolds. In that case, and under the additional hypothesis that each component of the boundary $\partial\overline{M}$ is a torus or a Klein bottle, the same Mostow's Rigidity Theorem as in Theorem 2 guarantees that the hyperbolic metric on M is unique up to isometry.

There is a small possible overlap between the two conclusions of Theorem 3, but no possible overlap in the case of hyperbolic geometry. In particular, non-trivial spheres, projective planes, tori or Klein bottles appear as topological obstructions to the existence of a hyperbolic geometric structure, and of several of the other geometric structures. We then benefit from two earlier pieces of work: one is the Kneser-Milnor [19, 23] unique factorization of a three-dimensional manifold into *prime manifolds* that contain no essential spheres or projective planes; the other one is the Waldhausen-Jaco-Shalen-Johannson [13, 14] canonical splitting (originally developed for completely different purposes) of a prime manifold into pieces that, either contain no essential tori or Klein bottles, or admits one of the seven non-hyperbolic geometries. In practice, this reduces the problem of the topological classification of three-dimensional manifolds to the isometric classification of hyperbolic three-dimensional manifolds. Three-dimensional hyperbolic geometry is still very rich, but our discussion of Ford domains in §2 should give an idea of the powerful techniques that are available in this field.

Thurston proved Theorem 3 in many cases, in particular when M is non-compact (which includes the case of knot complements considered in §2), in the late nineteen seventies. He also conjectured Theorem 3 in its full generality, which was then known as the Thurston Geometrization Program until Grigori Perelman proved it around 2000. Neither Thurston nor Perelman provided a complete exposition of their proofs, but they circulated partial preprints [48, 50, 51, 31, 32] and gave enough lectures to enable others to fill in all the details; see for instance [15, 29, 30, 7, 8, 18, 25, 26].

4. A broader perspective: using geometry to prove results in topology and algebra

The Geometrization Program took place in, and contributed to, a broader trend which in the last quarter of the twentieth century saw a closer integration between topology, differential geometry, dynamical systems and group theory.

We already indicated how the combination of Thurston's Hyperbolization Theorem and Mostow's Rigidity Theorem translates topological problems to hyperbolic geometry questions, and can be used to prove theorems in knot theory. This interaction between topology and geometry occurs, not just in the consequence of these results, but also in the novel ideas introduced by Thurston for the proof of his Hyperbolization Theorem.

Indeed, the flexibility of topology comes with the curse of a very large number of degrees of freedom. Geometry can be used to introduce some rigidity in a topological situation, in order to make it easier to handle. As an example, consider the 2–dimensional analogue of knot theory which studies, in a surface S, all simple (namely smooth and without self-intersection) closed curves in S up to deformation. There is of course an overwhelming abundance of simple closed curves, and of deformations between them. However, we can take advantage of the following consequence of the Uniformization Theorem in complex analysis: if the topology of the surface S is complicated enough that it does not belong to a small finite number of exceptions such as the plane or the torus, the surface S can be endowed with a hyperbolic metric d. Once such a hyperbolic metric is chosen, every simple closed curve can be deformed to a *unique* simple closed curve that is *geodesic*, namely provides the shortest arc between any two of its points that are sufficiently close to each other (a hyperbolic geodesic is thus the hyperbolic equivalent of a straight line). This provides a one-to-one correspondence between, simple closed curves considered up to deformation on the one hand, and simple closed geodesics on the other hand. This greatly simplifies the original problem by eliminating the need to consider deformations, provided that we restrict attention to a very specific type of simple closed curves.

Thurston took advantage of this construction to introduce a certain completion $\mathcal{ML}(S)$ of the set $\mathcal{S}(S)$ of simple closed curves in the surface S considered up to deformation. The elements of $\mathcal{ML}(S)$ are measure-theoretic, or probabilistic, generalizations of simple closed geodesics and are called *measured geodesic laminations*.

The space $\mathcal{ML}(S)$ is endowed with a natural topology (for which, rather surprisingly, it is homeomorphic to a euclidean space $\mathbb{R}^n$) and with a rescaling operation. In particular, given a sequence in $\mathcal{S}(S)$, it makes sense to talk of the limit of this sequence or, after suitably rescaling, of the asymptotic direction of this sequence in $\mathcal{ML}(S)$.

This method of taking limits of objects that are only defined up to deformation was a real conceptual breakthrough. It played a critical rôle in Thurston's work on surface diffeomorphisms [49] and on three-dimensional hyperbolic geometry [46, 47, 52]. Together with similar rigidification techniques, it also provided the impetus and technical tools for much subsequent work by the low-dimensional geometry and topology community.

At about the time when Thurston was pioneering the use of geometry to prove results in topology, Mikhail Gromov [12] was translating insights from geometry to abstract group theory. This comes from another situation where one considers closed curves up to deformation in a topological space M, namely in the definition of the fundamental group $\pi_1(M)$ of M. The Milnor-Švarc Lemma [37, 24] asserts that, for a compact space M, the large scale properties of the fundamental group $\pi_1(M)$ are essentially the same as those of the universal cover $\widetilde{M}$. In particular, this has far reaching consequences when M is a compact riemannian manifold of negative curvature. Building on the insights provided by the geometry, Gromov was able to identify the key algebraic features that yield these consequences, and to develop a purely algebraic theory of "groups that behave like fundamental groups of negatively curved manifolds" (now called *Gromov hyperbolic groups* or *negatively curved groups*). This gave an important boost to the field of combinatorial group theory, further enhanced by the rich families of examples provided by the Geometrization Theorem.

The Geometrization Program also greatly energized another area of mathematics. We already mentioned how Poincaré had inserted two- and three-dimensional hyperbolic geometry into the world of complex analysis. In the century that followed, the connection had become a little more tenuous (however, see for instance [1, 22]), but was greatly invigorated by Thurston's Hyperbolization Theorem. Indeed, Thurston's original proof combined both the complex analytic and the hyperbolic geometric aspects of kleinian groups. Conversely, the three-dimensional point of view provided strong tools and insights for the corresponding complex analytic problems. These insights were pushed one step further, first by Thurston and then by Dennis Sullivan, to the dynamics of rational maps on Riemann surfaces; see for instance Sullivan's "dictionary" [44] between the theories of kleinian groups and of complex dynamics.

Thurston's Geometrization Program provided great results and tools that were used to solve many topological problems. However, its even more lasting impact may be the integration and cross-fertilization between numerous branches of mathematics that it triggered: topology, geometry, complex analysis, combinatorial group theory, dynamical systems, etc. From a sociological point of view, mathematics historians may trace the germs of these developments to the Berkeley mathematical school of the late nineteen sixties, where the same group of people were working on topology, dynamical systems and rigid geometry. However, it is Bill Thurston's extraordinary talent that initiated this technical and conceptual revolution, which led to one of the most productive periods in mathematics.

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