New Geometric Splittings of Classical Knots
and
the Classification and Symmetries of Arborescent Knots

Francis Bonahon & Laurence C. Siebenmann

Department of Mathematics
University of Southern California
Los Angeles, CA 90089-2532, U.S.A.

Mathématique, Bâtiment 425
Université de Paris-Sud
91405 Orsay Cedex, France

Version: January 26, 2016

Copyright: Francis Bonahon and Laurent Sienbenmann, 1979-2010

Preface to the 30-th year edition

Dedicated to José Montesinos
on the occasion of his 65-th birthday

We began writing this monograph in 1979. It originally started as an article for the conference proceedings [**BroT**], but it soon developed into a book project of its own as the size of the manuscript was rapidly increasing. The two articles [$\mathbf{BonS}_1$] and [$\mathbf{BonS}_2$] both originated as appendices to this book, as did the unpublished part of the preprint [$\mathbf{BonS}_3$] that is dedicated to crystallographic groups. As time passed by, we eventually became bogged down in the process, in large part because of the overwhelming number of pictures and diagrams (about 640 in the current version) and because of the rapid changes in technology occurring at the same time. The book started as a camera-ready typescript with ink drawings, but we soon had to move to the new norm of TEX and computer drawn pictures. In particular, the book was used as developing ground for Sweet-TEX [$\mathbf{Sie}_3$], a TEX preprocessor for *Macintosh* computers created by the second author; this had the unfortunate outcome that many files became obsolete when *Apple* completely redesigned its operating system. The fact that the authors became geographically separated, and eventually drifted towards different scientific interests, did not help. By the late 1980s, we had collected the then-current state of the manuscript into an "official version", that we made freely available by request. However, very few pictures were inserted in the text, it lacked bibliographic references, and it incorporated very different styles. By all accounts, it was pretty much unreadable to anybody but the authors. The next 20 years saw no change in the state of the manuscript, until recently.

On the other hand, this work seems to have been reasonably influential. For instance, a quick Internet search returns about 100 articles and books referring to it (under many different titles!). We cannot help noting this fact with a certain satisfaction, in particular if one takes into account that these references point to an unpublished document that was difficult to access even as a preprint. Most of the theoretical results underpinning this work had appeared in [$\mathbf{BonS}_1$] in a very general form, but there clearly was a need for a precise description of their applications to knot theory, and of the algorithmic methods that we had developed. Combined with persistent exhortations from various colleagues and friends, this provided a strong incentive to resume the task of putting the text in a form suitable for publication.

What follows is an extensively edited version of the text in its 1988 form. Many events have occurred in low-dimensional topology in the 30 years that have elapsed since we started this project, the most important one being the completion of the geometrisation program for 3–dimensional manifolds and orbifolds. With the corresponding hindsight and perspective, there are many parts that we would probably write in a very different way right now. However, we decided to preserve the historical data, and to remain faithful to the original. We limited our editing to locally improving the clarity of the exposition, and we respected some of the stylistic idiosyncrasies of the text. Updates and references to more recent work are indicated in footnotes.

Several people have helped us with the technical aspects of the enterprise, in particular in the early years for typing and picture drawing. This includes Valerie Siviter, Bernadette Barbichon, Bernard Thomas and Banwari Lal Sharma. The majority of the pictures of the current version were drawn by Julien Roger. We are very grateful to all of them, as well as to Rob Kirby, Cameron Gordon and Andrew Ranicki for pushing us to prepare this edition.

While preparing this revised version, the first author and Julien Roger were partially supported by the grant DMS-0604866 from the U.S. National Science Foundation.

A large part of the mathematics in this book can be traced back to a series of lectures that José Montesinos gave at Orsay in the spring of 1976. In particular, the first author fondly remembers the many meetings that he had with José in the *Jardin du Luxembourg* in Paris, where José suggested as a possible research topic the classification of what we call here Montesinos knots. This became the dissertation [**Bon**$_1$], and consequently launched the first author's career as a mathematician (in addition to providing material for several chapters in this monograph). It is a pleasure to dedicate this book to José Montesinos as an acknowledgement of our debt, and as a way to celebrate his 65-th birthday and his many contributions to 3–dimensional topology.

December 2009

Introduction

I was led to the consideration of the forms of knots by Sir W. Thomson's[1] *Theory of Vortex Atoms, and consequently the point of view which at least at first I adopted was that of classifying knots by the number of their crossings...*

The subject is very much more difficult and intricate than at first sight one is inclined to think, and I feel that I have not succeeded in catching the key-note. When that is found, the various results here given will, no doubt, appear in their real connection with one another, perhaps even as immediate consequences of a thoroughly adequate conception of the question.

P.J. Tait [**Tai**$_1$], 1877.

A century after the Scottish physicist P.G. Tait wrote these words, it is still a challenging task for the topologists of this day to acquire a theoretical understanding of knotting adequate to explain even the 630 or so (connected) knots tabulated in the last century by Tait, T.P. Kirkman and C.N. Little [**Tai**$_1$**, Tai**$_2$**, Tai**$_3$**, Kirk**$_1$**, Kirk**$_2$**, Lit**$_1$**, Lit**$_2$**, Lit**$_3$**, Lit**$_4$]. These XIX-th century tabulations listed connected knots of $\leqslant 11$ crossings excepting the 11 crossing connected knots having non-alternating projection. Since then, J.H. Conway [**Conw**] has tabulated the missing (non-alternating) 11 crossing connected knots plus the links of $\leqslant 10$ crossings and very recently A. Caudron [**Cau**] has listed the 11 crossing links; concerning tabulations, see also [**Has**$_1$**, Has**$_2$].

This is the task that we shall attempt here, and accomplish in part. To Thomson and Tait, our understanding would be disturbing inasmuch as it is more molecular than atomic; indeed the first phenomena we explain are described by integrally weighted trees a bit reminiscent of the organic molecule of the paraffin series.

A ***knot*** K in the 3–sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is a collection of smoothly and disjointly embedded circles, namely K is a closed 1–submanifold of S^3. Note that our terminology is somewhat non-standard, in the sense that we do not require knots to be connected, nor oriented. A ***link*** is a knot with $\geqslant 2$ components. Two knots (S^3, K_1), (S^3, K_2) are ***isomorphic*** or "the same" if there is a degree $+1$ (namely orientation-preserving) diffeomorphism $f\colon S^3 \to S^3$ with $f(K_1) = K_2$. Since a degree $+1$ diffeomorphism of S^3 is isotopic to the identity [**Cer**], this is equivalent to the property that K_1 can be deformed to K_2 in S^3 by a smooth isotopy.

In his remarkable heuristic work [**Conw**] on knot classification, J.H. Conway made much use of 2–spheres smoothly embedded in S^3 and cutting K transversely in 4 points. We call these ***Conway spheres*** for the knot (S^3, K). Figure 0.1

[1]= Lord Kelvin, or more precisely Baron Kelvin of Largs. See [**Tho**]

provides examples of such Conway spheres, obtained by capping off each of the dotted curves by one disk located in front of the sheet of paper and another disk in the back.

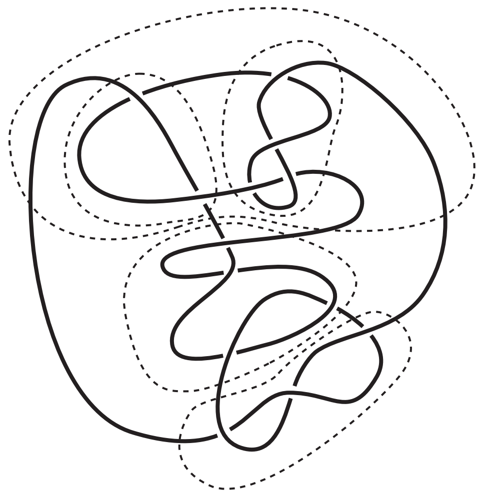


FIGURE 0.1.

A knot (S^3, K) is ***arborescent*** if there exists a finite collection $F_1, \ldots, F_n$ of disjoint Conway spheres such that, if N is the closure of any component of $S^3 - \bigcup_{i=1}^{n} F_i$, then the pair $(N, K \cap N)$ takes the simple form of Figure 0.2 after suitable isotopic deformation in S^3. In this figure, N is S^3 with the interior of a finite collection of disjoint balls deleted and $K \cap N$ is the intersection of N with two standard circles.

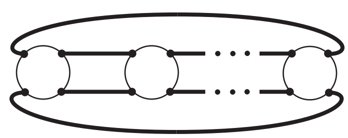


FIGURE 0.2.

For example the Conway spheres dotted in Figure 0.1 reveal that knot to be arborescent. In other words, arborescent knots are those that can be broken up into (almost) trivial pieces by Conway spheres.

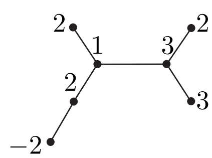
It was Conway who singled out this class and first studied their projections [**Conw**]. He found that over 75% of the knots tabulated in the XIX–th century, including nearly 90% of all knots and links of $\leqslant 10$ crossings, are arborescent; see also [**Cau**].

These arborescent knots are called ***algebraic*** by Conway because, in the framework of [**Conw**], they are obtained from finitely many copies of the 1–crossing tangle by applying a sequence of "sum" and "product" operations; see Chapter 14. Unfortunately, the name of algebraic knot is also used to describe the iterated torus knots that occur as links of algebraic complex singularities; see for instance [**Bra, Bur$_1$, Bur$_2$, Zar, EisN**]. The two families have very little in common, in fact just the $\{2, n\}$ $\{3, 4\}$ and $\{3, 5\}$ torus knots, as will be proved in Theorem A.8 in the Appendix. For this reason, we prefer to use the word "arborescent", which reflects the fact that the internal classification of these knots can be expressed in terms of trees.

When just one Conway 2–sphere is required in our definition, the arborescent knot is said to be a ***rational knot*** or a ***two-bridge knot***. H. Schubert classified these in 1956 [**Sch**$_3$]. Rational knots already account for about 35% of the knots tabulated in the XIX–th century.

In [**Sch**$_3$], Schubert provides two very distinct proofs of the classification of rational knots. One lengthy proof is very geometric, and shows that the Conway sphere separating the pair (S^3, K) in two rational tangles is unique up to isotopy respecting the knot. The second proof, significantly shorter, is based on H. Seifert's observation that the 2–fold branched covering of a rational knot is a lens space $L(p, q)$ and on the classification of these lens spaces.

To a large extent, our approach to arborescent knots combines these two points of view. José Montesinos [**Mon**$_1$] showed that the 2–fold branched cover of an arborescent knot is a graph manifold, in the sense of Waldhausen [**Wal**$_2$]. Using a $\mathbb{Z}_2$–equivariant version of Waldhausen's classification of sufficiently large graph manifolds [**Wal**$_2$], we manage to classify (prime) arborescent knots by certain "almost canonical" weighted planar trees. For example the arborescent knot of Figure 0.1 is classified by the tree



The edges of the tree are dual to a family of Conway 2–spheres which is uniquely determined up to isotopy respecting the knot.

This classification in fact participated in the first proof (by K. Perko [**Perk**$_4$], assisted by Caudron [**Cau**]) that the 11–crossing knots listed in [**Conw**] are all distinct. It also leads to a calculation of the symmetry groups of most arborescent knots, given in Chapter 16.

Our work is theoretically more interesting here than might be suggested by this "chemical" or "botanical" classification of arborescent knots. Indeed, from any (prime) knot (S^3, K) we manage to extract an arborescent part well-defined up to isotopy (see Chapter 3). This is because the same techniques that we developed for the classification of arborescent knots can be used for a $\mathbb{Z}_2$–equivariant version of the characteristic splitting of 3–manifolds along incompressible tori discovered by K. Johannson [**Joh**$_2$], W. Jaco and P. Shalen [**JacS**$_2$] (and originally suggested by F. Waldhausen [**Wal**$_4$]). We also make use of Thurston's Geometrisation Theorem [**Thu**$_1$**, Thu**$_2$] to strengthen our results by introducing additional rigidity to the pieces of our decomposition that are not arborescent.

This theory was originally developed in terms of actions of $\mathbb{Z}_2$ on 3–dimensional manifolds [**Bon**$_1$]. We then found generalisations for finite group actions on 3–manifolds, and more generally for orbifolds [**BonS**$_1$**, BonS**$_2$]. However, we use here a more down-to-earth point of view where all the data can be seen right away in S^3.

It is a pleasure to acknowledge helpful comments of Michel Boileau and Alain Caudron, who have been engaged in related work [**Boi**$_1$**, BoiS, BoiZ, Cau**]. This

work was also greatly inspired by lectures given by José Montesinos at Orsay in 1976, and by Bill Thurston's Princeton lectures and lecture notes [**Thu**$_1$] on hyperbolic structures.

Contents

Part 1

Characteristic splittings

CHAPTER 1

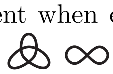
Conway's analysis of knot projections: some secrets of its success

This chapter is a historical and heuristic excursion motivating several major themes of this monograph. Although we will show the reader how to read our observations between the lines of history, it is possible that the numbered theorems of this chapter have never been clearly perceived before.

1.1. Knot projections

A ***knot*** in the 3-dimensional euclidean space $\mathbb{R}^3$, or in the 3–dimensional sphere $S^3 = \mathbb{R}^3 \cup \infty$, is a compact 1–dimensional submanifold K of $\mathbb{R}^3$ or S^3. We do not require K to be connected, and a ***link*** will be a knot with at least two connected components. Note that these conventions may not necessarily agree with those used by other authors. In general, we will not be concerned with the possible orientations of the knot K. However, the orientation of the ambient manifold $\mathbb{R}^3$ or S^3 will play a critical rôle.

Given a knot K in $\mathbb{R}^3$, for most directions of projection to $\mathbb{R}^2 \subset \mathbb{R}^3$, the projected image of K is an immersed 1–manifold in $\mathbb{R}^2$ with no singularities worse than double points of transverse self-intersection .

The projection plane $\mathbb{R}^2$ will habitually be viewed as the plane of the page you are reading. The immersed 1–manifold can be thought of as a quadrivalent graph Γ in the plane whose vertices are the above double points. (A graph is quadrivalent when each vertex is adjacent to exactly 4 ends of edges.) Here are examples: .

To retrieve the knot K from Γ up to isotopy of K in the 3–sphere $S^3 = \mathbb{R}^3 \cup \infty$, we need only to indicate at each vertex which strand lies above the other: Each ***crossing*** (= double point) becomes  or . Call Γ with this extra data a ***knot projection*** .

For a given crossing number (= number of vertices in Γ), there are clearly only finitely many knot projections, up to isotopy of $\mathbb{R}^2 \cup \infty = S^2$. Kirkman was expert in finding them (see [**Kirk**$_1$, **Kirk**$_2$, **Kirk**$_3$, **Kirk**$_4$]), Tait and Little attempted to organise the resulting knots into isotopy equivalence classes ([**Tai**$_1$, **Tai**$_2$, **Tai**$_3$, **Lit**$_1$, **Lit**$_2$, **Lit**$_3$, **Lit**$_4$]). They had a number of standard isotopies, which they applied to these projected knots: flyping, concealed flyping, etc... Compare [**Conw**] and Chapter 14. Unfortunately the knots that they listed as distinct were at the time only conjecturally so and, indeed, some were not distinct [**Perk**$_2$]. Astonishingly, in the 1950s, W. Haken (see [**Hak**$_3$, **Hak**$_1$, **Hak**$_2$, **Sch**$_4$, **SchuS**]) found an algorithmic method which is capable of deciding whether two individual knots are isotopic; a missing step in his method has been completed in the last few years (see [**Hemi,**

Wal$_5$]). However, it remains an important problem to make Haken's method, or some other method, usable in practice. It was not Haken's uniform method, but a fortuitous combination of algebraic invariants, notably Alexander polynomials and Reidemeister linking numbers in irregular branched covers, combined with our classification of arborescent knots, that recently let K. Perko complete the job of distinguishing the knots listed in the XIX-th century [**Perk**$_2$][1].

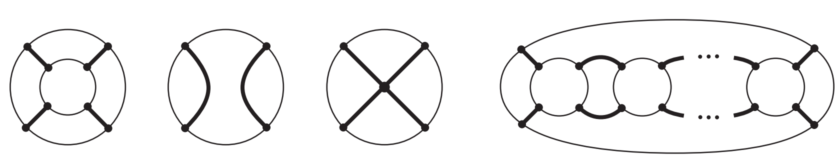
It should be added that, having clumsy methods capable (in principle) of listing and distinguishing individual knots, does not of itself constitute a *good* understanding of any one knot, nor is it a priori useful at all for any infinite class of knots. For finite groups in place of knots this was obviously the initial state of affairs when the theory of finite groups began; so it is no state in which to leave geometric knot theory.

J.H. Conway's heuristic analysis of knot projections [**Conw**] constituted the beginnings of an understanding of what Tait, Kirkman and Little had seen. We now reconstruct part of his analysis in simple terms that will suggest our intrinsic (projection-independent) analysis of knots in later chapters.

Conway considers the class $\mathcal{J}$ of Jordan curves (namely embedded circles) in $\mathbb{R}^2$ that cut the knot projection Γ transversely, avoiding the double points (vertices). Each C in $\mathcal{J}$ specifies a 2–sphere $F \subset S^3$ up to isotopy respecting K: Writing $\mathbb{R}^3$ as $\mathbb{R}^2 \times \mathbb{R}$, it is the cylinder $C \times [-r, r]$ (where r is sufficiently large) with nearly horizontal 2–disc caps added. For disjoint curves $C_1, \ldots, C_n$ in $\mathcal{J}$ we get disjoint 2–spheres $F_1, \ldots, F_n$ with $F = \bigcup_i F_i$ well-defined up to isotopy; we just insist that $r_i < r_j$ whenever C_i sits inside of C_j.

Like all knot tabulators, Conway was assuming that the knot K is neither a disjoint union of two knots separated by an embedded sphere in S^3, e.g. , nor a connected sum of two non-trivial knots, e.g. . Indeed, celebrated connected sum theories of Kneser, Seifert and Schubert reduce the classification of knots to those of this type; see §2.1. With this assumption, he was assured (after isotopy of K) that Γ is ***irreducible*** in the sense that Γ is connected and that the only Jordan curves in $\mathcal{J}$ meeting Γ in two points bound a vignette  in $S^2 = \mathbb{R}^2 \cup \infty$. (Note that the connectedness of Γ is actually implied by the second condition, except in the case when Γ consists of two disjoint embedded circles.)

Thus Conway set out to analyse the class of all curves $C \in \mathcal{J}$ cutting Γ in 4 points. We call these ***Conway circles***. By the rule given above, they determine ***Conway spheres*** for the knot pair (S^3, K), namely 2–spheres in S^3 transversely cutting K in 4 points.



(a) (b) (c) (d)

FIGURE 1.1.

Consider a family $C_1, \ldots, C_n$ of disjoint Conway circles subject to the condition that no closure D of a component of $S^2 - \bigcup_{i=1}^{n} C_i$ gives a pair $(D, \Gamma \cap D)$ as in

[1](Added 2009) See [**HTW, Hos**] for a more current update on knot tabulations.

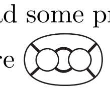
Figure 1.1(a), where D is an annulus, or as in Figure 1.1(b) where D is a disc. The number n of Conway circles in such a family is easily seen to be bounded in terms of the number c of crossings of the knot projection Γ (in fact, $n \leqslant 2c - 3$). We can therefore impose that, in addition, the family $C_1, \ldots, C_n$ is ***maximal*** for the above property. Namely, it cannot be enlarged to a family $C_1, \ldots, C_n, C_{n+1}, \ldots, C_{n+k}$ with $k > 0$ such that the closure of no component of $S^2 - \bigcup_{i=1}^{n+k} C_i$ is as in Figures 1.1(a) or (b).

The ***arborescent part*** of the projection Γ is defined as the union A_Γ of all closed-up components D of $S^2 - \bigcup_{i=1}^{n} C_i$ such that the pair $(D, D \cap \Gamma)$ is of one of the forms (c), (d) of Figure 1.1. Actually, in the case of Figure 1.1(d), the component D must have only 3 boundary components by maximality of $\bigcup_{i=1}^{n} C_i$.

In the above paragraph, we began to use the phrase "closed-up component". In this case, this just means the closure of a component of $S^2 - \bigcup_{i=1}^{n} C_i$. More generally, if F is a codimension 1 submanifold of a manifold M, a ***closed-up component*** of $M - F$ is a component of the manifold with boundary obtained by splitting M along F. Formally, this new manifold can be defined as the union of $M - F$ and of the unit normal bundle of F, with the obvious topology. When, as happens in most cases in this monograph, every component of F separates M, a closed-up component of $M - F$ is of course just the closure of a component of $M - F$.

Theorem 1.1 below shows that this arborescent part A_Γ is well-defined, namely is independent of the original collection of curves C_i, up to isotopy of S^2 respecting Γ.

The projection is ***arborescent*** when the arborescent part A_Γ is equal to the whole sphere $S^2 = \mathbb{R}^2 \cup \infty$. A knot is ***arborescent***, as defined in the introduction, precisely if *some* projection is arborescent as defined here; see Chapters 3, 12 and 14.

Conway was undoubtedly aware of the following elementary planar result (see [**Conw**, §2]). Even Kirkman had some premonitions in terms of flaps , which are closely related to the picture  of Figure 1.1(d).

THEOREM 1.1. *If from the interior of A_Γ we delete and continue to delete Conway circles C_i as long as the closed-up components of A_Γ minus the remaining C_i are of the form* (c) *or* (d) *of Figure* 1.1*, then the union $G_\Gamma \subset S^2$ of all remaining C_i among $C_1, \ldots, C_n$ is well-defined up to isotopy of S^2 respecting* Γ.

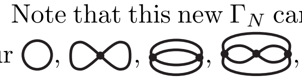
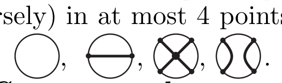
In particular, A_Γ itself is well-defined up to isotopy of S^2 respecting Γ.

One of our major results (Theorem 3.3) is an analogue of this theorem for knots, which makes no explicit reference to a knot projection. Actually, Theorem 1.1 can be obtained as a corollary of the slightly more general form of Theorem 3.3 proved in Chapters 7 and 9.

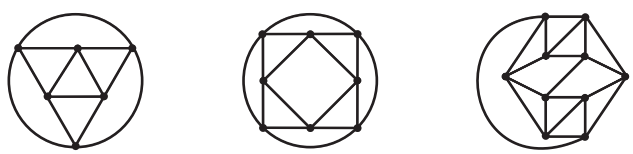
Here are a few indications of how to deduce Theorem 1.1 from the results of Chapters 7 and 9, anticipating these two chapters and with the terminology used there. Consider the knot pair $(S^3 - V, \Gamma - V)$ where V is the interior of a tubular neighbourhood of the vertex set of Γ. Note that any family F of disjoint pairwise incompressible Conway 2–spheres in this pair arises from a family of disjoint Conway circles in S^2, up to isotopy respecting Γ. (Hint: First isotop F so that the number of components of $F \cap \mathbb{R}^2$ cannot be further reduced by an isotopy.) Also observe that, if two families of disjoint Conway circles in (S^2, Γ) have in their respective complements no component as in Figure 1.1(a) or 1.1(b), then they are isotopic respecting Γ if and only if the connected components of their complements induce the same partition of the vertex set of Γ. Then apply Theorems 7.1 and 9.1 to the knot pair $(S^3 - V, \Gamma - V)$, using Criterion 9.5.

Surely there are satisfactory 2–dimensional proofs of Theorem 1.1. One that seems to work well is in close parallel with our proof of Theorem 3.3. Thoroughly understanding this parallel should help the reader understand both results better.

The pair $(A_\Gamma, \Gamma \cap A_\Gamma)$ can be simply classified using a weighted planar tree, defined in Chapter 14. Better, we will gain a good 3–dimensional understanding of this part of the knot. To be more precise, let $\widehat{G}_\Gamma \subset \mathbb{R}^3$ be the family of disjoint Conway 2–spheres for (S^3, K) associated to the family of Conway circles $G_\Gamma \subset \mathbb{R}^2$ offered by Theorem 1.1. The union $\widehat{A}_\Gamma$ of those closed-up components of $S^3 - \widehat{G}_\Gamma$ which contain part of $A_\Gamma - S^2$ gives a so-called "arborescent pair" $(\widehat{A}_\Gamma, K \cap \widehat{A}_\Gamma)$ that will be satisfactorily analysed up to pair isomorphism in Chapters 12–13.

To study a closed-up component N of $S^2 - G_\Gamma$ that is not contained in A_Γ, it is convenient to shrink the closed-up components of $S^2 - N$ to points to get a modified quadrivalent graph Γ_N in S^2. Note that this new Γ_N can be any quadrivalent planar graph distinct from these four , and such that every Jordan curve meeting Γ_N (transversely) in at most 4 points bounds a disk in S^2 meeting Γ_N in one of the four ways: . A graph in S^2 with the above properties will be called a ***Conway graph***.

The best known Conway graphs are those of Turk's head type $(a \times b)^*$ with ab vertices, $a \geqslant 2$, $b \geqslant 3$, made from an outer circle and a successively inscribed regular b–gons, as in Figure 1.2. When the product ab factors uniquely, Conway writes $(ab)^*$ for $(a \times b)^*$ with $a \leqslant b$, e.g. 6^* for $(2 \times 3)^*$ and 8^* for $(2 \times 4)^*$. Up to 10 vertices, there is just one other Conway graph, the graph 10^{**} of Figure 1.2, making a total of just five, namely: $(2 \times 3)^*$, $(2 \times 4)^*$, $(3 \times 3)^*$, $(2 \times 5)^*$, 10^{**}.



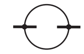
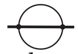
$(2 \times 3)^*$ $(2 \times 4)^*$ 10^{**}

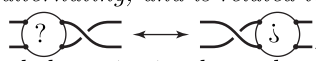
FIGURE 1.2.

There are two more of 11 crossings, but eight new ones of 12 crossings [**Cau**]. The modesty of this collection of graphs let Conway give his extension of the XIX-th century tabulations.

To emphasise the dependence of Theorem 1.1 on the knot projection, we note that any finite quadrivalent graph $\Gamma \subset \mathbb{R}^2$ can arise as the projection of a knot $K \subset \mathbb{R}^3$ that is ***trivial***, namely that is the boundary of a collection of disjoint smoothly embedded 2–discs. (Just lay down string from above on $\mathbb{R}^2$ following around the immersed curves of the projection Γ.) Thus, there is grave difficulty in drawing conclusions about the knot (S^3, K) from an analysis of the above sort.

Nevertheless, it is an experimental fact that a good projection of a knot reveals a lot of intrinsic structure. For example, the following unproved conjecture has been an axiom for all tabulators up to and including Conway.

CONJECTURE 1.2 (Tait's Alternating Projection Conjecture[2]). *Suppose that a knot projection on S^2 is alternating, in the sense that no vignette*  *nor*  *can be seen. Suppose also (as always) that the projection graph is connected and irreducible in the sense that a circle meeting it in two points*  *bounds a vignette* .

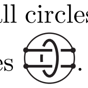
Then there exists no projection of the same knot with fewer crossings. Further any other projection of the same knot with the minimal number of crossings is also alternating, and is related to the first by a sequence of simple isotopic flyping moves *, and perhaps the global flype that similarly rotates the whole projection by π about an axis in the plane.*

This conjecture should be regarded as an empirical discovery of Tait's. He believed it easy to verify (see [**Tai**$_1$**, Lit**$_4$]). Note that it made his organisation of Kirkman's lists of alternating knot projections into knot equivalence classes a routine matter. By 1930, the conjecture had been recognised as such and was proved for the trivial knot by C. Bankwitz [**Ban**]. (For connected alternating knots, the knot determinant $A(-1)$ is in absolute value greater than or equal to the crossing number of an alternating projection as above.)

Although Tait's conjecture is beautiful and sweeping, it seems safe to counterconjecture that, as the crossing number increases, the proportion of alternating knots among all knots dwindles to zero.[3]

In Chapter 3, we will manage to restructure Conway's analysis to make it intrinsic to the knot. Although we will abandon projections and work in three dimensions, our result can be partially interpreted in terms of a projection as follows.

For any knot K in S^3 that is simple for Schubert (the precise definition will be given in Chapter 2, and means that the complement $S^3 - K$ contains no non-trivial 2–sphere or torus), there exists a suitable projection Γ such that, for the family G_Γ of Conway circles of Theorem 1.1, a certain subfamily G^*_Γ of G_Γ gives a family $\widehat{G}^*_\Gamma$ of Conway 2–spheres for (S^3, L) that is ***characteristic***, namely is invariant under automorphisms of (S^3, K) up to isotopy of S^3 respecting K.

More precisely, say that a Conway circle C for the knot projection $\Gamma \subset S^2$ ***bounds a rational tangle projection*** when it bounds a disk D such that the pair $(D, D\cap\Gamma)$ is the union of vignettes  and of one vignette . (Compare [**Conw**] and §1.2 below.) Then, *for a suitable projection* of K, G^*_Γ will be obtained from the family G_Γ provided by Theorem 1.1 by deleting all circles bounding rational tangle projections and certain circles bounding vignettes . The reader can return to verify these statements after reading Chapter 3, where a full definition of $\widehat{G}^*_\Gamma$ is given (it is called G there).

[2](Added 2009) The Tait Conjecture is now a theorem. The first part of the conjecture, related to the number of crossings, was proved by L. Kauffman [**Kau**], K. Murasugi [**Mur**], and Thistlethwaite [**Thi**], using the Jones polynomial. The full conjecture was obtained by W. Menasco and M. Thistlethwaite [**MenT**]. The results of this monograph are actually one of the ingredients of the Menasco-Thistlethwaite proof, together with polynomial knot invariants and cut-and-paste geometric topology.

[3](Added 2009) This has now been proved by the combination of results of A. Stoimenow [**Stoi**], C. Sundberg and M. Thistlethwaite [**SunT**].

In the next section, we shall see that at least some of the abandoned components of G_Γ that bound rational tangle projections can also give characteristic Conway spheres for the knot (S^3, K).

1.2. Rational tangle substitutions

Next we propose to show that, under certain conditions, additional components of the arborescent part A_Γ that are rational tangle projections tend to give characteristic Conway spheres for a knot described by a given knot projection Γ. The argument is based on hyperbolic geometry, mainly Thurston's hyperbolic Dehn surgery [**Thu**$_1$, Chap. 5] (see also [**BenP**, Chap. E]), and the proof will be deferred to Chapter 6. It is rather unfortunate that the results we obtain are mainly theoretical, and often difficult to work out in practice to get explicit information (but see Chapter 4, and compare [**Ril**$_1$**, Ril**$_2$]). If there is some day a way to make these arguments more explicit, it will surely have an impact already on tabulated knots. See Chapter 6 for more discussion.

We first start by closely examining rational tangle projections, and their relationship with the pieces of knot they describe. This analysis is largely inspired by [**Conw, Sch**$_3$**, Sie**$_1$].

For a knot projection $\Gamma \subset S^2$, define a ***tangle projection*** as a compact surface $S \subset S^2$ together with the following extra information: Each component C of ∂S transversely meets Γ in 4 points, and one of the four components of $C - \Gamma$ is singled out and oriented by, say, marking it with an arrow running parallel to it. See Figure 1.3 for an example, and compare our formal definition of tangles in §§8.1 and 12.1.

This definition is easily related to the one in [**Conw**]. Indeed, any tangle in Conway's sense gives a tangle in the above sense with a disc as underlying surface S, and with the arrow joining the South-West string to the South-East string.

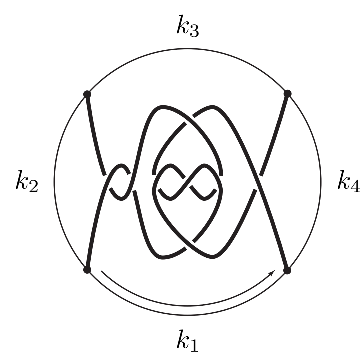
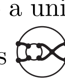


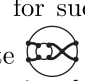
FIGURE 1.3.

When the knot projection $\Gamma \subset S^2 \subset S^3$ describes a knot $K \subset S^3$, we are free to assume that K coincides with Γ outside of a small neighbourhood of its vertices, and to require that the boundaries of tangle projections for Γ avoid this small neighbourhood of the vertex set. Then, a tangle projection with underlying surface S specifies a submanifold M of S^3, up to isotopy respecting K. Namely ∂M consists of the Conway 2–spheres determined by the Conway circles ∂S (with $S^2 \cap \partial B = \partial S$), and M is the closed-up component of $S^3 - \partial M$ containing S.

Consider two tangle projections S, S' for knot projections Γ, Γ', describing the knots K, K' and with associated 3–submanifolds $M, M' \subset S^3$. (We will usually denote by the same letter a tangle projection and its underlying surface.) These two tangle projections are said to be ***equivalent*** when there is a degree $+1$ pair isomorphism φ: $(M, K \cap M) \to (M', K' \cap M')$ such that $\varphi(\partial S) = \partial S'$ and φ sends arrow to arrow.

Here, when we say that φ is an isomorphism, this notion will depend on the category that we are working with. We are mostly interested in the category DIFF of differentiable manifolds and differentiable maps, and in the category PL of piecewise manifolds and piecewise maps. We could also use the category TOP of topological manifolds and continuous maps, provided we restrict attention to locally flat submanifolds, namely submanifolds admitting collar neighbourhoods. Standard smoothing results in dimension 3 [**Moi, KirbS**] enable one to freely go back and forth between these three categories. In general, the PL category is more convenient for cut-and-paste constructions, whereas the DIFF category is better adapted to hyperbolic geometry arguments. To return to the statement that φ: $(M, K \cap M) \to (M', K' \cap M')$ is a ***degree* $+1$** (or ***orientation-preserving***) ***pair isomorphism***, we mean that φ is a homeomorphism from M to M' if we are in the TOP category, a piecewise linear homeomorphism if we are in the PL category, or a diffeomorphism if we are in the DIFF category; that φ sends $K \cap M$ to $K' \cap M'$; and that it sends the orientation of M to the orientation of M'. If instead φ sends the orientation of M to the opposite of the orientation of M', then it has ***degree* -1**. In particular, note that the degree ± 1 refers to the orientations of the 3–manifolds M, M', and not to orientations of the knots K, K' or of the surfaces S, S'.

A tangle projection for a knot projection Γ is ***rational*** if its underlying surface D is a disc and can be split along a family of Conway circles into a union of several annuli (possibly none) and of one disc, respectively meeting Γ as  and .

Given a disc $D \subset S^2$ whose boundary meets the knot projection Γ in 4 points, here is a quick algorithm to decide whether D comes from a rational tangle projection or not (the arrows are clearly irrelevant here): If no closed-up component of $D - \Gamma$ is a triangle bounded by an arc in ∂D and two edges of $\Gamma \cap D$, the answer is no. Otherwise, for such a triangle T, a regular neighbourhood of $T \cup \partial D$ in D offers a vignette  ; then the closed-up complement D' of this vignette in D comes from a rational tangle projection if and only if so does D. The proof is straightforward. By induction on the number of double points of $D \cap \Gamma$, this clearly offers a solution to the problem considered.

The terminology is motivated by the property that the equivalence class of a rational tangle projection is classified by a certain rational number, possibly infinite, called its ***slope*** or ***type*** (see Proposition 1.3). This property was first observed by Conway [**Conw**], although it is already implicit in Schubert's classification of 2–bridge knots [**Sch$_3$**].

The rule assigning a slope $m \in \mathbb{Q} \cup \infty$ to a rational tangle projection will satisfy the properties indicated in Figure 1.4, where it is understood that the vignettes 

in Figure 1.4(b)–(f) represent the same portion of the knot projection. Here, we adopt the convention that $\infty = -\infty = \infty \pm 1 = \frac{1}{0}$ and $0 = \frac{1}{\infty}$. Note that this uniquely determines the slope by induction, once we have chosen a decomposition

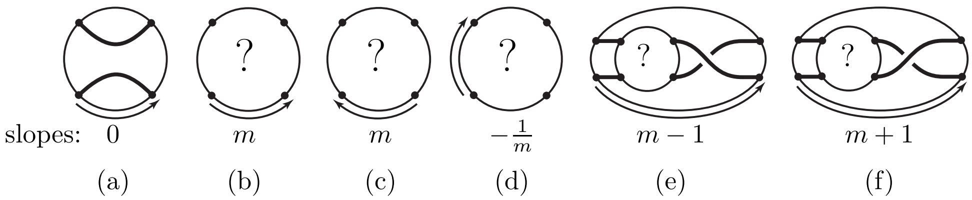
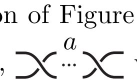
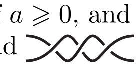


FIGURE 1.4.

of the rational tangle projection into annuli  and one disc . For instance, the slope of the rational tangle projection of Figure 1.3 is readily seen to be

$$\frac{-1}{\frac{-1}{1+1-1}+1+1}+1+1-1=0,$$

which is also the slope of the rational tangle projection of Figure 1.4(a). The reader may enjoy untangling the strings in Figure 1.3, and check that these two rational tangles are indeed isomorphic.

For practical computation, it is also useful to note the relation of Figure 1.5, which is an immediate consequence of the ones in Figure 1.4. Here,  with $a \in \mathbb{Z}$ denotes a right-handed half-twists if $a \geqslant 0$, and $-a$ left-handed half-twists if $a \leqslant 0$, for instance  if $a = +3$, and  if $a = -3$.

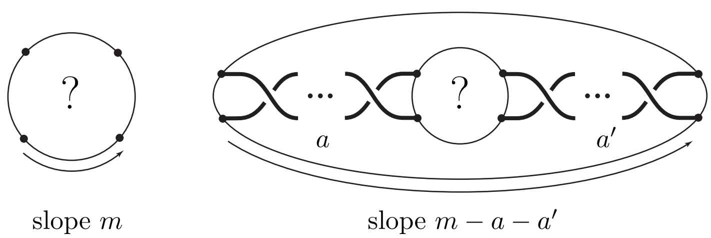


FIGURE 1.5.

We need to check that this slope is independent of the chosen decomposition of the rational tangle projection, and that it is an invariant of its equivalence class. This is done in the next statement.

PROPOSITION 1.3. *There is a unique rule which associates a slope $m \in \mathbb{Q} \cup \infty$ to each rational tangle projection, which is invariant by isomorphism of rational tangle projections, and which satisfies the relations of Figure* 1.4. *Two rational tangle projections are equivalent if and only if they have the same slope.*

PROOF. We will first give a more intrinsic construction of the slope $m \in \mathbb{Q} \cup \infty$ (compare §8.1 to really see m as a slope).

Consider a knot projection $\Gamma \subset S^2$, describing a knot $K \subset S^3$ which is assumed to coincide with Γ outside of a small neighbourhood of the vertices of this graph. Then consider a rational tangle projection D for Γ, defining a knot pair $(B, K \cap B)$ bounded by the Conway sphere in S^3 associated to the Conway circle $\partial D \subset S^2$.

By progressively untangling the strings, observe that the knot pair $(B, K \cap B)$ is isomorphic to the knot pair associated to the "simplest" rational tangle projection of Figure 1.4(a), namely to the knot pair drawn on Figure 1.6(a). It therefore admits a (unique) double branched cover, which is a solid torus $\widetilde{B} \cong S^1 \times D^2$; the easiest way to see this last point is to consider $\mathbb{Z}_2$ acting on $S^1 \times D^2$ by the rotation τ of angle π represented on Figure 1.6(b), and to check that the pair made of the quotient $(S^1 \times D^2)/\tau$ and of the image of the fixed point set of τ is isomorphic to the pair of Figure 1.6(a). See also §A.1 in the Appendix for basic facts about double branched covers.

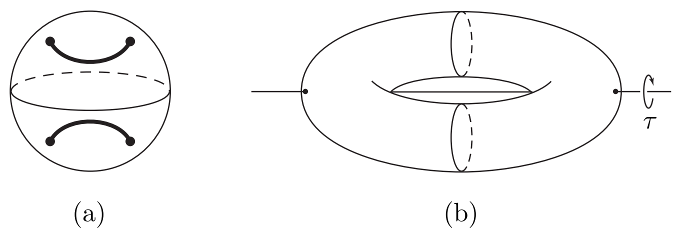


FIGURE 1.6.

Let k_1 be the component of $\partial D - \Gamma$ marked by the arrow, and let k_2, k_3, k_4 denote the other components of $\partial D - \Gamma$, in this order, starting from k_1 and in the direction opposite to the one specified by the arrow (see Figure 1.3). The preimage of each arc k_i in the double branched cover $\widetilde{B}$ is a closed curve $\widetilde{k}_i$ on the boundary torus $\partial\widetilde{B}$. Note that $\widetilde{k}_1$ is parallel to $\widetilde{k}_3$ on $\partial\widetilde{B}$, that $\widetilde{k}_2$ is parallel to $\widetilde{k}_4$, and that $\widetilde{k}_1$ meets $\widetilde{k}_2$ in exactly one point. Orient $\widetilde{B}$ by the orientation of $B \subset \mathbb{R}^3$, orient $\partial\widetilde{B}$ as boundary of $\widetilde{B}$, and orient $\widetilde{k}_1$ and $\widetilde{k}_2$ so that the algebraic intersection number $[\widetilde{k}_1] \cdot [\widetilde{k}_2] \in \mathbb{Z}$ of cycles in $H_1(\partial\widetilde{B};\mathbb{Z})$ is $+1$.

As $[\widetilde{k}_1]$ and $[\widetilde{k}_2]$ generate $H_1(\partial\widetilde{B})$, let $p[\widetilde{k}_1] + q[\widetilde{k}_2]$ generate the kernel of the homomorphism $H_1(\partial\widetilde{B}) \to H_1(\widetilde{B})$ induced by the inclusion map. Then, we associate the rational $m = \frac{q}{p} \in \mathbb{Q} \cup \infty$ to the rational tangle projection considered. In other words, m is the slope of the 1–dimensional kernel of $H_1(\partial\widetilde{B};\mathbb{R}) \to H_1(\widetilde{B};\mathbb{R})$ in $H_1(\partial\widetilde{B};\mathbb{R})$ identified to $\mathbb{R}^2$ by the base $\{[\widetilde{k}_1], [\widetilde{k}_2]\}$. Note that m is independent of the choice of orientations for $\widetilde{k}_1$ and $\widetilde{k}_2$ (provided $[\widetilde{k}_1] \cdot [\widetilde{k}_2] = +1$).

By construction, this slope m is clearly invariant by equivalence of rational tangle projections. We have to check that the rule thus defined to associate a slope $m \in \mathbb{Q} \cup \infty$ to each rational tangle projection satisfies the properties stated in Figure 1.4.

The relations described by Figure 1.4(a) and Figures 1.4(b)(c)(d) are easy to check.

For the relation presented by Figures 1.4(b)(e), let D' denote the tangle projection of Figure 1.4(e) and consider the tangle projection $D \subset D'$ of Figure 1.4(b). Then, denoting by primes the data associated to D' as above, the knot pairs $(B, K \cap B)$ and $(B', K \cap B')$ can be chosen so that $B \subset B'$. The closed-up complement of $\widetilde{B}$ in $\widetilde{B}'$ is now a collar neighbourhood of $\partial\widetilde{B}'$ in $\widetilde{B}'$, which induces a projection $\partial\widetilde{B} \to \partial\widetilde{B}'$. One then readily checks that, for suitable choices of orientations, this projection sends $[\widetilde{k}_2]$ to $[\widetilde{k}'_2]$ and $[\widetilde{k}_1]$ to $[\widetilde{k}'_1] - [\widetilde{k}'_2]$. The relations of Figures 1.4(b)(e) now immediately follow.

The proof is similar for Figures 1.4(b)(f).

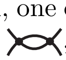
The intrinsic nature of this construction makes it clear that equivalent rational tangles have the same slope.

It remains to see that, conversely, two rational tangle projections D and D' are equivalent when they have the same slope $m = \frac{q}{p} \in \mathbb{Q} \cup \infty$ (see [**Conw**] or Chapters 12–13 for alternative proofs). Let K, B, $\widetilde{B}$, $\widetilde{k}_1$, $\widetilde{k}_2$ be associated to D as above and let primes distinguish similar data for D'.

The pair $(B, K \cap B)$ is such that there is a union $k \subset \partial B$ of two disjoint arcs with $\partial k = K \cap \partial B$ for which $K \cap B$ is obtained by pushing $k - \partial k$ inside B, up to isotopy fixing ∂B. The preimage of k in $\partial \widetilde{B}$ consists of two closed curves $\widetilde{k}$, each of which is in the kernel of $H_1(\partial \widetilde{B}) \to H_1(\widetilde{B})$, and therefore homologous to $\pm(p[\widetilde{k}_1] + q[\widetilde{k}_2])$.

Let $k' \subset \partial B'$ and $\widetilde{k}' \subset \partial \widetilde{B}'$ be similarly defined from D'.

We begin with an orientation-preserving isomorphism $\psi\colon (\partial B, K \cap \partial B) \to (\partial B', K' \cap \partial B')$ sending ∂D to $\partial D'$ and arrow to arrow. Lift ψ to $\widetilde{\psi}\colon \partial \widetilde{B} \to \partial \widetilde{B}$. After perturbation of ψ, we can assume that $\widetilde{\psi}(\widetilde{k})$ meets $\widetilde{k}'$ transversely.

As D and D' have the same slope $\frac{q}{p}$, each component of $\widetilde{\psi}(\widetilde{k})$ has algebraic intersection number 0 with each component of $\widetilde{k}'$. Since $\partial \widetilde{B}' - \widetilde{k}'$ consists of two open annuli, one concludes that at least one closed-up component of $\partial \widetilde{B}' - \widetilde{k}' \cup \widetilde{\psi}(\widetilde{k})$ is a digon , and gives a similar digon between k' and $\psi(k)$ in $\partial B'$. After a succession of intersection reductions using such digons, one eventually makes $k' \cup \psi(k)$ be the boundary of two disjoint digons  in $\partial B'$, which yields an isotopy fixing $K' \cap \partial B'$ between k' and $\psi(k)$.

Since K and K' are respectively obtained by pushing k and k' inside of B and B', we can use this isotopy to extend ψ to $\psi\colon B \to B'$ sending K to K' (and still sending ∂D to $\partial D'$ and matching arrow to arrow). This proves that D and D' are equivalent when they have the same slope, and concludes the proof of Proposition 1.3. $\square$

After this preamble on rational tangle projections, we are now ready to prospect for additional characteristic Conway spheres in knots.

Consider a knot projection Γ where the characteristic family G_Γ of Conway spheres provided by Theorem 1.1 is particularly simple in the following sense: The arborescent part A_Γ consists of disjoint rational tangle projections and G_Γ is completely contained in A_Γ. In other words, Γ is obtained from a Conway graph Γ_0 by replacing a small neighbourhood of each of its vertices by a rational tangle projection. In saying this, we have of course neglected the arrow which, by definition, is part of the data forming a rational tangle projection. For each component A' of A_Γ, there are 8 possible choices of arrow on $\partial A'$, and each choice defines a slope $m \in \mathbb{Q}$, namely the slope of the rational tangle projection so defined.

Assuming that $A \cup \Gamma \subset \mathbb{R}^2$ in $S^2 = \mathbb{R}^2 \cup \infty$, there is a natural convention to choose arrows in a coherent way so that these slopes are well-defined: We give the unique black and white checkerboard colouring to the regions of $\mathbb{R}^2$ cut out by the Conway graph Γ_0 such that the infinite region is white. For each component A_v of A_Γ corresponding to a vertex v of Γ_0, choose an arrow marking a component of $\partial A_v - \Gamma$ located in a white region of $\mathbb{R}^2 - \Gamma_0$. There are still four possible choices of such an arrow for each vertex v of Γ_0 but, because of the equivalences of Figure 1.4(b)(c)(d), they all define the same slope $m_v \in \mathbb{Q}$.

In this situation, we will say that the knot projection Γ is obtained from the Conway graph $\Gamma_0 \subset \mathbb{R}^2$ by ***substituting a rational tangle projection*** of slope m_v at each vertex v of Γ_0.

We can slightly extend this construction by allowing ourselves to substitute a rational tangle projection  of slope 0 or ∞ at some vertices of Γ_0, although such vignettes do not give components of the arborescent part of the knot projection so obtained. Observe however that we do not gain much generality by doing so; indeed, such a substitution  gives the same knot as a substitution , of the type previously considered.

THEOREM 1.4. *For every Conway graph $\Gamma_0 \subset \mathbb{R}^2 = S^2 - \infty$ and every vertex v of Γ_0, there is a finite subset $E_v(\Gamma_0)$ of $\mathbb{Q} \cup \infty$ with the following property: Let Γ be a knot projection obtained by substituting a rational tangle projection of slope $m_v \in (\mathbb{Q} \cup \infty) - E_v(\Gamma_0)$ at each vertex v of Γ_0. Similarly, let Γ' be obtained from another Conway graph Γ'_0 by substituting a rational tangle projection of slope $m'_{v'} \notin E_{v'}(\Gamma'_0)$ at each vertex v' of Γ'_0. Then the two knots respectively described by these knot projections are degree $+1$ isomorphic if and only if there is a degree ± 1 isomorphism θ: $S^2 \to S^2$ sending Γ_0 to Γ'_0 and such that:*

(1) *if θ respects the colours of the components of $S^2 - \Gamma_0$ and $S^2 - \Gamma'_0$, then $m'_{\theta(v)} = m_v$ for every vertex v of Γ_0;*
(2) *otherwise, $m'_{\theta(v)} = -\frac{1}{m_v}$ for every $v \in \Gamma_0$.*

Theorem 1.4 can be rephrased as follows. Given a Conway graph Γ_0, call a rational tangle substitution ***sufficiently complicated*** when, for each vertex v of Γ_0, the corresponding slope m_v is not in the exceptional (finite) set $E_v(\Gamma_0)$. Then, consider the class of knots that are obtained by sufficiently complicated rational tangle substitutions on some Conway graph. Theorem 1.4 asserts that the only isomorphisms between knots in this class are the obvious ones (use Proposition 1.3).

The proof of this result will be given in Chapter 6 (see Theorem 6.11), and will be based on hyperbolic geometry. In light of known examples (see for instance Chapter 4), it seems reasonable to conjecture that the exceptional sets $E_v(\Gamma_0)$ can be relatively small, for instance with no more than 4 or 5 elements. See §6.2 for more discussion of $E_v(\Gamma_0)$.

CHAPTER 2

Known factorisations: connected sums and companionship

We propose to discuss classical first steps in the geometrical analysis of knots. This includes the Kneser-Haken-Milnor theory of connected sums of 3–dimensional manifolds, and the analysis of satellite knots initiated by Schubert. We will actually give our own version of Schubert's decomposition of a satellite knot into its companions, in a form that we believe is particularly convenient.

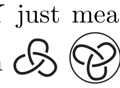
2.1. Connected sums and characteristic decompositions along tori

Knot theory can be understood as the analysis of pairs (S^3, K), where K is a knot in the 3–sphere S^3. If, as we began doing in Chapter 1, we split the pair (S^3, K) along surfaces which are transverse to the knot, we obtain new pairs (M, L) where M is a 3–dimensional manifold with boundary and where L is a 1–dimensional submanifold of M. Our systematic reliance on such splitting constructions makes it worth introducing a new term to describe these objects.

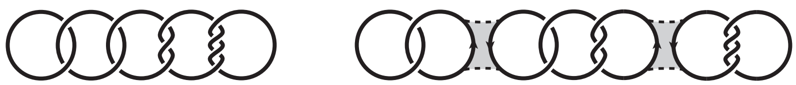
A ***knot pair*** is a pair (M, K) where M is an oriented connected compact 3–manifold with (possibly empty) boundary, and where K is a proper 1–dimensional submanifold of M. Recall that the properness property means that $\partial K = K \cap \partial M$. In general, we do not assume that a preferred orientation of K has been chosen.

Eventually, we will focus attention on the case where M is the complement of finitely many disjoint balls in S^3 but, initially, M can be any compact connected oriented 3–manifold. However, we will progressively need to impose additional restrictions.

The theory of connected sums of 3–manifolds initiated by H. Kneser [**Kne**] (see also [**Hak$_1$, Hak$_2$, Mil, Hemp**]) can be applied to $M - K$ to show that, without essential loss of generality, one can assume that $M - K$ is ***irreducible***, namely that every 2–sphere in $M - K$ is the boundary of a 3–ball in $M - K$. Indeed, this theory tells us much more: there always exists a finite (unordered) collection of knot pairs $(M_1, K_1), \ldots, (M_s, K_s)$, well defined up to isomorphism and permutations, so that $(M, K) \cong (M_1, K_1) \# \ldots \# (M_s, K_s)$ where # denotes connected sum *avoiding* the knots K_i and, for each index i, either $M_i - K_i$ is irreducible or (M_i, K_i) is isomorphic to $(S^2 \times S^1, \varnothing)$.

For $M = S^3$, the fact that every (smooth) 2–sphere in S^3 bounds a 3–ball [**Ale**] shows that this irreducibility of $S^3 - K$ just means that no 2–sphere in $S^3 - K$ separates two distinct components as in . If K is a link, one then calls it ***unsplittable***. Naturally enough, only unsplittable links (S^3, K) are tabulated. (One exception: the 2–component link  is tabulated by Conway.)

There is a celebrated pairwise version of this theory of connected sums, which was initiated by H. Seifert [**Sei**$_2$] and H. Schubert [**Sch**$_1$]. An arbitrary knot K in S^3 with $S^3 - K$ irreducible is uniquely factored as a pairwise connected sum (see Figure 2.1(b)) of knots $(S^3, K_1), \ldots, (S^3, K_s)$, where each pair (S^3, K_i) is ***pairwise irreducible*** in the sense that any 2–sphere in S^3 transversely cutting K_i in two points necessarily bounds a 3–ball B in S^3, such that $(B, K_i \cap B)$ is isomorphic to the standard pair (B^3, B^1). For example, the knot in Figure 2.1(a) has three factors as in Figure 2.1(b) from which it is retrieved by a simple sort of band connected sum (dotted in Figure 2.1(b)) that is sufficiently specified by the correspondence of components given by the two bands (making string orientation correspond).



(a) (b)

FIGURE 2.1.

Because of these facts, which were accepted without proof in the XIX-th century, only pairwise irreducible knots have been tabulated.

This pairwise connected sum theory is *subsumed* (for knots in S^3) by a more recent theory concerning embedded 2–tori. This, too, was initiated by Schubert [**Sch**$_2$], but reached a satisfactory state only recently through the work of F. Waldhausen [**Wal**$_4$], K. Johannson [**Joh**$_1$**,** **Joh**$_2$], W.H. Jaco and P.B. Shalen [**JacS**$_1$**,** **JacS**$_2$].

We now pause to describe this theory of embedded 2–tori, for a number of reasons which, taken together, seem compelling:

(1) it subsumes the pairwise connected sum theory above;
(2) it had a role in suggesting our analysis of Conway 2–spheres (see Chapter 3);
(3) we can offer an attractive formulation of it for knots in S^3 as a unique factorisation into characteristic companion knots (a rather similar factorisation of irreducible $\mathbb{Z}$–homology 3–spheres played a crucial role in [**Sie**$_2$]);
(4) this factorisation greatly facilitates the subsequent analysis of Conway 2–spheres, as presented in Chapter 3.

Consider an irreducible orientable 3–manifold X, not necessarily compact. Shortly, X will be the complement $M - K$ of the knot K in a knot pair (M, K). A 2–torus F embedded in X is ***trivial*** if it is either *compressible* or *peripheral* as defined below.

The torus F is ***compressible*** if there exists a 2–disc D in X such that $D \cap F = \partial D$ is essential in F, namely represents a non-trivial element of $\pi_1(F)$. This D is called an ***effective compression disc*** for F.

It is ***peripheral*** if some closed-up component of $X - F$ is isomorphic to $T^2 \times [0, \infty)$, with T^2 the standard 2–torus.

The manifold X is called ***geometrically atoroidal***, or simply ***atoroidal***, if it contains no non-trivial 2–torus.

REMARK 2.1. By the irreducibility of X, a compressible torus $F \subset X$ bounds either a solid torus, isomorphic to $B^2 \times S^1$, or a 3–ball from which a knotted wormhole has been drilled out. To see this, consider a regular neighbourhood U of $F \cup D$ in X and note that one component of its boundary ∂U is a 2–sphere.

REMARK 2.2. The 3–manifold X is called ***algebraically atoroidal*** if it satisfies a similar condition for arbitrary continuous maps of $S^1 \times S^1 \to X$ in place of embeddings. Beware that other authors may use the term atoroidal for algebraically atoroidal. It is a deep fact that, provided X is Haken (namely contains a closed incompressible surface), geometrically atoroidal is equivalent to algebraically atoroidal unless X is a Seifert fibre space over S^2 with 3 exceptional fibres or over the disk B^2 with 2 exceptional fibres. These Seifert fibre spaces are always *geometrically* atoroidal. But they are not *algebraically* as there is an essential singular 2–torus over a suitable figure eight drawn in the base space. (See [**Wal**$_5$] and the singular torus theorems of [**Feu**$_1$**, Feu**$_2$**, Joh**$_2$**, JacS**$_2$**, Sco**$_1$**, Sco**$_2$].)

REMARK 2.3. A knot K in S^3 is pairwise irreducible whenever its complement $X = S^3 - K$ is irreducible and atoroidal, except when K is the knot .

PROOF OF REMARK 2.3. Suppose S is a 2–sphere that cuts K transversely in 2 points, and does not bound a 3–ball B in S^3 with $(B, K \cap B) \cong (B^3, B^1)$. Both points of $K \cap S$ lie on a single component K_1 of K, since the intersection number of any circle with S is zero. Now the regular neighbourhood N of $K_1 \cup S$ has two 2–tori as boundary. Using the property that $S^3 - K$ is atoroidal, one easily reconstructs the pair (S^3, K) and checks that K must be . □

The version of the characteristic submanifold theorem of Johannson and Jaco-Shalen [**Joh**$_1$**, JacS**$_1$**, Joh**$_2$**, JacS**$_2$] that ignores the manifold boundary can be stated as follows. For us the manifold X involved will usually be $int(M) - K$ where (M, K) is a knot pair.

THEOREM 2.4 (Characteristic Torus Decomposition). *Let X be an irreducible orientable 3–manifold isomorphic to the interior of a compact manifold. Up to isotopy in X, there exists a unique compact surface T in X satisfying the following properties:*

(a) *Each component of T is a non-trivial 2–torus in X.*
(b) *Each component V of $X - T$ is either a Seifert fibre space (namely has a foliation by circles, with finite holonomy; see* §A.1 *in the Appendix) or else is atoroidal.*
(c) *If any component of T is deleted, Property* (b) *fails.* □

The surface T of Theorem 2.4 is the ***characteristic* 2*–torus family*** of X.

The uniqueness of T entails that it is preserved up to isotopy by every automorphism of X. This property justifies the name "characteristic". (Recall that, in an abstract group, a subgroup is called characteristic when it is preserved by every automorphism of the group.)

W. Thurston has proved the astounding theorem that whenever an atoroidal component V in (b) is not a Seifert fibre space, it then has a complete hyperbolic metric of finite volume, assuming that V is a Haken manifold; see [**Thu**$_2$**, Mor**$_1$]. The last assumption is conjecturally unnecessary[1].

As impressive as it is, Theorem 2.4 has no impact on the existing tabulations of connected knots. Indeed, the two simplest connected prime knots with a non

[1](Added 2009) This Geometrisation Conjecture is now a theorem, by work of R.D. Hamilton, G.Y. Perelman and others; see [**Pere**$_1$**, Pere**$_2$**, Pere**$_3$**, KleiL, CaoZ, ChoK, ChoLN, ChoEtAl, MorT**$_1$**, MorT**$_2$].

trivial torus in their complement are the cables  and  of the trefoil; they seem to have no projection with < 13 crossings. Readers interested mostly in the tabulated connected knots thus have good reason to skip on to Chapter 3.

On the other hand, there is a generous handful of links of $\leqslant 10$ crossings with a non-trivial torus in their complement. They are drawn in Figure 2.2) with either weighted tree notations for them from Chapter 12 or non-graphical notations from Conway [**Conw**].

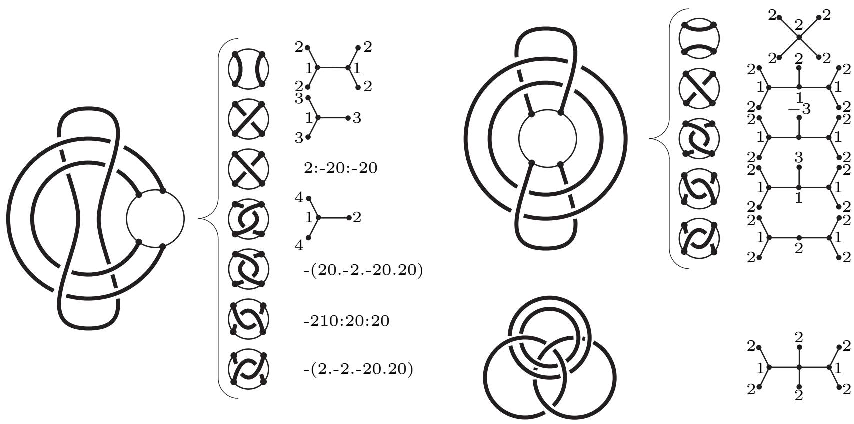


FIGURE 2.2.

It is unsettling to see such varied notations for links that are so similar. One practical remedy for this is to factorise such links into their canonical companions as will be described below. It then becomes no more necessary to tabulate such "composite" links than it is to tabulate links that are non-trivial connected sums of others.

We now introduce elementary *splitting* and *splicing constructions*, which convert Theorem 2.4 into a *unique factorisation* of any knot pair (M, K) such that $M - K$ is irreducible and $H_1(M; \mathbb{Q}) = 0$. This factorisation will split (M, K) into knot pairs called its *characteristic companions*[2].

2.2. The splitting construction

Suppose that we are given a knot pair (M, K), together with a surface $T \subset M - K$ whose components $T_1, \ldots, T_r$ are all 2–tori. At this point, M can be any compact 3–manifold, possibly with boundary, but must satisfy the crucial hypothesis that $H_1(M, \mathbb{Q}) = 0$. Note that this implies that all boundary components of M are 2–spheres. The components of K can be either circles or intervals.

The closure N_j of any component of $M - T$ gives rise to a knot pair (M_j, K_j) as follows. We simply plug every (torus) component of $\partial N_j - \partial M$ with a solid torus. Clearly the way we glue on the solid tori to N_j matters but, by good fortune, our hypothesis that $H_1(M, \mathbb{Q}) = 0$ provides a canonical way to do the gluing (up to automorphism of the solid tori). Namely, we insist that the meridian of each solid torus glued on be null-homologous in $M - N_j$ (when pushed outwards). This

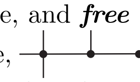
[2](Added 2009) See also [**Bud**$_1$, **Bud**$_2$] for a more recent account.

meridian is determined up to sign since, by an exact homology sequence calculation, *both* closed-up components of the complement of any 2–torus T_i in M_+ behave *homologically* like solid tori, where M_+ is the closed 3-manifold obtained from M by plugging all boundary components with 3–balls.

The knot K_j in M_j consists of old components $K \cap N_j$, plus new components that are the cores of the solid tori $M_j - \text{int}(N_j)$.

The knot pairs (M_j, K_j) deriving from the closed-up components N_j of $M - T$ are called the ***companions*** of (M, K) for the torus family T.

The way the closed-up components N_j of $M - K$ fit together in M is recorded by the ***companionship tree*** Γ for the torus family T, defined as follows: The vertices v_j correspond to the closed-up components N_j of $M - T$, and the edges e_i are associated to the components T_i of T. The edge e_i joins the vertices v_j and v_k when the corresponding component T_i of T separates the closed-up components N_j and N_k of $M - T$. Moreover, each vertex v_j has one ***free bond*** for each closed component of $K \cap N_j$, namely a germ of edge emanating from v_j but leading to no other vertex. The fact that this graph is a tree, is a consequence of $H_1(M, \mathbb{Q}) = 0$.

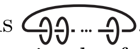
Note that our trees are not exactly 1–dimensional simplicial complexes inasmuch as the vertices have ***bonds***, namely "germs of edges", that may or may not lead to another vertex. The reader can readily formalise this notion of graph. A bond of such a graph is ***tied*** when it is contained in an edge, and ***free*** otherwise. The ***valence*** of a vertex is its number of bonds. For example,  has three vertices (with respective valences 4, 3 and 1), two edges and four free bonds. We do not symmetrically allow free edges; namely each edge ties precisely two bonds.

Thus, each vertex v_j of our companionship tree has one free bond for each circle component of $K \cap N_j$, and one tied bond for each torus component of ∂N_j, or equivalently for each new component of the knot K_j in the companion knot pair (M_j, K_j).

For an edge e_i joining vertices v_j and v_k, there is a preferred isomorphism $K_{ji} \cong K_{ki}$ (up to isotopy) between the knot components associated to the bonds tied by e_i, namely between the new knot components $K_{ji} \subset K_j$ and $K_{ki} \subset K_k$ arising from T_i in the above splitting construction. This isomorphism is the one making string orientations correspond so that in the algebraic linking number $\ell(K_{ji}, K_{ki}) \in \mathbb{Z}$ in M is equal to $+1$ when K_{ji} and K_{ki} are pushed near T_i, so that they can be considered as simultaneously lying in M. A short-hand way to specify this identification is to draw arrows coherently on the circles K_{ji} and K_{ki}.

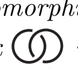
In practice, one should draw the new components K_{ji} and K_{ki} very near T_i in M, each null-homologous on its side of T_i in $M - T_i$. Then orient both curves so that $\ell(K_{ji}, K_{ki}) = +1$ when we push each to the opposite side of T_i.

We are most interested in this splitting construction in the case where T is the characteristic 2–torus family of Theorem 2.4 in the complement $X = M - K$ of a knot pair (M, K). In this situation, the knot pairs (M_j, K_j) are the ***characteristic companions*** of (M, K), and Γ is the ***characteristic companionship tree*** for (M, K).

Then, for each j, the complement $M_j - K_j$ of the knot pair (M_j, K_j) is irreducible and is either atoroidal or Seifert fibred (or both). Under the additional hypothesis that no component of ∂M meets the knot K in 1 or 2 points, each companion (M_j, K_j) for which $M_j - K_j$ is Seifert fibred has empty boundary $\partial M_j = \varnothing$, and the fibration can be chosen to a extend to a Seifert fibration of M_j in which each component of K_j is a circle fibre. Note that a component of K_j may be an ***infinitely singular*** fibre, for which nearby fibres are meridians .

We summarise the properties of this characteristic splitting in the following statement.

LEMMA 2.5. *Let (M, K) be an irreducible knot pair with $H_1(M, \mathbb{Q}) = 0$, and such that no component of ∂M meets the knot K in 1 or 2 points. Then its characteristic companionship tree Γ and its characteristic companion pairs (M_j, K_j) are such that:*

(i) *(M_j, K_j) is irreducible, and is either atoroidal or Seifert fibred;*
(ii) *(M_j, K_j) is not isomorphic to the knot pair formed by the trivial knot*  *or by the Hopf link*  *in S^3, unless the splitting is trivial in the sense that $(M_j, K_j) = (M, K)$ and Γ is reduced to the vertex v_j;*
(iii) *if the edge e_i joins the vertices v_j and v_k and if the corresponding knot pairs (M_j, K_j) and (M_k, K_k) are Seifert fibred, the Seifert fibrations cannot be chosen so that they induce fibrations on the boundary tori ∂V_{ji} and ∂V_{ki} that match under the gluing map φ_{jk}.*

Conversely, if the companions and companion tree associated to a family $T \subset M - K$ of tori satisfies the above three conditions, then T is the characteristic family of Theorem 2.4.

PROOF. Condition (i) is identical to Condition (b) of Theorem 2.4. Condition (ii) comes from the fact that every torus in the characteristic family T is non-trivial, and that no two components of T can be parallel by the Minimality Condition (c) of Theorem 2.4. The same Minimality Condition (c) clearly implies Condition (iii).

For the converse statement, the only thing to check is that Conditions (ii)-(iii) imply the Minimality Condition (c) of Theorem 2.4. This follows from the classical property that, for any nontrivial torus in a Seifert manifold, the Seifert fibration can always be chosen so that the torus is a union of fibres; see [**Wal**$_2$, Satz 2.8]. □

We are leaving as an exercise to the reader the task of translating this third condition (iii) of Lemma 2.5 in terms of Seifert invariants of the Seifert fibration; compare the Appendix.

We are mostly interested in knot pairs (M, K) that are contained in the 3–sphere S^3. Since its boundary must consist of 2–spheres, M is then isomorphic to the closure of the complement of finitely many disjoint balls in S^3. Quite satisfactorily, the companions (M_j, K_j) provided by our splitting construction are also embeddable in S^3.

LEMMA 2.6. *Let (M_1, K_1), (M_2, K_2), ..., (M_r, K_r) be the characteristic companions of the irreducible knot pair (M, K) with $H_1(M, \mathbb{Q}) = 0$. If M is embeddable in the 3–sphere S^3, then so are the M_j.*

PROOF. Consider such a companion (M_j, K_j), associated to the closed-up component N_j of $M - T$, where T is the characteristic 2–torus family of Theorem 2.4. The boundary spheres are clearly irrelevant here, so it is convenient to plug all sphere boundary components of M, N_j and M_j with 3–balls, which gives 3–manifolds $M^+ \cong S^3$, N_j^+ and M_j^+, respectively. We can disregard the knots K and K_j.

Applying Sublemma 2.7 below to $N = N_i^+ \subset M^+ \cong S^3$ then shows that M_j is contained in $M_j^+ = \widehat{N} \cong S^3$. □

Sublemma 2.7. *Let N be a connected 3–submanifold of S^3 whose boundary consists of tori, and let $\widehat{N}$ be obtained from N by gluing a solid torus V_i along each component T_i of ∂N, in such a way that the meridian curve of V_i in $\partial V_i = T_i$ is homologous to 0 in $S^3 - \mathrm{int}(N)$. Then $\widehat{N}$ is isomorphic to S^3.*

Proof of Sublemma 2.7. We proceed by induction on the number of components of ∂N, starting the induction with the trivial case where $\partial N = \varnothing$.

Assume that the property is proved for every submanifold of S^3 with fewer boundary components than N, and that N has at least one boundary component T_1. By [**Ale**], at least one of the two closed-up components of $S^3 - T_1$ is a solid torus U_1.

If $\mathrm{int}(U_1)$ is disjoint from N, the solid tori U_1 and V_1 have the same meridian curves in T_1, and we can therefore arrange that $V_1 = U_1$. Then $\widehat{N}$ is obtained from $N' = N \cup U_1 \subset S^3$ by gluing solid tori along its boundary components, one fewer than for N. By induction hypothesis, it follows that $\widehat{N} \cong S^3$.

If $\mathrm{int}(U_1)$ meets N, then N is contained in the solid torus U_1. Choose a different embedding $\varphi\colon U_1 \to S^3$ with image an unknotted solid torus $\varphi(U_1)$, for which the closure U_1' of $S^3 - \varphi(U_1)$ is a solid torus. In addition, the embedding can be chosen so that it sends a longitude of U_1, namely a curve in T_1 that is homologous to 0 in $S^3 - \mathrm{int}(U_1)$, to a longitude of $\varphi(U_1)$. We then conclude by applying the previous case to $\varphi(N)$ instead of N. □

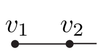
In particular, the characteristic companions of a knot (S^3, K) in S^3 are knots (S^3, K_j).

In §2.4 we will present an inverse construction, called *splicing*, for the splitting construction that we just described. However, we first give a few examples of companions and companionship trees.

2.3. Examples of characteristic companions and companionship trees

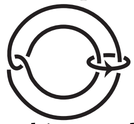
In all these examples, we will ask the reader to believe us when we claim that, for each of the companion pairs (M_j, K_j) indicated, the complement $M_j - K_j$ is either atoroidal or Seifert fibred. These claims can be *a posteriori* justified using the results of later chapters, in particular Chapters 4, 5 and 8.

Example 2.8. The characteristic companionship tree of the doubled trefoil knot 

is , where v_1 and v_2 respectively correspond to  and .

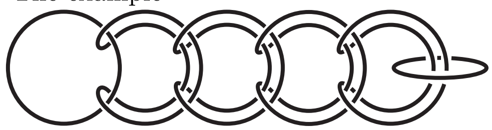
The 6 negative half twists seen on the band of the second knot are worth explaining. The general reason for them is clear: The boundary of a Seifert surface for the trefoil knot seems to twist -6 times compared with a horizontal band: . Note that this is also -2 times the linking number in S^3 of the two edges of this horizontal band. This linking number is quickly calculated for *any* projection of a connected knot, as follows: orient the knot and add up the signs of the crossings: $+1$ for , -1 for .

EXAMPLE 2.9. The characteristic companions of

 are  and 

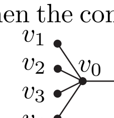
and the characteristic companionship tree has the form .

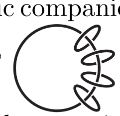
EXAMPLE 2.10. The example



was suggested by Pierre Soury [**Sou₁, Sou₂, Sou₃**]. It has characteristic companionship tree  where each vertex corresponds to a copy of the Borromean rings

In fact, Soury was using companionship trees to create many links with unknotted components. He was probably the first to take this sort of tree seriously. The Lacan school of psychoanalysis was contemplating links as paradigms of complex relationships; see in particular the years 1973-80 of J. Lacan's seminar [**Lac₁, Lac₂, Lac₃, Lac₄, Lac₅, Lac₆, Lac₇**].

EXAMPLE 2.11. Let (S^3, K_1), (S^3, K_2), (S^3, K_3) and (S^3, K_4) be four connected knots, each with $S^3 - K_i$ atoroidal and not Seifert fibred. Then the connected sum $(S^3, K_1\#K_2\#K_3\#K_4)$ has characteristic companionship tree 

where the vertex v_0 corresponds to the "key chain"  with the free bond of v_0 associated to the big circle, and where the other v_i with $i > 0$ each correspond to (S^3, K_i). See also Remark 2.12.

As a further exercise, the reader can give the characteristic companions of the links in Figure 2.2, and observe that they suffice to distinguish these links.

2.4. The splicing construction

It should be clear that the companionship tree for a family T of 2–tori in the complement of a knot pair (M, K) with $H_1(M, \mathbb{Q}) = 0$ has been loaded with so much information that the knot pair (M, K) and the torus family T in $M - K$ can be recovered from it. To avoid any misunderstanding, we make quite explicit what the starting materials are for this inverse construction called ***splicing***.

The splicing construction starts with any finite collection of knot pairs (M_j, K_j) with $H_1(M_j, \mathbb{Q}) = 0$, together with a tree Γ where each vertex v_j is associated to one of these knot pairs (M_j, K_j), and where each bond of v_j is associated to a circle component of K_j. In addition, for every edge e_i binding a bond of the vertex v_j to a bond of a vertex v_k, we are given an identification between the components $K_{ji} \subset K_j$ and $K_{ki} \subset K_k$ associated to these bonds. This identification $K_{ji} \cong K_{ki}$

is only defined up to isotopy and, in practice, specified by matching orientations on K_{ji} and K_{ki}.

For each vertex v_j of Γ, let N_j be obtained from M_j by removing the interior of a regular neighbourhood V_{ji} of each component K_{ji} of K_i that corresponds to a tied bond of v_j. The boundary torus ∂V_{ji} comes with a preferred meridian μ_{ji} and longitude λ_{ji}, respectively generating the kernels of the homomorphisms $H_1(\partial V_{ji}) \to H_1(V_{ji})$ and $H_1(\partial V_{ji}) \to H_1\big(M - \mathrm{int}(V_{ji})\big)$. Choose consistent orientations on μ_{ji} and λ_{ji} so that their algebraic intersection number is $\mu_{ji} \cdot \lambda_{ji} = +1$ on ∂V_{ji}, oriented as boundary of V_{ji}; there are two possible choices for such orientations, which both provide the same identification $\mu_{ji} \cong \lambda_{ji}$ up to isotopy. In addition, by projection in V_{ji}, the longitude λ_{ji} has a natural identification with the core K_{ji} of V_{ji}, well-defined up to isotopy.

Now, suppose that the edge e_i ties two bonds of v_j and v_k, respectively associated to the components $K_{ji} \subset K_j$ and $K_{ki} \subset K_k$. We then have identifications $\mu_{ji} \cong \lambda_{ji} \cong K_{ji} \cong K_{ki} \cong \lambda_{ki} \cong \mu_{ki}$, where the middle isomorphism is part of the splicing data. By projection, we also have natural isomorphisms $\partial V_{ji} \cong \mu_{ji} \times \lambda_{ji}$ and $\partial V_{ki} \cong \mu_{ki} \times \lambda_{ki}$. We can then glue the manifolds N_j and N_k along their boundary components $\partial V_{ji} \cong \mu_{ji} \times \lambda_{ji}$ and $\partial V_{ki} \cong \mu_{ki} \times \lambda_{ki}$ by the gluing map $\varphi\colon \partial V_{ji} \to \partial V_{ki}$ which, for the above identifications $\mu_{ji} \cong \lambda_{ji} \cong \lambda_{ki} \cong \mu_{ki}$, translates as the map $(x, y) \mapsto (y, x)$. Note that φ is orientation-reversing, so that the orientations of N_j and N_k match to give an orientation on the glued manifold.

Performing this gluing operation for all the edges e_i of Γ, the manifolds N_j are now glued together to form a 3–manifold M. Let K be the image in M of the union of the $N_j \cap K_j$. We now have a knot pair (M, K). We already observed that M is oriented by the orientations of the M_j. An immediate computation shows that $H_1(M, \mathbb{Q}) = 0$.

The splicing construction is specially designed to be the inverse of the splitting construction of §2.2, provided the knot pairs (M_j, K_j) satisfy the following additional conditions of Lemma 2.5.

Remark 2.12 (Compare Example 2.11). Let the splicing tree Γ be  where the knot pair associated to the central vertex is the link  in S^3, with the two components marked with arrows corresponding to the two tied bonds, and where the other two vertices correspond to connected knots (S^3, K_1) and $S^3, K_2)$, respectively oriented by arrows and . The resulting spliced knot (S^3, K) then is none other than the pairwise connected sum $K_1 \# K_2$ in S^3, defined using the orientations indicated:

K_1 K_2 $K_1 \# K_2$

This example illustrates the need to specify the isomorphisms $K_{ij} \cong K_{ki}$ in the splicing data. Indeed it is well known that the pairwise connected sum of two connected knots depends on the choice of matching orientations on each knot, at least for non-invertible knots such as the knot 8_{17} of Examples 16.20 and 18.11. Recall that a knot (S^3, K) is ***invertible*** if there exists a degree $+1$ automorphism of (S^3, K) which reverses the orientation of K.

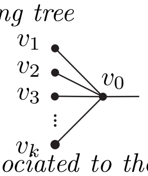
Remark 2.13. On the other hand, the simplest knots have so much symmetry that very often, for splicing to be well-defined, the arrows and even the specification of the bound components are immaterial. Thus the link of Example 2.9 is unambiguously obtained by splicing one component of its first companion to one component of the second companion, with no need to specify which components and identifications. Indeed, for each of its two companions (S^3, K_1) and (S^3, K_2), there exists a degree $+1$ isomorphism of (S^3, K_i) exchanging its two components, and another one reversing the orientation of each component.

When M is embeddable in S^3, we saw in Lemma 2.6 that the splitting construction applied to the knot pair (M, K) provides knot pairs (M_j, K_j) where each M_j is embeddable in S^3. However, the converse is far from being true. For instance, we can take the crude example where the companionship tree is •——•, and where the companions are non-trivial connected knots in S^3. Then, in the knot pair $(M, \varnothing)$ provided by the splicing construction, the manifold M is a $\mathbb{Z}$–homology sphere but is not a genuine 3–sphere since the gluing torus T is an incompressible 2–torus in it.

Thus we have a technical problem to solve: Given a splicing construction presented by a tree Γ, such that each vertex knot pair (M_j, K_j) is embeddable in S^3, give an effective criterion to decide when the resulting knot pair (M, K) has $M \cong S^3$.

A solution is offered by the following statement.

Lemma 2.14. *Consider the splicing tree*

$\Gamma_0 =$ 

where the knot pairs (M_i, K_i) associated to the vertices v_i are such that $M_i \cong S^3$. Let (M, K) be the result of splicing according to this data. In particular, K is connected and can be identified to the component K_0^* of $K_0 \subset M_0$ corresponding to the free bond. Suppose in addition that, for each $i = 1, \ldots, k$, the connected knot (M_i, K_i) is non-trivial. Then:

(1) *M is isomorphic to S^3 precisely if the k circles $K_0 - K_0^*$ bound k disjoint discs in M_0.*
(2) *(M, K) is isomorphic to the trivial knot (S^3, S^1) precisely if the $(k+1)$ circles K_0 bound $(k+1)$ disjoint discs in M_0.*

Proof. The proof is an exercise using the Loop Theorem (see [**Hemp**]) and innermost disc arguments, exploiting the fact that the complement in M_j of a tubular neighbourhood of K_j has an incompressible torus boundary when $i > 0$. □

We now give a recipe based on this lemma to decide whether a general splicing tree Γ yields a pair (M, K) with $M \cong S^3$. It gradually simplifies Γ until a decision is reached.

Algorithm 2.15.

(0) Suppress from each K_j all components that are not tied bonds of Γ. This makes $K = \varnothing$ without changing M. Also, replace each M_i by the closed manifold M_i^+ obtained from M_i by plugging each (sphere) boundary component with a 3–ball. This replaces M by the similarly defined closed

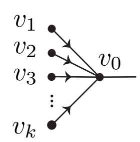
manifold M^+, and reduces the embeddability problem to deciding whether $M^+ \cong S^3$.

(1) Consider the twigs of Γ, namely the univalent vertices •—, and mark with an arrow •→ those in which the bond corresponds to a knotted curve.

(2) If Γ is one of •, •——• or •→—•, then $M^+ \cong S^3$ and M is embeddable in S^3. If Γ is •→←•, then $M^+ \not\cong S^3$ and M is not embeddable in S^3. In any other case we suppress the twigs •— lacking arrows, without affecting M.

After (0), we apply (1) and (2) as long as possible. When none apply, and we still do not know whether $M^+ \cong S^3$, we can go on to (3) below.

(3) Consider a subgraph

$\Gamma_0 =$ 

of Γ such that, for every vertex $v_j \in \Gamma_0$, each bond of v_j in Γ also appears in Γ_0. Apply Lemma 2.14 to Γ_0, with the M_i replaced by M_i^+. If, in the knot pair (M_0, K_0) associated to the central vertex v_0 of Γ_0, the k components of $K_0 - K_0^*$ do not bound k disjoint disks in M_0, then $M \not\cong S^3$ and M is not embeddable in S^3. If, at the other extreme, the $k+1$ components of K_0 bound $k+1$ disjoint disks in M_0, replace Γ_0 by •— in Γ and return to (2). Otherwise, replace Γ_0 by •→ and return to (2).

Note that the only information needed to apply Algorithm 2.15 is that we know which subknots of the knot pairs (M_j, K_j) associated to the vertices of Γ are trivial. (A ***subknot*** is obtained by deleting some, namely $\geqslant 0$, components of K_j; ***trivial*** means bounding disjoint discs in M_j.)

2.5. Conclusion

The chief purpose of this chapter has been to reduce the classification of arbitrary knots in S^3 to the study of those knots (S^3, K) that are ***simple for Schubert***, in the sense that (S^3, K) is irreducible and (geometrically) atoroidal. Let us recapitulate how the foregoing accomplishes this.

As a first easy step, the Kneser connected sum theory showed that the classification of arbitrary knots in S^3 is equivalent to the classification of knots (S^3, K) such that $S^3 - K$ is irreducible.

Now consider two knots (S^3, K) and (S^3, K') of this sort. For them, we have defined characteristic companionship trees Γ and Γ', whose vertices correspond to characteristic companion knots (S^3, K_i) and $(S^3, K'_{i'})$, respectively, which each are either simple for Schubert or Seifert fibred (or both), and whose edges specify identifications between some components of these knots. More precisely, an edge e_k joining the vertices associated to (S^3, K_i) and (S^3, K_j) in Γ specifies an isomorphism $\theta_k\colon K_{ik} \to K_{jk}$ of a circle in K_i to a circle in K_j, this θ_k being specified up to isotopy; similarly in Γ'.

Theorem 2.4, and its corollary the uniqueness of the splitting construction of §2.2, show that $(S^3, K) \cong (S^3, K')$ with degree $+1$, precisely if there exists an isomorphism

$$f\colon \coprod_i (S^3, K_i) \longrightarrow \coprod_{i'} (S^3, K'_{i'})$$

of degree +1 on each 3–sphere, making the string identifications specified by the edges of Γ and Γ' match in the following sense: For each edge e_k of Γ specifying an identification θ_k between circle components L_1 and L_2 of two distinct K_i, there is an edge $e'_{k'}$ of Γ' that specifies an identification $\theta'_{k'}$ between $f(L_1)$ and $f(L_2)$, and $\theta'_{k'} \circ f_{|L_1} = f_{|L_2} \circ \theta_k$ up to isotopy.

Thus to decide whether (S^3, K) is degree +1 isomorphic to (S^3, K') it suffices to know, for every pair of vertices $v_i \in \Gamma$ and $v'_{i'} \in \Gamma'$:

(a) whether there is a degree +1 pair isomorphism $(S^3, K_i) \to (S^3, K_{i'})$ of the characteristic companions respectively associated to v_i and $v_{i'}$ and, if so,

(b) which isomorphisms $K_i \to K'_{i'}$ of 1–manifolds are realised by such pair isomorphisms.

Note that, given an answer to (a), Question (b) amounts to knowing something about the automorphisms of (S^3, K_i), namely their effect on the components of K_i. Also, from a practical point of view, one does not need to consider all pairs $\{v_i, v'_{i'}\}$, but only those where the two vertices are in correspondence by a combinatorial isomorphism between the trees Γ and Γ'.

If (S^3, K_i) is Seifert fibred, Waldhausen classification of Seifert manifolds [**Wal**$_2$] tells how to answer (a) and (b) systematically. If (S^3, K_i) is not Seifert fibred, it is simple for Schubert and, in case (S^3, K_i) appear in knot and link tables along with their symmetry groups, then questions (a) and (b) are answered.

Thus the question of isomorphism of the two knots (S^3, K) and (S^3, K') can be settled by studying the isomorphism and symmetry problems posed by their characteristic companions. These are exactly the problems that we address in following chapters.

CHAPTER 3

The arborescent part of a knot

In the preceding chapter, we explained how the study of an arbitrary knot in S^3 can be reduced to the study of knots (S^3, K) that are simple for Schubert, in the sense that $S^3 - K$ is irreducible and (geometrically) atoroidal.

This section will describe how one can use Conway 2–spheres to cut out certain characteristic parts of such a knot (S^3, K), notably an arborescent part A. This arborescent part A will be a codimension 0 submanifold of S^3 whose boundary consists of Conway 2–spheres; and A will be well-defined up to isotopy of S^3 respecting K. The definition is so designed that A is all of S^3 precisely if the knot (S^3, K) is arborescent as defined in the Introduction or in Chapter 1, namely if it is algebraic in the sense of Conway.

The proof that A is well-defined, which will be given in Chapter 7, is reasonably short. However, further study of the structure of A (see Chapters 9, 11 and following) requires patient geometrical analysis of surfaces (in Chapters 8 and 10). In this chapter, we concentrate on definitions, constructions, examples, and statements of results.

3.1. Characteristic splittings

Consider any knot pair (M, K). By a ***surface*** F in the knot pair (M, K), we mean a proper 2–submanifold of M, which cuts K nicely in the sense that $\partial F \cap K = \varnothing$ and that the intersections of K and F in M are transverse, with local model the z–axis cutting the xy–plane in $\mathbb{R}^3$. Remember that the property that the submanifold F is ***proper*** means that $F \cap \partial M = \partial F$.

A ***boundary surface*** F in the knot pair (M, K) is a codimension 0 submanifold of ∂M such that $K \cap \partial F = \varnothing$.

Let F be a surface or a boundary surface in the knot pair (M, K). Then a ***pairwise compression*** 2–***disc*** for F in (M, K) is a 2–disc $D \subset M$ with $\partial D = (F - K) \cap D$ so that D meets K in $\leqslant 1$ point, transversely. We say that D is ***ineffective*** or ***futile*** if ∂D is the boundary of a 2–disc $D' \subset F$ meeting K in as many points as does D. Otherwise, D is ***effective***.

The surface F is ***pairwise incompressible*** in (M, K) if:

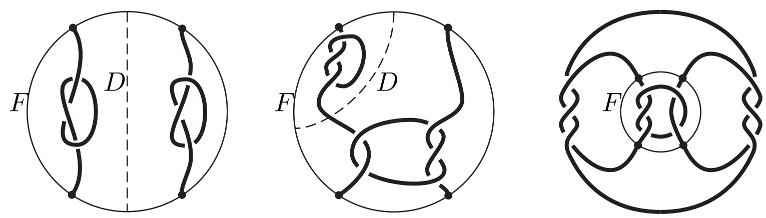
(1) there exists no effective compression disc for F in (M, K);
(2) no component of F is the boundary of a 3–ball B in M such that $K \cap B$ is empty or is an arc unknotted in B.

One easily sees that *a surface is pairwise incompressible if and only if each of its components is pairwise incompressible*. This only requires an innermost circle argument for Condition (1), and compare Lemma 7.2 for Condition (2).

An argument of (Kneser and) Haken provides us with the following:

Parallelism Principle. *For any pairwise irreducible knot pair (M, K) there exists an integer N such that, if F is a closed pairwise incompressible surface in (M, K) with $\geqslant N$ components, then at least two of its components are* ***pairwise parallel****, namely separated by a closed-up component of $(M, K) - F$ which is a product with the interval, respecting K.*

The reader will find a proof of this result in [**Hemp**, p.140] for the particular case when $K = \varnothing$, which readily extends to the general case.



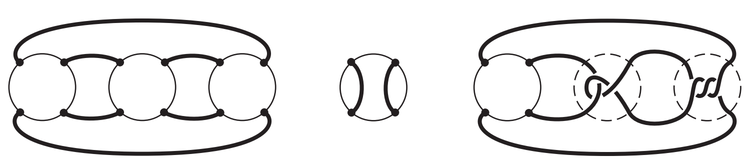
(a) (b) (c)

Figure 3.1.

In a knot (S^3, K), we now consider pairwise incompressible ***Conway spheres***, namely 2–spheres cutting K transversely in four points. For example, in Figures 3.1(a) and 3.1(b) the Conway sphere F is pairwise compressible by a dotted 2–disc D. We will prove in Chapter 8 that the Conway sphere F of Figure 3.1(c) is pairwise incompressible.

Suppose now that $K \subset S^3$ is a knot which is simple for Schubert. According to Haken's Parallelism Principle mentioned above, there exists a maximal finite collection $F_1, \ldots, F_n$ of disjoint pairwise incompressible Conway spheres in S^3 no two of which are pairwise parallel. This maximal family will of course be empty if (S^3, K) contains no pairwise incompressible Conway sphere.

Let N be the closure in S^3 of a component of $S^3 - \bigcup_i F_i$. We call N ***elementary*** if the knot pair $(N, K \cap N)$ is obtained from the pair of Figure 3.2(a) by plugging 0, 1, 2 or 3 of its three holes with a ***rational tangle pair***, namely a knot pair isomorphic to the pair of Figure 3.2(b). Thus N has 3, 2, 1 or 0 Conway spheres on its boundary. Figure 3.2(c) gives an example of such an elementary N, with one boundary component.



(a) (b) (c)

Figure 3.2.

The ***arborescent part*** A of the knot (S^3, K) is the union of the closed-up components of $S^3 - \bigcup_i F_i$ that are elementary.

THEOREM 3.1 (proved in Chapter 7). *For a knot (S^3, K) which is simple for Schubert, the arborescent part A so constructed is well-defined in the sense that, up to isotopy of S^3 respecting K, it is independent of the family F we started with.*

As a consequence, the arborescent part A is invariant under all automorphisms of (S^3, K), up to isotopy respecting K. Namely A is *characteristic* for (S^3, K).

The reader should be warned that the surface $F = \bigcup_i F_i$ itself is in general not characteristic in (S^3, K), as the 90°–rotation θ in Figure 3.3 shows.

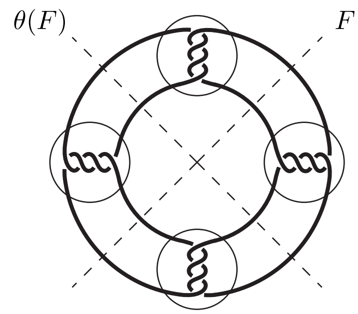


FIGURE 3.3.

We will be able to give a very satisfactory analysis of the arborescent part $(A, K \cap A)$ of the knot (S^3, K) in terms of integrally weighted planar trees in Chapters 12–13. The non-arborescent part $A^* = S^3 - \mathrm{int}(A)$ of the knot is much more mysterious; we shall begin to study $(A^*, K \cap A^*)$ in Chapters 4–6 using hyperbolic geometry.

The first step in studying $(A, K \cap A)$ and the complementary pair $(A^*, K \cap A^*)$ is to extract from F a characteristic family of Conway spheres in it. For A^* the result is optimal.

THEOREM 3.2 (proved in Chapter 7). *The surface $F \cap A^*$ is well-defined in $(A^*, K \cap A^*)$, up to isotopy respecting K.*

By a more lengthy argument, we will show in Chapter 9, that some of the components of $F \cap A$ form a collection of Conway 2–spheres which is characteristic in $(A, K \cap A)$. We now formulate this result so as to include Theorems 3.1 and 3.2.

Let a ***Montesinos pair*** be a knot pair built from a pair of the type shown in Figures 3.4(a) or 3.4(b) (with any number of holes), by plugging some of the holes with rational tangle pairs. In particular, every elementary pair is also a Montesinos pair. Conversely, observe that a Montesinos pair can be split into elementary pairs along a family of Conway spheres (consider in particular the sphere Σ of Figure 3.4(b)) and, as a consequence, is arborescent.

We will extend this definition of Montesinos pairs in Chapter 8 to allow more circle components in the pair of Figure 3.4(b), but this has no influence on the present discussion as we restrict attention to knots which are simple for Schubert. It was J. Montesinos who identified in [**Mon**$_1$] the 2–fold branched coverings of these knot pairs as Seifert fibre spaces, and the covering translation as a fibre preserving automorphism giving a certain reflection of the base; see also [**Mon**$_2$] and the Appendix.

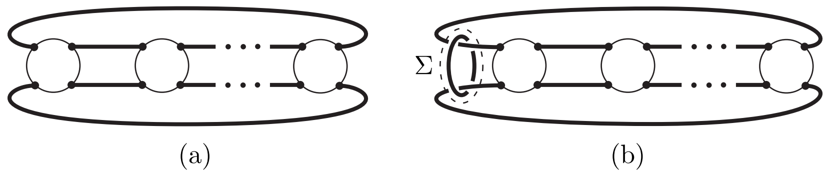


FIGURE 3.4.

From the collection of Conway spheres $F_1, \ldots, F_n$ in (S^3, K) delete, and continue to delete, members in the interior of A as long as the remaining F_i split A into Montesinos pairs.

THEOREM 3.3 (proved in Chapter 9). *When this process of deletion can proceed no further, the union G of the remaining Conway 2–spheres among $F_1, \ldots, F_n$ is well-defined by (S^3, K), up to isotopy of S^3 respecting K.*

By maximality of F, each closed-up component (M, L) of $(S^3, K) - F$ has the following property: Any pairwise incompressible Conway sphere in (M, L) is pairwise parallel to a boundary component. By analogy with the term "simple for Schubert" introduced in Chapter 2, we will say that any knot pair (M, L) with the above property is ***simple for Conway***.

Then, summarising the properties of G, we can rephrase Theorem 3.3 in the following form, which is readily seen to be equivalent.

THEOREM 3.4. *Given a knot (S^3, K) that is simple for Schubert there is, up to isotopy respecting K, a unique surface $G \subset S^3$ such that:*

(a) *the components of G are pairwise incompressible Conway spheres, no two of which are pairwise parallel;*
(b) *each closed-up component N of $S^3 - G$ gives a pair $(N, K \cap N)$ that either is simple for Conway, or else is a Montesinos pair;*
(c) *when any component is omitted from G, Property* (b) *fails.* □

Theorems 3.3 and 3.4 admit much generalisation. In [**BonS**$_1$], we have worked out one generalisation, that is a characteristic submanifold theorem for compact 3–orbifolds. It includes Theorems 3.3 and 3.4 by associating to each knot pair (M, K) the (unique) 3–orbifold with underlying topological space M and with K as singular set, where the isotropy group is generated at each point by a rotation of π; see [**Thu**$_1$, Chapter 13] [**BonS**$_1$**, BonS**$_2$] for the language of orbifolds.

Each *open* component $(N, K \cap N)$ of $(S^3, K) - G$ that is in the non-arborescent part A^* admits a fascinating hyperbolic structure revealed by Thurston's Orbifold Geometrisation Theorem, applied to the orbifold associated to $(N, K \cap N)$ as above (see §5.3 for more discussion). Indeed, Thurston proves that N can be endowed with what we call a *π–hyperbolic structure*, namely a complete metric space structure on N making the pair $(N, K \cap N)$ locally isometric to the pair $(\mathbb{H}^3/\rho, \gamma/\rho)$ where ρ is a π–rotation around a geodesic γ of the hyperbolic 3–space $\mathbb{H}^3$. Such a π–hyperbolic structure also amounts to a complete Riemannian metric of constant curvature -1 on the double branched covering $\widetilde{N}$ of the pair $(N, K \cap N)$ for which the covering involution is an isometry. (This $\widetilde{N}$ will have as many cusps as N has Conway spheres in its frontier.)

We will discuss these π–hyperbolic structures in more detail in Chapter 5, and will give several examples in Chapter 4. One cardinal fact is that such a π–hyperbolic structure on the pair $(N, K \cap N)$ is unique up to isotopy of N respecting $K \cap N$. Thus, any geometric invariant of this metric turns out to be a *topological* invariant of the pair $(N, K\cap N)$, and consequently of the knot (S^3, K). For instance, the (finite) volume of the hyperbolic manifold $\widetilde{N}$ gives such an invariant. See also the characteristic markings we will define in Chapters 17. Other applications of these π–hyperbolic structures will be found in Chapters 5 and 6.

3.2. An example

Let us give an example illustrating Theorems 3.1, 3.3 and 3.4. Justifying the corresponding statements will require results from later chapters so, at this point, we can only direct the reader to the appropriate assertions.

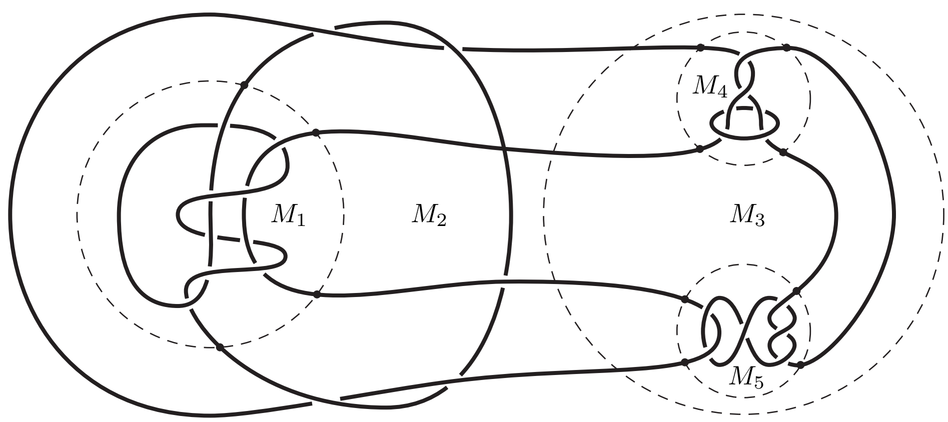
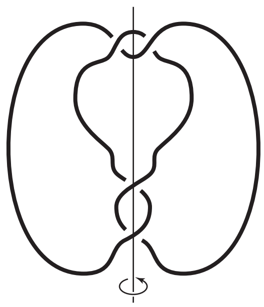
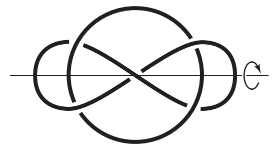


FIGURE 3.5.

Consider the knot (S^3, K) drawn in Figure 3.5. Here, a maximal family F consists of four Conway spheres (dotted in Figure 3.5). The arborescent part A lies in the largest dotted sphere and is the union of three elementary pairs $(M_3, K\cap M_3)$, $(M_4, K \cap M_4)$ and $(M_5, K \cap M_5)$. The surface G of Theorem 3.3 is F minus the boundary of M_5. The non-arborescent part is the union of two (closed-up) π–hyperbolic pieces $(M_1, K\cap M_1)$ and $(M_2, K\cap M_2)$. They were constructed to be π–hyperbolic: Indeed, M_1 is the quotient of the complete hyperbolic figure eight knot complement (see [**Thu**$_1$, Chap. 3]) by the 180° rotation illustrated in Figure 3.6(a), in such a way that $K \cap M_1$ is the projection of the axis of the rotation; similarly, M_2 is the quotient of the (complete hyperbolic) Whitehead link complement by the rotation of Figure 3.6(b). Thurston's construction of hyperbolic metrics on these knot complements readily shows that they can be chosen to be respected by the involutions shown (see also §4.2). In each case, the string arises from the rotation axis; to see that the quotients are the open 3–ball int(B^3) and $S^2\times]0, 1[$, respectively, note that the quotients of the tubular neighbourhoods of the knots are balls.



(a)



(b)

FIGURE 3.6.

To prove that F is actually maximal and that $G = F - \partial M_5$ is the characteristic family of Conway spheres defined by Theorems 3.3 and 3.4, we need to use results from later chapters:

Each component of F is pairwise incompressible and each closed-up component of $(S^3, K) - F$ is simple for Conway, by application of Theorem 5.2 to the π–hyperbolic pieces, and of Theorem 8.15 to the elementary Montesinos pieces. Moreover, no closed-up component of $(S^3, K) - F$ gives a pairwise parallelism between two components of F by Theorem 5.7 and our classification of Montesinos pairs in Chapter 11. This proves that F is really a maximal family of pairwise incompressible Conway spheres.

Further, by Theorems 5.2 and 8.15 again, each closed-up component of $(S^3, K) - F$ is pairwise irreducible, is simple for Schubert and contains no non-trivial annulus avoiding K. By an innermost circle argument, it follows that (S^3, K) is simple for Schubert, and the hypotheses of Theorems 3.1 and 3.4 are consequently satisfied. Also, Theorem 5.7 asserts that the π–hyperbolic pieces M_1 and M_2 are not Montesinos pairs. Thus, the arborescent part of the knot is really the union of M_3, M_4 and M_5 as announced. That G satisfies the Mimimality Condition (c) of Theorem 3.4 is proved by our analysis of this characteristic splitting in Chapter 9, in particular Criterion 9.5, together with our classification of "necklaces" in Montesinos pairs in Chapter 10 (compare also Chapters 12 and 13).

3.3. Practical search for characteristic Conway spheres

Given any knot (S^3, K) that is simple for Schubert, it can be a challenging problem to *find practically and rigorously* the characteristic arborescent part A of Theorem 3.1 and also, in the non-arborescent part A^*, the characteristic surface $F \cap A^*$ of Theorem 3.2, which we have just succeeded doing for one interesting example. (As mentioned above, the problem of finding the characteristic surface $G \cap A$ in A is solved in Chapters 9 or 12–13.)

This problem is easily reduced to the two following steps:

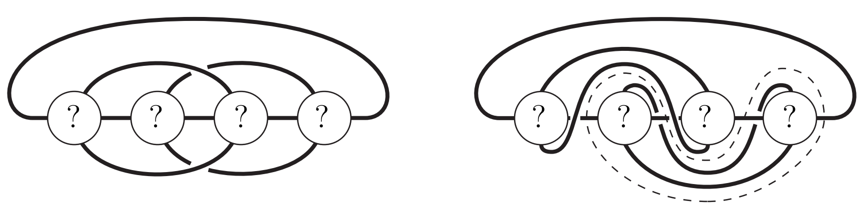
(1) Find a family G' of disjoint pairwise incompressible Conway spheres in (S^3, K) such that each closed-up component of $(S^3, K) - G'$, either is simple for Conway, or is a Montesinos pair.
(2) Given G' as in (1), identify exactly which closed-up components of $(S^3, K) - G'$ are Montesinos pairs.

Certainly, Problem (1) can *in principle* be solved by the kind of algorithm developed by W. Haken in the 1950s; see [**Hak$_3$, Hak$_1$, Hak$_2$, Sch$_4$, SchuS**]. However, his algorithm is still too cumbersome to be very useful in practice, although the techniques of branched surfaces may eventually succeed in making it more workable (see [**FloO, Oer, JacO**]).

It should also be noted that, still at least theoretically, Haken's algorithm can be used to answer Problem (2) *if the family G' found in Problem (1) is non-empty*, namely if (S^3, K) contains a pairwise incompressible Conway sphere. Indeed, we will later prove, in Proposition 7.4 and Theorem 8.15(b) (compare also Propositions 8.14 and 8.20), the following statement: Consider a pairwise irreducible knot pair (N, L) with pairwise incompressible non-empty boundary, such that every pairwise incompressible Conway sphere in (N, L) is pairwise parallel to a boundary component; then (N, L) is a Montesinos pair if and only if it contains a pairwise incompressible surface which is, either an annulus avoiding L, or a disc transversely cutting L in two points, and which is not trivially obtained by pushing inside of N the interior of a subsurface of ∂N. As the latter condition is precisely the kind of statement whose validity can be decided by Haken's algorithm, applying this algorithm to each closed-up component (N, L) of $(S^3, K) - G'$ solves Problem (2).

As indicated above, Haken's algorithm is in general not easy to work out in practice. In the case of alternating knots, there is however a very efficient way to solve Problem (1), which was developed by W. Menasco in [**Men$_1$**] and [**Men$_2$**]. Menasco's algorithm, which was originally inspired by Haken's methods, runs as follows.

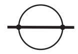
Starting from an alternating knot projection $\Gamma \subset S^2$ describing (S^3, K), Menasco singles out two kinds of Conway spheres in (S^3, K) which can be read directly from Γ; these are called the ***normal Conway spheres*** of type (I) or (II), respectively, associated to the projection Γ. The normal Conway spheres of type (I) are exactly those associated to Conway circles of Γ, as in Chapter 1. The normal Conway spheres of type (II) are less easy to see, and occur when Γ can be decomposed as in Figure 3.7(a); the normal Conway sphere defined (up to pairwise isotopy) by such a decomposition is the one associated to the dotted Conway circle in the other projection of (S^3, K) shown in Figure 3.7(b). (This Conway sphere can also be directly seen on Figure 3.7(a) as a plane parallel to the projection plane.) These normal Conway spheres were already encountered by Conway; see Figure 10 of [**Conw**].



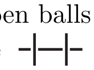
(a) (b)

FIGURE 3.7.

Then, Menasco proves:

THEOREM 3.5 ([**Men**$_1$, **Men**$_2$]). *Let $\Gamma \subset S^2$ be an alternating knot projection which is irreducible, namely such that any circle cutting Γ in two points bounds a vignette* . *Then:*

(a) *The knot (S^3, K) described by Γ is simple for Schubert.*
(b) *Every family of disjoint pairwise incompressible Conway spheres in (S^3, K) can be pairwise isotoped to a family of normal Conway spheres.*
(c) *A normal Conway sphere of type* (II) *is always pairwise incompressible. A normal Conway sphere of type* (I) *is pairwise incompressible if and only if the corresponding Conway circle does not bound a rational tangle projection.*

Actually, Menasco's theorem readily extends to alternating ***knot pair projections***, consisting of a compact surface $S \subset S^2$, of an immersed 1–submanifold $\Gamma \subset S$ with only double point singularities, and of some crossing information at the double points of Γ. Such a knot pair projection readily describes a knot pair (M, K) where M is the complement of disjoint open balls in S^3. A knot pair projection is ***alternating*** when it contains no vignette  nor  .

Given a knot pair projection, one can define normal Conway spheres of type (I) or (II) in the corresponding knot pair in exactly the same way as for a knot projection. Then, the conclusions of Theorem 3.5 still hold if the knot projection $\Gamma \subset S^2$ is replaced by any alternating knot pair projection (S, Γ') where Γ' meets each component of S in an even number of points, and if the knot (S^3, K) is replaced by the knot pair (M, K') determined by (S, Γ'). This extension of Theorem 3.5 is easily checked by a simple scrutiny of Menasco's proofs. When $\partial(S, \Gamma)$ consists of Conway circles, it can also be *deduced* from Theorem 3.5 by enlarging $\Gamma' \subset S$ to a suitable alternating knot projection $\Gamma \subset S^2$ (exercise!).

Menasco's results now readily provide a solution to Problem (1) for alternating knots. Indeed, consider a knot (S^3, K) described by an irreducible alternating projection $\Gamma \subset S^2$. Start with a family C of disjoint Conway circles for Γ that is maximal for the properties that no circle of C bounds a rational tangle projection, and that no two components of C are parallel respecting Γ. The family C defines a family G_1' of normal Conway spheres of type (I). Then, consider the closed-up components of $(S^2, \Gamma) - C$ which are of the type of Figure 3.7(a), where the inside of the bubbles can be empty, and let G_2' be the union of the corresponding normal Conway spheres of type (II). (A closed-up component of $(S^2, \Gamma) - C$ can sometimes be seen in the form of Figure 3.7(a) in several ways, and therefore give several normal Conway spheres of type (II); arbitrarily choose one such normal sphere in this case.)

We claim that $G' = G_1' \cup G_2'$ is a solution to Problem (1) for (S^3, K). Indeed, each component of G' is pairwise incompressible by Theorem 3.5. Also, each closed-up component of $(S^3, K) - G'$ that is adjacent to a type (II) component of G' is a Montesinos pair (see Figure 3.7(b), and note that the bubbles there either are empty or are rational tangle projections). And all other closed-up components of $(S^3, K) - G'$ are simple for Conway by the above mentioned extension of Theorem 3.5 to knot pair projections. Thus, G' has the properties sought in Problem (1).[1]

To conclude this section we should also mention the following consequence of a result of T. Kobayashi [**Kob**]: Assume that the knot (S^3, K) admits a bridge

[1](Added 2009) See also [**MenT**] for Problem (2) for alternating knots.

presentation with n bridges; namely that $K \subset S^3 - \infty = \mathbb{R}^3$ can be isotoped so that the restriction to K of the projection to the first coordinate of $\mathbb{R}^3$ is a Morse function with n maxima (and n minima). Then, the surface G of Theorems 3.3 and 3.4 has at most $3(n-2)$ components. This statement follows by applying Kobayashi's result on Heegaard splittings to the 2–fold branched cover of the knot K.[2]

[2](Added 2009) See also more recent work by M. Scharlemann and J. Schultens [**SchaS**] who, among other things, show that the splitting of Theorem 3.4 can have at most $n-1$ Montesinos pieces.

Part 2

π–hyperbolic structures

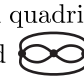
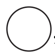
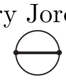
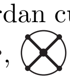
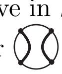
CHAPTER 4

π–hyperbolicity: first examples

Our study, outlined in the last chapter, of pairwise incompressible Conway spheres in a knot (S^3, K) which is simple for Schubert pushes the limit of our understanding into certain characteristic parts $M \subset S^3$ bounded by Conway spheres. The interior of the knot pair $(M, K \cap M)$ admits a complete metric that is π–hyperbolic in the sense that it is locally isometric to $\big(\mathbb{H}^3/\rho, \mathrm{axis}(\rho)/\rho\big)$, where ρ is a rotation of angle π in the hyperbolic 3–space $\mathbb{H}^3$. This chapter is devoted to the description of a few examples of such π–hyperbolic metrics. The reader who is not too familiar with the hyperbolic space $\mathbb{H}^3$ may find it helpful to first read the beginning of §5.1 for an introduction.

4.1. Conway graphs and hyperbolic polyhedra

When considering the arborescent part of a knot pair projection in §1.1, we saw that each closed-up component of its complement is associated to some Conway graph, and defines a knot pair in S^3. We propose to construct π–hyperbolic metrics of the interior of these knot pairs.

Remember from Chapter 1 that a ***Conway graph*** is a connected quadrivalent graph Γ embedded in $S^2 \subset S^3$, distinct from , ,  and , and such that every Jordan curve in S^2 transversely meeting Γ in $\leqslant 4$ points bounds a vignette , ,  or . A Conway graph defines a knot pair projection $(S, \Gamma \cap S)$ where S is the complement of a family of disjoint open vignettes  around the vertices of Γ. We are interested in the knot pair (M, K) described by this knot pair projection. Note that the interior of (M, K) is isomorphic to $(S^3 - V, \Gamma - V)$ where V is the set of vertices of Γ.

Proposition 4.1. *For any Conway graph $\Gamma \subset S^2 \subset S^3$ with set of vertices V, the pair $(S^3 - V, \Gamma - V)$ admits a finite volume π–hyperbolic structure.*

Proof. In [**And$_1$, And$_2$**], E.M. Andreev characterised which combinatorial types and which dihedral angles can be realised by convex polyhedra in the hyperbolic 3–space $\mathbb{H}^3$ with finite volume and with acute angles. In particular, if Γ is a Conway graph in S^2 and if $B^3_+ = (\mathbb{R}^2 \times \mathbb{R}_+) \cup \infty$ is the upper hemisphere of $S^3 = \mathbb{R}^3 \cup \infty$ delimited by $S^2 = \mathbb{R}^2 \cup \infty$, Andreev showed that there is a proper embedding of $B^3_+ - V$ in $\mathbb{H}^3$ such that: Each face, namely each closed-up component of $S^2 - \Gamma$, is sent to a planar hyperbolic polygon with vertices at infinity; each edge, namely each component of $\Gamma - V$, is sent to a complete geodesic of $\mathbb{H}^3$, and the images of the adjacent faces locally form a dihedron of angle $\frac{\pi}{2}$ near this geodesic. In addition, the image of $B^3_+ - V$ in $\mathbb{H}^3$ has finite volume.

Equip $B^+ - V$ with the metric induced by this embedding, and extend it (uniquely) to a path metric on $S^3 - V$ that is invariant by the reflection through

S^2. (A path metric on a topological space is a function assigning a length to each path in this space, and satisfying certain obvious condition so that the function $d(x, y)$, defined as the minimum of the lengths of path joining x to y, is a distance compatible with the topology of the space; see [**Gro**].) We claim that this metric is π–hyperbolic, namely locally isometric to $\mathbb{H}^3/\rho$ for some π–rotation ρ of $\mathbb{H}^3$.

On $S^3 - S^2$, the metric is clearly locally isometric to $\mathbb{H}^3$.

Near a point of $S^2 - \Gamma$, this metric is locally isometric to the double of a half-space delimited by a hyperbolic plane of $\mathbb{H}^3$. As this double is isometric to $\mathbb{H}^3$, the metric is again locally isometric to $\mathbb{H}^3$ near such a point.

Near $\Gamma - V$, however, our space is locally isometric to the double of a dihedron of angle $\frac{\pi}{2}$ in $\mathbb{H}^3$, delimited by two hyperbolic planes P_1 and P_2 meeting orthogonally in $\mathbb{H}^3$. If these two planes are chosen so that their intersection is the axes of the π–rotation ρ, one readily sees that this double is isometric to $\mathbb{H}^3/\rho$.

The metric we have constructed is therefore π–hyperbolic. It is clearly complete. Its volume is twice the volume of $B^3_+ - V$, and is therefore finite. □

For the Turk's head graphs $(2 \times k)^*$ of §1.1, it is not difficult to find the hyperbolic polyhedron promised by Andreev's theorem in the above proof. Indeed, take for $\mathbb{H}^3$ the model of the unit ball in $\mathbb{R}^3$ (see §5.1), and place the vertices of the two k–gon faces of $(2 \times k)^*$ on two circles parallel to and at equal euclidean distance d from the equator (this distance d is to be specified later), arranging that one k–gon is rotated by $\frac{\pi}{k}$ with respect to the other. Fill in the edges of $(2 \times k)^*$ by geodesics on the unit sphere S^2. Each face of $(2 \times k)^*$ now lies on a unique circle, which is the frontier of a unique hyperbolic plane in $\mathbb{H}^3$. For each face, delete from $\mathbb{H}^3$ the open half-space delimited by this hyperbolic plane and containing the face of $(2 \times k)^*$ in question. This gives a hyperbolic polyhedron with faces at infinity.

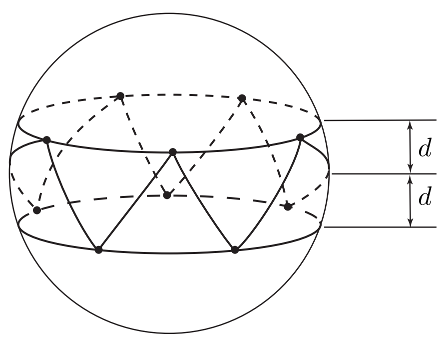
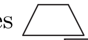
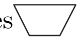


Figure 4.1.

A quick euclidean geometry argument readily shows that there is a unique value of d for which the dihedral angles of this polyhedron are all equal to $\frac{\pi}{2}$. Indeed, the hyperbolic dihedral angles between two hyperbolic planes P_1, P_2 in $\mathbb{H}^3$ is equal to the euclidean angle between the two circles delimiting these planes P_1 and P_2 in the unit sphere S_2 bounding $\mathbb{H}^3$. Near every vertex, we have angles  for d small and angles  for d near 1, and we consequently have right angles  for some value of d in-between.This value of d is unique by monotonicity of the angles of the trapezoids, expressed as a function of d.

As a matter of fact, such variational techniques are at the core of Andreev's proof [**And**$_1$, **And**$_2$].

We next give a similar use of hyperbolic polyhedra to put a π–hyperbolic metric on the knot pair $(S^3 - V, \Gamma - V)$ associated to another quadrivalent graph Γ embedded in S^3 and to its vertex set V. Here, Γ is not planar any more but we will exploit its symmetry.

Consider the 1–skeleton Γ of a symmetric 4–simplex Δ^4 centred at $0 \in \mathbb{R}^4$, and embed Γ by radial projection in the unit 3–sphere $S^3 \subset \mathbb{R}^4$. Topologically, Γ is the embedded graph of Figure 4.2. We want to show that $(S^3 - V, \Gamma - V)$ has a π–hyperbolic structure, where V is the vertex set of Γ.

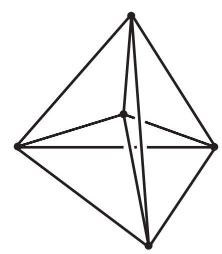


FIGURE 4.2.

To do this, consider the quotient of S^3 by the full symmetry group $\mathfrak{S}_5$ of (S^3, Γ). It is a copy of a tetrahedral fundamental domain Q with vertices v_0, v_1, v_2, v_3, where v_i is the image of the centre of an i–face of Δ^4 (so that the centre of any i–face projects to the same point as v_i in the quotient space $S^3/\mathfrak{S}_5$). Let n_{ij} denote the number of copies of Q in S^3 that are adjacent to the edge v_iv_j. By inspection, one finds that $n_{01} = 6$, $n_{02} = 4$, $n_{03} = 6$, $n_{12} = 4$, $n_{13} = 4$, and $n_{23} = 6$.

Assume that we are given an identification of $Q - v_0$ with a hyperbolic tetrahedron in $\mathbb{H}^3$ with a vertex at infinity, so that the two faces adjacent to the edge v_iv_j make an angle of $\frac{2\pi}{n_{ij}}$, except at v_0v_1 where they make an angle of $\frac{\pi}{n_{01}} = \frac{\pi}{6}$. Extend this metric to a path metric on $S^3 - V = \mathfrak{S}_5(Q - v_0)$ by the action of $\mathfrak{S}_5$. The same argument as before shows that this metric is complete, is locally isometric to $\mathbb{H}^3$ outside of $\Gamma - V = \mathfrak{S}_5(v_0v_1 - v_0)$, and is locally isometric to $\mathbb{H}^3/\rho$ near $\Gamma - V$ (where ρ is still a π–rotation of $\mathbb{H}^3$).

Andreev's theorem for the tetrahedron (see [**Luo**] for this case) abstractly asserts the existence of a hyperbolic tetrahedron with the dihedral angles indicated, but it is very easy to construct it "by hand". For this, it is convenient to consider the upper half space model for $\mathbb{H}^3$. Take v_0 to be ∞, and choose 3 points w_1, w_2, w_3 forming a euclidean triangle of angles $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$ in the plane $\mathbb{R}^2$ bounding $\mathbb{H}^3$. Choose v_1 on the geodesic v_0w_1, high enough so that the hyperbolic plane P cutting v_0w_1 orthogonally at v_1 also meets v_0w_2 and v_0w_3 and define v_2 and v_3 as the respective intersections of this plane with v_0w_2 and v_0w_3 (see Figure 4.3); note that this hyperbolic plane P is just a euclidean hemisphere centred at w_1. Then the hyperbolic tetrahedron $v_0v_1v_2v_3$ has the dihedral angles required except perhaps along v_2v_3. Observe that this dihedral angle at v_2v_3 is approximately $\frac{\pi}{2}$ when v_1 is close to ∞, and is $\frac{\pi}{6}$ at the critical value of v_1 for which $v_3 = w_3$. Thus, there is a (unique) choice of v_1 which gives the hyperbolic tetrahedron wanted.

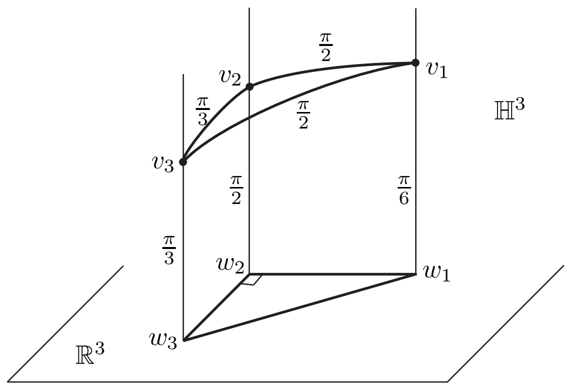


FIGURE 4.3.

As indicated before, this defines a complete π–hyperbolic metric on $(S^3 - V, \Gamma - V)$ where Γ is the embedded graph of Figure 4.2. This metric has finite volume since so does the hyperbolic tetrahedron which we used.

This example has two "platonic" companions (each with far more vertices). Namely we can replace Γ by the 1–skeleton of the cellulation of S^3 by 8 cubes or by 120 dodecahedra. The associated tessellating hyperbolic tetrahedron differs from the one just built only in that the dihedral angle at the edge $v_2 v_3$ is not $\frac{\pi}{3}$ but $\frac{\pi}{4}$ or $\frac{\pi}{5}$, respectively.

4.2. Hyperbolic Dehn surgery along the figure-eight knot

In this section, we consider the knots obtained by gluing a rational tangle pair to the knot pair $(M_1, K \cap M_1)$ of Figure 3.5, and analyse which ones admit a π–hyperbolic metric. This stems from the observation that the double branched coverings of these knots are obtained by Dehn surgery along the figure-eight knot, and we rely heavily on Thurston's analysis of these manifolds in [**Thu**$_1$, §4].

Consider a knot (S^3, K) and its exterior M, namely $M = S^3 - \mathrm{int}(W)$ is the closed-up complement of a tubular neighbourhood W of K in S^3. The operation of gluing a solid torus V to M along ∂M gives a new closed 3–manifold. A manifold of this type is said to be obtained by ***Dehn surgery along*** K.

Up to degree $+1$ isomorphism, such a manifold $M(\mu, \lambda)$ is characterised by a pair of coprime integers (μ, λ), defined only modulo simultaneous sign reversal, which are called the ***Dehn surgery coefficients*** (see for instance [**Rol**, §9F]). These are defined as follows: Choose in $H_1(\partial M)$ a meridian m and a longitude l for K, namely such that $[l] = 0$ in $H_1(M)$ and $[m] = 0$ in $H_1(W)$, oriented so that their intersection number is $[m] \cdot [l] = +1$ in $\partial M = \partial W$ oriented as boundary of W. Then, a meridian of V is homologous to $\mu[m] + \lambda[l]$ on $\partial M = \partial V$.

When K is the figure-eight knot, Thurston gives in [**Thu**$_1$, §4] a subtle but explicit construction of a hyperbolic structure on $M(\mu, \lambda)$ for (μ, λ) outside of the rectangle $|\mu| \leqslant 4$, $|\lambda| \leqslant 1$. (Beware that Thurston's coefficients (μ, λ) differ from ours by a sign reversal.) We are going to exploit this analysis to give many examples of π–hyperbolic knots.

From the π–rotation τ of Figure 3.6, we get an involution on the exterior M of the figure-eight knot. This gives a knot pair (N, L) where $N = M/\tau$ and L is the image of the fixed point set of τ. If we identify the quotient S^3/τ to S^3 so that the axis of τ projects to the union of the z–axis union the point ∞, the image $B = W/\tau$ of the tubular neighbourhood W of K is the stretched ball represented in Figure 4.4(a); compare Figure 3.6. In particular, in Figure 4.4(a), N is the closed-up complement of the ball B in S^3, and L is the intersection of N with the z–axis union ∞. Unknotting B in S^3, we can then draw (N, L) as in Figure 4.4(b).

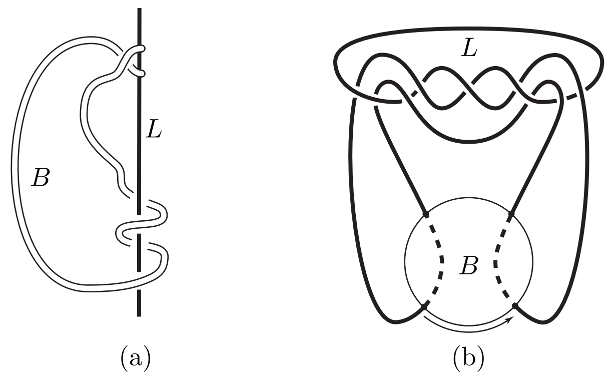


FIGURE 4.4.

From the knot pair projection for (N, L) given by Figure 4.4(b), we can obtain a knot projection by substituting to B a rational tangle pair projection, in the sense of §1.2. The resulting knot $L_{\frac{q}{p}}$ in S^3 depends only on the slope $\frac{q}{p}$ of the rational tangle projection substituted, defined using §1.2 and the arrow shown in Figure 4.4(b).

Consider the double branched covering of $(S^3, L_{\frac{q}{p}})$. The preimage of N in this space is the double branched covering of (N, L), namely M equipped with the involution τ. Also, the preimage of the rational tangle pair substituted to B is a solid torus V (compare Figure 1.6). Thus, the double branched covering of $(S^3, L_{\frac{q}{p}})$ is a manifold $M(\mu, \lambda)$ obtained by Dehn surgery along the figure-eight complement, with covering involution coinciding with τ on the knot exterior M.

How are the slope $\frac{q}{p}$ and the Dehn surgery coefficient (μ, λ) related? Remember that $\frac{q}{p}$ is defined as follows (see §1.2): Choose a basis $[\widetilde{k}_1]$, $[\widetilde{k}_2]$ for $H_1(\partial M)$ such that, in the description of (N, L) by the knot pair projection of Figure 4.4(b), $\widetilde{k}_1$ projects onto the arc k_1 of $\partial N \cap \mathbb{R}^2$ marked by the arrow and $\widetilde{k}_2$ projects to an adjacent arc k_2 of $\partial N \cap \mathbb{R}^2$, where $\mathbb{R}^2$ is the projection plane; moreover, the intersection number $[\widetilde{k}_1] \cdot [\widetilde{k}_2]$ is $+1$ on $\partial M = \partial V$ oriented as boundary of the solid torus V. Then $\frac{q}{p}$ is defined by the property that the kernel of $H_1(\partial V) \to H_1(V)$ is generated by $p[\widetilde{k}_1] + q[\widetilde{k}_2]$.

If m and l are the meridian and longitude of the figure-eight knot, it follows from the original construction of (N, L) that the orientations can be chosen so that $[\widetilde{k}_2] = m$, and then $[\widetilde{k}_1] = -l + km$ for some $k \in \mathbb{Z}$. So $\lambda = -p$ and $\mu = pk + q$. To determine the precise value of k, one could scrutinise the construction of (N, L) more closely. The easiest way is however to observe that the double branched covering $M(\lambda,\mu)$ of $L_{\frac{q}{p}}$ in S^3 has infinite H_1 if and only if $(\mu,\lambda)=\pm(0,1)$; on the other hand, the complement of L_{-4} contains a non-trivial torus (see Figure 4.5) lifting to two non-separating tori in the double branched covering of this knot. Thus $\frac{q}{p}=-4$ corresponds to $(\lambda,\mu)=\pm(0,1)$, which gives $k=+4$.

Therefore, the relation between the slope $\frac{q}{p}$ and the surgery coefficients (μ,λ) is that $\frac{\mu}{\lambda}=-\frac{q}{p}-4$.

Substituting various rational tangle projections in Figure 4.4, we get in particular the knots illustrated in Figure 4.5 and 4.6.

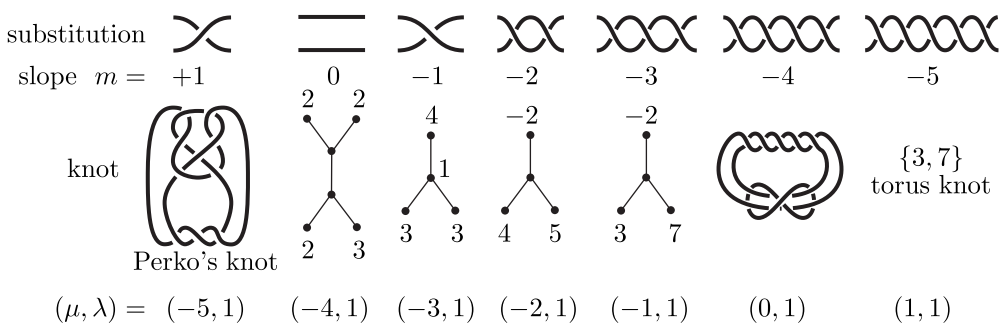


FIGURE 4.5.

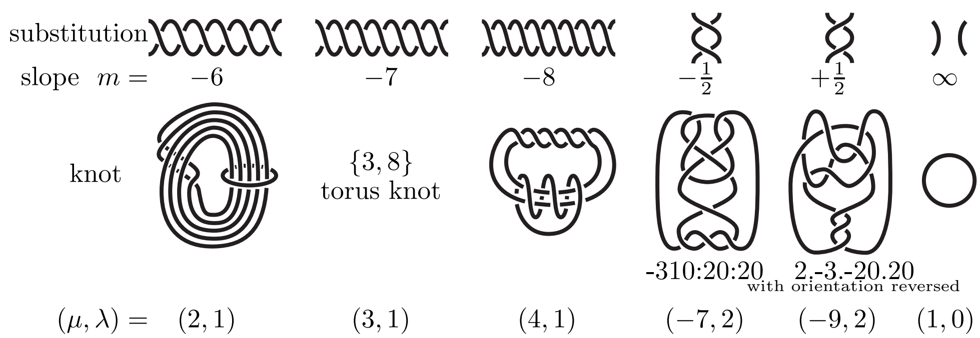


FIGURE 4.6.

For substitutions of slopes $\frac{q}{p}=0,\,-1,\,-2,\,-3$, the knot $L_{\frac{q}{p}}$ is arborescent and is described in Figure 4.5 by the weighted planar tree that is going to characterise it in the plumbing calculus to be developed in Chapter 12. Observe that $L_{\frac{q}{p}}$ is not simple for Schubert if $\frac{q}{p}=-4$ or -8, and is Seifert fibred for $\frac{q}{p}=-5,\,-6,\,-7$ and ∞. We will see in §5.2 that these properties prevent these $(S^3, L_{\frac{q}{p}})$ from admitting π–hyperbolic structures. On the other hand, we will prove that all other $(S^3, L_{\frac{q}{p}})$ are π–hyperbolic.

The other substitutions that appear in Figures 4.5 and 4.6 are the remaining ones for which we were sure to get a knot with $\leqslant 11$ crossings, which could consequently be identified in Conway's tabulations. The first one L_1 is rather notorious. This 10 crossing knot was doubly listed by Little and Conway, as Perko discovered in [**Perk**$_2$]. The projection illustrated is Conway's 3:−20:−20, and the other projection listed by Conway is 21:−20:−20 modulo reversal of the orientation of S^3.

(These two projections also appear as 10_{161} and 10_{162} in [**Rol**].) The 11–crossing knots $L_{-\frac{1}{2}}$ and $L_{+\frac{1}{2}}$ are tabulated by Conway as −310:20:20 for the first one, and 2.−3.20 with orientation of S^3 reversed for the second one.

As announced, we are now going to prove that all $(S^3, L_{\frac{q}{p}})$ with $\frac{q}{p} \neq 0, -1, -2, -3, -4, -5, -6, -7, -8, \infty$ are π–hyperbolic. This amounts to showing that their double branched coverings $M(\mu,\lambda)$ admit a hyperbolic metric that is invariant by the covering involution τ. Observe that the above restriction of $\frac{q}{p}$ amounts to exclude the $M(\mu,\lambda)$ with $|\mu| \leqslant 4$ and $|\lambda| \leqslant 1$.

In [**Thu**$_1$, §4], Thurston explicitly constructs hyperbolic structures on the $M(\mu,\lambda)$ with (μ,λ) outside of the rectangle $|\mu| \leqslant 4$, $|\lambda| \leqslant 1$. For this, he starts from the complement $M_\infty = S^3 - K \cong \mathrm{int}(M)$ of the figure-eight knot (S^3, K), which he constructs from two copies of a symmetric hyperbolic tetrahedron with vertices at infinity in $\mathbb{H}^3$. This defines a finite volume hyperbolic metric on M_∞. Then, Thurston modifies this metric by deforming these two hyperbolic tetrahedra to get some incomplete hyperbolic metric on $S^3 - K$, whose completion is identified to $M(\mu,\lambda)$ for suitable deformations.

It is a curious fact that, although $\tau\colon M_\infty \to M_\infty$ can be seen as an explicit isometric involution exchanging the two hyperbolic tetrahedra building M_∞, the same is not true for $\tau\colon M(\mu,\lambda) \to M(\mu,\lambda)$. Indeed, the two tetrahedra are not isometric any more in the latter case. So, we have to turn to more elaborate arguments to make τ: $M(\mu,\lambda) \to M(\mu,\lambda)$ an isometry.

An alternative to prove this would be to apply Thurston's Geometrisation Theorem for orbifolds, as discussed in §5.3, to conclude that $M(\mu,\lambda)$ admits some geometric structure that is invariant by τ. For (M, L) outside of the rectangle $|\mu| \leqslant 4$, $|\lambda| \leqslant 1$, we know by [**Thu**$_1$, §4] that $M(\mu,\lambda)$ admits a hyperbolic structure, and this compels the τ–invariant geometric structure to be hyperbolic.

However, instead of using this difficult theorem which is not yet accessible to the public[1], we favour an *ad hoc* proof which uniquely relies on the easier hyperbolisation theorem for manifolds (see [**Thu**$_2$**, Thu**$_3$**, Thu**$_4$**, Thu**$_5$**, Mor**$_1$]).

So, consider the hyperbolic metric on $M(\mu,\lambda)$ constructed in [**Thu**$_1$, §4]). By construction, the core C of the solid torus $V = M(\mu,\lambda) - \mathrm{int}(M)$ is geodesic for this metric.

Now, Mostow's rigidity theorem [**Mos**$_1$**, Mos**$_2$] gives an isometric involution τ' of $M(\mu,\lambda)$ which is homotopic to τ. We will show that τ' is conjugate to τ, which will complete the proof.

First, observe that τ' sends the closed geodesic C to itself. Indeed, since $\tau(V) = V$, the closed geodesic $\tau(C)$ is homotopic and thus equal to C. Therefore, adjusting M and τ by an isotopy of $M(\mu,\lambda)$, we can assume that $\rho'(M) = \rho(M) = M$.

Now, τ and τ' induce two automorphisms of M. Lemma 4.2 below, applied to $\varphi = (\tau'\tau^{-1})_{|M}$, shows that these two automorphisms of M are homotopic. Taking this lemma for granted, this concludes the proof that τ and τ' are conjugate in $M(\mu,\lambda)$. Indeed, a theorem of Tollefson [**Tol**$_3$], using the hypothesis that $H^1(M) \neq 0$ (to construct a τ–equivariant hierarchy for M), then asserts that these restrictions of τ and ρ' to M are conjugate by an automorphism isotopic to the identity. An easy argument on the solid torus $V = M(\mu,\lambda) - \mathrm{int}(M)$ now extends this conjugation to a conjugation between τ and τ' in all of $M(\mu,\lambda)$. Using this conjugation to

[1](Added 2009) See [**CooHK, BoiP, BoiMP, BoiLP**] for current expositions.

modify the hyperbolic metric on $M(\mu,\lambda)$, we have now managed that this metric is invariant by τ, and thus induces a π–hyperbolic structure on the knot $(S^3, L_{\frac{q}{p}})$.

To complete the proof that $(S^3, L_{\frac{q}{p}})$ is π–hyperbolic, we however need to prove the following lemma, whose proof was temporarily left aside.

LEMMA 4.2. *If φ is a self homotopy equivalence of M which extends to a map ψ: $M(\mu,\lambda) \to M(\mu,\lambda)$ that is homotopic to the identity, then φ itself is homotopic to the identity.*

PROOF. We first show that we can assume ψ periodic. Indeed, the hyperbolic structure on the figure-eight knot complement M_∞ can easily be chosen so that the boundary torus ∂M of $M \subset M_\infty$ is horospherical (see §5.2 for definitions). By Mostow's Rigidity Theorem [**Mos**$_1$**, Mos**$_2$] we can then assume that φ is the restriction of an isometry of M_∞.

In particular, φ is now periodic. (The isometry group of a finite volume hyperbolic manifold is easily seen to be compact and discrete, and therefore finite; see for instance [**Thu**$_1$, §5.7.4] or Corollary 5.12). The map ψ extending φ can then be chosen to be also periodic, by an easy argument on the solid torus $M(\mu,\lambda) - \mathrm{int}(M)$.

We claim that the fact that ψ is homotopic to the identity compels its order p to be 1. Indeed, lifting a homotopy of ψ to the identity up to the universal covering $\mathbb{H}^3$ of $M(\mu,\lambda)$, we get $\widetilde{\psi}$: $\mathbb{H}^3 \to \mathbb{H}^3$ covering ψ and fixing the sphere at infinity S^2_∞ of $\mathbb{H}^3$ (see §5.1 for the definition of this sphere at infinity). Also, $\widetilde{\psi}^p$ is an isometry of $\mathbb{H}^3$ since it lifts the identity of $M(\mu,\lambda)$, and must consequently be the identity of $\mathbb{H}^3$ as it fixes the sphere at infinity. We now have an order p homeomorphism $\widetilde{\psi}$ of the closed ball $\mathbb{H}^3 \cup S^2_\infty$ that fixes its boundary S^2_∞; it must be the identity by M.H.A. Newman's theorem [**New, Smi, Dre**]. In particular, ψ is the identity and the lemma is proved (see [**ConM**] for a more general argument, and compare Lemma 16.7). □

This completes the proof that the knot $(S^3, L_{\frac{q}{p}})$ obtained by plugging a rational tangle projection of slope $\frac{q}{p}$ to the knot pair projection of Figure 4.4 admits a π–hyperbolic structure if (and only if) $\frac{q}{p}$ is different from 0, -1, -2, -3, -4, -5, -6, -7, -8 and ∞.

We conclude this section by an observation. Since the figure-eight knot is amphicheiral, $M(\mu,\lambda)$ is isomorphic to $-M(\mu,\lambda)$ (namely $M(\mu,\lambda)$ with the opposite orientation). In particular, the knots $(S^3, L_{\frac{q}{p}})$ and $(-S^3, L_{-8-\frac{q}{p}})$ have the same double branched coverings. On the other hand we believe that these knots are never isomorphic.

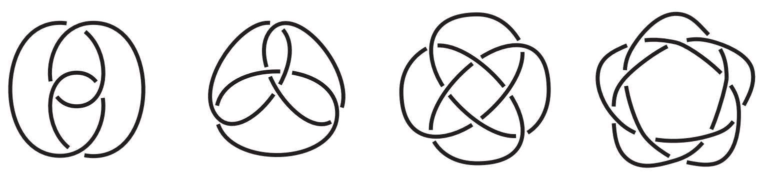
This is the case for the non π–hyperbolic knots listed in Figures 4.5 and 4.6. Indeed, we will show in §8.3 and in the Appendix that the arborescent knots shown are simple for Schubert and are not Seifert fibred, and (S^3, L_{-4}) is clearly not amphicheiral since one of its components is a $\{2,5\}$ torus knot.

It is also the case for all but finitely many slopes $\frac{q}{p}$ giving π–hyperbolic knots. Indeed, we will show in Chapter 6 that, for all but finitely many $\frac{q}{p}$ and $\frac{q'}{p'}$, any degree ± 1 isomorphism $(S^3, L_{\frac{q}{p}}) \to (S^3, L_{q'/p'})$ can be pairwise isotoped so as to send $(N, L) \subset (S^3, L_{\frac{q}{p}})$ to $(N, L) \subset (S^3, L_{\frac{q'}{p'}})$. In particular, because there is no degree -1 isomorphism $(N, L) \to (N, L)$ (one component of L forms a $(2,5)$ torus knot), the two knots $(S^3, L_{\frac{q}{p}})$ and $(-S^3, L_{\frac{q'}{p'}})$ cannot be (degree +1) isomorphic in this case.

This gives an infinite, although imperfectly specified, family of pairs of distinct knots $(S^3, L_{\frac{q}{p}})$ and $(-S^3, L_{-8-\frac{q}{p}})$ which have the same double branched covering. On the other hand, $(S^3, L_{\frac{q}{p}})$ and $(-S^3, L_{-8-\frac{q}{p}})$ are never mutation equivalent when they are π–hyperbolic, since they are then simple for Conway (see Corollary 5.3). Thus, we get a negative answer to [**Kirb**, Problem 1.22]. J.-M. Montesinos has independently noticed examples of this sort.

4.3. Turk's head knots

In this section, we again use Thurston's hyperbolic Dehn surgery on the figure-eight knot to get π–hyperbolic structures for the alternating knots with Turk's head projection $(2 \times k)^*$ of Chapter 1 with $k \geqslant 4$ (see Figure 4.7). Observe that $(2 \times 2)^*$ and $(2 \times 3)^*$ respectively give the figure-eight knot and the Borromean rings; by Theorem 5.7, these arborescent knots cannot admit a π–hyperbolic structure.



$(2 \times 2)^*$ $(2 \times 3)^*$ $(2 \times 4)^*$ $(2 \times 5)^*$

FIGURE 4.7.

To construct these π–hyperbolic structures, we are going to exploit the symmetry of the $(2 \times k)^*$ knot by the $\frac{2\pi}{k}$–rotation σ_k with axis perpendicular to the projection plane. The quotient S^3/σ_k is topologically S^3, and contains the quotient K of the knot and the image L of the fixed point set of σ_k, the two linked as in Figure 4.8 (independent of k).

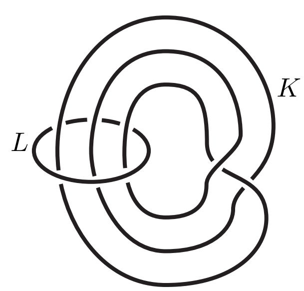


FIGURE 4.8.

To get a π–hyperbolic metric on the Turk's head knot $(2 \times k)^*$, it clearly suffices to find a metric on $S^3 = S^3/\sigma_k$ for which $S^3 - K \cup L$ is locally isometric to $\mathbb{H}^3$, $(S^3 - K, L)$ is locally isometric to $(\mathbb{H}^3/\rho_k, \text{axis}(\rho_k))$, and $(S^3 - L, K)$ is locally isometric to $(\mathbb{H}^3/\rho_2, \text{axis}(\rho_2))$, where ρ_n denotes any $\frac{2\pi}{n}$–rotation of the hyperbolic space $\mathbb{H}^3$. Indeed, such a metric on S^3/σ_k will lift to a path metric on S^3 that is locally isometric to $\mathbb{H}^3$ outside of the knot $(2 \times k)^*$ and is locally isometric to $\mathbb{H}^3/\rho_2$ near this knot, namely to a π–hyperbolic metric for $(2 \times k)^*$. A metric of the type we seek for S^3/σ_k, namely locally isometric to some $\mathbb{H}^3/\rho_n$ near each point, will be called a hyperbolic metric with rotational singularities in Chapter 5, where we will consider some general properties of these metrics.

Now, a trick is to observe that K and L can be interchanged by an ambient isotopy in S^3. Thus, if we pass to the 2–fold branched covering along K (not L), then L lifts to become the figure-eight knot K_0 in S^3. The metric we seek for S^3/σ_k would then lift to a path metric on S^3 so that (S^3, K_0) is locally isometric to $(\mathbb{H}^3/\rho_k, \text{axis}(\rho_k))$.

It turns out that such metrics are readily provided by Thurston's analysis [**Thu**$_1$, §4] of the hyperbolic Dehn surgery on the figure-eight knot. Indeed, by deformation of the hyperbolic metric on the knot complement M_∞, Thurston obtains *hyperbolic orbifolds* $M(\mu, \lambda)$ where the integers μ, λ are not necessarily coprime. An introduction to orbifolds can be found in [**Thu**$_1$, §13] or [**BonS**$_1$] (see also §5.3), but we can elementarily describe these $M(\mu, \lambda)$ as follows: The $M(\mu, \lambda)$ with μ, λ coprime are just the hyperbolic manifolds that we encountered in §4.2, obtained by gluing a solid torus V to the exterior M of the figure-eight knot. In the general case, let d be the greatest common divisor of $|\mu|$ and $|\lambda|$. Then the hyperbolic orbifold $M(\mu, \lambda)$ is topologically the manifold $M(\frac{\mu}{d}, \frac{\lambda}{d})$, but equipped with a (path) metric which is locally isometric to $\mathbb{H}^3/\rho_d$, in such a way that the set of points corresponding to the axis of ρ_d is exactly the core of the solid torus V glued on M to construct $M(\frac{\mu}{d}, \frac{\lambda}{d})$.

In particular, $M(k, 0)$ is S^3 together with a metric making (S^3, K_0) locally isometric to $(\mathbb{H}^3/\rho_k, \text{axis}(\rho_k))$, and Thurston obtains such $M(k, 0)$ for all $k \geqslant 4$. (The case $k = 4$ is not explicit in [**Thu**$_1$, §4], but was explained to us by Bill Thurston in a private conversation.)

To get the wanted metric on S^3/σ_k, we need to arrange that the above metric on (S^3, K_0) is invariant by the covering involution σ of the 2–fold branched covering $S^3 \to S^3/\sigma_k$ of $(S^3/\sigma_k, L)$. Although we could use for that the arguments used in §4.2, it turns out that this fact is immediately granted by Thurston's construction (at least for $k > 4$). Indeed, Thurston decomposes $S^3 - K_0$ into two hyperbolic tetrahedra (with vertices at infinity), in such a way that the completion of $S^3 - K_0$ with the induced metric gives (S^3, K_0) with the wanted metric. One observes that these hyperbolic tetrahedra can be chosen so that σ maps each of them onto itself by a π–rotation, and thus induces an isometry of (S^3, K_0).

Thus, we get for $k \geqslant 4$ the metric we wanted on $S^3/\sigma = S^3/\sigma_k$, which lifts to a π–hyperbolic structure for the Turk's head knot $(2 \times k)^*$ in S^3.

CHAPTER 5

Hyperbolic metrics with rotational singularities

In Chapters 3 and 4, we introduced the notion of π–hyperbolic structures for knot pairs, and gave a few examples. In the present chapter, we investigate some general properties of these structures. In fact, our discussion will cover hyperbolic structures with rotational singularities of angle $\frac{2\pi}{k}$ for $k = 2, 3, \dots$, which we already briefly encountered in §4.3 and whose analysis does not give much more trouble than do π–hyperbolic structures. However, §5.3 will suggest that π–hyperbolic structures are the singular hyperbolic structures most suited to knot theory.

5.1. Definitions

Before going further, it may be useful to recall a few facts about the hyperbolic n–space $\mathbb{H}^n$. See [**Thu₁, Thu₆, BenP, Rat, Bon₄**] for more background material.

One description of the hyperbolic space $\mathbb{H}^n$ views it as the open unit ball $\mathrm{int}(B^n)$ in $\mathbb{R}^n$, equipped with the Riemannian metric which, at $x \in \mathrm{int}(B^n)$, is $2/(1-\|x\|^2)$ times the euclidean metric of $\mathbb{R}^n$. Another model for $\mathbb{H}^n$, isometric to the previous one, is the upper half-space $\mathbb{R}^{n-1} \times]0,\infty[$ equipped with the metric which is x_n^{-1} times the euclidean metric at the point $x = (x_1, \dots, x_n)$. These open ball and upper half-space models for $\mathbb{H}^n$ are isometric by an inversion through a suitable sphere in $\mathbb{R}^n \cup \infty$.

Also, the hyperbolic space $\mathbb{H}^n$ has many isometries. For instance, in the upper half-space model, any homothety or euclidean isometry of $\mathbb{R}^n$ respecting $\mathbb{R}^{n-1}$ gives such an isometry of $\mathbb{H}^n$. A less obvious example is provided by any inversion of $\mathbb{R}^n \cup \infty$ through a sphere centred at some point of $\mathbb{R}^{n-1}$. Further, in either model, the full isometry group is the ***Möbius group*** generated by the inversions in the spheres perpendicular to the frontier of the model. From this, we can see that the isometry group acts transitively on $\mathbb{H}^n$, and even on the space of points of $\mathbb{H}^n$ equipped with an orthonormal basis for their tangent space. Thus, up to scaling by a positive constant, the hyperbolic metric is the unique Riemannian metric invariant under this group. The constant has been chosen so that the sectional curvature is -1. A celebrated theorem of Cartan-Hadamard asserts that $\mathbb{H}^n$ is the unique simply connected complete Riemannian manifold with constant sectional curvature -1 (see for instance [**Wol, CheE, Car**]).

Using the isometry group of $\mathbb{H}^n$, the geodesics of this Riemannian manifold are easily determined. For instance, in the upper half space model $\mathbb{R}^{n-1} \times]0,\infty[$, they are precisely the traces of the circles and lines of $\mathbb{R}^n$ meeting $\mathbb{R}^{n-1}$ orthogonally. This leads to a *natural* compactification of $\mathbb{H}^n$ by a ***sphere at infinity*** S^{n-1}_∞ defined by adding an end point to each geodesic ray issued from a given point x. Indeed, one easily checks that this compactification is independent of the base point x. (Hint: For any geodesic ray issued from x and any $x' \in \mathbb{H}^n$, there is a unique geodesic ray issued from x' that stays at bounded distance from the first one.) In particular, the action of the isometry group of $\mathbb{H}^n$ continuously extends to S^{n-1}_∞. This sphere at infinity can of course be described as the unit sphere $S^{n-1} = \partial B^n$ for the unit ball model, and as $\mathbb{R}^{n-1} \cup \infty$ for the upper half space model $\mathbb{R}^{n-1} \times]0, \infty[$ for $\mathbb{H}^n$.

Any isometry of $\mathbb{H}^n$ thus continuously acts on the ball $\mathbb{H}^n \cup S^{n-1}_\infty$ and, by Brouwer's Fixed Point Theorem, must fix some point of this ball. By considering their fixed point set, this separates the isometries of $\mathbb{H}^n$ different from the identity into three disjoint classes, invariant by conjugation:

(a) The ***elliptic*** isometries are those fixing a point of $\mathbb{H}^n$. In dimension 3, the degree $+1$ elliptic isometries are thus rotations around geodesics.
(b) The ***loxodromic*** isometries are those acting by (non-trivial) translation on some geodesic. In dimension 3 and degree $+1$, these are glide-rotations along this geodesic.
(c) Lastly, the ***parabolic*** isometries are those fixing a single point of S^{n-1}_∞. In the upper half-space model for $\mathbb{H}^n$, there are conjugated to (fixed point free) euclidean isometries of this model.

By convention, the identity is considered to be simultaneously elliptic, loxodromic and parabolic.

After this brief survey of the hyperbolic space, we can now define our hyperbolic structures with rotational singularities.

For every integer $k \geqslant 2$, choose an isometric rotation ρ_k of angle $\frac{2\pi}{k}$ around a geodesic of the hyperbolic 3–space $\mathbb{H}^3$. Any two choices for ρ_k are conjugated by an isometry of $\mathbb{H}^3$, so that the quotient metric space $\mathbb{H}^3/\rho_k$ is actually well defined up to isometry.

Consider a pair (M, K) where K is a 1–dimensional submanifold of an orientable 3–manifold M, and where M and K are without boundary but not necessarily compact. A ***singular hyperbolic structure (with rotational singularities)*** on (M, K) is a complete metric space structure on M such that, for each point $x \in M$, there exists an isometry from a neighbourhood U of x to an open subset V of some $\mathbb{H}^3/\rho_k$ sending $U \cap K$ to $V \cap \text{axis}(\rho_k)$. A point $x \in K$ where this metric is locally isometric to $\mathbb{H}^3/\rho_k$ is said to be ***singular of order*** k or $\frac{2\pi}{k}$***–singular*** ; the second terminology is motivated by the fact that $\mathbb{H}^3/\rho_k$ is isometric to the space obtained by identifying by ρ_k the two faces of a hyperbolic dihedron of angle $\frac{2\pi}{k}$.

The above definition holds if we are working in the category TOP of topological manifolds. For the differentiable category DIFF, note that $\mathbb{H}^3/\rho_k$ has a natural differentiable structure: The exponential map gives a preferred identification $\mathbb{H}^3 \cong \mathbb{C} \times \text{axis}(\rho_k)$ (unique up to isometries of $\mathbb{C}$); then the map $(z, t) \mapsto (z^k, t)$ from $\mathbb{C} \times \text{axis}(\rho_k)$ to itself provides a homeomorphism $\mathbb{H}^3/\rho_k \cong \mathbb{C} \times \text{axis}(\rho_k) \cong \mathbb{R}^3$ well-defined up to diffeomorphism of $\mathbb{R}^3$ and therefore defines a preferred differentiable structure on $\mathbb{H}^3/\rho_k$. If we are working in the differentiable category DIFF, we will therefore insist in the definition of singular hyperbolic structures that the local isometries $(M, K) \cong (\mathbb{H}^3/\rho_k, \text{axis}(\rho_k))$ be diffeomorphisms.

The piecewise linear category PL is not well adapted to this discussion, and we will not use it in this chapter.

In the language of orbifolds (see [**Thu**$_1$, §13] or [**BonS**$_1$]) a singular hyperbolic structure on (M, K) amounts to an identification of M with the underlying topological space of a complete hyperbolic 3–orbifold for which K corresponds to the singular set of this orbifold. The hyperbolic 3–orbifolds that can occur in this way are precisely those for which all the isotropy groups are cyclic, acting by hyperbolic rotations.

Also, given such a singular hyperbolic structure on the connected pair (M, K), a theorem due to Poincaré provides a discrete group $\mathcal{G}$ of isometries of $\mathbb{H}^3$ such that (M, K) is isometric to $(\mathbb{H}^3/\mathcal{G}, (\operatorname{Fix}\mathcal{G})/\mathcal{G})$, where $\operatorname{Fix}\mathcal{G}$ denotes the set of points fixed by *some* non-trivial element of $\mathcal{G}$; see [**Sin, Wol**] or [**Thu**$_1$, §3]. The completeness of the metric of M is here essential.

In this situation, assume that $\mathcal{G}$ is finitely generated; this occurs for instance if (M, K) has ***finite topological type***, namely is homeomorphic to the interior of a (compact) knot pair. Then, the celebrated Selberg Lemma [**Sel**] provides a finite index normal subgroup $\mathcal{G}_0$ of $\mathcal{G}$ which has no torsion; indeed, any finitely generated matrix group has this property. As $\mathcal{G}_0$ is discrete and torsion-free, it acts freely on $\mathbb{H}^3$, and $\mathbb{H}^3/\mathcal{G}_0$ is a hyperbolic manifold (with no singularities). Thus, M is isometric to the quotient of the hyperbolic manifold $\mathbb{H}^3/\mathcal{G}_0$ by the finite group of isometries $\mathcal{G}/\mathcal{G}_0$.

We collect these two remarks in a statement.

PROPOSITION 5.1. *Consider a complete singular hyperbolic pair (M, K). Then, there exists a discrete group $\mathcal{G}$ of isometries of $\mathbb{H}^3$ and an isometry $M \to \mathbb{H}^3/\mathcal{G}$ sending K to $\operatorname{Fix}(\mathcal{G})/\mathcal{G}$. Moreover, if (M, K) has finite topological type, M is isometric to the quotient of a hyperbolic manifold by a finite group of isometries.* □

We have already encountered many examples of singular hyperbolic structures in Chapters 3 and 4. In addition to π–hyperbolic structures, we constructed in §4.3 a pair $(S^3, K \cup L)$ admitting for every $k \geqslant 4$ a hyperbolic structure that is π–singular along the circle K and $\frac{2\pi}{k}$–singular along the circle L.

5.2. Topological restrictions

The existence of a singular hyperbolic structure entails very strong restrictions on the topology of the pair. These are very classical, but we give proofs since none seems available for this context[1].

To state these restrictions, let us introduce the following definition: A closed surface F embedded in the singular hyperbolic pair (M, K) is ***topologically spherical*** when F is a sphere and the respective orders of the points of $F \cap K$ are $\varnothing$, $\{n, n\}$, $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$, with $n \geqslant 2$. Similarly, F is ***topologically euclidean*** when it is one of the following types:

(1) F is a torus avoiding K.
(2) F is a sphere and the respective orders of the points of $F \cap K$ are $\{2, 2, 2, 2\}$, $\{2, 3, 6\}$, $\{2, 4, 4\}$ or $\{3, 3, 3\}$.

In other words, F is topologically spherical (resp. euclidean) if and only if the pair $(F, F \cap K)$ admits a 2–dimensional singular spherical (resp. euclidean) structure with the same orders of singularities as the hyperbolic structure of (M, K) at the points of $F \cap K$ (the definition of singular spherical or euclidean structures being the same as that of singular hyperbolic structures); this immediately follows from the classification of quotients of S^2 and $\mathbb{R}^2$ by properly discontinuous groups of isometries (see for instance [**Thu**$_1$, §13.3]).

[1](Added 2009) See the nice survey [**Sco**$_3$] for more proofs, as well as [**Bon**$_3$].

THEOREM 5.2. *Let the interior of the (compact) knot pair (M, K) admit a singular hyperbolic structure. Then (M, K) contains no sphere meeting K in one point, every topologically spherical surface in (M, K) is pairwise compressible (and therefore meets K in 0 or 2 points), and every pairwise incompressible topologically euclidean surface in (M, K) is pairwise parallel to a boundary component.*

COROLLARY 5.3. *A knot pair whose interior admits a π–hyperbolic structure is simple for Schubert and simple for Conway.* □

Corollary 5.3 provides a useful tool to prove in practice that some splittings of a given knot are actually the characteristic splittings defined in Chapters 2 or 3 (see for instance our discussion of the example of Figure 3.5).

PROOF OF THEOREM 5.2. Let $\mathcal{G}$ be a discrete subgroup of isometries of $\mathbb{H}^3$ such that $\mathrm{int}(M)$ is isometric to $\mathbb{H}^3/\mathcal{G}$. Let $\widetilde{K}$ denote the preimage of K in $\mathbb{H}^3$, consisting of a family of disjoint geodesics.

Considering the regular covering $(\mathbb{H}^3 - \widetilde{K}) \to (M - K)$, one readily sees that M contains no sphere meeting K in one point and that $M - K$ is irreducible. Now, consider a spherical surface F with $F \cap K \neq \varnothing$, and let $\widetilde{F}$ be a component of its preimage. As $(F, F \cap K)$ is topologically spherical, $\widetilde{F}$ is a sphere (by an Euler characteristic argument, for instance). In particular, it meets $\widetilde{K}$ in an even number of points and $F \cap K$ therefore consists of exactly 2 points; thus (M, K) contains no spherical surface meeting K in > 2 points. Considering in $\mathbb{H}^3$ a hyperbolic half-plane bounded by a component of the (geodesic) preimage of K, an outermost arc argument provides a disc Δ in M such that $\partial\Delta$ is the union of the arc $\Delta \cap K$ and of an arc component of $\Delta \cap F$ (which may also contain closed components in the interior of Δ). Using the irreducibility of $M - K$, it easily follows that F bounds a 3–ball B in M^3 such that $B \cap K$ is an unknotted arc, namely that F is pairwise compressible.

This proves the statement of Theorem 5.2 concerning topologically spherical surfaces.

Consider now a pairwise incompressible topologically euclidean closed surface F in $\mathrm{int}(M, K)$. Choose a component $\widetilde{F}$ of its inverse image in $\mathbb{H}^3$ and let $\mathcal{G}_0$ be the stabiliser of $\widetilde{F}$ in $\mathcal{G}$.

The surface $\widetilde{F}$ is a proper submanifold of $\mathbb{H}^3$ (namely the intersection of $\widetilde{F}$ with any compact subset of $\mathbb{H}^3$ is compact). Moreover, it is incompressible: Otherwise $\widetilde{F}$ would admit, by the Equivariant Loop Theorem [**MeeY$_1$, MeeY$_2$, MeeY$_3$**], a $\mathcal{G}$–equivariant compression disc which itself would provide an effective pairwise compression disc for F in (M, K). Thus, $\widetilde{F}$ is isomorphic to S^2 or $\mathbb{R}^2$.

The projection $\widetilde{F} \to F$ is a regular branched covering, with translation group $\mathcal{G}_0$. Because of the hypothesis that F is topologically euclidean, there is a regular branched covering $T \to F$ whose branching indices above the points of F are the same as those of $\widetilde{F} \to F$, and such that T is a 2–torus. Now, $\widetilde{F} \to F$ factors through an unbranched covering $\widetilde{F} \to T$. In particular, $\widetilde{F}$ is isomorphic to $\mathbb{R}^2$ and $\mathcal{G}_0$ has a rank 2 abelian subgroup of finite index. Using this algebraic property of $\mathcal{G}_0$, it is now an easy (and classical) exercise to show that $\mathcal{G}_0$ fixes some point x of the sphere at infinity S^2_∞.

Let $\mathcal{G}_x$ be the stabiliser of x in $\mathcal{G}$. If we arrange that $x = \infty$ in the upper half space model for $\mathbb{H}^3$, then $\mathcal{G}_x$ is induced by a group of euclidean isometries of $\mathbb{R}^3$

respecting the plane $\mathbb{R}^2$ bounding $\mathbb{H}^3$. A group of isometries which is (conjugated to one) of this type is said to be ***parabolic***.

The quotient $(\mathbb{H}^3/\mathcal{G}_x, \mathrm{Fix}(\mathcal{G}_x)/\mathcal{G}_x)$ topologically is a product $(G,P)\times]0,\infty[$, where $G = \mathbb{R}^2/\mathcal{G}_x$ and P is the image of the fixed points of $\mathcal{G}_x$ in $\mathbb{R}^2$. Observe that F is pairwise incompressible in it. By [**Wal**$_3$, Proposition 3.1] (or compare Chapter 8), F is pairwise isotopic to $(G,P)\times 1$ in $(G,P)\times]0,\infty[$. In particular, (G,P) is isomorphic to $(F, F\cap K)$, and $\mathcal{G}_0 = \mathcal{G}_x$.

Adding to $\widetilde{F}$ the fixed point x, we get a topological sphere $\widetilde{F}\cup x$ in $\mathbb{H}^3 \cup S^2_\infty$. (Use the fact that x is the unique limit point of any orbit of the parabolic group $\mathcal{G}_x$ acting on $\mathbb{H}^3\cup S^2_\infty$.) This topological sphere bounds a piece V of the ball $\mathbb{H}^3\cup S^2_\infty$. Since $\mathcal{G}_0 = \mathcal{G}_x$, either $gV = V$ or $V\cap gV = \varnothing$ for every $g\in\mathcal{G}$. Thus, F splits $\mathrm{int}(M)$ into two pieces, one of which is $(V-x)/\mathcal{G}_0$.

As $(V-x)/\mathcal{G}_0$ is also a closed-up component of $\mathbb{H}^3/\mathcal{G}_0 - F$, it is a collar isomorphic to $(F, K\cap F)\times [1,\infty[$ as a pair. Thus F is pairwise parallel to a boundary component of (M,K).

This concludes the proof of Theorem 5.2. □

Define the ***volume*** of a singular hyperbolic pair as the volume of its set of non-singular points (which is an honest Riemannian manifold). In view of the Rigidity Theorem 5.11 to be proved in §5.4, it is important to know when a singular hyperbolic pair has finite volume. The following result shows that this volume finiteness is equivalent to certain topological properties of the knot pair.

THEOREM 5.4. *Let the interior of the (compact) knot pair (M,K) admit a singular hyperbolic structure.*

If this singular hyperbolic structure has finite volume, then each boundary component of (M,K) is topologically euclidean.

Conversely, if all components of $\partial(M,K)$ are topologically euclidean, the singular hyperbolic metric on $\mathrm{int}(M,K)$ *has finite volume except in the following cases:*

(i) *M is a solid torus, and K either is empty or is the core of M.*
(ii) *(M,K) is a rational tangle pair, and the hyperbolic metric is only π–singular.*
(iii) *$(M,K)\cong (F, K\cap F)\times[0,1]$ for some topologically euclidean surface F in M.*

PROOF. A proof of Theorem 5.4 for the case when $K=\varnothing$ is well-known, and appears for instance in [**Thu**$_1$, §5]. We will use a parallel argument in the general case.

The basic ingredient is the celebrated Margulis Lemma:

LEMMA 5.5 (Margulis Lemma). *There is a universal constant μ_0 with the following property: For any discrete subgroup $\mathcal{G}$ of isometries of $\mathbb{H}^3$ and for any $x\in\mathbb{H}^3$, the subgroup of $\mathcal{G}$ generated by isometries moving x at distance $\leqslant \mu_0$ is almost abelian, namely contains an abelian subgroup of finite index.* □

A proof of this Margulis Lemma can be found in [**Thu**$_1$, §5] (compare also [**Jør**]). This result also generalises to complete Riemannian n–manifolds of bounded curvature if "nilpotent" replaces "abelian"; see [**BusK**] for a proof.

We will apply this Margulis Lemma to a group $\mathcal{G}$ such that $\mathrm{int}(M)$ is isometric to $\mathbb{H}^3/\mathcal{G}$ (see Proposition 5.1).

Easy fixed point considerations show that the almost abelian subgroups $\mathcal{G}_1$ of $\mathcal{G}$ are of one of the following two types:

(a) $\mathcal{G}_1$ respects a geodesic of $\mathbb{H}^3$, and is generated by a finite order rotation respecting this geodesic and a glide-rotation along it (each possibly trivial); note that the axis of the rotation can be orthogonal to the geodesic if the rotation angle is π.

(b) $\mathcal{G}_1$ is parabolic, namely conjugated to a group of euclidean isometries of the upper half-space model for $\mathbb{H}^3$.

In particular, a non-elliptic element of $\mathcal{G}$ is contained in a unique maximal almost abelian subgroup of $\mathcal{G}$. (Recall that the elliptic isometries of $\mathbb{H}^3$ are those fixing some point of $\mathbb{H}^3$.)

This observation leads us to consider, for μ_0 as in Lemma 5.5, the set W of points $x \in \mathbb{H}^3$ that are moved at distance $\leqslant \mu_0$ by some non-elliptic element. From the above analysis, one finds that each component of W is, either a tubular neighbourhood of a geodesic of $\mathbb{H}^3$, consisting of all points at hyperbolic distance $\leqslant \varepsilon$ from this geodesic, for some constant ε, or a ***horoball***, conjugated to the part of the upper half-space model lying above a horizontal euclidean plane.

This gives a natural decomposition of $\mathrm{int}(M)$ into the ***thin part*** $M_{\mathrm{thin}}(\mu_0) = W/\mathcal{G}$, and the ***thick part*** $M_{\mathrm{thick}}(\mu_0)$ defined as the closure of $\mathrm{int}(M) - M_{\mathrm{thin}}(\mu_0)$. By construction, each component of $M_{\mathrm{thin}}(\mu_0)$ is, either a tubular neighbourhood of a closed geodesic, or the quotient of a horoball B of $\mathbb{H}^3$ by its parabolic stabiliser in $\mathcal{G}$.

LEMMA 5.6. *The singular hyperbolic metric on* $\mathrm{int}(M,K)$ *has finite volume if and only if the thick part* $M_{\mathrm{thick}}(\mu_0)$ *is compact.*

PROOF OF LEMMA 5.6. Choose in $M_{\mathrm{thick}}(\mu_0)$ a maximal family L of points at distance $\geqslant \mu_0$ from each other. Observe that, by definition of $M_{\mathrm{thick}}(\mu_0)$, the ball of radius $\frac{1}{2}\mu_0$ around $x \in L$ has a volume bounded from below, in terms of the order of singularities of the hyperbolic metric along the (finitely many) components of K.

If $\mathrm{int}(M)$ has finite volume, it follows that L is finite. By maximality of L, $M_{\mathrm{thick}}(\mu_0)$ is now covered by the finitely many balls of radius μ_0 centred at the points of L. As a consequence, $M_{\mathrm{thick}}(\mu_0)$ is compact.

Conversely, assume that $M_{\mathrm{thick}}(\mu_0)$ is compact. Then, $\partial M_{\mathrm{thick}}(\mu_0) = \partial M_{\mathrm{thin}}(\mu_0)$ has finitely many components, so that the number of components of $M_{\mathrm{thin}}(\mu_0)$ is also finite. Also, for each component of $M_{\mathrm{thin}}(\mu_0)$ which is the quotient of a horoball B by its parabolic stabiliser $\mathcal{G}_B$ in $\mathcal{G}$, the quotient $\partial B/\mathcal{G}_B$ is compact; an easy calculation in the upper half-space model then shows that $B/\mathcal{G}_B$ has finite volume, bounded by the area of $\partial B/\mathcal{G}_B$. Since the components of $M_{\mathrm{thin}}(\mu_0)$ which are not of this type are compact, it follows that $M_{\mathrm{thin}}(\mu_0)$ has finite volume. Consequently, so does $\mathrm{int}(M) = M_{\mathrm{thick}}(\mu_0) \cup M_{\mathrm{thin}}(\mu_0)$.

This proves Lemma 5.6. □

We are now ready to prove Theorem 5.4.

If $\mathrm{int}(M,K)$ has finite volume, we showed in the proof of Lemma 5.6 that each end of $\mathrm{int}(M)$ is isometric to the quotient of a horoball B by a parabolic group $\mathcal{G}_B$. Topologically, $B/\mathcal{G}_B$ is a product $\cong (\partial B/\mathcal{G}_B) \times [0,\infty[$, respecting the singularity orders (consider the foliation of B by vertical lines in the upper half-space model).

It follows that the corresponding component of $\partial(M, K)$ is isomorphic to the topologically euclidean surface $\partial B/\mathcal{G}_B$. This proves that, if $\operatorname{int}(M, K)$ has finite volume, then every boundary component of (M, K) is topologically euclidean.

Conversely, consider a boundary component F of (M, K) that is topologically euclidean.

If F is pairwise compressible, one easily sees, surgering F along an effective compression disc and then using the pairwise irreducibility of (M, K) (Theorem 5.2), that either M is a solid torus and K is empty or is the core of M, or (M, K) is a rational tangle pair. These are Cases (i) and (ii) of Theorem 5.4.

Otherwise, let F' be obtained by pushing F inside of $\operatorname{int}(M, K) \cong \mathbb{H}^3/\mathcal{G}$; and let $\widehat{F}'$ be a component of the preimage of F' in $\mathbb{H}^3$. Since F' is pairwise incompressible, the argument used in the proof of Theorem 5.2 shows that the stabiliser of $\widetilde{F}'$ in $\mathcal{G}$ is parabolic. Also, if V is the piece of $\mathbb{H}^3 \cup S^2_\infty$ bounded by the topological sphere made of $\widehat{F}'$ and of its limit point $x \in S^2_\infty$, the quotient of $V - x$ by the stabiliser of $\widehat{F}'$ gives a pairwise parallelism $W \cong (F', K \cap F') \times [0, \infty[$ between $(F', K \cap F')$ and an end of $\operatorname{int}(M, K)$. Observe that, for t sufficiently large, the part $F' \times [t, \infty[$ of W is contained in the image of $M_{\text{thin}}(\mu_0)$ in $\operatorname{int}(M)$; this comes from the elementary observation that, given a euclidean translation h of the upper half-space model, the hyperbolic distance from y to $h(y)$ tends to 0 when the height of y tends to $+\infty$. Also, if the component of $\partial(M, K)$ adjacent to this parallelism $F' \times [0, \infty[$ is not F, note that (M, K) is isomorphic to $(F, K \cap F) \times [0, 1]$.

Thus, we have proved the following: For each topologically euclidean component F of $\partial(M, K)$, either F is adjacent to a component of $M_{\text{thin}}(\mu_0) \subset \operatorname{int}(M)$, or (M, K) is one of the exceptions (i), (ii), (iii) of Theorem 5.4.

This readily proves that, if all components of $\partial(M, K)$ are topologically euclidean, either (M, K) is one of the exceptions (i), (ii), (iii), or $M_{\text{thick}}(\mu_0)$ is compact. In the latter case, Lemma 5.6 asserts that $\operatorname{int}(M, K)$ has finite volume.

This concludes the proof of Theorem 5.4. □

As a byproduct of the proof of Theorem 5.4, we can define a natural compactification of a singular hyperbolic pair with finite volume.

Indeed, with the notation of Theorem 5.4 and assuming the volume of $\operatorname{int}(M, K)$ finite, we showed that each end of $\operatorname{int}(M, K)$ has a neighbourhood isometric to the quotient of a horoball B by its parabolic stabiliser $\mathcal{G}_B$. The horoball B has a natural foliation by geodesics going to its asymptotic point in S^2_∞ (these geodesics are just vertical straight lines in the upper half-space model). In the quotient $B/\mathcal{G}_B \subset \operatorname{int}(M)$, this foliation gives a foliation of $B/\mathcal{G}_B$ by proper lines joining $\partial B/\mathcal{G}_B$ to infinity. Adding an end point to each such line, one gets a natural compactification of $B/\mathcal{G}_B$ by a copy S of $\partial B/\mathcal{G}_B$. Observe that $(B/\mathcal{G}_B) \cup S$ has a natural structure of differentiable manifold with boundary.

Doing this for all ends of $\operatorname{int}(M, K)$, we can thus use the singular hyperbolic structure on $\operatorname{int}(M, K)$ to construct a differentiable manifold (M^+, K^+) whose interior is naturally identified to $\operatorname{int}(M, K)$. Of course, (M^+, K^+) is isomorphic to (M, K) by standard collaring theorems in dimension 3 but, in general, there is no isomorphism $(M^+, K^+) \to (M, K)$ extending the identification between their interiors. For instance, although every geodesic of the foliation of $B/\mathcal{G}_B$ converges to a single point of ∂M^+, there is no reason why it should have a single limit point in ∂M. We will require the hyperbolic metric to be tame near ∂M, by introducing the following definition.

Consider a (compact) knot pair (M, K), possibly with boundary. A ***finite volume singular hyperbolic structure*** on the knot pair (M, K) consists of:

(i) a singular hyperbolic metric with finite volume on its interior $\mathrm{int}(M, K)$;
(ii) an isomorphism $(M, K) \to (M^+, K^+)$ (for the category DIFF or TOP we are working in), respecting the identification $\mathrm{int}(M, K) = \mathrm{int}(M^+, K^+)$, between (M, K) and the natural compactification (M^+, K^+) of $\mathrm{int}(M, K)$ defined as above by using its singular hyperbolic structure.

This definition is mostly designed for the following property to be satisfied: Let (M, K) and (M', K') be two knot pairs, each equipped with a finite volume singular hyperbolic structure. Then, *any isometry* $\mathrm{int}(M, K) \to \mathrm{int}(M', K')$ *extends to an isomorphism* $(M, K) \to (M', K')$.

A last topological restriction, which is important in view of the determination of the arborescent part of a knot as in Chapter 3, is the following:

THEOREM 5.7. *Let the interior of the knot pair (M, K) admit a finite volume singular hyperbolic structure. Then (M, K) cannot be Seifert fibred.*

Also, (M, K) cannot be presented as a Montesinos pair in the sense of Chapter 3, with possibly one ring R, in such a way that the points of $K - R$ are π–singular.

PROOF. Write $\mathrm{int}(M)$ as the quotient of $\mathbb{H}^3$ by a discrete group $\mathcal{G}$ of isometries (Proposition 5.1).

Assume (M, K) Seifert fibred, in search of a contradiction, and lift this fibration to a foliation of $\mathbb{H}^3$. First observe that any two leaves of this foliation stay at bounded distance from each other in $\mathbb{H}^3$, and consequently have the same set of cluster points in the sphere at infinity S^2_∞.

If one leaf of this foliation is compact, its follows that so are all other leaves. But the antipodal map on generic fibres gives an involution of $\mathbb{H}^3$ whose fixed point set consists of circles, which is absurd by elementary Smith theory.

Thus, all leaves are non compact. Considering the subgroup of $\mathcal{G}$ generated by the monodromy of a fibre of $\mathrm{int}(M) = \mathbb{H}^3/\mathcal{G}$, one sees that the lift of this fibre is asymptotic to 1 or 2 points of S^2_∞. Since these 1 or 2 points are asymptotic to all leaves of $\mathbb{H}^3$, they must be respected by $\mathcal{G}$. We conclude that $\mathcal{G}$ is either parabolic or generated by some glide rotation along a geodesic of $\mathbb{H}^3$, contradicting the hypothesis that $\mathbb{H}^3/\mathcal{G}$ has finite volume.

This concludes the proof that (M, K) cannot be Seifert fibred.

We will use this first part to prove the second statement of Theorem 5.7. Assume, in search of a contradiction, that (M, K) is a Montesinos pair, possibly with a ring R, and that the points of $K - R$ are π–singular. Then, M admits a 2–fold cover N branched along $K - R$. By an observation of Montesinos [**Mon**$_1$] (see also the Appendix), N admits a Seifert fibration for which the preimage L of R consists of 0, 2 fibres. On the other hand, the singular hyperbolic metric on (M, K) gives a finite volume singular hyperbolic metric on (N, L), contradicting the existence of this Seifert fibration. □

5.3. Existence theorems

Thurston announced in 1981 that Theorems 5.2, 5.4 and 5.7 admit a converse provided the knot string K is not empty. This is part of his more general Orbifold

Geometrisation Conjecture (see [**Thu$_2$, Sco$_3$, BonS$_1$**]) which, when specialised to orbifolds with rotational singularities, reduces to:

Conjecture 5.8 (Geometrisation Conjecture for knot pairs). *Let (M, K) be a knot pair where each component K_i of K is labelled by an integer $k_i \geqslant 2$. Assume that (M, K) contains no sphere meeting K in 1 point, that every (topologically) spherical surface in (M, K) is pairwise compressible, and that the components of $(\partial M, \partial K)$ are euclidean (for the obvious definition of "spherical" and "euclidean" taking the labels k_i into account). Assume moreover that every closed pairwise incompressible euclidean surface in (M, K) is pairwise parallel to a boundary component. Then the interior of (M, K) is isomorphic to some $(X/\mathcal{G}, (\mathrm{Fix}\,\mathcal{G})/\mathcal{G})$ such that the stabiliser of a point of $\mathrm{Fix}\,\mathcal{G}$ located above the component K_i of K is cyclic of order k_i, and where X and $\mathcal{G}$ are of one of the following types:*

1. *$\mathcal{G}$ is a finite covolume discrete group of isometries of the hyperbolic 3–space $X = \mathbb{H}^3$, namely (M, K) has a finite volume singular hyperbolic structure.*
2. *X is a Seifert manifold, $\mathcal{G}$ is finite and respects the fibration. An example of this case is provided by elementary pairs as defined in Chapter* 3 *(see* [**Mon$_1$**] *and the Appendix).*
3. *$\mathcal{G}$ is a finite isometry group of $X = T^3$ or $T^2 \times \mathbb{R}$, and respects no circle fibration of X. An example of this type occurs for the figure eight knot labelled by* 3*; see the discussion just above Proposition* 5.10.

Thurston's difficult proof of this conjecture (for $K \neq \varnothing$) is based on his hyperbolisation theorem for Haken manifolds [**Thu$_1$, Thu$_2$, Mor$_1$, Kap, Ota$_1$, Ota$_2$**] and on an analysis of the deformations and degenerations of such hyperbolic structures under Dehn surgery[2]. However, for the case needed by Chapter 3, we feel of interest to give a proof using only Thurston's simpler manifold hyperbolisation theorem.

Theorem 5.9. *Let (M, K) be a knot pair such that $M \subset S^3$ and $(\partial M, \partial K)$ consist of Conway spheres. Assume that (M, K) is pairwise irreducible, simple for Schubert and simple for Conway. When $\partial M = \varnothing$, assume moreover that (M, K) contains a pairwise incompressible surface. Then, exactly one of the following holds:*

(a) *(M, K) admits a finite volume complete π–hyperbolic structure.*
(b) *(M, K) is elementary, as defined in Chapter* 3.

Proof of Theorem 5.9. Let $\widehat{M}$ be the double branched covering of (M, K), which exists since $M \subset S^3$ and $(\partial M, \partial K)$ consists of Conway spheres. Conclusion (a) is then equivalent to the existence of a complete non-singular hyperbolic structure of finite volume on $\mathrm{int}(\widehat{M})$ for which the covering involution τ is an isometry.

By pairwise irreducibility of (M, K), $\widehat{M}$ is irreducible (use [**KimT**] or the Equivariant Sphere Theorem of [**MeeY$_1$, MeeY$_2$, MeeY$_3$**]). Also, when $\partial M \neq \varnothing$, the pairwise incompressible surface F lifts to a surface $\widehat{F} \subset \widehat{M}$, which is incompressible in $\widehat{M}$ by the Equivariant Loop Theorem [**MeeY$_1$, MeeY$_2$, MeeY$_3$**]. As a consequence, $\widehat{M}$ is a Haken manifold.

[2](Added 2009) It is only relatively recently that detailed expositions of the proof of this conjecture have become available. A proof in the full generality of the conjecture appears in [**BoiLP**]. The slightly simpler case of knot pairs, as in the above statement of Conjecture 5.8, was already fully expounded in [**CooHK, BoiP, BoiMP**]

We now borrow a lemma from [**Bon**$_1$, Lemme 3.1], which is also implicit in [**Tol**$_2$]; see [**BonS**$_1$, Proposition 11] or [**MeeS**] for generalisations. This result asserts that one of the following holds (possibly both):

(i) Every incompressible torus in $\widehat{M}$ is parallel to a boundary component (namely $\widehat{M}$ is atoroidal);
(ii) $\widehat{M}$ admits a Seifert fibration preserved by τ.

The rough lines of the argument of [**Bon**$_1$] and [**Tol**$_2$] run as follows: Assuming (i) fails, there exists a non-peripheral incompressible torus T in $\widehat{M}$. After a sequence of cut-and-paste isotopies, this torus can be chosen so that $T-\tau(T)$ and $\tau(T)-T$ consist of annuli. Then, a regular neighbourhood U of $T\cup\tau(T)$ admits a fibration by circles that is preserved by τ. Each component of $(\partial U)/\tau$ is a Conway sphere or a torus avoiding K. It then follows from the hypotheses on (M,K) that each component of $M-U$ is a solid torus or a collar of a boundary component of M. The circle fibration of U now easily extends to a Seifert fibration of M preserved by τ, and (ii) holds.

First consider the case where (ii) holds. In [**Mon**$_1$], Montesinos studied which fibre-preserving involutions τ of a Seifert manifold $\widehat{M}$ give a quotient $\widehat{M}/\tau$ embeddable in S^3. For these, either the pair $(\widehat{M}/\tau, \mathrm{Fix}\tau)$ is a Montesinos pair, as defined in Chapter 3 or allowing more rings as in §8.2, or $\widehat{M}/\tau$ is S^3 minus a collection of fibres of a Seifert fibring of S^3 and K consists of fibres of this fibration. Among such, only elementary Montesinos pairs satisfy the hypotheses of Theorem 5.9. Indeed, we will prove in Chapter 8 (see Theorem 8.15(b)) that non-elementary Montesinos pairs always contain a non-trivial Conway sphere or a non-trivial torus avoiding the knot. The possibility that (M,K) is Seifert fibred is excluded by the hypothesis that it contains a pairwise incompressible surface $\widehat{F}$ which is not a torus disjoint form K; indeed, Waldhausen's classification of non-trivial surfaces in Seifert manifolds [**Wal**$_2$] shows that such a surface would have to be transverse to the fibration, which is impossible because $H_1(M)=H_1(S^3)=0$.

Therefore, (b) holds in Case (ii).

In Case (i), Thurston's Hyperbolisation Theorem [**Thu**$_2$] asserts that $\mathrm{int}(\widehat{M})$ admits either a complete hyperbolic structure of finite volume or a Seifert fibration. When $\mathrm{int}(\widehat{M})$ is hyperbolic, Mostow's theorem [**Mos**$_1$] provides an involution τ' which is isometric on $\mathrm{int}(\widehat{M})$ and homotopic to τ. By [**Tol**$_3$], there exists φ isotopic to the identity such that $\tau'=\varphi\tau\varphi^{-1}$ (the results of [**Tol**$_3$] are stated with the hypothesis that $H^1(\widehat{M})\neq 0$ but, when $\partial\widehat{M}=\varnothing$, only the existence of $\widehat{F}$ is needed). Changing the hyperbolic structure via φ, τ is now isometric, which gives a π–hyperbolic structure on (M,K).

Finally, we have to consider the case where $\widehat{M}$ is Seifert fibred. Because we are in Case (i), Waldhausen's classification [**Wal**$_3$] of incompressible surfaces in $\widehat{M}$ shows that the Seifert fibration can be chosen to be transverse to $\widehat{F}$, so that $\widehat{F}$ splits $\widehat{M}$ into two bundles with fibre the interval $[0,1]$. Applying [**Tol**$_1$, Theorem 3], one can easily deform the Seifert fibration to make it invariant under τ, in which case the discussion of Case (ii) enables us to conclude. □

If K is a knot in S^3 such that (S^3,K) is singular hyperbolic, Corollary 5.3 and Thurston's Hyperbolisation Theorem [**Thu**$_2$**, Mor**$_1$**, Kap, Ota**$_1$**, Ota**$_2$] show that

$S^3 - K$ admits a finite volume hyperbolic structure. Indeed, $S^3 - K$ cannot be Seifert fibred, by Theorem 5.7.

The converse is certainly not true, in that sense that, if $S^3 - K$ is hyperbolic, not every labelling of the components of K can be realised by a singular hyperbolic structure on (S^3, K). For instance, although the figure-eight knot has a hyperbolic complement [**Thu**$_1$, §3], it does not admit any π– or $\frac{2\pi}{3}$–hyperbolic structure: Its 2–fold branched covering is a lens space (see the Appendix). We also saw in §4.3 that its 3–fold branched covering is the 2–fold branched covering of the Borromean rings, and in particular admits a Seifert fibration with orbit space $\mathbb{RP}^2$ and two exceptional fibres, each with order 2 monodromy; beware that the order 3 covering translation cannot respect the fibres (compare Theorem 16.18 and §18.1).

PROPOSITION 5.10. *Label by an integer* $k_i \geqslant 2$ *each component* K_i *of a knot* (S^3, K) *whose complement has a complete hyperbolic structure. Then Thurston's Orbifold Geometrisation Theorem* 5.8 *implies that* (S^3, K) *admits a hyperbolic structure that is* $\frac{2\pi}{k_i}$*–singular near* K_i*, except when one of the following holds:*

(a) (S^3, K) *contains a pairwise incompressible Conway sphere meeting only components of* K *labelled by* 2.
(b) (S^3, K) *is an elementary (Montesinos) pair and at most one* k_i *is* $\neq 2$*, corresponding to the ring of the pair (if any) as defined in Chapter* 8.
(c) (S^3, K) *is the figure eight knot labelled by* $k = 3$*, or the two-component link of Figure* 4.8 *with respective labels* 2 *and* 3.

Note that there is only one such exception when all k_i are $\geqslant 3$.

PROOF. Assume that (a) does not hold. The sphere S^3 contains no sphere meeting K in 1 or 3 points for homological reasons and, since $S^3 - K$ is hyperbolic, (S^3, K) is simple for Schubert and pairwise irreducible (see Theorem 5.2). Thus Thurston's Orbifold Geometrisation Theorem applies and provides one of three things:

(i) the singular hyperbolic structure required;
(ii) a homeomorphism $(S^3, K) \cong (X/\mathcal{G}, (\mathrm{Fix}\mathcal{G})/\mathcal{G})$, for some finite group $\mathcal{G}$ respecting a Seifert fibration of X, such that the order of the stabiliser of a point of $\mathrm{Fix}\,\mathcal{G}$ is the label k_i of the corresponding component of K. The argument of [**Mon**$_1$] straightforwardly extends to this situation to show that Case (b) then holds;
(ii) the last possibility is a homeomorphism $(S^3, K) \cong (\mathbb{R}^3/\mathcal{G}, (\mathrm{Fix}\mathcal{G})/\mathcal{G})$ for some crystallographic group $\mathcal{G}$ respecting no direction at infinity. By enumeration of these groups ([**Dun**$_1$**, Dun**$_2$**, Burk**$_1$**, Burk**$_2$**, ITC, BonS**$_3$]), this only yields the two labelled knots of Case (c). □

5.4. The Rigidity Theorem for singular hyperbolic structures

This section is devoted to proving the following uniqueness property for singular hyperbolic structures.

THEOREM 5.11 (Rigidity Theorem). *Let* (M, K) *and* (M', K') *be two connected knot pairs, each equipped with a finite volume singular hyperbolic structure. Assume* $K \neq \varnothing$*. Then, every isomorphism* $\varphi\colon (M, K) \to (M', K')$ *respecting the order of the singular points (namely sending a* $\frac{2\pi}{k}$*–singular point to a* $\frac{2\pi}{k}$*–singular point) is pairwise isotopic to an isometry.*

Here φ_0, φ_1: $(M, K) \to (M', K')$ are ***pairwise isotopic*** if they are isotopic through a family of isomorphisms φ_t: $(M, K) \to (M', K')$, $0 \leqslant t \leqslant 1$, each sending K to $K' = \varphi_t(K)$. In the proof of Theorem 5.11, finding an isometry $(M, K) \to (M', K')$ will be an easy consequence of Mostow's Rigidity Theorem [**Mos**$_2$]. The difficult part of the proof will actually be to show that this isometry is pairwise isotopic to φ.

Before going any further, let us mention some important corollaries of Theorem 5.11.

COROLLARY 5.12. *For a connected finite volume singular hyperbolic knot pair (M, K) with $K \neq \varnothing$, the group $\pi_0 \operatorname{Aut}(M, K)$ of isotopy classes of automorphisms of (M, K) is finite.*

PROOF OF COROLLARY 5.12 (ASSUMING THEOREM 5.11). By Theorem 5.11, it suffices to show that the isometry group of (M, K) is finite.

An isometry is completely determined by the image of a point of the thick part $M_{\text{thick}}(\mu_0)$ and of the image of an orthogonal basis for its tangent space. Since $M_{\text{thick}}(\mu_0)$ is compact by Lemma 5.6, it follows that this isometry group is compact.

Also, two isometries of $\operatorname{int}(M, K) \cong \mathbb{H}^3/\mathcal{G}$ which are sufficiently close on the compact $M_{\text{thick}}(\mu_0)$ lift to isometries of $\mathbb{H}^3$ which act similarly on the components of the preimage of K. As the end points of these preimages are dense on the sphere at infinity S^2_∞, these two isometries coincide on S^2_∞ and thus on $\mathbb{H}^3$. This proves that the isometry group of (M, K) is discrete, and therefore finite (compare [**Thu**$_1$, §5.7]). □

COROLLARY 5.13. *Consider two degree +1 isomorphisms φ, φ': $(M, K) \to (M', K')$ between connected finite volume singular hyperbolic knot pairs, which coincide on $S \cap \partial K$ for some component S of ∂M with $S \cap \partial K \neq \varnothing$. Then φ and φ' are pairwise isotopic.*

PROOF OF COROLLARY 5.13 (ASSUMING THEOREM 5.11). Without loss of generality, we can clearly assume that $(M, K) = (M', K')$, that $\varphi' = \operatorname{Id}$ and, by Theorem 5.11, that φ is an isometry of (M, K). We want to show that φ is the identity.

Identify $\operatorname{int}(M, K)$ to $\mathbb{H}^3/\mathcal{G}$ as in Proposition 5.1. As in the proof of Theorem 5.7, the boundary component S defines a parabolic subgroup of $\mathcal{G}$, up to conjugation in $\mathcal{G}$. For a suitable identification of $\mathbb{H}^3$ with the upper half-space model $\mathbb{R}^2 \times]0, \infty[$, this parabolic subgroup $\mathcal{G}_\infty$ is generated by (euclidean) rotations around lines $x \times]0, \infty[$ of $\mathbb{R}^2 \times]0, \infty[$, where x ranges over all points of a linear lattice L in $\mathbb{R}^2$. These lines $x \times]0, \infty[$ are the geodesic components of the preimage of $K - \partial K$ that are asymptotic to the point $\infty \in S^2_\infty$ in $\mathbb{H}^3$.

Now, the isometry φ lifts to a euclidean isometry of this upper half space model $\mathbb{R}^2 \times]0, \infty[$ which has degree +1 and acts trivially on $L/\mathcal{G}_\infty \cong S \times \partial K$. It easily follows that φ is the identity. □

Observe that, in Corollary 5.13, the condition that φ and φ' have the same degree is absolutely necessary. Indeed, if (M, K) is constructed from some graph $\Gamma \subset S^2 \subset S^3$, namely if (M, K) is obtained from (S^3, Γ_0) by removing regular neighbourhoods of the vertices of Γ_0 (compare §1.2 and §4.1), the reflection through S^2 gives a degree −1 automorphism of (M, K) fixing K.

PROOF OF THE RIGIDITY THEOREM 5.11. By Proposition 5.1, the interiors of (M, K) and (M', K') are respectively isometric to the quotients of $\mathbb{H}^3$ by two discrete groups of isometries $\mathcal{G}$ and $\mathcal{G}'$. A first step is to note that, because $\varphi\colon (M,K) \to (M',K')$ respects the orders of singular points, it induces an isomorphism $\varphi_*\colon \mathcal{G} \to \mathcal{G}'$, well-defined modulo inner automorphisms of $\mathcal{G}$ and $\mathcal{G}'$. This is a consequence of the following remark.

LEMMA 5.14. *For every component K_i of K where the hyperbolic structure of* $\operatorname{int}(M,K)$ *is $\frac{2\pi}{k_i}$–singular, choose a meridian $m_i \in \pi_1(M-K)$ of K_i. Then $\mathcal{G}$ is isomorphic to the quotient of $\pi_1(M-K)$ by the subgroup normally generated by the $(m_i)^{k_i}$.*

PROOF OF 5.14. If $\widetilde{K}$ denotes the preimage of K in $\mathbb{H}^3$, just note that $\pi_1(\mathbb{H}^3 - \widetilde{K})$ is normally generated by the meridians of the components of $\widetilde{K}$, and that each of these meridians projects in $\pi_1(M-K)$ to a conjugate of some $(m_i)^{k_i}$. □

Now, as $\mathcal{G}$ and $\mathcal{G}'$ have finite covolume, Mostow's Rigidity Theorem [**Mos**$_1$**, Mos**$_2$**, Thu**$_1$**, BenP**] provides an isometry $\widetilde{\psi}$ of $\mathbb{H}^3$ such that $\varphi_*(\alpha) = \widetilde{\psi}\alpha\widetilde{\psi}^{-1}$ for every $\alpha \in \mathcal{G}$. In particular, $\widetilde{\psi}\mathcal{G}\widetilde{\psi}^{-1} = \mathcal{G}'$ and $\widetilde{\psi}$ therefore induces an isomorphism $\psi\colon (M,K) \to (M',K')$. We will show that φ and ψ are pairwise isotopic.

The proof of Theorem 5.11 will be achieved by the following two claims.

ASSERTION 5.15. *The isomorphism φ can be pairwise isotoped so that $\theta = \psi^{-1}\varphi$ is periodic.*

ASSERTION 5.16. *The periodic isomorphism θ is then the identity.*

We begin by proving the easier of these two assertions, namely Assertion 5.16. The argument is very similar to the one used in the proof of Lemma 4.2 (compare also [**ConM**] or Lemma 16.7).

PROOF OF ASSERTION 5.16. Lift θ to $\widetilde{\theta} : \mathbb{H}^3 \to \mathbb{H}^3$ (use Lemma 5.14 and start by lifting it to the complement $\mathbb{H}^3 - \widetilde{K}$ of the preimage of K in $\mathbb{H}^3$). Modulo inner automorphism, the conjugation by $\widetilde{\theta}$ induces the same automorphism of $\mathcal{G}$ as $\widetilde{\psi}^{-1}\widetilde{\varphi}$, namely the identity. The lifting $\widetilde{\theta}$ can therefore be chosen so that it commutes with every element of $\mathcal{G}$. In particular, if θ is of order n, $\widetilde{\theta}^n$ is an element of the centre of $\mathcal{G}$ and $\widetilde{\theta}$ is therefore periodic as $\mathcal{G}$ has trivial centre (easy exercise in the isometry group of $\mathbb{H}^3$).

A standard argument considering end points of quasi-geodesics (see for instance [**Mos**$_1$**, Mos**$_2$] [**Thu**$_1$, §5.9]) provides an extension of $\widetilde{\theta}$ to a homeomorphism of the compact ball $\mathbb{H}^3 \cup S^2_\infty$, where S^2_∞ is the sphere at infinity of $\mathbb{H}^3$. Moreover, since $\widetilde{\theta}$ commutes with the elements of $\mathcal{G}$, the restriction $\widetilde{\theta}_{|S^2_\infty}$ fixes the dense subset of points fixed by some elements of $\mathcal{G}$ and is therefore the identity. Thus, $\widetilde{\theta}$ is a periodic homeomorphism of a 3–ball that fixes the boundary. It now follows from Newman's theorem [**New, Smi, Dre**] that $\widetilde{\theta}$ is the identity, which proves Assertion 5.16. □

PROOF OF ASSERTION 5.15. The proof is quite simple when $\partial K = \varnothing$. Indeed, by Thurston's hyperbolisation theorem, $\operatorname{int}(M) - K$ then admits a finite volume hyperbolic structure. Mostow's theorem [**Mos**$_1$**, Mos**$_2$] provides an isometry θ' of $\operatorname{int}(M)-K$ that is homotopic, and therefore isotopic by [**Wal**$_3$, Theorem 7.1], to the restriction of θ. Since the isometry group of $\operatorname{int}(M)-K$ is finite (see Corollary 5.12 or [**Thu**$_1$, §5.7]), θ' is periodic. Using a regular neighbourhood of K, the assertion easily follows in this case.

The proof in the general case involves the same ideas via a doubling argument. For this, we need a first lemma.

LEMMA 5.17. *Let the knot pair* (M, K) *admit a finite volume singular hyperbolic structure. Then every pairwise incompressible annulus* A *embedded in* $M - K$ *with* $\partial A \subset \partial M$ *is pairwise boundary parallel, namely obtained from a boundary surface by pushing its interior inside of* M *(up to pairwise isotopy).*

PROOF OF LEMMA 5.17. Let $\mathrm{int}(M, K)$ be isometric to $\mathbb{H}^3/\mathcal{G}$, and let $\mathcal{G}_A$ be the image of $\pi_1(A) \subset \pi_1(M - K)$ in $\mathcal{G}$ (see Lemma 5.14). By pairwise incompressibility of A, $\mathcal{G}_A$ is non-trivial and fixes exactly one point $x \in S^2_\infty$; also, $\mathcal{G}_A$ respects a component $\widehat{A}$ of the preimage of A in $\mathbb{H}^3$.

Considering a fundamental domain, $\widehat{A} \cup x$ is a topological sphere in $\mathbb{H}^3 \cup S^2_\infty$; in particular, it bounds a piece N of $\mathbb{H}^3 \cup S^2_\infty$. As in the proof of Theorem 5.2, the consideration of the annulus $\widehat{A}/\mathcal{G}_A$ in the solid torus $\mathbb{H}^3/\mathcal{G}_A$ now shows that $(N - x)/\mathcal{G}_A$ provides a pairwise parallelism between A and an annulus in ∂M. □

REMARK 5.18. There is an alternative proof of Lemma 5.17 which makes it a purely combinatorial corollary of Theorems 5.2 and 5.7. Compare Proposition 7.4.

Now, consider the double (DM, DK) constructed by gluing two copies of (M, K) along $\partial M - \partial_T M$, where $\partial_T M$ consists of the torus components of ∂M (avoiding K). Let τ denote the involution of DM exchanging its two halves.

LEMMA 5.19. *The interior of* $DM - DK$ *admits a finite volume hyperbolic metric for which* τ *is an isometry.*

Such a τ–invariant hyperbolic structure can also be interpreted as a hyperbolic-structure-with-totally-geodesic-boundary on $(M - \partial_T M) - K$.

PROOF OF LEMMA 5.19. Using Lemma 5.17 and Theorem 5.2, one easily sees that (DM, DK) is simple for Schubert or, equivalently, contains no pairwise incompressible non-peripheral torus. Also, $DM - DK$ cannot be Seifert fibred: Indeed, by [**Wal**$_2$, Satz 2.8], the incompressible surface $(\partial M - \partial_T M) - K$ could be assumed to be transverse to the fibration, and the $[0, 1]$–fibration so induced on $M - K$ would reveal non boundary parallel annuli, contradicting Lemma 5.17. Then, Thurston's hyperbolisation theorem asserts the existence of a finite volume hyperbolic structure on $\mathrm{int}(DM - DK)$.

By Mostow's theorem [**Mos**$_1$**, Mos**$_2$], τ is homotopic to an isometric involution τ'. A result of J. Tollefson already quoted [**Tol**$_3$], in fact easy to prove "by hand" in this case, then provides an isotopy conjugating τ to τ' and concludes our proof of Lemma 5.19. □

If $D\theta\colon (DM, DK) \to (DM, DK)$ denotes the double of θ, Mostow's Rigidity Theorem provides an isometry $(D\theta)'$ of $\mathrm{int}(DM - K)$ that is homotopic to the restriction of $D\theta$. As τ is isometric and commutes with $D\theta$, it must also commute with $(D\theta)'$ by uniqueness in the conclusion of Mostow's theorem. Thus, $(D\theta')$ is the double $D\theta'$ of some isomorphism θ' of $(M - \partial_T M) - K$. Also, $D\theta'$ is periodic by finiteness of the isometry group of $\mathrm{int}(DM - DK)$. By adjustment of θ' near K, it is now easy to construct a periodic isomorphism θ'' of the compact pair (M, K) such that the doubles $D\theta$ and $D\theta''$ are pairwise isotopic in (DM, DK).

To conclude the proof of the Rigidity Theorem 5.11, we must show that θ and θ'' are isotopic in (M, K). This will be accomplished by Proposition 5.20 below, applied to the surface $\partial M - \partial_T M$ in (DM, DK); note that (DM, DK) cannot fibre over S^1 with $\partial M - \partial_T M$ a union of fibres as this would reveal annuli in (M, K) contradicting Lemma 5.17.

PROPOSITION 5.20. *Let F be a closed surface embedded in the connected knot pair (M, K) such that $M - K$ is irreducible and $F - K$ is incompressible in $M - K$. Let ψ be an isomorphism of (M, K) preserving F and pairwise isotopic to the identity. Then (at least) one of the following holds:*

(i) *There is a pairwise isotopy from ψ to the identity which respects F.*
(ii) *(M, K) fibres over S^1 with F a union of fibres, and ψ is connected to the identity by the composition of a pairwise isotopy respecting F and a fibration-preserving pairwise isotopy.*

PROOF OF PROPOSITION 5.20. Clearly, an iterative argument reduces the proof to the case where F is connected. Therefore, we can and do assume henceforth that F is connected.

First, consider the case where the isotopy from ψ to the identity fixes ∂M and a tubular neighbourhood U of K. To deal with this case, we will make use of arguments from [**Wal**$_3$].

Let M_0 denote $M - \mathrm{int}(U)$. The isotopy gives by restriction a homotopy $H\colon M_0 \times [0, 1] \to M_0$ fixing ∂M_0 (namely $H_{|\partial M_0)\times[0,1]}$ is the projection onto ∂M_0). Conversely, it is proved in [**Wal**$_3$, §7] that the existence of such a homotopy implies the existence of an isotopy from $\psi_{|M_0}$ to Id. The proof of this fact in pages 81–85 of [**Wal**$_3$] makes use of a hierarchy for M_0. If the first surface of the hierarchy is chosen to be $F_0 = F \cap M_0$, a careful examination of the proof shows that the isotopy of $\psi_{|M_0}$ it provides, either fixes $F_0 \cup \partial M_0$ and therefore extends to a pairwise isotopy of type (i), or directly gives a pairwise isotopy of type (ii) in some case where $\partial M_0 = \varnothing$. (Note that, when F_0 is closed, one must use the arguments of Case 4 (the most elaborate case) of the proof of [**Wal**$_3$, Theorem 7.1].)

Now, consider the general case where the pairwise isotopy may not fix $K \cup \partial M$. Without loss of generality, we assume that ψ and the original pairwise isotopy fix ∂M, the arc components of K and the closed components of K avoided by F. Also, we may require that the pairwise isotopy is "monotonic" (namely conjugated to a rotation) on the other components of K. With the analysis of the previous case, the proof of Proposition 5.20 will be completed by the following affirmation.

CLAIM 5.21. *In the above situation, either the isotopy fixes K or Conclusion* (ii) *of Proposition* 5.20 *holds.*

The proof of Claim 5.21 will make use of the following notion: Consider two surfaces F_0 and G_0 in a 3–manifold M_0, such that $\partial F_0 = \partial G_0 \subset \partial M_0$ and the intersection of F_0 and G_0 is transverse (including near the boundary). A ***rel ∂ blister*** (or a product region) between F_0 and G_0 is an embedded copy P of $\Sigma \times [0, 1]/\sim$, where Σ is a compact surface, the relation $\sim$ pinches $(\partial\Sigma) \times [0, 1]$ by collapsing each arc $* \times [0, 1]$ to a point, and $F_0 \cap \partial P$ and $G_0 \cap \partial P$ respectively correspond to $\Sigma \times 0$ and $\Sigma \times 1$; moreover, it is required in the definition that $F_0 \cap \mathrm{int}(P) = \varnothing$ and that $G_0 \cap \mathrm{int}(P)$ may be non-empty only when Σ is a disc. A fundamental lemma of Waldhausen [**Wal**$_3$, Proposition 5.4] asserts that *there exists such a blister as soon as F_0 and G_0 are incompressible and homotopic in M_0 by a homotopy fixing their boundary.*

Proof of Claim 5.21. Assume that ψ does not fix K. By hypothesis, it is anyway isotopic fixing K to an isomorphism ψ' of (M, K) that is the identity outside of a tubular neighbourhood U of K and "monotonically winds around K" as in Figure 5.1.

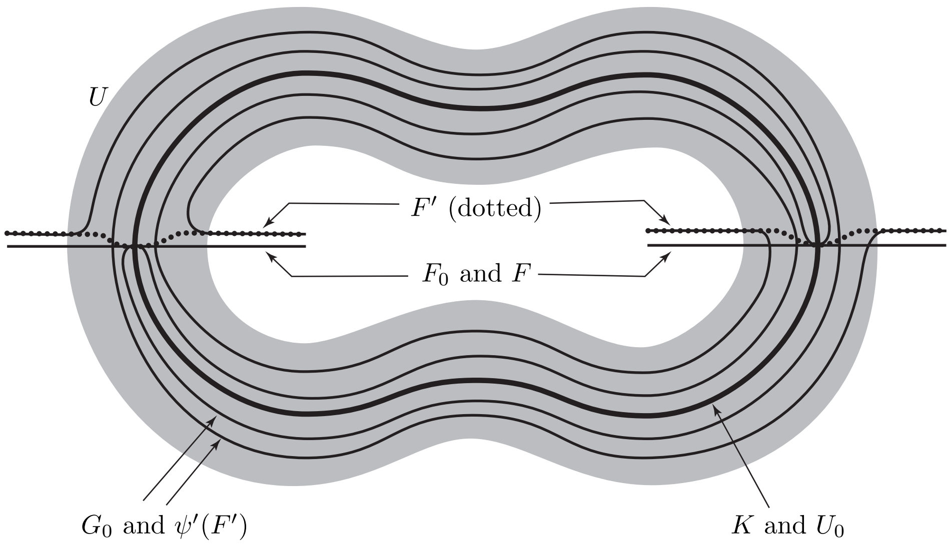


Figure 5.1.

By hypothesis, ψ' is not the identity on at least one component of U. Let F' be the image of F by a small isotopy fixing a very small tubular neighbourhood $U_0 \subset U$ of K, such that "ψ' shifts in the direction of F'" near at least one point of $F \cap K$ (as on the left side of Figure 5.1).

By construction the two surfaces F and $\psi'(F')$ are isotopic fixing U_0. It therefore follows from [**Wal**$_3$, Proposition 5.4] that there is in $M_0 = M - \mathrm{int}(U_0)$ a rel ∂ blister between $F_0 = F \cap M_0$ and $G_0 = \psi'(F') \cap M_0$.

Each closed-up component of $M_0 - F_0 \cup G_0$ that is contained in U meets $\partial M_0 - \partial F_0$ or has non-connected intersection with F_0 or G_0, and therefore cannot be a rel ∂ blister. There are exactly two closed-up components P_1 and P_2 of $M_0 - F_0 \cup G_0$ meeting $M_0 - U$ where P_1, say, contains the "obvious parallelism" $\cong (F - U) \times [0, 1]$ between $F - U$ and $F' - U$. As ψ' shifts in the direction of F' near at least one point of $F \cap K$, the intersection $P_1 \cap F_0$ is not connected (consider Figure 5.1) and P_1 cannot be a rel ∂ blister. Therefore, the rel ∂ blister between F_0 and G_0 is necessarily P_2. It easily follows that the winding directions of ψ' near all points of $F \cap K$ are non-trivial and agree with a normal orientation of F (forbidding in particular the situation of Figure 5.1) and that $P_2 - U \cong (F - U) \times [0, 1]$. Therefore, $M - U$ fibres over S^1 with fibre $F - U$ and (M, K) consequently fibres over S^1 with fibre F.

Since the winding directions of ψ' agree with a normal orientation of the fibre F, there is a fibration preserving pairwise isotopy from ψ to a ψ_1 that is pairwise isotopic to the identity by an isotopy with smaller degree on K (and still $\psi_1(F) = F$ and $\psi_1|K = \mathrm{Id}$. By induction, it eventually follows that (ii) holds for ψ.

This completes the proof of Claim 5.21, of Proposition 5.20 and consequently of the Rigidity Theorem 5.11. □

Remark 5.22. The reader may have noticed that we proved a stronger form of Proposition 5.20. Indeed, without any modification of the proof, the hypothesis that ψ is isotopic to the identity can be replaced by the weakest assumption that ψ is connected to the identity by a homotopy $H\colon M\times[0,1]\to M$ with $H^{-1}(K\cup\partial M) = (K\cup\partial M)\times[0,1]$ (compare [**Wal**$_3$, Theorem 7.1]).

We will make further use of Proposition 5.20 and of blisters in Chapters 8 and 16. However, blisters will occur in a slightly different form, which we pause here to describe.

In a knot pair (M,K), consider two surfaces F and G in general position (namely the intersections $F\cap G$ and $(\partial F)\cap(\partial G)$ are transverse). A ***pairwise blister*** is a copy P of $\Sigma\times[0,1]/\sim$ embedded in M with the following properties:

(a) $\sim$ pinches $k\times[0,1]$, with k a compact 1–submanifold of $\partial\Sigma$, by collapsing to one point each arc $x\times[0,1]$ with $x\in k$.
(b) $P\cap K$ is a finite collection of fibres $x\times[0,1]$ with x in the interior of the compact surface Σ.
(c) $\Sigma\times 0$, $\Sigma\times 1$ and the closure of $(\partial\Sigma-k)\times[0,1]$ are respectively contained in F, G and ∂M.
(d) $F\cap\operatorname{int}(P)=\varnothing$,
(1) $G\cap\operatorname{int}(P)$ can be non-empty only when Σ is a disc, $P\cap K=\varnothing$ and $P\cap\partial M$ is connected (possibly empty).

Proposition 5.23 (Blister Lemma). *In a knot pair (M,K) with $M-K$ irreducible and $\partial M-\partial K$ incompressible in $M-K$, consider two surfaces F and G intersecting transversely such that $F-K$ and $G-K$ are incompressible in $M-K$. If F and G are isotopic, then there exists a pairwise blister between F and G.*

Proof. When $\partial M=\varnothing$, the existence of a pairwise blister is easily deduced from the existence of rel ∂ blisters, making the isotopy from F to G "monotonic" on the components of K as in the proof of Claim 5.21.

When $\partial M\neq\varnothing$, consider the double (DM,DK) obtained by gluing two copies of (M,K) along their boundaries, and let τ be the involution of (DM,DK) that exchanges these two copies. By the previous case, there is a pairwise blister P between the doubles DF and DG. If $\tau(P)\cap P=\varnothing$, it immediately gives a pairwise blister between F and G. Otherwise, $\tau(P)=P$ as τ respects DF and DG. By incompressibility of $\partial M-\partial K$, the intersection of $\partial M=\operatorname{Fix}(\tau)$ with P is incompressible and ∂–incompressible; by [**Wal**$_3$, Lemma 3.4] it is therefore vertical for the product structure of the blister P. The intersection of P with one half of DM now provides a pairwise blister between F and G in (M,K). $\square$

CHAPTER 6

π–hyperbolic structures and characteristic rational tangles

Consider a knot (S^3, K) that is simple for Schubert. In Chapter 3, we singled out in (S^3, K) a certain characteristic family G of pairwise incompressible Conway spheres splitting (S^3, K) into pieces which are, either Montesinos pairs, or (finite volume) π–hyperbolic knot pairs. We want to go one step further and find new characteristic Conway spheres in the closed-up components of $(S^3, K) - G$. For Montesinos pieces, this will be extensively done in the next chapters (in particular Chapter 10) in the course of our classification of arborescent knot pairs. This chapter is devoted to discussing the case of the π–hyperbolic pieces; of course, the characteristic Conway spheres we are looking for are going to be pairwise compressible (see Theorem 5.2).

6.1. Short geodesic arcs

The very first device to find such characteristic Conway spheres is provided by the Margulis Lemma encountered as Lemma 5.5 in §5.2.

Consider a finite volume π–hyperbolic knot pair (M, K). In §5.2, we gave a natural decomposition of $\mathrm{int}(M, K)$ into a "thin part" $M_{\mathrm{thin}}(\mu)$ and a "thick part" $M_{\mathrm{thick}}(\mu)$. Remember that here the constant μ is chosen to satisfy the conclusions of the Margulis Lemma 5.5 and that, for some isometric identification $\mathrm{int}(M) \cong \mathbb{H}^3/\mathcal{G}$, the thin part $M_{\mathrm{thin}}(\mu)$ consists of the images of points $x \in \mathbb{H}^3$ that are moved at distance $\leqslant \mu$ by some non-elliptic element of $\mathcal{G}$.

As seen in §5.2, the closure in (M, K) of each component of $M_{\mathrm{thin}}(\mu) \subset \mathrm{int}(M)$ is of one of the following types:

(a) a collar neighbourhood of a boundary component in (M, K);
(b) a solid torus V and $V \cap K$ either is empty or is the core of V; this solid torus can possibly be reduced to its core when this one has length μ;
(c) a rational tangle pair, or an arc joining K to itself and of length $\mu/2$.

Clearly, if (M', K') is another π–hyperbolic knot pair, any isometry $(M, K) \to (M', K')$ sends $M_{\mathrm{thin}}(\mu)$ to $M'_{\mathrm{thin}}(\mu)$. Thus, by our Rigidity Theorem 5.11,

Proposition 6.1. *Let (M, K) and (M', K') be two finite volume π–hyperbolic knot pairs, and let μ be as in the Margulis Lemma 5.5. Then, any isomorphism $\varphi\colon (M, K) \to (M', K')$ can be pairwise isotoped to send $M_{\mathrm{thin}}(\mu)$ to $M'_{\mathrm{thin}}(\mu)$.* □

Corollary 6.2. *The boundaries of the rational tangle components of $M_{\mathrm{thin}}(\pi)$ form a characteristic family of Conway spheres in (M, K).* □

The cores of the components of $M_{\mathrm{thin}}(\mu)$ which are solid tori avoiding K also give characteristic curves in (M, K). The other components of $M_{\mathrm{thin}}(\mu)$, which are just tubular neighbourhoods of components of $\partial(M, K)$ or K, are less interesting.

Unfortunately, Corollary 6.2 is not very useful in practice. This comes in particular from the indeterminacy in the choice of a constant μ satisfying the conclusions of the Margulis Lemma 5.5. Indeed, if μ is chosen too small, the characteristic family provided by Corollary is going to be empty in most cases (compare however §6.2). The set of such μ is obviously an interval $]0, \mu_0[$, but the lower estimates for μ_0 which are known so far are still very small. (See [**BusK**]; the best estimate seems to be obtained from Jorgensen's inequality [**Jør**].) It remains an important problem to determine a good approximation of μ_0, so as to make Corollary 6.2 more convenient[1].

Lacking such a good approximation, we can however try another method.

Indeed, first focus attention on the case when $M_{\text{thin}}(\mu)$ has rational tangle components, and consider more closely the shape of one of them, V. As we saw in §5.2, V is the projection in $\text{int}(M) = \mathbb{H}^3/\mathcal{G}$ of a tubular neighbourhood of a geodesic g of $\mathbb{H}^3$. Moreover, the stabiliser of g in $\mathcal{G}$ is generated by a glide rotation along g and a π–rotation whose axis meets orthogonally g. From this, it follows that $\mathcal{G}g$ is $\mathcal{G}k$ for some arc $k \subset g$ joining the axes of two distinct π–rotations of $\mathcal{G}$, and meeting them orthogonally. Also, the restriction $k \to p(k)$ of the projection $p\colon \mathbb{H}^3 \to \text{int}(M)$ is an embedding, and V is a regular neighbourhood of $p(k)$ respecting K as in Figure 6.1.

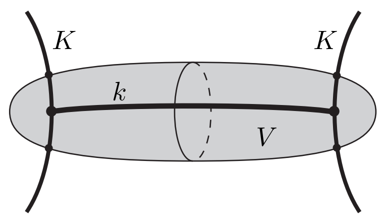


FIGURE 6.1.

Conversely, consider two π–rotations R_1 and R_2 of $\mathcal{G}$, whose axes a_1 and a_2 have distinct endpoints on the sphere at infinity S^2_∞. In $\mathbb{H}^3$, there is a shortest arc k joining a_1 to a_2; this k is geodesic, and meets a_1 and a_2 orthogonally. Observe that the orbit of k under the group generated by a_1 and a_2 is a whole geodesic g of $\mathbb{H}^3$, and that the composition $a_1 a_2$ is a glide-rotation along g. Thus, we are almost in the same situation as before. The only difference is that the projection $k \to p(k)$ may not be injective, so that the image of a small tubular neighbourhood of g does not necessarily give a rational tangle pair $(V, K \cap V)$ in (M, K) as in Figure 6.1.

In this situation, the Margulis Lemma ensures that $k \to p(k)$ is injective when k has length $\leqslant \mu/2$. Actually, the cores of the rational tangle components of $M_{\text{thin}}(M)$ are exactly all such $p(k)$ with k of length $< \mu$.

When there is no such k of length $< \mu/2$, namely when the Margulis Lemma does not yield any characteristic rational tangles, we can however consider the arcs which are shortest among these k.

PROPOSITION 6.3. *In the finite volume π–hyperbolic pair (M, K) with interior isometric to $\mathbb{H}^3/\mathcal{G}$, consider the family $\mathfrak{K}$ of geodesic arcs in $\mathbb{H}^3/\mathcal{G}$ joining two axes*

[1](Added 2009) See [**Mey**] for the current best estimate for μ_0 which, surprisingly, has seen no improvement in 20 years.

of π–rotations of $\mathcal{G}$ and meeting these axes orthogonally. Then, for any $\lambda > 0$ there are only finitely many arcs of length $\leqslant \lambda$ in $\mathfrak{K}$ modulo the action of $\mathcal{G}$.

PROOF. First observe that, for $k \in \mathfrak{K}$, its image $p(k)$ by the projection $p\colon \mathbb{H}^3 \to \mathrm{int}(M)$ cannot be contained in a component of the thin part $M_{\mathrm{thin}}(\mu)$ that is adjacent to an end of $\mathrm{int}(M)$, namely in the quotient of a horoball of $\mathbb{H}^3$ by its parabolic stabiliser in $\mathcal{G}$. Indeed, the preimage of $p(k)$ would then provide a geodesic of $\mathbb{H}^3$ that is completely contained in this horoball, which is absurd.

Choose a compact W of $\mathbb{H}^3$ so that $p(W)$ contains the complement of these cusp components of $M_{\mathrm{thin}}(\mu)$. By the above remark, there is for each arc $k \in \mathfrak{K}$ an element $g \in \mathcal{G}$ such that gk meets W. Also, since $\mathcal{G}$ acts discontinuously on $\mathbb{H}^3$, there are only finitely many arcs of length $\leqslant \lambda$ in $\mathfrak{K}$ which meet W. Therefore, there are only finitely many elements of length $\leqslant \lambda$ in $\mathfrak{K}/\mathcal{G}$. □

In particular, there are finitely many projections $p(k)$ in $\mathrm{int}(M)$ of arcs $k \in \mathfrak{K}$ of length $\leqslant \lambda$. Each of these $p(k)$ is an arc in M, with possible self-intersections, joining K to itself. It is always possible to perturb $p(k)$ to an embedded arc k^* joining K to itself; now, a regular neighbourhood B of k^* in (M, K) gives a rational tangle pair $(B, K \cap B)$ in (M, K).

Clearly, for each $k \in \mathfrak{K}$, there are only finitely many embedded arcs k^*, up to pairwise isotopy in (M, K), which are obtained by perturbation of $p(k)$. Therefore, for each $\lambda > 0$, we can consider the finite family $\mathfrak{B}_\lambda$ of pairwise isotopy classes of rational tangle pairs $(B, B \cap K)$ in (M, K) which are obtained as above: Namely, B is a regular neighbourhood of an embedded arc k^* obtained by perturbation of $p(k)$ for some $k \in \mathfrak{K}$ of length $\leqslant \lambda$.

Then, our Rigidity Theorem 5.11 readily shows that $\mathfrak{B}_\lambda$ is characteristic in the following weak sense.

PROPOSITION 6.4. *Consider two finite volume π–hyperbolic pairs (M, K) and (M', K') and, for $\lambda > 0$, the above defined finite families $\mathfrak{B}_\lambda$ and $\mathfrak{B}'_\lambda$ of (pairwise isotopy classes of) rational tangle pairs in (M, K) and (M', K'), respectively. Then, for any $B \in \mathfrak{B}_\lambda$ and any isomorphism $\varphi\colon (M, K) \to (M', K')$, the rational tangle pair $\varphi(B)$ is in $\mathfrak{B}'_\lambda$.* □

Of course, Proposition 6.4 requires a rather precise determination of the π–hyperbolic structure of (M, K) and (M', K') to be very useful. However, there are numerous examples of explicit π–hyperbolic structure where this argument can be effectively applied; compare Chapter 4, [**Thu$_1$, Ril$_1$, Ril$_2$**].

Also observe that, if λ is not too large, the projection $p(k)$ of an arc $k \in \mathfrak{K}$ of length $\leqslant \lambda$ is very likely to have few self-intersection points, which simplifies the construction of the family $\mathfrak{B}_\lambda$ and gives a stronger form of Proposition 6.4. For instance, an easy cut-and-paste argument readily shows:

PROPOSITION 6.5. *With the data of Proposition 6.4, assume that $\partial K = \varnothing$ and that λ is the minimum of the lengths of the elements of $\mathfrak{K}$ (this minimum is realised, by Proposition 6.4). Then, any two distinct arcs of length λ of $\mathfrak{K}$ have disjoint interiors. As a consequence, the projections $p(k)$ of the $k \in \mathfrak{K}$ of length λ form a characteristic finite family of arcs joining K to itself, whose interiors are embedded and pairwise disjoint.* □

6.2. Rational tangle substitutions

In a knot projection with minimal crossing number, one often sees rational tangle projections of moderately large complexity which give rational tangle pairs that seem to be characteristic. In this section, we give a few reasons why these rational tangle pairs could be characteristic, using hyperbolic geometry. In particular, we will prove Theorem 1.4 of §1.2. Unfortunately, our arguments are more qualitative that quantitative, and crucial specific questions remain unsettled.

The argument we are going to use is just a generalisation of the one of §4.2. It is based, via double branched coverings, on Thurston's theory of hyperbolic Dehn surgery [**Thu**$_1$, §5] [**BenP**].

Consider a compact 3–manifold M whose boundary consists of tori. The operation of ***Dehn surgery*** (also called ***Dehn filling***) consists of gluing solid tori $\cong S^1 \times D^2$ to M along some components of ∂M to get a new 3–manifold M_+.

As already indicated in §4.2 (see [**Rol**] for instance), the manifold M_+ obtained by such a Dehn surgery is completely determined by its types at each component T of ∂M. This ***type*** of the Dehn surgery at T is defined as the kernel of $H_1(T) \to H_1(V)$ if a solid torus V is glued along T in the construction, and is empty otherwise.

In practice, one chooses a basis $\{k_T, l_T\}$ for $H_1(T)$, and the type at T is described by $m_T \in \mathbb{Q} \cup \{\infty, \varnothing\}$ defined as follows: m_T is the symbol $\varnothing$ if no solid torus V is glued at T; otherwise, m_T is the rational $\frac{q}{p} \in \mathbb{Q} \cup \infty$ such that $pk_T + ql_T$ generates the kernel of $H_1(T) \to H_1(V)$.

Suppose that, moreover, the interior of M is equipped with a finite volume hyperbolic structure. Then, by deformation of this structure, Thurston shows in [**Thu**$_1$, §5] (see also [**BenP**]) that "most" Dehn surgeries on M give a hyperbolic manifold. More precisely, he proves:

THEOREM 6.6. *Let M be a compact 3–manifold, whose interior admits a finite volume hyperbolic structure. Then, for any component T of ∂M, there is a finite set $E_T(M)$ of 1–dimensional subspaces of $H_1(T)$ with the following property: For any Dehn surgery on M whose type at each T does not belong to $E_T(M)$ (and can in particular be empty), the interior of the manifold M_+ so obtained admits a finite volume hyperbolic structure.*

Moreover, for M_+ as above, the cores of the solid tori $M_+ - M$ are the shortest closed geodesics of $\text{int}(M_+)$. □

In the second half of Theorem 6.6, the statement that the cores of the components of $M_+ - M$ are the shortest geodesics of $\text{int}(M_+)$ of course means that they consist of all closed geodesics of length $\leqslant \lambda$ of $\text{int}(M_+)$ for some $\lambda > 0$. Some precise estimates on the lengths of these cores in terms of the Dehn surgery slopes can be found in [**NeuZ**].

We will apply this theorem to the situation of §1.2, slightly generalised.

For this, consider a ***knot pair projection*** (S, Γ), consisting of a (compact) subsurface S of the sphere S^2, and of a 1–submanifold Γ immersed in S whose only singularities are transverse double points, each equipped with an indication telling which strand lies above the other one. Such a knot pair projection describes a knot pair (M, K) as follows. Cap off each component of ∂S with two disks, one above and one under S^2. This gives a family S_+ of 2–spheres in S^3 with $S_+ \cap S^2 = \partial S$, and M is taken to be the closed-up component of $S^3 - S_+$ containing S. The knot string $K \subset M$ is obtained by perturbing Γ according to the crossing information.

Suppose in addition that each boundary component of S meets Γ in 4 points, so that the boundary of the knot pair (M, K) consists of Conway spheres.

If we glue rational tangle projections to (S, Γ) along some components of its boundary, we get a new knot pair projection (S_+, Γ_+) which is said to be obtained from (S, Γ) by ***rational tangle substitution***. The knot pair (N_+, L_+) described by (S_+, Γ_+) contains (N, L) described by (S, Γ) in a natural way and we will show that, for most rational tangle substitutions, (N, L) is characteristic in (N_+, L_+).

By Proposition 1.3, the knot pair (N_+, L_+) depends only on (S, Γ) and on the slopes of the rational tangle projections substituted. Recall that these slopes $\in \mathbb{Q} \cup \infty$ are defined as soon as each boundary component of (S, Γ) is equipped with an arrow as in Figure 1.3. At a boundary component of S where no rational tangle projection is glued, we say that we are performing there a rational tangle substitution of slope $\varnothing$, where $\varnothing$ is just a standard symbol recalling the empty substitution.

When (the interior of) (N, L) has a finite volume π–hyperbolic structure, we have the following analogue of Theorem 6.6.

PROPOSITION 6.7. *Consider a finite volume π–hyperbolic knot pair (N, L) described by a knot pair projection (S, Γ). Equip each boundary component of (S, Γ) with an arrow as in* §1.2 *so as to define slopes for rational tangle substitutions. Then, for each component C of $\partial(S, \Gamma)$, there is a finite set $E_C(S, \Gamma) \subset \mathbb{Q} \cup \infty$ with the following property: For any rational tangle substitution on (S, Γ) whose type at each C is not a slope in $E_C(S, \Gamma)$ (but can be $\varnothing$), the knot pair (N_+, L_+) described by the knot pair projection so obtained admits a finite volume π–hyperbolic structure.*

Moreover, let $\mathfrak{K}$ denote the set of geodesic arcs in $\operatorname{int}(N_+, L_+)$ *joining L_+ to itself and meeting L_+ orthogonally, as in* §6.1. *Then the closed-up components of $(N_+, L_+) - (N, L)$ are regular neighbourhoods of arcs of $\mathfrak{K}$, and these geodesic arcs are the shortest ones in $\mathfrak{K}$.*

PROOF. The π–hyperbolic structure on (N, L) lifts to a finite volume hyperbolic structure on (the interior of) its double branched covering M. Also, the double branched covering M_+ of (N_+, L_+) is obtained by Dehn surgery on M (compare §4.2). Thus, we are in the situation of Theorem 6.6, which provides us with a set $E_C(S, \Gamma)$ of slopes for each component C of $\partial(S, \Gamma)$, with the following property: If (S_+, Γ_+) is obtained from (S, Γ) by a rational tangle substitution whose type at each C is not a slope of $E_C(S, \Gamma)$, then (the interior of) M_+ admits a finite volume hyperbolic structure; moreover, the cores of the solid tori $M_+ - M$ are the shortest geodesics of M_+.

To complete the proof, we clearly only have to show that this hyperbolic structure on M_+ can be chosen so as to make the covering involution τ an isometry. But this is done by the *ad hoc* argument we developed in §4.2. □

REMARK 6.8. Although a mere reference to §4.2 is certainly faster, there is another proof that τ can be made an isometry which is probably more conceptual. It consists in checking that Thurston's proof of Theorem 6.6 in [**Thu**$_1$, §5] can be made $\mathbb{Z}/2$–equivariant. This actually amounts to a straightforward extension of Theorem 6.6 to orbifolds, by consideration of the space of representations of $\mathcal{G}$ in the isometry group of $\mathbb{H}^3$, where $\mathcal{G}$ is such that $\operatorname{int}(N, L)$ is isometric to $\mathbb{H}^3/\mathcal{G}$.

REMARK 6.9. There is substantial evidence that the exceptional sets of slopes $E_C(S,\Gamma)$ of Proposition 6.7 can be chosen to be relatively small, perhaps with no more than 4 or 5 elements. This is indeed the case for all examples known. Also, [**CuGLS**] seems to provide some progress towards this conjecture[2].

Combining Proposition 6.7 with our Rigidity Theorem 5.11, we immediately get:

COROLLARY 6.10. *Consider two knot pair projections (S,Γ) and (S',Γ') whose associated knot pairs (N,L) and (N',L') admit a finite volume π–hyperbolic structure, and such that ∂S and $\partial S'$ have the same number of components. After labelling these boundary components by arrows, let $E_C(S,\Gamma)$ and $E_{C'}(S',\Gamma') \subset \mathbb{Q}\cup\infty$ be the exceptional sets of slopes associated to each component C of ∂S and C' of $\partial S'$ by Proposition 6.7.*

Then, for any rational tangle substitution on (S,Γ) (resp. $S',\Gamma')$) whose type at each C (resp. C') is not a slope in $E_C(S,\Gamma)$ (resp. $E_{C'}(S',\Gamma')$), any degree ±1 isomorphism φ: $(N_+,L_+)\to(N'_+,L'_+)$ between the resulting knot pairs can be pairwise isotoped so as to send $(N,L)\subset(N_+,L_+)$ to $(N',L')\subset(N'_+,L'_+)$. □

In other words, if (N_+,L_+) is obtained from (N,L) by a sufficiently complicated rational tangle substitution, then (N,L) is characteristic in (N_+,L_+).

These results can be considerably sharpened in the case when the knot pair projection (S,Γ) has no crossing, namely arises from a Conway graph Γ_0 in S^2 as defined in Chapter 1. In particular, we can get rid in this case of the condition on the number of boundary components.

For such a Conway graph $\Gamma_0\subset S^2$, the knot pair projection (S,Γ) and the knot pair (N,L) it describes are respectively obtained from (S^2,Γ_0) and (S^3,Γ_0) by removing regular neighbourhoods of the vertices of Γ_0. In particular, (N,L) has a degree -1 involution τ fixing L, arising from the reflection of S^3 through S^2.

We are now ready to prove Theorem 1.4 of §1.2, slightly extended to allow rational tangle substitutions of type $\varnothing$. For a Conway graph $\Gamma_0\subset\mathbb{R}^2=S^2-\infty$, we adopt the convention of §1.2 to assign slopes to rational tangle substitutions, using the checkerboard colouring of $S^2-\Gamma_0$ that is white near ∞.

THEOREM 6.11. *For every Conway graph $\Gamma_0\subset\mathbb{R}^2=S^2-\infty$ and every vertex v of Γ_0, there is a finite subset $E_v(\Gamma_0)$ of $\mathbb{Q}\cup\infty$ with the following property:*

Consider two such Conway graphs Γ_0 and Γ'_0. Perform at each vertex v of Γ_0 a rational tangle substitution whose slope is not in $E_v(\Gamma_0)$ (but can be $\varnothing$). Similarly perform at each vertex v' of Γ'_0 a rational tangle substitution whose slope is not in $E_{c'}(\Gamma_0)$. Then the knot pairs described by the resulting knot pair projections are degree $+1$ isomorphic if and only if there is a degree ±1 isomorphism θ: $S^2\to S^2$ sending Γ_0 to Γ'_0 such that:

(i) *if θ respects the colours of the components of $S^2-\Gamma_0$ and $S^2-\Gamma'_0$, then $m'_{\theta(v)}=m_v$ for every vertex v of Γ_0 ;*
(ii) *otherwise, $m'_{\theta(v)}=-\frac{1}{m_v}$ for every vertex v of Γ_0 (with the convention that $-\frac{1}{\varnothing}=\varnothing$).*

PROOF. If (S,Γ) is the knot pair projection associated to Γ_0, the exceptional set of slopes $E_v(\Gamma_0)\subset\mathbb{Q}\cup\infty$ are of course the $E_C(S,\Gamma)$ of Proposition 6.7 and Corollary 6.10, where C is the component of ∂S corresponding to v.

[2](Added 2009) See [**Ago, Lack, LacM**] for current results in this direction.

In the situation of the theorem, let (N_+, L_+) and (N'_+, L'_+) denote the knot pairs described by the rational tangle substitutions. By construction, (N_+, L_+) contains the knot pair (N, L) obtained from (S^3, Γ_0) by removing a regular neighbourhood of the vertices of Γ_0. Similarly for (N', L') in (N'_+, L'_+). By Proposition 1.3, it clearly suffices to show that any degree $+1$ isomorphism $\varphi\colon (N_+, L_+) \to (N'_+, L'_+)$ can be pairwise isotoped to send N to N' and $N \cap S^2$ to $N' \cap S^2$.

First assume that Γ_0 and Γ'_0 have the same number of vertices. Then Corollary 6.10 asserts that φ can be pairwise isotoped so that $\varphi(N) = N'$. Remember that (N, L) has a degree -1 involution τ induced by the reflection of S^3 through S^2, and that (N', L') similarly has a reflection τ'. By Corollary 5.13, the involutions τ and $\varphi_{|N}^{-1}\tau'\varphi_{|N}$ of (N, L) are pairwise isotopic since they both fix L. By [**Tol**$_3$] or an easy *ad hoc* argument, it follows that φ can be pairwise isotoped so that $\tau = \varphi_{|N}^{-1}\tau'\varphi_{|N}$ in addition to $\varphi(N) = N'$. In particular, $\varphi(N \cap S^2) = N' \cap S^2$ by fixed point considerations. This proves Theorem 6.11 when Γ_0 and Γ'_0 have the same number of vertices.

In the general case, assume for instance that Γ'_0 has at least as many vertices as Γ_0. By Proposition 6.7 and the Rigidity Theorem 5.11, φ can be pairwise isotoped so that $\varphi(\partial N) \subset \partial N'$. We want to show that $\varphi(N) = N'$.

Observe that $\varphi(N, L)$ is obtained by rational tangle substitution on (N', L'). By the first case, the automorphism $\varphi_{|N}\tau\varphi_{|N}^{-1}$ of $\varphi(N, L)$ is pairwise isotopic to one respecting N', $N' \cap S^2$, and each component of $N' \cap S^2 - L$. As its degree is -1, this is possible only if all slopes of this rational tangle substitution are $\varnothing$, namely if $\varphi(N) = N'$. This proves that Γ_0 and Γ'_0 necessarily have the same number of vertices, and completes the proof of Theorem 6.11 (and Theorem 1.4). □

Part 3

The arborescent part of a knot
(the proofs)

CHAPTER 7

The arborescent part is characteristic

This chapter is devoted to proving the assertion of the title, which is Theorem 3.1.

The argument is rather direct and elementary. It simultaneously proves the somewhat stronger result that the potentially π–hyperbolic part is characteristic, namely Theorem 3.2.

For a first reading, it is probably best to restrict attention, as in Chapter 3, to a knot (S^3, K) that is simple for Schubert. However, since the proofs work in considerably greater generality without requiring any modification, we pause here to set out, using language and facts from Chapter 3, four hypotheses that amply suffice. (See also the end of this chapter).

Hypotheses. We consider knot pairs (M, K) such that:

(1) $M \subset S^3$.
(2) There is no 2–sphere in (M, K) cutting K in 1 or 3 points.
(3) (M, K) is pairwise irreducible, namely contains no pairwise incompressible 2–sphere meeting K in 0 or 2 points.
(4) There is no 2–torus in $M - K$ that is pairwise incompressible in (M, K).

Hypotheses (2) and (3) prevent ∂M from having a 2–sphere component cutting K in $\leqslant 3$ points, unless the corresponding component M_1 of M is such that $(M_1, K \cap M_1) \cong (B^3, \varnothing)$ or (B^3, B^1). Also, Hypotheses (3) and (4) prevent $\partial M - K$ from having a 2–torus component unless the corresponding component M_1 of M is a solid torus for which $K \cap M_1$ is its core or is empty. But these are the only restrictions on $(\partial M, \partial K)$.

The only 2–tori in $M - K$ allowed by these hypotheses are the inevitable pairwise compressible ones, namely: boundaries of tubular neighbourhoods of circles of K or of circles in $M - K$, and boundaries of balls with empty wormhole, where the ball lies in $M - K$.

One readily verifies that a knot pair (M, K) with $M \subset S^3$ has a 2–fold branched cover precisely if K meets each boundary component in an even number of points. (Hint: For any compact pair (N, L) with N an n-manifold and L a codimension 2 submanifold, a 2–fold branched cover exists precisely if L is zero in $H_{n-2}(N, \partial N; \mathbb{Z}_2)$.) Thus the $\mathbb{Z}_2$–equivariant arguments first exploited to prove Theorems 3.1 and 3.2 in [**Bon**$_1$] do not have the full generality of the arguments of this section.

In (M, K) we consider a family F of disjoint pairwise incompressible Conway 2–spheres $F_1, \ldots, F_n$ that is maximal for the property that no closed-up component X of $M - F$ is a pairwise parallelism $\cong (S^2, 4 \text{ points}) \times [0, 1]$ and meets F.

If F is empty we say that (M, K) is ***simple for Conway***, by analogy with the term simple for Schubert introduced in Chapter 2. However let the reader be warned that the article [**Conw**] suggests that Conway would not in general be content to call such a pair simple until some rational tangle pairs have been extracted (see the discussion in §1.2 and Chapter 6).

The knot pair (M, K) decomposes as the union of the arborescent part A, union of the elementary closed-up components of $M - F$, and of its closed-up complement $A^* = M - \mathrm{int}(A)$. This section is devoted to proving that the surface $G = F \cap A^*$, which was called F^* in Chapter 3, is characteristic. This surface G is the union of all those F_i in F which lie on the boundary of one or two *non-elementary* closed-up component of $M - F$.

THEOREM 7.1. *In the above situation, G is well-defined in (M, K) up to pairwise isotopy.*

In the proof we will encounter *Conway disks* as well as Conway spheres. A ***Conway disk*** in a knot pair (M, K) is a 2–disc D embedded in M with $\partial D \cap K = \varnothing$, such that D cuts K in two points, transversely. Conway discs often occur as half of a Conway sphere. As with compression discs, there is no *a priori* assumption that $\partial D \subset \partial M$.

We shall need:

LEMMA 7.2. *In a pairwise irreducible knot pair (M, K), consider a family F of disjoint pairwise incompressible closed surfaces. Then, for any closed-up component N of $M - F$, the pair $(N, K \cap N)$ is pairwise irreducible.*

PROOF. Consider a sphere S in N cutting K in 0 or 2 points. It bounds a ball B in M so that the pair $(B, K \cap B)$ is isomorphic to the standard linear pair $(B^3, \varnothing)$ or (B^3, B^1). If $B \not\subset N$, then B contains a component F' of F.

Since it is well known that B^3 contains no nonempty incompressible surface [**Ale**], it is sufficient now to establish the

SUBLEMMA 7.3. *In the linear pair (B^3, B^1) there is no pairwise incompressible nonempty closed surface F'.*

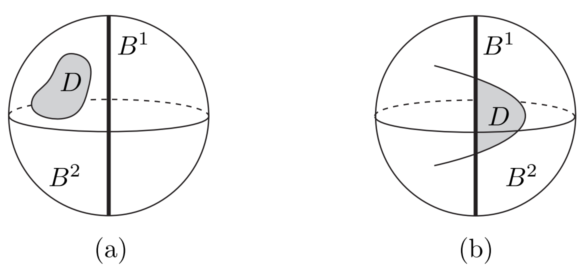


FIGURE 7.1.

PROOF. Choose a linear 2–disc B^2 bisecting B^3 and containing B^1. Seeking a contradiction, we suppose F' exists. By a pairwise isotopy moving F', make the intersection $F' \cap B^2$ transverse and minimal, in the sense that that the number of components of $F' \cap B^2$ cannot be reduced by any pairwise isotopy. Necessarily, $F' \cap B^2 \neq \varnothing$ by [**Ale**], since B^2 splits B^3 into two halves, the interior of each isomorphic to an open ball. Consequently, there must somewhere be a configuration as in Figure 7.1(a) or (b), involving a 2–disc D in B^2 with $\mathrm{int}(D)\cap F' = \varnothing$. In each case, there appears a compression disc for F' that avoids B^1: This is the disc D in Case (a), and is constructed from two copies of the "half-disc" D in Case (b). In both cases, one then uses the pairwise incompressibility of F' and the Alexander-Schoenflies theorem [**Ale**] to show that this cannot happen: In case (a), $F' \cap B^2$ could be reduced by pairwise isotopy, and, in case (b), a component of F' would be a pairwise compressible sphere meeting B^1 in two points. □

Sublemma 7.3 concludes the proof of Lemma 7.2. □

Proof of Theorem 7.1. We have to show that, if G' is constructed by the same recipe as G (primes will distinguish data corresponding to G'), then G and G' are pairwise isotopic.

By a pairwise isotopy moving G', we can make $G \cap G'$ contain as many Conway spheres as possible, say $G_1, \ldots, G_k$. Then we make G and G' meet transversely outside $G_1, \ldots, G_k$, in a number of circles that is minimal for pairwise isotopy fixing $G_1, \ldots, G_k$. In this situation we aim to show that $G = G' = G \cap G'$.

The 1–dimensional part of $G \cap G'$ is now ***pairwise essential*** in $(G, K \cap G)$ and in $(G', K \cap G')$, in the sense that no intersection circle is the boundary of a disc in G or in G' meeting K in $\leqslant 1$ point. One needs the pairwise incompressibility of G and G' together with the pairwise irreducibility of (M, K) to verify this. By elimination, one concludes that each closed-up component of $G - G'$ or $G' - G$ is therefore a Conway sphere or disc, or an annulus avoiding K.

Let N be a closed-up component of $M - G$. Now $G' \cap N$ is pairwise incompressible in $(N, K \cap N)$. Indeed if $D' \subset N$ is a pairwise compression disc for $G' \cap N$, the pairwise incompressibility of G' in (M, K) shows there is a disc $D_1 \subset G'$, with $\partial D_1 = \partial D$, meeting K in as many points as D' (namely $\leqslant 1$). An innermost disc argument on D_1 then reveals that D_1 cannot leave N (as the innermost disc would have to be a Conway disc).

The main burden of the proof of Theorem 7.1 lies in a proposition which, when applied first to $(N, K \cap N)$, shows that $G' \cap \mathrm{int}(N) = \varnothing$. (Lemma 7.2 shows that $(N, K \cap N)$ is pairwise irreducible.)

Proposition 7.4. *Given (M, K) as for Theorem 7.1, suppose that every pairwise incompressible Conway sphere in (M, K) is pairwise parallel to a boundary component, and that (M, K) is connected but not elementary. Let $\partial_{\mathrm{C}} M$ consist of the components of ∂M that are Conway spheres. Consider a pairwise incompressible surface F in (M, K) with $\partial F \subset \partial_{\mathrm{C}} M$, such that each component of F is a Conway disc or an annulus disjoint from K.*

Then, such an F necessarily lies in a pairwise collar of the boundary of (M, K), namely in a neighbourhood U of ∂M such that $(U, U \cap K) \cong (\partial M, K \cap \partial M) \times [0, 1]$.

Remark 7.5. It follows that each component of F is pairwise boundary parallel in (M, K). However, it is convenient to neither use nor prove this fact until the next Chapter 8.

We momentarily postpone the proof of Proposition 7.4 to complete the proof of Theorem 7.1. The proposition certainly shows that $G' \subset (A \cup G)$.

Now the property that $G' \subset G$ can only fail if G' meets $\mathrm{int}(A)$. Suppose this, seeking a contradiction. It implies that a (non-elementary!) closed-up component N' of $A'^{*} - G'$ meets $\mathrm{int}(A)$.

Then, by Proposition 7.4 applied to N' (in place of M), we must have $N' \subset A$. Furthermore, on making $F \cap \text{int}(A)$ meet N' minimally, the same proposition shows that N' lies entirely in an elementary closed-up component Z of $A - F$. (Recall that F is the maximal family of Conway spheres we started with to define G.) Since Z is simple for Conway by construction of F, we have that $Z = N'$, a contradiction.

This proves that $G' \subset G$. Symmetrically $G \subset G'$, so that $G = G'$ and the proof of Theorem 7.1 is complete, assuming Proposition 7.4. □

The proof of Proposition 7.4 requires a little preparation.

LEMMA 7.6 (for instruction only). *A pairwise compressible Conway sphere Σ in the unknot (S^3, S^1) is the boundary of a regular neighbourhood B of an embedded arc J with $\partial J = J \cap S^1$, as in Figure* 7.2(a). *Equivalently, Σ bounds a ball B such that $(B, K \cap B)$ is a rational tangle pair, as in Figure* 7.2(b).

PROOF. The proof is straightforward. □

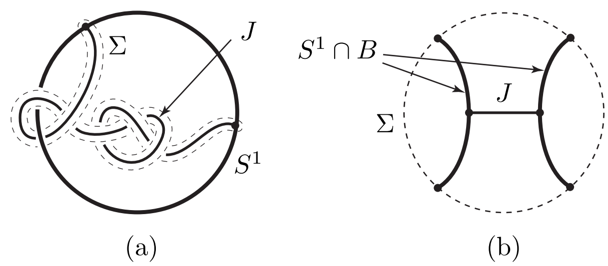


FIGURE 7.2.

Furthermore, we already know that every Conway 2–sphere in (S^3, S^1) is pairwise compressible (use Sublemma 7.3).

LEMMA 7.7. *Consider a pairwise compressible Conway sphere Σ in a pairwise irreducible connected knot pair (M, K) with no 2–spheres cutting K in 1 or 3 points. Then either Σ bounds a ball B giving a rational tangle $(B, K \cap B)$, or the following holds: Σ splits (M, K) into two pieces, and one of these pieces is obtained from the pair in Figure* 7.3 *by gluing in some knot pair (M_0, K_0) along the 2–sphere Σ'.*

Furthermore, the complement of $\text{int}(M_0, K_0)$ *in (M, K) is isomorphic to the unknotted pair (B^3, B^1). As a consequence, M_0 contains all of ∂M and all of K except for an arc.*

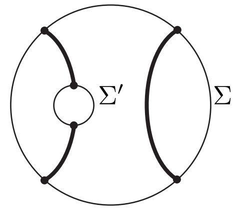


FIGURE 7.3.

Proof of Lemma 7.7. Let D be an effective pairwise compression disc for Σ in (M, K). Our hypotheses assure that ∂D cuts Σ into two Conway discs Δ_1 and Δ_2, and that $D \cap K = \varnothing$. Now $\Delta_1 \cup D$ and $\Delta_2 \cup D$ are two separating spheres by pairwise irreducibility. Hence Σ also splits M into two pieces.

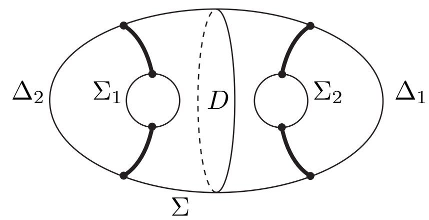


Figure 7.4.

Denote by N the closed-up component of $M - \Sigma$ containing D, and let U be a regular neighbourhood of $\Sigma \cup D$ in N. Clearly, the pair $(U, K \cap U)$ is as in Figure 7.4. The boundary ∂U consists of three spheres Σ, Σ_1, Σ_2, the latter two each meeting K in two points. By pairwise irreducibility of (M, K), both of Σ_1, Σ_2 bound balls B_1, B_2 in M each meeting K in an unknotted arc. If B_1 and B_2 are both disjoint from $\mathrm{int}(U)$, then $(N, K \cap N)$ is a rational tangle pair. If one ball, say B_1, meets U, then it contains U and therefore Σ_2; so B_2 can be chosen disjoint from $\mathrm{int}(U)$ (use Sublemma 7.3). This gives the desired result. □

Proof of Proposition 7.4. Consider a component F_0 of F. We want to find a pairwise collar neighbourhood $N_0 \cong \partial_{\mathrm{C}} M \times [0, 1]$ of $\partial_{\mathrm{C}} M$ in (M, K), which contains F_0 in its interior and such that $F \cap N_0$ consists of certain whole components of F and cylindrical pieces (that are a product with $[0, 1]$ in N_0). Clearly this result, applied successively at most as many times as F has components, yields Proposition 7.4.

The proof of this result, which we call 7.4* or ***the inductive form of*** 7.4, distinguishes cases.

Proof of 7.4* when F_0 is a Conway disc. Let U be a regular neighbourhood, respecting K and $F - F_0$, of the union of F_0 and of the component S of $\partial_{\mathrm{C}} M$ containing ∂F_0. The pair $(U, K \cap U)$ is elementary with three boundary components as described in Figure 7.5. In it, S_1 and S_2 are two Conway sphere boundary components of U disjoint from F_0, and S is a third containing ∂F_0.

We know by hypothesis that each S_i is either pairwise compressible or pairwise parallel to a component of $(\partial M, \partial K)$.

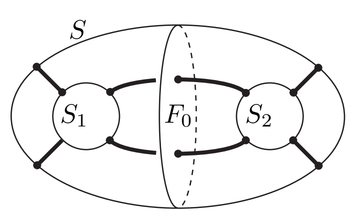


Figure 7.5.

Assertion 7.8. *One of S_1, S_2 is pairwise parallel to S.*

The assertion implies the desired result. Indeed, suppose that $E_i \cong S^2 \times [0,1]$ gives a pairwise parallelism in (M,K) between S_i and $S \subset \partial_C M$. Then, by a connectivity argument, E_i contains U and so also F_0; adding to E_i a collar neighbourhood of $\partial_C M - S$ respecting K and F, we clearly get the collaring N_0 required to prove 7.4*.

Proof of Assertion 7.8. We suppose the contrary and seek a contradiction in every case.

By hypothesis, either S_i is pairwise parallel to the boundary by a parallelism E_i disjoint from U, or S_i is pairwise compressible.

Suppose the second alternative, and consider an effective pairwise compression disc D for it. It is impossible for D to be on the same side of S_i as U. For, as F_0 is pairwise incompressible, an innermost disc argument would let us make D disjoint from F_0, which leaves D in a collar neighbourhood N respecting K of S_i in U. This is absurd, as the reader will readily see by, for instance, a fundamental group argument in $N - K$. Thus D is contained in $M - \text{int}(U)$. In view of Lemma 7.7, we can now conclude that S_i is the boundary of a ball E_i with $S_i = U \cap E_i$, so that $(E_i, K \cap E_i)$ is a rational tangle pair.

We have now defined E_i in every case $i = 1, 2$, and $(E_i, K \cap E_i)$ is isomorphic either to a product $(S^2, \text{four points}) \times [0,1]$ or to a rational tangle pair. We form $U \cup E_i \cup E_2$. This has empty frontier in M, so is all of M. Hence (M,K) is an elementary pair, against hypothesis.

This final contradiction completes the proof of the inductive form 7.4* of Proposition 7.4, in the case when F_0 is a Conway disc. □

Proof of 7.4* when F_0 is an annulus. Let U again be a regular neighbourhood, respecting K and $F - F_0$, of the union of F_0 and the components of $\partial_C M$ that meet ∂F_0. The pair $(U, K \cap U)$ is described respectively in Figure 7.6 or 7.7, according to whether the two circles ∂F_0 lie in a single component S or in two components S, S' of $\partial_C M$.

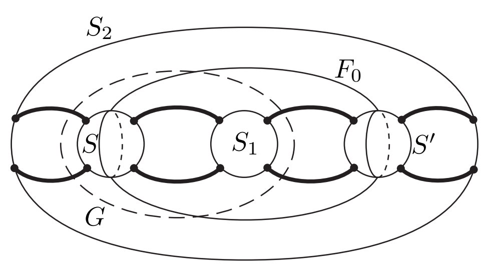


Figure 7.6.

In the case of Figure 7.6, we reach a contradiction in all circumstances by an argument similar in part to the one above for the case where F_0 is a Conway disc. Indeed, following that line of argument, one shows that the component E_i of $M - \text{int}(U)$ containing S_i, either is a pairwise parallelism of S_i to a component of $\partial_C M$, or gives a rational tangle pair $(E_i, K \cap E_i)$. Also $M = U \cup E_1 \cup E_2$ as before. The end of the proof has a new twist. We consider the Conway sphere G in Figure 7.6 that has S_1 and S on one side and S_2 and S' on the other; it splits (M,K) into two elementary pairs. By Lemma 7.7, this G is pairwise incompressible;

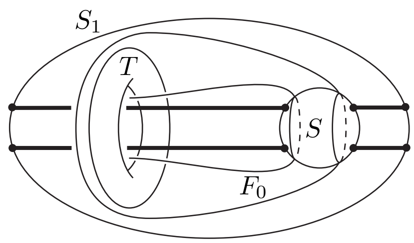


FIGURE 7.7.

the presence of boundary on each side assures this. Thus G is pairwise boundary parallel. It follows that one of the two elementary pairs is a mere parallelism; hence (M, K) is itself elementary, a final contradiction proving 7.4* in the case when F_0 is an annulus, as in Figure 7.6.

The case of Figure 7.7 requires some fresh arguments using the hypothesis that $M \subset S^3$.

By Hypothesis (4), the torus boundary component T of U admits an effective pairwise compression disc D in (M, K). This disc D can be made disjoint from the pairwise incompressible surface F. Also, D cannot lie in U near ∂D: If it did, $F \cap D = \varnothing$ would imply that D is trapped in a collar neighbourhood of T in U. Therefore, as T separates M, the disc D lies entirely outside $\text{int}(U)$. Adding a regular neighbourhood of D to U and applying the pairwise irreducibility of (M, K), we conclude that T bounds a solid torus V with $V \cap U = T$ and that $K \cap V$ either is empty or is the core of V.

Now, $U \cup V$ is visibly a lens space with two holes. Since $M \subset S^3$ by Hypothesis (1), we must have just a 3–sphere with two holes. Hence $U \cup V$ is constructed by plugging the torus boundary component of U in Figure 7.7 in the standard fashion.

In the case when $V \cap K = \varnothing$, we conclude that $U \cup V$ is a pairwise collar neighbourhood of S in (M, K). Adding to it a small pairwise collar neighbourhood of $\partial_C M - S$ respecting K and $F - F_0$ we get a collar N_0 of $\partial_C M$ proving 7.4*.

To conclude, we need to prove

ASSERTION 7.9. $V \cap K = \varnothing$

PROOF OF ASSERTION 7.9. We suppose $V \cap K \neq \varnothing$ and seek a contradiction.

We already know that $V \cap K \neq \varnothing$ implies $V \cap K$ is a circle at the core of V. Thus $K \cap (U \cup V)$ is as in Figure 7.8.

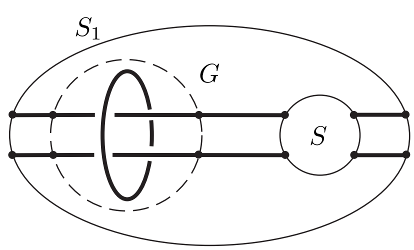


FIGURE 7.8.

In this figure, consider the dotted Conway sphere G (it is the result of "surgering" T with a Conway disc). Note that G bounds a 3–ball B in M such that $(B, K \cap B)$ is an elementary pair.

The Conway sphere G is not pairwise boundary parallel. Indeed, if it were, the parallelism would necessarily be to S (as there is no boundary of M on the other side of G). This would show that $(M, K) \cong (B, K \cap B)$ which is elementary. Absurd!

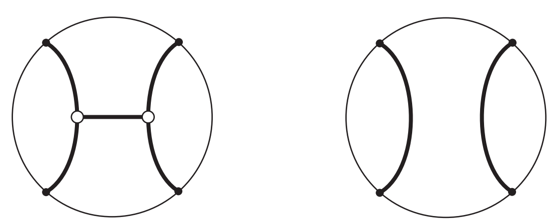
On the other hand, G is pairwise incompressible in (M, K) by Lemma 7.7.

The two properties of G just established violate our hypothesis that (M, K) is simple for Conway. So we have the final contradiction to complete the proof of the assertion that $V \cap K = \varnothing$, and with it the proof of Proposition 7.4 via 7.4*. □

This also concludes the proof of Theorem 7.1. □

REMARK 7.10. The Hypothesis (1) that $M \subset S^3$ can readily be weakened. For example, it suffices to assume that $H_1(M; \mathbb{Z})$ has no torsion, or more specifically that M contains no multiply punctured lens space with finite $H_1 \neq 0$. See the one intervention of this hypothesis, in the proof of Proposition 7.4, for the case where F_0 is an annulus. We leave the verification as an exercise. (Alternatively, Hypothesis (1) could be entirely suppressed by broadening the definition of an elementary pair.)

REMARK 7.11. It is interesting, and not difficult, to somewhat weaken Hypothesis (2) on the pair (M, K) while slightly modifying Theorem 7.1. Indeed if we allow 2–spheres in (M, K) that meet K in 3 points provided that they are pairwise boundary parallel, then Theorem 7.1 still holds provided the definition of elementary pairs is enlarged as follows. Allow substitution in the 3–holed pair of Figure 3.2, with pairs of Figure 7.9(a) below in the same way that rational tangle pairs of Figure 7.9(b) may be substituted.



(a) (b)

FIGURE 7.9.

In Figure 7.9(a), the pair has three boundary components, all 2–spheres.

This result is general enough to apply to any compact pair (M, K) with $M \subset S^3$ after a battery of preliminary reductions involving successive splitting along families of 2–spheres in (M, K) that meet K in 0, 1, 2, 3 points respectively; and finally a splitting along 2–tori in $M - K$ (as in Chapter 2). The splitting along 2–spheres meeting K in 2 points is the well-known pairwise connected sum factorisation; and the other splittings along 2–spheres are easier.

This last generalisation of Theorem 7.1 permits one to study arbitrary knotted graphs in S^3 (in place of knotted circles); one begins by deleting fattened vertices.

CHAPTER 8

Pairwise incompressible surfaces in Montesinos pairs

This chapter is devoted to a major technical step towards our classification of arborescent knots, namely to the classification of pairwise incompressible surfaces in Montesinos pairs. There are more such surfaces than first meet the eye, and their classification requires patience.

We have here adopted an elementary viewpoint. The reader interested in more modern proofs may prefer the arguments of [**BonS**$_1$] based on minimal surface theory (and valid in greater generality).

It will be helpful to use part of Conway's language of tangles [**Conw**]. A long preamble is devoted to it.

8.1. The language of tangles

When a knot in S^3 is split along a family of Conway 2–spheres, each resulting piece (M, K) is a compact pair with M embeddable in S^3 and such that each component of ∂M is a 2–sphere intersecting K in four points. We will use coordinates on such Conway 2–spheres to permit some careful cataloguing of surface boundaries.

For this purpose, following Conway [**Conw**], we shall make use of a model 2–sphere S^2 in $\mathbb{R}^3$ which is symmetric by reflection in the three coordinate planes of $\mathbb{R}^3$, as in Figure 8.1. On the equatorial circle $S^1 \subset S^2$ in the xy–plane, are marked four standard points of the form $(\pm 1, \pm 1, 0)$. Conway thinks of the positive y–axis as pointing northwards so that the points form the four diagonal points $P^0 = \{\mathrm{NE}, \mathrm{NW}, \mathrm{SW}, \mathrm{SE}\}$ of the compass in cyclic order.

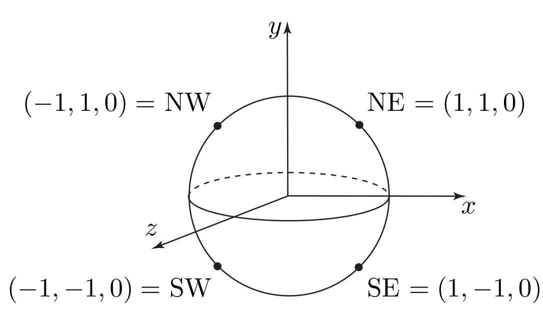


FIGURE 8.1.

We orient the above standard 2–sphere S^2 as the boundary of the 3–ball that it bounds in $\mathbb{R}^3$. In particular, the projection from the front hemisphere (consisting of those $(x, y, z) \in S^2$ with $z \geqslant 0$) to the xy–plane preserves orientation.

Following Conway we define a (k–valent) ***tangle***[1] to be a knot pair (M, K) whose boundary consists of k Conway spheres $F_1, \ldots, F_k$ parametrised by specifying orientation-preserving isomorphisms $\theta_i\colon (F_i, K \cap F_i) \to (S^2, P^0)$, $i = 1, \ldots, k$, to the standard pair above. Writing $\theta = \big(\theta_i\big)_{i=1,\ldots,k}$ we denote the tangle by $(M, K; \theta)$, or simply by (M, K) if the choice of θ is evident.

A ***tangle isomorphism*** $\varphi\colon (M, K; \theta) \to (M', K'; \theta')$ between two tangles is simply a pair isomorphism $\varphi\colon (M, K) \to (M', K')$ such that

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ \theta_i \downarrow & & \downarrow \theta'_i \\ S^2 & \xrightarrow{\text{Id}} & S^2 \end{array}$$

commutes for all i. In other words, φ makes the boundary parametrisations agree.

The sign of isomorphism of tangles will be $\equiv$.

It is very convenient that certain knot pairs (M, K) in $S^3 = \mathbb{R}^3 \cup \infty$ can be given a canonical tangle structure: namely those such that, for each boundary component S of ∂M, the pair $(S, K \cap S)$ is related to (S^2, P^0) by mere translation and homothety (scaling). This motion is unique; it gives the parametrisation of S if it has degree $+1$; otherwise we compose this motion with reflection ρ across the xy–plane to parametrise S.

This will let us *represent tangles* $(M, K; \theta)$ *by planar diagrams* as in Figure 8.2. Specifically, we arrange that the equators of the boundary components of M be in the xy–plane $z = 0$ and that projection to the xy–plane $\mathbb{R}^2$ immerse K into $M \cap \mathbb{R}^2$ (so as not to cross these equators), with simple double points only. Then, indicating over- and under-crossings, we have specified the tangle by a diagram in the xy–plane as in Figure 8.2, where we see $(\partial M) \cap \mathbb{R}^2$ as four circles.

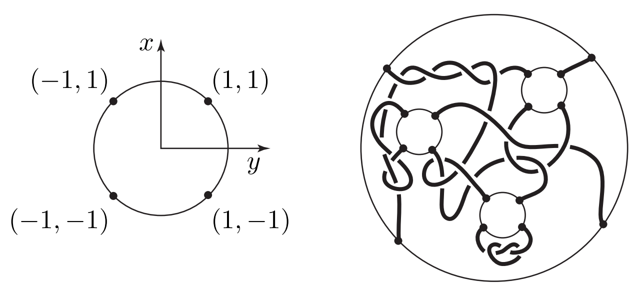
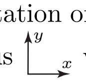


FIGURE 8.2.

For such diagrammatic representation of tangles we agree that the page is the xy–plane with the axes directed thus 

while the z–axis points upwards from the paper to the eye.

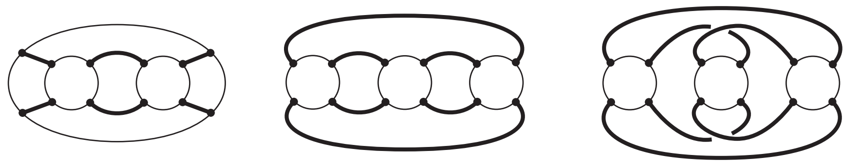
The reader can verify that every tangle is tangle isomorphic to one presented diagrammatically as above.

This simple-minded way of denoting tangles is less evolved than Conway's in [**Conw**] but will suffice until our graphical notations appear in Chapter 12.

Here are two pitfalls involving tangles and pairs, of which the reader should beware.

[1](Added 2009) The word "tangle" has taken various meanings in the past 30 years, in particular in quantum topology. We are here restricting our terminology to Conway's original definition.

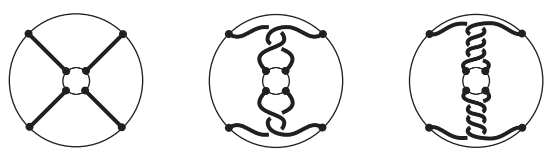
(1) In Figure 8.3, the tangle (a) is not isomorphic to the tangle (b) but rather to (c).



(a) (b) (c)

FIGURE 8.3.

(2) In Figure 8.4, the tangle (a) is tangle isomorphic to (b), but when the total space of each is naturally identified to $S^2 \times [0,1]$, the string in (a) is not isotopic fixing boundary to the string in (b). (This is a consequence of the fact that, if G_3 denotes the space of degree $+1$ continuous maps $S^2 \to S^2$, then $\pi_1(G_3) = \pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2 \neq 0$). An auxiliary fact is that the automorphisms of pair (a) fixing boundary are isotopic to the identity; see [**Wal**$_3$] or Theorem 16.8.) On the other hand, the string in (c) is isotopic fixing boundary to position (a), because $2\pi_1(\mathrm{SO}(3)) = 0$. It is a pleasant trick to perform this isotopy physically!



(a) (b) (c)

FIGURE 8.4.

The example in (2) reveals that *isotopy fixing boundary can be a finer equivalence relation than tangle isomorphism.* This example is typical. One can show that for knot pairs (M, K_1), (M, K_2), etc..., where M is a specific 3–sphere with holes, the equivalence of pair isomorphism fixing ∂M is the same as the equivalence generated by isotopy fixing ∂M and enough modifications $(a) \mapsto (b)$ of Figure 8.4, made in collars of the boundary components. (Prove this by showing that any isomorphism of M fixing ∂M is isotopic to the identity moving ∂M.)

It is important that there is a natural way to ***glue*** or ***match*** a tangle (M, K) to another (M', K') along a specified boundary sphere S of M and S' of M'. The result is a new tangle $(M \cup_\psi M', K \cup_\psi K')$, where the identification $\psi\colon S \to S'$ is the unique orientation-reversing isomorphism that, in terms of the given parametrisations of S and S' by S^2, is the reflection ρ across the xy–plane.

When (M', K') is monovalent we describe the above process by saying that we ***plug*** (M', K') ***into*** (M, K) ***at*** $S \subset \partial M$.

Figure 8.5 gives an example of matching where ψ is a mere translation. More generally, for tangles represented diagrammatically, $\psi\colon S \to S'$ is always just composed of translation, homothety, and (when orientation dictates it) the reflection ρ across the xy–plane.

In certain tangles (M, K), it will be crucial to specify how surfaces meet the boundary spheres. Thus we need a good understanding of the *curves* in the model

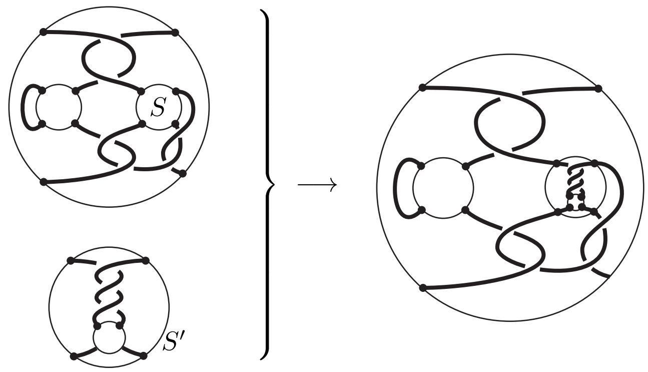


FIGURE 8.5.

(S^2, P^0) that are *pairwise essential*. We explain these terms. By a ***curve*** in (S^2, P^0) we mean a connected compact 1–submanifold of S^2 whose boundary (perhaps empty) is its intersection with P^0. By ***pairwise essential*** we mean that no component bounds a 2–disc meeting P^0 in at most one point.

Choosing our terminology in accord with [**HatT**], we let $\mathcal{R}$ be the group of isometries of $\mathbb{R}^2$ generated by the rotations of angle π around each point of $\mathbb{Z}^2$, and consider the pair $(\mathbb{R}^2/\mathcal{R}, \mathbb{Z}^2/\mathcal{R})$. It admits a homeomorphism θ to (S^2, P^0). We can require that this homeomorphism preserves orientation, that it sends the square of Figure 8.6 onto the front hemisphere of S^2 delimited by the plane $\mathbb{R}^2 \times 0$, and that the image of $(0, 0)$ is $(-1, -1)$. Then this homeomorphism is unique up to pairwise isotopy. Use it to identify (S^2, P^0) and $(\mathbb{R}^2/\mathcal{R}, \mathbb{Z}^2/\mathcal{R})$.

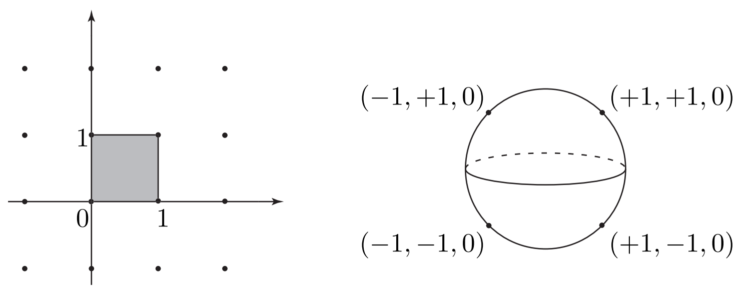


FIGURE 8.6.

If we are working in the category of piecewise linear manifolds and maps, note that $\mathbb{R}^2/\mathcal{R}$ is naturally a polyhedron; we can and do insist that θ be a piecewise linear isomorphism.

In the differentiable category, note that $\mathbb{R}^2/\mathcal{R}$ is not naturally a smooth manifold at $\mathbb{Z}^2/\mathcal{R}$; it is rather a smooth orbifold shaped like a pillow with four corners $\mathbb{Z}^2/\mathcal{R}$. But we can and do insist that θ be smooth away from these corners, and at each corner locally send emanating linear rays onto smooth arcs. There is in fact a standard choice for θ, the one making the composition $\mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2/\mathcal{R} \xrightarrow{\theta} S^2$ holomorphic (use for example the Riemann mapping theorem on squares in $\mathbb{R}^2$).

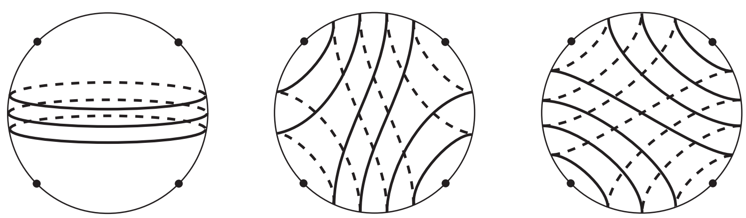
A line with rational or infinite slope in $\mathbb{R}^2$ projects to $S^2 = \mathbb{R}^2/\mathcal{R}$ onto a pairwise essential curve, which we call ***linear*** or ***straight***. Conversely, any pairwise essential curve is pairwise isotopic to such a linear curve. We let the reader devise a proof (compare the proof of Proposition 1.3). The (well-defined) corresponding slope is called the ***slope*** of the curve considered.

Note that disjoint *closed* curves with the same slope are parallel, separated by an annulus avoiding P^0.

More generally, the ***slant*** of a system C of several disjoint pairwise essential closed curves in (S^2, P^0) is the pair (p, q) in $\mathbb{Z}^2/\pm$ (= pairs up to sign), where $\frac{q}{p}$ in $\mathbb{Q} \cup \infty$ is the slope of these parallel curves and the greatest common divisor of p and q is the number of components of C.

For another interpretation of p and q, assume that the intersection of C with $S^2 \cap (\mathbb{R}^2 \times \{0\})$ (namely with the boundary of the "square") cannot be reduced by any pairwise isotopy; then the intersection of C with a "vertical" or "horizontal" side of the square consists respectively of $|p|$ and $|q|$ points. Figure 8.7 gives some examples.

These definitions extend in the obvious way to a system of curves on the boundary of a tangle.



slant = $(3, 0)$ $(2, 4)$ $(4, -3)$

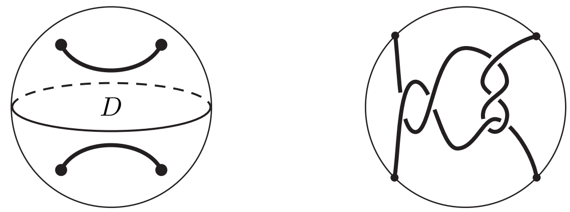
FIGURE 8.7.

Note that the reflection ρ through the xy–plane sends slant (p, q) to slant $(p, -q) = (-p, q) \in \mathbb{Z}^2/\pm$.

The simplest tangles are probably those for which the underlying pair consists of two unknotted and unlinked arcs in a ball, as in Figure 8.8(a). These are *rational tangles*, in Conway's terminology. In such a pair, there exists a disc D which is not pairwise boundary parallel, shown in Figure 8.8(a), and an easily proved fact is that D is unique up to pairwise isotopy (see also Theorem 8.3). The slant (α, β) in $\mathbb{Z}^2/\pm$ of ∂D, or equivalently its slope $\frac{\beta}{\alpha}$ in $Q \cup \infty$, is called the ***type*** or ***slope*** of the rational tangle. For example, the type of the rational tangle of Figure 8.8(b) is $(12, 5)$. (Unknot the strings, or compare §1.2.) Conversely, it follows from the uniqueness of D that a rational tangle is classified up to equivalence by its type.

Observe that this definition of the slope of a rational tangle is entirely compatible with the definition of the slope of a rational tangle projection given in §1.2.

After this bulky preamble concerning tangles, we are more than ready to study Montesinos pairs and tangles, the natural building blocks of the arborescent part of a knot.



(a) (b)

FIGURE 8.8.

8.2. Surfaces in hollow Montesinos pairs

A ***hollow Montesinos tangle*** is one isomorphic to a tangle represented diagramatically in Figure 8.9; this figure gives one ***model*** or ***standard*** tangle for each number ($\geqslant 0$) of boundary components.

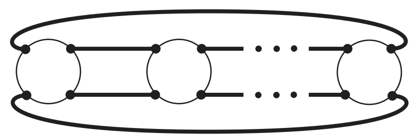


FIGURE 8.9.

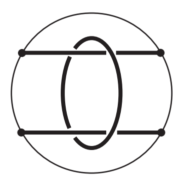


FIGURE 8.10.

A ***model Montesinos tangle*** is one constructed from a model hollow Montesinos tangle of Figure 8.9 by plugging in some rational tangles, as defined in §8.1, and some so-called ***ring tangles*** shown in Figure 8.10. A ***Montesinos pair*** is a knot pair isomorphic to the underlying pair of a model Montesinos tangle. Similarly, a ***Montesinos tangle*** (no epithet) is a tangle that admits an isomorphism to a model Montesinos tangle; such an isomorphism is said to give a ***presentation*** of the Montesinos tangle (or pair).

Observe that this definition of a Montesinos pair is slightly more general than the one introduced in Chapter 3, inasmuch as it allows it to have several ***rings*** (= the circles in the ring tangles of a presentation of the Montesinos pair).

Note that a presentation of a Montesinos pair offers, beyond a tangle structure, some internal parametrised Conway 2–spheres, one for each tangle plugged into the hollow tangle.

Two presentations of the same Montesinos pair are said to be ***equivalent*** if the resulting inclusions of model hollow Montesinos pairs are the same up to pairwise isotopy.

Recall that, when the contrary is not specified, we usually require that any surface F in a knot pair (M, K) is ***properly embedded*** (namely $F \cap \partial M = \partial F$)

and meets K transversely. Since we are mainly dealing with the case where M is the complement of disjoint balls in S^3, this will compel F to be orientable by the following fact.

FACT 8.1. *Every properly embedded surface F in a 3–manifold M is orientable provided $H_1(M, \mathbb{Z}_2) = 0$.*

PROOF. To begin with, $H_1(M; \mathbb{Z}_2) = 0$ implies M is orientable. If F were non-orientable, an orientation-reversing path γ in it could be deformed in M to meet F transversely in one point, contradicting $[\gamma] = 0 \in H_1(M; \mathbb{Z}_2)$. □

An important object in the model hollow Montesinos pair (M, K) shaded in Figure 8.9 is the ***band*** in Figure 8.11(a) (it is intimately related to the plumbing bands of Chapter 12). This band B admits an $[0, 1]$–fibration, compatible with the boundary, for which K corresponds to the $\{0, 1\}$–fibration. We shall see in Proposition 10.1 that, when ∂M has $\geqslant 3$ components, B is an invariant of the pair (M, K) in the sense that it is characteristic, namely preserved up to pairwise isotopy by every automorphism of (M, K).

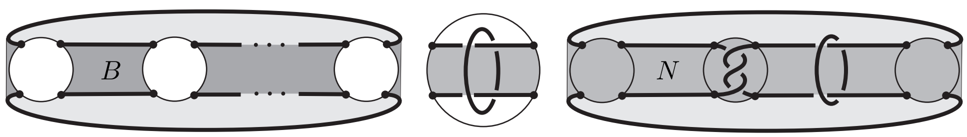


FIGURE 8.11.

In a model Montesinos pair (M, K) derived from a hollow Montesinos pair (M_0, K_0), the ***band*** B is the union of the band B_0 of (M_0, K_0) and, in any ring tangle, of the square indicated in Figure 8.11(b). The ***necklace*** N is the union of B with all of ∂M_0 except the boundary of any ring tangle. Figure 8.11(c) gives an example.

In a presented Montesinos pair, the *band* and *necklace* come via the given isomorphism with a model Montesinos pair.

To study the structure of Montesinos pairs, a first step is to find, up to pairwise isotopy, all the surfaces which cannot be "simplified" in ways we now explain.

In Chapter 3, we defined what it means for a surface F in a knot pair (M, K) to be pairwise incompressible. Now we define *pairwise boundary-incompressibility* (∂–incompressibility).

A ***∂–compression disc*** D for F in a knot pair (M, K) is a 2–disc in $M - K$ such that $\partial D = D \cap (F \cup \partial M)$ and ∂D consists of an arc $\partial_+ D$ in F and an arc $\partial_- D$ in ∂M. This disc D is called ***futile*** if there exists a 2–disc D' in $F - K$ such that $\partial D'$ consists of the arc $\partial_+ D$ and of an arc in∂M.

The same definition applies to a boundary surface F if we replace ∂M above by $\partial M - \text{int}(F)$.

A connected surface F in a knot pair (M, K) is ***pairwise boundary parallel*** (∂–***parallel***) if for some closed-up component M_0 of $M - F$, there is an isomorphism $F \times [0, 1] \to M_0$ sending $F \times 0$ to F and $(F \cap K) \times [0, 1]$ onto $K \cap M_0$. In other words, F is pairwise boundary parallel when it is obtained from a boundary surface $F' \subset \partial M$ by "pushing" $\text{int}(F')$ inside M.

A surface (or boundary surface) F in a knot pair (M, K) is ***pairwise ∂–incompressible*** if

(1) Every pairwise ∂–compression disc for F in (M, K) is futile.
(2) No component of F is a 2–disc meeting K in $\leqslant 1$ point and pairwise ∂–parallel.

∂–incompressibility is a technical notion that has little independent significance for us in view of a lemma (Lemma 8.19, proved where first needed, towards the end of this chapter) which establishes :

FACT 8.2. *Let (M, K) be a pairwise irreducible knot pair, each of whose boundary components is a Conway 2–sphere. If F is a pairwise incompressible surface in (M, K) with no component pairwise ∂–parallel, then F is ∂–incompressible.*

A surface (or boundary surface) F in a knot pair (M, K) is ***pairwise essential*** if it is both pairwise incompressible and pairwise ∂–incompressible.

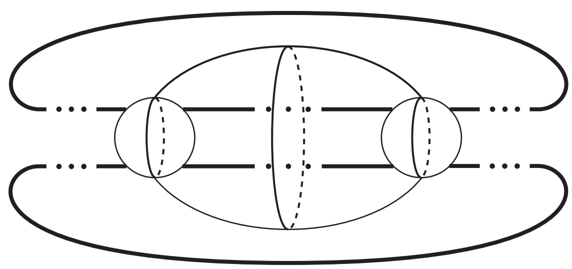


FIGURE 8.12.

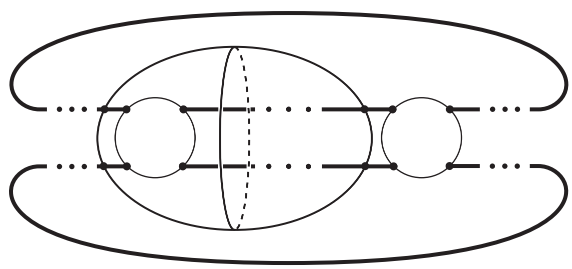


FIGURE 8.13.

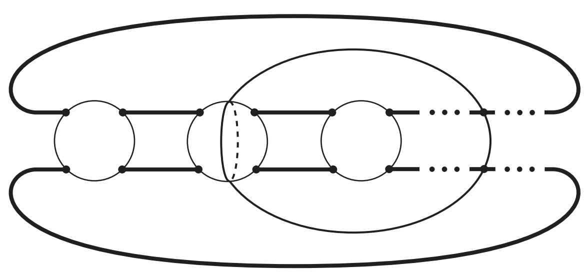


FIGURE 8.14.

In a hollow Montesinos pair (M, K), there are two classes of surfaces which seem to be pairwise essential, called *vertical* and *horizontal*. The components of a ***vertical surface*** are annuli, Conway spheres and Conway discs of the type described respectively in Figures 8.12, 8.13, and 8.14. As a reminder of the terminology, remark that the slants of the boundary of a vertical surface are all of type $(0, q)$, and that its intersection with the band B is vertical for the $[0, 1]$–fibration of B. Note that some vertical Conway spheres are pairwise parallel to the boundary.

The ***horizontal surfaces*** are less easy to see. To construct a typical one, begin with a surface F_0 consisting of p parallel disjoint copies of the horizontal sphere with holes Σ of Figure 8.15.

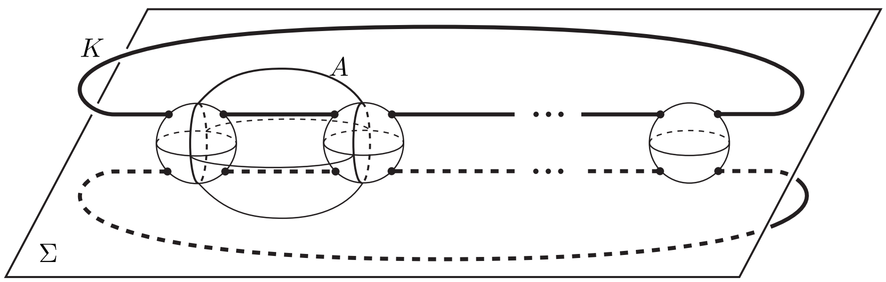


FIGURE 8.15.

Choose a regular neighbourhood of the band B (respecting F_0); its frontier in M is made up of disjoint vertical annuli. Near each such annulus A we permit a modification of F_0 by a fractional twist as follows.

Let $U \cong S^1 \times [0, 1] \times [0, 1]$ be a collar neighbourhood of A (with $A = S^1 \times \frac{1}{2} \times [0, 1]$) in M, respecting F_0 and ∂M. Erase $F_0 \cap U$ and replace it by $\alpha(F_0 \cap U)$ where α, as indicated in Figure 8.16, is an automorphism of U respecting (not fixing) $U \cap \partial M$ and $(\delta U) \cap F_0$. (Here δ indicates the topological frontier in M). This α can be a product with the identity on the second $[0, 1]$–factor of U above; its activity on the other two (forming an annulus) is a twist as represented in Figure 8.16.

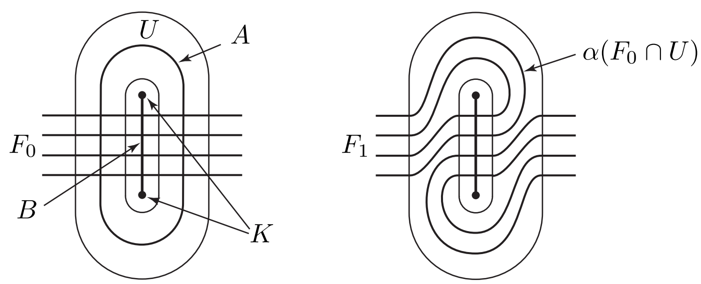


FIGURE 8.16.

We can explain the terminology by indicating that, as we will see in the Appendix, the double branched covering $\widehat{M}$ of a Montesinos pair (M, K) admits a Seifert fibration. A vertical surface in (M, K) can be pairwise isotoped so that its preimage in $\widehat{M}$ is a union of fibres, whereas, after pairwise isotopy, the preimage of a horizontal surface is everywhere transverse to the fibre. See also [**BonS**$_1$].

We can now classify all the essential surfaces in hollow Montesinos pairs.

THEOREM 8.3. *Consider the model hollow Montesinos tangle* (M, K) *with* $k \geqslant 1$ *boundary components. Then:*

(1) *Every pairwise essential surface* F *in the pair* (M, K) *is pairwise isotopic to a vertical surface, or to a horizontal surface as defined above.*
(2) *For a horizontal surface, there exist numbers* $p, q_1, \ldots, q_k \in \mathbb{Z}$ *with* $\sum_{i=1}^{k} q_i = 0$ *such that the slant of* ∂F *on the i-th boundary component is* (p, q_i).
(3) *Conversely, every horizontal or vertical surface is pairwise essential and every set of numbers* $p, q_1, \ldots, q_k$ *with the above condition defines a horizontal surface unique up to pairwise isotopy.*

COROLLARY 8.4. *A hollow Montesinos pair with non-empty boundary is pairwise irreducible, and it is pairwise* ∂*–irreducible as soon as it has at least two boundary components.* □

REMARK 8.5. The hollow Montesinos pair (M, K) with empty boundary, consisting of $M = S^3$ and of two unknotted unlinked circles K, is exceptional in that it is not pairwise irreducible.

PROOF OF THEOREM 8.3. Let F be a pairwise essential surface in (M, K). Assume for the while that ∂F is pairwise essential in $(\partial M, \partial K)$; we shall show in the course of the proof that (M, K) is pairwise irreducible, and deduce at the end that this assumption always holds.

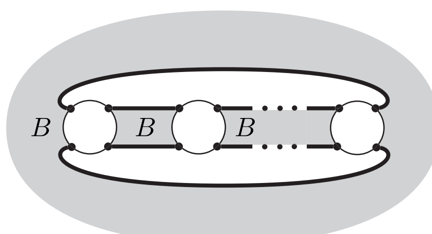


FIGURE 8.17.

Consider the band B in (M, K), namely the union of the squares represented on Figure 8.17. After a pairwise isotopy, we may assume that the intersection of B and F is transverse and that its number of components cannot be reduced by any pairwise isotopy of F.

CLAIM 8.6. *In* B*, the components of* $B \cap F$ *are of the types* (a) *and* (b) *of Figure* 8.18.

PROOF OF CLAIM 8.6. We need to prove that the types (c), (d), (e) and (f) are excluded by the above assumptions.

If one component of $B \cap F$ is of type (c), it bounds in B a disc D whose interior can be assumed, without loss of generality, disjoint from F. Since F is pairwise incompressible, ∂D bounds in F a disc D' which does not meet K. Lemma 8.7 below shows that the sphere $D \cup D'$ bounds a ball in $M - K$; so there would then exist a pairwise isotopy of F which reduces its intersection with B, contradicting our assumption; see Figure 8.19(c). We interrupt to give:

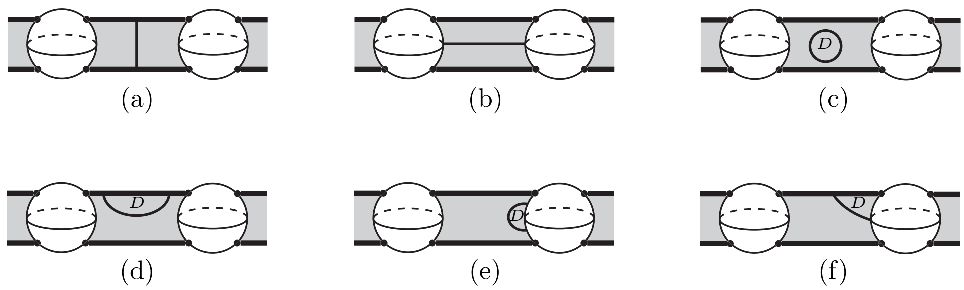


FIGURE 8.18.

LEMMA 8.7. *The manifold $M - K$ is irreducible.*

PROOF OF LEMMA 8.7. Embed $M - K$ in S^3. Each sphere in $M - K$ bounds a ball on each side in S^3. The sphere does not meet the frontier $K \cup \partial M$ of $M - K$ in S^3 (which is connected since $k \geqslant 1$). Hence one of the balls contains $K \cup \partial M$ and the other lies in $M - K$. □

If one component of $B \cap F$ is of type (d), let D be the half disc of the picture. Again, we can assume $F \cap \operatorname{int}(D)$ empty. From two copies of D, we can now construct a disc D' which does not meet K and such that $D' \cap F = \partial D'$. The curve $\partial D'$ bounds a disc D'', disjoint from K, in the pairwise incompressible surface F and the sphere $D' \cup D''$ bounds a ball in the irreducible manifold $M - K$. But this implies that a component of F is a pairwise compressible sphere as in Figure 8.19(d)), which is excluded.

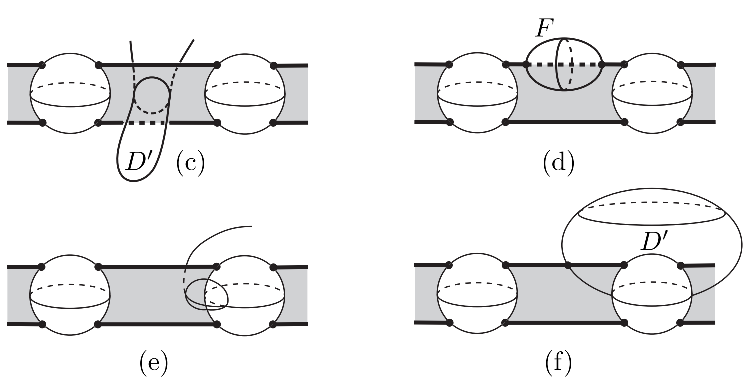


FIGURE 8.19.

If one component of $B \cap F$ is of type (e), let D be the disc shown on Figure 8.18(e). Since F is pairwise ∂–incompressible, there exists a disc D' in F whose boundary is the union of the arc $F \cap \partial D$ and of an arc $D' \cap \partial M$ in ∂M. The interior of D' may intersect B; nevertheless there exists a disc D'' in D' such that $\partial D''$ is the union of the two arcs $D'' \cap \partial M$ and $D'' \cap B$ with disjoint interiors. Then the arc $D'' \cap \partial M$ joins one side of $B \cap \partial M$ to itself without meeting B anywhere else. But this implies the existence of a pairwise isotopy of F near ∂M which reduces the cardinality of $B \cap \partial F$; if this isotopy is followed by another one which eliminates the closed components of $B \cap F$ as in Case (c), this process would decrease $B \cap F$ and therefore contradict our assumptions.

If one component of $B \cap F$ is of type (f), the same construction as for (d) reveals a component D' of F which is a disc meeting K in exactly one point. Since we assumed ∂F was essential at the beginning of the proof of Theorem 8.3, $\partial D'$ bounds a disc D'' in ∂M which meets K in two points. But then, the sphere $D' \cup D''$ meets K in 3 points, which is impossible by a counting argument.

This ends the proof of Claim 8.6. □

Next, consider the behaviour of F on a boundary component S of M. The band B meets S in two intervals I_1, I_2 (drawn vertically in Figure 8.20 below).

CLAIM 8.8. *Following along a circle of $F \cap S$, one cuts I_1 and I_2 alternatively (or one cuts neither).*

PROOF OF CLAIM 8.8. Supposing this is not so, we will cut the same interval I_i twice in a row. This could happen only in three ways illustrated in Figure 8.20(a), (b), (c).

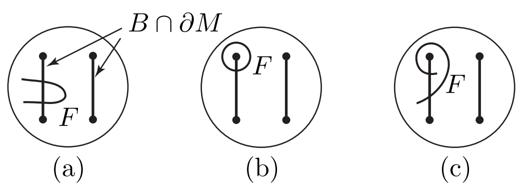


FIGURE 8.20.

In view of Claim 8.6, Case (a) would allow a reduction of the number of components of $F \cap B$ by a pairwise isotopy of (M, K).

Case (b) is excluded since ∂F was assumed to be pairwise essential in $(\partial M, \partial K)$.

Case (c) implies the occurrence of Case (a), indeed for the same arc $I_i \subset S \cap B$. □

Claims 8.6 and 8.8 reveal that, if one square of B contains p horizontal arcs of $F \cap B$ (as in Figure 8.18(b)), then so does every other square. Consequently, at each boundary component of ∂M, the slope of ∂F is of the form (p, q_i). Here $p = 0$ is admissible; in this case the squares of B may have varying numbers of vertical arcs of $B \cap F$.

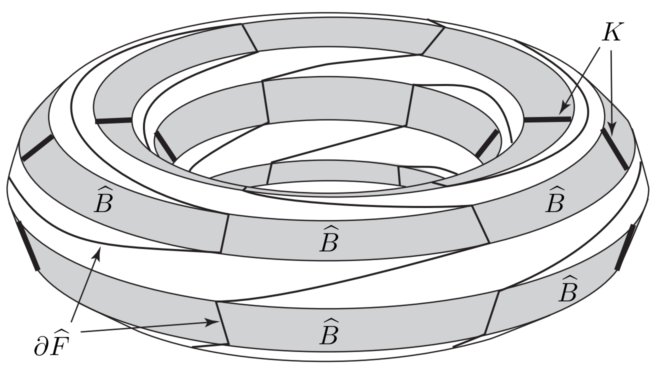


FIGURE 8.21.

Let $\widehat{M}$ be the manifold obtained by cutting M open along $B-K$ and let $\widehat{F} \subset \widehat{M}$ be the surface corresponding to F. The manifold $\widehat{M}$ is a solid torus, and the band B lifts to a union $\widehat{B}$ of disjoint longitudinal annuli.

Claim 8.9. *The surface $\widehat{F}$ is incompressible in $\widehat{M}$.*

Proof of Claim 8.9. If C is a curve in $\widehat{F}$ which is null-homotopic in M, the pairwise incompressibility of F implies that C bounds in F a disc D which does not meet K. The disc D cannot meet B since no component of $B \cap F$ is closed and $\partial(B \cap F) = (\partial B \cap \partial F) \cup (F \cap K)$. Therefore, D is contained in $\widehat{M}$. □

By [**Wal**$_2$, Lemma 2.3], every component of $\widehat{F}$ is therefore a meridian disc, a boundary parallel disc or a boundary parallel annulus in the solid torus $\widehat{M}$.

Claim 8.10. *Every component of $\widehat{F}$ is, either a meridian disc, or an annulus whose boundary components are parallel to the circles $\partial\widehat{B}$.*

Proof of Claim 8.10. If a component of $\widehat{F}$ was a boundary parallel disc D, its boundary ∂D could not meet $\partial\widehat{B}$ since, by Claims 8.6 and 8.8, no closed-up component of $\partial\widehat{F} - \partial\widehat{B}$ joins a component of $\partial\widehat{B}$ to itself.) Nor could ∂D be in $\widehat{B}$ (as every component of $B \cap F$ is vertical or horizontal), nor could ∂D be in $\partial\widehat{M} - \widehat{B}$ (as ∂F is pairwise essential in $(\partial M, \partial K)$).

If a component of $\widehat{F}$ is a boundary parallel annulus which meets $\partial\widehat{B}$, it admits a ∂–compression disc D such that $J = D \cap \partial\widehat{M}$ is an arc in $\partial\widehat{M} - \widehat{B} \subset \partial M$ and joins two different components of $\partial\widehat{F}$. The disc D cannot be an effective pairwise ∂–compression disc for F in (M, K), as F is pairwise ∂–incompressible; there exists therefore a disc D' in F such that $\partial D'$ is the union of $D \cap F$ and $D' \cap \partial M$, and $D' \cap K$ is empty. Then $D' \cap B$ consists of arcs with boundary in $\partial D'$; and there consequently exists an arc in $D' \cap \partial M$ that joins the two ends of a component of $B \cap F$ and whose interior is disjoint from B. But this would give an arc in $\partial\widehat{F}$ joining a component of $\partial\widehat{B}$ to itself and meeting $\partial\widehat{B}$ nowhere else, which is excluded by Claims 8.6 and 8.8. □

When the components of $\widehat{F}$ are annuli whose boundary does not meet $\partial\widehat{B}$, F is clearly vertical (after pairwise isotopy).

When the components of $\widehat{F}$ are meridian discs, $\partial\widehat{F}$ can be described in the following way: Its intersection with each component of $\partial\widehat{B}$ consists of $2p$ horizontal arcs, p of them in each half determined by the inverse image of K; without loss of generality, these $2p$ arcs may be assumed to be in a standard position which is fixed in advance. The intersection of $\partial\widehat{F}$ with the inverse image of a component of ∂M consists of $2p$ arcs wrapping $\frac{q_i}{2p}$ times around the axis of this annulus, where (p, q_i) is the slant of ∂F on the considered component of ∂M. Since $\partial\widehat{F}$ is null-homotopic in $\widehat{M}$ ($\partial\widehat{F}$ is a meridian of the solid torus $\widehat{M}$), it follows that $\sum_{i=1}^{k} q_i = 0$. Note that the slopes characterise F up to pairwise isotopy, and that, conversely, every set of slopes of the above type defines such a surface. Besides, it is easy to see that F is horizontal (after pairwise isotopy).

Recall that we assumed, at the beginning of the proof, that ∂F was pairwise essential in $(\partial M, \partial K)$. But the above study implies that (M, K) is pairwise irreducible as soon as $\partial M \neq \varnothing$, and a straightforward argument shows that every pairwise incompressible surface in a pairwise irreducible knot pair has pairwise essential boundary. This hypothesis therefore turns out to be unnecessary.

To complete the proof of Theorem 8.3, we need to prove that vertical and horizontal surfaces in (M, K) are all pairwise essential.

A vertical surface F splits (M, K) into two hollow Montesinos pairs (M_1, K_1) and (M_2, K_2), each with an obvious presentation. Moreover, each (M_i, K_i) has $\geqslant 2$ boundary components, unless F is an annulus joining two adjacent boundary components of (M, K). Also, ∂F has only infinite slope(s) in $(\partial M_i, \partial K_i)$.

A pairwise (∂–)compression disc D for F is also a properly embedded surface in one of the (M_i, K_i)'s, say (M_1, K_1). From what we have just proved, it then follows that either D is pairwise isotopic in (M_1, K_1) to a vertical or horizontal surface, or D is pairwise inessential in (M_1, K_1); in this last case, D is necessarily pairwise boundary parallel (it clearly admits no effective (∂–)compression disc).

Every vertical disc in (M_1, K_1) meets K_1 in 2 points. Thus D cannot be pairwise isotopic to a vertical surface.

If D is pairwise isotopic to a horizontal surface in (M_1, K_1), then (M_1, K_1) is a rational tangle of slope 0, and F was an annulus contained in ∂M_1. Also, ∂D and ∂F have different slopes in $(\partial M_1, \partial K_1)$; indeed one is zero and the other one infinite. Consequently ∂D meets each of the two components of ∂F in $\geqslant 2$ points, which is incompatible with D being a (∂–)compression disc.

Therefore, D is pairwise parallel to a disc $D' \subset \partial M_1$ such that $\partial D = \partial D'$ and cutting K_1 in as many points as D. By considering $D' \cap F$, it follows easily that D is futile.

This completes the proof that a vertical surface in a hollow Montesinos pair is pairwise essential.

If F is horizontal, it splits M into a manifold $\widetilde{M}$ naturally constructed from a linear $[0,1]$–bundle $F' \widetilde{\times} [0,1]$, with orientable total space and base a possibly non-orientable surface F', by taking a 1–submanifold J of $\partial F'$ and making the folding identification on $J \times [0,1] \subset F' \widetilde{\times} [0,1]$ that glues (x, t) to $(x, 1-t)$ for $(x, t) \in [0,1] \times [0,1]$; the string K corresponds to $J \times \frac{1}{2}$. The part F'' of $\partial \widetilde{M}$ arising from F is the quotient of the $\{0,1\}$–bundle $F' \widetilde{\times} \{0,1\} \subset F' \widetilde{\times} [0,1]$; it clearly consists of two copies of F (recall F is orientable). The following lemma shows that F is pairwise essential in (M, K), and therefore completes the proof of Theorem 8.3. $\square$

LEMMA 8.11. *In the above manifold $\widetilde{M}$, the boundary surface $\widetilde{F}$ is pairwise incompressible and ∂–incompressible in $(\widetilde{M}, J \times \frac{1}{2})$.*

PROOF. We resort to a covering trick. The manifold $\widetilde{M}$ admits a 2–fold covering branched along $K = J \times \frac{1}{2}$, namely the double $D(F' \widetilde{\times} [0,1])$ of $F' \widetilde{\times} [0,1]$ along $J \times [0,1]$, where the covering translation is the antipodal map on the interval fibres composed with the reflection in $J \times [0,1]$ exchanging the halves of the double.

Consider a pairwise compression or ∂–compression disc D for F'' in $(\widetilde{M}, K) = (\widetilde{M}, J \times \frac{1}{2})$. Its preimage $\widehat{D}$ in $D(F' \widetilde{\times} [0,1]) \cong (DF') \widetilde{\times} [0,1]$ consists of one or two pairwise compression or ∂–compression discs for $(DF') \widetilde{\times} [0,1]$ (one if D meets K). But, since the maps $\pi_1\big((DF') \widetilde{\times} \{0,1\}\big) \to \pi_1\big((DF') \widetilde{\times} [0,1]\big)$ and $\pi_1\big((DF') \times \{0,1\}, (\partial DF') \widetilde{\times} \{0,1\}\big) \to \pi_1\big((DF') \widetilde{\times} [0,1], (\partial DF') \widetilde{\times} [0,1]\big)$ are injective for every choice of base points, all such (∂–)compression discs are futile. In particular, it is then a homological absurdity for $\widehat{D}$ to meet $K = J \times \frac{1}{2} \subset D(F \widetilde{\times} [0,1])$.

Thus $\widehat{D}$ consists of two discs disjoint from K, each futile; it follows easily that D is futile. □

8.3. Surfaces in general Montesinos pairs

Extending Theorem 8.3 we now study pairwise essential surfaces in any Montesinos tangle.

Consider a model Montesinos tangle (M, K) obtained by plugging rational tangles and ring tangles into a hollow Montesinos tangle (M_0, K_0). A surface F in (M, K) is called ***horizontal*** if its intersection with M_0 is horizontal in the hollow Montesinos pair (M_0, K_0), as defined in §8.2, while F meets each rational tangle in ($\geqslant 0$) parallel discs avoiding K as in Figure 8.22(a), and F meets each ring tangle in ($\geqslant 0$) parallel discs as in Figure 8.22(b). On the other hand, F is called ***vertical*** if $F \cap M_0$ is vertical in (M_0, K_0) as defined in §8.2 while F meets each rational tangle in discs avoiding K and F meets each ring tangle in parallel annuli as in Figure 8.22(c); we also require for vertical surfaces that no component of $F \cap M_0$ be pairwise parallel in (M_0, K_0) to the boundary of a rational tangle of (M, K).

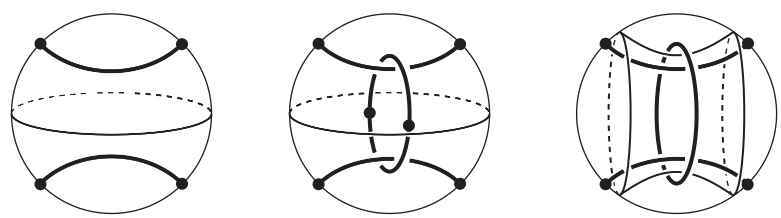


FIGURE 8.22.

Note that there cannot exist any horizontal surface if (M, K) contains a rational tangle of slope ∞, and that a vertical surface is necessarily disjoint from every rational tangle of finite slope.

Similar definitions apply to *presented* Montesinos pairs using the given isomorphism with a model pair.

Clearly vertical surfaces are made up of Conway spheres, Conway discs, annuli disjoint from K and tori disjoint from K; tori occur precisely if there are $\geqslant 2$ ring tangles.

Ordering arbitrarily, label the boundary components of (M, K) by $i = 1, \ldots, d$, the rational tangles by $j = 1, \ldots, t$ and the ring tangles by $k = 1, \ldots, r$. Write (M_j, K_j) for the j–th rational tangle and $\frac{\beta_j}{\alpha_j}$ or (α_j, β_j) for its type, with the convention that $\alpha_j \geqslant 0$.

PROPOSITION 8.12. *Consider any model (or presented) Montesinos pair* (M, K)*, with the terminology introduced above.*

(a) *The non-empty horizontal surfaces, viewed up to pairwise isotopy, correspond bijectively to sequences of* $d + 1$ *integers* $p, q_1, \ldots, q_d$ *with* $p > 0$ *satisfying the conditions that* $\sum_{i=1}^{d} \frac{q_i}{p} = \sum_{j=1}^{t} \frac{\beta_j}{\alpha_j}$ *and that* α_j *divides* p *for every* $j = 1, \ldots, t$ *(so none exist if some* $\alpha_j = 0$*). These* $d + 1$ *integers corresponding to a horizontal surface* F *are determined by the two following properties:*

(*) *On the i–th boundary component of M, the slant of ∂F is (p, q_i); the integer p is thus the number of horizontal segments into which F cuts any component of the band B of (M, K). This first property suffices if $d > 0$.*

(**) *The slant of F on the boundary of the j–th rational tangle is $(p, n_j\beta_j)$ where $n_j = \frac{p}{\alpha_j}$.*

(b) *The number of components of a horizontal surface is the greatest common divisor of the integers p, q_1, …, q_d introduced in* (a). *The components of a horizontal surface are related by a series of pairwise parallelisms; in particular two disjoint connected horizontal surfaces are pairwise parallel.*

(c) *If $d = 0$ (namely if $\partial M = \varnothing$), there is at most one connected non-empty horizontal surface, up to pairwise isotopy; the condition for the existence of one is $\sum_{j=1}^{t} \frac{\beta_j}{\alpha_j} = 0$. If $d = 1$, there is exactly one such surface. In these two cases, the integer p is the least common multiple of α_1, …, α_t. If $d > 1$, there are infinitely many connected horizontal surfaces.*

Proof of Proposition 8.12(a). Given integers $p, q_1, \ldots, q_d$ as described in (a), our Theorem 8.3 applied to the hollow tangle (M_0, K_0) assures the existence of a surface F determining these numbers; recall that the slope of F on the boundary component of M_0 at the i–th rational tangle is $-\frac{\beta_i}{\alpha_i}$, and that it is 0 at a ring tangle.

To complete the proof of (a), we must satisfy ourselves that two horizontal surfaces F and F', presented in the standard manner and corresponding to distinct integer vectors p, q_1, …, q_d and p', q'_1, …, q'_d are not pairwise isotopic in M. This is obvious when $\partial M \neq \varnothing$ (namely when $d > 0$) because ∂F and $\partial F'$ are not pairwise isotopic in $(\partial M, \partial K)$. On the other hand, if $\partial M = \varnothing$, something stronger will be proved by (c), to which we defer. □

Proof of Proposition 8.12(b) and (c). The greater common divisor δ of the numbers p, q_1, …, q_d is easily seen to be the number of components of the intersection of the corresponding horizontal surface F with the necklace N. But $\pi_0(F \cap N) \cong \pi_0(F)$ by inclusion, which proves the first part of (b). For the second assertion in (b), note that the numbers p, q_1, …, q_d are also associated to the horizontal surface obtained by taking δ parallel copies of the horizontal surface corresponding to $\frac{p}{\delta}$, $\frac{q_1}{\delta}$, …, $\frac{q_d}{\delta}$.

Property (c) is an easy consequence of (b). □

Proposition 8.13. *With the data of Proposition* 8.12(a)*, the Euler characteristic $\chi(F)$ of a horizontal surface F in a model Montesinos pair (M, K) is given by*

$$\chi(F) = 2p - (d + t)p + \sum_{j=1}^{t} n_j = p\Big(2 - d - \sum_{j=1}^{t}\big(1 - \tfrac{1}{\alpha_j}\big)\Big).$$

As a consequence,

(i) *A horizontal surface is a 2–sphere precisely if $d = 0$ and either* 0 *or* 2 *of the rational tangles have non-integral slopes.*

(ii) *A horizontal surface is a 2–disc precisely if $d = 1$ and at most one slope is non-integral.*

(iii) *A horizontal surface is an annulus if and only if, either $d = 2$ and every rational tangle has integral slope, or $d = 1$ and exactly two rational tangles* *have slopes congruent to* $\frac{1}{2}$ *modulo* 1*, while all other rational tangles have integral slope.*

PROOF. Splitting F along the necklace we see that F can be reconstructed from $2p$ discs by making $(d+t)p$ pairwise identifications of arcs on the disc boundaries, then attaching to this the $\sum_{j=1}^{t} n_j$ discs in the rational tangles. This leads to the Euler characteristic formula.

If we have a horizontal 2–sphere F then

$$2 = \chi(F) = p\Big(2 - \sum_{1}^{t}\big(1 - \tfrac{1}{\alpha_j}\big)\Big)$$

where, by Proposition 8.12(c), the integer p is the least common multiple of α_1, …, α_t. Clearly no more than three α_j can be $\geqslant 2$. Then a case-by-case analysis using $\sum_{j=1}^{t} \frac{\beta_j}{\alpha_j} = 0$ yields only the solution advertised. Note that the non-integral slopes must be of the form $\frac{\beta}{\alpha}$ and $-\frac{\beta}{\alpha}$ modulo the integers.

If we have a horizontal disc F, the formula yields

$$1 = \chi(F) = p\Big(1 - \sum_{j=1}^{t}\big(1 - \tfrac{1}{\alpha_j}\big)\Big].$$

Clearly no more than one α_j can be $\geqslant 2$.

For a horizontal annulus F,

$$0 = \chi(F) = p\Big(2 - d - \sum_{j=1}^{t}\big(1 - \tfrac{1}{\alpha_j}\big)\Big].$$

with $d > 0$. The only possibilities are those indicated. □

We continue to deal with a presented Montesinos pair (M, K) with d boundary components, constructed by plugging t rational tangles and r ring tangles into the hollow Montesinos pair (M_0, K_0). To classify vertical surfaces in (M, K), it is useful to consider the connected horizontal surface Σ_0 in (M_0, K_0) with slope 0 on each boundary component. It is a sphere with $d+t+r$ punctures, shown in Figure 8.15. Choose a reflection τ of Σ_0 through its 1–submanifold $B_0 \cap \Sigma_0$, as in Figure 8.23.

This symmetry under τ lets us in effect look at F in just half of Σ_0.

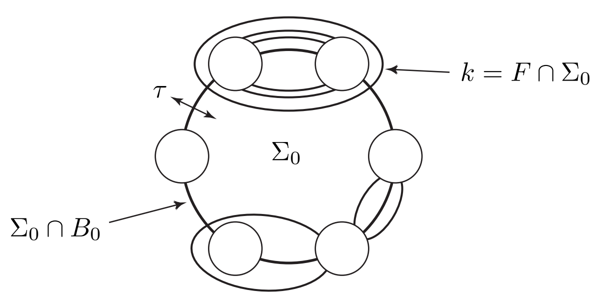


FIGURE 8.23.

PROPOSITION 8.14.

(a) *For the above data, the rule* $F \mapsto F \cap \Sigma_0$ *defines a one-to-one correspondence between, on the one hand, the vertical surfaces in* (M, K) *considered up to isotopy respecting* M_0 *and, on the other hand, those essential 1–submanifolds* k *of* Σ_0*, defined modulo isotopy respecting the segments* $\Sigma_0 \cap B_0$*, that enjoy the following properties:* k *is preserved by* τ *up to isotopy respecting* $\Sigma_0 \cap B_0$*; it meets* $\Sigma_0 \cap B_0$ *transversely; for each component* C *of* $\partial\Sigma_0$ *in a rational tangle of finite slope,* $k \cap C = \varnothing$ *and no component of* k *is parallel to* C *; the number of intersection points of* ∂k *with any ring tangle is a multiple of* 4.

(b) *If no rational tangle has infinite or integral slope, then two vertical surfaces* F *and* F' *are pairwise isotopic (if and) only if they are pairwise isotopic respecting* M_0.

(c) *Up to pairwise isotopy respecting* M_0*, there are finitely many connected vertical surfaces precisely if there is* $\leqslant 1$ *ring tangle.*

(d) *A vertical surface cannot be pairwise isotopic to a horizontal surface.*

Note that, in (a), the condition that k is preserved by τ up to isotopy respecting $\Sigma_0 \cap B_0$ is independent of the choice of τ.

Proof of Proposition 8.14(a). Giving a vertical surface F up to pairwise isotopy respecting M_0 is readily seen to be equivalent to giving $F_0 = F \cap M_0$ up to pairwise isotopy in (M_0, K_0).

If k is an essential 1–submanifold of Σ_0 with the announced properties, its isotopy class respecting $B_0 \cap \Sigma_0$ is determined by giving the intersection of each of its components with $(B_0 \cap \Sigma_0) \cup \partial\Sigma_0$.

In case k is $F_0 \cap \Sigma_0$, this k is therefore characterised by giving the intersection of each component of F_0 with $K_0 \cup \partial M_0$. Consequently, when two vertical surfaces F, F' are pairwise isotopic respecting M_0, the associated 1–submanifolds k, k' are equivalent as announced in (a).

Conversely, if $F \cap \Sigma_0$ are isotopic respecting $B_0 \cap \Sigma_0$, it is clear from the definition of vertical surfaces that F_0 and F_0' are pairwise isotopic in (M_0, K_0). Indeed $F_0 \cap (B_0 \cup \partial M_0)$ is easily recovered from $F_0 \cap \Sigma_0$, and each component of F_0 is obtained by making suitable gluings of two discs along $F_0 \cap (B_0 \cup \Sigma_0)$.

Lastly, it is easy to check that every 1–submanifold of Σ_0 with the required properties is the intersection of Σ_0 with a vertical surface (use the above reconstruction of F_0 from $F \cap \Sigma_0$). □

Proof of Proposition 8.14(b). Our proof of (b) anticipates later results in this section (namely Theorem 8.15 and Corollary 8.18); in principle, this result should be delayed. However, here is our proof. (One can alternatively use fundamental groups.)

It is convenient to consider, in the union M_ρ of M_0 and of all the ring tangles, the horizontal surface Σ_ρ that is the union of Σ_0 and of the discs of Figure 8.22(b). The reflection τ of Σ_0 extends to a reflection of Σ_ρ across the intersection of Σ_ρ with the band B of (M, K), which respects the $2s$ points $K \cap \Sigma_\rho$; we still denote this reflection by τ.

For any vertical surface F in (M, K) the intersection $F \cap \Sigma_\rho$ is a 1–submanifold which is pairwise essential in $(\Sigma_\rho, K \cap \Sigma_\rho)$, since $F \cap \Sigma_0$ was essential in Σ_0. Also $F \cap \Sigma_\rho$ is preserved by τ up to isotopy respecting $B \cap \Sigma_\rho$.

Given two vertical surfaces F and F', an easy argument using blisters and the 2–dimensional version of Proposition 5.23 shows that, in the surface Σ_ρ, the 1–submanifolds $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$ are pairwise isotopic respecting $(B \cup K) \cap \Sigma_\rho$ precisely if, in Σ_0, the 1–submanifolds $F \cap \Sigma_0$ and $F' \cap \Sigma_0$ are isotopic respecting $B \cap \Sigma_0$. By Proposition 8.14(a) the vertical surface F is therefore classified, up to isotopy respecting M_0, by $F \cap \Sigma_\rho$ modulo isotopy respecting $B \cap \Sigma_\rho$ and $K \cap \Sigma_\rho$.

Consider now two vertical surfaces F and F' such that $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$ are not pairwise isotopic respecting $B \cap \Sigma_\rho$ and $K \cap \Sigma_\rho$. To prove (b), we need to show that F and F' cannot be isotopic.

After a pairwise isotopy of F respecting M_ρ and keeping F vertical, we may assume that there is no 2–dimensional pairwise blister between $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$ in $(\Sigma_\rho, K \cap \Sigma_\rho)$, namely no closed-up component of $\Sigma_\rho - F \cup F'$ that is a pairwise parallelism between a part of $F \cap \Sigma_\rho$ and a part of $F' \cap \Sigma_\rho$. This would be false with Σ_0 in place of Σ_ρ.

If F and F' were pairwise isotopic in (M, K), a closed-up component of $M - F \cup F'$ would be a pairwise blister; here in order to apply Proposition 5.23, we need the fact that every vertical surface is pairwise essential which, under the hypotheses of (b), will be proved in Theorem 8.15. But, reconstructing any closed-up component P of $M - F \cup F'$ from closed-up component(s) of $\Sigma_\rho - F \cup F'$, one readily sees that one of the following holds:

(1) P is a solid torus and, either it contains a ring in its interior, or $P \cap F$ consists of at least two annuli (remember the absence of blisters between $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$).
(2) $(P, K \cap P)$ is a Montesinos pair, naturally presented, so that the rational tangles in $(P, K \cap P)$ have the same type as in (M, K); moreover ∂F, $\partial F'$ and $F \cap F'$ have infinite slope on the boundary.

Assume, in search of a contradiction, that P is a pairwise blister between F and F'. Then, $(P, K \cap P)$ certainly contains no ring; also $F \cap \partial P$ and $F \cap \partial P'$ are connected. This shows that P cannot be of type (1), whence P must be of type (2).

Now we know that the blister $(P, K \cap P)$ satisfies (2). Since it is a Montesinos pair, $F \cap P$ can only be a Conway sphere or a Conway disc.

If $F \cap P$ is a Conway sphere, the knot pair $(P, K \cap P)$ contains many essential annuli (namely curves in $F \cap P$ crossed with $[0, 1]$) and has two boundary components. Then, by Theorem 8.15 and Proposition 8.13, the natural presentation of $(P, K \cap P)$ is hollow with two boundary components. Hence the horizontal surface $P \cap \Sigma_\rho$ would be a 2–dimensional blister (an annulus) in $(\Sigma_\rho, K \cap \Sigma_\rho)$ between $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$, a contradiction.

If $F \cap P$ is a Conway disk, then $(P, K \cap P)$ contains a pairwise essential disk D whose boundary meets $\partial(F \cap P)$ in two points, so that the slope of ∂D is an integer. Again, Theorem 8.15 and Proposition 8.13 imply that there is no rational tangle in $(P, K \cap P)$, and $P \cap \Sigma_\rho$ provides a 2–dimensional blister between $F \cap \Sigma_\rho$ and $F' \cap \Sigma_\rho$ in $(\Sigma_\rho, K \cap \Sigma_\rho)$, which was excluded.

Therefore, no closed-up component P of $M - F \cup F'$ can be a pairwise blister between F and F' in (M, K). By Proposition 5.23, it follows that F and F' are not pairwise isotopic, which ends the proof of Proposition 8.14(b). □

Proof of Proposition 8.14(c). Note that a vertical surface F is connected precisely if $F \cap \Sigma_\rho^*$ is connected, where Σ_ρ is the surface defined in the proof of (b) and where Σ_ρ^* is either one of the two closed-up components of $\Sigma_\rho - B$. Now, (c) is an easy consequence of the fact that F is classified, modulo isotopy respecting M_0, by $F \cap \Sigma_\rho^*$ modulo isotopy respecting $B \cap \Sigma_\rho^*$; see the proof of Proposition 8.14(b). □

Proof of Proposition 8.14(d). Suppose that the vertical surface $F \neq \varnothing$ is pairwise isotopic to a horizontal surface G. Clearly $\partial G = \varnothing = \partial F$ so $\partial M = \varnothing$. Also as G exists there are no rational tangles of slope ∞. Thus we are dealing with a 2–torus disjoint from K or with a Conway sphere. Hence G meets rings and F does not. This contradicts F and G being pairwise isotopic. □

We are now well advanced in the proof of the following conceptually simple result, parallel to a celebrated one of Waldhausen for Seifert fibre spaces [**Wal**$_2$].

Theorem 8.15.

(a) *In a pairwise irreducible presented Montesinos (M, K), every pairwise essential surface is pairwise isotopic to a vertical or horizontal one.*
(b) *Conversely, every horizontal surface is pairwise essential and, provided no rational tangle has infinite type, so is every vertical surface.*

When (M, K) is pairwise reducible, a somewhat similar statement holds for essential spheres meeting the knot in $\leqslant 2$ points, namely:

Theorem 8.16. *In a presented Montesinos pair (M, K) suppose that there exists a pairwise essential 2–sphere F meeting K in $\leqslant 2$ points. Then there exists a possibly different vertical or horizontal pairwise essential 2–sphere F' meeting K in no more points than F.*

Remark 8.17. In general, it is not possible to require in Theorem 8.16 that F' be pairwise isotopic to F. For instance, Figure 8.24 provides an example of a pairwise essential sphere Σ meeting the knot in $\leqslant 2$ points which is not pairwise isotopic to any vertical or horizontal surface.

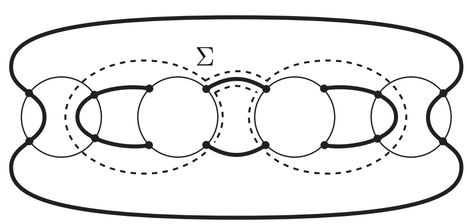
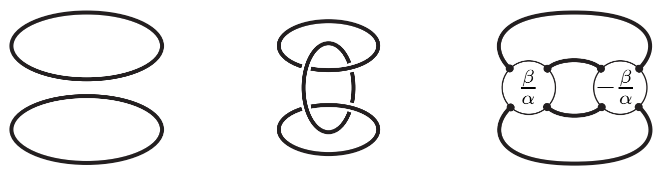


Figure 8.24.

Combining Theorems 8.16 and 8.15(a) (which still require proof) with Proposition 8.13 (already proved) and noting that, when every rational tangle has finite slope, a vertical surface cannot be a sphere meeting K in $\leqslant 2$ points, nor a disc meeting K in $\leqslant 1$ points, we get the following:

Corollary 8.18. *Suppose that the presented Montesinos pair (M, K) is restricted in the sense that no rational tangle is of integral or infinite type. Then (M, K) is pairwise irreducible, unless it is one of the three (reducible) presented* *pairs of Figure* 8.25. *Also, ∂M is pairwise incompressible in (M, K) unless it is a rational tangle pair, with the presentation $d = 1$, $r = 0$, $t \leqslant 1$.* □

Note that the pair of Figure 8.25(c) is pair isomorphic to that of Figure 8.25(a), by untangling the strings.



(a) (b) (c)

FIGURE 8.25.

A partial converse for Corollary 8.18 will be provided by Proposition 8.20, which in particular shows that every pairwise irreducible Montesinos pair admits a presentation without any rational tangle of infinite slope.

PROOF OF THEOREMS 8.15(A) AND 8.16. We prove Theorems 8.15(a) and 8.16 simultaneously. Consider a pairwise essential surface F in (M, K) such that one of the following two conditions hold:

(i) either (M, K) is pairwise irreducible and the number of components of $F \cap \partial M_0$ cannot be reduced by any pairwise isotopy of F (here we aim to prove Theorem 8.15(a));
(i) or F is a sphere meeting K in $\leqslant 2$ points, and there exists no pairwise essential sphere F' such that $\#(F' \cap K) \leqslant \#(F \cap K)$ and $F' \cap (K \cap \partial M_0)$ has fewer components than $F \cap (K \cup \partial M_0)$ (here we aim to prove Theorem 8.16).

In either case we propose to show that F can be pairwise isotoped respecting M_0 to a vertical or horizontal surface. Clearly, this will prove Theorems 8.15(a) and 8.16.

Consider first the case where the presentation of (M, K) has no ring tangle. If (M_j, K_j), $1 \leqslant j \leqslant t$, denotes the j–th rational tangle, let F_j be $F \cap M_j$ for $0 \leqslant j \leqslant t$.

An easy argument shows that F_j is pairwise incompressible for every $j \geqslant 0$. Indeed, in Case (i), an effective pairwise compression disk would provide a pairwise isotopy of F reducing $F \cap \partial M_0$ (using the pairwise irreducibility of (M, K)). In Case (ii), surgering F along an effective compression disc for F_j would provide two spheres cutting $K \cup \partial M_0$ in fewer components than F ; but, by Condition (ii), these two spheres must be pairwise compressible and F would also be so, which would contradict our hypotheses.

To make use of Theorem 8.3, we need the following result.

LEMMA 8.19. *Let (M, K) be a pairwise irreducible pair whose boundary consists of Conway spheres and tori disjoint from K. Let F be a pairwise ∂–compressible surface in (M, K). Then F is pairwise boundary parallel, and is either a Conway disc or an annulus disjoint from K.*

PROOF OF LEMMA 8.19. Waldhausen proves the case where $K = \varnothing$ as [**Wal**$_2$, Lemma 2.11]. His argument also applies if there is an effective pairwise ∂–compression disc D for F meeting a torus boundary component of M.

Otherwise there exists such a D meeting a Conway sphere in ∂M. Then all the possibilities for $D \cap \partial M$ are exhausted in Figures 8.26, 8.27 and 8.28. (Observe that ∂F is pairwise essential in $(\partial M, \partial K)$, and therefore splits it into Conway spheres, Conway discs and tori and annuli avoiding ∂K.) A straightforward argument, using the pairwise incompressibility of F and the pairwise irreducibility of (M, K) then shows that the cases of Figure 8.26 cannot occur at all, since otherwise the arc $D \cap F$ would be pairwise inessential in F. In the cases of Figures 8.27 and 8.28, F is respectively a pairwise boundary parallel Conway disc or annulus (see the pictures). $\square$

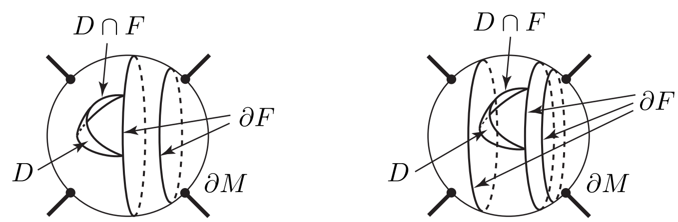


FIGURE 8.26.

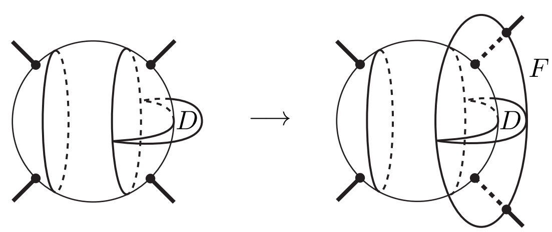


FIGURE 8.27.

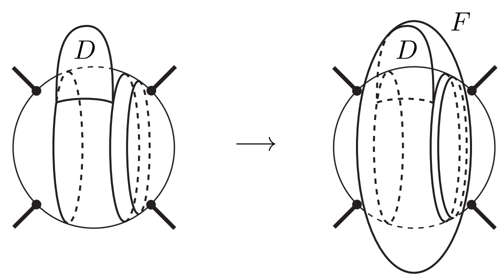
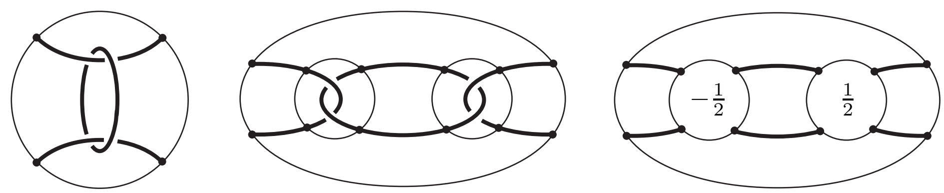


FIGURE 8.28.

Let us return to the proofs of Theorems 8.15(a) and 8.16. By Conditions (i) and (ii), no component of F_j can be pairwise boundary parallel in (M_j, K_j) and, by Lemma 8.19, F_j is consequently pairwise essential in the hollow (presented) Montesinos pair (M_j, K_j), for $0 \leqslant j \leqslant t$. The isotopy of F to a vertical or horizontal surface is now obtained by applying Theorem 8.3 to each F_j.

This proves Theorems 8.15(a) and 8.16 when (M, K) contains no ring tangle.

Now suppose the presentation of (M, K) has a ring tangle. Note that the ring pair is also the Montesinos pair obtained by plugging two rational tangles of types $(2, 1)$ and $(2, -1)$ in the hollow Montesinos tangle with three boundary components; to see this, just rotate the picture by 90° as in Figure 8.29. Thus the analysis of the previous case shows that the ring tangle is pairwise irreducible, and that pairwise essential surfaces in it are, either pairwise boundary parallel, or vertical or horizontal for the presentation of Figure 8.29(c). For this new presentation, there is exactly one vertical surface, which is the Conway disk of Figure 8.22(b), and one horizontal surface, which is the annulus of Figure 8.22(c) (compare Propositions 8.12 and 8.13(iii)).



(a) (b) (c)

FIGURE 8.29.

The same argument as in the previous case, using Lemma 8.19, the above result on ring tangles, and Theorems 8.15(a) applied to the complement of the ring tangles, now provides a pairwise isotopy respecting M_0 from F to a vertical or horizontal surface.

This ends the proof of Theorems 8.15(a) and 8.16. □

PROOF OF THEOREM 8.15(B). Every horizontal surface splits (M, K) into two pieces, each of them containing an entire component of K in its interior. It follows that every horizontal surface which is a sphere meeting K in $\leqslant 2$ points is pairwise essential.

Now consider a connected horizontal surface F that is not such a sphere. The above considerations on the closed-up components of $M - F$ show that F is not pairwise boundary parallel. Therefore, by Lemma 8.19, we just need to prove that F is pairwise incompressible.

When no ring tangle was plugged into (M, K), the proof that F is pairwise essential is verbatim the same as for a hollow Montesinos pair in Theorem 8.3.

Otherwise, let (M_i, K_i), $t+1 \leqslant i \leqslant t+r$, denote the ring tangles ($r \geqslant 1$), and consider the union

$$(M', K') = (M, K) - \bigcup_{i=t+1}^{t+r} \operatorname{int}(M_i, K_i)$$

of (M_0, K_0) and of the rational tangles. We have shown that $F \cap M'$ is pairwise essential in (M', K'). Also by Theorem 8.15(a), the surface $F \cap M_i$ is pairwise essential in (M_i, K_i), for $t+1 \leqslant i \leqslant t+r$.

If D is a pairwise compression disc for F, it can be pairwise isotoped until ∂D is completely contained in M' or in a ring tangle; for this, use the pairwise ∂–incompressibility of $F \cap M'$ and of $F \cap M_i$ in (M', K') and (M_i, K_i). Now, D meets $\bigcup_{i=t+1}^{t+r} \partial M_i$ only in its interior (and meets it transversely).

Let D' be a disc that is an innermost closed-up component of $D - \bigcup_i \partial M_i$. Assume for instance that $D' \subset M'$. Then $\partial D'$ is not pairwise essential in $(\partial M', \partial K')$: Otherwise, since it is disjoint from $F \cap \partial M'$, it would be pairwise isotopic to one of its components. Then, by pairwise incompressibility of $F \cap M'$ in (M', K'), a component of $F \cap M'$ would be a disc avoiding F'. But F would then be a sphere meeting K in 2 points, which has been excluded. Since $\partial D'$ is pairwise inessential in $(\partial M', \partial K')$, it is now possible to find a new pairwise compression disc D_1 for F, with $\partial D_1 = \partial D$, that meets $\bigcup_i \partial M_i$ in less components that D. A similar argument holds if $D' \subset M_i$.

By the above argument, we may henceforth assume that D is completely contained in M' or in one M_i. By pairwise incompressibility of $F \cap M'$ and $F \cap M_i$ in (M', K') and (M_i, K_i), it follows that D is futile.

This ends the proof that a horizontal surface in (M, K) is pairwise essential.

Now, we want to show that, when no rational tangle has infinite or integral slope, every vertical surface is pairwise essential.

Consider for instance a *vertical annulus* G (avoiding K). By construction, each closed-up component of $M - G$ is, either a solid torus meeting ∂M along an annulus and containing at least one ring, or a naturally presented Montesinos pair meeting ∂M in two Conway discs and such that ∂G has infinite slope on the boundary. For homological reasons, an effective compression disc D for G could only be contained in a closed-up component of $M - G$ of the second type, where, by Theorem 8.15(a), D can be chosen to be horizontal (since no rational tangle has infinite slope). But this would contradict the fact that $\partial D \cap \partial G = \varnothing$. Hence G is pairwise incompressible.

As G is not pairwise boundary parallel, Lemma 8.19 proves that it is pairwise essential.

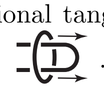
The proof of essentiality is quite similar for vertical Conway spheres and discs. For a vertical torus G', it is slightly different: By construction, G' bounds a solid torus V in M, such that $V \cap K$ consists of $\geqslant 2$ rings. For homological reasons, a pairwise compression disc D for G' is contained in the closure of $(M - V)$ and meets K in an even number of points (thus 0!). Were D effective, surgering G' along D would provide a pairwise essential sphere avoiding K, which is excluded. The vertical torus G' is therefore pairwise incompressible. This completes the proof of Theorem 8.15(b). □

In view of Corollary 8.18, it will often be convenient (see the next two sections) to insist that any presentation used for a Montesinos pair (M, K) be ***restricted*** in the sense that no rational tangle of infinite or integral slope appears.

We know by Corollary 8.18 that if (M, K) has restricted presentation, it is almost always pairwise irreducible. Conversely,

PROPOSITION 8.20. *A pairwise irreducible presented Montesinos pair* (M, K) *admits a new presentation that is restricted, unless it is the pair of Figure* 8.30 *presented with* $r \geqslant 2$ *rings and a rational tangle of integral slope* $q \neq 0$.

PROOF. For the given presentation of (M, K), there is necessarily $\leqslant 1$ rational tangle of slope ∞, by pairwise irreducibility.

Suppose there is one rational tangle of slope ∞. Then, there exists no ring since any ring could slip free  . By considering vertical 2–spheres (dotted

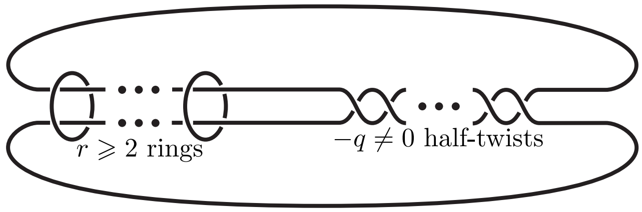


FIGURE 8.30.

in Figure 8.31) cutting the knot in 2 points each, we see that (M, K) is just a rational knot or a rational tangle pair, and as such, admits a presentation without any rational tangle of infinite type.

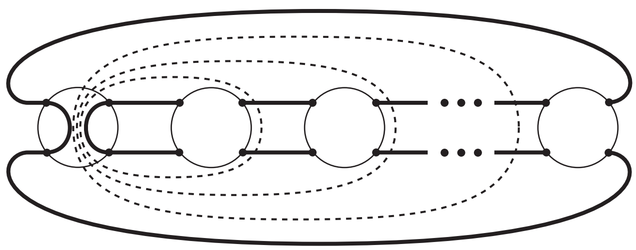


FIGURE 8.31.

Now, the equivalences of Figures 8.32, 8.33, 8.34 and 8.35 easily show that any presented Montesinos pair without rational tangle of infinite type admits a new presentation where no rational tangle of infinite or integral type appear, unless it is one of the exceptional presented pairs advertised. It may be useful to remember the rational tangle relations of Proposition 1.3 and Figure 1.4, and the fact that a rational tangle of integer slope $m \in \mathbb{Z}$ corresponds to a horizontal band with $-m$ right-handed half-twists. $\square$

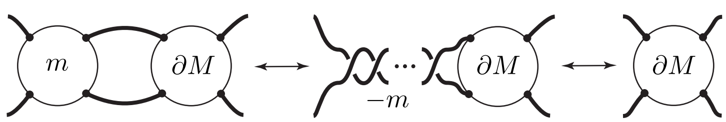


FIGURE 8.32.

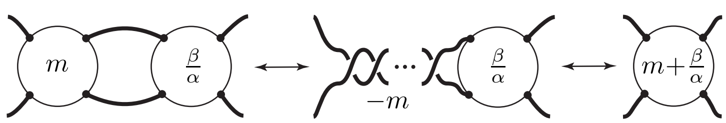


FIGURE 8.33.

Theorem 8.21 below will play a crucial role in the following sections. In fact, much of our analysis of pairwise essential surfaces in Montesinos pairs was necessary (and designed) to prove it.

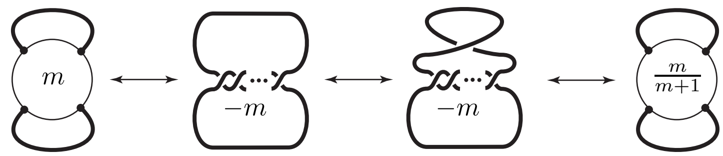


FIGURE 8.34.

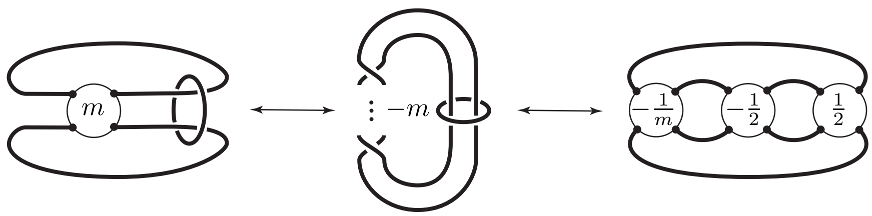


FIGURE 8.35.

THEOREM 8.21 (Verticalisation Theorem). *In a pairwise irreducible Montesinos pair (M, K), let F be a pairwise essential surface which consists of Conway spheres, Conway discs, and annuli avoiding K. Then (M, K) admits a presentation as a Montesinos pair for which F is vertical.*

PROOF. Choose an arbitrary presentation of (M, K). By Theorem 8.15, this presentation can be modified by pairwise isotopy so that F is vertical or horizontal.

If F is vertical, the property is proved. Otherwise, the presentation of (M, K) can be chosen to be restricted (since the exceptions of Proposition 8.20 contain no horizontal surface).

Proposition 8.13 now lists all possibilities.

If F is a horizontal Conway sphere and (M, K) has two rings, then there are no rational tangles and (M, K) is presented as in Figure 8.36(a). Rotating the picture then provides another presentation with two rings and no rational tangles for which F is vertical, as in Figure 8.36(b).

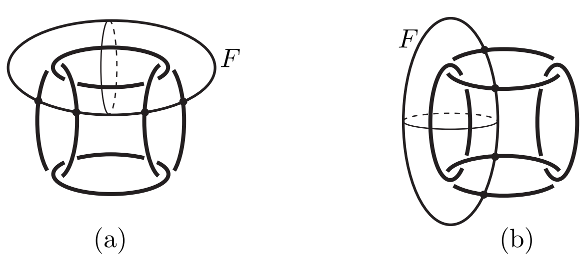
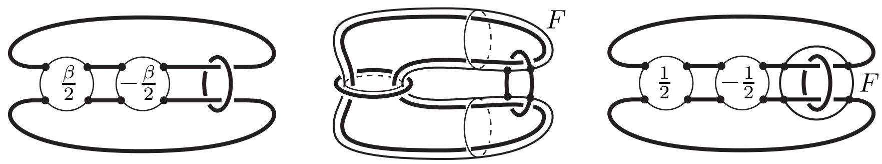


FIGURE 8.36.

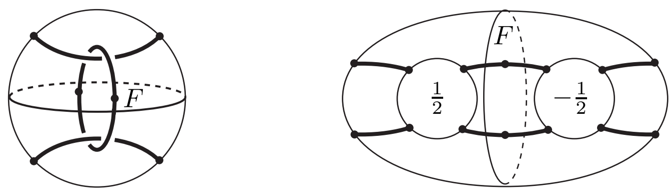
If F is a horizontal Conway sphere and there is only one ring, then (M, K) is presented as in Figure 8.37(a) so that, using the relation of Figure 8.33, F is the Conway sphere represented in Figure 8.37(b). Rotating the picture, (M, K) admits a new presentation for which F is vertical, as in Figure 8.37(c).



(a) (b) (c)

FIGURE 8.37.

If F is a horizontal Conway disk, (M, K) is presented as in Figure 8.38(a). Rotating the picture as in Figure 8.29 provides a new presentation for which F is vertical, as in Figure 8.38(a).



(a) (b)

FIGURE 8.38.

If F is a horizontal annulus disjoint from K, there are two possibilities. The first one is that the pair (M, K) is presented with two boundary components and no ring tangles or rational tangles. The second possibility, using the move of Figure 8.32, is that (M, K) is presented with one boundary component, no ring, and two rational tangles of slope $\frac{1}{2}$.

In the first case, (M, K) is isomorphic to $(S^2, 4 \text{ points}) \times [0, 1]$ in such a way that F corresponds to $C \times [0, 1]$ for some closed curve C. Modifying the presentation of (M, K) by an automorphism of $(S^2, 4 \text{ points}) \times [0, 1]$ that respects the two factors, we can arrange that the curve C has infinite slope in the Conway sphere $(S^2, 4 \text{ points})$, in which case the annulus F is now vertical, as in Figure 8.39.

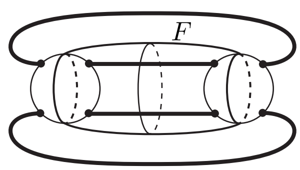
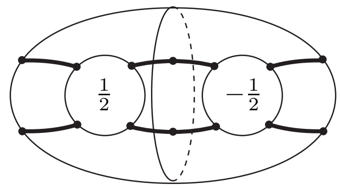


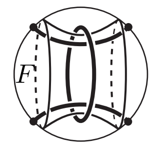
FIGURE 8.39.

In the second case, after converting one slope $\frac{1}{2}$ to $-\frac{1}{2}$ by the move of Figure 8.32 and rotating the picture, (M, K) admits a presentation with one ring and no rational tangle for which the annulus F is vertical. See Figure 8.40. $\square$

In Chapter 3, we met Montesinos pairs as building blocks of the arborescent part of a knot that is *simple for Schubert*. The following result tells which Montesinos pairs actually occur there.



(a)



(b)

FIGURE 8.40.

PROPOSITION 8.22. *A pairwise irreducible restricted Montesinos pair* (M, K) *with rational tangles of types* (α_i, β_i), $i = 1, \ldots, t$, *fails to be simple for Schubert precisely if it has* $\geqslant 2$ *rings or all of the three following conditions hold:*

(i) (M, K) *has no ring and no boundary;*
(ii) *the set of the* α_i *is one of the following:* $\{2, 3, 6\}$, $\{2, 4, 4\}$, $\{3, 3, 3\}$, $\{2, 2, 2, 2\}$;
(iii) $\sum_i \frac{\beta_i}{\alpha_i} = 0$.

PROOF. By Theorem 8.15, (M, K) is simple for Schubert if and only if it contains no vertical and no horizontal torus. A vertical torus exists precisely when (M, K) contains $\geqslant 2$ rings. By the Euler characteristic formula of Proposition 8.13, one verifies that Conditions (i), (ii) and (iii) together are equivalent to the existence of a horizontal torus. □

EXERCISE 8.23 (compare Chapter 12). Show that, up to degree ± 1 isomorphism, the asserted exceptions to Proposition 8.22 constitute exactly 4 knots (and links). Find their respective companionship trees, in the sense of §2.2.

CHAPTER 9

Splitting an arborescent pair into Montesinos pairs

A knot pair (M, K) is called ***arborescent*** (or ***algebraic*** in Conway's sense) if it is connected, if it can be split into elementary pairs (as defined in Chapter 3) by cutting along a finite collection F of disjoint Conway spheres $F_1, \dots, F_n$, and if M embeds in S^3 (or equivalently each F_i separates M).

Consider an arborescent pair (M, K) as above which is also pairwise irreducible. For example, if we are given a knot (S^3, L) that is simple for Schubert, then (M, K) could be the arborescent part of (S^3, L) singled out in Chapters 3 and 7.

By suppressing members from the family F in (M, K), we can derive a family G of disjoint Conway spheres such that:

(i) Each closed-up component N of $M - G$ gives a Montesinos pair $(N, K \cap N)$, as defined in Chapter 8 (possibly with several rings).
(ii) If any component of G is suppressed, Property (i) fails.

This chapter is devoted to proving the following statement.

THEOREM 9.1. *Up to pairwise isotopy, G is characterised by Properties* (i) *and* (ii) *alone.*

Clearly, this theorem and Theorem 7.1 together prove Theorems 3.3 and 3.4, concerning characteristic Conway spheres in a knot that is simple for Schubert.

Before starting the proof of Theorem 9.1, observe that G is necessarily pairwise incompressible. Indeed, a closed-up component N of $M - G$ would otherwise give a rational tangle pair $(N, K \cap N)$, by Corollary 8.18, and $G - \partial N$ would consequently still satisfy Property (i), contradicting the Minimality Property (ii).

The proof of Theorem 9.1 will use the Verticalisation Theorem 8.21 at a crucial point (see the Barrier Lemma 9.4). We shall also need the following two easy lemmas on Montesinos pairs.

LEMMA 9.2. *Consider a knot pair (M, K) that is split by a Conway sphere F into two Montesinos pairs (M_1, K_1) and (M_2, K_2). Suppose that for some presentations of (M_1, K_1) and of (M_2, K_2) as Montesinos pairs, their bands B_1 and B_2 (see Chapter* 8*) coincide in F, so that $B_1 \cap F =$ two arcs $= B_2 \cap F$. Then (M, K) is a Montesinos pair with $B_1 \cup B_2$ as band.* □

The proof is trivial.

We will say that two closed surfaces F, F' in a pairwise irreducible knot pair (M, K) ***intersect minimally*** if their intersection is transverse and if there is no obvious way of reducing $F \cap F'$ by a pairwise isotopy. That is, if:

(a) Each component of $F \cap F'$ is pairwise essential in F and F'.
(b) There is no pairwise blister between F and F', in the sense of Proposition 5.23. In other words, when (M, K) and F' are split along F giving,

say, $(\widehat{M}, \widehat{K})$ and $\widehat{F}'$ in it, then no component $\widehat{F}'_1$ of $\widehat{F}'$ is pairwise boundary parallel and has $\partial \widehat{F}'_1 \neq \varnothing$.

LEMMA 9.3 (Crossing Lemma). *Let (M, K) be a restricted model Montesinos pair with nonempty boundary, which contains a vertical Conway sphere F that is not pairwise boundary parallel. Then, there exists another vertical Conway sphere S in (M, K) that intersects F non-trivially and minimally.*

PROOF. This is a straightforward exercise using our cataloguing of vertical surfaces in Proposition 8.14. Figure 9.1 attempts to illustrate an example. □

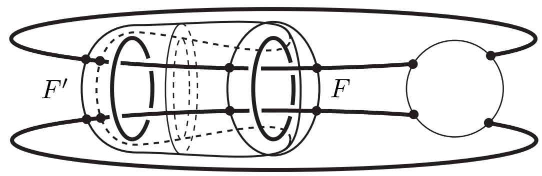


FIGURE 9.1.

The most crucial step towards Theorem 9.1 is the

LEMMA 9.4 (Barrier Lemma). *With the data of Theorem 9.1, there does not exist an incompressible Conway 2–sphere S in (M, K) that intersects G nontrivially and minimally.*

PROOF OF LEMMA 9.4. Suppose such an S exists. By elimination, each closed-up component of $S - G$ is a Conway disc or an annulus disjoint from K.

Consider two closed-up components S_1 and S_2 of $S - G$ that are adjacent on S, namely with $S_1 \cap S_2 \neq \varnothing$. The closed-up components (M_1, K_1) and (M_2, K_2) of $(M, K) - G$ which contain S_1 and S_2 are distinct, *because M embeds in S^3*. Also, since S intersects G minimally, each S_i is pairwise incompressible and not pairwise boundary parallel in (M_i, K_i). Therefore, S_i is vertical in (M_i, K_i) for a suitable presentation as a Montesinos pair, by Theorem 8.21 (this is the most crucial intervention of Chapter 8 in this chapter).

On the component G_0 of G that separates M_1 and M_2, the traces of the bands B_1 for (M_1, K_1) and B_2 for (M_2, K_2) are disjoint from the circle $S_1 \cap G_0 = S_2 \cap G_0$. As a consequence, these two bands can be isotoped so as to coincide on G_0 (see Figure 9.2). Now, Lemma 9.2 asserts that gluing (M_1, K_1) and (M_2, K_2) along G_0 yields a Montesinos pair. Since this contradicts the minimality property (ii) of G (we can suppress G_0), S cannot exist. Thus Lemma 9.4 is proved. □

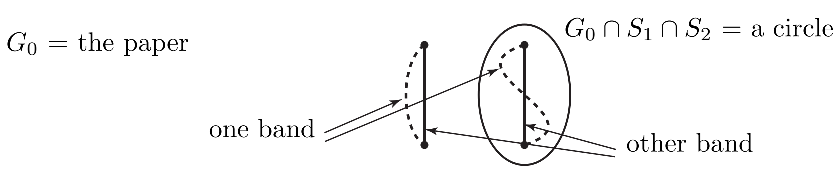


FIGURE 9.2.

Proof of Theorem 9.1. Suppose that a family G' of Conway 2–spheres in (M, K) has the same properties (i) and (ii) as G. By pairwise isotopy we make $G \cap G'$ contain as many Conway spheres as possible, say $G_1 \cup \cdots \cup G_k = G_*$. Then by a further pairwise isotopy fixing G_*, we can make $G - G_*$ and $G' - G_*$ meet minimally in a certain number of circles.

By Lemma 9.4, one has $(G - G_*) \cap (G' - G_*) = \varnothing$, which means $G \cap G' = G_*$.

If $G' \not\subset G$, there must be a closed-up component N of $M - G$ that contains a component G'_j of $G' - G_*$. Use Theorem 8.21 to make $N \cap G'$ vertical for some presentation of $(N, K \cap N)$. Then apply Lemma 9.3 to G'_j, in the closed-up component of $N - (G' - G'_j)$ which contains it, to find a Conway sphere S meeting G'_j non-trivially but not $G \cup G' - G'_j$, and meeting $G \cup G'$ minimally. One readily checks that the intersection of S and G' is also minimal as in (M, K). Then Lemma 9.4 yields a contradiction, and we conclude that $G' \subset G$.

Symmetrically, $G \subset G'$ and thus $G = G'$, which completes the proof of Theorem 9.1. □

In practice, the characterisation of the surface G by Theorem 9.1 is not very convenient, as the Minimality Condition (ii) is in general difficult to check. However, if one scrutinises the proof of Theorem 9.1, one can give a very useful criterion to decide when a family of Conway spheres, which splits (M, K) into Montesinos pairs, is actually the characteristic family of Theorem 9.1.

Criterion 9.5. Let G be a family of disjoint Conway spheres splitting the pairwise irreducible arborescent pair (M, K) into Montesinos pairs. Assume that:

(a) No closed-up component of $(M, K) - G$ is a rational tangle pair.
(b) For any two adjacent closed-up components N_1, N_2 of $M - G$, the Montesinos pairs $(N_1, K \cap N_1)$ and $(N_2, K \cap N_2)$ do not admit restricted presentations whose bands B_1 and B_2 agree on the Conway sphere $N_1 \cap N_2$.

Then, G is the characteristic family characterised by Theorem 9.1.

Proof. First scrutinise the proof of Lemma 9.4 to check that Properties (a) and (b) are sufficient for G to satisfy the conclusion of this lemma. Also, no closed-up component N of $M - G$ can be a pairwise collar $(N, K \cap N) \cong (S^2, P^0) \times [0, 1]$ by Condition (b). Then note that this is all we need in the proof of Theorem 9.1 to show that G is pairwise isotopic to any surface G' satisfying the hypotheses of Theorem 9.1. □

The interest of this criterion will become clearer in Chapter 10 where we shall see that it suffices to check Property (b) for just one restricted presentation of $(N_1, K \cap N_1)$ and $(N_2, K \cap N_2)$. Indeed, we shall prove that the trace $B_i \cap \partial N_i$ of the band is, up to pairwise isotopy, independent of the restricted presentation of $(N_i, K \cap N_i)$, unless this pair is a collar $\cong (S^2, P^0) \times [0, 1]$ or a ring pair. When $(N_i, K \cap N_i)$ is a collar, Property (b) clearly fails. When it is a ring pair, there are precisely two possibilities for $B_i \cap \partial N_i$, up to pairwise isotopy.

Part 4

The classification of Montesinos pairs

CHAPTER 10

Characteristic bands and necklaces in Montesinos pairs

These bands and necklaces, defined in Chapter 8, will constitute the backbone of our classification of arborescent knots in Chapters 11–13, and of our study of their automorphisms in Chapter 16. The main result of this chapter is Theorem 10.5, which proves that the necklaces of most Montesinos pairs are characteristic.

We begin with the case of hollow Montesinos pairs, presented without ring or rational tangles.

10.1. The case of hollow Montesinos pairs

PROPOSITION 10.1. *In a model hollow Montesinos pair* (M, K) *with* $d \geqslant 3$ *boundary components, the band* B *is invariant under all automorphisms (up to pairwise isotopy).*

Further, if $\varphi\colon (M, K) \to (M', K')$ *is a pair isomorphism to any other restricted Montesinos pair (restricted means presented without rational tangles of integral or infinite slope), then* (M', K') *is necessarily presented as a hollow Montesinos pair and* $\varphi(B)$ *is pairwise isotopic to the band* B' *for* (M', K').

REMARK 10.2. Note that this is hopelessly false for 1 and 2 boundary components (Figure 10.1), because enough automorphisms of a given boundary Conway sphere then extend to automorphisms of (M, K).

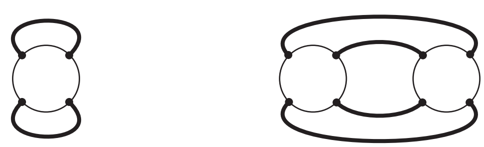


FIGURE 10.1.

PROOF OF PROPOSITION 10.1. We immediately attack the stronger statement.

CLAIM 10.3. *The presentation of* (M', K') *as a Montesinos pair is necessarily hollow.*

PROOF OF CLAIM 10.3. Certainly (M', K') has no ring tangle, since $K \cong K'$ contains no circle component. Thus if (M', K') is not hollow, it contains a rational tangle of non-integral slope. Then by Theorem 8.15(b), the pair can be split along two essential Conway spheres to give, as one piece, the Montesinos pair of

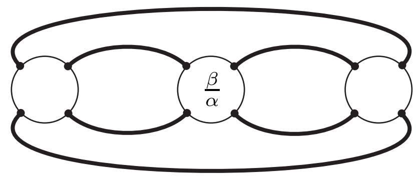


FIGURE 10.2.

Figure 10.2. This piece has exactly two boundary components and is not homeomorphic to (Conway sphere) $\times [0, 1]$, since by Theorem 8.15 and Proposition 8.13 it contains just one essential (vertical) annulus disjoint from K.

But, by Theorem 8.15(a) applied to $(M, K) \cong (M', K')$, the pair (M', K') does not have this property. Thus (M', K') is presented as a hollow Montesinos pair as claimed. □

Consider a small regular neighbourhood U in M of the band B. Its topological frontier δU in M consists of d vertical essential annuli.

By Theorem 8.3, we can make $\varphi(\delta U)$ vertical in (M', K') by a pairwise isotopy (horizontal is out of question since $d \geqslant 3$). Thus $\varphi(\delta U)$ can be isotoped to coincide with $\delta U'$ where U' is a regular neighbourhood of the band B' of (M', K'). Then $\varphi(U) = U'$, as one easily checks.

By a pairwise isotopy we can further arrange that the arcs $\varphi(B \cap \partial M)$ coincide with $B' \cap \partial M'$. The point is that, in any Conway disc, there is, up to pairwise isotopy, a unique arc joining the two distinguished points (see Figure 10.3).

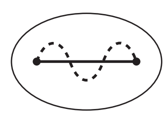


FIGURE 10.3.

Now we have $\varphi(U) = U'$ and $\varphi(\partial B) = \partial B'$. To conclude we make $\varphi(B) = B'$ by a pairwise isotopy of (M', K') moving only points in $\text{int}(U')$. The argument is very much like the proof that any two discs in S^3 with common boundary $S^1 \subset S^3$ are pairwise isotopic in (S^3, S^1); these remaining details can therefore be left to the reader. Proposition 10.1 now is proved. □

10.2. General Montesinos pairs

We next examine necklaces in general Montesinos pairs. Unfortunately there is some misbehaviour that we must avoid (or cope with).

Let us review and adjust our terminology. A Montesinos pair (M, K) is a pair presented in a certain way explained in Chapter 8. For this presentation, a necklace is defined, which we now call the ***presented necklace*** (or *the* necklace). We will describe as *a* necklace for the restricted pair (M, K) what is the presented necklace for a possible different restricted presentation of (M, K) as Montesinos pair. Similarly for bands.

Thus Proposition 10.1 says that a hollow Montesinos pair with $\geqslant 3$ boundary components has only one necklace (up to pairwise isotopy). This is clearly false for many Montesinos pairs. For instance, in Figures 10.4–10.7, the presented necklaces of pairs (e_1), (e_2) and (e_5) *are not invariant under pair automorphism.* Likewise in the double-along-boundary D(e_3) of (e_3), which is also the link of Figure 8.36. Note that (e_5) is the three Borromean rings, which are permutable by automorphisms. The same misbehaviour obviously occurs for (e_2) with one rational substitution, which is a rational tangle pair. It also occurs for suitable rational substitutions in (e_1) and (e_2); the study of symmetry of rational knots tells which (see Chapter 16).

Here is a second pathology. A Montesinos pair (M, K) may admit two necklaces N_1, N_2 (each from a presentation of (M, K) as in Chapter 8) which *are not related by a pair automorphism.* This happens rather trivially for any rational knot since it can be obtained by admissible rational substitutions in (e_1) and (e_2); see §11.2. The interesting examples of the second pathology arise from the pair isomorphism (e_3)≅(e_4); and from the double of this D(e_3)≅ D(e_4); and from the pair isomorphism (e_6)≅(e_7) when $\frac{\beta'}{\alpha'} = -\frac{\alpha}{\beta+\frac{\beta_1+\beta_2}{2}\alpha}$. The reader should check that (e_6)≅(e_7) by rotating (e_7) by $\frac{\pi}{2}$ (the case $\beta_1 = 1$, $\beta_2 = -1$ is sufficiently general). In all these examples the two necklaces in question are not even homeomorphic.

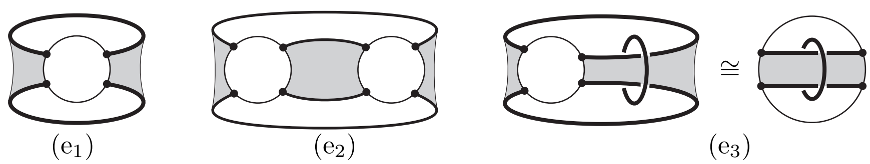


FIGURE 10.4.

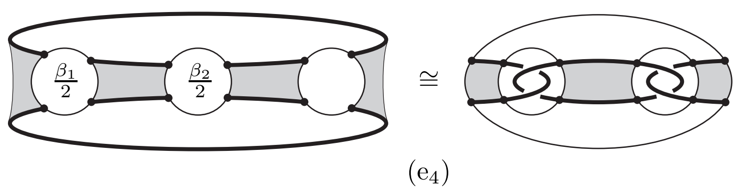


FIGURE 10.5.

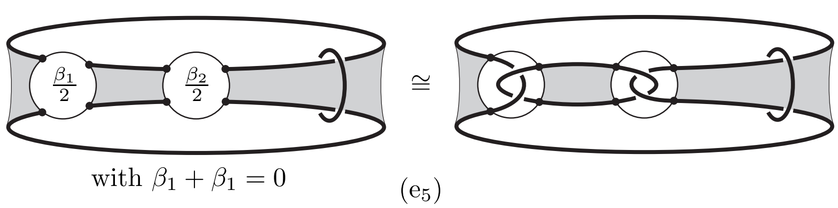


FIGURE 10.6.

We conjecture that there is no further misbehaviour of either sort. The remainder of this section and the symmetry theorem of M. Boileau [**Boi₁, BoiZ, BurZ**] for closed Montesinos pairs with 3 rational substitutions leave this unproved only for the cases of closed "elliptic" Montesinos pairs with 3 rational substitutions of

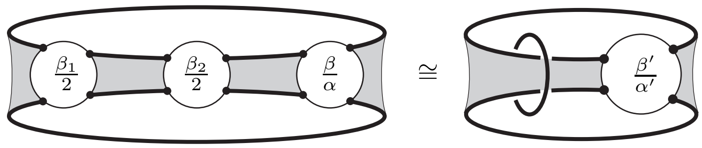


FIGURE 10.7.

slopes $\frac{\beta_1}{\alpha_1}$, $\frac{\beta_2}{\alpha_2}$, $\frac{\beta_3}{\alpha_3}$ where $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$. To settle them, one needs only extend Boileau's symmetry results to these pairs.[1]

We begin by classifying necklaces of the ring pair.

PROPOSITION 10.4. *Up to pairwise isotopy, the ring pair has exactly two necklaces, namely those given by* (e$_3$) *and* (e$_4$) *of Figures* 10.4 *and* 10.5.

PROOF. By Proposition 8.12, Theorem 8.15 and Corollary 8.18, the ring pair (M, K) has exactly two pairwise essential surfaces (up to pairwise isotopy) that are connected and not pairwise parallel to boundary, namely a Conway disc D and an annulus A in $M - K$.

Let N be a necklace for the ring pair, associated to a presentation of (M, K) as a restricted Montesinos pair. By Theorem 8.15, we can pairwise isotop the surfaces D and A so that they each are either vertical or horizontal for the presentation considered. Since their boundary curves have different slopes in ∂M, these two surfaces D and A cannot be simultaneously vertical, so that at least one of them is horizontal.

If D is horizontal then, by Proposition 8.13, the presentation is as in (e$_3$) of Figure 10.4, the band in the presented necklace N is a square. If A is horizontal then, by Proposition 8.13, the presentation is as in (e$_4$) of Figure 10.5 and N consists of two Conway spheres and three squares.

In the first case, the annulus A is vertical. Then the argument proving Proposition 10.1 now shows that, up to pairwise isotopy, (M, K) has only one necklace whose band is a square as in (e$_3$).

In the second case, the Conway disk D is vertical. The two spheres S_1 and S_2 in the necklace N are characteristic because, up to isotopy, they are determined by the property that $S_1 \cup S_2$ is the frontier of a regular neighbourhood of $\partial M \cup D$ in M. Applying Proposition 10.1 to the complement of the balls bounded by S_1 and S_2, we find that, up to pairwise isotopy, this necklace is the only one containing Conway spheres.

The proof of Proposition 10.4 is now complete. □

THEOREM 10.5. *Let* (M, K) *and* (M', K') *be two restricted Montesinos pairs presented with necklaces* N *and* N'. *Assume that the following holds:*

(i) (M, K) *contains a vertical Conway sphere, and*
(ii) (M, K) *is not pair isomorphic to the ring pair, nor to the double of the ring pair, nor to the Borromean rings, nor to the thickened Conway sphere* $(S^2, P^0) \times [0, 1]$.

Then, every pair isomorphism ρ: $(M, K) \to (M', K')$ *can be pairwise isotoped so that* $\rho(N) = N'$.

[1](Added 2009) This is now a consequence of Thurston's Geometrisation Theorem for Orbifolds. See [**Sak**].

REMARK 10.6. If (M, K) has d boundary components, r ring tangles and t rational tangles, then Condition (i) is equivalent to the inequality $t + 2r + 2d \geqslant 4$. Also, by Theorems 8.15(b) and 8.21, it is equivalent to the existence of a pairwise incompressible Conway sphere in (M, K).

PROOF OF THEOREM 10.5. We begin with:

ASSERTION 10.7. *The theorem holds if* (M, K) *is presented as an elementary pair, namely derived by rational tangle substitutions in a hollow Montesinos pair with* 3 *holes.*

PROOF OF ASSERTION 10.7. Our hypothesis (i) then guarantees that M has at least one boundary component; let $\partial_0 M$ be one. Then all the possibilities (a), (b), (c) for (M, K) and N are listed in Figure 10.8.

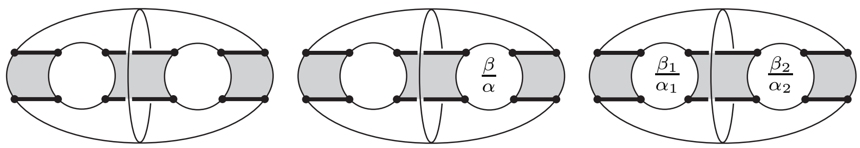


FIGURE 10.8.

In case (a), the pair is hollow and the result follows from Proposition 10.1.

In all three cases, (M, K) clearly contains a unique vertical Conway disc D with boundary in $\partial_0 M$, as indicated in Figure 10.8. It is pairwise essential by Theorem 8.15(b) and it is the only pairwise essential Conway disc by Theorem 8.15(a) and Proposition 8.13.

Hypothesis (ii) and Proposition 8.13 prevent $\varphi(D)$ from being horizontal in (M', K'). Hence, by Theorem 8.15, the unique pairwise essential Conway disc $\varphi(D)$ is vertical. This shows that (M', K') is presented as an elementary Montesinos pair.

We now see that φ sends the Conway spheres in N to the Conway spheres in N' (after isotopy), since D and $\varphi(D)$ determine these spheres (see proof of Proposition 10.4).

Then the application of Proposition 10.1 to the complement of the rational tangles gives $\varphi(N) = N'$ as required. $\square$

Let us now prove Theorem 10.5 in full generality.

Choose a family F of disjoint vertical Conway spheres $F_1, \ldots, F_n$ in (M, K) so that each closed-up component M_i of $M - F$ gives an elementary pair (M_i, K_i), with $K_i = K \cap M_i$, that is not a collar $\cong (S^2, P^0) \times [0, 1]$. (Recall that the ring pair is an elementary pair.) Note that $N_i = (N \cap M_i) \cup \partial M_i$ is a necklace for (M_i, K_i). Denote by B the band of N, and by B_i the band of N_i; clearly $B_i = B \cap M_i$.

By (ii), Theorem 8.15, and Proposition 8.13, the surface $F' = \varphi(F)$ is vertical in (M', K') (with respect to N') after pairwise isotopy, and has the properties of F.

Thus we have elementary Montesinos pairs (M'_i, K'_i) defined for $F' = \varphi(F)$ (so that $M'_i = \varphi(M_i)$), with necklaces $N'_i = (N' \cap M'_i) \cup \partial M'_i$ and bands $B'_i = B' \cap M'_i$ where B' is the band in N'.

Our aim is to show that there is a pairwise isotopy of (M', K') respecting F', after which $\varphi(N) = N'$. For this, it suffices to show that, for all i, the necklace $\varphi(N_i)$ is pairwise isotopic in (M'_i, K'_i) to N'_i.

We begin by showing this for *some* i, exploiting the uniqueness and near-uniqueness of the N_i proved in Assertion 10.7 and Proposition 10.4.

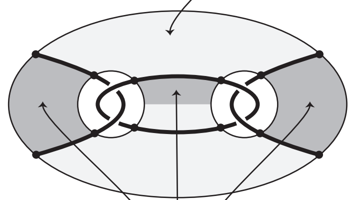


FIGURE 10.9.

Seeking a contradiction, suppose this is false for all i. Then all (M_i, K_i) are pair isomorphic to ring pairs. Since a ring pair has, as boundary, just one Conway sphere, we conclude that, either there is just one block M_i (contradicting (ii)), or else there are exactly two blocks M_1, M_2. The presence of two blocks also leads to a contradiction, as follows. In M_i, the band B_i and the distinct band $\varphi^{-1}(B'_i)$ are as in Figure 10.9 after a pairwise isotopy of φ, by Proposition 10.4. In particular the intersection $\partial M_i \cap \big(B_i \cup \varphi^{-1}(B'_i)\big)$ is a quadrangle made up of 2 vertical and 2 slope zero arcs. Since these quadrangles match up in M_1 and M_2, we conclude that either (M, K) is the Borromean rings (namely is (e_5) of Figure 10.6), or else (M, K) is the double of (M_1, K_1). Either case contradicts Hypothesis (ii).

We have now shown by *reductio ad absurdum* that for some i, the necklaces N_i and N'_i are pairwise isotopic in (M'_i, K'_i).

The same is therefore true for any index j in place of i, such that M_j is adjacent to M_i, namely such that $M_i \cap M_j \neq \varnothing$. Indeed, two necklaces N_a, N_b with bands B_a, B_b in an elementary pair (M^*, K^*) are the same up to pairwise isotopy if one arc component of $\partial M^* \cap B_a$ coincides with an arc component of $\partial M^* \cap B_b$ (use Proposition 10.4 and Assertion 10.7).

By connectivity of M, the same is true for every index. We conclude that $\varphi(N) = N'$ after a pairwise isotopy, as required to prove Theorem 10.5. □

Among the Montesinos pairs which were ruled out by Condition (ii) of Theorem 10.5, we already considered the case of the ring pair in Proposition 10.4. We now give the classification of necklaces for the remaining pairs, namely the Borromean rings, the double of the ring pair and the thickened Conway sphere.

PROPOSITION 10.8. *The Borromean rings (see* (e_5) *of Figure* 10.6*) have exactly three necklaces; for any one of the three components exactly one of these necklaces is disjoint from it.*

PROOF. We begin by showing that any two necklaces of the Borromean rings are related by a pair automorphism. This follows from:

ASSERTION 10.9. *Every restricted presentation* (M, K) *of the Borromean rings as a Montesinos pair is of the form* (e_5) *of Figure* 10.6.

PROOF OF ASSERTION 10.9. By Proposition 8.13, it suffices to show that (M, K) contains a horizontal Conway sphere. Applying Theorem 8.15 to the presentation of (e_5), we see that (M, K) contains exactly three (pairwise isotopy classes of) pairwise essential Conway spheres S_1, S_2, S_3, which can be mutually positioned as indicated in Figure 10.10.

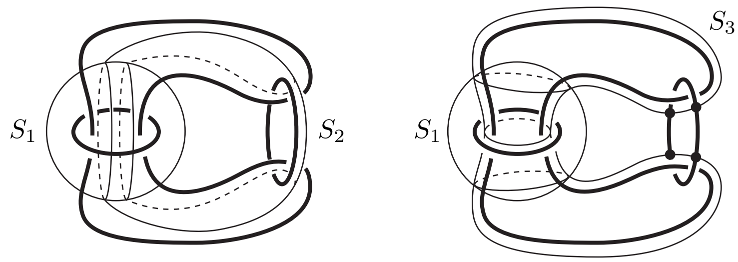


FIGURE 10.10.

If S_1 is not horizontal, then, by Theorem 8.15, it is vertical (after pairwise isotopy) and splits (M, K) into two Montesinos pairs (M', K') and (M'', K''). Now the surfaces $S_2 \cap M'$ and $S_3 \cap M'$ in (M', K') are, respectively, two Conway discs and one annulus disjoint form K'; both are pairwise essential and their boundary slopes are certainly distinct. Hence one of these two surfaces, say $S_i \cap M'$ is horizontal in (M', K'), and so S_i is horizontal in (M, K). □

ASSERTION 10.10. *The necklace N for the presentation* (e_5) *of the Borromean rings (M, K) in Figure* 10.6 *is invariant (up to pairwise isotopy) under all automorphisms of (M, K) respecting the component K_1 of K disjoint from N.*

PROOF OF ASSERTION 10.10. Let φ be such an automorphism. It is no loss of generality to assume that φ respects all three components of K, for we can compose φ with an automorphism respecting K_1 and N that permutes the other two components K_2, K_3 of K.

Split (M, K) and N along a vertical Conway sphere S into one pair (M', K') of type (e_3) with necklace $N' = (M' \cap N) \cup \partial M'$ and one (M'', K'') of type (e_4) with necklace N'' (see Figure 10.5). The Conway sphere S is respected by φ up to pairwise isotopy, because the three Conway spheres in (M, K) are distinguished by which component of K they meet. Thus an application of Proposition 10.4 in (M', K') and (M'', K'') proves Assertion 10.10. □

We can now complete the proof of Proposition 10.8. By symmetry of the Borromean rings (M, K), it suffices to show (keeping the notation of Assertion 10.10 just proved) that any necklace N^* for (M, K) disjoint from K_1 is pairwise isotopic to the necklace N of Assertion 10.10. By Assertion 10.9, the necklace N^* is $\varphi(N)$ for some automorphism φ of (M, K). Obviously, $\varphi(K_1) = K_1$. Thus Assertion 10.10 shows that $N^* = \varphi(N)$ is pairwise isotopic to N. □

PROPOSITION 10.11. *The double of the ring pair, namely the closed Montesinos pair*  *that can be presented with two rings and no rational tangles, has three necklaces up to pairwise isotopy. Two of these are annuli, while the third one consists of* 4 *squares and* 4 *Conway spheres.*

PROOF. We leave this as an exercise (easier than Proposition 10.8); one can exploit the two pairwise essential Conway spheres in (M, K). □

PROPOSITION 10.12. *Up to pairwise isotopy, every necklace of the thickened Conway sphere* $(S^2, P^0) \times [0,1]$ *is of the form* $k \times [0,1]$ *where* k *consists of two disjoint arcs in* S^2 *with* ∂k.

PROOF. By Theorem 8.15 and Proposition 8.14, the hollow Montesinos pair with two boundary components is the only presented Montesinos pair (M, K) that contains infinitely pairwise isotopy classes of annuli disjoint from K, and where no component of K is closed. This observation shows that every presentation of $(S^2, P^0) \times [0,1]$ is hollow.

The result then follows from the classification of essential annuli in $M - K$, and from the arguments used in the proof of Proposition 10.1. □

This completes our discussion of necklaces in restricted Montesinos pairs.

If we are interested in pairwise irreducible Montesinos pairs, this leaves only one case unsettled, by Proposition 8.20, namely the one of a closed Montesinos pair (M, K) presented with $r \geqslant 2$ ring tangles and one rational tangle (M_1, K_1) of integer slope $e \in \mathbb{Z} - 0$. In this case, we define the necklace N of (M, K) as follows: Its intersection with the complement of the rational tangle (M_1, K_1) is the band of the necklace of this restricted Montesinos pair; and $N \cap M_1$ is a band of (M_1, K_1) meeting the boundary along two arcs of infinite slope. Thus, this necklace is a twisted band avoiding the rings.

Then, the proof of Theorem 10.5 readily gives:

PROPOSITION 10.13. *The conclusion of Theorem* 10.5 *still holds if one allows* (M, K) *or* (M', K') *to be presented with* $r \geqslant 2$ *ring tangles and one rational tangle of integer slope* $e \in \mathbb{Z} - 0$*, where the necklace is defined as above.* □

CHAPTER 11

The classification of Montesinos pairs

The main result of this chapter is Theorem 11.7, which classifies pairwise irreducible Montesinos pairs in terms of certain data vectors[1].

11.1. The necklace-preserving classification

First we give a rather trivial but helpful classification theorem which corresponds, via the 2–fold branched covering as in the Appendix, to Seifert's *fibred* classification of Seifert fibre spaces [$\mathbf{Sei}_1$] (compare Theorem A.2).

This classification is so trivial because it considers only isomorphisms preserving necklaces. Thereby it gains some generality, however, since we can treat arbitrary Montesinos pairs.

Given a presented Montesinos tangle (M, K), we denote by (M_ρ, K_ρ) the tangle derived from (M, K) by deleting the rational tangles. It still contains the ring tangles, so in general it is not the hollow Montesinos tangle (M_0, K_0) from which (M, K) was built; but, deleting from K_ρ the circles in the ring tangles to define L_ρ, we clearly get a hollow Montesinos tangle (M_ρ, L_ρ), with $\partial M_\rho = S_1 \cup \cdots \cup S_t$.

Data. Consider Montesinos tangles (M, K), (M', K') presented with necklaces N, N' and bands B, B', respectively. Consider also an isomorphism of pairs φ: $(M, K) \to (M', K')$ such that $\varphi(N) = N'$. (By Theorem 10.5, the condition that $\varphi(N) = N'$ is often realisable via a pairwise isotopy of φ.) We list the Conway 2–spheres in N as $S_1, \ldots, S_t$ and note that the $S'_i = \varphi(S_i)$, $i = 1, \ldots, t$, are the Conway 2–spheres in N'.

Proposition 11.1 (for the above data).

(a) *There is an integer n_i such that φ sends any slope $\frac{q}{p}$ in S_i to slope $\frac{q}{p} + n_i$ in S'_i. Hence, if S_i bounds a rational tangle of type $\frac{\beta_i}{\alpha_i}$ (namely if the slope in $S_i \subset \partial M_\rho$ of its effective compression 2–disc is $-\frac{\beta_i}{\alpha_i}$), then $\varphi(S_i) = S'_i$ bounds a rational tangle of type $\frac{\beta'_i}{\alpha'_i} = \frac{\beta_i}{\alpha_i} - n_i$.*

(b) *The sum $\sum_{i=1}^{t} n_i$ is equal to zero.*

(c) *If $S_1, \ldots, S_t$ are listed going around the necklace in the cyclic order of the presentation of the Montesinos pair (M, K) (the direction and starting point of the order being immaterial), then the same is true of $S'_i, \ldots, S'_t$ up to reversal of the cyclic order.*

(d) *(M, K) and (M', K') have the same number of ring tangles plugged in.*

The proof will rest on:

[1](Added 2009) See [**BoiS, Zie, BurZ**] for other proofs of the classification of Montesinos knots.

LEMMA 11.2 (See Remark 11.5 for the differentiable category DIFF). *Any degree* ± 1 *isomorphism* φ *of the standard Conway sphere* (S^2, P^0) *of* §8.1 *is pairwise isotopic to an isomorphism* ψ *which is linear, in the sense that it lifts to an affine-linear map* $\widetilde{\psi}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *with* $p\widetilde{\psi} = \psi p$ *where* $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathcal{R} \cong S^2$ *is the branched covering map of* §8.1.

PROOF OF LEMMA 11.2. In the model Conway sphere consider the equatorial "square" $S^1 = S^2 \cap (\mathbb{R}^2 \times 0)$ whose four vertices are the four branch points P_0 of p. Its four sides are *linear* in the sense that they are covered by lines in $\mathbb{R}^2$, namely the horizontal and vertical lines, respectively, passing through the points of the lattice $\mathbb{Z}^2$.

One can pairwise isotop φ to some φ', such that the sides of the square $\varphi'(S^1)$ are in the same sense *linear*, and $\varphi'\colon S^1 \to \varphi(S^1)$ maps each side of the square linearly. Indeed, one can do this by comparing the wiggly square $\varphi(S^1)$ to the linear square S^* whose sides have the same slopes: one simplifies the intersection $S^* \cap S^1$ by a classical transversality and blister suppression procedure.

Now φ' is covered by a (wiggly) isomorphism $\widetilde{\varphi}' : \mathbb{R}^2 \to \mathbb{R}^2$ carrying adjacent right angled 2–dimensional squares with vertices in $\mathbb{Z}^2$ onto adjacent 2–dimensional linear parallelograms, and $\widetilde{\varphi}'$ is already linear on their sides. Isotopy of $\widetilde{\varphi}'$ on each square (fixing boundary) completely linearises $\widetilde{\varphi}'$ and induces a further pairwise isotopy of φ' to the desired linear automorphism ψ. In the topological or piecewise linear categories, we can then conclude with a standard Alexander isotopy. See Remark 11.5 for the differentiable category. □

COROLLARY 11.3. *The group* $\pi_0 \operatorname{Aut}_\pm(S^2, P^0)$ *of pairwise isotopy classes of degree* ± 1 *automorphisms of the Conway sphere* (S^2, P^0) *is naturally isomorphic to the discrete group of linear automorphisms of* $\mathbb{R}^2/\mathcal{R}$ *(identified to* S^2 *in* §8.1*).*

PROOF OF COROLLARY 11.3. The lemma establishes that the inclusion-induced map of the linear automorphisms $\operatorname{Aut}_{\text{lin}}(\mathbb{R}^2/\mathcal{R}) \to \pi_0 \operatorname{Aut}_\pm(S^2, P^0)$ is surjective. It is also injective; indeed if a linear automorphism is pairwise isotopic to the identity, it fixes all slopes and the 4 points P^0, hence is covered by the identity map of $\mathbb{R}^2$. □

A linear automorphism ψ of (S^2, P^0) acts on the slopes $m \in \mathbb{Q} \cup \infty$ by a linear fractional map of the form $m \mapsto \frac{c+dm}{a+bm}$, when the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ is the linear part of $\widetilde{\psi}\colon \mathbb{R}^2 \to \mathbb{R}^2$ lifting ψ. Note that this matrix necessarily has integer entries and determinant ± 1 as $\widetilde{\psi}$ respects $\mathbb{Z}^2$, and that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is only defined modulo multiplication by ± 1. Thus, Corollary 11.3 immediately gives the following structure theorem.

COROLLARY 11.4. *Each degree* ± 1 *automorphism of* (S^2, P^0) *acts on the slopes by linear fractional map, and this defines an exact sequence*

$$1 \to V_4 \to \pi_0 \operatorname{Aut}_\pm(S^2, P^0) \to PGL_2(\mathbb{Z}) \to 1$$

where $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *is the subgroup defined by automorphisms of* (S^2, P^0) *that lift to (integral) translations of* $\mathbb{R}^2$*.* □

The subgroup V_4 of (pairwise isotopy classes of) automorphisms of (S^2, P^0), respecting each slope, deserves some special attention, since we will encounter it again and again. It has 4 elements and is known as the ***Klein 4–group*** (or

Viergruppe). For the standard embedding of (S^2, P^0) in $\mathbb{R}^3$, these four elements are respectively represented by the identity ι and the π–rotations ξ, η, ζ whose axes are the x–, y–, z–axes, respectively (see Figure 11.1).

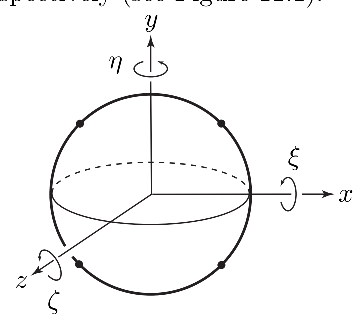


FIGURE 11.1.

REMARK 11.5. In the smooth category DIFF of differentiable manifolds and maps, Lemma 11.2 is not quite true as stated. Indeed, there is no homeomorphism $\mathbb{R}^2/\mathcal{R} \to S^2$ for which the linear homeomorphisms of $\mathbb{R}^2/\mathcal{R}$ correspond to diffeomorphisms of S^2. But, if we delete the four singular points, the lemma and its proof become valid, although the Alexander isotopy must be replaced by something more subtle [**Sma, Mun**]. The corollaries are valid as stated, but their proof uses the (easily proved) fact that restriction gives an isomorphism $\pi_0 \operatorname{Aut}_\pm(S^2, P^0) \to \pi_0 \operatorname{Aut}_\pm(S^2 - P^0)$ of groups of degree ± 1 diffeomorphisms up to DIFF isotopy. When linear maps of (S^2, P^0) are exploited henceforth, such refinements of proofs for the DIFF case will be left to the reader.

PROOF OF PROPOSITION 11.1. The Conway spheres S_i and S_i' are both parametrised, in the tangles (M_ρ, K_ρ) and (M_ρ', K_ρ'), by the standard Conway sphere (S^2, P^0), which is branched covered by $\mathbb{R}^2$ in a standard way (see §8.1). Thus each restriction $\varphi_{|S_i}\colon S_i \to S_i'$ is identified to an automorphism of the standard Conway sphere $\varphi_i\colon S^2 \to S^2$ respecting the four points P^0.

The key fact is that, inasmuch as φ maps the necklace N to N' and hence the band B to B', the two vertical segments in S^2 joining the branch points are respected by φ_i (possibly interchanged, though). In particular, φ_i sends slope ∞ to slope ∞ and therefore, by Corollary 11.4, it sends each slope $m \in \mathbb{Q} \cup \infty$ to $m + n_i$ for some $n_i \in \mathbb{Z}$ (independent of m).

This proves the first statement of (a). The second follows; we note that the minus signs are the consequences of the reflection in the xy–plane of $\mathbb{R}^3$ relating the two parametrisations of S_i (or S_i'), namely the one from the hollow tangle and the one from the rational tangle (see §8.1).

In the hollow Montesinos tangle (M_ρ, L_ρ), where L_ρ is K_ρ minus the rings, consider a standard horizontal 2–sphere with holes F, whose boundary slopes are all 0. In view of (a), the image $\varphi(F)$ is a surface F' in (M_ρ', L_ρ') with boundary slope n_i in S_i'. By Theorem 8.3, the sum of the slopes is zero; recall that this is just a homological fact seen by splitting open along the band B'. This proves (b).

To show (c), it suffices to observe that the cyclic order as described (modulo orientation reversal) is an intrinsic topological property of the necklace N. To see this, consider the nerve of the covering of N by the closed-up components of the complement of the *non-manifold points* (these components are squares and spheres).

As for (d), just note that the number of ring tangles equals the number of string components that are disjoint from the necklace. □

Next, we convert Proposition 11.1 into an elementary classification theorem, in several ways. Given a Montesinos tangle (M, K), presented as usual, we enumerate the Conway spheres $S_1, \ldots, S_t$ in N going from left to right in the standard hollow Montesinos tangle.

To each S_i we associate a generalised slope m_i as follows: If S_i is the boundary of a rational tangle, let m_i be the slope $\frac{\beta_i}{\alpha_i} \in \mathbb{Q} \cup \infty$ of this rational tangle; if S_i is a boundary component of M, let m_i be the symbol $\varnothing$ (of course recalling the empty set).

The ***raw data vector*** for (M, K) is $(r; m_1, \ldots, m_t)$ where r is the number of ring tangles plugged in.

The ***fractional data vector*** is $(r; e_0; \bar{m}_1, \ldots, \bar{m}_t)$ where:

(1) $\bar{m}_i \in \mathbb{Q}/\mathbb{Z}$ is the class of m_i if $m_i \in \mathbb{Q}$ is a genuine rational number;
(2) $\bar{m}_i = m_i$ if m_i is one of the symbols $\varnothing$ or ∞;
(3) $e_0 = -\sum_{i=1}^{t} m_i$ if *all* m_i lie in $\mathbb{Q}$;
(4) e_0 is the symbol $\varnothing$ otherwise, namely when some m_i is equal to $\varnothing$ or to ∞.

By convention, e_0 is 0 when $t = 0$, namely when the necklace N has no Conway spheres. Note that the class $\bar{e}_0 \in \mathbb{Q}/\mathbb{Z}$ of e_0 (when this makes sense) is determined by $\bar{m}_1, \ldots, \bar{m}_t$.

The ***normalised data vector*** is $(r; e; \widehat{m}_1, \ldots, \widehat{m}_t)$ where

(1) if $m_i \in \mathbb{Q}$, then $\widehat{m}_i$ is the unique number in the interval $\left]-\frac{1}{2}, \frac{1}{2}\right]$ that is congruent to m_i modulo $\mathbb{Z}$;
(2) $\widehat{m}_i = m_i$ otherwise, namely when m_i is $\varnothing$ or ∞;
(3) $e = \sum_{i=1}^{t} \left(\widehat{m}_i - m_i\right) \in \mathbb{Z}$ if *all* $m_i \in \mathbb{Q}$ are rational numbers, keeping the convention that $e = 0$ when $k = 0$;
(4) $e = \varnothing$ otherwise, namely if some m_i is $\varnothing$ of ∞.

From a given Montesinos presentation with data vectors as above we get by reflection in the xy–plane another presentation that is degree -1 isomorphic to the first. The *raw* and *fractional* data vectors of the latter are respectively $(r; -m_1, \ldots, -m_r)$ and $(r; -e_0; -\bar{m}_1, \ldots, -\bar{m}_r)$, with the convention that $-\infty = \infty$ and $-\varnothing = \varnothing$. Note that the *normalised* data vector will misbehave if some $\widehat{m}_i$ is equal to $\frac{1}{2}$.

The ***cyclic permutations*** of $\mathbb{Z}_t = \mathbb{Z}/t\mathbb{Z}$ are those of the form $x \mapsto x + k$; the ***dihedral permutations*** are generated by cyclic permutations and $x \mapsto -x$. Similarly for permutations of *any linearly ordered set* of t elements in place of $\mathbb{Z}_t$, since it is in natural bijective correspondence with $\mathbb{Z}_t$.

THEOREM 11.6. *Consider two presented Montesinos pairs (M, K) and (M', K') with data as above, and with primes distinguishing the data for the second pair. There exists a degree $+1$ pair isomorphism $\varphi : (M, K) \to (M', K')$ sending necklace N to necklace N' precisely if any one of the three (obviously equivalent) conditions on data vectors holds:*

(i) *$r = r'$, $t = t'$, and the raw data vector $(r; m_1, \ldots, m_t)$ can be modified to coincide with $(r'; m'_1, \ldots, m'_t)$ by changes of the following three sorts:*
   (a) *rearrange $m_1, \ldots, m_t$ by a dihedral permutation;*

(b) *if* m_i *and* m_j *are in* $\mathbb{Q}$ *(and* $i \neq j$ *) replace them by* $m_i + 1$ *and* $m_j - 1$*;*

(c) *if* $m_i \in \mathbb{Q}$ *and* $m_j \notin \mathbb{Q}$ *replace them by* $m_i \pm 1$ *and* m_j*.*

(ii) *The fractional data vectors differ only by a dihedral permutation of* $\bar{m}_1$*, ...,* $\bar{m}_t$*.*

(iii) *The normalised data vectors differ only by a dihedral permutation of* $\widehat{m}_1$*, ...,* $\widehat{m}_t$*.*

Furthermore, one can insist in (i)*,* (ii) *or* (iii) *that the permutation* σ *of indices* 1*, ...,* t *be related to the isomorphism* φ *by the property that* $\varphi(S_i) = S'_{\sigma(i)}$*.*

Proof. Given a pair isomorphism $\varphi : (M, K) \to (M', K')$ with $\varphi(N) = N'$, there is a unique permutation σ of indices 1, 2, ..., t so that $\varphi(S_i) = S'_{\sigma(i)}$. By Proposition 11.1, the equivalent conditions (i), (ii) and (iii) hold for this σ.

Conversely, suppose that Condition (i) holds for a certain dihedral permutation σ of indices 1, ..., t. We must build the isomorphism φ satisfying $\varphi(S_i) = S'_{\sigma(i)}$.

There is visibly an isomorphism of *hollow tangles* $\varphi_\sigma\colon (M_\rho, K_\rho) \to (M'_\rho, K'_\rho)$ sending N to N' and satisfying $\varphi_\sigma(S_i) = S'_{\sigma(i)}$. It extends to φ as desired precisely if $m_i = m'_{\sigma(i)}$.

Condition (i) assures that there exist integers n_1, ..., n_t such that $m_i + n_i = m'_{\sigma(i)}$ and $n_1 + \cdots + n_t = 0$. (We shall say that, in case $m_i = \varnothing$ or ∞, the "sum" $m_i + n_i$ is m_i.) As $n_1 + \cdots + n_r = 0$, there visibly exists a pair automorphism g of (M_ρ, K_ρ), respecting the band B, so that each restriction $g_{|S_i}\colon S_i \to S_i$ sends slope x to $x + n_i$; indeed g can be composed of many "twists", each supported on a regular neighbourhood $\cong B^2 \times [0,1]$ in M_ρ of some component $\cong [0,1] \times [0,1]$ of the band B, where on $B^2 \times [0,1]$ the twist is the product of the identity $\mathrm{id}_{[0,1]}$ and of an automorphism of B^2 as illustrated in Figure 11.2 (or its inverse).

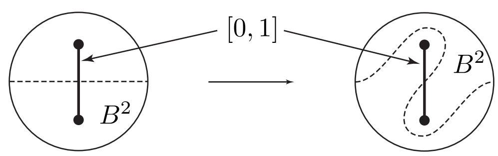


Figure 11.2.

Then $\varphi_\sigma g_{|S_i}\colon S_i \to S'_{\sigma(i)}$ sends slope m_i to $m_i + n_i = m'_{\sigma(i)}$. Hence the pair isomorphism $\varphi_\sigma g\colon (M_\rho, K_\rho) \to (M'_\rho, K'_\rho)$. extends to φ as desired. □

11.2. Classification of pairwise irreducible Montesinos pairs

We will now use our analysis in Chapter 10 to pass from the necklace preserving classification of Theorem 11.6 to the general isomorphism classification of pairwise irreducible Montesinos pairs.

Recall from Chapter 8 that a presentation of a Montesinos pair is *restricted* when all rational tangles have finite non-integral slope. By Corollary 8.18, a Montesinos pair admitting a restricted presentation is almost always pairwise irreducible, with a few obvious exceptions. Conversely, we proved in Proposition 8.20 that a pairwise irreducible Montesinos pair admits a presentation which either is restricted, or has $\geqslant 2$ rings and a unique rational tangle, of integral non-zero slope.

When (M, K) is closed (namely when $M = S^3$) and admits a presentation with no ring and two rational tangles of respective slopes $\frac{\beta_1}{\alpha_1}$ and $\frac{\beta_2}{\alpha_2}$, then untangling the first of these two tangles gives another presentation with (no ring and) only one rational tangle, of a slope $\frac{\beta}{\alpha}$. To be more precise, an easy computation on the slopes, using Corollary 11.4, gives

$$\frac{\beta}{\alpha} = \frac{\alpha_1\beta_2 + \beta_1\alpha_2}{\alpha_1'\beta_2 + \beta_1'\alpha_2}$$

where α_1' and β_1' are such that $\alpha_1\beta_1' - \alpha_1'\beta_1 = 1$. The corresponding knots in S^3 are the so-called ***rational knots*** or ***two-bridge knots***. They were first classified by H. Schubert [**Sch**$_3$] in 1956, who identified their 2–fold branched covering as the lens space $L(\beta, \alpha)$: Indeed, it turns out that two such Montesinos knots, presented with no ring and exactly one rational tangle of respective slopes $\frac{\beta}{\alpha}$ and $\frac{\beta'}{\alpha'}$, are degree $+1$ isomorphic precisely when their 2–fold branched coverings $L(\beta, \alpha)$ and $L(\beta', \alpha')$ are degree $+1$ isomorphic, namely when $\beta' = \beta$ and $\alpha' \equiv \alpha^{\pm 1} \bmod \beta$.

Similarly, a Montesinos pair presented with one boundary component, no ring and exactly one rational tangle is a rational tangle pair, and thus admits a presentation with no rational tangle.

We can now state our classification. Start with a pairwise irreducible Montesinos pair (M, K). Choose some presentation for (M, K), and consider the fractional data vector $(r; e_0; \bar{m}_1, \ldots, \bar{m}_t)$ which we have associated to this presentation. Choosing the presentation *restricted* when possible (see Proposition 8.20), and simplifying it as above for rational knots and rational tangle pairs, we can make this data vector satisfy:

(i) $r \in \mathbb{N}$ (including $r = 0$);
(ii) all generalised slopes $\bar{m}_i$ are in $(\mathbb{Q}/\mathbb{Z} - 0) \cup \varnothing$, except for data vectors $(r; e_0; 0)$ with $r \geqslant 2$ and $e_0 \in \mathbb{Z} - 0$;
(iii) $e_0 = \varnothing$ when some $\bar{m}_i$ is $\varnothing$; otherwise, $e_0 \in \mathbb{Q}$ and its class in $\mathbb{Q}/\mathbb{Z}$ is equal to $-\sum_{i=1}^{t} \bar{m}_i$; moreover, $e_0 = 0$ when there is no m_i;
(iv) the vectors $(0; 0;)$, $(1; 0;)$, $(0; e_0; \bar{m}_1, \bar{m}_2)$, with $\bar{m}_1 \neq \varnothing$, are excluded.

THEOREM 11.7. *The above rule determines a bijective correspondence between the degree $+1$ pair isomorphism classes of pairwise irreducible Montesinos pairs (M, K) and the fractional data vectors $(r; e_0; \bar{m}_1, \ldots, \bar{m}_t)$ as above, taken up to dihedral permutation of the $\bar{m}_i$, together with the exchanges:*

$$(1; \varnothing; \varnothing) \leftrightarrow (0; \varnothing; \tfrac{1}{2}, \tfrac{1}{2}, \varnothing)$$
$$(2; 0;) \leftrightarrow (0; 0; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$
$$(1; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}) \leftrightarrow (0; -\tfrac{\alpha}{\beta}; \tfrac{1}{2}, \tfrac{1}{2}, \overline{\tfrac{\alpha}{\beta}})$$
$$(0; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}) \leftrightarrow (0; -\tfrac{\beta}{\alpha'}; \overline{\tfrac{\beta}{\alpha'}}) \text{ when } \alpha' \equiv \alpha^{\pm 1} \bmod \beta.$$

PROOF. We first check that the above exchanges of fractional data vectors actually reflect some pair equivalences of presented Montesinos pairs. To see this, note that the first exchange corresponds to the equivalence between the two presentations of the ring pair shown as (e_3)–(e_4) in Figures 10.4 and 10.5. The two fractional data vectors of the second exchange both give the double of the ring pair, and the equivalence is obtained by doubling the previous one. The third equivalence appears as (e_6)–(e_7) in Figure 10.7, and involves any Montesinos knot obtained by plugging the ring tangle with a rational tangle. The last equivalence follows from the classification of rational knots (see our earlier discussion in this section); it can easily be described by untangling the first rational tangle of the presented Montesinos pair with raw data vector $(0; \frac{\beta}{\alpha}, 0)$. This gives a new presentation with raw data vector $(0; 0, \frac{\beta}{\alpha'})$ where $\alpha\alpha' \equiv 1 \bmod \beta$ as an easy slope computation shows.

Conversely, let (M, K) and (M', K') be two pairwise irreducible Montesinos pairs, and let N and N' be their respective necklaces for the presentations that were used to associate fractional data vectors to them. Assume that there exists a degree $+1$ isomorphism $\varphi\colon (M, K) \to (M, K')$. We want to show that their fractional data vectors differ only by the equivalence started in Theorem 11.7.

In Chapter 10, we showed that φ can be modified so that $\varphi(N) = N'$ whenever the presented (M, K) contains a vertical Conway sphere and is not the ring pair nor its double (see Theorem 10.5 and Propositions 10.8, 10.12 and 10.13). In this case, it follows from Theorem 11.6 that the two fractional data vectors associated to (M, K) and (M', K') differ only by a dihedral permutation of their generalised slopes.

For the ring pair or its double, we can decide for (M, K) and (M', K') to prefer the presentations with rational tangles of slope $\frac{1}{2}$, using the pair equivalence reflected by the first two exchanges of Theorem 11.7. One then concludes the proof as in the previous cases, using Propositions 10.4 and 10.11.

It remains to settle the cases when the presented (M, K) and (M', K') admit no vertical Conway spheres, namely when their associated fractional data vectors are of type $\big(0; -\frac{\beta}{\alpha}; \overline{\frac{\beta}{\alpha}}\big)$, $\big(1; -\frac{\beta}{\alpha}; \overline{\frac{\beta}{\alpha}}\big)$ or $\big(0; -\sum_{i=1}^{3} \frac{\beta_i}{\alpha_i}; \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_1}{\alpha_1}}\big)$. First note that, by Theorem 8.15, these Montesinos pairs contain no pairwise incompressible Conway sphere. This class of Montesinos pairs is therefore disjoint from the ones considered previously in the proof.

On the other hand, their internal classification was completed by Montesinos, who observed in [**Mon**$_1$] that their 2–fold branched covering is a Seifert manifold, and is sufficient to distinguish them. More precisely, the 2–fold branched covering of the Montesinos knot associated with $(0; e_0; \overline{\frac{\beta}{\alpha}})$, $(1; e_0; \overline{\frac{\beta}{\alpha}})$ or $\big(0; e_0; \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_1}{\alpha_1}}\big)$ is the Seifert manifold with type $(\mathrm{Oo}0|e_0 + \frac{\beta}{\alpha}; (\alpha, \beta))$, $(\mathrm{On}0|e_0 + \frac{\beta}{\alpha}; (\alpha, \beta))$ or $(\mathrm{Oo}0|e_0 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_3}; (\alpha_1, \beta_1)(\alpha_2, \beta_2)(\alpha_3, \beta_3))$ in Seifert's terminology (see [**Sei**$_1$**, Orl**] and §A.2). The classification of these Seifert manifolds asserts that they are degree $+1$ isomorphic precisely when their data vectors differ only by a permutation of the Seifert invariants (α_i, β_i) of the exceptional fibres, together with a possible application of the last two exchanges of Theorem 11.7 (see [**OVZ, Orl**]). As these data vectors have $\leqslant 3$ slopes, any permutation of their slopes is dihedral, and this completes the proof of Theorem 11.7. □

Part 5

The classification of arborescent knots and pairs

CHAPTER 12

The plumbing calculus

We develop in this chapter our plumbing calculus to classify arborescent knots (and pairs) in terms of certain integrally weighted planar trees. The corresponding classification theorems are stated; half of their proofs, namely the fact that weighted trees related by certain moves give rise to isomorphic knots will be manifest, while the converse, which depends on our analysis of incompressible surfaces in Chapter 8, will be completed only in Chapter 13.

12.1. Plumbing of surfaces and tangles

This section goes back to first principles; the reader is only supposed to understand the language of tangles from §8.1. The most important result to be proved here is that any arborescent knot in S^3 can be expressed as a connected sum of knots that are formed by plumbing bands, according to a special sort of integrally weighted planar tree called canonical.

The plumbing operation takes two surfaces F_1 and F_2 embedded in S^3, with non-empty boundary and possibly non-orientable, and glues them together to obtain a new embedded surface, well-defined up to isotopy in S^3. To specify this gluing operation, we need to select on each surface F_i, with $i = 1, 2$, a ***plumbing patch***, namely an embedding $p_i : [0,1] \times [0,1] \to F_i$ sending the two sides $[0,1] \times \{0,1\}$ to the boundary of F_i and the rest of the square $[0,1] \times [0,1]$ in the interior of the surface.

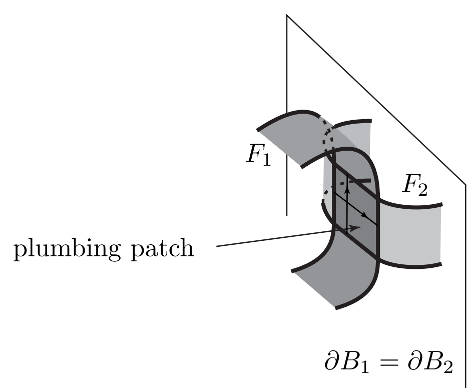


FIGURE 12.1.

Decompose S^3 into the union of two balls B_1 and B_2 meeting along their boundary. Then isotop each surface F_i in S^3 so that $F_i \subset B_i$ and so that $F_i \cap \partial B_i$ is equal to the plumbing patch $p_i([0,1] \times [0,1])$. This can be done so that the map

$p_i : [0,1]\times[0,1] \longrightarrow F_i \subset B_i$ has degree $+1$ for the boundary orientation of ∂B_i, and so that $p_1(x,y)$ coincides with $p_2(y,x)$ on $\partial B_1 = \partial B_2$ for every $(x,y) \in [0,1]\times[0,1]$. When this is done, the union $F_1 \cup F_2$ forms a new surface F with boundary. One easily checks that, up to isotopy in S^3, this $F \subset S^3$ depends only on the embedded surfaces $F_1 \subset S^3$ and $F_2 \subset S^3$ and on the plumbing patches p_1 and p_2. We will say that the embedded surface F is obtained by ***plumbing*** F_1 ***and*** F_2 ***along the plumbing patches*** p_1 ***and*** p_2.

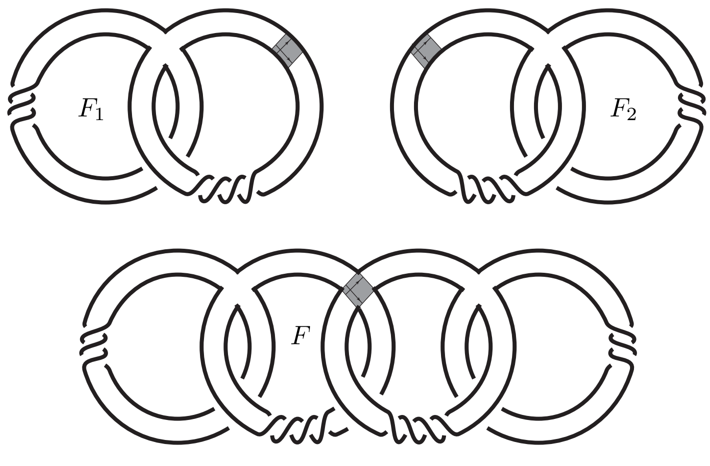


FIGURE 12.2.

Observe that this construction is also well-defined if the plumbing patches $p_i : [0,1]\times[0,1] \longrightarrow F_i$ are given only up to isotopy of $[0,1]\times[0,1]$. In practice, it is therefore sufficient (and convenient) to simply specify their images $p_i([0,1]\times[0,1]) \subset F_i$ together with the induced orientations on a "core" $p_i([0,1]\times y)$ and on a "fibre" $p_i(x\times[0,1])$.

Figure 12.2 offers an example, where $\partial B_1 = \partial B_2$ corresponds to the sheet of paper, and where the ball B_1 sits behind this sheet of paper. .

Observe that it is essential that the F_i lie in different balls B_i, so as to prevent any further knotting.

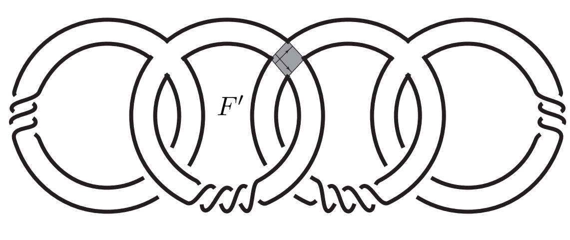


FIGURE 12.3.

REMARK 12.1. An alternative rule for plumbing surfaces with patches would be to require the maps $p_i : [0,1]\times[0,1] \to F_i \subset B_i$ to have degree -1 (as opposed to $+1$) for the boundary orientation of ∂B_i, as in Figure 12.3. It is a remarkable fact that, as suggested by Figure 12.3, the surface F' obtained by plumbing according to this second rule has a boundary $\partial F'$ which is isotopic in S^3 to the boundary ∂F of the surface F obtained by plumbing according to the first rule. (Hint: Push F_1 across the plumbing patch; see also our discussion of the plumbing of tangles below.) Thus, there are two natural ways to plumb surfaces, although the knots $(S^3, \partial F) \cong (S^3, \partial F')$ produced are the same. On the other hand, the plumbed surfaces may indeed be distinct up to isotopy in S^3. For instance, this is the case in the example of Figures 12.2 and 12.3, as proved by [**HatT**]. Indeed, Hatcher and Thurston classify in [**HatT**] the Seifert surfaces of the corresponding rational knot $(S^3, \partial F)$ up to isotopy respecting the knot; considering the action of the symmetry group of $(S^3, \partial F)$ (as computed in Chapter 15) on the corresponding 8 classes of Seifert surfaces, we can conclude that F is not isotopic to F' in S^3.

On the other hand, the different ways of plumbing always yield surfaces that are isotopic in the 4–ball B^4 keeping the surface boundaries in $S^3 = \partial B^4$ (exercise).

Remark 12.2. K. Murasugi has considered a more general sort of plumbing of surfaces where the "square" plumbing patch is replaced by a $2k$–gon, $k = 1$, 2, 3, ... with alternate sides in the surface boundary. This ***Murasugi sum*** has remarkable properties; for example, it preserves the property that the knot complement should fibre over S^1 with fibre the interior of the surface mentioned; see [**Mur, Gab$_1$, Gab$_2$**].

Remark 12.3. The above plumbing construction is defined in the TOP of PL categories. In the DIFF category, it provides a surface with corners in its boundaries. These corners can easily (and must) be smoothed by a further step in the construction to remain in the DIFF category.

We now consider a related operation on tangles. Let two tangles $(M_1, K_1; \theta_1)$ and $(M_2, K_2; \theta_2)$, as defined in §8.1, be given as well as a choice of boundary components $S_1 \subset \partial M_1$ and $S_2 \subset \partial M_2$. Recall that the tangle structures θ_1 and θ_2 specify parametrisations of each S_i by the standard Conway sphere (S^2, P^0). The ***plumbing construction*** associates to this data the tangle $(M, K; \theta)$ defined as follows. The knot pair (M, K) is obtained by gluing (M_1, K_1) to (M_2, K_2) by identifying $S_1 \cong (S^2, P^0)$ to $S_2 \cong (S^2, P^0)$ through the map corresponding to the reflection $(x, y, z) \longmapsto (y, x, z)$ of the standard Conway sphere in $(S^2, P^0) \subset \mathbb{R}^3$. The tangle structure θ is defined by restriction of θ_1 and θ_2 to the remaining boundary components.

Recall that the *matching* or *gluing* construction of §8.1 is similarly defined but uses reflection in the plane $z = 0$ rather than the plane $x = y$.

As the terminology indicates, this plumbing of tangles bears a precise relation to the plumbing of surfaces described above. This relationship can be described as follows.

Given a tangle $(M, K; \theta)$ with $M \subset S^3$, we can always choose a surface $F \subset M$ (possibly non-orientable) whose boundary ∂F consists of K and of arcs of slope ∞ in ∂M. Then enlarge K to a knot $K_+ \subset S^3$ by adding slope zero arcs in ∂M, and enlarge the surface F to $F_+ \subset S^3$ with $\partial F_+ = K_+$ by adding the lower hemisphere H^2_- (the square $z \leqslant 0$) of each boundary component of M; Figure 12.4(a) offers an example. Finally regard each H^2_- as a plumbing patch in F_+ using the "core" and "normal" orientations of Figure 12.5(b) (so that the orientation of the plumbing patch coincides with the boundary orientation of ∂M). The example becomes that of Figure 12.4(b), a band with a plumbing patch. Thus, from tangle (M, K) and

F, we get (S^3, K_+) and F_+ with $\partial F_+ = K_+$, where F_+ is a surface with boundary, having plumbing patches as in Figure 12.4(b).

Conversely, slitting open S^3 along all the plumbing patch interiors and parametrising boundaries as Figure 12.5 suggests, we retrieve the tangle (M, K) (up to isomorphism) together with the surface F. It is now a pleasant exercise to verify that plumbing of surfaces with plumbing patches in S^3 corresponds under the above rules to plumbing of tangles.

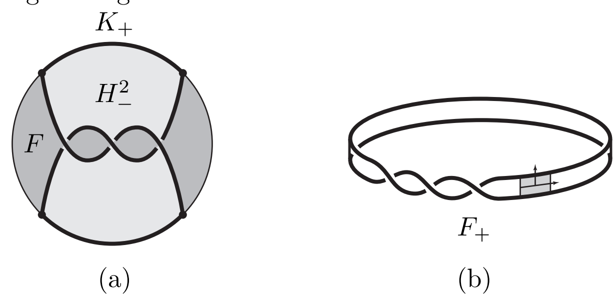


FIGURE 12.4.

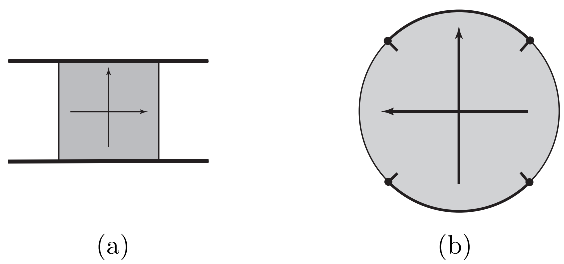


FIGURE 12.5.

Plumbing of tangles is also defined for graphs of tangles. Consider a graph Γ in which each vertex is associated to a tangle, and the bonds of a vertex correspond to the boundary components of the tangle while each edge of Γ consists of a pair of *distinct* bonds (that may however belong to the same vertex). As always, each bond lies in 1 or 0 edges; it is called ***tied*** or ***free*** respectively. One performs one plumbing identification for each edge (in any order) making boundary identifications in the disjoint sum of all the vertex tangles. There results a well-defined tangle whose boundary components are the free bonds in the entire graph Γ.

Henceforth we shall be using the language of plumbing of tangles rather than of surfaces. Admittedly the reader may find it helpful to think in terms of plumbing of surfaces, which is easily visualised spatially but less easy to explain with 2–dimensional diagrams.

To facilitate these diagrammatic explanations we now elaborate some of the conventions of §8.1 for representing tangles.

Recall that, by conventions in §8.1, certain knot pairs (M, K) with $M \subset S^3 = \mathbb{R}^3 \cup \infty$ carry natural tangle structures: namely, any pair such that each component of $(\partial M, \partial K)$ is a round Conway 2–sphere in $\mathbb{R}^3$ related to the standard one by a (unique) motion composed of translation and homothety (scaling); the tangle parametrisation is this motion, composed, if orientation dictates, with reflection in the $xy-$plane. This convention for parametrising a component S of ∂M will henceforth be generalised in case one arc of the equator of S (parallel to the $xy-$plane) is singled out and oriented by an arrow, for instance as in Figure 12.6(a); the degree $+1$ parametrisation is then to be further composed with the unique rigid rotation carrying this oriented arc to the linear arc in the standard Conway sphere (S^2, P^0) which has slope 0 and joins $(0,0,0)$ to $(1,0,0)$; see Figure 12.6(b), and compare §1.2.

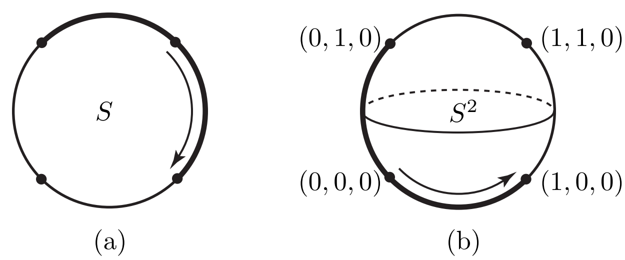


FIGURE 12.6.

When there is some ambiguity, e.g. when the Conway sphere S separates two knot pairs, we shall place the arrows on the side of S occupied by the tangle. Then, each side of S can be labelled by an arrow to specify tangle structures on the two knot pairs adjacent to S.

Note that when an arc is thus labelled on each side of S, the two tangles are plumbed precisely when the arrows label adjacent arcs and emanate from their common point, as in Figure 12.7 (allowing rotations and reflections of this picture).

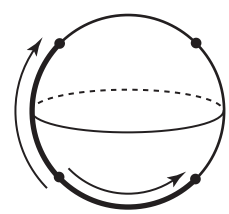


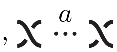
FIGURE 12.7.

Such a tangle has a diagrammatic presentation on paper in case the Conway spheres making up ∂M are all centred on the $xy-$plane and perpendicular projection p to this plane immerses K into $N = \mathbb{R}^2 \cap M$ with simple double points as the only singularities. Let us use the term ***tangle projection*** in this context for the planar data consisting of $(N, p(K))$ together with the overcrossings indicated at the double points, and in addition an arrow labelling as above an arc of each of the equators ∂N. A tangle projection with $N \subset S^2$ determines its tangle up to isomorphism.

Similar data, where N is replaced by any isomorphic submanifold of $S^2 = \mathbb{R}^2 \cup \infty$ *possibly with wiggly boundary*, will still be called a tangle projection since it similarly determines a tangle up to isomorphism.

12.2. Plumbing according to a weighted planar tree

The plumbing operation for graphs of tangles has interesting formal properties involving a special sort of Montesinos tangle called *atomic*. We define a Montesinos tangle to be ***atomic*** if all of the tangles plugged in are rational tangles of integral slope.

Figure 12.8 provides one ***model*** n***–valent atomic tangle*** for each sequence of n integers $a_1, \ldots, a_n$, with $n \geqslant 1$; in it,  with $a \in \mathbb{Z}$ represents a positive half-twists, for instance  if $a = 3$ and  if $a = -3$. (This notion will be slightly extended in §12.5, so as to allow the presence of rings.)

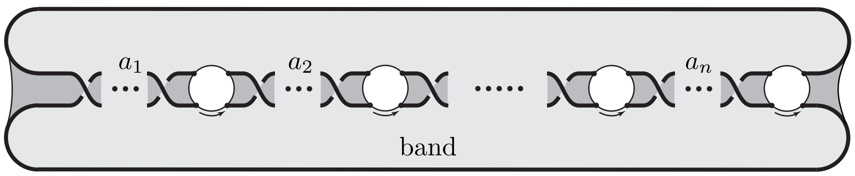


FIGURE 12.8.

This tangle will be denoted by the weighted graph of Figure 12.9 which is also called ***atomic***. It has one vertex, n free bonds, and n weights $a_1, \ldots, a_n$ attached to the n sectors between these bonds. The n bonds and the n integral weights correspond respectively to the n boundary components and the n groups of twists, the order left to right in Figure 12.8 corresponding to counterclockwise order in Figure 12.9, starting at a_1. It is perhaps helpful to geometrically specify the correspondence by choosing a standard continuous map of the underlying 3–manifold onto the graph so that the preimage of the end point of any bond is the corresponding Conway sphere.

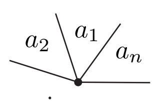


FIGURE 12.9.

The surface with boundary containing the knot, indicated in Figure 12.8 is called the ***twisted band*** or simply the ***band***; note that for $n \geqslant 3$ it is the characteristic band of the underlying hollow Montesinos pair provided by Proposition 10.1.

In terms of surfaces with plumbing patches, the atomic tangle of Figure 12.10 together with its band correspond to the twisted band of Figure 12.10, with the plumbing patches indicated.

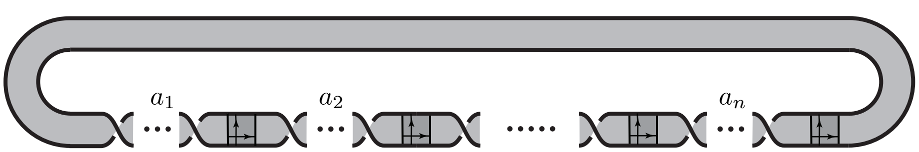
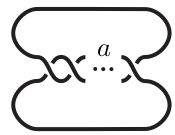


FIGURE 12.10.

Similarly, we define for each $a \in \mathbb{Z}$ a ***model closed atomic tangle of weight*** a, which is the knot



symbolised by $\overset{a}{\bullet}$, and define its twisted band as in Figure 12.8.

Note that any isomorphism of two such one-vertex "atomic" graphs, preserving the order of bonds and weights up to cyclic permutation, is realised by a band-preserving tangle isomorphism of the corresponding atomic tangles. Thus, viewed up to isotopy in the plane, a weighted one-vertex graph as in Figure 12.9 determines an atomic tangle, up to tangle isomorphism respecting twisted bands and also respecting the correspondence of boundary components to graph bonds.

We now combine this notation for (model) atomic tangles, with the plumbing operation for tangles.

Let a ***weighted planar tree*** Γ consist of:

(i) an abstract tree, with free bonds allowed as in Chapter 2;
(ii) a cyclic ordering of the bonds of each vertex of Γ;
(ii) an integral weight assigned to each angular sector between adjacent bonds (define these in terms of (ii)!).

An easy way to specify this data is to embed it in the plane, in such a way that the cyclic order of bonds and integer weights at a vertex correspond to the counterclockwise order; the embedding is then unique up to isotopy of the plane. This also explains the terminology.

Convention. An angular sector that is blank is understood to carry weight 0.

The plumbing operation on tangles then associate a tangle $(M, K; \theta)$ to each such weighted planar tree Γ. We begin by considering, for each vertex v_i, the atomic graph consisting of v_i and of its adjacent bonds and weights, and then the atomic tangle $(M_i, K_i; \theta_i)$ associated to this atomic weighted graph. The tangle $(M, K; \theta)$ is then constructed by plumbing the $(M_i, K_i; \theta_i)$ together in such a way that, if an edge e joins two vertices v_i and v_j, the corresponding atomic tangles $(M_i, K_i; \theta_i)$ and $(M_j, K_j; \theta_j)$ are plumbed along the boundary components $S_i \subset \partial M_i$ and $S_j \subset \partial M_j$ respectively associated to the two bonds leading to the edge e. The fact that Γ is a tree (with no loop) guarantees that the resulting tangle $(M, K; \theta)$ embeds in S^3.

For instance, the tangle associated to the weighted planar tree of Figure 12.11 is the knot represented in Figure 12.12.

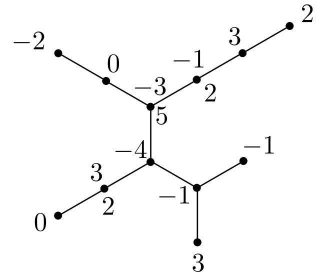


Figure 12.11.

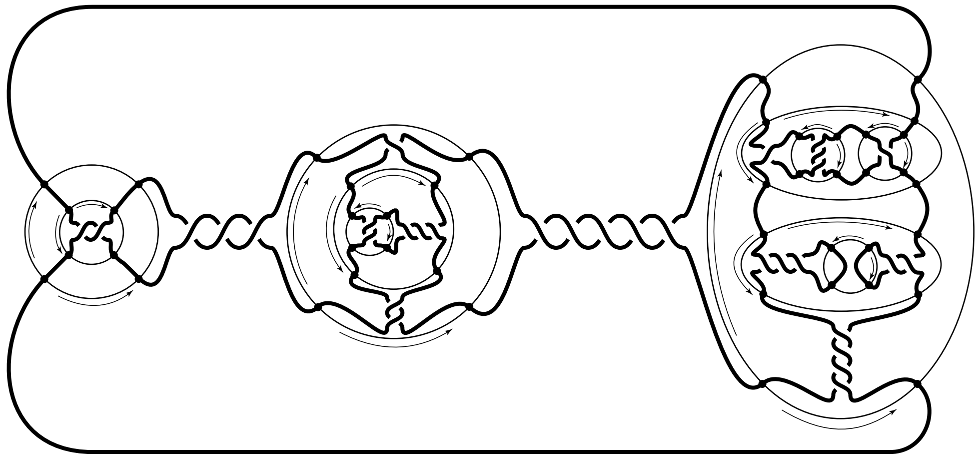


FIGURE 12.12.

The same construction can equivalently be described in terms of plumbing of twisted bands with plumbing patches as in Figure 12.10. The case of Figure 12.11 is illustrated in Figure 12.13, which arguably is aesthetically more pleasing.

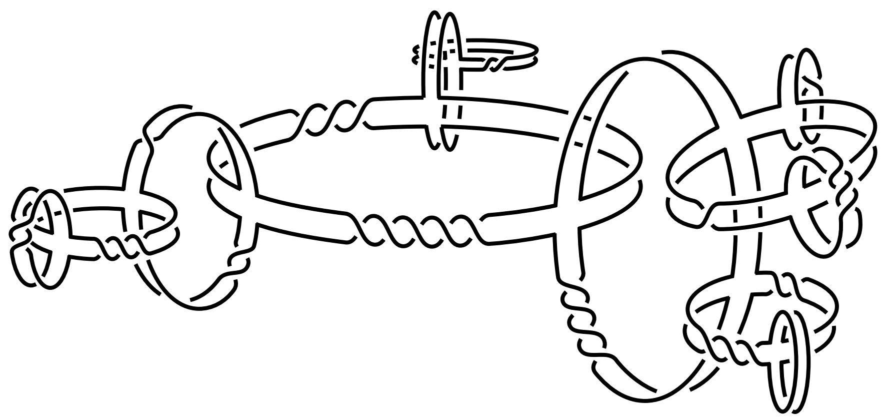


FIGURE 12.13.

Define an ***arborescent tangle*** as one whose underlying knot pair (M, K) is arborescent in the sense defined in Chapter 9; namely M is connected, embeds in S^3 and if (M, K) can be split up along a family of disjoint Conway spheres into pairs that are hollow Montesinos pairs (see §8.2).

Since every atomic tangle gives an arborescent pair, it is clear that every knot pair that results from plumbing according to a weighted planar tree is arborescent. The first theorem of this chapter asserts the converse.

THEOREM 12.4. *Every arborescent tangle is (isomorphic to one) produced by plumbing according to a weighted planar tree.*

PROOF. This will follow from well-known facts about projections of braids.

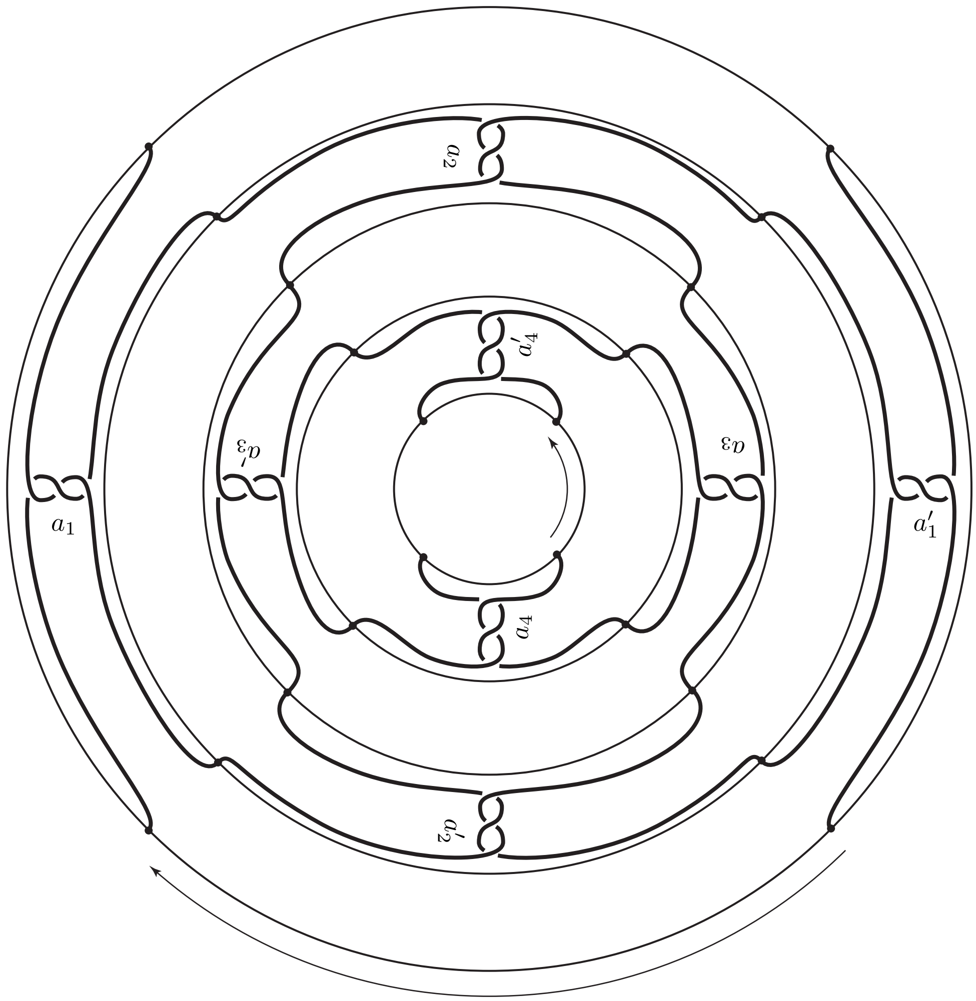


FIGURE 12.14. (Note the clockwise rotations of 90°)

As hollow Montesinos pairs underlie atomic tangles, we see that a weakened version of the theorem is trivially true, namely: Every arborescent tangle can be produced from tangles that are either atomic or have underlying pair a thickened Conway sphere $\cong (S^2, P^0) \times [0,1]$, by plumbing these according to a graph.

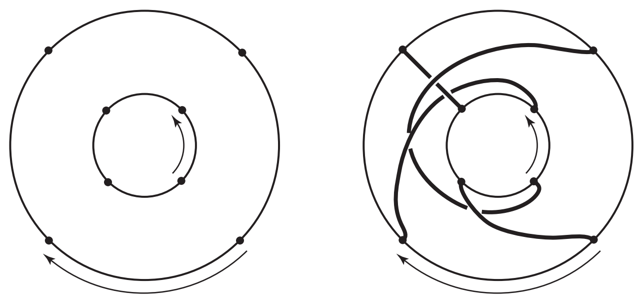
Thus it suffices to prove the theorem for arborescent tangles whose underlying pair is $\cong (S^2, P^0) \times [0,1]$. Let us call these ***braid tangles***.

ASSERTION 12.5. *Every braid tangle $(M, K; \theta)$ is isomorphic to one obtained by plumbing according to a weighted planar tree of the form* $\frac{a_1'}{a_1}\!\!-\!\!\frac{a_2'}{a_2} \cdots \frac{a_{4k}'}{a_{4k}}$*, namely to a tangle of the type represented in Figure* 12.14 *(for $k = 1$, $a_i = 3 = a_i'$).*

PROOF OF ASSERTION 12.5. Certainly, we can assume that (M, K) is a standard thickening $(S^2, P^0) \times [0,1] \subset \mathbb{R}^3$ of the unit sphere, and that the boundary parametrisation θ is standard on $S^2 \times 1$. On $S^2 \times 0$ however, θ may be any degree

-1 automorphism of S^2 respecting the four points P^0. Since all degree -1 automorphisms of S^2 are isotopic (not respecting P^0), we can use an isotopy to change θ and K near $S^2 \times 0$ so that, afterwards,

(i) on $S^2 \times [0,1]$, the parametrisation is as indicated in Figure 12.15(a), which (intentionally) is the same as in Figure 12.14, and
(ii) K has no critical point for the radius function (projection to $[0,1]$), which means that K is a four string braid in $S^2 \times [0,1]$.



(a) (b)

FIGURE 12.15.

By further general position (transversality) arguments, we can successively arrange that, in addition,

(iii) the projection of K to $\mathbb{R}^2$ has no critical points for the radius function (projection $S^1 \times [0,1] \to [0,1]$), and
(iv) this projection has isolated double points only, namely simple crossings only, as in Figure 12.15(b).

At this point, K has a projection isotopic to one as in Figure 12.14, and we can give it exactly such a projection. Beware that many of a_i, a_i' may be zero. For instance, the graph will be $\overset{0}{\underset{0}{\bullet}}\!\!-\!\!\overset{0}{\underset{1}{\bullet}}\!\!-\!\!\overset{-1}{\underset{1}{\bullet}}\!\!-\!\!\overset{-1}{\underset{-1}{\bullet}}$ in the example of Figure 12.15(b).

This completes the proof of the Assertion 12.5, and of Theorem 12.4. □

12.3. Moves on weighted planar trees

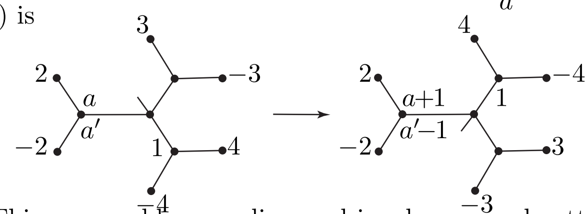
Let Γ be an integrally weighted planar tree. We list alterations of Γ that are intended to leave the resulting pair unchanged (up to pair isomorphism).

Flips. This first set of moves essentially deals with the cyclic order of weights and bonds around vertices.

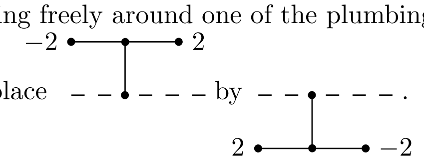
(F_1) Reverse the cyclic order of bonds at every vertex of the graph.
(F_2) Reverse the cyclic order at one vertex, and at each vertex at even distance from it in the graph.
(F_3) Replace $- - -\overset{a}{\underset{a'}{\bullet}}\!\!-\; - -$ by $- - \overset{a\mp 1}{\underset{a'\pm 1}{\bullet}}\!\!-\; - -$, where the cyclic order of bonds is reversed at all vertices lying to the right of the vertex shown, and at odd distance from this vertex.

In our description of the flip (F_3), only one vertex, one bond and two (adjacent) weights of the tree are visible. The dashed line $- -$ indicates any continuation to the right, namely a (connected, possibly empty) weighted subtree attached to the bond shown. The other dashed line $- - -$ represents any continuation to the left, namely a sequence of consecutive weights, free bonds and weighted subtrees attached to the vertex shown. We will use similar conventions in our description of further moves.

For example the original tree might be as simple as $\overset{a}{\underset{a'}{-\bullet-}}$, but a more typical example of (F_3) is



Ring move. This move seldom applies, and involves an unknotted component of the link moving freely around one of the plumbing bands, whence the name.

(R) Replace 

Boundary moves. These moves occur near a free bond of the graph, corresponding to a boundary component of the associated tangle pair.

(B_1) Replace $- - - \bullet \overset{a}{\bullet}-$ by $- - - \bullet-$.

(B_2) Replace $- - - \overset{a}{\bullet}-$ by $- - - \overset{0}{\bullet}-$.

Arithmetic moves. This next set of moves changes the weights of vertices in the tree, and is closely related to arithmetic properties of continued fractions and of the group $SL_2(\mathbb{Z})$. They are labelled by two integers; the first one refers to the weight of the main vertex involved, the second one to its valence. Note the precise location of the weights, which is often critical.

(0.1) Replace $\overset{0}{\bullet}\!-\!\overset{a}{\bullet}\!-\!\bullet - - -$ by $\bullet - - -$.

(0.2) Replace $- - - \overset{a}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\underset{b}{\bullet} - -$ by $- - \overset{a+b}{\bullet} - -$, where the cyclic orders of bonds are preserved in the left hand side subgraph $- - -$ and reversed anywhere else. In particular, the order of the components of $- -$ is reversed.

(1.0) Exchange $\overset{+1}{\bullet}$ and $\overset{-1}{\bullet}$.

(1.1) Replace $\overset{\pm1}{\bullet}\!-\!\overset{a}{\bullet} - - -$ by $\overset{a\mp1}{\bullet} - - -$.

(1.2) Replace $- - \overset{a}{\bullet}\!-\!\underset{\pm1}{\bullet}\!-\!\overset{b}{\bullet} - - -$ by $- \overset{a\mp1}{\bullet}\!-\!\overset{b\mp1}{\bullet} - - -$.

(2.0) Exchange $\overset{+2}{\bullet}$ and $\overset{-2}{\bullet}$.

(2.1) Replace $\overset{\pm2}{\bullet}\!-\!\overset{a}{\bullet} - - -$ by $\overset{\mp2}{\bullet}\!-\!\overset{a\mp1}{\bullet} - - -$.

(2.2) Replace $- - \overset{a}{\bullet}\!-\!\overset{\pm2}{\bullet}\!-\!\underset{b}{\bullet} - - -$ by $- \overset{a\mp1}{\bullet}\!-\!\underset{\mp2}{\bullet}\!-\!\underset{b\mp1}{\bullet} - - -$.

This list of moves is convenient to work with, but very redundant in the sense that Flips (F_1) and (F_2) easily follow from (F_3), whereas all arithmetic moves are simple consequences of the three moves (0.1), (0.2) and (1.2).

There is one more, somewhat troublesome, arithmetic move (0.1^*) that we shall introduce after the following proposition.

Proposition 12.6. *For a weighted planar tree, the above moves do not affect (up to pair isomorphism) the pairs associated by plumbing.*

Complement 12.7. *If* Γ *and* Γ' *are weighted planar trees related by one of the above moves, and* (M, K) *and* (M', K') *are the tangles respectively associated to* Γ *and* Γ' *by plumbing, our proof of Proposition* 12.6 *will yield a preferred pair* *isomorphism* $\varphi : (M, K) \to (M', K')$. *Its effect on boundary will be evident; in particular, except for the boundary moves* (B_1) *and* (B_2), *the pair isomorphism* φ *will preserve slopes on all boundary components, namely lie in the Klein* 4*–group* V_4 *introduced at the beginning of Chapter* 11. *Thus, we can say that these moves leave the tangle unchanged modulo boundary mutations, namely mutations along Conway spheres that are pairwise boundary-parallel.*

Proof of Proposition 12.6 and Complement 12.7. We begin by introducing some notation.

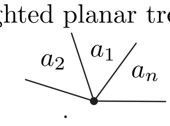
If A is a tangle and $\alpha \in V_4$, we let αA be the same tangle with all boundary parametrisations altered by α. Recall from Chapter 11 that the Viergruppe V_4 is made up of the π–rotations ξ, η, ζ about the three axes of $\mathbb{R}^3$, and of the identity ι.

Also, let $-A$ denote the tangle obtained by reversing ambient orientation and correspondingly changing the parametrisation of every boundary Conway sphere by the reflection $(x, y, z) \mapsto (x, y, -z)$ in the equatorial "square" of the standard Conway sphere. Observe that $-(\alpha A) = \alpha(-A)$ for $\alpha \in V_4$. Also, if a weighted planar tree Γ yields the tangle A by plumbing, then plumbing according to the weighted planar $-\Gamma$ obtained by reversing the sign of all integer weights gives the tangle $-A$.

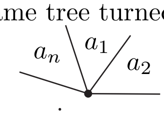
After these preliminaries, we can now begin the proof of Proposition 12.6 and Complement 12.7.

If A is atomic, we have a natural tangle isomorphism $\xi A \equiv A$ by $\pi-$rotation about a horizontal axis in Figure 12.8.

If A is associated to the weighted planar tree



and A' is associated to the same tree turned over



we have a tangle isomorphism $A' \equiv \eta A$, illustrated in Figure 12.16, that reverses the cyclic order of boundary components.

Next note that if A is obtained by plumbing the tangles B and C, then the identity map constitutes a *pair* isomorphism from A to the tangle A' obtained by plumbing ξB to ηC. Of course, the boundary parametrisations differ by ξ on B and by η on C.

Taken together, these observations prove invariance under Flips (F_1) and (F_2).

To prove the invariance under (F_3), the reader also will have to use the tangle isomorphism of Figure 12.17 (its pairs are to be viewed as parts of the atomic pairs appearing in (F_3)).

The ring move (R) is a consequence of the tangle isomorphism of Figure 11. To get this isomorphism we just slide the ring around; the dotted spheres are not respected. The insides of the left and right hand spheres contain respectively what – – and – – – stand for in the ring move.

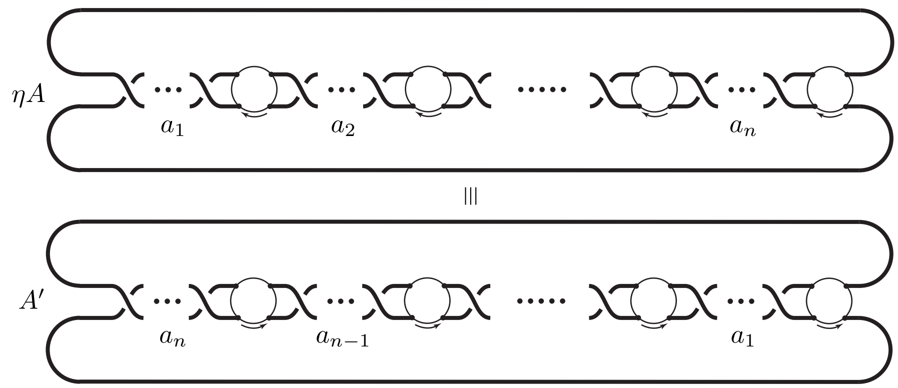


FIGURE 12.16.

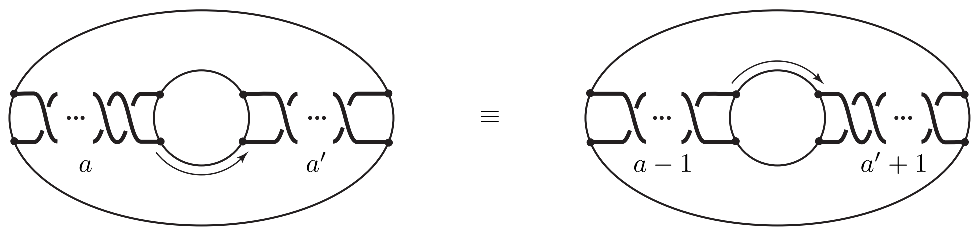


FIGURE 12.17. the flip (F_3)

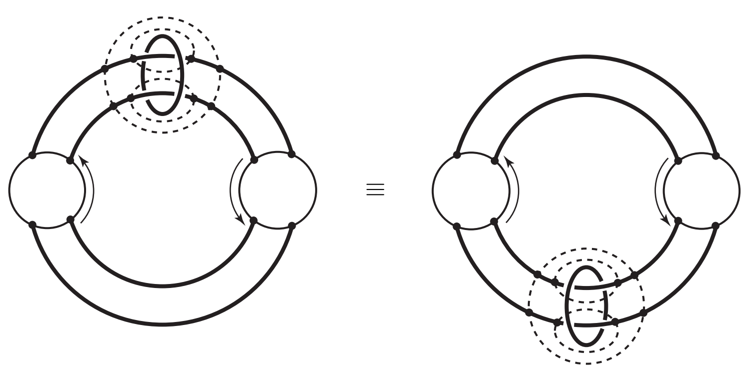


FIGURE 12.18. the ring move (R)

The boundary moves (B_1) and (B_2) follow readily from the pair isomorphism of Figure 12.19, which fixes (say) the left-hand Conway sphere (namely respects its parametrisation).

Regarding the arithmetic moves, recall that, strictly speaking, we need only check (0.1), (0.2) and (1.2).

For (0.1) use the tangle isomorphism of Figure 12.20.

For (0.2), use that of Figure 12.21.

For move (1.2), use Figure 12.22.

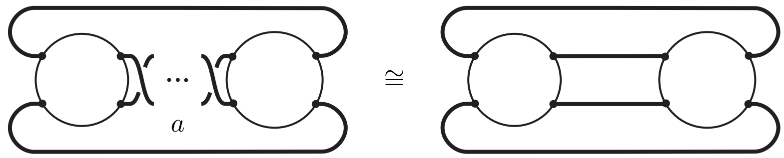


FIGURE 12.19. the boundary moves (B_1) and (B_2)

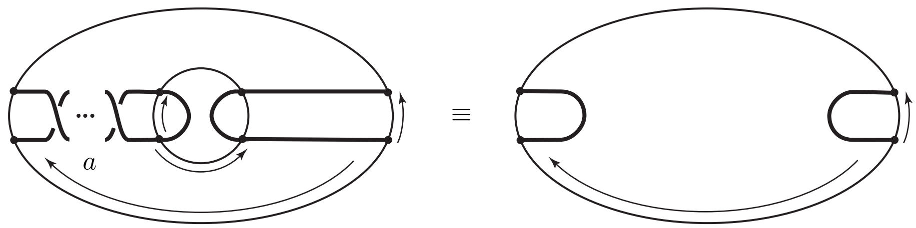


FIGURE 12.20. the arithmetic move (0.1)

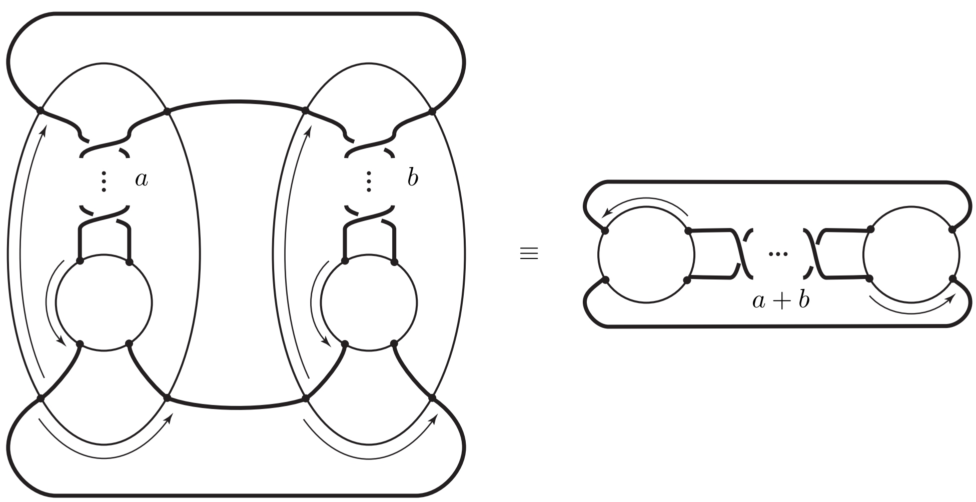
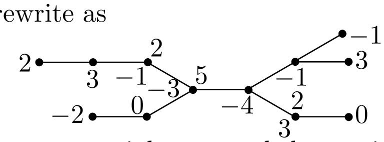


FIGURE 12.21. the arithmetic move (0.2)

For good measure, we also give figures for the moves (1.0), (1.1), (2.0), (2.1), (2.2). They show, incidentally, that the pair isomorphisms are tangle isomorphisms in these cases, as they were for (R), (0.1), (1.1), in contrast to the flips, boundary moves, and (0.2).

This ends the proof of Proposition 12.6 and Complement 12.7. □

As an illustration of Proposition 12.6, consider the weighted planar tree of Figure 12.11, which we rewrite as



Applying Flip (F_3) to move weights around the vertices, one obtains

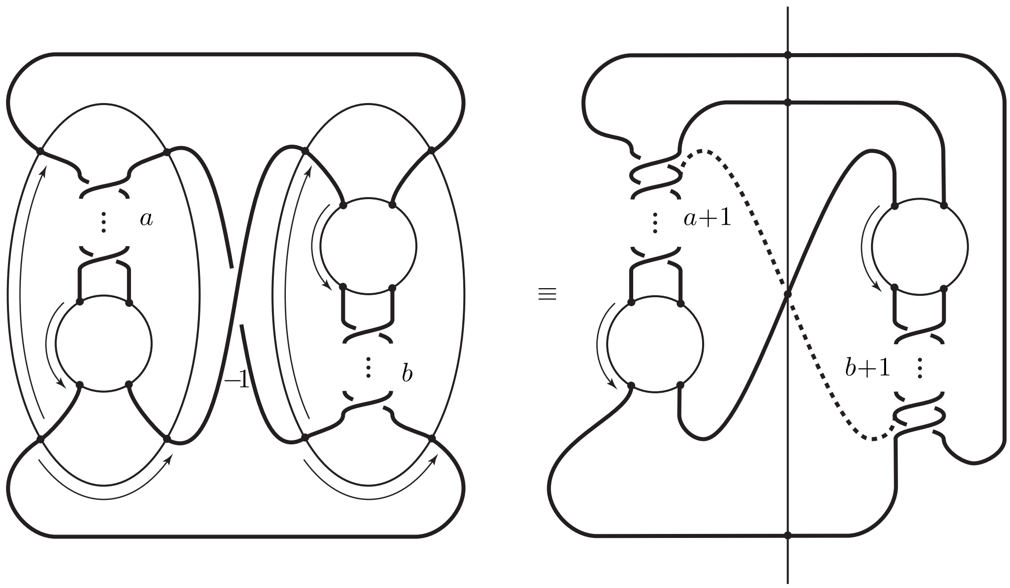


FIGURE 12.22. the arithmetic move (1.2)

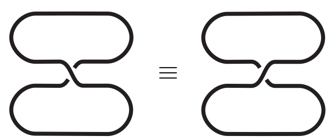


FIGURE 12.23. the arithmetic move (1.0)

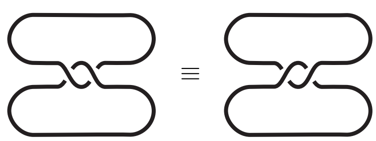


FIGURE 12.24. the arithmetic move (2.0)

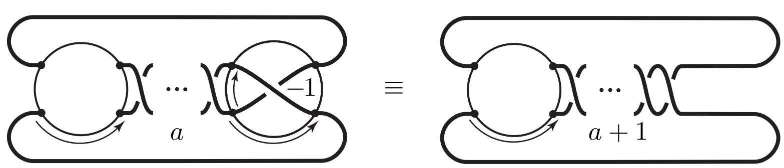
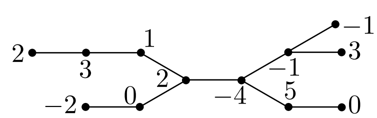


FIGURE 12.25. the arithmetic move (1.1)



Then, the arithmetic moves (1.2), (0.2), (1.1) and (0.1) give

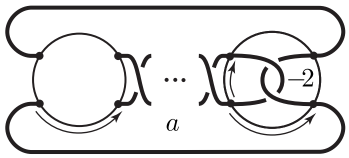


FIGURE 12.26. the arithmetic move (2.1)

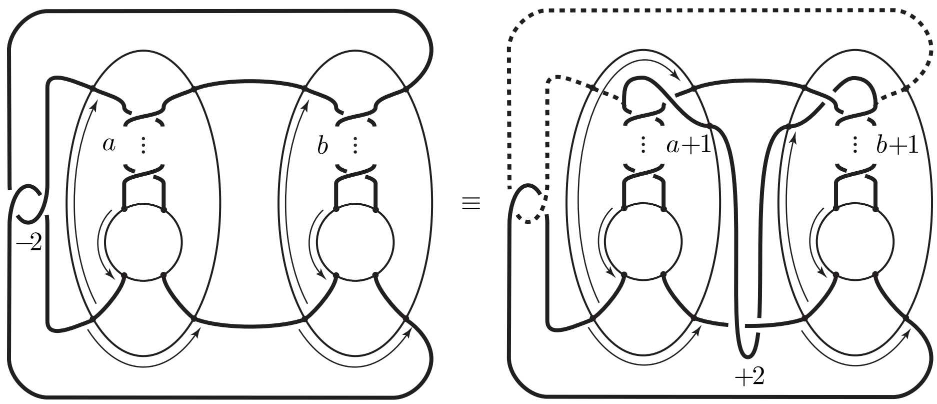
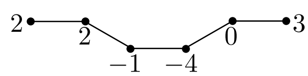


FIGURE 12.27. the arithmetic move (2.2)



A further application of (0.2) turns this tree into $\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\overset{-1}{\bullet}$, which itself can be converted to $\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}$ and finally $\overset{2}{\bullet}$ by (1.1) and (0.2).

What this manipulation proves is that the knot of Figures 12.12 and 12.13 is the same as the boundary of an unknotted band with 2 half-twists, namely the Hopf link!

There is one final, somewhat troublesome, sort of arithmetic move in the calculus of arborescent pairs in S^3.

Connected sum move.

(0.1*) Replace the weighted planar tree $\Gamma = \overset{0}{\bullet}\!\!-\!\!\!-\!\!\bullet$ – – – , where the central vertex has valence $\geqslant 3$, by the family of disjoint weighted planar trees obtained from Γ by erasing the two vertices shown as well as their adjacent edges, and by adding k copies of $\bullet\!\!-$ if there were k free bonds attached to the central vertex.

Note that the family associated to Γ by Move (0.1*) consists of at least two weighted planar trees.

This move is justified by the following observation.

LEMMA 12.8. *If the family of weighted planar trees* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ *is associated to* Γ *by Move* (0.1*)*, the arborescent pair* (M, K) *defined by the weighted planar tree* Γ *is a connected sum of the tangle pairs* (M_i, K_i) *defined by the* Γ_i.

PROOF. See Figure 8.31. □

Note that reconstructing the knot pair (M, K) from the (M_i, K_i) requires that we know some additional information, such as on which knot components the connected sum takes place and in which direction. For this reason, Move (0.1^*) is in general not reversible.

So far our moves have only been defined for weighted planar trees. However, it should be clear that they also apply to disjoint union of weighted planar trees, by operating on one component at a time leaving the others unchanged.

It is a deep fact (to be proved in Chapter 13) that two weighted planar trees which give isomorphic arborescent pairs are always related by a sequence of moves and inverse moves from our calculus of arborescent pairs (including Move (0.1^*)). In the next two sections we present important elementary steps in the proof.

12.4. Canonical trees for arborescent pairs

The above moves show that an arborescent pair can be associated to arbitrarily large weighted planar trees. If these trees are going to be of any use, it therefore appears necessary to impose a certain number of restrictions. The weighted planar trees satisfying these conditions will be called canonical, since we will see that they classify the arborescent pair up to a limited number of moves.

We list these conditions for canonicity.

Weight Condition.

(W) At each vertex of Γ, at most one weight is non-zero.

This property can clearly be attained by a finite sequence of applications of Flip (F_3). We consequently assume it.

The remaining conditions for canonicity concern only the ***abstract weighted tree*** Γ_0 ***underlying*** Γ, defined by forgetting the embedding of Γ in the plane. The weights are now assigned to the vertices of Γ_0, by the Weight Condition (W).

Free Bond Condition.

(F) Every free bond of Γ_0 belongs to a vertex of weight 0 and with valence $\geqslant 3$, unless Γ is $\overset{0}{\bullet}\!\!-\!\!-$ or $-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-$.

Clearly this Free Bond Condition can be realised by applications of the Boundary Moves (B_1) and (B_2).

Form a weighted graph Γ_0' from Γ_0 by deleting all its vertices of valence $\geqslant 3$. Each component of Γ_0' is a ***stick*** of Γ_0. It is topologically an interval, and has 0, 1 or 2 free bonds at its ends. In particular, there are three possible types of sticks: $\overset{a_1}{\bullet}\!\!-\!\!-\!\!\overset{a_2}{\bullet}\!\!-\!\!- \cdots -\!\!\overset{a_n}{\bullet}$, $\overset{a_1}{\bullet}\!\!-\!\!-\!\!\overset{a_2}{\bullet}\!\!-\!\!- \cdots -\!\!\overset{a_n}{\bullet}\!\!-\!\!-$ or $-\!\!-\!\!\overset{a_1}{\bullet}\!\!-\!\!-\!\!\overset{a_2}{\bullet}\!\!-\!\!- \cdots -\!\!\overset{a_n}{\bullet}\!\!-\!\!-$ with $n \geqslant 1$.

Stick Condition.

(S) On every stick of Γ_0, the weights are non-zero and of alternating sign, unless the whole tree Γ_0 is one of $\overset{0}{\bullet}$, $\overset{0}{\bullet}\!\!-\!\!-$ or $-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-$. The end weights a_1 and a_n of a stick are not ± 1, unless Γ_0 is $\overset{\pm 1}{\bullet}$.

The last condition for canonicity comes with two options.

Positive Canonicity Condition.

(P) None of the sticks of Γ_0 is $\overset{-1}{\bullet}$, $\overset{-2}{\bullet}$, $\overset{-2}{\bullet}\!\!-\!\!-$ or $-\!\!-\!\!\overset{-2}{\bullet}\!\!-\!\!-$.

Negative Canonicity Condition.

(N) None of the sticks of Γ_0 is $\overset{+1}{\bullet}$, $\overset{+2}{\bullet}$, $\overset{+2}{\bullet\!\!-\!\!-}$ or $-\!\!-\!\!\overset{+2}{\bullet}\!\!-\!\!-$.

A weighted planar tree is ***positively canonical*** or $(+)$–***canonical*** if it satisfies the Weight Condition (W), the Free Bond Condition (F), the Stick Condition (S) and the Positive Canonicity Condition (P). It is ***negatively canonical*** or $(-)$–***canonical*** if it satisfies the Weight Condition (W), the Free Bond Condition (F), the Stick Condition (S) and the Negative Canonicity Condition (N).

The reason to keep these two canonicity conditions is that, if the arborescent pair (M, K) is associated to the weighted planar tree Γ, the tangle pair $(-M, K)$ obtained by reversing the orientation of M is associated to the weighted planar tree $-\Gamma$ obtained by reversing the signs of all the weights of Γ. Then Γ is $(+)$–canonical if and only if $-\Gamma$ is $(-)$–canonical.

THEOREM 12.9. *There exists an effective algorithm which, for any weighted planar tree Γ, alters Γ by a sequence of moves of the calculus of arborescent pairs to produce a collection of positively (or negatively) canonical weighted planar trees.*

COROLLARY 12.10. *Every arborescent pair is obtained by pairwise connected sum operations from arborescent pairs associated to positively (or negatively) canonical weighted planar trees.*

PROOF. Combine Theorem 12.4, Proposition 12.6, Lemma 12.8 and Theorem 12.9. □

PROOF OF THEOREM 12.9. As far as the Weight Condition (W) and the Free Bond Condition (F) are concerned the algorithm is obvious, as we have already observed. Hence we can assume that (W) and (F) hold. All later steps will preserve these properties.

The Stick Condition (S) is more problematic. The following observation will simplify our task: The arithmetic moves clearly make sense for *abstract* integrally weighted trees such as the tree Γ_0, where the integral weights are associated to the vertices. Let Γ be an integrally weighted planar tree (satisfying (W)) and let Γ_0 be its underlying abstract weighted tree. Then, any arithmetic move on Γ_0 can be realised by a sequence of moves applied to Γ which together preserve (W), and more precisely by a sequence of flips followed by an arithmetic move of the same type.

It follows that it will suffice to give an algorithm which, for any integrally weighted abstract tree Γ_0 satisfying the Free Bond Condition (F), provides a sequence of arithmetic moves making Γ_0 canonical in the sense that it satisfies (F), (S) and (P) (although it is now perhaps not connected). Here is such an algorithm.

Apply the following moves as much as possible: (B_1), (B_2), (0.1), (0.1^*), (0.2), (1.1), and

(i) Move (1.2) $-\,-\overset{a}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\pm1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{b}{\bullet}-\,-\,- \longrightarrow -\,-\overset{a\mp1}{\bullet}\!\!-\!\!-\!\!\overset{b\mp1}{\bullet}-\,-\,-$ when $a \neq 0$ has sign $\pm$, or when the vertex at left is of valence $\geqslant 3$;

(ii) the inverse $-\,-\overset{a}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{b}{\bullet}-\,-\,- \longrightarrow -\,-\overset{a-1}{\bullet}\!\!-\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!-\!\!\overset{b-1}{\bullet}-\,-\,-$ of Move (1.2) if the two vertices indicated have valence $\leqslant 2$ and weights $a, b > 1$;

(iii) the inverse $-\,-\overset{a}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{b}{\bullet}-\,-\,- \longrightarrow -\,-\overset{a+1}{\bullet}\!\!-\!\!-\!\!\overset{+1}{\bullet}\!\!-\!\!-\!\!\overset{b+1}{\bullet}-\,-\,-$ of Move (1.2) if the two vertices indicated have valence $\leqslant 2$ and weights $a, b < -1$.

CLAIM 12.11. *After finitely many steps, we reach a family of weighted planar trees where none of the moves* (B_1), (B_2), (0.1), (0.1^*), (0.2), (1.1) *and* (i), (ii), (iii) *above apply.*

Proof. Each such move reduces the complexity $(X, 2Y+Z) \in \mathbb{N}^2$ where $\mathbb{N}^2$ is endowed with the lexicographic order, where X is the number of sticks of Γ_0 excluding whole components $\bullet\!\!-\!\!-$, where Y is the sum of the absolute values of the weights of all vertices of valence $\leqslant 2$, and where Z is the number of vertices excluding components $\bullet\!\!-\!\!-$. (Actually, most moves reduce the complexity $2Y+Z \in \mathbb{N}$, with the exception of certain cases of (1.1). The components $\bullet\!\!-\!\!-$ are excluded because (0.1*) can create many of these.)

Therefore, the process must be finite. □

When none of these moves is applicable (so that, in particular, Condition (F) is preserved), it is easily seen that the Stick Condition (S) is now satisfied by all the components of Γ_0 except those of the form $\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{a}{\bullet}$, which we convert to $\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{a\pm1}{\bullet}$, and eventually to $\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{-1}{\bullet}$ and $\overset{1}{\bullet}$, by the inverse of (1.2) and by (1.1).

To complete the algorithm it suffices to apply (1.0), (2.0), (2.1), (2.2) to eliminate all unwanted sticks of the form $\overset{-1}{\bullet}$, $\overset{-2}{\bullet}$, $\overset{-2}{\bullet\!\!-\!\!-}$ or $\overset{-2}{-\!\!\bullet\!\!-}$. □

Note that the Ring Move (R) does not apply to canonical weighted planar trees, and that Flip (F_3) usually perturbs the Weight Condition (W). This leads us to modify these moves as follows.

The Flip (F_3) is replaced by two distinct moves.

Modified flips.

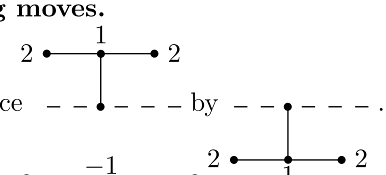
(F_3') Replace $- - -\overset{a}{\bullet}- -$ by $- - -\underset{a}{\bullet}- -$ and, if a is odd, reverse the cyclic order of weights and bonds at all vertices of the subgraph $- -$ lying at odd distance from the vertex shown.

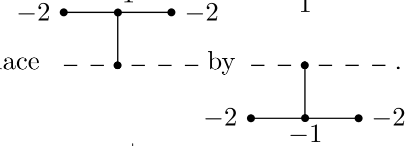
(F_3'') If Γ can be split as $- - -\,\!\!\mid\!\!\bullet\!\!\mid\,- -$, reverse the cyclic order of weights and bonds at all vertices of the subgraph $- -$ lying at odd distance from the vertex shown.

The fact that these new Flips (F_3') and (F_3') do not modify the associated arborescent pair easily follows from iterating (F_3) and combining it with the Boundary Move (B_2).

There are two forms of the modified ring move, one for positively canonical trees and one for negatively canonical trees.

Modified ring moves.

(R^+) Replace  .

(R^-) Replace  .

These two moves ($R^{\pm}$) are easy consequences of (R) and (2.1).

Then, our main classification theorem, whose proof will occupy most of Chapter 13, states:

Theorem 12.12 (Classification Theorem for arborescent pairs). *Plumbing according to two positively (or negatively) canonical weighted planar trees* Γ *and* Γ'

gives arborescent pairs (M, K) *and* (M', K') *which are degree* $+1$ *isomorphic precisely when* Γ *and* Γ' *can be deduced from each other by a sequence of flips* (F_1), (F_2), (F'_3), (F''_3) *and of modified ring moves* (R^+) *(or* (R^-)*). Moreover,* (M, K) *is always pairwise irreducible unless* Γ *is* $\overset{0}{\bullet}$.

Observe that, up to isomorphism of weighted planar trees, there are only finitely many $(+)$–canonical weighted planar trees that are obtained by moves (F_1), (F_2), (F'_3), (F''_3) and (R^+) from a given one. Thus, the combination of Theorem 12.4, Lemma 12.8, Theorem 12.9 and Theorem 12.12 gives an effective algorithm to decide when an arborescent pair (M, K) is pairwise irreducible and when two pairwise irreducible arborescent pairs are isomorphic.

In the next two sections §12.5 and §12.6, we will improve Theorem 12.12 by giving a very efficient method for deciding in practice when two canonical trees are equivalent by flips and modified ring moves.

We conclude this section with a pleasant exercise. So far, we have not made much distinction between connected knots and links with several components. However, one may be interested in the question of quickly computing the number of components of the knot K of an arborescent pair (M, K) obtained by plumbing according to a weighted planar tree Γ. Here is a recipe:

(i) Assuming Γ satisfies the weight condition (W) without loss of generality, consider the weighted tree Γ_2 that consists of the combinatorial tree underlying Γ with each vertex weighted by the mod 2 reduction of its weight in Γ. Thus, Γ_2 is the mod 2 reduction of the weighted tree Γ_0 which we considered earlier.

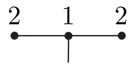
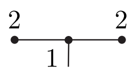
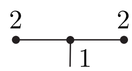
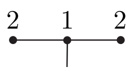
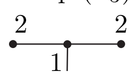
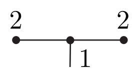
(i) Observe that the arithmetic moves (0.1), (0.1*), (0.2), (1.1) and (1.2) make sense for such trees with weights in $\mathbb{Z}_2$, and apply them as much as possible. One eventually obtains a collection Γ'_2 of weighted trees which either have all vertices of valence $\geqslant 3$, or are among $\overset{0}{\bullet}$, $\overset{1}{\bullet}$, $\overset{0}{\bullet}\!\!-$ or $-\!\!\overset{0}{\bullet}\!\!-$.

Proposition 12.13. *With the above data, let* n_f *be the number of free bonds of* Γ *(or* Γ'_2 *), and let* n_0 *be the number of components* $\overset{0}{\bullet}$ *in* Γ'_2*. Then the number of arc components of* K *is* $2n_f$*; and the number of circle components is* n_0 *if* $n_f > 0$ *and* $n_0 + 1$ *otherwise.*

Proof. Observe that the number of components of K is unchanged when one replaces any weight a of Γ by some a' with $a \equiv a'$ mod 2, or when one permutes two bonds of a given vertex. From this point the proof will be left as an exercise. □

12.5. Abbreviated trees

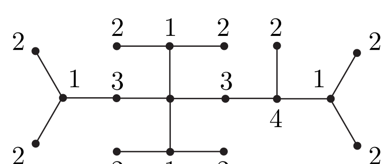
In view of Theorem 12.12, we are confronted to the following practical problem: Given two $(\pm)$–canonical weighted planar trees Γ and Γ', decide whether or not it is possible to pass from one to the other by a sequence of moves (F_1), (F_2), (F'_3), (F''_3), $(\mathrm{R}^\pm)$. Of course, it is always possible to write down the (finite) list of all weighted planar trees obtained from Γ by a sequence of such moves, up to weighted planar tree isomorphism, and then to check whether or not Γ' is isomorphic to one of the elements of this list. This naïve method is quite sufficient for "small" trees, but turns out to be rather clumsy as soon as the tree reaches a reasonable size. The main purpose of this section and of the following one is to describe a more efficient method to solve this practical problem. Meanwhile, we will make a first step towards the proof of Theorem 12.12, by introducing a new kind of weighted planar tree, which will be better adapted to the proof of this theorem to be given in Chapter 13.

A first idea to simplify this search of move equivalences between (+)–canonical weighted planar trees is to try to *get rid of the modified ring move* (R^+). Indeed, when a subtree  occurs in a (+)–canonical tree Γ, it can be considered as attached to the vertex of Γ which is adjacent to it, since its location in the plane is irrelevant for the plumbing construction by move (R^+). The same property holds for subtrees  and  deduced from the previous one by application of Flip (F_3). We will henceforth refer to any subtree of Γ of such type ,  or  as a ***ring subtree*** of Γ.

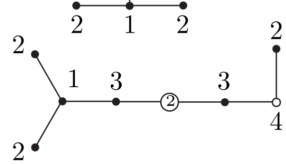
By the above considerations, it is natural to simplify Γ in the following way: Erase each ring subtree that is attached to a vertex of valence $\geqslant 3$ of Γ, together with the corresponding bond of this vertex. Then, for each vertex v of the resulting weighted planar tree, keep track of the number $r \in \mathbb{N}$ of the ring subtrees that are attached to v in Γ by making this vertex hollow and inscribing the integer r inside it. Thus we get a weighted planar tree Γ_* of a new type, consisting of a combinatorial tree embedded in the plane, of weights $a \in \mathbb{Z}$ located in some of its angular sectors, and of a new weight $r \in \mathbb{N}$ attached to each vertex and called the ***ring number*** of this vertex. This weighted planar tree Γ_* is the ***abbreviated*** (+)–***canonical tree associated to*** Γ.

Of course, the ring number is in general 0 for most vertices of Γ_*. To avoid clumsy notation, we shall denote each vertex of Γ_* with ring number 0 by a solid "black" dot of the type we have used so far (with nothing inside). Similarly, we denote a vertex with ring number 1 by a small hollow dot, with no indication of ring numbers.

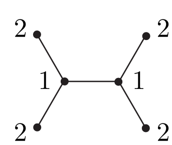
Here is a sample example: The weighted planar tree

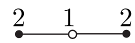


is abbreviated as



The reader may have noticed that the one case where the above construction does not really make sense is when erasing one ring subtree destroys another one. This occurs when Γ is



up to a few flips (F_3'). In this case, we let the abbreviated tree Γ_* be 

. In all other cases, the ring subtrees of Γ are disjoint and there is no ambiguity in the definition of Γ_*.

The reader may wonder why, when defining the abbreviated tree Γ_*, we restricted attention to ring subtrees of Γ *which were attached to vertices of valence* $\geqslant 3$. There are primarily two reasons for this. The first one is that a move (R^+) applied at a vertex of valence $\leqslant 2$ can be decomposed as a product of flips (F_3'); therefore, the consideration of ring subtrees attached to vertices of valence $\leqslant 2$ is not useful. The second reason is that this convention facilitates the characterisation of those weighted planar trees with ring numbers which can occur in this way (compare Proposition 12.16).

These abbreviated trees are more than an *ad hoc* trick to simplify the calculus of weighted planar trees. Indeed, it is possible to develop a whole plumbing calculus based on these weighted planar trees with a ring number $r \in \mathbb{N}$ attached to each of their vertices. The basic idea is to enlarge the notion of atomic tangle. Given such a weighted planar tree Γ with ring numbers, we associate to each vertex of Γ of the type shown on Figure 12.28 a ***model atomic tangle (with rings)*** as in Figure 12.29. This fixes a natural correspondence between the bonds of the vertex and the boundary components of the associated atomic tangle, and Γ therefore specifies a way to plumb all these atomic tangles together. This associates a knot pair (M, K), and even a tangle, to Γ.

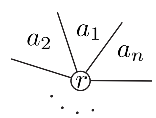


FIGURE 12.28.

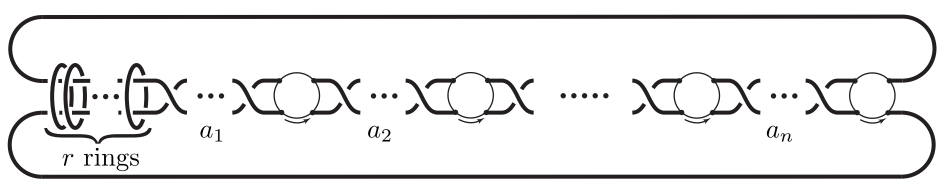


FIGURE 12.29.

Observe that this plumbing construction coincides with the one we have previously used when all rings numbers were 0. Also, if Γ_* is the abbreviated tree associated to the (+)–canonical tree Γ, it is readily seen from Figure 12.18 that plumbing according to Γ and Γ_* gives the same arborescent tangle (or knot pair).

From the moves of §12.3, one readily deduces some moves of weighted planar trees with ring numbers which do not change the knot pairs obtained by plumbing construction. For instance, the flips (F_1), (F_2) and (F_3) straightforwardly apply to any vertex of such a tree Γ, whatever its ring number may be. (But beware that arithmetic moves require ring numbers to be 0.) If Γ moreover satisfies the Weight Condition that there is at most one non-zero weight at each vertex, then one similarly defines flips (F_3') and (F_3'') to be applied to any vertex of Γ.

One readily checks:

LEMMA 12.14. *Let Γ and Γ' be two $(+)$–canonical weighted planar trees with associated abbreviated trees Γ_* and Γ'_*, respectively. Then, it is possible to pass from Γ to Γ' by a sequence of moves* (F_1), (F_2), (F'_3), (F''_3) *and* (R^+) *if and only if one can pass from Γ_* to Γ'_* by a sequence of flips* (F_1), (F_2), (F'_3) *and* (F''_3). □

In particular, Theorem 12.12 can be rephrased as:

THEOREM 12.15. *Let Γ and Γ' be two $(+)$–canonical weighted planar trees, with associated abbreviated trees Γ_* and Γ'_*. Then, the two knot pairs (M,K) and (M',K') respectively associated to Γ and Γ' (or, equivalently, to Γ_* and Γ'_*) by the plumbing construction are (degree $+1$) isomorphic if and only if Γ_* can be made isomorphic to Γ'_* by a sequence of flips* (F_1), (F_2), (F'_3) *and* (F''_3).

This is the form under which we will prove Theorem 12.12 in Chapter 13. Indeed, abbreviated trees turn out to be better adapted to relate the results of previous chapters (in particular Chapters 9, 10 and 11) to the plumbing description of an arborescent pair.

Let an ***abbreviated*** $(+)$–***canonical tree*** be a weighted planar tree with ring numbers that is obtained by abbreviating a $(+)$–canonical weighted planar tree (without ring numbers). Theorem 12.15 (together with Corollary 12.10) asserts that pairwise irreducible arborescent pairs, up to isomorphism, are in one-to-one correspondence with abbreviated $(+)$–canonical trees different from $\overset{0}{\bullet}$, up to flips (F_1), (F_2), (F'_3) and (F''_3). We therefore want to characterise these abbreviated $(+)$–canonical trees among all weighted planar trees with ring numbers. This is clearly an easy exercise using the definition of $(+)$-canonical trees and of the process of abbreviation.

An abbreviated $(+)$-canonical graph Γ must first satisfy the

Weight Condition.

(W) At each vertex of Γ, at most one weight is non-zero.

The remaining conditions concern only the ***abstract weight tree*** Γ_0 ***underlying*** Γ obtained by forgetting the embedding in the plane; namely Γ_0 consists of the combinatorial graph underlying Γ together with the ring number and weight attached to each vertex.

It will be convenient to say that a vertex of Γ_0 (or Γ) is ***essential*** when it has valence $\geqslant 3$ or ring number $\geqslant 1$. In particular, a non-essential vertex has ring number 0 and is drawn solid black, according to our conventions.

Free Bond Condition.

(F) Every free bond of Γ_0 belongs to an essential vertex of weight 0, unless Γ_0 is $\overset{0}{\bullet}\!\!-$ or $-\!\!\overset{0}{\bullet}\!\!-$.

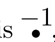
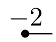
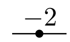
The third condition involves the ***sticks*** of Γ_0, namely the components of the weighted graph obtained from Γ_0 by removing all essential vertices together with all bonds (and all their weights and ring numbers).

Stick Condition.

(S) On every stick of Γ_0, the weights are non-zero and of alternating sign, unless Γ_0 is one of $\overset{0}{\bullet}$, $\overset{0}{\bullet}\!\!-$ or $-\!\!\overset{0}{\bullet}\!\!-$. The end weights a_1 and a_n of a stick are not ± 1, unless Γ_0 is $\overset{\pm 1}{\bullet}$.

Also, we also have a *positive canonicity condition* coming from the analogous condition in the definition of (+)-canonical weighted planar trees.

Positive Canonicity Condition.

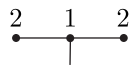
(P) None of the sticks of Γ_0 is , ,  or  .

So far, all these conditions correspond to conditions which already appeared in the definition of (+)–canonicity for weighted planar trees. The new condition comes from the definition of the process of abbreviation, which yields the following:

Abbreviation Condition.

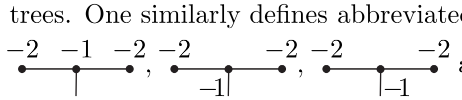
(A) A vertex of Γ_0 of ring number 1 has valence $\geqslant 2$, and Γ_0 is not $\overset{a}{②}$. Every ring subtree of Γ is adjacent to a non-essential vertex.

Proposition 12.16. *A weighted planar tree with ring numbers is an abbreviated (+)–canonical tree, namely is obtained by abbreviating a (+)–canonical tree, if and only if it satisfies the above conditions* (W)*,* (F)*,* (S)*,* (P) *and* (A)*.*

Proof. Clearly, an abbreviated (+)–canonical tree satisfies these conditions. Conversely, given a weighted planar tree with ring numbers Γ_*, we reconstruct a weighted planar tree Γ by, at each vertex of Γ_*, erasing the ring number $r \in \mathbb{N}$ and attaching to this vertex r subtrees  . By inspection, one readily checks that Γ is (+)-canonical if Γ satisfies conditions (W), (F), (S), (P) and (A). □

Note that the above characterisation of abbreviated (+)–canonical trees would have been more troublesome if, when defining these, we had been allowed to remove ring subtrees attached to vertices of valences $\leqslant 2$.

This concludes our discussion of the interplay between (+)–canonical weighted planar trees and abbreviated (+)–canonical trees.

There is of course an analogous theory for (−)–canonical weighted planar trees. One similarly defines abbreviated (−)–canonical trees by removing subtrees  and adding ring numbers. Among weighted planar trees with ring numbers, these abbreviated (−)–canonical trees are characterised by the above properties (W), (F), (S), (A), together with an obvious ***Negative Canonicity Condition*** (N) replacing (P).

12.6. Detecting flip equivalences

In this section we give a rather efficient algorithm to decide whether two abbreviated (+)–canonical trees are flip equivalent or not.

The algorithm is based on the following observation: Applying one of the flips (F_1), (F_2), (F_3') or (F_3'') to an abbreviated (+)–canonical tree Γ gives another abbreviated tree Γ' *together with a preferred isomorphism* $\varphi : \Gamma_c \to \Gamma_c'$ between the combinatorial trees Γ_c and Γ_c' respectively underlying Γ and Γ'. Similarly for a weighted planar tree isomorphism. Let us call such a φ ***an elementary degree*** $+1$ ***isogeny from*** Γ to Γ'.

Now, consider two abbreviated (+)–canonical trees Γ and Γ', together with their combinatorial trees Γ_c and Γ_c'. By the above observation, Γ and Γ' are flip equivalent if and only if there exists a combinatorial isomorphism $\varphi : \Gamma_c \to \Gamma_c'$ which can be decomposed as a product of elementary degree $+1$ isogenies. Namely, if there exists a sequence of abbreviated trees $\Gamma = \Gamma^{(0)}$, $\Gamma^{(1)}, \ldots$, $\Gamma^{(n)} = \Gamma'$ such that each $\Gamma^{(i+1)}$ is obtained by application of a flip to $\Gamma^{(i)}$ and such that φ is equal to the product $\varphi_n \ldots \varphi_1$ of the corresponding elementary degree $+1$ isogenies $\varphi_i : \Gamma_c^{(i-1)} \to \Gamma_c^{(i)}$. An isomorphism $\varphi : \Gamma_c \to \Gamma'_c$ which can be so decomposed as a product of elementary degree $+1$ isogenies will be called a ***degree*** $+1$ ***isogeny from*** Γ ***to*** Γ'. Note that a degree $+1$ isogeny respects weights and ring numbers. At each vertex, a degree $+1$ isogeny also respects, up to orientation reversal, the cyclic ordering of bonds specified by Γ and Γ'; we will henceforth call such a cyclic ordering defined up to orientation reversal a ***dihedral ordering***.

We will introduce a notion of degree -1 isogeny later in Chapter 16. These are also combinatorial isomorphism, but usually do not respect the weights. In this section, we will often say "isogeny" for short instead of "degree +1 isogeny", since there is no ambiguity so far.

With the above data, the problem of deciding whether Γ and Γ' are flip equivalent can now be split into two steps:

(a) List all isomorphisms $\varphi : \Gamma_c \to \Gamma'_c$ which respect weights, ring numbers and dihedral orderings of bonds.
(b) Given such a $\varphi : \Gamma_c \to \Gamma'_c$, decide whether it is a (degree $+1$) isogeny or not.

The first step (a) is easily accomplished, at least when the sizes of Γ_0 and Γ'_0 are moderate. We consequently emphasise the second problem (b).

An initial remark is that, once we are given the isomorphism $\varphi : \Gamma_c \to \Gamma'_c$, Problem (b) is independent of ring numbers. We can therefore forget all data concerning these ring numbers, or equivalently assume that all of them are 0.

Let us state our algorithm to solve Problem (b). We are given two (+)–canonical trees Γ and Γ' (without ring numbers, since we have decided to neglect these), and a combinatorial isomorphism $\varphi : \Gamma_c \to \Gamma'_c$ respecting weights and dihedral orderings of bonds. We want to decide whether or not φ is a (degree $+1$) isogeny.

Our strategy will be to progressively modify Γ' by flips, composing φ with the corresponding isogenies, so that φ respects the cyclic order of weights and bonds at as many angular sectors of Γ as possible. The reason why this strategy will eventually succeed in deciding whether φ is an isogeny or not stems from a "magical" fact, to be proved as Proposition 12.18 below, which can be paraphrased as follows: *When there is no obvious way to go one step further in this construction, there is actually a good reason for this in the sense that φ is not an isogeny.*

If all vertices of Γ have valence $\leqslant 2$, a succession of flips (F'_3) readily shows that φ is an isogeny. So, assume that there exists a vertex v_1 of valence $\geqslant 3$ in Γ.

Then, list the vertices of Γ as $v_1, v_2, \ldots, v_n$, in such a way that each v_i is adjacent to a v_j with $j < i$.

When Γ is of type – – –•– – , we require a further property for this indexing of the vertices of Γ. Define a ***cut*** of Γ as a subgraph – – – of Γ occurring for some decomposition of Γ as – – –•– – . We then require that v_j does not belong to any cut disjoint from $\{v_j ; j < i\}$ whenever this is possible, namely unless the complement of $\{v_j ; j < i\}$ is precisely a union of cuts.

ALGORITHM 12.17. Modifying Γ' by flips (F_1) and/or (F'_3) if necessary, we first make φ respect the cyclic order of weights and bonds at v_1.

Now, let $i \geqslant 2$ and assume as induction hypothesis that we have succeeded in making φ respect the cyclic order of weights and bonds at each v_j with $j < i$.

If $i = n + 1$, φ comes from a weighted planar tree isomorphism and is consequently an isogeny. Otherwise, there are two cases.

CASE 1 *The isomorphism φ respects the cyclic order of bonds at v_i.* (In particular, this automatically holds when v_i has valence $\leqslant 2$.)

Then, φ induces a unique correspondence between the angular sectors at v_i and those at $\varphi(v_i)$ that respects the cyclic order of these. (The order preserving condition is relevant only when v_i has valence 2.) Modifying Γ' by a flip (F_3') if necessary, we can now assume that this correspondence respects the weights of these angular sectors. Moreover, this flip (F_3') can be chosen so that all $\varphi(v_j)$ with $j < i$ lie in the subgraph of Γ' fixed by this move, so that the induction hypothesis is still satisfied; note that we used here the fact that the v_j with $j \leqslant i$ are the vertices of a subtree of Γ, which comes from our choice of the indexing of the vertices of Γ.

Now, φ respects the cyclic order of weights and bonds at all v_j with $j \leqslant i$. Go one step further in the induction.

CASE 2 *φ reverses the cyclic order of bonds at v_i.*

We are stuck. We then make another attempt, starting from a different Γ'.

If Γ' can be split as $- - - \bullet - -$, so that $\varphi(v_1)$ is in $- - -$ and $\varphi(v_i)$ is in $- -$, choose such a decomposition of Γ' so that the cut $- -$ is minimal for this property. Then, let $\widehat{\Gamma}'$ be obtained from Γ' by a flip (F_3'') reversing cyclic orders at the vertices of $- -$ located at odd distance from the vertex shown.

Otherwise, define $\widehat{\Gamma}'$ by performing on Γ' a flip (F_2) that respects the cyclic order at $\varphi(v_1)$.

In both cases, let $\widehat{\varphi} : \Gamma_c \to \widehat{\Gamma}'_c$ denote the composition of φ with the induced isogeny.

Again, we try to make $\widehat{\varphi}$ respect the cyclic order of weights and bonds at as many vertices as possible, by modifying $\widehat{\Gamma}'$ (and $\widehat{\varphi}$ accordingly) by a sequence of flips (F_3') keeping $\widehat{\varphi}(v_1)$ in the subgraph they fix. So, reapply the algorithm right from the beginning with $\widehat{\Gamma}'$ and $\widehat{\varphi}$ in place of Γ' and φ. This leads to two subcases:

SUBCASE 2A *After a certain number of modifications as in Case* 1*, $\widehat{\varphi}$ eventually respects the cyclic order of weights and bonds at all v_j with $j \leqslant i$.*

Then, replace Γ' by $\widehat{\Gamma}'$ and φ by $\widehat{\varphi}$, and go one step further in the induction.

SUBCASE 2B *The algorithm applied to $\widehat{\Gamma}'$ and $\widehat{\varphi}$ again leads to Case* 2 *for some v_j with $j \leqslant i$.*

We are again stuck. However, we will prove in Proposition 12.18 below that φ cannot be an isogeny in this case, and this concludes the algorithm.

PROPOSITION 12.18. *The above process is actually an algorithm to solve Problem* (b)*. Namely, φ cannot be an isogeny when Subcase* 2b *holds at some point.*

PROOF. First consider the simple case when Γ satisfies the following property: Any two free bonds of Γ which are attached to the same vertex are adjacent bonds of this vertex. Then, we do not have to worry about cuts when defining the v_i, and flips (F_3'') are irrelevant (they coincide with flips (F_2)). Note that Γ' also satisfies this property as φ respects dihedral orders.

Assume that we are in the situation of Subcase 2b, which we can summarise as follows. We have three weighted planar trees Γ, Γ', $\widehat{\Gamma}'$ and two combinatorial isomorphisms $\varphi : \Gamma_c \to \Gamma'_c$ and $\widehat{\varphi} : \Gamma_c \to \widehat{\Gamma}'_c$ such that:

(a) The composition $\widehat{\varphi}\varphi^{-1} : \Gamma'_c \to \widehat{\Gamma}'_c$ is an isogeny which can be decomposed as a product of elementary isogenies $\psi_n \dots \psi_1\psi_0$ such that: ψ_0 is induced by a flip (F_2) preserving order at the vertex $v'_1 = \varphi(v_1)$; and each ψ_k with $k > 0$ is induced by a flip (F'_3) (possibly trivial) such that $\psi_{k-1} \dots \psi_0(v'_0)$ is in the subgraph fixed by this flip.
(b) For some $i > 1$, φ respects the cyclic order of weights and bonds at each vertex v_k with $k < i$, and reverses the cyclic order of the bonds of v_i.
(c) For some $j > 1$, $\widehat{\varphi}$ respects the cyclic order of weights and bonds at each v_k with $k < j$, and reverses the order of the bonds of v_j.

Suppose now, in search of a contradiction, that φ is an isogeny, namely can be decomposed as a product of elementary isogenies induced by a series of flips. We will first simplify this series of flips by a suitable reordering.

Indeed, observe the following commutativity property of flips. Let Γ_1 be a weighted planar tree, and w_1 and w_2 be two of its vertices. Apply a flip (F'_3) at w_1 to get a weighted planar tree Γ_2, and let Γ_3 be obtained from Γ_2 by a flip (F'_3) applied at the image of w_2 by the corresponding elementary isogeny. If $w_1 \neq w_2$ then, up to weighted planar tree isomorphism, Γ_3 is also obtained by applying, first a flip (F'_3) to Γ_1 at w_2, then another flip (F'_3) to the tree so obtained at the vertex image of w_1 by the corresponding isogeny. When $w_1 = w_2$, note that Γ_3 is obtained from Γ_1 by a single flip (F'_3) at w_1, or by a flip (F_2), or by a weighted planar tree isomorphism.

One has a similar commutativity property for flips (F'_3) and flips (F_2). Observe also that the composition of two flips (F_2) amounts to a flip (F_1) or a weighted planar tree isomorphism.

Using these commutativity properties, φ can thus be decomposed as a product $\varphi_0\varphi_1 \dots \varphi_n$ of elementary isogenies φ_k such that: φ_0 either is a weighted planar tree isomorphism or is induced by a flip (F_1) or (F_2); for $k \geqslant 1$, φ_k either is a weighted planar tree isomorphism or is induced by a flip (F'_3) performed at the vertex $\varphi_{k+1} \dots \varphi_n(v_k)$, image of v_k by the previous elementary isogenies. Also, when φ_k is induced by a flip (F'_3), composing this flip (F'_3) with a suitable flip (F_2) and using the above commutativity properties, we can also require that the subgraph fixed by this flip contains the vertex $\varphi_{k+1} \dots \varphi_n(v_1)$; by our choice of indexing for the vertices of Γ, note that this fixed subgraph also contains all $\varphi_{k+1} \dots \varphi_n(v_s)$ with $s < k$.

This last condition implies in particular that the isogeny $\varphi_1 \dots \varphi_n$ respects the cyclic order of bonds at v_1. Since φ also respects this order, so does φ_0. Therefore, either φ_0 is a weighted planar tree isomorphism or it is induced by a flip (F_2) respecting the bond order at v_1. This alternative splits the argument into two cases.

CASE A φ_0 *is a weighted planar tree isomorphism.*

We first prove by induction that each φ_k with $k < i$ is a weighted planar tree isomorphism.

For this, assume as induction hypothesis that φ_s is a weighted planar tree isomorphism for all $s < k$. As the image of v_k is always in the subgraph fixed by each flip (F'_3) corresponding to a φ_t with $t > k$, the isogeny $\varphi_{k+1} \dots \varphi_n$ respects the cyclic order of weight and bonds at v_k. Since so does $\varphi = \varphi_0 \dots \varphi_{k-1}\varphi_k\varphi_{k+1} \dots \varphi_n$, it follows that φ_k respects the cyclic order of weight and bonds at $\varphi_{k+1} \dots \varphi_n(v_k)$, and therefore is not induced by a flip (F_3') performed at this vertex. Consequently, φ_k is a weighted planar tree isomorphism as announced.

Recall that $\varphi_{i+1} \ldots \varphi_n$ respects the cyclic order of bonds at v_i, since it is induced by a sequence of flips (F_3') fixing the successive images of this vertex. The isomorphism φ_i respects the cyclic order of bonds at $\varphi_{i+1} \ldots \varphi_n(v_i)$ as it is either a weighted planar tree isomorphism or induced by a flip (F_3') at this vertex. And the weighted planar tree isomorphism $\varphi_0 \ldots \varphi_{i-1}$ preserves the cyclic order of bonds at $\varphi_i \ldots \varphi_n(v_i)$. We now reach a contradiction with the hypothesis that $\varphi = \varphi_0 \ldots \varphi_n$ reverses the cyclic order of bonds at v_i.

Thus, Case A cannot hold.

Case B φ_0 *is induced by a flip* (F_2).

In this case, we consider $\widehat{\varphi}$ to get a contradiction. Indeed, using the form of the isogeny $\widehat{\varphi}\varphi^{-1}$ and the commutativity of flips, we readily see that $\widehat{\varphi}$ is induced by a series of flips (F_3') fixing the successive images of v_1. We then reach, as in Case A, a contradiction with the fact that $\widehat{\varphi}$ reverses the cyclic order of bonds at v_j. Therefore, Case B cannot hold either.

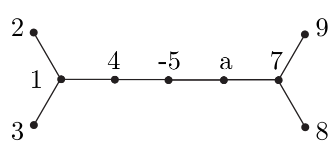
This proves that our assumption that φ is an isogeny is contradictory, and concludes the proof of Proposition 12.18 in the case when no vertex of Γ carries two non-adjacent free bonds.

It remains to consider the case when Γ can be split as $- - - \!\!\!\bullet\!\!\! - -$ so that the two cuts $- - -$ and $- -$ are non-trivial. In this case, let Γ_* be obtained from Γ by erasing all cuts which do not contain the vertex v_j occurring in Subcase 2.2. Let Γ_*' be the similarly defined weighted subtree of Γ', so that φ induces an isomorphism between the combinatorial trees underlying Γ_* and Γ'. Also, modifying Γ' by flips, we can assume without loss of generality that $\Gamma'' = \Gamma'$ and φ' is the identity.

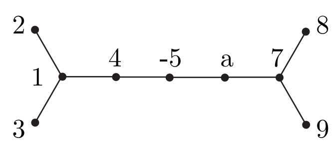
Observe that each flip performed on Γ' induces on Γ_*' a flip or a weighted planar tree isomorphism. For instance, a flip (F_3'') either fixes Γ_*' or induces a flip (F_2) on it. So, φ_* is an isogeny if φ is an isogeny. On the other hand, by our choice of vertex indexing, the vertices of Γ_* are the v_s with $k \leqslant s \leqslant \ell$ for some k, ℓ. Note that v_i and v_j both belong to Γ_*, and that φ respects the cyclic ordering of weights and bonds at v_k. It easily follows that the restriction to Γ_* of our algorithm to decide whether φ is an isogeny is precisely the algorithm to decide whether φ_* is an isogeny (starting from v_k. Since we reached Subcase 2.2 for both φ and φ_*, our analysis of the first case shows that φ_* is not an isogeny. Therefore, φ cannot be an isogeny.

This concludes the proof of Proposition 12.18. □

Let us illustrate this algorithm by an example. Consider the two weighted planar trees

$\Gamma =$ 

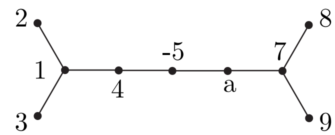
and

$\Gamma' =$ 

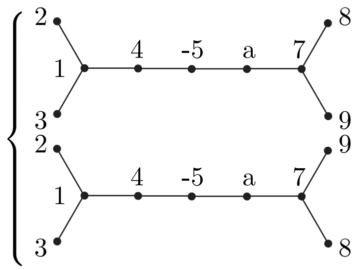
We want to know whether the obvious isomorphism $\varphi \colon \Gamma_c \to \Gamma_c'$ respecting weights is an isogeny.

Label the vertices of Γ as $v_1, \dots, v_9$ so that their weights are respectively 1, 2, 3, 4, -5, a, 7, 8, 9. Clearly, φ respects weights and cyclic orders at all v_j with $j < 7$, and reverses the order of the bonds at v_7. Thus, we reach Case 2 of the algorithm with $i = 7$.

At this point, following the algorithm, we apply a flip (F_2) to get

$\widehat{\Gamma}' =$ 

and an isomorphism $\widehat{\varphi} : \Gamma_c \to \widehat{\Gamma}'_c$ respecting weights. Applying the algorithm, we modify $\widehat{\Gamma}'$ by a flip (F'_3) fixing $\widehat{\varphi}(v_1)$ applied first at $\widehat{\varphi}(v_4)$, then at $\widehat{\varphi}(v_6)$, to get

$\widehat{\Gamma}'' =$ 

if a is even

if a is odd.

and a combinatorial isomorphism $\widehat{\varphi}' : \Gamma_c \to \widehat{\Gamma}''_c$

If a is odd, $\widehat{\varphi}'$ respects cyclic orderings and φ: $\Gamma_c \to \Gamma'_c$ is an isogeny.

If a is even, we are in Subcase 2B of the algorithm, and φ is not an isogeny.

12.7. The plumbing calculus of arborescent tangles

Having stated our classification of pairwise irreducible arborescent pairs in terms of weighted planar trees (to be proved in Chapter 13), we develop in this section a similar calculus for arborescent tangles.

Indeed, the plumbing construction associates a tangle to each weighted planar tree Γ, and we have actually observed in §12.3 how the moves of the calculus of arborescent pairs affect this tangle. Except for the boundary moves, these moves modify the boundary parametrisations by no more than composition with four elements of the Klein Viergruppe V_4 of slope preserving automorphisms of the standard Conway sphere (S^2, P^0). Recall that V_4 consists of the identity ι and of the π−rotations ξ, η, ζ around the x−, y− and z−axes, respectively (see Figure 12.4).

To take these boundary parametrisations into account, it is convenient to slightly augment the data involved in a weighted planar tree. We allow ourselves to label the tip of any free bond —• − − − of a weighted planar tree with any element $\gamma \in V_4$, thus γ—• − − −; this, we decide, causes the parametrisation of the corresponding Conway sphere boundary to be altered by γ. Labelling by the identity ι is equivalent to not labelling at all. We can now operate on a labelled free bond γ—• − − − by any $\alpha \in V_4$, getting $\alpha\gamma$—• − − − (recall that V_4 is abelian).

12.7.1. Moves for arborescent tangles. We can now give a list of moves which do not alter the associated tangle. Maintaining the nomenclature of the above calculus for arborescent pairs, we indicate (using italics) only the changes required in it. A weighted planar tree with free bonds labelled by elements of V_4 as explained above is given.

Flips.

(F_1) Reverse the cyclic order of bonds at every vertex of the tree and *apply ζ to every free bond.*

(F_2) Reverse the cyclic order at one vertex on the tree and at every vertex at even distance from it; also *apply ξ to every free bond of a vertex at even (or zero) distance, and apply η to every free bond at odd distance.*

(F_3) Replace $- - - \overset{a}{\underset{a'}{\bullet}}\!\!-\; - - -$ by $- - \overset{a \pm 1}{\underset{a' \mp 1}{\bullet}}\!\!-\; - -$, where the cyclic order of bonds and eights is reversed at all vertices lying to the right of the vertex shown, and at odd distance from this vertex. *Also, apply ξ to all free bonds in $- -$ that are attached to a vertex at even distance from the vertex shown, and η to those at odd distance.*

Ring move (R). Unchanged, as well as the modified ring moves ($R^{\pm}$).
Boundary moves. Omitted!
Arithmetic moves. Unchanged, except for

(0.2) Replace $- - - \overset{a}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\underset{b}{\bullet} - -$ by $- - \overset{a+b}{\bullet} - -$, where the cyclic orders of bonds are reversed in the right hand side subgraph $- -$ *and where ζ is applied to every free bond there*, while nothing is changed in the left hand side subgraph $- - -$.

12.7.2. Canonical trees for arborescent tangles. As for the calculus of arborescent pairs, we now define for weighted planar trees with free bonds labelled by elements of V_4 a notion of canonicity with respect to the calculus of arborescent tangles. We are content to only point out the differences with the calculus of arborescent pairs.

Canonicity is defined by three conditions (W), (S), and (P) or (N).

The Weight Condition (W) is unchanged; but the Free Bond Condition (F) is absent since the present calculus has no boundary moves.

The Stick and Positivity Conditions (S) and (P) change somewhat because there now may exist sticks having a bond that is free in Γ. They become:

(S) On any stick the weights are non-zero except for end vertices that have a bond free in Γ_0 and for the cases $\Gamma_0 = \overset{0}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$, $\Gamma_0 = \overset{0}{\bullet}$. The non-zero weights along any stick are of alternating sign. No end vertex of a stick has weight ± 1 unless it has a bond free in Γ_0, or $\Gamma_0 = \overset{\pm 1}{\bullet}$.

(P) There are no sticks in Γ_0 of the form $\overset{-1}{\bullet}$, $\overset{-2}{\bullet}$, $\overset{-2}{\bullet}\!\!-$ or $-\!\!\overset{-2}{\bullet}\!\!-$ except ones having a bond that is free in Γ_0.

The negativity condition (N) is similarly defined.

By the same argument as for Theorem 12.9 and Corollary 12.10, one readily proves:

THEOREM 12.19. *There exists an effective algorithm which, for any weighted planar tree Γ with free bonds labelled by elements of V_4, alters Γ by a sequence of moves of the calculus of arborescent tangles to produce a collection of positively (or negatively) canonical weighted planar trees.* □

COROLLARY 12.20. *Every arborescent tangle is obtained by pairwise connected sum operations from arborescent tangles associated to positively (or negatively) canonical weighted planar trees (with labels in V_4 at free bonds).* □

12.7.3. Modified flips for arborescent tangles. As in the calculus of arborescent pairs, the Weight Condition (W) is not respected by Flip (F_3). We consequently replace it by

(F_3') Replace $- - - \overset{a}{\bullet} - -$ by $- - - \underset{a}{\bullet} - -$ where, if a is odd, the cyclic order of bonds and weights is reversed at all vertices of the subtree $- -$ lying at odd distance from the marked vertex. *Moreover, still when a is odd, the free bonds in $- -$ have their label in V_4 modified by ξ for those attached to a vertex at even distance, and by η for those at odd distance.*

Since no boundary moves are allowed, flip (F_3'') disappears.

We can now state our main classification theorem of arborescent tangles, which is exactly the counterpart of Theorem 12.12 and will be proved in §13.4.

THEOREM 12.21 (Classification Theorem for arborescent tangles). *Consider two positively (or negatively) canonical weighted planar trees Γ and Γ', with free bonds labelled by elements of V_4. Plumbing according to Γ and Γ' gives isomorphic arborescent tangles if and only if Γ and Γ' can be deduced from each other by a sequence of flips* (F_1)*,* (F_2)*,* (F_3') *and of modified ring moves* (R^+) *(or* (R^-)*).*

To make this theorem more practical, we can introduce abbreviated trees in exactly the same manner as in §12.3, in such a way that two canonical weighted planar trees (with labels in V_4) are equivalent by flips (F_1), (F_2), (F_3') and modified ring moves ($R^{\pm}$) if and only if their abbreviated trees are equivalent by flips (F_1), (F_2), (F_3').

The algorithm of §12.5 easily extends to this context, and is even somewhat simpler. To describe it, consider two abbreviated canonical trees Γ and Γ' (with labels in V_4) and an isomorphism $\varphi : \Gamma_c \to \Gamma'_c$ between their underlying combinatorial trees which respects weights, ring numbers and bond dihedral orderings (but not necessarily labels in V_4). We want to decide whether φ is an isogeny.

If Γ has no free bond, the data and flip equivalences are exactly the same as for the calculus of arborescent pairs. Thus we can use in this case the algorithm of §12.5.

If Γ has some free bond, label its vertices as $v_1, v_2, \ldots, v_n$ so that v_1 bears this free bond f and each v_{i+1} is adjacent to v_i. Modifying Γ' by suitable flips (F_1) or (F_2), we can assume that the label (in V_4) of $\varphi(f)$ in Γ' is the same as the label of f in Γ. As in §12.5, progressively modify Γ' by a succession of flips (F_3') of the calculus of arborescent tangles, so as to make φ respect the cyclic order of weights and bonds at as many v_i as possible. Further, choose each of these flips(F_3'), so that the free bond $\varphi(f)$ is in the subgraph fixed by this flip; observe in particular that the label of $\varphi(f)$ is then unchanged by these flips.

After a finite number of such modifications, either we succeed in making φ an isogeny induced by an isomorphism of weighted planar trees (with labels in V_4, or we reach a situation where the hypotheses of Proposition 12.22 below are satisfied, in which case this statement proves that φ is not an isogeny. This clearly completes the extended algorithm.

PROPOSITION 12.22. *With the above data, assume that there is an $i \geqslant 1$ such that φ respects the cyclic orders of weight and bonds at each vertex v_j with $j < i$, and that the labels of the free bond f in Γ and of $\varphi(f)$ in Γ' are identical. Assume moreover that:*

(a) *either φ reverses the cyclic order of bonds at v_i;*

(b) *or* φ *does not respect the label in* V_4 *of some free bond of a vertex* v_j *with* $1 \leqslant j < i$.

Then φ *is not an isogeny.*

PROOF. Similar to that of Proposition 12.18. Exercise. □

12.8. The calculus of marked arborescent pairs

A marking is something weaker than a tangle structure: Each Conway sphere parametrisation is replaced by an essential circle (up to isotopy). Why need we consider them at all?

For a knot (S^3, K) which is simple for Schubert, we defined in Chapters 3 and 7 an arborescent part $(A, K \cap A)$ which is well-defined up to pairwise isotopy, and is thus a geometric invariant of this knot. Very often, it is possible to sharpen this invariant by singling out some extra structure on the boundary of this arborescent part. In most cases, we will have to be content with data weaker than a tangle structure, called boundary markings. We will indicate in Chapter 17 several standard ways in which one can select such preferred boundary markings for the arborescent part of a knot.

We add that markings again prove more suitable than tangle structures in the study of knot automorphisms in Chapters 15–18.

We now formulate the classification of marked arborescent pairs in terms of weighted planar trees; the proof of the corresponding classification theorem is deferred to §13.3.

Consider a Conway sphere (S, P); namely, P consists of 4 distinct points of the 2–sphere S. By definition, a ***marking*** of (S, P) is a pairwise isotopy class of 1–submanifolds k of S made of two disjoint arcs with $\partial k = P$. Equivalently, a marking can be defined as a pairwise isotopy class of connected curves in $S - P$ separating P into twice two points. Indeed one easily goes from one definition to the other by considering a closed curve separating the two arcs; compare Figure 9.2.

When (S, P) is identified to the standard Conway sphere (S^2, P^0), we saw in §9A that a marking has a well-defined *slope* which is an element of $\mathbb{Q} \cup \infty$. Moreover, two distinct markings have different slopes, and every element of $\mathbb{Q} \cup \infty$ is the slope of some marking. Thus, markings of $(S, P) \cong (S^2, P^0)$ are in one-to-one correspondence with their slopes in $\mathbb{Q} \cup \infty$.

We define a ***marked knot pair*** as the data of a knot pair (M, K) together with a marking C_i on each of its boundary component S_i. We usually denote such a marked arborescent pair by $(M, K; C)$ when $C = \bigcup C_i$.

A tangle naturally defines a marked knot pair by considering its underlying knot pair together with, on each boundary component, the marking that has slope 0 for the identification of this boundary component with the standard Conway sphere (S^2, P^0) specified by the tangle structure. In particular, this enables us to associate a marked arborescent pair to each planar tree with integral weights in its angular sectors, via plumbing construction as in §12.1.

12.8.1. Moves for marked arborescent pairs. In Complement 12.7, we observed that most moves of the calculus of arborescent pairs induce pair isomorphisms which are slope preserving on the boundary, for the tangle structure inherited from plumbing construction. In particular, these moves do not affect the marked arborescent pairs associated to the corresponding weighted planar trees.

For convenience, we recall here the list of these moves, referring to §12.3 for their precise description.
Flips (F_1), (F_2), and (F_3).
Ring move (R) (or modified ring moves (R^+) and(R^-)).
Arithmetic moves (0.1), (0.2), (1.0), (1.1), (1.2), (2.0), (2.1), (2.2), and (0.1*).

On the other hand, the boundary moves of §12.3 do change the markings of the corresponding arborescent pairs. In the plumbing calculus of marked arborescent pairs, they are replaced by the following:
Boundary moves

(B_1) Replace $- - - \bullet\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-$ by $- - - \bullet\!\!-$.
(B_2) Replace $- - - \overset{a}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ by $- - - \overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$.

The fact that these two boundary moves do not change the associated marked arborescent pairs is still a consequence of the pair isomorphism of Figure 12.19. (Also note that (B_1) is a consequence of (B_2) and (0.2).)

12.8.2. Canonical trees for marked arborescent pairs. In the calculus of marked arborescent pairs, canonicity for a weighted planar tree Γ is to be defined by four conditions (W), (S), (F), and (P) or (N).

The Weight Condition (W) is the same as in the calculus of arborescent pairs, namely that at most one weight is non-zero at each vertex.

The Stick Condition (S) is slightly different, because of the new form of boundary moves. Recall that Γ_0 is obtained from Γ by forgetting its embedding in the plane.

(S) On any stick of Γ_0, the weights are non-zero and of alternating signs, except for sticks $-\!\!\overset{0}{\bullet}\!\!-$ which have one bond free in Γ_0, and unless Γ_0 is $\overset{0}{\bullet}$, $\overset{0}{\bullet}\!\!-$, $\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-$, or $-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-$. No end vertex of a stick has weight ± 1 unless Γ is $\overset{\pm 1}{\bullet}$.

The Free Bond Condition (F) is also modified.

(F) Any vertex which is adjacent to a stick $-\!\!\overset{0}{\bullet}\!\!-$ (whose other bond is free in Γ_0) has weight 0.

The Positive Canonicity Condition (P) for (+)–canonical trees, or the Negative Canonicity Condition (N) for (−)–canonical trees, are the same as for the calculus of unmarked arborescent pairs.

THEOREM 12.23. *There exists an effective algorithm which, for any weighted planar tree Γ, alters Γ by a sequence of moves of the calculus of marked arborescent pairs to produce a collection of weighted planar trees that are positively (or negatively) canonical for this calculus.*

PROOF. The algorithm is approximately the same as that for the calculus of unmarked arborescent pairs. One should just add the following modifications:

$- - - \overset{a}{\bullet}\!\!-\!\!\overset{\pm 1}{\bullet}\!\!-$ becomes $- - \overset{a \mp 1}{\bullet}\!\!-\!\!\underset{\mp 1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-$ by the inverse of (1.2), and then $- - \overset{a \mp 1}{\bullet}\!\!-$ by the boundary move (B_1).

Replace $\overset{\pm1}{\bullet}$ by $\overset{0}{\bullet}\!-\!\underset{\mp1}{\bullet}\!-\!\overset{\pm1}{\bullet}\!-\!\underset{\mp1}{\bullet}\!-\!\overset{0}{\bullet}$ by the inverse of the boundary move (B_1) applied twice, then by $\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ by an application of (1.2), and finally by $\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ using (0.2).

Turn $\overset{\pm1}{\bullet}$ into $\overset{\pm1}{\bullet}\!-\!\overset{\mp1}{\bullet}\!-\!\overset{0}{\bullet}$ by the inverse of (B_1), and then to $\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ by (1.1).

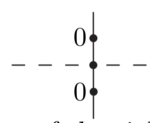
Replace $- - -\overset{a}{\bullet}\!-\!\overset{\pm2}{\bullet}$ by $- - -\overset{a}{\bullet}\!-\!\overset{\pm2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ by the inverse of (B_1), then by $- - -\overset{a\mp1}{\bullet}\!-\!\overset{\mp2}{\bullet}\!-\!\overset{\mp1}{\bullet}\!-\!\overset{0}{\bullet}$ using (2.2), and eventually by $- - -\overset{a\mp1}{\bullet}\!-\!\overset{\mp2}{\bullet}$ with (B_1).

Finally, $\overset{\pm2}{\bullet}$ becomes $\overset{\pm2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ using the inverse of (B_1), then $\overset{\mp2}{\bullet}\!-\!\overset{\mp1}{\bullet}\!-\!\overset{0}{\bullet}$ by (2.1), and $\overset{\mp2}{\bullet}$ by (B_1). □

COROLLARY 12.24. *Every marked arborescent pair is obtained by pairwise connected sum operations from marked arborescent pairs associated to weighted planar trees that are positively (or negatively) canonical for the calculus of marked arborescent pairs.* □

12.8.3. Modified flips for marked arborescent pairs. Recall that flips are moves preserving canonicity. The flips for arborescent pairs must be modified for marked arborescent pairs.

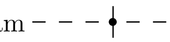
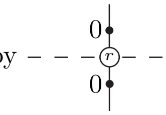
Flip (F_3) must be modified to respect the Weight Condition (W), and is replaced by two flips (F_3') and (F_3'). Flip (F_3') is the same as is the calculus of (unmarked) arborescent pairs. Flip (F_3'') is different, owing to the new form of boundary moves.

(F_3'') If Γ can be split as  , reverse the cyclic order of bonds and weights at all vertices of the right hand side subgraph – – lying at odd distance from the one of the 3 vertices shown that is adjacent to this subgraph.

We can now state our classification theorem of pairwise irreducible marked arborescent pairs, which will be proved in §13.3.

THEOREM 12.25 (Classification Theorem for marked arborescent pairs). *Two marked arborescent pairs obtained by plumbing according to two positively (or negatively) canonical weighted planar trees* Γ *and* Γ′ *are isomorphic if and only if* Γ *and* Γ′ *can be deduced from each other by a sequence of flips* (F_1)*,* (F_2)*,* (F_3')*,* (F_3'') *and of modified ring moves* (R^+) *(or* (R^-)*).*

For practical applications of this result, it is necessary to have an efficient method to detect when two canonical weighted planar trees are flip equivalent. The method we have developed for unmarked arborescent pairs, by the introduction of abbreviated trees in §12.5 and by the algorithm of §12.6, straightforwardly extends to the framework of marked pairs. The abbreviated (+)–canonical trees for this calculus of marked arborescent pairs are weighted planar trees with ring numbers, which satisfy the above Conditions (W), (S), (F), (P) (defining sticks as in §12.5), together with the same Abbreviation Condition (A) as for unmarked pairs in §12.5.

To adapt the algorithm of §12.6, one just needs to replace any diagram  by 

because of the new form of Flip (F_3''). Beware however that ring numbers are not completely negligible in this case, as Flip (F_3'') takes them into account.

CHAPTER 13

Classification of arborescent knots and pairs (the proofs)

This chapter is devoted to proving the main classification theorems stated in Chapter 12, which assert that pairwise irreducible pairs (or tangles, or marked pairs) are classified by certain canonical weighted planar trees, modulo flip equivalences. After the work accomplished in Chapters 9, 10 and 11, what remains of the proof is now combinatorial in nature and involves almost no fresh geometry.

The success of the plumbing calculus can be attributed to Lemma 13.1, below, together with elementary facts about continued fractions. Recall that a ***braid tangle*** $(M, K; \theta)$ is one such that the knot pair (M, K) is pair isomorphic to the thickened Conway sphere $(S^2, P^0) \times [0, 1]$. This product structure defines, up to pairwise isotopy, a preferred degree -1 identification $(S_+, S_+ \cap K) \to (S_-, S_- \cap K)$ between the two boundary components S_+ and S_- of M. Through the parametrisation θ by the standard Conway sphere (S^2, P^0), it therefore defines a degree -1 element of $\pi_0 \operatorname{Aut}(S^2, P^0)$, whose action on slopes is described by a determinant -1 element of $\mathrm{PGL}_2(\mathbb{Z})$, according to Corollary 11.4. Presently, Lemma 13.1 will relate this element of $\mathrm{PGL}_2(\mathbb{Z})$ to the plumbing construction of Chapter 12.

Before stating this result, we introduce some notation. Let $[a_1, \ldots, a_n]$ denote the continued fraction

$$[a_1, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}$$

and let $[[a_1, \ldots, a_n]]$ denote the alternating continued fraction

$$[[a_1, \ldots, a_n]] = [a_1, -a_2, \ldots, (-1)^{n-1} a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}.$$

LEMMA 13.1. *Let $(M, K; \theta)$ be the arborescent braid tangle associated by plumbing construction to the linear weighted planar tree* $\overset{a_1'}{\underset{a_1''}{\bullet}} \; \overset{a_2'}{\underset{a_2''}{\bullet}} \cdots \overset{a_n'}{\underset{a_n''}{\bullet}}$ *, and let S_- and S_+ be the two components of ∂M respectively corresponding to the left and right free bonds of the above linear tree. Then the pair (M, K) admits a product structure inducing an identification $(S_+, S_+ \cap K) \to (S_-, S_- \cap K)$ whose action on the slopes is described by*

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z})$$

where, if $a_i = a'_i + a''_i$ denotes the total weight of the i–th vertex,

$$\begin{aligned} D_n/C_n &= [[a_n, a_{n-1}, \ldots, a_1]] \\ A_n/C_n &= -[[a_1, a_2, \ldots, a_n]] \\ A_n D_n - B_n C_n &= -1. \end{aligned}$$

Note that C_n/A_n is precisely the slope image of the slope 0, and that the above relations completely determine the element of $\mathrm{PGL}_2(\mathbb{Z})$ considered.

Proof. One proceeds by induction on n. When $n = 1$, one readily sees on the model 2–valent atomic tangle that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ a_1 & 1 \end{pmatrix}.$$

In general, composing plumbing maps and identifications corresponding to each building block, one gets

$$\begin{aligned} \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ a_n & 1 \end{pmatrix} \\ &= \begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

In particular, by induction,

$$D_n/C_n = (a_n - D_{n-1}/C_{n-1})^{-1} = [[a_n, a_{n-1}, \ldots, a_1]].$$

On the other hand, applying this formula to

$$\begin{aligned} \begin{pmatrix} -D_n & B_n \\ C_n & -A_n \end{pmatrix} &= \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 0 \\ a_n & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ a_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ a_1 & 1 \end{pmatrix} \end{aligned}$$

immediately yields $A_n/C_n = -[[a_1, a_2, \ldots, a_n]]$ as claimed. □

The second chief ingredient of our proof of Theorems 12.12, 12.23 and 12.25 is the following immediate consequence of the uniqueness of decompositions in continued fractions (see for instance [**HarW**]):

Lemma 13.2. *Let a_1, a_2, …, a_n be a sequence of non-zero integers of alternating signs, such that a_1 and a_n are both distinct from ± 1. Then the rational number $[[a_1, \ldots, a_n]]$ is not zero, has absolute value $\leqslant \frac{1}{2}$, and completely determines the sequence a_1, a_2, …, a_n.*

Conversely, every non-zero rational number in the interval $[-\frac{1}{2}, +\frac{1}{2}]$ is equal to $[[a_1, \ldots, a_n]]$ for some (unique) sequence of non-zero integers a_1, …, a_n of alternating signs, with a_1, $a_n \neq \pm 1$. □

13.1. Rephrasing the classification of Montesinos pairs

As a first step towards our classification of arborescent pairs, we will rephrase in terms of the plumbing calculus the classification of pairwise irreducible Montesinos pairs which we gave in Chapter 11, adopting the language of abbreviated trees developed in §12.5. Thus we will prove the restricted version of Theorem 12.12 for Montesinos pairs.

Consider an abbreviated $(+)$–canonical tree Γ for the calculus of arborescent pairs, as defined in §12.5. Such a tree is ***stellar*** when at most one of its vertices is essential, namely has valence $\geqslant 3$ or ring number $\geqslant 1$. In other words, Γ is stellar when it is star-shaped and when only its central vertex can have ring number $\geqslant 1$.

Given such a stellar (abbreviated $(+)$–canonical) tree Γ, we will associate to Γ a fractional data vector analogous to the fractional data vectors used in Chapter 11 to classify Montesinos pairs. It will turn out that the pair (M, K) obtained by plumbing according to Γ is a Montesinos pair, whose associated fractional data vector in the sense of Theorem 11.7 is precisely the one just defined for Γ (up to a few details in some special cases, owing to the respective conventions adopted in Chapters 11 and 12).

Remember that we decided to denote by solid black dots (without any indication of ring numbers) the vertices of Γ which have ring number 0, and by hollow small dots those with ring number 1. We will adopt the same convention for the weighted tree Γ_0 obtained from Γ by forgetting the embedding in the plane; this Γ_0 consists of the combinatorial tree underlying Γ together with, attached to each vertex, the ring number and (total) weight of this vertex.

First consider the case when Γ_0 is not a stick $\overset{a_1}{\bullet}\!\!-\!\!\!-\!\!\overset{a_2}{\bullet}\!\!-\; \cdots \;-\!\!\overset{a_n}{\bullet}$, so that Γ_0 (and Γ) has a well-defined "central" vertex: This ***central vertex*** is by definition the unique one with valence $\geqslant 3$ or ring number $\geqslant 1$, except in the degenerate cases when Γ is $-\!\!\overset{0}{\bullet}\!\!-$, $\overset{0}{\bullet}\!\!-$ or $\overset{0}{\bullet}$, where the central vertex is of course the only existing vertex. Index the bonds around this central vertex by $i = 1, \ldots, k$ in the cyclic order prescribed by Γ. Then the ***fractional data vector*** $(r; e_0; \bar{m}_1, \ldots, \bar{m}_k)$ associated to Γ equipped with this bond indexing is defined as follows:

(1) The number $r \in \mathbb{N}$ is the ring number of the central vertex.
(2) If the i–th bond of the central vertex is a free bond of Γ, $\bar{m}_i$ is the symbol $\varnothing$. Otherwise, this i–th bond is adjacent to a stick $-\!\!\overset{a_1}{\bullet}\!\!-\!\!\!-\!\!\overset{a_2}{\bullet}\!\!-\; \cdots \;-\!\!\overset{a_n}{\bullet}$ of Γ_0, and $\bar{m}_i \in \mathbb{Q}/\mathbb{Z}$ is defined as the mod 1 reduction of the alternating continued fraction $m_i = [[a_1, a_2, \ldots, a_n]]$; note that this makes sense as $m_i \neq \infty$ by Lemma 13.2 and by the Stick Condition (S) imposed to Γ (see §12.5).
(3) The data e_0 is the symbol $\varnothing$ when Γ has a free bond, namely when some $\bar{m}_i$ is $\varnothing$. Otherwise e_0 is the rational number $e - \sum_i m_i$ where $e \in \mathbb{Z}$ is the weight of the central vertex.

When Γ is linear, namely when Γ_0 is a stick $\overset{a_1}{\bullet}\!\!-\!\!\!-\!\!\overset{a_2}{\bullet}\!\!-\; \cdots \;-\!\!\overset{a_n}{\bullet}$ with $n \geqslant 2$, one similarly defines a fractional data vector by making one of the two end vertices play the role of the central vertex in the previous construction. In other words, when choosing the left end vertex as central vertex, the associated fractional data vector is $(0; e_0; -e_0)$ where $e_0 = a_1 - [[a_2, \ldots, a_n]] = [[a_1, \ldots, a_n]]^{-1}$ and $-e_0 \in \mathbb{Q}/\mathbb{Z}$ is the mod $\mathbb{Z}$ class of e_0. Of course, this fractional data vector depends on the end vertex chosen to be "central".

We claim that this fractional data vector characterises Γ modulo flip equivalences, namely modulo the equivalence relation generated by the Flips (F_1), (F_2), (F'_3), (F''_3) of the calculus described in §12.5.

LEMMA 13.3. *Given a stellar abbreviated $(+)$–canonical tree Γ for the calculus of arborescent pairs, the flip equivalence class of Γ is characterised by the fractional data $(r; e_0; \bar{m}_1, \ldots, \bar{m}_k)$ associated to Γ as above, taken up to dihedral permutation* *of $\bar{m}_1, \ldots, \bar{m}_k$, together with the exchange:*

$$(0; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}) \longleftrightarrow (0; -\tfrac{\beta}{\alpha'}; \overline{\tfrac{\beta}{\alpha'}}) \text{ when } \alpha\alpha' \equiv 1 \text{ mod } \beta.$$

Proof. As Γ is stellar, it is immediate that its flip equivalence class is characterised by:

(i) the weighted tree Γ_0 obtained by forgetting the embedding in the plane;
(ii) when Γ has a well-defined central vertex, the cyclic ordering of the bonds of this vertex, taken modulo orientation reversal.

To prove Lemma 13.3, first consider the case when Γ has a well-defined central vertex. If $m_i = [[a_1, a_2, \ldots, a_n]]$ is the rational number associated to a stick $\overset{a_1}{-\bullet}\overset{a_2}{-\bullet} \cdots \overset{a_n}{-\bullet}$ of Γ_0, it follows from Lemma 13.2, together with the Stick Condition (S) and the Positivity Condition (P) of §12.5 that m_i is in the interval $\left]-\frac{1}{2}, \frac{1}{2}\right]$. Thus, m_i is completely determined by its class $\bar{m}_i \in \mathbb{Q}/\mathbb{Z}$, and the weights a_1, a_2, $\ldots$, a_n can therefore be recovered from $-m_i \in \mathbb{Q}/\mathbb{Z}$ by Lemma 13.2 (and the Stick Condition). Noting that the central weight $e \in \mathbb{Z}$ is determined by the property that $e = 0$ when $e_0 = \varnothing$, and $e = e_0 + \sum_i m_i$ otherwise, Lemma 13.3 immediately follows in this case.

When Γ has no well-defined central vertex, namely when Γ_0 is a stick $\overset{a_1}{\bullet}\overset{a_2}{-\bullet-} \cdots \overset{a_n}{-\bullet}$, the two choices of end vertices yield two fractional data vectors $(0; -\frac{\beta}{\alpha}; \overline{\frac{\beta}{\alpha}})$ and $(0; -\frac{\beta}{\alpha'}; \overline{\frac{\beta'}{\alpha'}})$ where $\frac{\beta}{\alpha} = -[[a_1, a_2, \ldots, a_n]]^{-1}$ and $\frac{\beta'}{\alpha'} = -[[a_n, a_{n-1}, \ldots, a_1]]^{-1}$. By Lemma 13.2, the Stick Condition (S) and the Positivity Condition (P), both $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ are in the interval $\left]-\frac{1}{2}, +\frac{1}{2}\right]$. Also, by elementary properties on continued fractions (see Lemma 13.1), one has $\beta = \beta'$ and $\alpha\alpha' \equiv 1$ mod β. Thus the exchange appearing in the statement of Lemma 13.3 precisely reflects this ambiguity of choosing an end vertex of Γ to define its fractional data vector. The proof of Lemma 13.3 in the case considered now follows immediately from Lemma 13.2, using the Stick Condition. □

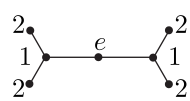
What are the fractional data vectors $(r; e_0; \bar{m}_1, \ldots, \bar{m}_k)$ which can actually occur for a stellar (+)–canonical tree Γ? By inspection, one readily sees that the precise conditions are:

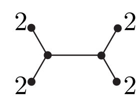
(i) $r \in \mathbb{N}$.
(ii) Each $\bar{m}_i$ is in $(\mathbb{Q}/\mathbb{Z} - 0) \cup \varnothing$.
(ii) $e_0 = \varnothing$ when some $\bar{m}_i$ is $\varnothing$; otherwise, the class of e_0 in $\mathbb{Q}/\mathbb{Z}$ equals $-\sum \bar{m}_i$.
(iv) The following fractional data vectors are excluded:
  (a) $(0; e_0; \bar{m}_1, \bar{m}_2)$ with $\bar{m}_1 \neq \varnothing$;
  (b) $(0; e_0; -e_0)$ with $e_0 \in \mathbb{Z}$ or $e_0 \in \mathbb{Q} \cap [-2, +2]$;
  (c) $(0; -1; \ )$ and $(0; -2; \ )$;
  (d) $(1; e_0; \bar{m}_1)$, $(1; e_0; \ )$, $(2; e_0; \ )$.

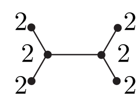
Only the exclusions in (iv) deserve some explanation: (a) is excluded because of the Free Bond Condition (when $\bar{m}_2 = \varnothing$) and because we decided to choose an extremity as the central vertex for linear trees. By construction, a data vector $(0; e_0, -e_0)$ with $e_0 \in \mathbb{Q}$ can only come from a linear tree with $\geqslant 2$ vertices; the exclusion of (b) then immediately follows from Lemma 13.2 and the Stick Condition (S). The exclusions (c) and (d) respectively reflect the Positivity Condition (P) and the Abbreviation Condition (A) imposed on abbreviated (+)–canonical trees (see §12.5).

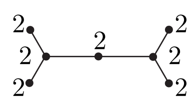
PROPOSITION 13.4. *Let Γ be a stellar abbreviated (+)–canonical tree, with associated fractional data vector $(r; e_0; \bar{m}_1, \dots, \bar{m}_k)$ as above, and assume that Γ is different from $\overset{0}{\bullet}$. Then the arborescent pair (M, K) obtained by plumbing according to Γ, as defined in* §12.5*, is the pairwise irreducible Montesinos pair classified by the same fractional data vector in the sense of Theorem* 11.7*, unless this one is $(0; e_0;\ )$ with $e_0 \neq -1, -2$, or $(r; e_0;\ )$ with $r \geqslant 3$ and $e_0 \neq 0$. For these two exceptions, (M, K) is respectively classified by $\left(0; \frac{e_0}{1+e_0}; -\frac{e_0}{1+e_0}\right)$ and $(r; e_0; \bar{0})$.*

Conversely, any pairwise irreducible Montesinos pairs is obtained by plumbing according to a (+)–canonical stellar tree, except for the ones classified by a fractional data vector $(2; e;\ )$ with $e \in \mathbb{Z} - 0$. These last Montesinos pairs are obtained by plumbing according to the (+)–canonical weighted planar trees:


if $e \neq 0, \pm 1, -2$


if $e = 1$


if $e = -1$


if $e = -2$.

PROOF. By the plumbing construction, a ***building block*** or ***plumbing block*** $(M_v, K_v) \subset (M, K)$ is naturally associated to each vertex v of Γ; namely, $(M_v, K_v)$is the copy of the corresponding model atomic tangle (possibly with rings) of §12.5 used in the construction. Note that the knot pair (M_v, K_v) comes with a preferred tangle structure inherited from the one of the model atomic tangle. If v has ring number $r \geqslant 0$ and weight $e \in \mathbb{Z}$, this tangle (M_v, K_v) can clearly be presented as a Montesinos tangle by plugging r ring tangles and one rational tangle of slope $-e$ to a hollow Montesinos tangle.

First consider the case when Γ has a well-defined central vertex w, and look at the corresponding building block (M_w, K_w).

Each closed-up component N of $M - M_w$ is the union of the building blocks associated to the vertices of some stick $\overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet} \cdots \overset{a_n}{\bullet}$ of the weighted tree Γ_0 (= Γ minus the embedding in the plane). From this, one readily sees that $(N, K \cap N)$ is a rational tangle pair. Thus the pair (M, K) is obtained by plugging rational tangles to the Montesinos tangle (M_w, K_w), and is consequently a Montesinos pair. To determine the fractional data vector associated to this Montesinos pair, we need to compute the slope on $\partial(M_w, K_w)$ of the pairwise essential discs of these rational tangles.

For this purpose, consider such a rational tangle $(N, L; \theta)$ obtained by plumbing according to a weighted planar tree which, disregarding the embedding in the plane, is of type $\overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet} \cdots \overset{a_n}{\bullet}$. As a matter of fact, the embedding of the tree (and weights) in the plane is clearly irrelevant, by application of Flip (F_1) of the plumbing calculus. The tangle $(N, L; \theta)$ can also be obtained by, first, plumbing according to $\overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet}\!\!- \cdots -\!\!\overset{a_n}{\bullet}\!\!-$, and then plugging a rational tangle of slope 0 into the braid tangle so obtained, along the boundary component corresponding to the right hand bond. Applying Lemma 13.2, it follows that $(N, L; \theta)$ is a rational tangle of slope $-[[a_1, a_2, \ldots, a_n]]^{-1}$. On the other hand, plumbing the tangle $(N, L; \theta)$ corresponds to plugging the tangle $(N, L; \tau\theta)$ where τ is the automorphism of the standard Conway sphere (S^2, P^0) in $\mathbb{R}^3$ defined by $\tau(x, y, z) = (y, x, -z)$ (compare the beginning of §12.1). As τ sends slope $\frac{q}{p}$ to slope $-\frac{p}{q}$, it follows that plumbing $(N, L; \theta)$ to some tangle amounts to plugging a rational tangle of slope $[[a_1, a_2, \ldots, a_n]]$ along the same boundary component.

After this, we are ready to complete the proof. Index the bonds of w by $i = 1$, $\ldots$, t, in the cyclic order prescribed by Γ, starting from the angular sector where the weight $e \in \mathbb{Z}$ of w sits (arbitrarily if $e = 0$ is omitted). Define m_i to be $\varnothing$ if the i–th bond of w is a free bond of Γ, and $m_i = [[a_1, a_2, \ldots, a_n]]$ when this i–th bond is adjacent to a stick $-\!\!\overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet}\!\!- \cdots -\!\!\overset{a_n}{\bullet}$ of Γ_0. By the above considerations, the pair (M, K) clearly has a presentation with raw data vector $(r; -e, m_1, m_2, \ldots, m_t)$ as defined in §11.1.

If some m_i is $\varnothing$, then $e = 0$ and (M, K) also has a restricted presentation with raw data vector $(r; m_1, m_2, \ldots, m_t)$ and fractional data vector $(r; \varnothing; \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_t)$ where $\bar{m}_i \in \mathbb{Q}/\mathbb{Z} \cup \infty$ is defined as usual, which is precisely the fractional data vector associated to Γ.

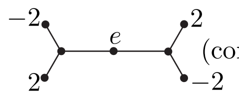
If all m_i are in $\mathbb{Q}$ but $t \geqslant 1$, then by Theorem 11.6 (M, K) has a presentation with raw data vector $(r; 0, m_1 - e, m_2, \ldots, m_t)$ and thus one with raw data vector $(r; m_1 - e, m_2, \ldots, m_t)$. The fractional data vector associated to this presentation then is $(r; e_0; \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_t)$ where $e_0 = e - \sum_i m_i$ and $\bar{m}_i \in \mathbb{Q}/\mathbb{Z}$ is the mod 1 reduction of m_i, which is the fractional data vector associated to Γ, as expected.

Lastly, when Γ consists of a single 0–valent vertex, so that there is no m_i, note that $r \neq 1$, 2 by the Abbreviation Condition (A) for abbreviated (+)–canonical trees. When $r \geqslant 3$, the fractional data vector $(r; e; 0)$ associated to the presentation of (M, K) is of the type required to classify (M, K) in Theorem 11.7. When $r = 0$, Γ is classified by $(r; e;\ )$ with $e \neq 0$, -1, -2; the equivalence of Figure 8.34 shows that (M, K) has a restricted presentation with raw data vector $\left(0; -\frac{e}{1+e}\right)$ and thus with fractional data vector $\left(0; \frac{e}{1+e}; -\overline{\frac{e}{1+e}}\right)$.

The only remaining case is now when Γ has no central vertex, namely is a linear stick with $\geqslant 2$ vertices. But the proof is exactly the same as above.

This completes the proof of the first part of Proposition 13.4.

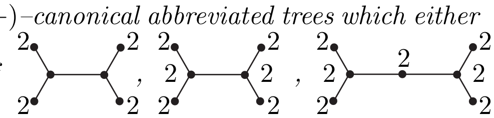
To prove the second statement of Proposition 13.4, it suffices to compare the respective domains to which fractional data vectors associated to pairwise irreducible Montesinos pairs or stellar (+)–canonical trees are restricted, using the exchanges of Theorem 11.7 if necessary. By inspection, one readily sees that the only pairwise irreducible Montesinos pairs that are not obtained by plumbing according to a (+)–canonical stellar tree are those classified by $(2; e;\ )$ with $e \in \mathbb{Z} - 0$. But these are clearly obtained by plumbing according to the weighted planar trees



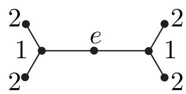
(compare Figure 12.18), and thus according to the (+)–canonical trees listed in Proposition 13.4 by using Moves (1.2) or (2.2) when possible. □

Combining Proposition 13.4, Lemma 13.3 and Theorem 11.7, we get:

Theorem 13.5. *The plumbing construction defines a one-to-one correspondence between the degree* $+1$ *isomorphism classes of pairwise irreducible Montesinos pairs, and the flip equivalence classes of* (+)*–canonical abbreviated trees which either are stellar or are flip equivalent to one of*



and



with $e \neq 0, \pm 1, -2$*, excluding the weighted tree* $\overset{0}{\bullet}$. □

13.2. Proof of the classification of arborescent pairs

Having proved in the previous section the restricted version of Theorems 12.12 and 12.15 for Montesinos pairs, we now pass to the general case.

Consider a (+)–canonical abbreviated tree Γ, and let (M, K) be the arborescent pair associated to Γ by plumbing construction. By construction a plumbing block $(M_v, K_v) \subset (M, K)$ is associated to each vertex v of Γ, and this pair (M_v, K_v) comes with a preferred tangle structure inherited from the corresponding model atomic tangle (with rings). Similarly, a Conway sphere $S_e \subset (M, K)$ separating two plumbing blocks is naturally associated to each edge e of Γ, namely S_e is the sphere separating the building blocks corresponding to the two end vertices of e.

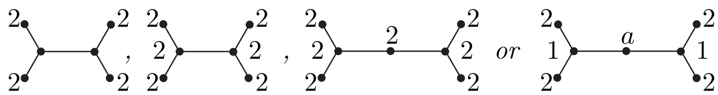
In a first step, we will recover the characteristic decomposition defined in Chapter 9 from this plumbing description of (M, K).

As usual, let Γ_0 denote the weighted tree obtained from Γ by forgetting all data concerning the embedding in the plane, and recall that a ***stick*** is a component of the weighted graph defined by removing from Γ_0 all essential vertices, namely those with valence $\geqslant 3$ or ring number $\geqslant 1$, together with their attached bonds and weights. A stick is ***open***, ***half-open*** or ***closed*** according as it has 2, 1 or 0 free bonds, respectively. (Note that a closed stick must be equal to Γ_0).

For each essential vertex v of Γ, let $\widehat{M_v} \subset M$ denote the union of the building block M_v together with the building blocks M_w associated with all vertices w which are contained in a half-closed stick of Γ_0 that is adjacent to v. Then, consider $(\widehat{M}_v^0, \widehat{K}_v^0) \subset (M, K)$ obtained from $(\widehat{M_v}, K \cap \widehat{M_v})$ by removing a small collar neighbourhood of its boundary, so that all $\widehat{M}_v^0$ are now pairwise disjoint. Finally let $\widehat{M}^0 \subset M$ denote the union of these $\widehat{M}_v^0$, with v ranging over all essential vertices of Γ.

By construction, and except in the degenerate case where Γ_0 is itself a stick, each closed-up component of $(M, K) - (\widehat{M}^0, K \cap \widehat{M}^0)$ is a thickened Conway sphere, namely is isomorphic to $(S^2, P^0) \times [0, 1]$. Indeed, such a closed-up component is obtained by thickening, either the union $\cong (S^2, P^0) \times [0, 1]$ of the blocks M_w corresponding to the vertices w of some open stick of Γ_0, or the Conway sphere S_e associated to an edge e of Γ_0 that joins two essential vertices of Γ_0.

Proposition 13.6. *With the above data, assume that* Γ_0 *is not one of the trees* $\overset{0}{\bullet}$ *,*



with $a \neq 0, \pm 1, -2$.

Then the arborescent pair (M, K) *is pairwise irreducible and, up to pairwise isotopy,* $\partial \widehat{M}^0 \subset M$ *consists of two parallel copies of the characteristic family of Conway spheres in* (M, K) *defined in Chapter* 9 *(splitting* (M, K) *into Montesinos pairs) and of a parallel copy of* ∂M*.*

Proof. When Γ is stellar, then $\partial \widehat{M}^0 = \varnothing$ and (M, K) is itself a pairwise irreducible Montesinos pair, by Proposition 13.4, so that the characteristic family of Chapter 9 is also empty. Thus Proposition 13.6 is already proved in this case, and we will henceforth assume that Γ is not stellar, or equivalently that $\partial \widehat{M}^0 \neq \varnothing$.

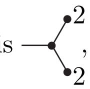
By construction, each $(\widehat{M}_v, \widehat{K}_v)$ is obtained by plumbing according to the weighted planar tree $\Gamma_v \subset \Gamma$ that consists of v, of the vertices contained in half-open sticks of Γ_0 adjacent to v, and of the weights and bonds attached to these vertices (including v). In particular, $(\widehat{M}_v, \widehat{K}_v)$ comes with a preferred tangle structure inherited from this plumbing description. As $(\widehat{M}_v^0, \widehat{K}_v^0)$ is obtained from $(\widehat{M}_v, \widehat{K}_v)$ by removing a small collar neighbourhood of its boundary, this induces a tangle structure on $(\widehat{M}_v^0, \widehat{K}_v^0)$ which is only defined up to pairwise isotopy.

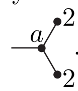
Let Γ_v^0 be defined by removing from Γ_v the weight of v (keeping all other data). Then the pairs $(\widehat{M}_v, \widehat{K}_v)$ and $(\widehat{M}_v^0, \widehat{K}_v^0)$ can also be obtained by plumbing according to Γ_v^0, by boundary move (B_2) of the plumbing calculus (extended to abbreviated trees). Also, the abbreviated tree Γ_v^0 is stellar, and is (+)–canonical for the calculus of arborescent pairs (see Proposition 12.16). By Theorem 13.5, it follows that $(\widehat{M}_v^0, \widehat{K}_v^0)$ is a pairwise irreducible Montesinos pair, not a rational tangle pair since Γ_v^0 is never $-\!\!\bullet$ by construction.

In particular, each closed-up component of $(M, K) - \partial \widehat{M}^0$ is pairwise irreducible, and has pairwise incompressible boundary by Corollary 8.18. By a now standard innermost circle argument (compare the proof of Theorem 8.16), it immediately follows that (M, K) is pairwise irreducible and that $\partial \widehat{M}^0$ is pairwise incompressible in (M, K).

To prove that the above splitting of (M, K) into Montesinos pairs is actually the characteristic splitting of Chapter 9, we will use Criterion 9.5. So, consider two components $\widehat{M}_v^0$ and $\widehat{M}_w^0$ of $\widehat{M}$ which are adjacent to each other, namely separated in (M, K) by a closed-up component $(W, W \cap K) \cong (S^2, P^0) \times [0, 1]$ of $M - \widehat{M}^0$. We need to show the following: For any presentation of $(\widehat{M}_v^0, \widehat{K}_v^0)$ and $(\widehat{M}_w^0, \widehat{K}_w^0)$ as restricted Montesinos pairs, the traces on ∂W of the corresponding bands do not project to pairwise isotopic pairs of arcs by the projection $(W, W \cap K) \cong (S^2, P^0) \times [0, 1] \to (S^2, P^0)$.

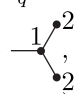
First of all, note that in the description of $(\widehat{M}_v^0, \widehat{K}_v^0)$ arising from plumbing construction, the band of this presentation intersects the boundary in arcs of slope ∞. Since Γ_v^0 is neither $-\!\!\bullet$ nor $-\!\bullet\!-$, $(\widehat{M}_v^0, \widehat{K}_v^0)$ is not a rational tangle pair nor a thickened Conway sphere (by Theorem 13.5). Thus, the necklace uniqueness Theorem 10.5 applies to show that the band of any other restricted presentation of $\widehat{M}_v^0, \widehat{K}_v^0)$ has slope ∞ on the boundary unless $(\widehat{M}_v^0, \widehat{K}_v^0)$ is isomorphic to the ring pair.

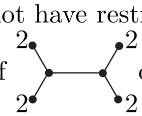
By Theorem 13.5, $(\widehat{M}_v^0, \widehat{K}_v^0)$ is a ring pair precisely when Γ_v^0 is 

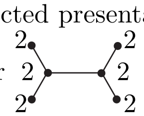
, and thus when the part of Γ_0 corresponding to Γ_v is of type 

. By Proposition 10.4, $(\widehat{M}_v^0, \widehat{K}_v^0)$ has in this case exactly one other necklace, up to pairwise isotopy, and the band of this necklace has slope $1-a$ on the boundary. To check this last slope computation, observe for instance that this is also the boundary slope of the unique pairwise essential annulus in $(\widehat{M}_v^0, \widehat{K}_v^0)$, and apply Proposition 8.12.

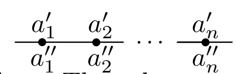
The same properties similarly hold for $(\widehat{M}_w^0, \widehat{K}_w^0)$.

When v and w are adjacent vertices of Γ, a curve of slope $\frac{q}{p}$ on $W \cap \widehat{M}_v^0$ corresponds through W to a curve of slope $\frac{p}{q}$ on $W \cap \widehat{M}_w^0$. Also, remember that neither Γ_v nor Γ_w can be flip equivalent to 

, by the definition of abbreviated (+)–canonical trees (compare Proposition 12.16). By inspection, it follows that $(\widehat{M}_v^0, \widehat{K}_v^0)$ and $(\widehat{M}_w^0, \widehat{K}_w^0)$ cannot have restricted presentations whose bands can fit through W unless Γ_0 is one of 

or 

, which precisely has been excluded.

The remaining case is when v and w are separated in Γ_0 by a stick 

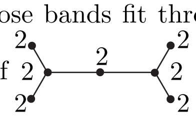
, whose left and right free bonds are respectively adjacent to v and w. Then, by Lemma 13.1, a curve of slope $\frac{q}{p}$ on $W \cap \widehat{M}_w^0$ corresponds through the collar W to a curve of slope $\frac{q'}{p'}$ on $W \cap \widehat{M}_v^0$ with

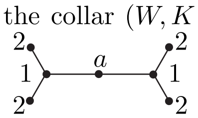
$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

where $D/C = [[a_n, a_{n-1}, \ldots, a_1]]$, $A/C = -[[a_1, a_2, \ldots, a_n]]$ and $AD - BC = -1$. Searching when $\frac{q}{p}$ and $\frac{q'}{p'}$ can both be in $\mathbb{Z} \cup \infty$, one readily checks that the only possibilities allowed by Lemma 13.2, together with the Stick Condition (S) and the Positivity Condition (P) on Γ, are:

(i) $n = 1$, $\frac{q}{p} = \frac{q'}{p'} = -1$, and $a_1 = +2$;

(ii) $n = 1$, $\frac{q}{p} = \frac{q'}{p'} = 0$, and $a_1 \neq 0, \pm 1, -2$.

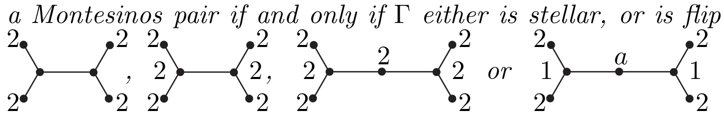
Therefore, the only cases when $(\widehat{M}_v^0, \widehat{K}_v^0)$ and $(\widehat{M}_w^0, \widehat{K}_w^0)$ could have restricted presentations whose bands fit through the collar $(W, K \cap W)$ would be the cases when Γ_0 is one of 

and 

with $a \neq 0, \pm 1, -2$, but these two possibilities have been excluded by hypothesis.

This proves that the conditions of Criterion 9.5 are satisfied, and thus that $\partial \widehat{M}^0$ actually consists of two parallel copies of the characteristic collection of Conway 2–spheres studied in Chapter 9, together with a parallel copy of ∂M. $\square$

Corollary 13.7. *Let (M, K) be the arborescent pair obtained by plumbing according to a (+)–canonical abbreviated weighted planar tree Γ. Then (M, K) is* *a Montesinos pair if and only if* Γ *either is stellar, or is flip equivalent to one of*



with $a \neq 0, \pm 1, -2$.

Proof. The arborescent pair (M, K) is a Montesinos pair if and only if the characteristic family of Chapter 9 is empty. It therefore suffices to combine Proposition 13.6 and the second part of Proposition 13.4. $\square$

Corollary 13.8. *Let* (M, K) *and* (M', K') *be arborescent pairs respectively obtained by plumbing according to the two abbreviated* $(+)$*–canonical trees* Γ *and* Γ'. *Define* $\widehat{M}^0 \subset M$ *and* $\widehat{M}'^0 \subset M'$ *as above. Then any degree* ± 1 *isomorphism* φ: $(M, K) \to (M', K')$ *can be pairwise isotoped so that* $\varphi(\widehat{M}^0) = \widehat{M}'^0$.

Proof. When neither Γ nor Γ' are one of the exceptional trees of Proposition 13.6, it suffices to combine this result with Theorem 9.1.

When Γ is $\overset{0}{\bullet}$, (M, K) is pairwise reducible. By Proposition 13.6 and 13.4, Γ is also necessarily of this type, which proves the statement.

When Γ is flip equivalent to one of the remaining exceptional trees, (M, K) is a pairwise irreducible Montesinos pair by Proposition 13.4. By Corollary 13.7 and Theorem 13.5, it follows that Γ' is flip equivalent to Γ. Moreover, recall that (M, K) then is the Montesinos pair classified by the fractional data vector $(2; e; 0)$ with $e \neq 0$, as defined for Theorem 11.7. In particular, the classification of pairwise incompressible surfaces in Montesinos pairs we gave in Chapter 8 characterises $\partial\widehat{M}^0$ as two parallel copies of the unique pairwise incompressible Conway sphere in (M, K). Since the same property holds for $\partial\widehat{M}'^0$ in (M', K'), this completes the proof. $\square$

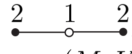
The next step of our proof will be to show that the decomposition of (M, K) into plumbing blocks (M_v, K_v) is, in general, characteristic. This will be proved in Proposition 13.10 below. For later reference, we first note the following slightly modified form of Proposition 11.1.

Lemma 13.9. *Let* (N, L) *and* (N', L') *be two model atomic tangles (possibly with rings), and let* $S_1, \ldots, S_k$ *denote the boundary components of* N. *For any (degree* $+1$*) pair isomorphism* ψ: $(N, L) \to (N', L')$ *sending band to band, there exist integers* $n_1, \ldots, n_k$ *such that*

(i) *Each restriction* $\psi_{|S_i}$: $S_i \to \psi(S_i)$ *sends slope* $\frac{q}{p}$ *to slope* $\frac{q}{p} + n_i$.

(ii) $\sum_i n_i = e - e'$ *where* e *and* e' *are the sums of the weights of the weighted planar trees respectively corresponding to* (N, L) *and* (N', L').

Proof. Same as Proposition 11.1. $\square$

Proposition 13.10. *Let* (M, K) *and* (M', K') *be two arborescent pairs respectively obtained by plumbing according to two abbreviated canonical trees* Γ *and* Γ'. *Exclude the cases when* Γ *is stellar with central vertex of valence* 3 *and ring number* 0 *and with no free bond, and when* Γ *is* .

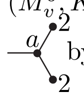
Then any (degree $+1$*) isomorphism* φ: $(M, K) \to (M', K')$ *can be pairwise isotoped so that it sends each plumbing block* (M_v, K_v) *of* (M, K) *to a plumbing block* $(M'_{v'}, K'_{v'})$ *of* (M', K'). *Furthermore, when* $\varphi(M_v) = M'_{v'}$, *the corresponding vertices* v *and* v' *have the same weights in* Γ *and* Γ'.

Proof. We first restrict attention to the case when Γ has $\geqslant 2$ vertices, and has at least one essential vertex. Then, by Corollary 13.7 and Theorem 13.5, so does also Γ'.

For every essential vertex v of Γ, let (M_v^0, K_v^0) be obtained from (M_v, K_v) by removing a small collar neighbourhood of the boundary. Let also $(\widehat{M}_v, \widehat{K}_v)$ and $(\widehat{M}_v^0, \widehat{K}_v^0)$ be defined as above Proposition 13.6, where the collars components of $\widehat{M}_v - M_v^0$ are chosen so that $\partial\widehat{M}_v^0 \subset \partial M_v^0$.

Similarly, define $M_{v'}'^0$,$\widehat{M}_{v'}'$ and $\widehat{M}_{v'}'^0$ for every essential vertex v' of Γ.

By Corollary 13.8, φ can first be pairwise isotoped so that it sends each $\widehat{M}_v^0$ to some $\widehat{M}_{v'}'^0$.

As in the proof of Proposition 13.4, the plumbing construction of $(\widehat{M}_v, \widehat{K}_v)$ gives a presentation of $(\widehat{M}_v^0, \widehat{K}_v^0)$ as a restricted Montesinos pair whose necklace N_v is the necklace of (M_v^0, K_v^0). Our hypotheses on Γ, together with Theorem 13.5, prevent $(\widehat{M}_v^0, \widehat{K}_v^0)$ from being one of the exceptions of the necklace uniqueness Theorem 10.5, except for the ring pair or the double of the ring pair. In fact, $(\widehat{M}_v^0, \widehat{K}_v^0)$ can certainly be a ring pair; in this case, the corresponding part of Γ_0 is 

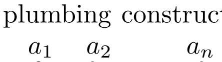
by Theorem 13.5 and N_v consequently has two Conway spheres.

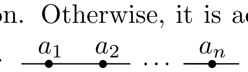
If $N_{v'}'$ is the necklace similarly defined in $M_{v'}' = \varphi(M_v)$, it then follows from Theorem 10.5, together with Proposition 10.4 for the ring pair and Proposition 10.11 for the double of the ring pair, that φ can be further pairwise isotoped so that $\varphi(N_v) = N_{v'}'$.

In particular, φ now sends each M_v^0 to some $M_{v'}'^0$ for any essential vertex v of Γ.

On each component S of ∂M_v^0, we can now distinguish two preferred pairwise isotopy classes of essential curves, namely what we called ***markings*** in §12.8. The first of these two markings, which is called the ***internal marking***, is simply defined by the trace of the band of the necklace N_v. It has slope ∞ for the tangle structure of (M_v^0, K_v^0) inherited from the identification of (M_v, K_v) with the corresponding model atomic tangle.

The second marking, which is called the ***external marking***, is only defined for components S of ∂M_v^0 which are not parallel copies of components of ∂M. Let W be the closed-up component of $M - \bigcup_w M_w^0$ that is adjacent to S. When $(W, K \cap W)$ is a rational tangle pair, this external marking is defined by the boundary of the pairwise essential disc in $(W, K \cap W)$ (unique up to pairwise isotopy by the analysis of Chapter 8). Otherwise, $(W, K \cap W)$ is a collar $\cong (S^2, P^0 \times I)$, separating M_v^0 from some M_w^0 and, by definition, the external marking of S corresponds through W to the internal marking of the boundary component $W \cap \partial M_w^0$ of M_w^0.

We have already encountered these external markings in the proofs of Propositions 13.4 and 13.6, and computed in terms of Γ their slopes for the tangle structure on (M_v^0, K_v^0). Indeed, given a component S of ∂M_v^0, a bond of the vertex v is naturally associated to S. If this bond is free bond of Γ, then the external marking of S is not defined. If this bond leads to another essential vertex w of Γ, then the slope of the external marking of S is clearly 0 by definition of the plumbing construction. Otherwise, it is adjacent to the left hand bond of a stick 

or 

of Γ_0. In these last cases, we computed the slope of the external marking of S in the proofs of Proposition 13.4 and 13.6, using Lemma 13.1: This slope is in both cases equal to the alternating continued fraction $m = [[a_1, a_2, \ldots, a_n]]$. In this case, observe that $m \in \left]-\frac{1}{2}, +\frac{1}{2}\right] - 0$ by Lemma 13.2 and the (+)–canonicity conditions satisfied by Γ.

One similarly defines internal and external markings for components of $\bigcup_{v'} \partial M'^{0}_{v'}$. Then, as φ sends necklaces N_v to necklaces $N'_{v'}$, the restriction $\varphi_{|S}\colon S \to \varphi(S)$ to each component S of $\bigcup_v \partial M_v^0$ sends internal marking to internal marking and, when defined, external marking to external marking.

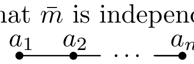
Let v be an essential vertex of Γ, namely one with bond $\geqslant 3$ or ring number $\geqslant 1$, and let $S_1, \ldots, S_k$ denote the components of ∂M_v^0. When defined, let m_i be the slope of the external marking of S_i, and let m'_i be the slope in $(M'^{0}_{v'}, K'^{0}_{v'}) = \varphi(M_v^0, K_v^0)$ of the external marking of $\varphi(S_i)$. Then, by Lemma 13.9, $m'_i = m_i$ mod 1 and, when all S_i have a well-defined external marking, $\sum_i (m'_i - m_i) = e' - e$ where e and e' are the respective weights of v and v' in Γ and Γ'. Since all m_i and m'_i are in the interval $\left]-\frac{1}{2}, +\frac{1}{2}\right]$, it follows from the first statement that each m'_i is equal to the corresponding m_i, and that $e = e'$ when v is not adjacent to a free bond of Γ (and consequently in any case since $e = e' = 0$ otherwise, by the Free Bond Condition on Γ and Γ').

Also, if W_i is the closed-up component of $M - \bigcup_w M_w^0$ adjacent to S_i, the number of plumbing blocks (M_s, K_s) contained in W_i and the weights of the corresponding vertices s can immediately be recovered from the slope m_i, by the above analysis combined with Lemma 13.2 and the Stick Condition (S) on Γ. The same property holds for the closed-up components of $M' - \bigcup M'^{0}_{w'}$. Therefore, one can pairwise isotop φ so that it sends each building block M_v to a building block $M'_{v'}$; moreover the weight of v in Γ_0 then is the same as the weight of v' in Γ'_0.

This completes the proof of Proposition 13.10 in the case when Γ has at least 2 vertices, and at least one essential vertex.

When Γ consists of a single vertex, the proof is an immediate consequence of Corollary 13.7 and Theorem 13.5.

It remains to consider the case when Γ_0 is a closed stick, namely when K is a rational knot in $M \cong S^3$. We rely on the main result of H. Schubert's 1956 paper on rational knots [**Sch**$_3$] which asserts the following: Up to pairwise isotopy, (M, K) contains a unique Conway sphere splitting it into two rational tangle pairs. If v is an end vertex of Γ, then the boundary S of the building block (M_v, K_v) clearly has this property of splitting (M, K) into two rational tangle pairs. Thus, Schubert's result implies that φ can be pairwise isotoped so that $\varphi(M_v)$ is the building block M'_v, associated to one of the two end vertices of Γ'.

Moreover, S has two preferred markings C_+ and C_-, respectively defined by the boundaries of the pairwise essential discs in (M_v, K_v) and its complement, and this defines an invariant $\bar{m} \in \mathbb{Q}/\mathbb{Z}$ as follows: Choosing a parametrisation of $(S, K \cup S)$ by the standard Conway sphere (S^2, P^0), so that C_+ has slope ∞, then $\bar{m}$ is the mod 1 reduction of the slope m of C_- for this parametrisation; indeed one readily sees, using Corollary 11.4, that $\bar{m}$ is independent of the choice of parametrisation. On the other hand, if Γ_0 is 

with v being the left-most vertex, a computation similar to the one used in the proof of Proposition 13.4 yields that this $\bar{m}$ is precisely the class of $m = [[a_1, a_2, \ldots, a_n]]$ in $\mathbb{Q}/\mathbb{Z}$. Since the same analysis holds in (M', K'), Lemma 13.2 now proves that Γ and Γ' have the same number of vertices and that the sequences of their weights, starting respectively from v and v', are identical. This completes the proof of Proposition 13.10 in the last case. $\square$

Proposition 13.10 establishes a weaker form of Theorem 12.15, namely: If (M, K) is obtained by plumbing according to the abbreviated $(+)$–canonical tree Γ, then the weighted tree (with ring numbers) Γ_0 obtained from Γ by disregarding its embedding in the plane *is a topological invariant of the arborescent pair* (M, K). Indeed, let (M', K') be the arborescent pair obtained by plumbing according to another abbreviated $(+)$–canonical tree Γ', and assume that there is a (degree $+1$) isomorphism φ: $(M, K) \to (M', K')$. Using Proposition 13.10, and Theorem 13.5 to deal with the exceptions of this statement, φ can be chosen so as to send each building block (M_v, K_v) to a building block $(M'_{v'}, K'_{v'})$, in such a way that the weights of v and v' in Γ and Γ' are equal. The rule $v \mapsto v'$ establishes a one-to-one correspondence between the vertices of Γ_0 and those of Γ'_0, which immediately extends to an isomorphism between the combinatorial trees underlying Γ_0 and Γ'_0. As this isomorphism preserves the weights of vertices and their ring numbers (= number of closed components of K_v or $K'_{v'}$), it defines an isomorphism $\bar{\varphi}$: $\Gamma_0 \to \Gamma'_0$ of weighted trees.

Given two abbreviated $(+)$–canonical trees Γ and Γ', recall that in §12.6 we defined a ***degree*+1 *isogeny*** $\Gamma \to \Gamma'$ as an isomorphism $\Gamma_0 \to \Gamma'_0$ between their underlying weighted trees which can be decomposed as a product of flip isomorphisms (namely elementary isogenies induced by flips).

PROPOSITION 13.11. *Under the hypotheses and conclusions of Proposition* 13.10*, the isomorphism* $\bar{\varphi}$: $\Gamma_0 \to \Gamma'_0$ *induced as above by* φ *is a degree* $+1$ *isogeny. In particular,* Γ *and* Γ' *are flip equivalent.*

We delay the proof of Proposition 13.11 a bit. First note that this statement completes the proof of Theorem 12.15. Indeed, let (M, K) and (M', K') be two arborescent pairs respectively obtained by plumbing according to $(+)$–canonical abbreviated weighted planar trees Γ and Γ', and suppose they are degree $+1$ isomorphic. If Γ is not one of the exceptions of Proposition 13.10, this statement and Proposition 13.11 prove that Γ and Γ' are flip equivalent. Otherwise, Corollary 13.7 shows that Γ and Γ' are both stellar, and the same result in this case is proved by Theorem 13.5.

Thus, the proof of Theorem 12.15 will be completed with that of Proposition 13.11. Before addressing this last step, let us recall a few facts from Chapter 12.

Let Γ and Γ' be two abbreviated $(+)$–canonical trees, deduced from each other by one of the flips (F_1), (F_2), (F'_3) or (F''_3). In this situation, we saw in Chapter 12 that there is a preferred *flip isomorphism* ψ: $(M, K) \to (M', K')$ between the corresponding arborescent pairs, sending plumbing block (M_v, K_v) to plumbing block $(M'_{v'}, K'_{v'})$. In fact, each of these blocks has a natural tangle structure $(M_v, K_v; \theta_v)$ or $(M'_{v'}, K'_{v'}; \theta'_{v'})$, inherited from the corresponding model atomic tangles (with rings). To describe how ψ behaves with respect to these tangle structures, let us distinguish cases according to the type of flip considered.

FLIP (F_1). Then, the restriction of ψ induces a tangle isomorphism $(M_v, K_v; \theta_v) \to (M'_{v'}, K'_{v'}; \zeta\theta'_{v'})$ for each building block, where $\zeta\theta'_{v'}$ denotes the composition of the boundary parametrisations $\theta'_{v'}$ with the automorphism $\zeta \in V_4$ of the standard Conway sphere (S^2, P^0). Recall here that the automorphism group V_4 consists of the identity ι and of the π–rotations ξ, η, ζ around the x–, y–, z–axis of the standard Conway sphere, respectively.

FLIP (F_2), applied at some vertex w of Γ. Then, ψ sends $(M_v, K_v; \theta_v)$ to the tangle $(M'_{v'}, K'_{v'}; \eta\theta'_{v'})$ if the vertex v is at even distance from w, and to the tangle $(M'_{v'}, K'_{v'}; \xi\theta'_{v'})$ otherwise.

FLIP (F'_3) $- - - \overset{a}{ⓡ} - - \longrightarrow - - - \underset{a}{ⓡ} - -$ applied at some vertex w of Γ. Then, ψ induces an isomorphism $(M_v, K_v; \theta_v) \cong (M'_{v'}, K'_{v'}; \theta'_{v'})$ for each vertex v contained in the left hand subgraph $- - -$ of Γ. For v in the right hand subgraph $- -$, ψ sends $(M_v, K_v; \theta_v)$ to $(M'_{v'}, K'_{v'}; \eta^a\theta'_{v'})$ if v is at odd distance from w, and to $(M'_{v'}, K'_{v'}; \xi^a\theta'_{v'})$ otherwise. Lastly, for $v = w$, the image $(M'_w, K'_w; \theta_w\psi^{-1}_{|\partial M_v})$ of $(M_w, K_w; \theta_w)$ is such that $\theta_w\psi^{-1}_{|\partial M_v}$ coincides with $\theta'_{w'}$ on the components of $\partial M'_{w'}$ corresponding to bonds adjacent to the left hand side $- - -$ of Γ, and with $\eta^a\theta'_{w'}$ on the other boundary components.

FLIP (F''_3). The automorphism ψ is well-defined only when we have fixed a decomposition of this flip as

$$- - - ⓡ - - \xrightarrow{(B_2)^{-1}} - - - \overset{a}{ⓡ} - - \xrightarrow{(F_2)} - - - \underset{a}{ⓡ} - - \xrightarrow{(B_2)} - - - ⓡ - -$$

(through weighted planar trees with ring numbers which are not (+)–canonical), with a odd. Let α denote an automorphism of the standard Conway sphere (S^2, P^0) which fixes the point $(0, 0, 0) \in P^0$ and sends slope $\frac{p}{q}$ to $\frac{p}{q}+a$; note that this specifies the pairwise isotopy class of α by Corollary 11.4. Then, for the vertex w where the flip is performed, the image of $(M_w, K_w; \theta_w)$ by ψ is some $(M'_{w'}, K'_{w'}; \theta_w\psi^{-1}_{|\partial M_v})$ so that the restriction of $\theta_w\psi^{-1}_{|\partial M_v}$ to a component S of $\partial M'_{w'}$ is pairwise isotopic to the restriction of:

(i) $\theta'_{w'}$ if the bond of w corresponding to S is adjacent to the left hand subgraph $- - -$.
(ii) $\eta\theta'_{w'}$ if this bond is adjacent to the right hand subgraph $- -$.
(iii) $\alpha\theta'_{w'}$ if this bond is the lower free bond.
(iv) $\eta\alpha^{-1}\theta'_{w'}$ if it is the upper free bond.

At other vertices v, ψ sends $(M_v, K_v; \theta_v)$ to some $(M'_{v'}, K'_{v'}; \theta_v\psi^{-1}_{|})$ so that $\theta_v\psi^{-1}_{|}$ is $\theta'_{v'}$ when v is in $- - -$, is $\eta\theta'_{v'}$ when v is in $- -$ and at odd distance from w, and is $\xi\theta'_{v'}$ when v is in $- -$ and at even distance from w.

Another ingredient of the proof of Proposition 13.11 is the following elementary remark.

LEMMA 13.12. *Let $(M, K; \theta)$ be one of the model atomic tangles (with or without rings) used in the plumbing calculus, and let φ be a (degree $+1$) pair automorphism of (M, K) with the following property: For each component S of ∂M, let $\theta_S : (S, K \cap S) \to (S^2, P^0)$ be the parametrisation specified by θ; then $\varphi(S) = S$ and the automorphism $\theta_S\varphi\theta_S^{-1}$ of (S^2, P^0) is an element of the Viergruppe V_4.*

Then $\theta_S\varphi\theta_S^{-1} \in V_4$ is independent of the component S. Furthermore, it is necessarily one of ι or ξ when ∂M has $\geqslant 3$ components.

PROOF. An element of V_4 is determined by its induced permutation of the 4 points of P^0. The result then immediately follows from connectivity considerations on the components on K. □

We are now ready to prove Proposition 13.11.

Proof of Proposition 13.11. We have to find a succession of alterations of Γ' by flips, defining a sequence of weighted planar trees $\Gamma^{(i)}$ and of preferred flip isomorphisms $\varphi^{(i)}\colon (M^{(i)}, K^{(i)}) \to (M^{(i+1)}, K^{(i+1)})$ between the corresponding arborescent pairs (sending plumbing block to plumbing block), such that $\Gamma^{(1)} = \Gamma$ and such that there is an n for which the composition $(\bar{\varphi}_1 \bar{\varphi}_2 \dots \bar{\varphi}_n)^{-1} \bar{\varphi}$ of the induced isomorphisms $\bar{\varphi}_i : \Gamma_0^{(i)} \to \Gamma_0^{(i+1)}$ comes from an isomorphism of weighted planar trees. Namely, $(\bar{\varphi}_1 \bar{\varphi}_2 \dots \bar{\varphi}_n)^{-1} \bar{\varphi}$ respects the cyclic orderings of weights and bonds at each vertex specified by Γ and $\Gamma^{(n)}$. We will define these $\Gamma^{(i)}$ stepwise and, to avoid cumbersome indexings, it will be convenient to continue to denote each $\Gamma^{(i)}$ of this sequence by Γ', and to replace at each step φ by $(\varphi_1, \varphi_2 \dots \varphi_i)^{-1} \varphi$.

If all vertices of Γ (and Γ') have valence $\leqslant 2$, the flip equivalence class of the weighted planar tree Γ is completely determined by the isomorphism class of the weighted tree Γ_0, and the conclusion of Proposition 13.11 immediately follows. Thus, we can henceforth assume that at least one vertex of Γ has valence $\geqslant 3$.

Then, write Γ_0 as an increasing union of subtrees $\Gamma_1, \dots, \Gamma_n$ where: Γ_1 consists of a valence $\geqslant 3$ vertex together with all its bonds; each Γ_{m+1} is the union of Γ_m, of another vertex v adjacent to Γ_m and of all the bonds of v. Our strategy for proving Proposition 13.11 will be to proceed stepwise. More precisely, we will inductively modify Γ' by a sequence of flips so that the following property holds for all m.

$(*)_m$ For each vertex v of Γ_m, the tree isomorphism $\bar{\varphi}\colon \Gamma_0 \to \Gamma_0'$ respects the cyclic orders of weights and bonds around v and $\bar{\varphi}(v) = v'$ respectively specified by Γ and Γ'. Moreover, the restriction of φ defines a tangle isomorphism $(M_v, K_v; \theta_v) \to (M(_{v'}, K'_{v'}; \theta'_{v'})$ between the corresponding building blocks, equipped with the tangle structure arising from the plumbing description.

To start the induction, we give

Procedure to assure $(*)_1$ Consider the (unique) vertex v of Γ_1. By connectivity considerations on the knot fragment K_v of the corresponding building block (M_v, K_v), the tree isomorphism $\bar{\varphi}$ respects the cyclic ordering of the bonds of v and $v' = \bar{\varphi}(v)$ *modulo orientation reversal*. After a possible application of Flip (F_1) to Γ', we can thus make $\bar{\varphi}$ respect these cyclic orders of bonds. Moreover, by use of Flip (F_3'), we can further arrange that $\bar{\varphi}$ sends any angular sector at v where the integral weight (if any) lies to a similarly weighted angular sector at v'.

Let $S_1, \dots, S_k$ denote the boundary components of M_v, and let S_i' be the component $\varphi(S_i)$ of $\partial M'_{v'}$. As v has valence $\geqslant 3$, φ sends the necklace of M_v to that of $M'_{v'}$, up to pairwise isotopy (Theorem 10.5). By Lemma 13.9 and because v and v' have the same weights, there consequently exist integers $n_1, \dots, n_k$ with $\sum_i n_i = 0$ such that the restriction $\varphi_{|S_i}\colon S_i \to S_i'$ sends slope $\frac{p}{q}$ to slope $\frac{p}{q} + n_i$, for the tangle structures on (M_v, K_v) and $(M'_{v'}, K'_{v'})$ arising by plumbing construction.

On an S_i (resp. S_i') which is not a boundary component of M (resp. M'), we singled out in the proof of Proposition 13.10 a preferred marking, called the external marking, whose slope is finite and lies in the interval $\left]-\frac{1}{2}, +\frac{1}{2}\right]$. As φ sends external marking to external marking, it follows that $n_i = 0$ when S_i is not a component of ∂M.

Assume that there are two distinct S_i and S_j with $n_i, n_j \neq 0$. By the above, these are necessarily components of ∂M. Just before starting this proof, we showed the following: There exists a weighted planar tree Γ'', and an isomorphism ψ from (M', K') to the corresponding arborescent pair (M'', K'') associated to Γ'' that sends plumbing block to plumbing block, with the following properties: Γ'' is obtained from Γ' by isomorphism or Flip (F_3') according as n_i is even or odd; for the tangle structures on $(M'_{v'}, K'_{v'})$ and $(M''_{v''}, K''_{v''}) = \psi(M'_{v'}, K'_{v'})$ defined by plumbing construction, the restriction $S'_k \to \psi(S'_k)$ of ψ sends slope $\frac{p}{q}$ to slope $\frac{p}{q}$ if $k \neq i$, j, to slope $\frac{p}{q} - n_i$ if $k = i$ and to slope $\frac{p}{q} + n_i$ if $k = j$. In this case, replacing Γ' by Γ'' and φ by $\psi\varphi$ changes n_i to 0, n_j to $n_i + n_j$ and keeps all other n_k unchanged.

Thus, after a finite number of such modifications, we reach a state where at most one n_i is $\neq 0$. Since $\sum_i n_i = 0$, it follows that all n_i are then 0. Thus, we may assume that the restriction of φ to each boundary component of M_v respects the slopes.

As v and v' have the same bonds, ring numbers and integral weights in Γ_0 and Γ'_0, the plumbing blocks $(M_v, K_v; \theta_v)$ and $(M'_{v'}, K'_{v'}; \theta'_{v'})$ are copies of the same model atomic tangle $(M^*_v, K^*_v; \theta^*_v)$. Thus the restriction $(M_v, K_v) \to (M'_{v'}, K'_{v'})$ of φ induces an automorphism φ^*_v of the pair (M^*_v, K^*_v). As $\bar\varphi$ respects the cyclic order of bonds and weights around v and v', φ^*_v respects the natural cyclic order of the components of ∂M^*_v. Moreover, if the weight of v is $\neq 0$, φ^*_v necessarily respects the pair of boundary components corresponding to the two bonds of v adjacent to this weight (as $\bar\varphi$ respects the angular sectors so defined) and φ^*_v therefore preserves each component of ∂M^*_v. When this weight is 0, it suffices to compose the identification $(M^*_v, K^*_v) \to (M_v, K_v)$ with a suitable automorphism of the tangle $(M^*_v, K^*_v; \theta^*_v)$ to make φ^*_v respect each boundary component.

So, φ^*_v preserves each boundary component S of (M^*_v, K^*_v). Moreover, its restriction to each S acts trivially on the slopes and, by Corollary 11.4, we can therefore pairwise isotop φ so that $\varphi^*_{v|S}$ corresponds to an element of V_4 for the parametrisation $(S, K \cap S) \cong (S^2, P^0)$ defined by θ^*_v. We now have the hypotheses of Lemma 13.12 and, since v has valence $\geqslant 3$, either φ^*_v fixes each boundary component S or each $\varphi^*_{v|S}$ coincides (up to isotopy) with $\xi \in V_4$ on $(S, K \cap S) \cong (S^2, P^0)$.

Going back to φ, the above statement means precisely that the restriction of φ induces a tangle isomorphism from $(M_v, K_v; \theta_v)$ to either $(M'_{v'}, K'_{v'}; \theta'_{v'})$ or $(M'_{v'}, K'_{v'}; \xi\theta'_{v'})$.

In the second case, namely when $\theta_v \varphi^{-1} = \xi\theta'_{v'}$, we will again have to perform a flip (F_2) at a vertex of Γ' adjacent to v'. This gives a new weighted planar tree Γ'', with an isomorphism ψ from (M', K') to the corresponding arborescent pair (M'', K'') sending plumbing block to plumbing block, whose restriction in particular induces a tangle isomorphism $(M'_{v'}, K'_{v'}; \theta'_{v'}) \to (M''_{v''}, K''_{v''}, \xi\theta''_{v''})$ with the notation now standard. Moreover, the isomorphism $\bar\psi\colon \Gamma'_0 \to \Gamma''_0$ respects the cyclic order of bonds and weights around v' and v''. Thus, replacing Γ' by Γ'' and φ by $\psi\varphi$ in this case, we can always arrange that φ induces a tangle isomorphism from $(M_v, K_v; \theta_v)$ to $(M'_{v'}, K'_{v'}; \theta'_{v'})$.

This completes the proof of $(*)_1$, namely the first step in the induction. □

To complete the induction we give

Procedure to assure $(*)_{m+1}$ assuming $(*)_m$, with $m \geqslant 1$ Consider the vertex v of Γ_{m+1} that is not in Γ_m, and let $S_1, \ldots, S_k$ denote the boundary components of the corresponding block (M_v, K_v), taken in the order specified by the plumbing construction, and so that S_1 corresponds to the bond of v that is adjacent to Γ_m. Similarly, let $S'_1, \ldots, S'_k$ be the boundary components of the blocks $(M'_{v'}, K'_{v'}) = \varphi(M_v, K_v)$ of (M', K'), taken in the order specified by the plumbing construction and starting from $S'_1 = \varphi(S_1)$.

By connectivity considerations on K_v and $K'_{v'}$, one readily sees that φ respects the orderings of the S_i and S'_i up to orientation reversal, namely that either $\varphi(S_i) = S'_i$ for all i, or $\varphi(S_i) = S'_{k+1-i}$ for all i. On the other hand, we already have some information on the restriction of φ to S_1. Indeed, S_1 is also a boundary component of some plumbing tangle $(M_w, K_w; \theta_w)$ with $w \in \Gamma_m$. If $w' = \bar{\varphi}(w)$, the induction hypothesis $(*)_m$ asserts that the restriction $S_1 \to S'_1$ of φ corresponds to the identity of (S^2, P^0) through the parametrisations θ_w and $\theta'_{w'}$. By definition of the plumbing construction, it follows that this same restriction $S_1 \to S'_1$ also corresponds to the identity through the parametrisations θ_v and $\theta'_{v'}$. In particular, φ sends the point $x \in S_1$ corresponding to $(0,0,0) \in (S^2, P^0)$ through θ_v to the point $x' \in S'_1$ corresponding to (0,0,0) through $\theta'_{v'}$. It therefore sends the boundary sphere S_k connected to x by a component of K_v to the boundary sphere of $M'_{v'}$ that is connected to x' by a component of $K'_{v'}$, namely to S'_k. Consequently, $\varphi(S_i) = S'_i$ for all i and $\bar{\varphi}$ respects the cyclic orderings of bonds around v and v'.

We have seen that, when one performs a flip (F'_3) or (F''_3) on Γ' at v', the corresponding flip isomorphism induces a tangle isomorphism on the plumbing blocks associated to the vertices of certain components of $\Gamma'_0 - v'$. Performing a suitable flip (F'_3) so that Γ_p is precisely in one such component of $\Gamma'_0 - v'$, one can thus arrange that $\bar{\varphi}$ respects the cyclic order of weights and bonds at v and $v' = \bar{\varphi}(v)$, without losing any of the properties of φ previously asserted.

We now want the restriction $(M_v, K_v) \to (M'_{v'}, K'_{v'})$ of φ to respect the slopes on the boundary for the natural tangle structures of these plumbing blocks. If v has valence $\geqslant 3$ or ring number $\geqslant 1$, we proceed as in the starting step $(*)_1$ of the induction, using Theorem 10.5 and Lemma 13.9 and modifying Γ' by a sequence of flips (F''_3). The only point to take care of is to avoid destroying the properties previously imposed to φ. But this clearly holds if we choose these flips so that the corresponding flip isomorphisms respect tangle structures on the plumbing blocks associated to vertices of $\Gamma'_p = \bar{\varphi}(\Gamma_p)$, and this can always be done.

In the remaining cases when v has ring number 0 and valence $\leqslant 2$, it turns out that the restriction $(M_v, K_v) \to (M'_{v'}, K'_{v'})$ of φ already respects slopes on the boundary. Indeed, we already know that this property already holds on S_1 by the induction hypothesis. Thus there is nothing to prove when v has valence 1. When M_v has one other boundary component S_2, Lemma 13.1 shows that a curve C_2 of slope $\frac{p}{q}$ on S_2 corresponds to a curve C_1 of slope $a - \frac{p}{q}$ on S_1 through the collar (M_v, K_v), where $a \in \mathbb{Z}$ is the weight of v. The curve C_1 is sent to $C'_1 = \varphi'(C_1)$ with the same slope $a - \frac{p}{q}$ on S'_1, and C'_1 corresponds to a curve C'_2 pairwise homotopic to $\varphi(C_2)$ on S'_2, whose slope is $a - (a - \frac{p}{q}) = \frac{p}{q}$, again by Lemma 13.1 and because v' has the same weight a as v.

Thus, we are assured in all cases that the restriction of φ to $(M_v, K_v) \to (M'_{v'}, K'_{v'})$ respects the slopes on boundary components. Also, $\bar{\varphi}$ respects the cyclic orders of bonds and weights at each vertex of Γ_{m+1}, and φ still satisfies the induction hypothesis $(*)_m$.

Let $(M^*_v, K^*_v; \theta^*_v)$ be the common model atomic tangle for (M_v, K_v) and $(M'_{v'}, K'_{v'})$, and let φ^*_v be the automorphism of the pair (M^*_v, K^*_v) thus induced by the restriction of φ to $(M_v, K_v) \to (M'_{v'}, K'_{v'})$. As in the proof of the initial step $(*)_1$ of the induction, φ^*_v respects each boundary component S of M^*_v (after a possible rearrangement by a tangle automorphism when the weight of v is 0). Also, $\varphi^*_{v|S}$ respects the slopes, and can therefore be assumed to be conjugate to an element of V_4 by the parametrisation θ_v^* (by Corollary 11.4). On the other hand, we know that φ_v^* fixes the component S_1 of ∂M_v^* corresponding to S_1 and S_1', as φ satisfies the induction hypothesis $(*)_m$. Therefore, by Lemma 14.12, φ_v^* fixes the whole boundary of M_v^* and by restriction φ induces a tangle isomorphism $(M_v, K_v; \theta_v) \to (M'_{v'}, K'_{v'}; \theta'_{v'})$. Thus φ now satisfies Condition $(*)_{m+1}$. $\square$

By induction, this completes the proof of Proposition 13.11. $\square$

As indicated before, the above proof of Proposition 13.11 completes the proof of Theorems 12.12 and 12.15, which together establish the classification of pairwise irreducible arborescent pairs in terms of canonical weighted planar trees or abbreviated canonical trees, respectively. $\square$

13.3. Proof of the classification of marked arborescent pairs

The proof of the classification Theorem 12.25 for marked arborescent pairs closely follows the one that we gave in the previous section for Theorem 12.15. We only sketch a proof of it, mostly pointing at the differences with §13.2.

Let Γ and Γ' be two abbreviated $(+)$–canonical trees for the calculus of marked arborescent pairs (see §12.8 and §12.5), and consider the two marked arborescent pairs $(M, K; C)$ and $(M', K'; C')$ associated to Γ and Γ' by plumbing construction, respectively. Assume that there is a (degree $+1$) isomorphism $\varphi : (M, K) \to (M', K')$ sending the marking C to C'. We want to show that Γ and Γ' are equivalent by a sequence of flips (F_1), (F_2), (F_3') and (F_3'') of the calculus of marked arborescent pairs (see §12.8).

For each essential vertex v of Γ, define $(\widehat{M}_v^0, \widehat{K}_v^0)$ as in §13.2. Namely, first consider the union $\widehat{M}_v$ of the building blocks corresponding to v and to all vertices w contained in half open sticks adjacent to v. Then, $(\widehat{M}_v^0, \widehat{K}_v^0)$ is obtained from $(\widehat{M}_v, K \cap \widehat{M}_v)$ by removing a small collar neighbourhood of the boundary.

Let $(\widehat{M}^0, \widehat{K}^0)$ denote the union of the $(\widehat{M}_v^0, \widehat{K}_v^0)$ when v ranges over all essential vertices of Γ.

Each boundary component S of $(\widehat{M}_v^0, \widehat{K}_v^0)$ now has two preferred markings. The ***internal marking*** is defined as before by the trace of the band of the building block (M_v, K_v).

Unlike in the case of unmarked pairs, the ***external marking*** of S is always defined. Indeed, consider the closed-up component N of $M - \widehat{M}^0$ containing S. If $(N, K \cap N)$ is a rational tangle pair or a thickened Conway sphere separating $\widehat{M}_v^0$ from some $\widehat{M}_w^0$, the external marking is defined as before by the essential disc in $(N, K \cap N)$ or by the internal marking of $N \cap \partial \widehat{M}_w^0$, respectively. Otherwise, $(N, K \cap N)$ is a thickened Conway sphere separating S from a boundary component S' of M; the external marking of S is then defined to correspond through $(N, K \cap N)$ to the restriction to S' of the marking C.

Lemma 13.1 enables us to compute the slope of this external marking of S for the tangle structure of $(\widehat{M}_v^0, \widehat{K}_v^0)$ induced by the one of the plumbing block (M_v, K_v). The only new feature with respect to the case of §13.2 is when the bond of v corresponding to S either is a free bond of Γ, or is adjacent to a stick $\overset{a_1}{\bullet}\,\overset{a_2}{\bullet}\cdots\overset{a_n}{\bullet}$ whose right hand bond is a free bond. The slope of the external marking is then clearly 0 in the first case, and the alternating continued fraction

$[[a_1, \ldots, a_n]]$ in the second one, by Lemma 13.1. Note that the slope is infinite when the stick is $\overset{0}{\text{—}\bullet\text{—}}$.

Thus, by Lemma 13.2 together with the stick condition imposed to Γ, the slope of the external marking of S determines the type of the corresponding bond of v, namely whether this bond is adjacent to a stick of Γ or not, and in the later case what the coefficient of this stick are.

Using these new data, it is now easy to get the following analog of Proposition 13.10.

PROPOSITION 13.13. *Let $(M, K; C)$ and $(M', K'; C')$ be two marked arborescent pairs, respectively obtained by plumbing according to two abbreviated trees Γ and Γ' that are $(+)$–canonical for the calculus of marked arborescent pairs (see §12.8). Exclude the cases when Γ is stellar with no free bond and with central vertex of valence 3 and ring number 0, and when Γ is $\overset{2}{\bullet}\text{—}\overset{1}{\circ}\text{—}\overset{2}{\bullet}$.*

Then any degree $+1$ isomorphism $\varphi\colon (M, K; C) \to (M', K'; C')$ can be pairwise isotoped so that it sends each plumbing block (M_v, K_v) of (M, K) to a plumbing block $(M'_{v'}, K'_{v'})$ of (M', K'). Further, when $\varphi(M_v) = M'_{v'}$, the corresponding vertices v and v' have the same weights and ring numbers in Γ and Γ'.

PROOF. When Γ admits an essential vertex, namely one with valence $\geqslant 3$ or ring number $\geqslant 1$, we only need to adapt the proof of Proposition 13.6 and 13.10 to take the boundary markings into account. This is a mere exercise using the external markings which we just define.

Thus, the only remaining cases are those when the weighted tree Γ'_0 is an open or half open stick. (The case of a closed stick is provided by Proposition 13.10.)

If Γ_0 is a half-open stick $\text{—}\overset{a_1}{\bullet}\text{—}\overset{a_2}{\bullet}\text{—} \cdots \text{—}\overset{a_n}{\bullet}$, the unmarked pair (M, K) is a rational tangle pair classified by the tree $\text{—}\bullet$ and, by Proposition 13.10, the weighted tree Γ'_0 underlying Γ' is also a half-open stick. Observe that the number $m = [[a_1, a_2, \ldots, a_n]]$ is determined by the following property: m is in $\left]-\frac{1}{2}, +\frac{1}{2}\right] \cup \infty$ and, for any tangle structure on (M, K) giving slope ∞ to the marking C, the boundary slope of the (unique) essential disc of (M, K) is congruent to m mod 1; compare the proof of Proposition 13.4, and use Corollary 11.3 to check that changing one such tangle structure to another one modifies slopes by only an integer. Also, m characterises the sequence of weights a_i of Γ, by Lemma 13.2 and the Stick Condition. As the same properties hold for $(M', K'; C')$ and Γ', it follows that $\Gamma'_0 = \Gamma_0$. Since the Conway spheres separating building blocks are parallel copies of the boundary, φ can now be pairwise isotoped to send building block to building block, and respect the weights.

The proof is similar when Γ is an open stick $\text{—}\overset{a_1}{\bullet}\text{—}\overset{a_2}{\bullet}\text{—} \cdots \text{—}\overset{a_n}{\bullet}\text{—}$. On each boundary component S of (M, K), we have two preferred markings: The first one is the restriction of C; the second one corresponds to the restriction of C to the other boundary component $\partial M - S$ through the thickened Conway sphere (M, K). Arithmetic considerations similar to the ones used in the previous case show that $\Gamma_0 = \Gamma'_0$, and the proof is easily completed. □

At this point, the proof of Proposition 13.11 straightforwardly extends to give Proposition 13.14 below. One just needs a few local modifications to take into account the new form of flip (F''_3).

PROPOSITION 13.14. *Under the hypotheses and conclusions of Proposition* 13.13*, the isomorphism* $\bar{\varphi}\colon \Gamma_0 \to \Gamma'_0$ *between the weighted trees underlying* Γ *and* Γ' *which is induced by* Γ *is a degree* $+1$ *isogeny.* □

The exceptions to Propositions 13.13 and 13.14 correspond to arborescent pairs without boundary (and therefore without boundary markings!). They consequently were already considered in §13.1. Thus, Propositions 13.13 and 13.14 complete the proof of the classification Theorem 12.25 for pairwise irreducible marked arborescent pairs.

13.4. The classification of arborescent tangles (proof)

The proof of the classification Theorem 12.21 for arborescent tangles closely follows the one of the classification of arborescent pairs given in §13.2. Again, we only sketch it.

Owing to the new form of the Stick Condition for trees that are canonical for the calculus of arborescent tangle, we need a new version of Lemma 13.2 to deal with the corresponding arithmetics, namely Lemma 13.15 below. Its proof is elementary (compare [**HarW**]).

LEMMA 13.15. *Consider the finite integral sequences* $a_1, \ldots, a_n$ *such that:*

(i) $a_i \neq 0$ *when* $i < n$.
(ii) *The signs of the* a_i *are alternating.*
(iii) $a_1 \neq \pm 1$ *except when* $n = 1$.

Then such sequences are uniquely determined by the two alternating continued fractions $[[a_1, \ldots, a_n]]$ *and* $[[a_1, \ldots, a_n, 0]]$ *associated to them. Moreover, these two alternating continued fractions are in the interval* $\left[-\frac{1}{2}, +\frac{1}{2}\right]$ *except when* $n = 1$ *and* $a_1 = 0$ *or* ± 1. □

We can now start the proof of Theorem 12.21. Let Γ and Γ' be two abbreviated (+)–canonical trees for the calculus of arborescent tangles, and let $(M, K; \theta)$ and $(M', K'; \theta')$ be the two tangles respectively obtained by plumbing according to Γ and Γ'. Assuming that there is a (degree +1) tangle isomorphism $\varphi\colon (M, K; \theta) \to (M', K'; \theta')$, we want to show that Γ and Γ' are equivalent by flips (F_1), (F_2) and (F'_3).

We can clearly restrict attention to the case when $\partial M \neq \varnothing$, as the other case is established by our analysis of §13.2.

For each essential vertex v of Γ, define $(\widehat{M}^0_v, \widehat{K}^0_v)$ as in §13.2 and §13.3. Namely $\widehat{M}^0_v$ is obtained from the building block M_v by, first adding the building blocks corresponding to all the vertices contained in half-open sticks adjacent to v, and then removing a small collar neighbourhood of the boundary from the submanifold so obtained.

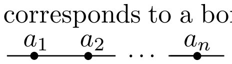
Let still $(\widehat{M}^0, \widehat{K}^0)$ denote the union of all $(\widehat{M}^0_v, \widehat{K}^0_v)$ when v ranges over all essential vertices of Γ.

As in §13.2 and §13.3, each boundary component S of $(\widehat{M}^0_v, \widehat{K}^0_v)$ carries an ***internal marking***, defined by the trace of the band of the plumbing block (M_v, K_v).

In contrast to the previous sections, the component S of $\partial(\widehat{M}^0_v, \widehat{K}^0_v)$ carries *one or two* external markings, according to the type of the closed-up component of $M - \widehat{M}^0$ containing S.

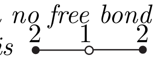
If $(N, K \cap N)$ is a rational tangle pair or a thickened Conway sphere separating $\widehat{M}_v^0$ from some $\widehat{M}_w^0$, S has only one vertical marking. This vertical marking is defined as in §13.2 by the essential disc in $(N, K \cap N)$ or the internal marking of $N \cap \partial \widehat{M}_w^0$, respectively.

Otherwise, $(N, K \cap N)$ is a thickened Conway sphere separating S from a boundary component S' of M, and S has a *vertical external marking* and a *horizontal vertical marking*. These are respectively defined to correspond through the product $(N, K \cap N)$ to the markings of S' which have slope ∞ and 0, for the tangle structure θ on (M, K).

The slope of these external markings, for the tangle structure of $(\widehat{M}_v^0, \widehat{K}_v^0)$ induced by the one of the plumbing blocks, are easily computed by use of Lemma 13.1. When S has only one external marking, its slope is determined as in §13.2. Otherwise, S corresponds to a bond of v which either is a free bond of Γ or is adjacent to a stick 

whose right hand bond is a free bond. In the first case, the horizontal and vertical external marking have respective slopes 0 and ∞. In the second case, Lemma 13.1, readily shows that the slope of the horizontal external marking is $[[a_1, \ldots, a_n]]$ and that the vertical one has slope $[[a_1, \ldots, a_n, 0]]$.

Taking these external markings into account and using Lemma 13.15, it is now an easy exercise to adapt the proof of Proposition 13.10 to get its following analog. (The case when Γ_0 is a stick requires a special treatment as in the proof of Proposition 13.13.).

PROPOSITION 13.16. *Let $(M, K; \theta)$ and $(M', K'; \theta')$ be two arborescent tangles, respectively obtained by plumbing according to two abbreviated trees Γ and Γ' that are $(+)$–canonical for the calculus of arborescent tangles. Exclude the cases when Γ is stellar with no free bond and with central vertex of valence* 3 *and ring number* 0*, and when Γ is* 

.

Then any degree $+1$ isomorphism $\varphi : (M, K; \theta) \to (M', K'; \theta')$ can be pairwise isotoped so that it sends each plumbing block (M_v, K_v) of (M, K) to a plumbing block $(M'_{v'}, K'_{v'})$ of (M', K'). Further, when $\varphi(M_v) = M'_{v'}$, the corresponding vertices v and v' have the same weights and ring numbers in Γ and Γ'. □

At this point, the proof of Proposition 13.11 can easily be adapted to give Proposition 13.17 below. Actually, the rigidity induced by tangle structures and the absence of Flip (F_3'') even make the proof simpler.

PROPOSITION 13.17. *Under the hypotheses and conclusions of Proposition* 13.16*, the isomorphism $\bar{\varphi}$: $\Gamma_0 \to \Gamma'_0$ between the weighted trees underlying Γ and Γ' which is induced by φ is a degree $+1$ isogeny, namely can be induced by a sequence of flips* (F_1)*,* (F_2) *and* (F_3') *of the calculus of arborescent tangles (see* §12.8*).* □

Again, the exceptions to Propositions 13.16 and 13.17 correspond to Montesinos tangles without boundary, and are consequently settled by §13.1 and 13.2. Thus, Propositions 13.16 and 13.13 complete the proof of the classification Theorem 12.21 for pairwise irreducible arborescent tangles. □

CHAPTER 14

Arborescent projections

This section classifies the arborescent knot projections (or parts thereof) that appeared in Chapter 1, using the language of weighted planar trees established in Chapter 12. Here the reader can learn many of the procedures one needs in practice to identify an arborescent knot given via a projection. In conclusion, two simple systems of linear notations for arborescent knot projections are proposed, and formulae to translate Conway's original notations [**Conw**] are given.

14.1. Plumbing tangle projections

At the end of §3.3, we introduced the notion of ***knot pair projection***. Recall that this consists of: a compact surface S in the 2–sphere S^2; an immersed 1–manifold $L \subset S$, meeting the boundary ∂S transversely, whose only singularities are transverse double points in $\text{int}(S)$; and crossing information at each double point of L, indicating which branch of L lies "over" the other. Thus $\mathcal{L} = (S, L; \text{crossing data})$. As in Chapter 1, each knot projection serves to describe a knot pair (M, K) where M is a 3–submanifold in $S^3 \supset S^2$ whose boundary consists of 2–spheres. Note that S is oriented (by S^2) and M is oriented by S^3 but L and K are not supposed oriented.

Remark 14.1. This $L \subset S \subset S^2$ was usually denoted Γ in Chapters 1 and 6. In this chapter, however, we prefer to reserve the letter Γ for weighted planar trees.

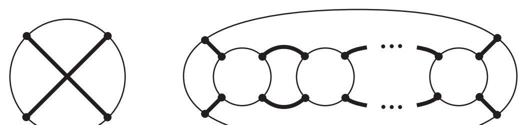
Such a knot pair projection is a ***Conway projection*** if each boundary component is a Conway circle, namely if L meets each component of ∂S in 4 points. As explained in Chapter 1, Conway projections describe knot pairs in S^3 whose boundary consists of Conway spheres. The diagrams in this book provide numerous examples of such Conway projections.

In §12.1, we also introduced some notational devices to specify a so-called tangle structure on the knot pair described by a Conway projection. If the underlying pair is (S, L), this was done by labelling, on each component C of ∂S, one of the four arcs $C - L$ by an arrow running parallel to it. The data consisting of the knot projection together with these arrow labellings is called a ***tangle projection***.

An ***isomorphism*** of knot pair (or tangle) projections is an isomorphism $(S, L) \to (S', L')$ of the corresponding underlying pairs, preserving the orientations from S^2 and all extra data. A ***flip-isomorphism*** is either an isomorphism as above, or a degree -1 pair isomorphism $(S, L) \to (S', L')$ reversing the signs of the crossing (and still respecting the arrow labellings for tangle projections). Observe that flip-isomorphisms induce degree $+1$ isomorphisms between the corresponding knot pairs or tangles.

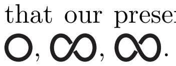
REMARK 14.2. We shall often use a pair symbol such as (S, L) to stand for a knot pair projection or a tangle projection whose underlying pair is (S, L), and rely on plain words to indicate the extra structure being entertained.

A Conway projection (S, L) is defined to be ***arborescent*** if there exists a family F of disjoint Conway circles in S such that each closed-up component of $S - F$ gives a Conway projection of one of the two types of Figure 14.1.



$k > 0$ boundary components

FIGURE 14.1.

Recall that in §1.1 we have already given an ad hoc definition of the term *arborescent* for knot projections (S^2, L) that are irreducible. That definition seems a little more restrictive, but Proposition 14.4 below will show that this is not so and that our present terminology is consistent, at least when (S^2, L) is distinct from .

We call a *tangle* projection ***arborescent*** when the underlying Conway projection is arborescent in the sense just defined.

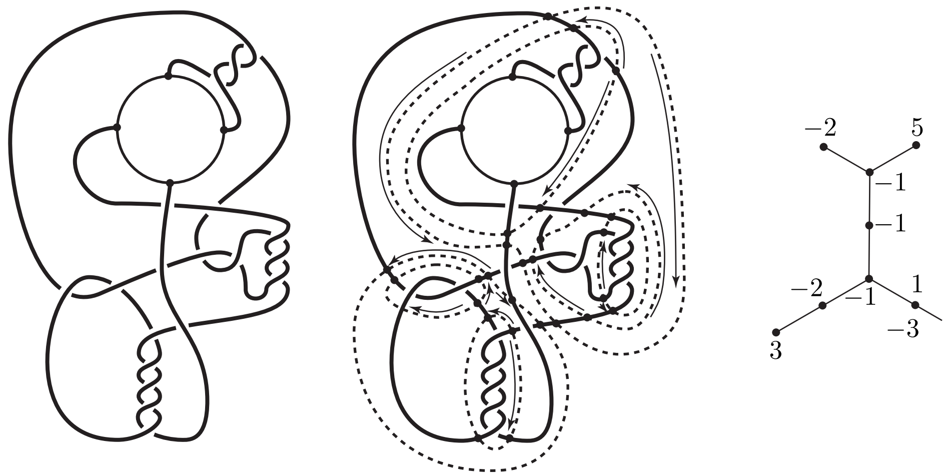
The plumbing of tangles described in §12.1 readily extends to tangle projections in the following way. Consider two tangle projections, with underlying knot pair projections (S_1, L_1) and (S_2, L_2), and select a component C_1 of ∂S_1 and a component C_2 of ∂S_2. Each of C_1 and C_2 has an orientation induced by the arrow of the tangle projection, and another orientation as boundary component of S_1 or S_2.

If these two orientations agree on both C_1 and C_2, or disagree on both C_1 and C_2, choose an isomorphism $\theta : (C_1, L_1 \cap C_1) \to (C_2, L_2 \cap C_2)$ such that the arrow of C_2 and the image by θ of the arrow of C_1 emanate from the same point, but in different directions (compare Figure 12.7). Then the ***plumbing of the two tangle projections*** is the tangle projection that consists of the knot pair projection $(S_1, L_1) \bigcup_\theta (S_2, L_2)$ (naturally oriented!), together with the arrow labellings coming from the original tangle projections. Observe that the resulting tangle projection is well-defined up to tangle projection isomorphism.

If the arrow orientation and boundary orientation agree on one of C_1 and C_2 and disagree on the other one, the plumbing of the two tangle projections is only defined up to flip isomorphism: First modify one of the tangle projections by a flip isomorphism, and then apply the previous construction.

Clearly, this construction is consistent with the tangle plumbing of §12.1, in the sense that the tangle described by the plumbing of two tangle projections is the tangle obtained by plumbing the associated tangles, along the boundary Conway spheres corresponding to the boundary circles selected in the construction.

In particular, this enables us to associate a tangle projection to any weighted planar tree Γ as in §12.1. Indeed, for each vertex of Γ that is as in Figure 12.9, choose a copy of the tangle projection of Figure 12.8 of the same section. Then associate to Γ the tangle projection obtained by plumbing these atomic tangle projections

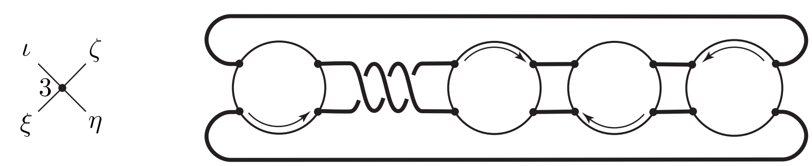


(a) (b) (c)

FIGURE 14.2.

together according to Γ. Observe that, for such an atomic tangle projection, the arrow and boundary orientations disagree on all boundary components. Thus, the tangle projection so associated to Γ is well-defined up to isomorphism. Figure 14.2 gives an example of such a plumbing construction according to a weighted planar tree.

This construction readily extends to weighted planar trees Γ whose free bonds bear labels in V_4 as in §12.7, by modifying the arrow labellings accordingly. For instance, the weighted planar tree of Figure 14.3(a) specifies the tangle projection of Figure 14.3(b).



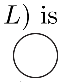
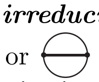
(a) (b)

FIGURE 14.3.

Since every atomic tangle projection is arborescent, plumbing as above certainly yields only tangle projections that are arborescent. The converse is

PROPOSITION 14.3. *Every arborescent tangle projection results by plumbing according to a weighted planar tree (no bond markings are needed).*

PROOF. Imitate the proof of the parallel 3–dimensional result, namely Theorem 12.4. □

Remember from §1.1 that a knot pair projection (S, L) is ***irreducible*** if every circle in S cutting L in $\leqslant 2$ points bounds a vignette  or  . A tangle projection is irreducible when so is its underlying knot pair projection.

We shall call a projection is ***efficient*** if it contains no vignette  .

The reader will perceive that the knot pair projection associated to a weighted planar tree Γ is left unchanged after alteration of Γ by the moves (1.0), (1.1), (0.1), (0.2) and (0.1*) of the calculus for arborescent pairs given in §12.3, *provided* that the projection is irreducible and efficient. (Beware this is not so for (1.2), (2.0), (2.1) and (2.2).) Also, there is a planar analog of Lemma 12.8 for knot pair projections, revealed clearly by Figure 8.31. This leads immediately to a (partial) analogue of our canonicity result for arborescent pairs, Corollary 12.10.

Proposition 14.4. *Any efficient irreducible arborescent knot pair projection is isomorphic to one obtained by plumbing atomic tangle projections according to a weighted planar tree Γ with the following properties:*

(a) *Γ has no valence $\leqslant 2$ vertex at which all weights are 0.*
(b) *Γ has no univalent vertex with weight ± 1, unless Γ has just one vertex.* □

Remark 14.5. Property (b) easily shows that, for irreducible knot projections distinct from  ,  ,  , the definition of arborescent given in §1.1 is equivalent to our present definition.

A weighted planar tree Γ which satisfies Conditions (a) and (b) of Proposition 14.4 will be called a ***tidy tree***.

The converse of Proposition 14.4 is true, namely

Proposition 14.6. *Every tidy tree Γ yields, by plumbing atomic tangle projections, an efficient irreducible knot pair projection.*

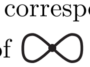
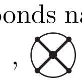
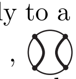
Proof. Exercise. If (S, L) is any one of the atomic projections plumbed, observe that it is irreducible and that the following property holds by Condition (a): For any arc $k \subset S$ with $\partial k \subset \partial S$ which meets L in $\leqslant 1$ point, transversely, k is pairwise boundary parallel in (S, L). □

We now have the following 2–dimensional analog of our classification Theorem 12.12 for arborescent pairs.

Theorem 14.7. *If two tidy trees Γ and Γ' yield, by plumbing, the same knot pair projection (S, L), and Γ is not among* $\overset{\pm1}{\bullet}$, $\overset{\pm1}{\bullet\!-}$, $\overset{0}{-\!\bullet\!-}$, $\overset{0}{\bullet\!-}$, *then after a possible modification of Γ by the simplest flip* (F_1) *of the calculus of arborescent pairs (*§12.3*), there exists an isomorphism of weighted planar trees $\Gamma \to \Gamma'$ respecting the correspondences of free bonds to the boundary circles of S.*

Proof. Let G be the family of the Conway circles in the interior of S corresponding to the edges of Γ, namely arising from plumbing. Let G' be similarly associated to Γ'.

In S, we momentarily crush each boundary circle to a point, getting from (S, L) a pair $(\widehat{S}, \widehat{L})$ consisting of a 2–sphere and a quadrivalent graph in it. Theorem 1.1 applies to this pair, and tells us that G above is pairwise isotopic to G'. Then we can and do assume that $G = G'$.

Let $X_1, \ldots, X_n$ be the closed-up components of $S - G$. Each X_i gives an atomic projection $(X_i, L \cap X_i)$ that corresponds naturally to a vertex of Γ and one of Γ'. Since $(X_i, L \cap X_i)$ is none of  ,  ,  ,  , there are exactly two ways to label boundary circles with arrows so as to produce an atomic tangle projection. Thus after at most a flip (F_1) on Γ as advertised, the arrow labellings arising from the plumbing for Γ and the plumbing for Γ' are the same on X_1. The plumbing rule that, on each plumbing circle, the two arrows emanate from the same point in opposite directions (compare Figure 12.7 and Figure 14.2) now assures by connectivity that this agreement of labellings holds on all the blocks $X_1, \ldots, X_n$. Then, Γ and Γ' are clearly isomorphic. □

REMARK 14.8. The above proof of Theorem 14.7 gives a very efficient way to find the tidy tree thus associated to a knot pair projection. Indeed, starting from the characteristic family of Conway circles characterised by Theorem 1.1, one just has to draw arrows in a coherent way so as to present this knot projection as a plumbing of atomic tangle projections. Figure 14.2 provides an example of the three steps in this procedure.

A salient conclusion to retain from the above discussion is:

COROLLARY 14.9. *Plumbing gives a natural bijection from the set of tidy trees, considered up to the equivalence generated by isomorphism and turning over (= Flip* (F_1)*), and the set of isomorphism classes of irreducible efficient arborescent knot pair projections.*

For tangle projections instead of knot pair projections, there are immediate analogs of Proposition 14.4, Theorem 14.7 and Corollary 14.9. They are straightforwardly obtained from these statements by replacing the notion of tidy tree by that of ***tidy tangle tree***. By definition, a tidy tangle tree is a weighted planar tree Γ with free bonds labelled by elements of V_4, which satisfies the following two conditions:

(a) Γ has no valence $\leqslant 1$ vertex at which all weights are zero, unless this vertex has a bond free in Γ.
(b) Γ has no univalent vertex with weight ± 1, unless Γ has just one vertex.

This establishes a one-to-one correspondence between the set of irreducible efficient arborescent tangle projections, up to isomorphism, and the set of tidy tangle trees, up to isomorphism and Flip (F_1) of the calculus of arborescent tangles (§12.7).

14.2. Linear notations for weighted planar trees

In view of the intimate relationship between weighted planar trees and arborescent tangle projections, a linearly ordered digital notation for such trees is of importance for tabulation, for preparing typescript without diagrams, and for computer programming. We propose two such notation systems.

14.2.1. First system. There is a very natural cyclic notation for a weighted planar tree, possibly with free bonds marked by elements of V_4, as follows. Position the tree as a graph in the plane so as to be a smooth 1–submanifold except at the vertices of valence $\geqslant 3$ where angles are $< \pi$; then cut open the plane along the graph. This is done for an example in Figure 14.4(a-b).

This cutting produces a hole in the plane (shaded in Figure 14.4(b)) whose boundary is topologically a circle. Going counterclockwise around this circle, one can record in cyclic order events corresponding to weighted sectors and (marked) free bonds, namely events of the four sorts in Figure 14.5 (where $a \in \mathbb{Z}$ and $\alpha \in V_4$).

If we denote these four models respectively by a, $\hat{a}$, $\check{a}$, $\hat{\alpha}$, the tree gets a linear notation. For example, the tree of Figure 14.4(a) is naturally denoted (up to cyclic

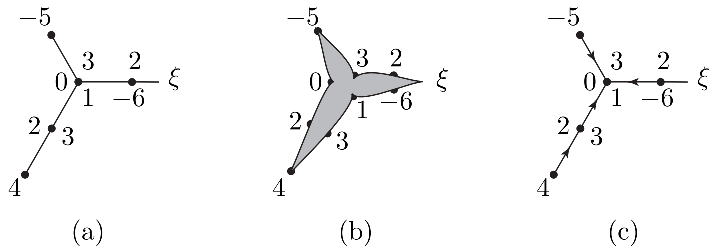


FIGURE 14.4.

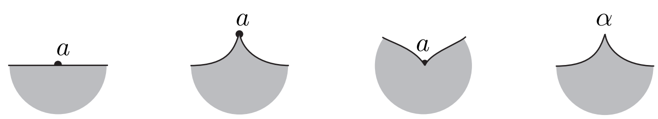


FIGURE 14.5.

reordering) by $\hat{4}3\check{1}\underline{6}\hat{\xi}2\check{3}\hat{\underline{5}}\check{0}2$. Here, a bar under a number is equivalent to a minus sign. Following the procedure in reverse, we retrieve (up to isomorphism) the weighted planar tree from this notation.

The simplicity of this system may make it suitable for computers.

14.2.2. Second system. Next we describe an alternative linear notation for these trees, which depends on the choice of a preferred base vertex. On each edge (not a free bond), we now put an arrowhead pointing toward this base vertex, as in Figure 14.4(c).

We cut open along the tree as before. This time, going counterclockwise around the boundary of the resulting hole in the plane, we record the weights of the angle sectors, the weights of the free bonds, and the orientations of the edges. The edge orientation of an edge is denoted by $\rangle$ if it coincides with the counterclockwise orientation of the boundary of the hole, and by $\langle$ otherwise. In addition, one can omit weights 0. For example, the notation for the graph of Figure 14.4(c) is $4\rangle 3\rangle 1\langle \underline{6}\xi 2\rangle 3\langle \underline{5}\rangle\langle 2\langle$.

It is usually most convenient to pick the base vertex to be polyvalent and in the so-called topological centre of the tree Γ (see Fact 18.9), and to begin listing weights and edge orientations from a point on the boundary of the hole that corresponds to the base vertex.

This system seems most promising for tabulations intended for the human eye.

14.3. Translating Conway's notations

Conway's original system of notation for arborescent knots and tangles [**Conw**], although similar to the one just described, differs subtly from it with regard to the orderings and signs. We try to clarify these differences.

Conway gave in [**Conw**] certain operations on monovalent tangle projections that let one derive any monovalent arborescent tangle projection from the tangle projection of Figure 14.6. He denotes this tangle projection by the integer n.

We recall some of Conway's operations on tangle projections. Let A and B be two monovalent tangle projections, as in Figure 14.7.

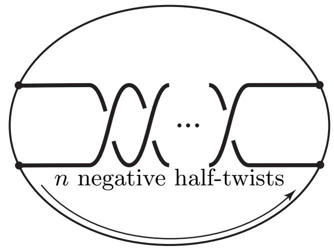


FIGURE 14.6.

Minus sign $A \mapsto -A$**.** The tangle projection $-A$ is obtained by converting overcrossings into undercrossings and *vice versa*. Expressed in three dimensions, this corresponds to the reflection of the knot strings in the projection plane.

Slope -1 **reflection** $A \mapsto \neg A$**.** If we use the old "scaling and translation" convention of §8.1 to define the parametrisation of boundary by the standard Conway circle  , this rule simply reflects A in a line of slope -1. Note that this operation changes over- to under-crossings, and conversely.

Addition $A, B \mapsto A{+}B$**.** Here the sum tangle $A{+}B$ is presented in Figure 14.7. This operation is clearly associative.

Closing-up $A \mapsto 1^*(A)$**.** This operation results in the knot projection (with no boundary) represented on the right hand side of Figure 14.7.

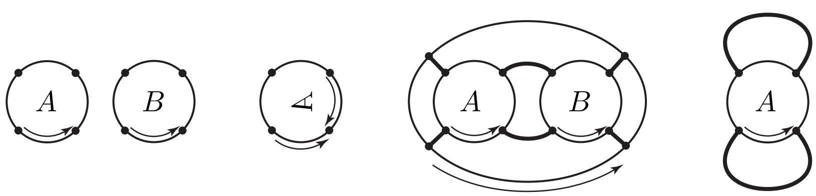


FIGURE 14.7.

Three more operations, represented in Figure 14.8, are derived from these.

Multiplication $AB = (\neg A) + B$**.** More generally, the product $A_1A_2 \ldots A_n$ is inductively defined by the property that $A_1A_2 \ldots A_n = (A_1A_2 \ldots A_{n-1})A_n$.

Comma operation $(A, B, \ldots, C) = \neg A + \neg B + \cdots + \neg C$**.**

Enclitic sign $A+ = \neg(\neg A + 1) = \neg(\neg A, -1)$**.** The operation $A- = \neg(\neg A - 1) = \neg(\neg A, +1)$ is similarly defined.

We now explain how to go from Conway's notation to our description of arborescent projections in terms of weighted planar trees.

From the outset, we note that Conway's integral tangle projection n is isomorphic to the ones that we denote by $\overset{-n}{\bullet\!\!-}\,\xi$ or $\overset{-n}{\bullet\!\!-}\,\eta$, but not to $\overset{-n}{\bullet\!\!-}$ nor to any tangle plumbed from a weighted planar tree without marked bonds. Indeed, the arrow in Figure 14.6 agrees with boundary orientation, whereas the reverse holds for any model atomic tangle projection as in Figure 12.8. To avoid this annoyance we now operate modulo flip-isomorphisms of tangle projections.

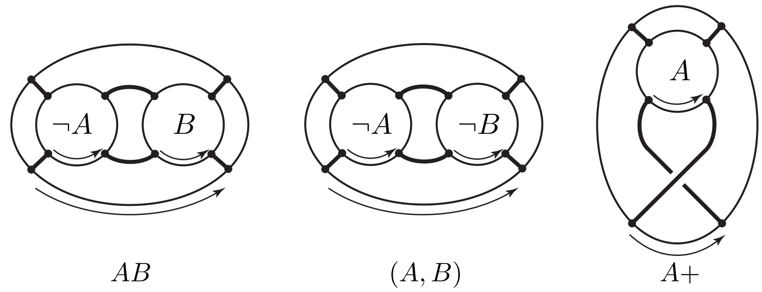
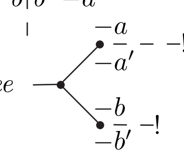
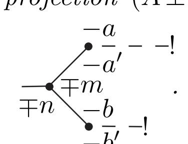


FIGURE 14.8.

PROPOSITION 14.10. *Let A, B be monovalent arborescent tangle projections described by the weighted planar trees* $-\overset{a'}{\underset{a}{\bullet}}- - -$ *and* $-\overset{b'}{\underset{b}{\bullet}}- -$ *, respectively. Let an exclamation mark ! applied to a subgraph $- - -$ indicate that all weights in the subgraph are multiplied by -1, and that the cyclic order of weights and bonds is reversed at all vertices of $- - -$ that are at even distance from the vertex to which the subgraph is attached. Then, up to flip-isomorphism,*

(1) *Conway's integer tangle projection n is (flip-isomorphic to the tangle projection) described by the weighted planar tree* $\overset{-n}{\bullet}\!-$ *;*
(2) *the tangle projection $-A$ is described by the weighted planar tree obtained from* $-\overset{a'}{\underset{a}{\bullet}}- - -$ *by reversing the sign of all weights;*
(3) *the knot projection $1^*(A)$ is described by the weighted planar tree $a + a'\bullet - - -$;*
(4) *the tangle projection AB is described by the weighted planar tree* $-\underset{b\,|\,b'}{\bullet}\!-\!\overset{-a}{\underset{-a'}{\bullet}}\; - -!$*;*
(5) *the tangle projection (A, B) is described by the weighted planar tree* 

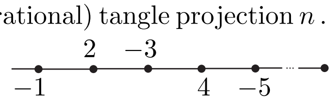
;
(6) *the tangle projection* $(A\overbrace{\pm\cdots\pm}^{total\ m}, B\overbrace{\pm\cdots\pm}^{total\ n})$ *is described by the weighted planar tree* 

.

The proof of Proposition 14.10 is left as an exercise to the reader.

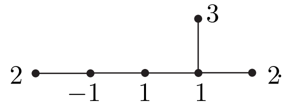
In his tabulations [**Conw**], Conway specifies arborescent knot projections by linear notations using his integral tangles n and the above operations. To be sure, there are a few more abbreviations and conventions to learn; for instance, he uses the same symbol A to denote the tangle projection A and the knot projection $1^*(A)$. Proposition 14.10 provides a simple algorithm to mechanically write down a weighted planar tree specifying the same projection.

Examples

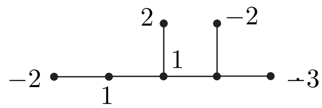
(1) Conway's (rational) tangle projection $n \dots 54321$ is described by the weighted planar tree 

$(-1)^n n$.

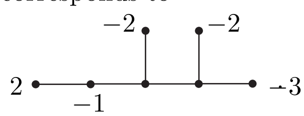
(2) The knot projection $1^*(211, 3, 2-)$, abbreviated by Conway as $211, 3, 2-$, corresponds to the plumbing tree



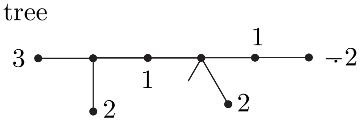
(3) Similarly, Conway's $(3, 2)(21, 2-)$ is described by the weighted planar tree



(4) Conway's knot projection $(3, 2)-(21, 2)$, which stands for $1^*\big((3, 2)\big(-(21, 2)\big)\big)$, corresponds to



(5) the tangle projection $(3, 2)1(21, 2)$ is associated to the weighted planar tree



For more information along these lines, see A. Caudron's extensive notes [**Cau**]. In fact, we benefited from several conversations with Caudron in the period when our weighted planar tree notations evolved from their origins in [**Sie**$_1$].

Part 6

Knot symmetries

CHAPTER 15

Knot symmetries and characteristic splittings

So far, we have discussed knots (S^3, K) where K carries no specified orientation (although S^3 does), and no preferred labelling of its components (strings). However, it is sometimes important to keep track of such an extra information. For instance, we have already briefly encountered such situation in Chapter 2, where the classification of all knots in S^3 was reduced to that of (finite collections of) knots which are *simple for Schubert* and which are equipped with certain isomorphisms among their string components.

However, it is immediate that this last problem for knots that are simple for Schubert would be solved by:

(a) a classification of such simple knots, ignoring string orientations;
(b) for each such knot (S^3, K) a determination of the action (up to isotopy) of its automorphisms on the components of K.

Towards the classification step, we have made significant progress in previous chapters. This chapter, as well as the following ones, addresses the automorphism problem.

More generally, we will study the ***symmetry group*** $\mathrm{Sym}(M, K)$ of a given knot pair (M, K) defined as the group of pairwise isotopy classes of degree ± 1 automorphisms of (M, K). The subgroup of $\mathrm{Sym}(M, K)$ that consists of degree $+1$ automorphisms is denoted by $\mathrm{Sym}^+(M, K)$.

In the present chapter, we exploit the characteristic splittings defined in Chapters 2 and 3, and show how the symmetry group of (M, K) can be reconstructed from the symmetry groups of the pieces defined by these splittings. Some of these results will be improved in Chapter 17 by the extraction of characteristic markings on the boundary of these pieces.

Our best results will be obtained in Chapter 16, where we calculate the symmetries of almost all arborescent knots (S^3, K) that are simple for Schubert (see [**Boi**$_1$**, BoiZ, Sak**] for the remaining cases[1]). The symmetry group of such an arborescent (S^3, L) is there determined as a set, but the exact sequences of Chapter 16 leave some group extension problems. We shall settle these in Chapter 18 by realising $\mathrm{Sym}(S^3, K)$ as a finite group of isometries of S^3, after isotopy of K. This also provides a good visualisation of these symmetries, which are more abstractly studied in Chapter 16.

15.1. Knot symmetries and characteristic companions

Consider a knot pair (M, K) with M *connected*. If this pair is ***splittable***, in the sense that $M - K$ contains a 2–sphere S that does not bound a 3–ball in

[1](Added 2009) See also [**Tur**] for another determination of the action of the symmetry group of a Montesinos knot on its set of orientations.

$M - K$, then there exist curious automorphisms of (M, K). First, one splits M at S into two pieces, and plugs the holes with two balls B_1, B_2 to get $(\widehat{M}, K)$; next one slides B_1, B_2 in $\widehat{M}$ isotopically around loops avoiding K to their original positions; the end of the isotopy provides an automorphism of (M, K) fixing S and K. These automorphisms are currently being sorted out; see [**Bon**$_2$, App. A][**Henr, HendL, Wic, HendM**]. Henceforth we avoid such automorphisms by assuming (M, K) to be ***unsplittable***.

Aiming to apply the characteristic companionship factorisation of Chapter 2, we assume $H_1(M; \mathbb{Z}) = 0$ (an assumption one could relax somewhat). Let Γ be its characteristic companionship tree as defined in Chapter 2, and denote the companion knots that are its vertices by (N_j, K_j); here $H_1(N_j; \mathbb{Z}) = 0$. (We are most interested in the case where $M \cong S^3$, and hence $N_j \cong S^3$.) Let $G \subset M - K$ be the characteristic family of non-trivial tori provided by Theorem 2.4 and defining Γ; recall each component of G is associated to an edge joining two vertices of Γ.

If φ is a degree ± 1 automorphism of (M, K), it is pairwise isotopic to φ' with $\varphi'(G) = G$ since G is characteristic. Now, φ' induces an automorphism $\widehat{\varphi}$ of $\coprod_j (N_j, K_j)$ that is compatible with Γ in the sense of §2.5. Let $\mathrm{Sym}_\Gamma \coprod_j (N_j, K_j)$ denote the group of pairwise isotopy classes of degree ± 1 automorphisms of $\coprod_j (N_j, K_j)$ that are compatible with Γ.

For some automorphisms φ with $\varphi(G) = G$, it may happen that the induced automorphism $\widehat{\varphi}$ of $\coprod_j (N_j, K_j)$ is pairwise isotopic to the identity. This occurs for ***Dehn twists along*** G, namely for every automorphism φ constructed in the following way: Given a regular neighbourhood $U \cong G \times [0, 1]$ of G (avoiding K) and an isotopy $t \mapsto \psi_t \in \mathrm{Aut}(G)$ with $\psi_0 = \psi_1 = \mathrm{Id}$, let φ be the identity outside of U and let $\varphi(x, t) = (\varphi_t(x), t)$ on $U \cong G \times [0, 1]$. Up to pairwise isotopy, the Dehn twist φ depends only on the normal orientation of G defined by the parametrisation $U = G \times [0, 1]$ and on the class of $t \mapsto \psi_t$ in $\pi_1(\mathrm{Aut}(G); \mathrm{Id})$.

Let $D(M, K)$ denote the subgroup on the elements of $\mathrm{Sym}(M, K)$ that are represented by Dehn twists along G.

THEOREM 15.1. *With the above data, the rule $\varphi \mapsto \widehat{\varphi}$ gives a well-defined group homomorphism R lying in an exact sequence*

$$0 \longrightarrow D(M, K) \longrightarrow \mathrm{Sym}(M, K) \xrightarrow{R} \mathrm{Sym}_\Gamma \coprod_j (N_j, K_j) \longrightarrow 0.$$

PROOF. To prove that the "restriction" map R is well-defined, consider another automorphism φ'' that is pairwise isotopic to φ and such that $\psi''(G) = G$. By Proposition 5.20, the automorphisms φ' and φ'' are pairwise isotopic by an isotopy preserving G, and φ'' therefore induces the same element of $\mathrm{Sym}_\Gamma \coprod_j (N_j, K_j)$ as φ'.

Any element in the kernel of R is represented by an automorphism that is the identity outside of a regular neighbourhood $U \cong G \times [0, 1]$ of G. By [**Wal**$_3$, Lemma 3.5], such an automorphism can itself be deformed to a Dehn twist by a (pairwise) isotopy fixing the complement of U. Thus $\mathrm{Ker}(R) = D(M, K)$.

The surjectivity of R is clear. □

Note that a Dehn twist along G can be pairwise isotopic to the identity even if it is constructed from a non-trivial element of $\pi_1(\mathrm{Aut}(G); \mathrm{Id})$. This occurs when a closed-up component (M_j, K'_j) of $(M, K) - G$ is (pairwise) Seifert fibred possibly with "infinitely singular" fibres as in §2.2. Indeed, orienting $G \cap M_j$ as boundary of M_j, let φ be the Dehn twist associated to the isotopy $t \mapsto \psi_t \in \operatorname{Aut}(G)$ that fixes $G - M_j$ and is an isotopic translation once around the non-singular (oriented) fibres of $G \cap M_j$. Clearly, φ is pairwise isotopic to the identity by an isotopy respecting the fibres of M_j. (This construction has used a coherent orientation of the Seifert fibres, whose existence is assured here by the condition $H_1(M) = 0 = H_1(N_j)$.)

Conversely, an argument very close to that used in the proof of Proposition 5.20, involving a pairwise essential surface in each closed-up component of $(M, K) - G$, shows that *an arbitrary Dehn twist along G* is pairwise isotopic to the identity (if and) only if it is a product of Dehn twists of the above type. In particular, $D(M, K) \cong \pi_1(\operatorname{Aut}(G); \mathrm{Id})$ when no closed-up component of $(M, K) - G$ is Seifert fibred.

To compute the group $D(M, K)$ more formally, choose a normal orientation for G. Via the above construction, this orientation determines a group homomorphism $\pi_1(\operatorname{Aut}(G); \mathrm{Id}) \to D(M, K)$. Note that since G consists of tori, there is also a canonical isomorphism $\pi_1(\operatorname{Aut}(G), \mathrm{Id}) \xrightarrow{\cong} H_1(G)$ which to an isotopy $t \mapsto \psi_t$ associates the homology class of the loops $t \mapsto \psi_t(X^0)$, where X^0 is any finite set with one point in each component of G.

For every closed-up component (M_j, K'_j) of $(M, K) - G$ that is Seifert fibred, pick out an (oriented) fibre C_{ji} on each component G_{ji} of ∂M_j. Let $\varepsilon_{ji} = +1$ or -1 according as the orientations of G and ∂M_j coincide on G_{ji} or not, and let V be the subspace of $H_1(G)$ generated by the elements $\sum_i \varepsilon_{ji}[C_{ji}]$, where (M_j, K'_j) ranges over all Seifert fibred closed-up components of $(M, K) - G$.

Now, the result italicised above can be rephrased as

PROPOSITION 15.2 (for the data of 15.1). *Given a normal orientation of G, the group of Dehn twists $D(M, K)$ is canonically isomorphic to $H_1(G)/V$.* □

COROLLARY 15.3. *$D(M, K)$ is finite precisely if either $G = \varnothing$, or G is connected and each of the two characteristic companion knots is Seifert fibred.* □

The group $\operatorname{Sym}_\Gamma \coprod_j (N_j, K_j)$ is completely analysed if we know for which $i \neq j$ the two pairs (N_i, K_i) and (N_j, K_j) are isomorphic (which is just a classification problem), and if the symmetry group of each (N_j, K_j) is already determined. Recall that, for each j, the pair (N_j, K_j) is either simple for Schubert or Seifert fibred (or both). In the second case, the determination of $\operatorname{Sym}(N_j, K_j)$ is provided by [**Wal**$_2$] or [**Wal**$_4$].

In case (N_j, K_j) is simple for Schubert, we will be discussing its symmetry in the rest of this chapter; alternatively, to calculate $\operatorname{Sym}(N_j, K_j)$, one can sometimes explicitly examine the complete hyperbolic structure on $N_j - K_j$, as in [**Ril**$_1$**, Ril**$_2$].

The reader should be warned that even when the extreme terms of the short exact sequence of Theorem 15.1 are known, there may be some subtle extension problems in determining the middle term $\operatorname{Sym}(M, K)$; see [**Sie**$_2$] for examples of these. Here is an example of a routine calculation with the exact sequence of Theorem 15.1.

Sample calculation

Consider the 2–component knot (S^3, K) whose companionship tree in the sense of §§2.2–2.4 is —•——•—, where the companions (N_1, K_1) and (N_2, K_2) associated to the two vertices are each isomorphic to the knot of Figure 15.1, where the splicing occurs along the components $K_{12} \subset K_1$ and $K_{21} \subset K_2$ singled out by the arrow, and where the identification $K_{12} \cong K_{21}$ used in the splicing respects the orientations specified by the arrow.

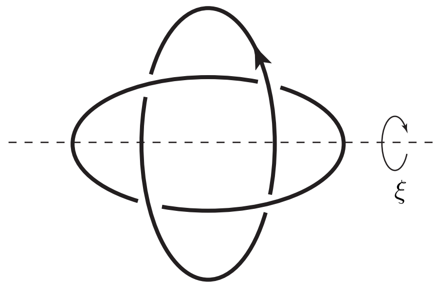


FIGURE 15.1.

The two pairs (N_1, K_1) and (N_2, K_2) are isomorphic Seifert fibred pairs and, using [**Wal**$_2$], one easily checks that

$$\mathrm{Sym}_\Gamma((N_1, K_1) \amalg (N_2, K_2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where one generator exchanges (N_1, K_1) and (N_2, K_2) respecting the arrows, and the other one is, on each factor, the π–rotation illustrated in Figure 15.1.

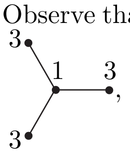
Let G be the (unique) non-peripheral incompressible torus in $S^3 - K$. The closed-up component of $(S^3, K) - G$ corresponding to (N_1, K_1) is Seifert fibred; if $C_{12} \subset G$ is one fibre of this fibration, then $\pm[C_{21}] = l_1 - 2m_1$ in $H_1(G)$, where l_1 and m_1 respectively represent a longitude and a meridian of the knot $K_{12} \subset N \cong S^3$, oriented by the arrows. (The meridian has linking $+1$ with the knot.) Similarly, with the obvious notations, $\pm[C_{21}] = l_2 - 2m_2$ in $H_1(G)$. Applying Proposition 15.2 and the fact that one has $l_2 = m_1$ and $l_1 = m_2$, it follows that

$$D(S^3, K) \cong H_1(G)/\langle [C_{12}], [C_{21}] \rangle \cong \mathbb{Z}_3.$$

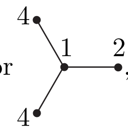
Thus, Theorem 15.1 defines an exact sequence

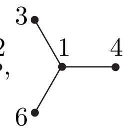
$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow \mathrm{Sym}(S^3, K) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 0.$$

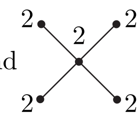
It is easy to see that his extension splits, and is therefore determined by the conjugacy action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{Z}_3$. This action of $\varphi \in \mathrm{Sym}_\Gamma((N_1, K_1) \coprod (N_2, K_2))$ on $D(M, K)$ is the identity or $x \to x^{-1}$ according as φ preserves or exchanges (N_1, K_1) and (N_2, K_2). Consequently, $\mathrm{Sym}(S^3, K) \cong D_6 \oplus \mathbb{Z}_2$, where D_6 is the dihedral group of order 6; this completes the calculation.

Observe that this knot is the arborescent knot classified by the weighted planar tree 

, which we have already encountered in Figure 2.2 of §2.1 and in Proposition 8.22. It is one of the four irreducible Montesinos knots presentable without rings that are not simple for Schubert. We shall see $\mathrm{Sym}(S^3, K)$ as a subgroup of the orthogonal group O(4) in Chapter 18.

EXERCISE 15.4. Perform similar calculation for 

, 

and 

(these are the other three!).

15.2. Knot symmetries and the arborescent part

Theorem 15.1 largely reduces the symmetry problem to the case of a knot pair (M, K) that is simple for Schubert. To study its symmetries we now use the analysis of Chapter 7 to extract its arborescent part A; this part is well-defined up to pairwise isotopy *under the hypothesis that* $M = S^3$ *or more generally that* $H_1(M; \mathbb{Z})$ *has no torsion.*

Consider a degree ± 1 automorphism ψ of (M, K); it can be pairwise isotoped, by Theorem 7.1, so that it preserves A.

THEOREM 15.5. *Restriction of* ψ *to* A *induces a well-defined homomorphism*

$$r \colon \mathrm{Sym}(M, K) \to \mathrm{Sym}(A, K \cap A).$$

This homomorphism r *is injective provided* $A \neq \varnothing$ *and* M *is connected.*

PROOF. The proof that the homomorphism is well defined is the same as for Theorem 15.1, using Proposition 5.20.

To prove the injectivity, choose a maximal family F' of disjoint pairwise incompressible Conway 2–spheres in $(M, K) - \mathrm{int}(A)$, no two pairwise parallel and none boundary parallel. By Theorem 7.1, the automorphism ψ can be pairwise isotoped fixing A to respect $F' \cup A$. Each component of $(M, K) - (F' \cup A)$ has a complete finite volume π–hyperbolic structure by Theorem 5.9. Then, the Rigidity Theorem 5.11 (inductively applied) shows that ψ is pairwise isotopic to the identity. □

EXERCISE 15.6. Consider the knot (S^3, K) of Figure 15.2, together with the Conway sphere F splitting the pair (S^3, K) into two pieces $(A, A \cap K)$ and $(B, B \cap K)$.

The pair $(A, A \cap K)$ is clearly arborescent, and not a rational tangle pair. The pair $(B, B \cap K)$ is π–hyperbolic; indeed, it is just the pair of Figure 4.4, and its double branched covering is the complement of the figure-eight knot. Consequently, F is pairwise incompressible and A is the arborescent part of (S^3, K).

We shall (easily) prove later that $\mathrm{Sym}(A, A \cap K)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by the two π–rotations ξ and η of Figure 15.3.

The rotation η extends to an automorphism of (S^3, K) as indicated in Figure 15.2, but ξ obviously does not (note that exactly one component of $B \cap K$ is a knotted arc in B). Consequently, it follows from Theorem 15.5 that $\mathrm{Sym}(S^3, K) \cong \mathbb{Z}_2$, generated by the rotation η of Figure 15.2.

A few useful results on the symmetries of π–hyperbolic pairs can be found in Chapters 1, 4, 5 and 6. See also [**Ril**$_1$**, Ril**$_2$]. The existing theory is not always helpful and is surely incomplete. On the other hand, we are going to give a rather satisfactory analysis of the symmetry group of the (marked) arborescent part of a knot.

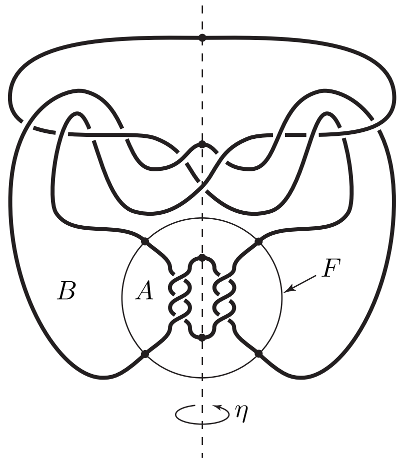


FIGURE 15.2.

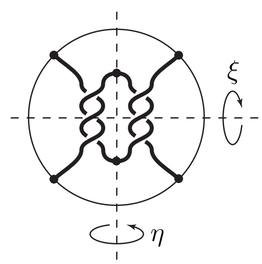


FIGURE 15.3.

Theorem 15.5 states in effect that $\mathrm{Sym}(A, K \cap A)$ is an upper bound for $\mathrm{Sym}(M^3, K)$. In the next chapter we proceed to a calculation (occasionally incomplete) of this upper bound, using the canonical graphs of Chapter 12.

We note from the outset that two improvements will be called for:

(i) When $\mathrm{Sym}(A, K \cap A)$ is infinite, a smaller, finite, upper bound is wanted; one is provided in Chapter 17, which involves boundary markings of $(A, K \cap A)$ determined by the rest of the knot.

(ii) $\mathrm{Sym}(A, K \cap A)$, when finite, is to be realised geometrically in Chapter 18 as a finite subgroup of the isometry group $\mathrm{O}(4)$ of S^3.

CHAPTER 16

Symmetries of arborescent knots and pairs

In this chapter, we analyse the symmetry group of a pairwise irreducible arborescent pair (M, K).

To prepare for the improvement (i) mentioned at the end of the previous chapter, sharpening the upper bound provided by Theorem 15.5, we assume (M, K) equipped with some boundary marking C as in §12.8, and will compute the ***symmetry group*** $\mathrm{Sym}(M, K; C)$ of the marked arborescent pair $(M, K; C)$, namely the subgroup of $\mathrm{Sym}(M, K)$ respecting the marking C.

Lacking a better choice for C, we can certainly choose it to be the ***vertical marking*** C_∞ defined as follows: Consider the characteristic splitting of (M, K) into Montesinos pairs studied in Chapter 9; then C_∞ is defined by the trace of the bands of restricted presentations of these Montesinos pieces.

FACT 16.1. *If* (M, K) *is neither a rational tangle pair nor a thickened Conway sphere, then* $\mathrm{Sym}(M, K; C_\infty) = \mathrm{Sym}(M, K)$.

PROOF. Combine the uniqueness of the splitting of Chapter 9 together with the uniqueness of necklaces proved in Chapter 10. □

Our computation of $\mathrm{Sym}(M, K; C)$ will be in terms of the abbreviated $(+)$–canonical weighted planar tree Γ used to classify the marked pair $(M, K; C)$ in §12.8 and §13.3. Remember that each vertex of Γ carries a ring number $r \in N$, usually inscribed inside of it as ⓡ. This notation is often simplified by denoting each vertex of ring number 0 by a solid black dot $\bullet$, and each vertex of ring number 1 by a small hollow dot $\circ$.

The planar tree also carries integral weights in the angular sectors of its vertices, and Γ satisfies the Weight Condition (W), the Stick Condition (S), the Free Valence Condition (F), the Positivity Condition (P) and the Abbreviation Condition (A) of §12.8.

We proved in §13.3 that the degree $+1$ isomorphism class of $(M, K; C)$ is classified by Γ up to flips (F_1), (F_2), (F'_3), (F''_3) of the calculus of marked arborescent pairs. But Proposition 13.13 actually proves a much sharper statement concerning the isomorphisms between such marked arborescent pairs. These properties will be the backbone of our analysis of $\mathrm{Sym}(M, K; C)$.

Indeed, given two marked arborescent pairs $(M, K; C)$ and $(M', K'; C')$ respectively obtained by plumbing according to two abbreviated $(+)$–canonical trees, and a degree ± 1 isomorphism $\varphi : (M, K; C) \to (M', K'; C')$ between them, Proposition 13.13 tells us that it is often possible to pairwise isotop φ so that it sends each plumbing block (M_v, K_v) of (M, K) to a plumbing block $(M'_{v'}, K'_{v'})$ of (M', K'). In this situation, the rule $v \mapsto v'$ defines an isomorphism $\psi \colon \Gamma_c \to \Gamma'_c$ between the combinatorial trees underlying Γ and Γ', respectively.

We define an ***isogeny*** as a combinatorial isomorphism $\Gamma_c \to \Gamma'_c$ which can occur in this way. More formally, an isogeny is a triple $(\Gamma, \Gamma'; \psi)$ where Γ and Γ' are two abbreviated $(+)$–canonical trees, and $\psi : \Gamma_c \to \Gamma'_c$ is an isomorphism between their underlying combinatorial trees, such that ψ can be realised by a degree ± 1 isomorphism φ: $(M, K; C) \to (M', K'; C')$ in the following sense: The marked arborescent pairs $(M, K; C)$ and $(M', K'; C')$ are respectively associated to Γ and Γ' by plumbing construction, and φ send the building block (M_v, K_v) of (M, K) to the building block $(M'_{\psi(v)}, K'_{\psi(v)})$ of (M', K'). An isogeny has ***degree*** $+1$ (resp. ***degree*** -1) if it can be realised by a degree $+1$ (resp. -1) isomorphism between the corresponding marked arborescent pairs.

We have already encountered a different definition of degree $+1$ isogenies in §12.6. There, we began by defining ***elementary degree*** $+1$ ***isogenies***. These were the preferred combinatorial isomorphisms $\psi\colon \Gamma_c \to \Gamma'_c$ that arise when Γ' is deduced from Γ by one of the flips (F_1), (F_2), (F'_3) and (F''_3) of the plumbing calculus, or by a weighted planar tree isomorphisms. Then, a degree $+1$ isogeny was defined as a product of such elementary degree $+1$ isogenies. We will prove in Lemma 16.2 below that this definition of degree $+1$ isogenies is equivalent to the one given above.

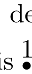
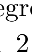
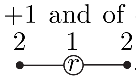
But before that, let us introduce the notion of ***elementary degree*** -1 ***isogeny***. If the marked pair $(M, K; C)$ is obtained by plumbing according to the abbreviated $(+)$–canonical tree Γ, reversal of the orientation of M gives the knot pair $(-M, K; C)$, which is obtained by plumbing according to the weighted planar tree $-\Gamma$ defined by reversing the sign of all weights of Γ. Let then Γ_* be the $(+)$–canonical tree obtained from $-\Gamma$ by performing on it as many of the arithmetic moves (2.1), (2.2), (1.0) and (2.0) as necessary. The three weighted planar trees Γ, $-\Gamma$ and Γ_* have the same underlying combinatorial tree, and $(\Gamma, \Gamma_*; \mathrm{Id})$ is clearly a degree -1 isogeny by construction. An ***elementary degree*** -1 ***isogeny*** is defined as any isogeny $(\Gamma, \Gamma_*; \mathrm{Id})$ so associated to a $(+)$–canonical tree Γ.

LEMMA 16.2. *Any degree* $+1$ *isogeny (in the above sense) is a composition of elementary degree* $+1$ *isogenies. A degree* -1 *isogeny is the product of one elementary degree* -1 *isogeny and of several degree* $+1$ *isogenies.*

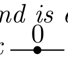
PROOF. Consider a degree $+1$ isogeny $(\Gamma; \Gamma'; \psi)$ induced by a degree $+1$ isomorphism $\varphi\colon (M, K; C) \to (M', K'; C')$ between the corresponding arborescent pairs, sending plumbing block to plumbing block. When Γ is not one of the exceptions of Proposition 13.13, we showed that ψ respects the weights of Γ and Γ'. Then, we proved in Proposition 13.14 that ψ is a product of degree $+1$ isogenies. But remember that the exceptions to Proposition 13.13 were excluded only to make φ send building block to building block, which we already have at hand. So, the arguments apply in all generality to show that any degree $+1$ isogeny is a product of elementary degree $+1$ isogenies, namely that our definition of degree $+1$ isogenies is the same as the one of §§12.5 and 12.8.

The second statement of Lemma 16.2 is immediate, as the composition of two degree -1 isogenies is a degree $+1$ isogeny. □

Note that a degree -1 isogeny in general does not respect the weights of the corresponding weighted trees, not even up to sign.

An isogeny may exceptionally be both of degree $+1$ and of degree -1. The simplest examples consist of $(\Gamma, \Gamma; \mathrm{Id})$ when Γ is ,  or . To analyse this pathology completely, let us call a weight 1 or 2 vertex of Γ ***active*** if an arithmetic move (1.0), (2.0), (2.1) or (2.2) is applied there in the standard transition from positively to negatively canonical form.

PROPOSITION 16.3. *For a positively canonical graph Γ, the following are equivalent:*

(a) *Some isogeny $(\Gamma, \Gamma'; \varphi)$ has both degrees $+1$ and -1.*
(b) *Every isogeny defined on Γ has both degrees ± 1.*
(c) *The identity Id_{Γ_c} has both degrees ± 1.*
(d) *Γ is related to $-\Gamma$ by flips and arithmetic moves of type* (1.0), (2.0), (2.1), (2.2) *of Chapter* 12.
(e) *For every vertex v of Γ, one of the following holds:*
   (i) *v is active;*
   (ii) *v has weight $n \in \mathbb{N}$ and is adjacent to $2n$ active vertices;*
   (iii) *v is attached to a stick* 

with a free bond (and consequently carries no weight).

PROOF. The proof is straightforward. □

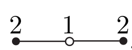
Given a canonical tree Γ the isogenies $(\Gamma, \Gamma; \psi)$, also called ***symmetries*** of Γ, constitute a group denoted by $\mathrm{Sym}(\Gamma)$. Its subgroup consisting of isogenies having degree $+1$ will be denoted by $\mathrm{Sym}^+(\Gamma)$.

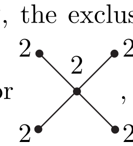
Given Γ, the group $\mathrm{Sym}(\Gamma)$ is rather easy to calculate explicitly. Although its definition is a bit subtle, it is clearly just a subgroup of the combinatorial automorphisms of the combinatorial tree Γ_c; further, in §12.6, we gave a perfectly practical procedure to decide if a combinatorial isomorphism is an isogeny.

THEOREM 16.4. *Let $(M, K; C)$ be a connected pairwise irreducible marked arborescent pair with abbreviated positively canonical tree Γ. Suppose that (M, K) is not the Borromean rings, not the double of the ring pair, nor a closed Montesinos pair presentable with no ring and (exactly) three non-integral tangles.*

Then, there exists a natural surjective group homomorphism $\gamma\colon \mathrm{Sym}(M, K; C) \to \mathrm{Sym}(\Gamma)$. By restriction, it gives a surjection $\gamma^+\colon \mathrm{Sym}^+(M, K; C) \to \mathrm{Sym}^+(\Gamma)$.

REMARKS 16.5.

(i) In terms of the positively canonical graph Γ, the exclusion of the three exceptions means that Γ is not 

, nor 

, nor a stellar tree with three branches and black vertices (and no free bonds).
(ii) For the Borromean rings and the double of the ring pair, Theorem 16.4 *fails*. The symmetries of the Borromean rings will be analysed in Theorem 16.18 below. The double of the ring pair has a non-trivial characteristic companionship tree, with Seifert fibred companions, so that its symmetry group is easily computed by Theorem 15.1 and Proposition 15.2 (using [**Wal**$_2$]).
(ii) Our method completely fails to calculate the symmetry groups of most Montesinos knots with three branches. Nevertheless, arguments involving planar reflection groups, enable M. Boileau and B. Zimmermann to show

that Theorem 16.4, and also the analysis of $\operatorname{Ker}\gamma^+$ in Theorem 16.8, still hold for most, and perhaps all, of these knots. See [**Boi**$_1$, **Boi**$_2$, **BoiZ**][1].

The proof of Theorem 16.4 will occupy us for a while.

Here is how γ is defined for Theorem 16.4. Recall that the plumbing construction of (M, K) according to Γ defines, for each edge e of Γ, a preferred Conway sphere S_e separating two building blocks in (M, K).

Given a degree ± 1 automorphism φ of $(M, K; C)$ we showed in Proposition 13.13 that φ can be pairwise isotoped so as to respect the family $\bigcup_e S_e$ of these preferred Conway spheres, and therefore to induce an isogeny from Γ to itself, namely a symmetry of Γ.

The map γ of Theorem 16.4 will associate this isogeny to the pairwise isotopy class of φ. That γ well-defined is proven by the following statement.

Claim 16.6. *Under the hypotheses of Theorem* 16.4*, let* φ' *and* φ'' *be two automorphisms of* $(M, K; C)$ *respecting the union of the Conway spheres* S_e*. If* φ' *and* φ'' *are pairwise isotopic, then they act similarly on the set of components of* $\bigcup_e S_e$ *and induce the same symmetry of* Γ*.*

Granting the well-definition of γ, the rest of Theorem 16.4 quickly follows. Indeed, γ is clearly a group homomorphism and its surjectivity is immediate from Chapter 12 (and our definitions) as is $\gamma\big(\mathrm{Sym}^+(M, K; C)\big) = \mathrm{Sym}^+(\Gamma)$. Thus, we need only prove this claim.

Proof of Claim 16.6. Let ψ be $\varphi'(\varphi'')^{-1}$. By definition, ψ is pairwise isotopic to the identity and we want to show that ψ maps each Conway sphere S_e to itself preserving orientation.

Let G be the characteristic collection, defined in Chapter 9, of pairwise incompressible Conway spheres, splitting (M, K) into Montesinos pairs. It can be constructed by picking out one element in each pairwise parallelism class containing a pairwise incompressible S_e not pairwise boundary parallel.

In fact, this choice can be made so that $\psi(G) = G$ after a pairwise isotopy of ψ respecting $\bigcup_e S_e$. Indeed, if G' is a component of G, let $S_1, \ldots, S_n$ be the components of $\bigcup_e S_e$ that are pairwise parallel to G'; the indexing can be chosen so that $S_1, \ldots, S_n$ occur in this order in (M, K), namely so that S_i and S_{i+1} are separated by a pairwise parallelism $\cong (S^2, 4\text{points}) \times [0, 1]$ avoiding the other S_e. Now, G' can be chosen (after pairwise isotopy), so that it lies in a median position with respect to the S_i, namely so that $G' = S_m$ if $n = 2m - 1$ is odd and G' lies between S_m and S_{m+1} if $n = 2m$ is even. If each component of G is chosen thus, then ψ clearly can be pairwise isotoped, respecting $\bigcup_e S_e$, so that $\psi(G) = G$.

By Proposition 5.20, the induced map $\psi_G : G \to G$ of ψ is pairwise isotopic to the identity. In particular, ψ maps each component G' of G to itself respecting orientation. It follows that ψ does the same for each S_e that is pairwise incompressible (this is trivial when S_e is pairwise parallel to a boundary component).

It remains to prove this even for the S_e that are pairwise compressible. The proof now splits into several cases.

Case 1: *(M, K) is not a closed Montesinos pair.*

Consider a closed-up component (M_v, K_v) of $(M, K) - \bigcup_e S_e$ (= a tangle plumbing block) that has $\geqslant 3$ boundary components or contains a ring.

[1](Added 2009) See also [**Sak**] for the remaining cases.

At least, one boundary component S of (M_v, K_v) is pairwise incompressible in (M, K) (possibly $S \subset \partial M$); otherwise, (M, K) would be a closed Montesinos pair. We have already seen that $\psi(S) = S$ and that ψ is pairwise isotopic to the identity by an isotopy respecting S. In particular, $\psi(M_v) = M_v$ and ψ fixes each point of $S \cap K$; a connectivity argument on K_v now proves that ψ preserves each component of ∂M_v.

We have thus shown that the isogeny $\Gamma \to \Gamma$ induced by ψ fixes each vertex with valence $\geqslant 3$ or ring number $\geqslant 1$, and each bond attached to it. An easy graph theoretic argument now shows that this isogeny fixes everything. (Remark: It even reveals that, if Γ contains at least one vertex of valence $\geqslant 3$, we did not need to consider vertices of ring number $\geqslant 1$ and valence $\leqslant 2$ in the previous step.) This proves Claim 16.6 in this first case.

Case 2: *(M, K) is a closed Montesinos pair, but not a rational knot.*

Here Γ has a unique vertex v with valence $\geqslant 3$ or ring number $r \geqslant 1$. Consider the necklace N_v of the plumbing block (M_v, K_v); then $\psi(N_v) = N_v$ after a pairwise isotopy of ψ respecting $\bigcup_e S_e$ (by Theorem 10.5, applied to the central block).

Assume, in search of a contradiction, that ψ acts non-trivially on Γ, and thus permutes the Conway spheres of N_v non-trivially. Then, consider two adjacent Conway spheres S_1 and S_2 of N_v and a component B_1 of the band of N_v that joins S_1 to S_2 (this B_1 is in fact unique when N has $\geqslant 3$ Conway spheres). Let F be the vertical Conway sphere that is a boundary component of a regular neighbourhood of $S_1 \cup S_2 \cup B_1$. Note that F is pairwise incompressible by Theorem 8.15 because we have excluded closed Montesinos pairs with no ring and $\leqslant 3$ branches. When $\psi(B_1) \neq B_1$, we know $\psi(F)$ is not pairwise isotopic to F, by our classification of vertical surfaces in Theorem 8.15, a contradiction. When ψ respects B_1, it must exchange S_1 and S_2 and therefore, after pairwise isotopy $\psi(F) = F$ but $\psi_f \colon F \to F$ is not pairwise isotopic to the identity. This would prevent ψ from being pairwise isotopic to the identity (use Proposition 5.20 if necessary), which again provides the contradiction sought.

Finally, to conclude the proof of Claim 16.6, we need to examine

Case 3: *(M, K) is a rational knot.*

Here all the S_e are pairwise parallel. Assume, in search of a contradiction, that the assertion of Claim 16.6 is false. One sees that the isogeny $\Gamma \to \Gamma$ induced by ψ must be an involution exchanging the end points of the combinatorial segment underlying Γ.

Thus after pairwise isotopy of ψ, there exists a Conway sphere S pairwise parallel to all the S_e's so that ψ maps S to itself with degree -1, exchanging the closed-up complementary balls B, B'. We shall show this contradicts G being pairwise isotopic to the identity.

For this, we first want to make ψ periodic by a pairwise isotopy. The arguments of §5.4 do this (see Assertion 5.15 in particular), but here is an *ad hoc* argument: First, choose an arbitrary parametrisation of $(S, S \cap K)$ by the standard Conway sphere (S^2, P^0). By Lemma 11.2, ψ can be pairwise isotoped to be linear on S (dropping to the PL or TOP category to avoid technicalities; see §11.1). The restriction $\psi_{|S} \colon S \to S$ exchanges two pairwise isotopy classes of essential closed curves C and C', namely those bounding the (unique) pairwise essential discs in $(B, K \cap B)$ and $(B', K' \cap B')$ avoiding K. These two curves are not pairwise isotopic; otherwise, Γ would be $\overset{0}{\bullet}$. It follows that $\psi_{|S}$ is periodic, and we easily make $\psi_{|S \cup K}$

periodic too. Note that the complement of $S \cup K$ consists of two open handlebodies U, U' of genus 2 that are exchanged by ψ. Finally ψ itself becomes periodic when we replace $\psi_{|U'}: U' \to U$ by the inverse of $\psi_{|U}: U \to U'$.

Next, we use a simple lemma of Giffen [**Gif**], which also follows from a stronger result of A. Borel (compare the proofs of Lemma 4.2 and Assertion 5.16, and [**ConM**]).

LEMMA 16.7. *Let ψ be a finite order automorphism, different from the identity, of a knot pair (M, K) with $K \neq \varnothing$ and $M - K$ irreducible. If ψ is pairwise isotopic to the identity, there is a non-trivial central element in $\pi_1(M - K)$. As a consequence, the pair (M, K) is Seifert fibred by* [**Wal**$_1$].

PROOF OF 16.7 (IN BRIEF). If f_t is the isotopy, we can chain together the isotopies f_t, $f f_t$, $f^2 f_t$, $\dots$, $f^{n-1} f_t$, where n is the order of f, to get a pairwise isotopy g_t, $0 \leqslant t \leqslant n$, from $g_0 =$ identity to $g_n = f^n =$ identity. Then, for any $x \in M - K$, the loop $g_t(x)$ traced by x is central in $\pi_1(M - K)$. With x chosen near K, this loop is clearly non-trivial in $\pi_1(U - K)$, with U an equivariant tubular neighbourhood of K. Then, the irreducibility of $M - U$ combined with Dehn's lemma assures that either this loop is non-trivial in $\pi_1(M - K)$ or $M - K$ is a solid torus. In both cases, $\pi_1(M - K)$ then has non-trivial centre. ☐

From this lemma, we conclude that the rational knot (M, K), which we are looking at in order to prove Claim 16.6 is Seifert fibred. But the only rational knots that are Seifert fibred are those with canonical graph $\Gamma = \overset{n}{\bullet}$ with $n \in \mathbb{Z}$, which are the $\{2, n\}$ torus knots; the 2–fold branched coverings show this (see [**Mon**$_1$**, Mon**$_2$] or the Appendix). This gives the desired contradiction because the canonical graph Γ of (M, K) was assumed to be topologically an interval. Thus Claim 16.6 is proved for rational knots.

This completes the proof of Claim 16.6, and thus of Theorem 16.4. ☐

Our next task is to evaluate the kernel of the epimorphism γ: $\mathrm{Sym}(M, K; C) \to \mathrm{Sym}(\Gamma)$ defined by Theorem 16.4.

When no symmetry of Γ has both degree $+1$ and -1 (compare with Proposition 16.3), there is a natural group homomorphism ε: $\mathrm{Sym}(\Gamma) \to \mathbb{Z}_2$ such that the composition $\varepsilon \circ \gamma$: $\mathrm{Sym}(M, K; C) \to \mathbb{Z}_2$ is just the degree map. It follows that, in this case, $\operatorname{Ker}\gamma = \operatorname{Ker}\gamma^+$. Otherwise, $\operatorname{Ker}\gamma^+$ has index 2 in $\operatorname{Ker}\gamma$.

By the above, we can now restrict attention to the determination of $\operatorname{Ker}\gamma^+$. For this, we now define a readily calculable group $\mathcal{K}(\Gamma)$, which will turn out to be isomorphic to $\operatorname{Ker}\gamma^+$ in most cases of interest.

The group $\mathcal{K}(\Gamma)$ is defined as follows for an arbitrary abbreviated weighted planar tree Γ of atomic tangles. By plumbing according to Γ, we obtain a marked pair $(M, K; C)$ made up of plumbing blocks (M_v, K_v) corresponding to the vertices of Γ, each equipped with a preferred necklace N_v coming from the necklace of the corresponding model atomic tangle. Let $N = N(\Gamma)$ be the union in M of all the necklaces N_v; it is called the ***conglomerate necklace***. By definition, $\mathcal{K}(\Gamma)$ is the group of isotopy classes of automorphisms of N that preserve each sphere and on each Conway sphere induce an element of the Viergruppe V_4 (or, equivalently, map each Conway sphere to itself with degree 1, preserving every marking).

Observe that $\mathcal{K}(\Gamma)$ is realised by a group of involutions of N. When Γ is connected, restriction to any parametrised Conway sphere injects $\mathcal{K}(\Gamma)$ into V_4. If

Γ has just one vertex of valence d, then $\mathcal{K}(\Gamma)$ is clearly V_4 for $d = 2$ and $\mathbb{Z}_2$ for $d \geqslant 3$. The general calculation of $\mathcal{K}(\Gamma)$ is given by Algorithm 16.10 below.

Theorem 16.8. *Under the hypotheses of Theorem* 16.4*, so that* $\gamma^+ \colon \mathrm{Sym}^+(M, K; C) \to \mathrm{Sym}^+\Gamma$ *is defined, assume moreover that all ring numbers of* Γ *are* $\leqslant 1$ *and that* Γ *does not contain any branch* $- - - \textcircled{r}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-$ *(corresponding to a vertical marking). Exclude the graphs* $-\!\overset{0}{\bullet}\!-$, $-\!\overset{0}{\bullet}$ *and* $\overset{n}{\bullet}$.

Then, there is a natural isomorphism $\mathrm{Ker}\,\gamma^+ \cong \mathcal{K}(\Gamma)$.

Proof of 16.8 when Γ has a single vertex. Consider a degree $+1$ automorphism φ of the marked atomic pair $(M, K; C)$. There exists a pairwise isotopy of φ to an automorphism φ' respecting the necklace N: If the ring number is $\neq 0$, or if the valence is $\geqslant 3$, this was proved in Chapter 10 (without using the marking). Otherwise, the argument of Proposition 13.13 gives the proof (using the marking and the exclusion of $-\!\overset{0}{\bullet}\!-$, $-\!\overset{0}{\bullet}$ and $\overset{n}{\bullet}$).

If φ represents an element of $\mathrm{Ker}\,\gamma^+$, then the restriction $\psi' : N \to N$ of φ' respects each Conway sphere and gives an element $[\psi']$ of $\mathcal{K}(\Gamma)$. Restriction to any one boundary Conway sphere shows that the rule $\varphi \to [\psi']$ gives a well-defined homomorphism $\theta : \mathrm{Ker}\gamma^+ \to \mathcal{K}(\Gamma)$.

Given $[\psi]$ in $\mathcal{K}(\Gamma)$, the automorphism ψ of N extends to one of $(M, K; C)$ by Proposition 11.1. Thus θ is surjective.

To show that θ is injective, suppose ψ' in the definition of θ is isotopic to the identity. Then, we can clearly pairwise isotop φ' to be the identity on N and also on a regular neighbourhood V of N (possibly with a ring as core), the following lemma completes the proof of Theorem 16.8 when Γ has one vertex. □

Lemma 16.9. *Any automorphism of* $B^2 \times S^1$ *or of* $(B^2, 0) \times S^1$ *fixing boundary is pairwise isotopic to the identity by an isotopy fixing the boundary.*

Proof of Lemma 16.9. Hint: For the first case, first make the automorphism respect a meridian disk of the solid torus $B^2 \times S^1$ (see Figure 16.1). For the second case, start by making the automorphism respect an annulus A stretching from the core $0 \times S^1$ to the boundary, split $B^2 \times S^1$ along this annulus, and then apply the first case to the resulting solid torus. □

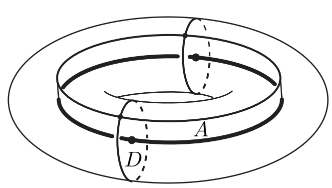


Figure 16.1.

Proof of Theorem 16.8 in the general case. There is a well-defined homomorphism $\theta^* : \mathcal{K}(\Gamma) \to \mathrm{Ker}\,\gamma^+$ as follows: For any automorphism ψ of the conglomerate necklace N representing an element of $\mathcal{K}(\Gamma)$, choose a (degree $+1$) automorphism φ of $(M, K; C)$ extending ψ. By the case just proved, φ is well-defined by $[\psi] \in \mathcal{K}(\Gamma)$, up to pairwise isotopy respecting each plumbing block (M_v, K_v).

To prove the surjectivity of θ^*, consider a (degree $+1$) automorphism φ of $(M, K; C)$ representing an element $[\varphi]$ of $\mathrm{Ker}\gamma^+$, namely respecting each plumbing block (M_v, K_v). In the proof of Proposition 13.10 and 13.13, we showed that in this situation the restriction of φ to each boundary component of (M_v, K_v) is slope preserving. By the proved case of Theorem 16.8 with one vertex, the restriction of φ to M_v respects its necklace N_v after pairwise isotopy. Then $[\varphi] \in \mathrm{Ker}\,\gamma^+$ is clearly in the image of θ^*.

We now have to prove that θ^* is injective.

If $\partial M \neq \varnothing$, this is clear by restriction to ∂M. If $\partial M = \varnothing$, argue as follows: In the construction of $\theta^*[\psi] = [\varphi]$, note that φ can be an involution, and not the identity if $\psi \neq \mathrm{Id}$. Indeed one can readily see this on the model atomic tangles of §12.5. Also one can verify it abstractly by extending ψ to an involution of a regular neighbourhood of N, then by hand over the remaining solid tori to an involution of (M, K). Then, Lemma 16.7 assures that either $[\varphi] \neq 0$, or else (N, K) is Seifert fibred, which, as proved by Theorem A.8 in the Appendix, is excluded by our hypotheses.

The proof of Theorem 16.8 is now complete. □

We now determine the group $\mathcal{K}(\Gamma)$ for an arbitrary abbreviated weighted planar tree Γ (not necessarily canonical). We know that $\mathcal{K}(\Gamma)$ is realised as a finite group of involutions ψ of the conglomerate necklace $N = \bigcup_v N_v$ so that, on each parametrised Conway sphere in N_v, the restriction of ψ is an element of V_4. In particular, $\mathcal{K}(\Gamma)$ is independent of ring numbers.

The determination of $\mathcal{K}(\Gamma)$ amounts to knowing which involutions of the N_v can be fitted together to give an involution of N in $\mathcal{K}(\Gamma)$. This is just a combinatorial problem involving some compatibility conditions which we analyse now.

Consider $\psi \in \mathcal{K}(\Gamma)$ and its restrictions ψ_v to N_v.

If the plumbing block M_v has $\geqslant 3$ boundary components and if $\psi \neq \mathrm{Id}$, then the restriction $\psi_{|S} : S \to S$ of ψ to each component S of $\partial M_v \subset N_v$ coincides with $\xi \in V_4 \subset \mathrm{Sym}(S^2, P^0)$, where the tangle structure of (M_v, K_v) specifies the identification of $(S, K \cap S)$ with the standard Conway sphere (S^2, P^0).

If M_v corresponds to a 2–valent vertex of Γ with total weight a, and if $\psi \in \mathcal{K}(\Gamma)$ coincides with $\alpha \in V_4$ on one component of ∂M_v, it acts as $\theta^e(\alpha) \in V_4$ on the other boundary component, where $\theta : V_4 \to V_4$ is the involution exchanging η and ζ; note that θ^a depends only on a mod 2.

Lastly, let S be a common boundary component of two adjacent blocks M_v and M_w. Then, by the atomic tangle structures for M_v and M_w, the restriction $\psi_S : S \to S$ gives two elements of V_4, and these are related by the involution σ of V_4 exchanging ξ and η.

The above observations lead to the following: If Γ and Γ' are two abbreviated weighted planar trees, a natural group isomorphism $\mathcal{K}(\Gamma) \to \mathcal{K}(\Gamma')$ is induced by any combinatorial isomorphism $\Gamma_0 \to \Gamma'_0$ between the underlying combinatorial graphs sending a vertex of Γ with total weight e always to a vertex of Γ' whose total weights e' is congruent to e modulo 2. In other words, only the weights mod 2 count for $\mathcal{K}(\Gamma)$, while the ring numbers and the cyclic orderings are entirely irrelevant. Thus $\mathcal{K}(\Gamma)$ is $\mathcal{K}(\Gamma_2)$ where Γ_2 is the underlying combinatorial graph Γ_0 together with, for each vertex, the value mod 2 of the integral weight of the corresponding vertex of Γ.

Also, if Γ and Γ' are related by an arithmetic move (0.2) or (1.2), one readily checks that there exists a natural isomorphism between the corresponding groups $\mathcal{K}(\Gamma)$ and $\mathcal{K}(\Gamma')$. Beware that this is sometimes false for the arithmetic moves (0.1) and (1.1).

Observe that the arithmetic moves (0.2) or (1.2) are well-defined on the weighted tree Γ_2. Our algorithm reduces Γ_2 by these moves to a form where the value of $\mathcal{K}(\Gamma_2)$ is obvious.

Algorithm 16.10. Let Γ_2 and $\mathcal{K}(\Gamma)$ be associated to any abbreviated weighted planar tree Γ as above. Performing moves (0.2) and (1.2) on Γ_2 sufficiently often, one necessarily reaches a graph Γ_2' (with weights in $\mathbb{Z}_2$) for which one of the following holds:

(a) Some edge of Γ_2' joins two vertices of valence $\geqslant 3$.
(b) Exactly one vertex of Γ_2' has valence $\geqslant 3$.
(c) Γ_2' is a line, namely is homeomorphic to a segment.

Then $\mathcal{K}(\Gamma) = 0$ in Case (a), $\mathbb{Z}_2$ in Case (b) and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in Case (c).

Proof. The first statement is an easy combinatorial fact. The second follows from $\mathcal{K}(\Gamma) \cong \mathcal{K}(\Gamma_2')$ and a direct calculation of the latter, using our preliminary observations. $\square$

It is also helpful to notice that $\mathcal{K}(\Gamma)$ is unchanged if any arm $- - - \bullet\!\!-\!\!-\!\!\overset{a_1}{\bullet}\!\!- \cdots -\!\!\overset{a_n}{\bullet}$ or $- - - \bullet\!\!-\!\!-\!\!\overset{a_1}{\bullet}\!\!- \cdots -\!\!\overset{a_n}{\bullet}\!\!-$ of Γ_2 that is just a stick (with weights in $\mathbb{Z}_2$) is lopped off at its attaching bond leaving $- - - \bullet\!\!-$.

We now propose to generalise the calculation of $\operatorname{Ker}\gamma^+$ in Theorem 16.8 to the more general situation of Theorem 16.4 in which γ^+ is defined. This is done in two steps; here is one:

Theorem 16.11. *Let the hypotheses of Theorem* 16.8 *be slightly relaxed so as to allow* $\delta_v \geqslant 0$ *branches* $- - -ⓡ\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$ *(corresponding to vertical markings) attached at each vertex* v *of* Γ *(while the ring numbers are still* $\leqslant 1$ *).*

Then, when Γ *is not* $-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$*, there is a splitting* $\operatorname{Ker}\gamma^+ \cong \mathcal{K}(\Gamma) \oplus \mathbb{Z}^\delta$ *where* $\mathcal{K}(\Gamma)$ *is determined by Algorithm* 16.10 *and where* $\delta = \sum_v \max(0, \delta_v - 1)$*; in general, the splitting is not natural.*

When Γ *is* $-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$, $\operatorname{Ker}\gamma^+$ *is only* $\mathcal{K}(\Gamma) \times \mathbb{Z}^\delta$ *as a set; algebraically, it is the semi-direct product of* $\mathcal{K}(\Gamma) \cong V_4$ *and* $\mathbb{Z}$ *defined by the relations* $a\xi a^{-1} = \xi$ *and* $a\eta a^{-1} = \xi$ *where* a *is the generator of* $\mathbb{Z}$.

In particular, $\operatorname{Ker}\gamma^+ \cong \mathcal{K}(\Gamma)$ *when at most one branch is attached at each vertex.*

Proof. It is convenient to consider the graph Γ' obtained from Γ by erasing each branch $- - -ⓡ\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$, keeping however the bond $- - -ⓡ\!\!-$ where it was attached. The pair (M, K) is still obtained by plumbing atomic tangles (M_v, K_v) according to Γ'. Note that, for the new tangle structure on (M, K), the marking C has slope ∞ on the components of ∂M corresponding to branches $- - -ⓡ\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-$ of Γ, but still has slope 0 on the other boundary components. Let $\partial_\infty M$ denote the union of the components of ∂M where the original marking C is vertical (slope ∞).

After pairwise isotopy, any automorphism of $(M, K; C)$ representing an element of $\operatorname{Ker}\gamma^+$ preserves each block (M_v, K_v) for Γ' and each component of ∂M_v; let φ be such an automorphism. The now standard argument based on necklaces (see the proof of Propositions 13.10 and 13.13) shows that φ is slope-preserving on each component of $\partial M_v - \partial_\infty M$. It then follows from Proposition 11.1 that, for every component S_{vi} of $M_v \cap \partial_\infty M$, there exists a number $n_{vi} \in \mathbb{Z}$ so that φ sends slope $\frac{p}{q}$ to slope $\frac{p}{q} + n_{vi}$ for the tangle structure of (M_v, K_v); moreover $\sum_i n_{vi} = 0$ for every vertex v.

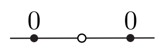
The rule sending φ to the n_{vi}'s defines a group homomorphism $\alpha\colon \operatorname{Ker}\gamma^+ \to \mathbb{Z}^\sigma$ where $\sigma = \sum_v \delta_v$, with image contained in the subspace $A \cong \mathbb{Z}^\delta$ determined by the relations $\sum_i n_{vi} = 0$ (one for each vertex v).

The kernel of α consists of the elements of $\operatorname{Ker}\gamma^+$ that respect each slope on the boundary. Then, the arguments proving Theorem 15.1 apply to show that $\operatorname{Ker}\alpha \cong \mathcal{K}(\Gamma)$.

The precise determination of the image of α in A is a bit subtle. However, we can already note that $\operatorname{Im}\alpha$ contains $2A \cong \mathbb{Z}^\delta \subset A$. Indeed, each collection of *even* numbers n_{vi} satisfying the conditions $\sum_i n_{vi} = 0$ for every v is realised by Dehn twists along suitably chosen vertical annuli in each (M_v, K_v). This proves that $\operatorname{Im}\alpha$ is itself isomorphic to $\mathbb{Z}^\delta$, and thus assures an exact sequence

$$0 \to \mathcal{K}(\Gamma) \to \operatorname{Ker}\gamma^+ \xrightarrow{\alpha} \operatorname{Im}\alpha \cong \mathbb{Z}^\delta \to 0.$$

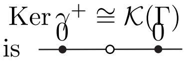
We want to prove that this exact sequence splits. Without loss of generality, we can restrict attention to the case where $\delta \neq 0$ and $\mathcal{K}(\Gamma) \neq 0$. Then fix a base point $* \in K \cap \partial_\infty M$, lying in a block (M_v, K_v) containing at least two components of $\partial_\infty M$.

Assuming moreover that Γ is not , the block (M_v, K_v) containing $*$ has at least three boundary components. Therefore $\mathcal{K}(\Gamma) \cong \mathbb{Z}_2$, and every automorphism φ of $(M, K; C)$ representing an element of $\operatorname{Ker}\gamma^+$ sends $*$ either to itself, or to the point of ∂K that is connected to $*$ by an arc of the marking C (by a connectivity argument on the necklace). It then follows that, for each $\psi \in \operatorname{Ker}\gamma^+$, there exists a unique $\widehat{\psi} \in \operatorname{Ker}\gamma^+$ having the same image by α and fixing the point $*$. The elements $\widehat{\psi}$ now define the splitting $\operatorname{Im}(\alpha) \to \operatorname{Ker}\gamma^+$ required.

Also, it is immediate that each such $\widehat{\psi}$ commutes with the normal subgroup $\mathcal{K}(\Gamma) \cong \mathbb{Z}_2$. Therefore the split exact sequence

$$0 \to \mathcal{K}(\Gamma) \to \operatorname{Ker}\gamma^+ \xrightarrow{\alpha} \mathbb{Z}^\delta \to 0$$

is trivial and $\operatorname{Ker}\gamma^+ \cong \mathcal{K}(\Gamma) \oplus \mathbb{Z}^\delta$.

When Γ is , the argument is quite similar, and we let the reader check that it leads to the claimed result. This concludes the proof of Theorem 16.11. □

The reader has been cheated in Theorem 16.11. He does not possess a map $\operatorname{Ker}\gamma^+ \to \mathbb{Z}^\delta$ (nor the other way around), and he does not know the effect of $\operatorname{Ker}\gamma^+$ on $(\partial M, \partial K)$. The following series of exercises give redress.

EXERCISES 16.12 (for data of Theorem 16.11, excluding $\Gamma =$ ).

(a) Let $N = N(\Gamma') \subset M$ be the conglomerate necklace for the pruned tree Γ' introduced in the proof of Theorem 16.11. Show that there is a natural injective homomorphism $\nu\colon \operatorname{Sym}(M, K; C) \to \operatorname{Sym}(N)$. Our aim is to calculate its image $\operatorname{Im}\nu$ effectively.

(b) Let C' be the marking of (M, K) associated to Γ', and let ν' be the restriction of ν to the finite subgroup $\operatorname{Sym}(M, K; C)$ of $\operatorname{Sym}(M, K; C)$

respecting C'. By Theorem 16.8, we have an exact sequence

$$0 \to \mathcal{K}(\Gamma) \to \operatorname{Im}\nu' \to \operatorname{Sym}(\Gamma) \to 0.$$

Using it, give an effective procedure to determine $\operatorname{Im}\nu'$.

(c) Show that $\operatorname{Im}\nu \subset \operatorname{Sym}(N)$ is generated by $\operatorname{Im}\nu'$ and $\nu(\operatorname{Ker}\gamma^+)$. We have yet to determine $\nu(\operatorname{Ker}\gamma^+)$.

(d) There is a restriction mapping $\rho\colon \operatorname{Sym}_0(N) \to \operatorname{Sym}(N - \partial_\infty M)$ where $\operatorname{Sym}_0(N) \subset \operatorname{Sym}(N)$ is the subgroup respecting each Conway sphere. Clearly $\nu(\operatorname{Ker}\gamma^+) \subset \operatorname{Sym}_0(N)$. Show that $\rho\nu(\operatorname{Ker}\gamma^+) \cong \mathcal{K}(\Gamma_*)$ where Γ_* is obtained from Γ by the following cutting procedure: At each vertex of Γ with at least two branches $- - -ⓡ\!\overset{0}{—\bullet—}$ attached, cut Γ like a pie along each such branch adding a free bond at the tip of each piece of pie (one can suppress all rings). Figure 16.2 illustrates and clarifies this, for a vertex with three such branches.

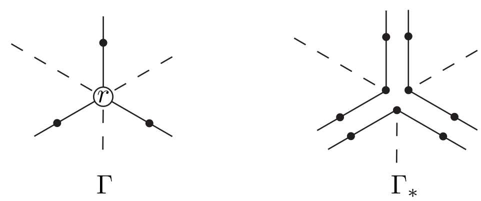


FIGURE 16.2.

Clearly, Γ_* depends strongly on the dihedral ordering of bonds at the vertices of Γ. Observe that for each component Γ_{**} of Γ_*, the group $\mathcal{K}(\Gamma_{**})$ is 0 or $\mathbb{Z}_2$ calculated by Algorithm 16.10, if $\mathcal{K}(\Gamma_*)$ is a known sum of $\mathbb{Z}_2$'s. If $N_* = N(\Gamma_*)$ is the conglomerate necklace for Γ_*, there is a natural inclusion $N - \partial_\infty M \to N_*$. By restriction, the group $\mathcal{K}(\Gamma_*) \subset \operatorname{Sym}_0(N_*)$ injects into $\operatorname{Sym}(N - \partial_\infty M)$; show that this identifies $\mathcal{K}(\Gamma_*)$ to $\rho\nu(\operatorname{Ker}\gamma^+)$.

(e) Show that $\nu(\operatorname{Ker}\gamma^+)$ is the subgroup of $\operatorname{Sym}_0 N$ consisting of those automorphisms $\psi\colon N \to N$ (up to isotopy) satisfying two conditions:
  (*) the restriction to $N - \partial_\infty M$ lies in $\mathcal{K}(\Gamma_*) \subset \operatorname{Sym}(N - \partial_\infty M)$;
  (**) for each plumbing block $(M_v, K_v) \subset (M, K)$ for Γ', the restriction $\psi_v : N_v \to N_v$ of ψ to the necklace $N_v = N \cap M_v$ satisfies the arithmetic condition of Proposition 11.1 for extension to an automorphism of $(M_v, K_v; C')$.

(f) For $M = S^3$, devise algorithms to determine from Γ the image of the natural homomorphism $\operatorname{Sym}(S^3, K) \to \operatorname{Sym}(S^3) \times \operatorname{Sym}(K)$.

THEOREM 16.13. *Let the hypotheses of Theorem 16.8 be relaxed once more, allowing ring numbers* $\geqslant 2$ *as well as branches* $- - -ⓡ\!\overset{0}{—\bullet—}$ *; exclude however the graphs* $ⓡ\overset{a_0}{—}\overset{a_1}{\bullet—}\cdots\overset{a_n}{—\bullet}$ *with* $r \geqslant 2$ *and* $n \geqslant 0$. *Consider, for every vertex* v *of* Γ *with ring number* $r_v \geqslant 0$, *the braid group* $B_v = \pi_0 \operatorname{Aut}(D^2, r_v \text{ interior points}; \operatorname{rel}\partial D^2)$. *(Note that* $B_v = 0$ *if* $r_v \leqslant 1$.*) Let* Γ_* *be the weighted graph that is obtained from* Γ *by replacing all ring numbers* $\geqslant 2$ *by* 1*; and let* $(M_*, K_*; C_*)$ *be the associated marked pair.*

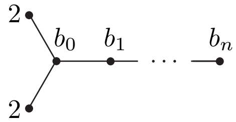
Then, there is an exact sequence

$$0 \to \prod_v B_v \to \operatorname{Ker}\gamma^+ \to \operatorname{Ker}\gamma_*^+ \to 0$$

where $\gamma_*^+ \colon \mathrm{Sym}(M_*, K_*; C_*) \to \mathrm{Sym}^+(\Gamma_*)$ *is the map defined by Theorem 16.4.*

Note that our hypotheses (and notably the exclusions) assure that Γ_* is canonical and satisfies the hypotheses of Theorem 16.4. Moreover, $\mathrm{Ker}\,\gamma_*^+$ is already determined by Theorem 16.4 and 16.11.

As for $\Gamma = \overset{a_0}{ⓡ}\!\!-\!\!\overset{a_1}{\bullet}\!\!- \cdots -\!\!\overset{a_n}{\bullet}$, note that Γ_* would have a canonical form of type



(see §12.5) so that $(M_*, K_*; C_*)$ is excluded by Theorem 16.4 (and γ_*^+ is undefined). Nevertheless, Theorem 16.13 will in effect be extended to this case in Proposition 16.15.

Proof of Theorem 16.13. For every vertex v of Γ with ring number $r_v \geqslant 2$, consider in the corresponding plumbing block (M_v, K_v) the closed-up complement V_v of a regular neighbourhood of the necklace N_v. The pair $(V_v, V_v \cap K)$ is isomorphic to $(D^2, r_v \text{ points}) \times S^1$, and ∂V_v is pairwise incompressible in (M, K).

Every automorphism φ of $(M, K; C)$ representing an element of $\mathrm{Ker}\,\gamma^+$ respects $\bigcup \partial V_v$ after pairwise isotopy, by uniqueness of necklaces (Chapter 10). Using Proposition 5.20, one sees that the "restriction" map to $M - \bigcup_v V_v$ then induces a well-defined homomorphism $\mathrm{Ker}\,\gamma^+ \to \mathrm{Ker}\,\gamma_*^+$ which associates to an automorphism ψ_v of $(D^2, r_v \text{ points})$ fixing ∂D, the automorphism that is $\psi_v \times \mathrm{Id}$ on V_v and is the identity outside of V_v. By uniqueness of Seifert fibrings up to isotopy [**Wal**$_2$], every automorphism of $(M, K; C)$ fixing the complement of $\bigcup_v V_v$ is obtained in this way, up to pairwise isotopy. We therefore have an exact sequence:

$$\prod_v B_v \to \mathrm{Ker}\,\gamma^+ \to \mathrm{Ker}\,\gamma_*^+ \to 0.$$

The fact that $\prod_v B_v$ injects into $\mathrm{Ker}\,\gamma^+$ follows (using Proposition 8.14) from its effect on pairwise essential surfaces in (M, K). Consider, for example, a family of Conway spheres ∂R where R is chosen so that $(M_v \cap R, K_v \cap R)$ consists of r_v disjoint ring pairs in $\mathrm{int}(M_v)$. □

Exercises 16.14 (for data of Theorem 16.13).

(a) Show that $\mathrm{Ker}\,\gamma^+$ splits naturally as a product $\prod_v B_v \times \mathrm{Ker}\,\gamma_*^+$, if and only if every element of $\mathrm{Ker}\,\gamma_*^+$ respects the orientation of each ring of (M_*, K_*) that corresponds to $\geqslant 2$ rings of (M, K). In particular, this happens when every vertex of Γ with ring number $\geqslant 2$ has valence $\geqslant 3$.

(b) Show that the exact sequence of Theorem 16.13 is always split exact (but not naturally). Similarly for Proposition 16.15 below.

(c) In the sense of Chapter 2, (M, K) does have a tree of characteristic companions, one of which is (M_*, K_*). Thus the results of Chapter 15 apply; compare what they give with Theorem 16.13.

(d) Johannson [**Joh**$_3$] showed, using general principles, that $\mathrm{Sym}^+(M, K; C)$ divided by the normal subgroup generated by Dehn twists along pairwise essential tori and annuli, is a finite group. Prove that its order is the product

$$|\mathcal{K}(\Gamma_*)| \times |\mathrm{Sym}^+\Gamma| \times \prod_v (r_v!)$$

where Γ_* was defined in Exercise 16.12(d).

Equipped with Theorems 16.4, 16.8, 16.11 and 16.13, we now consider the symmetries of most marked arborescent cases that were left aside hitherto.

Proposition 16.15. *Let* (S^3, K) *be a closed (presented) Montesinos pair with* $r \geqslant 1$ *rings and exactly one rational tangle of finite non-integral slope. Then there is an exact sequence*

$$0 \to B_r \to \mathrm{Sym}^+(S^3, K) \to V_4 \to 0$$

where B_r *is the braid group* $\pi_0 \operatorname{Aut}(D^2, r \text{ interior points}; \operatorname{rel} \partial D^2)$.

Proof. When $r \geqslant 2$, we have proved in Theorem 10.5 that the necklace of (S^3, K) is characteristic. Then, the arguments used to prove Theorems 16.4 and 16.13 apply straightforwardly to establish the desired sequence.

When $r = 1$, these arguments also give an exact sequence

$$0 \to B_1 = 0 \to \mathrm{Sym}_0^+(S^3, K) \to V_4 \to 0,$$

where Sym_0^+ is the subgroup of Sym^+ respecting the ring. This follows from

Claim 16.16. *All automorphisms* φ *respecting the ring, also respect the necklace up to pairwise isotopy.*

Proof of Claim 16.16. Put in an extra ring R' near the first and isotop φ respecting K to make φ respect the second ring R'. Then, by Theorem 10.5, isotop φ respecting $K \cup R'$ to make φ respect the necklace. □

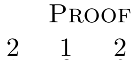
The next claim completes the proof of Proposition 16.15.

Claim 16.17. *Every automorphism of* (S^3, K) *respects the ring.*

Proof of Claim 16.17. The complement of the ring in K is a non-integral rational knot in S^3. Thus, the ring is characterised by the property that it is the unique component of K whose complement is not the trivial knot nor the Hopf link . In particular, the ring is characteristic. □

This completes the proof of Proposition 16.15. □

Theorem 16.18. *For the Borromean rings* (S^3, K)*,* $\mathrm{Sym}(S^3, K)$ *has order* 48 *and* $\mathrm{Sym}^+(S^3, K)$ *has order* 24.

Proof. Form (S^3, K) by plumbing atomic tangles according to the graph $\Gamma =$ 

. This yields a preferred presentation of (S^3, K) as a Montesinos pair, and thus a preferred necklace N.

The subgroup $\mathrm{Sym}_0(S^3, K)$ of $\mathrm{Sym}(S^3, K)$ respecting the ring is also the subgroup respecting the necklace N by Proposition 10.8. Thus, the proofs of Theorems 16.4 and 16.8 show that $\mathrm{Sym}_0(S^3, K)$ lies in an exact sequence

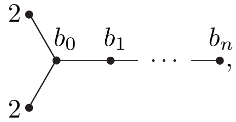
$$0 \to V_4 = \mathcal{K}(\Gamma) \to \mathrm{Sym}_0^+(S^3, K) \to \mathrm{Sym}(\Gamma) = \mathbb{Z}_2 \to 0.$$

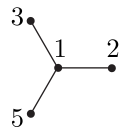
Hence $\mathrm{Sym}_0^+(S^3, K)$ has order 8 and $\mathrm{Sym}^+(S^3, K)$, being transitive on the three components of K, has order 24. As (S^3, K) is amphicheiral, $\mathrm{Sym}(S^3, K)$ has order 48. □

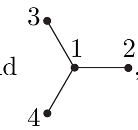
For Montesinos pairs that are Seifert fibred, the symmetry groups are determined by [**Wal**$_2$] (Seifert fibring uniqueness). This includes in particular those classified by the point graphs $\overset{n}{\bullet}$ and $\overset{n}{\circ}$ (namely Montesinos knots with just one integral rational tangle and $\leqslant 1$ ring). In these two cases, with however the restriction that $n \neq \pm 2$ in the second case, the degree $+1$ symmetry group is V_4 or $\mathbb{Z}_2$ according as n is even or odd. For $\overset{\pm 2}{\circ}$ the degree $+1$ symmetry group is, as expected, an extension of the dihedral group D_6 by $\mathbb{Z}/2$.

The case of the graph $\overset{n}{ⓡ}$, with $r \geqslant 2$, is also a straightforward application of Theorem 15.1 and [**Wal**$_2$], but beware that the answer comes in three versions: for n odd, $n = 0$, and n even $\neq 0$.

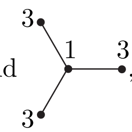
Our calculations for pairwise irreducible arborescent pairs have succeeded except for those whose canonical graph is a 3–branched star with black vertices (black means ring number 0). In fact, we have even succeeded for three clans of these:

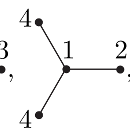


, with $n \geqslant 1$, by Proposition 16.15;


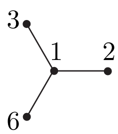
and


,

which are the torus knots $\{3, 5\}$, $\{3, 4\}$ (see the Appendix); and


,


,

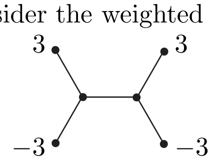
and


whose characteristic companions are Seifert fibred (see Chapter 2).

Beyond this, M. Boileau [**Boi**$_1$**, Boi**$_2$] has succeeded with a fourth and most populous clan: those whose classifying fractional data vector of §11.2 is of the form $(0; e_0; \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})$ with irreducible fractions $\neq 0$, ∞ such that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leqslant 1$. For the remainder, see further comments in [**Boi**$_1$**, Boi**$_2$**, BoiZ, BurZ**].[2]

We conclude with a few examples illustrating the results of this chapter. We will revisit these examples at the end of Chapter 18, to show how to realise the corresponding symmetries as rigid motions of S^3.

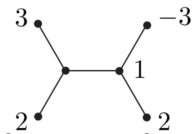
Example 16.19. First consider the weighted planar tree



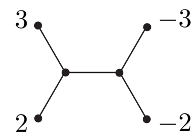
It has two obvious symmetries of degrees $+1$ and -1, gotten by reflection through a vertical or horizontal line. They generate $\mathrm{Sym}(\Gamma) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. No symmetry of Γ has both degree $+1$ and -1 (compare with Proposition 16.3) and the group $\mathcal{K}(\Gamma)$ found by Algorithm 16.10 is 0. Thus the symmetry group of the knot K associated to Γ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Example 16.20. The knot (S^3, K) associated to the $(+)$–canonical tree

[2](Added (2009) The remaining cases now follow from a combination of [**Sak**] and of the proof [**CooHK, BoiP, BoiMP, BoiLP**] of the Orbifold Geometrisation Conjecture discussed in Chapter 5.



is the knot labelled as 8_{17} in [**AleB, Rol**]. This tree has a degree -1 isogeny, factoring through the weighted planar tree

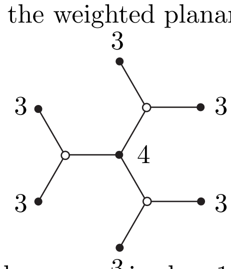


One easy checks that this is the only non-trivial symmetry of Γ, so that $\mathrm{Sym}(\Gamma) \cong \mathbb{Z}_2$. No graph symmetry has both degree $+1$ and -1, and Algorithm 16.10 shows that the kernel $\mathcal{K}(\Gamma)$ is trivial. Therefore, by Theorem 16.8, the symmetry group $\mathrm{Sym}(S^3, K)$ has exactly two elements, the identity and a symmetry reversing the orientation of S^3.

The fact that (S^3, K) has a degree -1 isomorphism means that the knot 8_{17} is ***amphicheiral***. By inspection, this symmetry reverses the orientation of the knot K, so that 8_{17} is ***negative amphicheiral***.

On the other hand, our computation shows that 8_{17} admits no degree $+1$ symmetry that reverses the orientation of the knot, namely that 8_{17} is not ***invertible***. This is the first knot in the tables to have this property. This was independently proved by A. Kawauchi [**Kaw**], using more algebraic techniques.

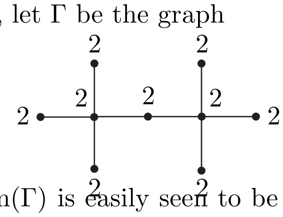
EXAMPLE 16.21. Let Γ be the weighted planar tree



The group $\mathrm{Sym}(\Gamma)$ of graph symmetries has 12 elements, and is generated by the dihedral group D_6 of rigid motions of $\mathbb{R}^2$ respecting the embedding of Γ in the plane, and by the involution ρ that moves precisely the vertices of weight 3. Also no symmetry has degree -1, and $\mathcal{K}(\Gamma) = 0$.

We conclude that the symmetry group $\mathrm{Sym}(S^3, K) = \mathrm{Sym}^+(S^3, K)$ has exactly 12 elements, and is actually isomorphic to $D_6 \times \mathbb{Z}_2$ since the involution ρ commutes with the dihedral group D_6.

EXAMPLE 16.22. Lastly, let Γ be the graph



Its symmetry group $\mathrm{Sym}(\Gamma)$ is easily seen to be the dihedral group D_4, while $\mathcal{K}(\Gamma) \cong \mathbb{Z}_2$. Moreover, the identity has both degree $+1$ and -1. Thus, $\mathrm{Sym}(S^3, K)$ consists of 16 different symmetries, half of them of degree $+1$.

CHAPTER 17

Natural markings of the arborescent part of a knot

The previous section gave a complete analysis of the symmetry groups of marked arborescent pairs that are pairwise irreducible. In the present section, we enquire more closely how this contributes to solving the more natural problem of finding the symmetry group of an arbitrary knot pair (M, K) that is simple for Schubert. We have shown in Theorem 15.5 that the symmetry group of the arborescent part A of (M, K) provides an upper bound for $\mathrm{Sym}(M, K)$, provided A is defined by Chapter 7 and is non-empty. Unfortunately, as Theorem 16.11 shows, $\mathrm{Sym}(A, K \cap A)$ is often infinite, while $\mathrm{Sym}(M, K)$ is necessarily finite when $\partial M = \varnothing$ by [**Joh**$_3$] (or by a combination of Thurston's Hyperbolisation Theorem [**Thu**$_2$] and Mostow's Rigidity Theorem [**Mos**$_1$]; see also [**Wal**$_4$]). To cope with this annoyance, we will give various devices to equip A with a boundary marking C respected by each symmetry of (M, K) such that $\mathrm{Sym}(A, K \cap A; C)$ be finite (by Theorem 16.4 and 16.8). This will provide a finite upper bound for $\mathrm{Sym}(M, K)$.

17.1. Markings from cusps

This device involves hyperbolic geometry. As such, it is not well adapted to the piecewise linear category PL, and is more suitable for the differentiable category DIFF or to the topological category TOP. It applies when we have in hand a π–hyperbolic metric on each of the pieces of the splitting of Chapter 7 that are abstractly known to admit such a structure, namely the components of $(M, K) - A \cup F$ with the notation of that chapter.

In this situation, each Conway sphere component $(S, K \cap S)$ of $\partial(A, K \cap A)$ gets a π–euclidean structure from the cusp of $M - (A \cup F)$ approaching S, well-defined up to scalar multiplication of the metric and pairwise isotopy (see §5.2). By our Rigidity Theorem 5.11, this π–euclidean structure is an invariant of $(M, K) - A \cup F$, and we are going to exploit this fact to define a natural boundary marking for the arborescent pair $(A, K \cap A)$.

Let us examine the geometry of any Conway sphere S equipped with a π–euclidean structure. It is isometric to $\mathbb{R}^2/\mathfrak{R}$, where $\mathfrak{R}$ is the group of euclidean isometries of $\mathbb{R}^2$ generated by the π–rotations around the points of a certain lattice L in $\mathbb{R}^2$. Any marking C_0 for S can be uniquely represented by a pair of geodesic arcs; these arcs are covered in $\mathbb{R}^2$ by a collection R_0 of straight arcs whose end points (only) lie in the lattice L and which we call ***rods***. This assigns to the marking C_0 a ***length*** and a ***direction*** in $\mathbb{RP}_1$, namely those of the rods in R_0.

Lemma 17.1 (Triangle Lemma). *For the above data, three distinct markings of shortest lengths have rods that always fit together to form acute* ($\leqslant 90°$) *angled triangles tessellating the plane. If all angles are* $< 90°$ *all other markings have strictly greater length. If one angle is equal to* $90°$*, the two shortest markings are* *uniquely defined and their rods cut up the plane into rectangles; also, there are two different third shortest markings and their rods are the two diagonals of these rectangles.* □

A pleasant corollary of Lemma 17.1 is that the π–euclidean Conway sphere $S = \mathbb{R}^2/R$ is isometric either to a tetrahedron in $\mathbb{R}^3$, not necessarily regular but with all four faces degree 1 isometric to the acute angled triangle met in the lemma, or else to the double of a rectangle of $\mathbb{R}^2$ (namely a flat pillow).

We now derive a marking C for $(A, K \cap A)$ from this natural π–euclidean structure on ∂A. On each component S of ∂A, it will be contrived to be distinct from the vertical marking C_∞ coming from necklaces for the natural splitting into Montesinos pairs (see the beginning of Chapter 16).

Consider the shortest markings distinct from $C_\infty \cap S$ and among these pick one whose direction is farthest from that of $C_\infty \cap S$ (for the angle metric of $\mathbb{RP}^1$). If this marking is unique, define it to be $C \cap S$.

If there is more than one such marking, then we make the

ASSERTION 17.2. *$C \cap S$ can be defined to be perpendicular to $C_\infty \cap S$, namely there are lines in $\mathbb{R}^2$ perpendicular to the lines over $C_\infty \cap S$ and containing many points of the lattice L.*

PROOF. Supposing there are two such markings, we are necessarily in a situation where the three shortest markings of S yield a tessellation of the plane $\mathbb{R}^2$ by congruent isosceles triangles with vertices the points of L. Also $C_\infty \cap S$ has direction in $\mathbb{RP}^1$ that is either the internal or the external bisector of the angle between two equal sides of the triangle. Then the perpendicular to the direction of $C_\infty \cap S$ is the other of these two bisectors; in particular it clearly corresponds to a marking. □

Application of this recipe to each component of ∂A defines a marking C of $(A, K \cap A)$ that is everywhere distinct from C_∞. Thus $\mathrm{Sym}(A, K \cap A; C)$ is finite by Theorems 16.4 and 16.8.

PROPOSITION 17.3. *With the above data, there is a well-defined "restriction" homomorphism* $\mathrm{Sym}(M, K) \to \mathrm{Sym}(A, K \cap A; C)$ *which is injective if* $A \neq \varnothing$.

PROOF. By Theorem 15.5, we just have to show that every degree ± 1 automorphism of (M, K) that preserves $A \cup F$ respects the marking C. But this follows easily from our Rigidity Theorem 5.11 together with the naturality of the definition of C. □

17.2. Markings of homological type

So far, π–hyperbolic structures remain difficult and/or expensive to find (see however [**Ril**$_1$, **Ril**$_2$, **Thu**$_1$] and our examples in Chapters 4, 5, 6). The above procedures consequently need to be supplemented by others that are simple and cheap. We now present two related procedures, with variants. Regrettably, although they are widely applicable, the markings produced do sometimes become vertical; for recalcitrant examples (S^3, K) with $A \neq \varnothing$, none will offer a finite upper bound for the finite group $\mathrm{Sym}(S^3, K)$. On the other hand, in favourable examples we will get several distinct upper bounds $\mathrm{Sym}(A, K \cap A; C_i)$, $i = 1, 2, \ldots, n$, say, for $\mathrm{Sym}(S^3, K)$ in $\mathrm{Sym}(A, K)$, and the intersection of all these will again be an upper bound, perhaps better than any one.

LEMMA 17.4. *Let (B, K) be a knot pair consisting of two arcs K in a 3–ball B. There exists a unique marking on the Conway sphere $(\partial B, \partial K)$, represented by two disjoint arcs C in ∂B with ends in ∂k, such that $K \cup C$ consists of two circles with linking number 0 in B.*

PROOF. Let K_1 and K_2 be two components of K. Using the fact that $B - K_2$ has the homology of a solid torus, one shows that there exists a surface F_1 in B which avoids K_2 and whose boundary is the union of K_1 and of an arc in ∂B. Similarly, there exists a surface F_2 avoiding K_1 and bounded by K_2 and an arc in ∂B.

We claim that $F_1 \cap \partial B$ and $F_2 \cap \partial B$ are disjoint after pairwise isotopy, and thus define the same marking C. Indeed, F_1 and F_2 can be pairwise isotoped so that their intersection is transverse and all intersection points of $F_1 \cap \partial B$ with $F_2 \cap \partial B$ have the same sign in ∂B. Then the 0-manifold $F_1 \cap F_2 \cap \partial B$, bounding the 1–manifold $F_1 \cap F_2$, has to be empty.

This establishes the existence of the marking desired. Conversely, given any other marking C' satisfying the conclusions of the lemma, there exists a surface F_1' bounded by the component of $K \cup C'$ containing K_1 and avoiding K_2. Now the above argument applies to show that $F_1' \cup \partial B$ represents the same marking as $F_2 \cap \partial B$, which proves the uniqueness of C. $\square$

Here is now a recipe to define, using Lemma 17.4, a marking C of the arborescent part A of a knot (S^3, K) that is simple for Schubert. For a component S of ∂A, consider the 3–ball B that it bounds in S^3 on the side opposite to A (namely so that leaving A at S we enter B). Then C is defined on S by Lemma 17.4 applied to $(B, K \cap B)$ or rather to (B, K') where K' is the two arcs of $K \cap B$ meeting ∂B.

An alternative recipe uses in place of B the complementary 3–ball $B' = S^3 - \overline{B}$.

A second alternative exploits the vertical marking C_∞ of A (see beginning of Chapter 16). At a component S of ∂A, we consider the ball B that it bounds in S^3 on the side opposite to A, and we apply Lemma 17.4 to the knot $(B \cap (K - \overline{A})) \cup (C_\infty \cap \mathrm{int}(B))$ in B.

Further variants are obtained by replacing Lemma 17.4 by the following lemma, which can be viewed as a generalisation:

LEMMA 17.5. *Let (M, K) be a compact knot pair with boundary a Conway sphere, that admits a unique 2–fold branched covering $\widetilde{W} \to W$. Then there is a unique marking C of $(\partial W, \partial K)$ such that each arc representing C has, as preimage in $\partial \widetilde{W}$, a loop bounding an integral cycle (or orientable surface) in $H_1(\widetilde{W})$.*

PROOF. There is a natural correspondence between the markings of the Conway sphere $(\partial W, \partial K)$ and the isotopy classes of connected essential curves in the torus $\partial \widetilde{W}$, which can themselves be interpreted as elements of the projective space of $H_1(\partial \widetilde{B}) \cong \mathbb{Z}^2$. Since the kernel of $H_1(\partial \widetilde{B}) \to H_1(\widetilde{B})$ is $\cong \mathbb{Z}$ by Poincaré duality, the result immediately follows. $\square$

REMARKS 17.6.

(a) The knot pair (W, K) admits a 2–fold branched covering precisely if K is zero in $H_1(W, \partial W; \mathbb{Z}_2)$. It is unique precisely if $H_1(W; \mathbb{Z}_2) = 0$. Thus our unique branched covering condition amounts to assuming that W is a $\mathbb{Z}_2$–homology ball.

(b) Lemma 17.4 and its proof can be generalised to $\mathbb{Z}_2$–homology 3–balls in place of B. Then, Lemma 17.5 gives the same marking of 17.4 in case K consists of two arcs. The proofs are left as an exercise.

(c) Applied to the ring tangle , Lemmas 17.4 and 17.5 give distinct markings, respectively of slope 0 and ∞.

CHAPTER 18

Knot symmetries as rigid motions of the 3–sphere

In this chapter, we return to symmetry groups of arborescent pairs. We have here a double aim.

The first one is to provide evidence for the Generalised Smith Conjecture[1]. This conjecture, which is actually included in the more general Orbifold Geometrisation Conjecture discussed in Chapter 6, asserts that any finite group of smooth automorphism of the 3–sphere S^3 is conjugate to the action of a subgroup of the group $\mathrm{O}(4)$ of isometries of S^3. A knot (S^3, K) that is simple for Schubert and is not Seifert fibred gives a good example to test this conjecture; indeed, it follows from Thurston's Hyperbolisation Theorem and Mostow's Rigidity Theorem that $\mathrm{Sym}(S^3, K)$ is realised by a finite group of automorphisms of S^3 respecting K (compare Chapter 6). The Generalised Smith Conjecture will then imply that, after an isotopy of K in S^3, there is a subgroup G of $\mathrm{O}(4)$ respecting K such that the canonical map $\mathcal{G} \to \mathrm{Sym}(S^3, K)$ is a group isomorphism. This section will provide a proof of this for arborescent knots, and even for marked arborescent pairs whose symmetry group has been determined to be finite in Chapter 16.

The second purpose of this section is more practical, and perhaps more important. The proof of the result stated above will provide an explicit pictorial method to realise symmetry groups in $\mathrm{O}(4)$. *A fortiori*, the reader will be able to determine for each arborescent knot (S^3, K) the algebraic structure of its symmetry group, together with the action of $\mathrm{Sym}(S^3, K)$ on the oriented components of K. Note that this last property solves the problem of deciding whether the knot is invertible, $\pm$–amphicheiral, etc.

In this section, we work in the differentiable or topological category DIFF or TOP; we avoid the piecewise linear category PL because we use a good deal of spherical (Möbius) geometry.

Theorem 18.1. *Let $(M, K; C)$ be a marked arborescent pair classified by the abbreviated $(+)$-canonical tree Γ. Assume that all ring numbers of Γ are $\leqslant 1$, and that at most one branch* $- - - ⓡ\!\!-\!\!\overset{0}{\bullet}\!-$ *(corresponding to a vertical marking) is attached at each vertex. Assume moreover that Γ is not stellar with exactly 3 branches and with all ring numbers 0, namely that (M, K) is not a closed Montesinos pair with no ring and 3 non-integral rational tangle, and that Γ is not one of* $\overset{0}{\bullet}\!-$ *and* $-\!\overset{0}{\bullet}\!-$.

Then M can be embedded in S^3 with ∂M a collection of round spheres, so that there exists a finite group $\mathcal{G}$ of isometries of S^3, respecting M, K and the marking C, for which the homomorphism $\mathcal{G} \to \mathrm{Sym}(M, K; C)$ so induced is a group isomorphism.

[1] (Added 2009) Now proved as part of the Orbifold Geometrisation Conjecture [**BoiLP, BoiP, CooHK, BoiMP**].

REMARKS 18.2.

(a) By Borel's theorem (see Lemma 16.7), $\mathcal{G}$ is maximal among finite order homeomorphism groups of the pair (M, K), provided that (M, K) is not Seifert fibred.

(b) The hypothesis that (M, K) is not a closed Montesinos pair with no ring and exactly three branches is for the most part unnecessary. Indeed, [**Boi$_1$, Boi$_2$, BoiZ**] prove that the results of Chapter 16 extend to *most* of these knots, and our proof of Theorem 18.1 will apply straightforwardly to them; more precisely, *most* means here all except those with fractional data vector (see §11.1) of the form $\big(0; e_0; \overline{\frac{\alpha_1}{\beta_1}}, \overline{\frac{\alpha_2}{\beta_2}}, \overline{\frac{\alpha_3}{\beta_3}}\big)$ where all α_i are > 1 and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$.

The proof of Theorem 18.1 that we present below constructs a subgroup $\mathcal{G} \subset \mathrm{O}(4)$ that is well-defined up to conjugacy. However we will not show that the properties ascribed to $\mathcal{G}$ in the statement of Theorem 18.1 suffices to determine $\mathcal{G}$ up to conjugacy in $\mathrm{O}(4)$. To clarify this uncertainty, we make the

CONJECTURE 18.3 (Symmetry Representation Conjecture). *For a knot (S^3, K) that is simple for Schubert, there is precisely one conjugacy class of finite subgroups $\mathcal{G} \subset \mathrm{O}(4)$ isomorphic to $\mathrm{Sym}(S^3, K)$ and respecting a knot $K' \subset S^3$ isotopic to K.*

There is a stronger conjecture for knots in S^3 that would account for this conjecture and much more. Given a knot K in S^3 that is simple for Schubert, consider the space $\mathcal{E}(K)$ of unparametrised circles in S^3 isotopic to K, and endow $\mathcal{E}(K)$ with the C^∞ topology.

CONJECTURE 18.4 (Best Position Conjecture). *There exists a "best" knot K^* in $\mathcal{E}(K)$ and a homotopy r_t, $0 \leqslant t \leqslant 1$, from $\mathrm{Id}_{|\mathcal{E}(K)}$ to a retraction $\mathcal{E}(K) \to \mathrm{O}(4)K^*$ that has (at least) these naturality properties:*

(i) *r_t is $\mathrm{O}(4)$-equivariant;*

(ii) *r_t fixes $\mathrm{O}(4)K^*$.*

Namely, $\mathcal{E}(K)$ deformation retracts to $O(4)K^*$ through an $\mathrm{O}(4)$–equivariant deformation.

From this statement, the reader can deduce the Symmetry Representation Conjecture and also Theorem 18.1. Note that, in the Seifert fibred case, the isometry group of (S^3, K^*) is a 1–dimensional Lie subgroup of $O(4)$.

When K is the trivial knot, this Best Position Conjecture is known to imply the Smale Conjecture; see [**Hat**]. Thus it is incredibly ambitious (if true). But as new methods mature it should be kept in mind.

18.1. The Borromean rings

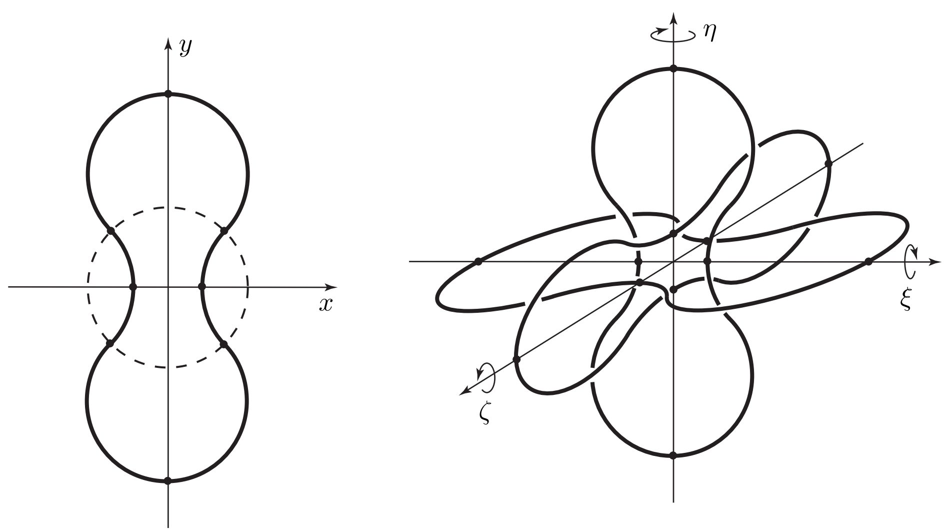
The proof of Theorem 18.1 for the Borromean rings is exceptionally simple, and provides an attractive introduction to the more difficult cases.

PROOF OF THEOREM 18.1 FOR THE BORROMEAN RINGS. In the xy–plane, choose a simple closed curve K_z which is respected by the reflections through the x– and y–axes and by the composition of the reflection $(x, y) \mapsto (y, x)$ with the inversion through the unit circle S^1. Also, we require that K_z meet the circle S^1 only at the four points $(\pm 1, \pm 1)$. For instance, take K_z to be the curve of Figure 18.1(a), a union of four circle arcs orthogonal to the unit circle (or a C^∞ smoothing of this);

then the sphere $(xy - \text{plane}) \cup \infty$ equipped with K_z looks like a tennis ball with K_z as its seam.

By cyclic permutation of the axes, we obtain two more curves K_x and K_y. The union $K = K_x \cup K_y \cup K_z$ is the Borromean link (up to isotopy) illustrated in Figure 18.1(b).

The isometry group $\mathcal{H}$ of $(\mathbb{R}^3, K)$ has order 24; it clearly is an index 2 subgroup of the 48 element isometry group of the cube $[-1, 1]^3$.



(a) (b)

FIGURE 18.1.

Interestingly, there is a larger group $\mathcal{G}$ of conformal (Möbius) automorphisms of $(\mathbb{R}^3 \cup \infty, K)$; namely the group $\mathcal{G}$ generated by $\mathcal{H}$ and by the composition of a $90°$ rotation about an axis with an inversion (reflection) in the unit sphere. The group $\mathcal{G}$ has order 48, since $\mathcal{H}$ is the index 2 subgroup that does not exchange 0 and ∞.

We stereographically identity $\mathbb{R}^3 \cup \infty$ to the unit sphere S^3 in $\mathbb{R}^4$, taking care to send the unit sphere in $\mathbb{R}^3$ onto an equatorial sphere of S^3. Then, $\mathcal{G}$ becomes an isometry group of (S^3, K).

We assert that the natural group homomorphism $\mathcal{G} \to \text{Sym}(S^3, K)$ is an isomorphism. By Theorem 16.18, the group $\text{Sym}(S^3, K)$ is also of order 48 so we only have to convince ourselves that $\mathcal{G}$ is injected. Borel's theorem (see Lemma 16.7) assured this; but it is an obvious fact since $\mathcal{G}$ acts faithfully on $H_3(S^3) \oplus H_1(K)$. $\square$

18.2. Degree +1 symmetries

In the general case, the proof of Theorem 18.1 heavily relies on the plumbing construction of (M, K) according to Γ. It gains some considerable complexity when $(M, K; C)$ admits degree -1 symmetries, at least when the associated positively and negatively canonical graphs are distinct. It is therefore convenient to initially limit attention to degree $+1$ symmetries, and thus prove the following weaker statement.

THEOREM 18.5. *With the data and hypotheses of Theorem 18.1, there exists an embedding of M in S^3 and a finite subgroup $\mathcal{G}^+$ of* SO(4) *for which restriction induces an isomorphism* $\mathcal{G}^+ \to \mathrm{Sym}^+(M, K; C)$.

The case where Γ consists of a single vertex that is black or white (namely has ring number 0 or 1) and has d free bonds and a weight $a \in \mathbb{Z}$, will be the starting point of the proof in the general case. For a while, we leave aside the degenerate case where $d = 0$.

Viewing S^3 as the unit sphere in $\mathbb{C}^2$, choose a small arc in $S^3 \cap \mathbb{R}^2$, centred at $(1, 0)$ and preserved by the involution $(z, z') \mapsto (z, -z')$ of $\mathbb{C}^2$. Let then B be the orbit of this arc by the action of the subgroup of $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ consisting of the automorphisms $R_t\colon (z, z') \mapsto (z\exp(2\pi \mathrm{i}t), z'\exp(\pi \mathrm{i}at))$ of $\mathbb{C}^2$ with $t \in \mathbb{R}$.

The set B is a ribbon in S^3 with unknotted central thread, and with a regularly distributed positive half-twists (see Figure 18.2, where $a = -3$). By construction, it is preserved by each R_t, by the π–rotation $\xi\colon (z, z') \mapsto (z, -z')$ around $S^1 \times 0$ (with S^1 the unit circle in C) and by the π–rotation $T\colon (z, z') \mapsto (\overline{z}, \overline{z}')$ around $S^3 \cap \mathbb{R}^2$.

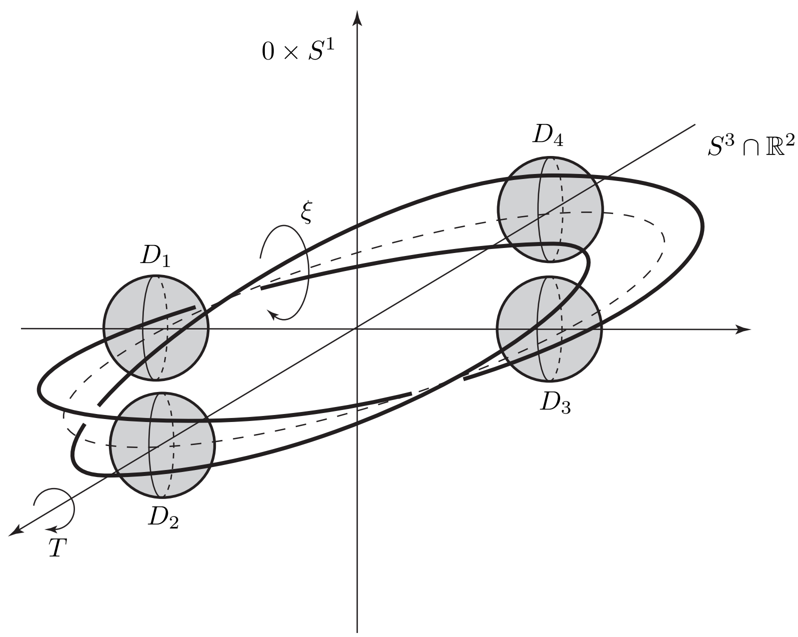


FIGURE 18.2.

Next, choose d disjoint round balls $D_1, D_2, \ldots, D_d$, respectively centred (for the metric of S^3) at the points $(\omega, 0)$, $(\omega^2, 0)$, $\ldots$, $(\omega^d, 0)$ where $\omega = \exp(\frac{2\pi \mathrm{i}}{d})$, with equal radii, so that each sphere ∂D_i meets ∂B in exactly 4 points as in Figure 18.2 (this is possible if the ribbon B has been chosen narrow enough). Note that the boundaries ∂D_i correspond to round spheres in $\mathbb{R}^3$, under the stereographic identification $S^3 \cong \mathbb{R}^3 \cup \infty$; however their centres in $\mathbb{R}^3$ differ from the points corresponding to the points $(\omega^i, 0)$.

Each pair $(D_i, \partial B \cap D_i)$ is a rational tangle pair. Equip each sphere ∂D_i with the marking C_i^* that has the same slope as the (unique) pairwise essential disc in $(D_i, \partial B \cap D_i)$.

The construction has been designed so that the marked pair $(M, K; C)$ classified by Γ is now isomorphic to the marked pair $(M^*, K^*; C^*)$, where M^* is the complement of $\bigcup_{i=1}^{d} \mathrm{int}(D_i)$ in S^3, where K^* is $\partial B \cap M^*$ or $(\partial B \cap M^*) \cup (0 \times S^1)$ according as the ring number of the vertex of Γ is 0 or 1, and where C^* is defined by the d markings C_i^*.

By construction, the marked pair $(M^*, K^*; C^*)$ is respected by the isometries $R_{\frac{k}{d}}$ with $k \in \mathbb{Z}$, by the π–rotation ξ around $S^1 \times 0$ and by d distinct π–rotations $R_{\frac{k}{2d}}\, T\, R_{\frac{k}{2d}}^{-1}$ around $R_{\frac{k}{2d}}(S^3 \cap \mathbb{R}^2)$ with $k \in \mathbb{Z}$. Let $\mathcal{G}^+$ denote the order $4d$ subgroup of $\mathrm{SO}(4)$ generated by these isometries.

Fix an isomorphism $(M, K; C) \cong (M^*, K^*; C^*)$. Then, evaluation on the set of components of ∂M^* defines a morphism from $\mathcal{G}^+$ onto the group $\mathrm{Sym}^+\Gamma$ (namely the dihedral group of order $2d$; see Chapter 16). Noting that the identity is the unique element of $\mathcal{G} \subset \mathrm{O}(4)$ that fixes one point of ∂K^*, one readily checks that the kernel of this morphism is isomorphic to $\mathbb{Z}_2$, generated by ξ when $d \geqslant 3$ and is isomorphic to $V_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by ξ and T, when $d = 1$ or 2. Using Theorems 16.4 and 16.8, we deduce that the natural homomorphism $\mathcal{G}^+ \to \mathrm{Sym}^+(M^*, K^*; C^*)$ is a group isomorphism. This proves Theorem 18.5 in the case where Γ consists of a single vertex with $d \geqslant 1$ free bonds.

When $d = 0$, the pair (M, K) is a $\{2, a\}$ torus knot, and $\mathrm{Sym}(M, K)$ is determined by the uniqueness of Seifert fibrations proved in [**Wal**$_2$] (see also our discussion of this case at the end of Chapter 16). Here are some details. When the ribbon B is constructed as in the previous case, (M, K) is isomorphic to $(S^3, \partial B)$. Define $\mathcal{G}^+$ to be generated by ξ and T if a is even, and by T alone if a is odd. Then, it follows from [**Wal**$_2$] that $\mathcal{G}^+ \cong \mathrm{Sym}^+(M, K)$ by the natural mappings. Note that ξ is pairwise isotopic to the identity when a is odd. □

Proof of Theorem 18.5 in the general case. Although we aim at constructing a finite group $\mathcal{G}$ of isometries of S^3, our construction will make use of ***degree*** ± 1 ***conformal automorphisms*** of S^3. Remember that the ***Möbius group***, consisting of these degree ± 1 conformal automorphisms, is the group generated by inversions through round spheres of $S^3 = \mathbb{R}^3 \cup \infty$. This includes all isometries of S^3, but many more diffeomorphisms. For instance, if we identify S^3 to $\mathbb{R}^3 \cup \infty$ by stereographic projection, the stabiliser of ∞ in the Möbius group is the group of similarities or $\mathbb{R}^3$, obtained by composition of an euclidean isometry with a homothety.

We exclude the cases already dealt with, namely the cases where (M, K) is the Borromean rings or where Γ consists of a single 0-valent vertex. Then, Theorems 16.4 and 16.8 describe $\mathrm{Sym}^+(M, K; C)$ by an exact sequence

$$0 \to \mathcal{K}(\Gamma) \to \mathrm{Sym}^+(M, K; C) \to \mathrm{Sym}^+\Gamma \to 0,$$

where $\mathcal{K}(\Gamma)$ is determined by Algorithm 16.10.

Assertion 18.6. *We can assume without loss of generality that* Γ *contains no branch* $\overset{0}{-\!\bullet\!-}$.

Proof of Assertion 18.6. Let Γ' be obtained from Γ by erasing each such branch $\overset{0}{-\!\bullet\!-}$, keeping however the bond where it was attached; the pair obtained by plumbing according to Γ' is naturally identified with (M, K), but comes equipped with a different marking C'. Now, Theorem 16.11, together with the hypothesis that there is at most one branch $\overset{0}{-\!\bullet\!-}$ at each vertex of Γ, shows that $\text{Sym}(M, K; C) = \text{Sym}(M, K; C')$. $\square$

By definition, the pair (M, K) is obtained by plumbing atomic tangles (M_i, K_i) (with possibly one ring) according to Γ. Let each pair (M_i, K_i) be equipped with the slope 0 marking C_i.

For each i, we have already constructed a marked pair $(M_i^*, K_i^*; C_i^*)$, embedded in S^3 and isomorphic to $(M_i, K_i; C_i)$, which is respected by a finite subgroup $\mathcal{G}_i^+$ of SO(4) such that the natural map $\mathcal{G}_i^+ \to \text{Sym}^+(M_i^*, K_i^*; C_i^*)$ is an isomorphism. (Here we use the fact that no vertex of Γ is of type $\overset{0}{\bullet\!-}$ or $\overset{0}{-\!\bullet\!-}$.) This construction makes use of a twisted ribbon B_i^* as in Figure 18.2. It will be convenient to slightly modify B_i^* and K_i^* so that, for each (round sphere) component Σ of ∂M_i^*, *the two arcs* $B_i^* \cap \Sigma$ *form a square contained in an equator of* Σ, *with sides of equal length* (for the metric of Σ induced by that of $S^3 \subset \mathbb{R}^4$). This can clearly be accomplished by a slight $\mathcal{G}_i^+$–equivariant modification of the picture near ∂M_i^*.

For the differentiable category DIFF, we impose an extra condition on K_i^* near ∂M_i^* in M_i^*, namely that K_i^* there be on geodesics (= great circles) normal to ∂M_i^*. This will allow us to fit together conformal copies of the M_i^* to get a smooth pair $\theta(M, K)$.

We shall also require a further hypothesis which will be crucial for our proof, namely that, *if* $(M_i, K_i; C_i)$ *is isomorphic to* $(M_j, K_j; C_j)$, *then the corresponding models* $(M_i^*, K_i^*; C_i^*)$ *and* $(M_j^*, K_j^*; C_j^*)$ *are the same.*

To construct the required embedding θ of (M, K) in S^3 first choose, for each i, a degree $+1$ isomorphism $\sigma_i\colon (M_i, K_i; C_i) \to (M_i^*, K_i^*; C_i^*)$. The embedding θ will have the property that, each $\theta\sigma_i^{-1}\colon M_i^* \to \theta(M_i)$ will be *conformal*, namely will be the restriction of a degree $+1$ conformal automorphism $S^3 \to S^3$.

Next we impose a compatibility condition among the isomorphisms σ_i. *For each Conway sphere* Σ *that separates a block* (M_i, K_i) *from a block* (M_j, K_j), we make an adjustment of σ_i (or of σ_j) near Σ so that $\alpha_{ij} = \sigma_j\sigma_i^{-1}{}_{|\sigma_i(\Sigma)}\colon \sigma_i(\Sigma) \to \sigma_j(\Sigma)$ is a degree -1 ***metric similarity***, namely such that the ratio $d\big(\alpha_{ij}(x), \alpha_{ij}(y)\big)/d(x, y)$ is independent of the points x, $y \in \sigma_i(\Sigma)$. This is possible because the equatorial square in $\sigma_i(\Sigma) \subset S^3$ formed by the slope ∞ and 0 markings in (M_i^*, K_i^*) maps up to pairwise isotopy to the similarly defined square in $\sigma_j(\Sigma)$, since a tangle plumbing was performed at Σ.

Let $\widehat{\alpha}_{ij}\colon S^3 \to S^3$ be the unique degree $+1$ conformal automorphism of S^3 that extends the metric similarity $\alpha_{ij}\colon \sigma_i(\Sigma) \to \sigma_j(\Sigma)$. Note that it sends M_i^* to the complement of M_j^*.

For the DIFF category, the above condition on α_{ij} must be reinforced by insisting that $\widehat{\alpha}_{ij}\sigma_i$ and σ_j *together define a smooth embedding into* S^3 *of a neighbourhood of* Σ *in* M. We can arrange this using DIFF collaring theorems.

We will construct θ stepwise. For this, it will be convenient to assume, without any loss of generality, that the indexing of the M_i is chosen so that each M_i with $i > 1$ be adjacent to some M_j with $j < 1$.

To start the induction, we define θ to be $\theta_1^*\sigma_1$ on M_1, where θ_1^* is an arbitrary degree $+1$ conformal automorphism of S^3. It will be convenient to further specify M_1 and θ_1^* later on.

Assuming θ defined on $\bigcup_{j<i} M_j$, with all compositions $\theta\sigma_j^{-1}$ degree +1 conformal, consider the (unique) Conway sphere Σ that is the intersection of M_i with some M_k such that $k < i$. Let θ_k^* be the unique extension of $\theta\sigma_k^{-1}$ to a conformal degree +1 automorphism of S^3. Then, define θ on M_i to be $\theta_k^*\widehat{\alpha}_{ik}\sigma_i$, noting that this rule agrees with $\theta_{|M_k}$ on Σ, and that $\theta\sigma_i^{-1}$ is degree +1 conformal.

Continuing thus, inductively define θ on all of M. Clearly, θ is the unique extension of $\theta_1^*\sigma_1$ to a degree +1 embedding $M \to S^3$ so that $\theta\sigma_i^{-1}$ is conformal for all i. Its image is the closed-up complement of finitely many round balls in S^3.

If we are working in the DIFF category, note that θ sends K to a smooth 1–dimensional submanifold of $\theta(M)$. This comes from our hypothesis that, in each model pair (M_i^*, K_i^*) and near each boundary component $\Sigma^* \subset \partial M_i^*$, the knot K_i^* locally coincides with geodesics that are orthogonal to the round sphere Σ^*. Indeed, the conformal extensions $\widehat{\alpha}_{ij} : S^3 \to S^3$ of the metric similarities $\alpha_{ij} : \sigma_i(\Sigma) \to \sigma_j(\Sigma)$ have the property that they send geodesics orthogonal to $\sigma_i(\Sigma)$ to geodesics orthogonal to $\sigma_j(\Sigma)$.

Then θ sends $(M, K; C)$ to a marked knot pair $\theta(M, K; C)$ contained in S^3.

Claim 18.7. *Every degree* +1 *symmetry of* $\theta(M, K; C)$ *is realised by restriction of a degree* +1 *conformal automorphism of* S^3.

Proof of Claim 18.7. Let ψ be a degree +1 automorphism of $\theta(M, K; C)$. After pairwise isotopy, it can be assumed to respect the decomposition into blocks $\theta(M_i, K_i)$ by Proposition 13.13.

Consider a restriction ψ_i: $\theta(M_i, K_i) \to \theta(M_j, K_j)$. We claim that it is pairwise isotopic to a conformal map ψ_i'. Indeed, we have proved in the proof of Propositions 13.10 and 13.13 that ψ_i sends the marking $\theta(C_i)$ to the marking $\theta(C_j)$. In particular, $\theta^{-1}\psi_i\theta$ realises an isomorphism $(M_i, K_i; C_i) \cong (M_j, K_j; C_j)$ and, by hypothesis, the model pairs $(M_i^*, K_i^*; C_i^*)$ and $(M_j^*, K_j^*; C_j^*)$ *coincide*. Thus, $\sigma_j\theta^{-1}\psi_j\theta\sigma_i^{-1}$ is a degree +1 automorphism of $(M_i^*, K_i^*; C_i^*) = (M_j^*, K_j^*; C_j^*)$ and, as such, is pairwise isotopic to an element ψ_i'' of $\mathcal{G}_i^+ = \mathcal{G}_j^+ \subset \mathrm{SO}(4)$. Therefore, ψ_i is pairwise isotopic to $\psi_i' = (\sigma_1\theta^{-1})\psi_i''(\theta\sigma_j^{-1})$, which, as a product of three conformal maps, is conformal.

When $M_i \cap M_j \neq \varnothing$, the restriction of ψ_i' and ψ_j' to $\theta(M_i \cap M_j) \to \psi\theta(M_i \cap M_j)$ coincide on the four points $\theta(K \cap (M_i \cap M_j))$. Since both maps are conformal, it follows that ψ_i' and ψ_j' coincide on $\theta(M_i \cap M_j)$. This proves that all the ψ_i' and ψ_j' are restrictions of a certain conformal map ψ' of S^3.

To conclude the proof of Claim 18.7, we now just need to check that ψ and ψ' are pairwise isotopic. But this follows from Theorems 16.4 and 16.8 since they induce the same element of $\mathrm{Sym}(\Gamma)$, and since $\psi'\psi^{-1}$ fixes each point of $K \cap \partial M_i$ and thus is trivial in $\mathcal{K}(\Gamma)$. □

Actually, we have proved a little more than stated in Claim 18.7. Let $\mathcal{G}^+$ be the group of degree +1 conformal automorphisms ψ' of S^3 such that:

(a) ψ' sends each block $\theta(M_i, K_i)$ to a block $\theta(M_j, K_j)$;
(b) when ψ' sends $\theta(M_i, K_i)$ to $\theta(M_j, K_j)$, then $\sigma_j\theta^{-1}\psi'\theta\sigma_i^{-1}$ is an element of the finite group $\mathcal{G}_i^+ = \mathcal{G}_j^+ \subset \mathrm{SO}(4)$ of automoprhisms of $(M_i^*, K_i^*; C_i^*) = (M_j^*, K_j^*; C_j^*)$.

We just proved that the natural map $\mathcal{G}^+ \to \mathrm{Sym}^+\theta(M, K; C)$ is surjective. An application of Theorem 16.8 shows that it is also injective.

To conclude our proof of Theorem 18.5, we need to arrange for $\mathcal{G}^+$ to consist of *isometries* of S^3 and not just of conformal automorphisms. This is provided by the following classical fact.

FACT 18.8. *Any compact group G of Möbius automorphisms of S^n is conjugate to a subgroup of the orthogonal group* $\mathrm{O}(n{+}1)$ *by a degree $+1$ Möbius automorphism.*

PROOF OF FACT 18.8. Taking the unit ball model B^{n+1} for the hyperbolic n-space $\mathbb{H}^{n+1}$ (see §5.1), the action of the Möbius group on $S^n = \partial B^n$ extends to an isometric action on $\mathbb{H}^n$. For instance, the inversion through a (round) hypersurface Σ of S^n extend by the inversion of $\mathbb{R}^{n+1} \cup \infty$ through the unique n-sphere cutting S^n orthogonally along Σ.

Any orbit of G in $\mathbb{H}^{n+1}$ has a well-defined barycentre, minimising the quadratic average of the distances to the points of this orbit. Conjugating G so that this barycentre is the centre 0 of the ball $B^{n+1} = \mathbb{H}^{n+1} \cup S^n$, the group G then is in the stabiliser of 0 in the isometry group of $\mathbb{H}^{n+1}$, which is the orthogonal group $\mathrm{O}(n+1)$. □

Fact 18.8 provides a degree $+1$ Möbius automorphism ψ conjugating $\mathcal{G}^+$ to a subgroup of SO(4).

Our inductive construction of θ made use of an arbitrary conformal automorphism θ_1^* of S^3, for which $\theta_{|M_1}$ was defined as $\theta_1^*\alpha_1$. If we replace θ_1^* by $\psi\theta_1^*$, this replaces θ by $\psi\theta$ and $\mathcal{G}^+$ by $\psi\mathcal{G}^+\psi_{-1} \subset \mathrm{SO}(4)$.

This concludes our proof of Theorem 18.5. □

For practical applications, however, we want to proceed more explicitly instead of relying on Fact 18.8. For this, we use the following:

FACT 18.9. *There exists a vertex or an edge of Γ that is characteristic, namely preserved by any symmetry of Γ.*

PROOF. Inductively define a sequence of subtrees Γ_k of the combinatorial tree Γ_c underlying Γ as follows: Γ_1 is obtained from Γ_c by erasing all free bonds; each Γ_{k+1} is obtained from Γ_k by removing all valence $\leqslant 1$ vertices together with their adjacent edges. The last non-empty Γ_k clearly consists of a single vertex or of two vertices joined by an edge. This singles out a vertex or edge which, by naturality of the construction, is respected by each combinatorial isomorphism of Γ_c and in particular by each symmetry of Γ. □

When Γ admits a characteristic vertex, one can assume that it corresponds to the first block (M_1, K_1). If, in the inductive construction of θ, we take θ_1^* to be the identity, so that θ coincides with σ_1 on M_1, then our group $\mathcal{G}^+$ is a subgroup of $\mathcal{G}_1^+ \subset \mathrm{SO}(4)$.

When Γ admits a characteristic edge, a Conway sphere Σ in $(M, K; C)$ is naturally associated to this edge by the plumbing construction, and we can choose subscripts so that $\Sigma \subset \partial M_1$. In this situation, select θ_1^* so that $\theta_1^*\alpha_1(\Sigma)$ is an equatorial sphere of S^3, and $\theta_1(\Sigma \cap K)$ forms an equatorial square in $\theta_1\Sigma$. Then, every element of $\mathcal{G}^+$ respects $\theta_1(\Sigma)$ and $\theta_1(\Sigma \cap K)$ and so is an isometry of S^3.

18.3. Degree ±1 symmetries

Proof of Theorem 18.1 in the presence of degree −1 symmetries. In this more general situation, our proof of Theorem 18.5 would encounter difficulties related to the active vertices of the positively canonical graph Γ classifying $(M, K; C)$. Recall that a vertex of Γ is called ***active*** when it is of type $\overset{2}{\bullet\!-}$ or $\overset{2}{-\!\bullet\!-}$, and each of its bonds leads to a vertex of ring number $\geqslant 1$ or of valence $\geqslant 3$. In other words, a vertex is active when an arithmetic move (2.1) or (2.2) of Chapter 12 is applied there to go from the positively canonical graph Γ classifying $(M, K; C)$ to its negatively canonical graph Γ^-. Now, it follows from the definition of symmetries of Γ that, under a degree -1 symmetry,

(a) Any active vertex is sent to another; in particular, its weight does not change sign.
(b) A non-active vertex of weight e is sent to another such of weight $-e + k$ where k is the number of adjacent active vertices.

Recall that the pair $(M, K; C)$ is obtained by plumbing atomic tangles (M_i, K_i) according to Γ. This construction provides, in each such $(M_i, K_i) \subset (M, K)$, a preferred band B_i^+, and a preferred slope 0 boundary marking C_i^+. This C_i^+ is also defined by C and the bands of the atomic tangles (M_j, K_j) adjacent to (M_i, K_i).

On the other hand, consider the negatively canonical graph Γ^- obtained by performing on each active vertex of Γ a move (2.1) or (2.2) of Chapter 12. Plumbing atomic tangles according to Γ^- yields a marked atomic pair which is naturally identified with $(M, K; C)$. Moreover, the plumbing blocks are naturally identified to the (M_i, K_i) as knot pairs, although not necessarily as tangles. These building blocks come equipped with a preferred band B_i^- and a preferred marking C_i^-, which may differ from B_i^+ and C_i^+.

In fact, the bands B_i^+ and B_i^- differ exactly at the active (M_i, K_i), namely at the blocks corresponding to active vertices of Γ. Figure 18.3 indicates their boundaries at such an active block; note that B_i^- has slope -1 in each boundary component parametrised for the atomic tangle (M_i, K_i) arising from Γ.

Since the markings C_j^+ and C_j^- are determined by C and the bands B_k^+ and B_k^- of the blocks adjacent to (M_j, K_j), this also reveals that: C_j^+ and C_j^- differ only when M_j is non-active and only at the components of ∂M_j which separate M_j from an active block (M_i, K_i); at such a boundary component of (M_j, K_j), the markings C_j^+ and C_j^- coincide respectively with the traces of B_i^+ and B_i^- indicated in Figure 18.3.

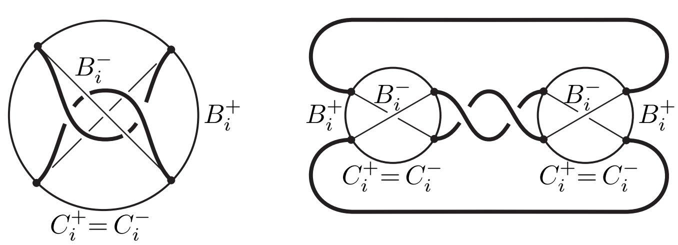


Figure 18.3.

It is convenient to introduce the following definition. Given two connected atomic pairs (N_k, L_k), with $k = 1, 2$, each equipped with two bands B_k^+ and B_k^- and two boundary markings C_k^+ and C_k^-, a degree ± 1 isomorphism $(N_1, L_1) \to (N_2, L_2)$ is ***admissible*** if

(a) either it has degree $+1$ and sends B_1^+, B_1^-, C_1^+, C_1^- respectively to B_2^+, B_2^-, C_2^+, C_2^-,

(b) or it has degree -1 and sends B_1^+, B_1^-, C_1^+, C_1^- respectively to B_2^-, B_2^+, C_2^-, C_2^+.

Then, we have shown in §13.3 and Chapter 16 that, under the hypotheses of Theorem 18.1, every degree ± 1 automorphism of $(M, K; C)$ is pairwise isotopic to an automorphism ψ of (M, K) mapping each (M_i, K_i) to another (M_j, K_j), by an isomorphism $(M_i, K_i) \to (M_j, K_j)$ that is admissible for the bands and markings defined above.

We now propose to choose for each block (M_i, K_i) a model (M_i^*, K_i^*) embedded in S^3, equipped with two bands B_i^{+*} and B_i^{-*} and two boundary markings C_i^{+*} and C_i^{-*}, so that there exists an admissible degree $+1$ isomorphism $\sigma_i\colon (M_i, K_i) \to (M_i^*, K_i^*)$. These models will satisfy the following two conditions.

Naturality Condition. Every admissible degree ± 1 isomorphism $(M_i^*, K_i^*) \to (M_j^*, K_j^*)$ is pairwise isotopic to the restriction of a (degree ± 1) conformal automorphism $S^3 \to S^3$.

Gluing Condition. If S is a common boundary component of M_i and M_j, then the map $\sigma_j \sigma_i^{-1}\colon \sigma_i(S, S \cap K) \to \sigma_j(S, S \cap K)$ of Conway spheres is pairwise isotopic to the restriction of a (necessarily unique) conformal degree $+1$ automorphism ψ of S^3.

If such models are given, then the arguments used for the proof of Theorem 18.5 (including of course the smoothing tricks for DIFF) straightforwardly apply to conclude the proof of Theorem 18.1 in full generality.

If the block (M_i, K_i) is *active*, the three markings induced by B_i^+, B_i^- and $C_i = C_i^+ = C_i^-$ on a boundary component S of M_i form a tetrahedral graph in that sense that each of these markings is represented by a pair of arcs such that the union of these three pairs of arcs is a tetrahedral graph on S (see Figure 18.3). For the corresponding model (M_i^*, K_i^*), the naturality condition requires the existence of a degree -1 conformal automorphism of $\sigma_i(S)$ respecting $C_i^* = C_i^{+*} = C_i^{-*}$ and exchanging the two markings induced by B_i^{+*} and B_i^{-*}.

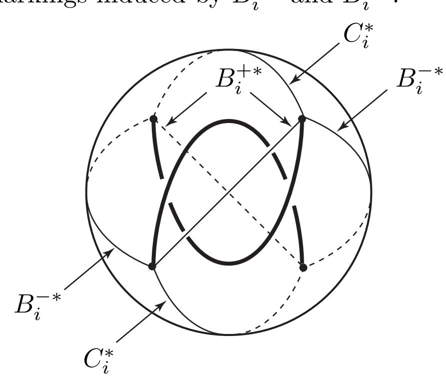


Figure 18.4.

This leads us to use the model shown in Figure 18.4 for an active block (M_i, K_i) with *connected* boundary. Here, the boundary ∂M_i^* is a round sphere where the markings defined by C_i^*, B_i^{+*} and B_i^{-*} form a regular tetrahedral graph. Moreover, K_i^* is chosen to be respected by the order 8 subgroup $\mathcal{G}_i$ of $O(3) \subset O(4)$ generated by the π–rotations respecting this graph (one for each pair of opposite edges) and by the reflection through a plane containing a component of C_i^* (and orthogonal to the other one).

We note that here (and coherently throughout this chapter) one could replace the regular tetrahedral graph by a slightly less regular type; what is essential here is that the two edges in C_i are perpendicular and that the other four edges are equal.

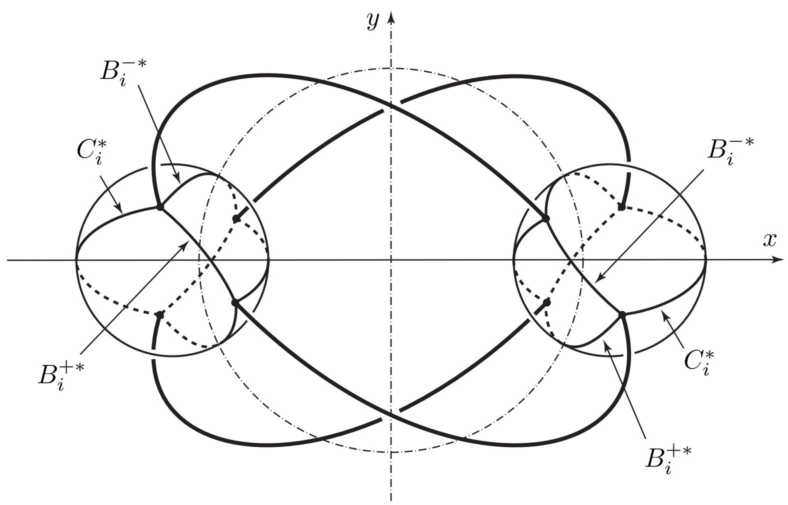


FIGURE 18.5.

For an active block (M_i, K_i) with two boundary components, we use a conformal model (M_i^*, K_i^*) as in Figure 18.5. The degree $+1$ conformal automorphisms of (M_i^*, K_i^*) form the group of order 8 consisting of the π–rotations about the three coordinate axes, of the (conformal) π–rotations about the three circles of intersection of the unit sphere with the coordinate planes, and finally of the identity and the free "antipodal" involution respecting all the spheres and planes mentioned. (The subgroup of this group respecting a boundary sphere is a Klein 4–group V_4.) The full group $\mathcal{G}_i^*$ of conformal automorphisms of (M_i^*, K_i^*) is the larger group of order 16 further generated by reflection in either of the two planes Σ_+, Σ_- derived by rotation of the xy-plane through angle $\pm\frac{\pi}{4}$ about the x-axis (note that this reflection symmetry dictates that the two boundary spheres of M_i^* be centred on the x-axis and that K_i^* and C_i^* lie in $\Sigma_+ \cup \Sigma_-$).

The two boundary spheres of M_i^* are round and B_i^{+*}, B_i^{-*}, C_i^* together describe conformally regular tetrahedral graphs on them (but these are not metrically regular in $\mathbb{R}^3$ as $\mathcal{G}_i^*$ would not allow it !).

Given that each string connects the two boundaries of the model, the features described above dictate all essential properties of the model. However it is helpful to observe that:

(a) If the boundary spheres are chosen very small, the tetrahedral graphs therein are (nearly) metrically regular and related by translation along the x–axis.
(b) By a conformal transformation of the model, one can arrange that: the boundary spheres are concentric about the origin; the tetrahedral graphs therein are metrically regular and dual to each other by radial projection; the string K_i^* (transformed) lies in two perpendicular planes through the origin and projects from the origin monotonically to four geodesic segments of either boundary sphere. This description clarifies the relationship with the model for one boundary component (Figure 18.4).

For any active block (M_i, K_i), the conformal model (M_i^*, K_i^*) proposed above (with 1 or 2 boundary components) is such that the group $\mathcal{G}_i$ of conformal automorphisms is visibly admissible with respect to the bands B_i^{+*}, B_i^{-*} and the marking $C_i^* = C_i^{\pm *}$.

Conversely, Theorems 16.4 and 16.8 show that any admissible automorphism of (M_i^*, K_i^*) is pairwise isotopic to an element of $\mathcal{G}_i$.

Thus the Naturality Condition holds for the conformal models of active blocks.

Now consider a *non-active block* (M_i, K_i). Here the two bands B_i^+ and B_i^- coincide, and will henceforth be denoted by B_i. The two markings C_i^+ and C_i^- differ only at the components of ∂M_i that are adjacent to active blocks.

For the corresponding model (M_i^*, K_i^*), the Gluing Condition and our previous construction of models for active blocks require the following: On each component S of ∂M_i that is adjacent to an active block, the component $\sigma_i(S)$ of ∂M_i^* is a round sphere in S^3 where the band $B_i^* = B_i^{+*} = B_i^{-*}$ and the markings C_i^{+*} and C_i^{-*} describe a tetrahedral graph in $\sigma(S)$, which is conformally equivalent to the regular tetrahedral graph.

On each boundary component S' of (M_i, K_i) that is not adjacent to an active block, the two markings C_i^+ and C_i^- coincide and describe with B_i a topological square. In the model M_i^*, the boundary sphere $\sigma_i(S')$ shall be a round sphere where C_i^{+*} (coinciding here with C_i^{-*}) and B_i^* shall describe a conformal square.

Let us now begin the construction of the model (M_i^*, K_i^*). Let $e \in \mathbb{Z}$ be the weight of the (non-active) vertex v_i of Γ corresponding to the plumbing block (M_i, K_i), and let $k \in \mathbb{N}$ denote the number of active vertices of Γ that are adjacent to v_i. Also, let $S_1, S_2, \ldots, S_d$ be the boundary components of M_i, occurring in this order on a necklace of (M_i, K_i).

Considering S^3 as the unit sphere in $\mathbb{C}^2$, choose d disjoint balls $D_1, \ldots, D_d$ in S^3, with equal radii and centred respectively at $(\omega, 0)$, $(\omega^2, 0)$, $\ldots$, $(\omega^d, 0)$ where $\omega = \exp\big(\frac{2\pi}{d} i\big)$, as in Figure 18.6. The model M_i^* will be $S^3 - \bigcup_{n=1}^d \operatorname{int}(D_n)$.

To define the band B_i^*, we first construct an object B which, topologically, is a disjoint union of d squares $B^{(1)}, \ldots, B^{(d)}$. For this, we introduce the number $a = \frac{e-k}{2}$ and, for $t \in \mathbb{R}$, the glide rotation

$$R_t \colon (z, z') \mapsto (z \exp(2\pi i t), z' \exp(a\pi i t))$$

on $S^3 \subset \mathbb{C}^2$.

The squares are defined inductively. In the circle $S^3 \cap \mathbb{R}^2$, choose a small arc λ centred at $(1, 0)$ and symmetric with respect to the π–rotation $\xi \colon (z, z') \mapsto (z, -z')$. Then the square $B^{(1)}$ is the union of the arcs $R_t(\lambda)$, with $0 \leqslant t \leqslant \frac{1}{d}$.

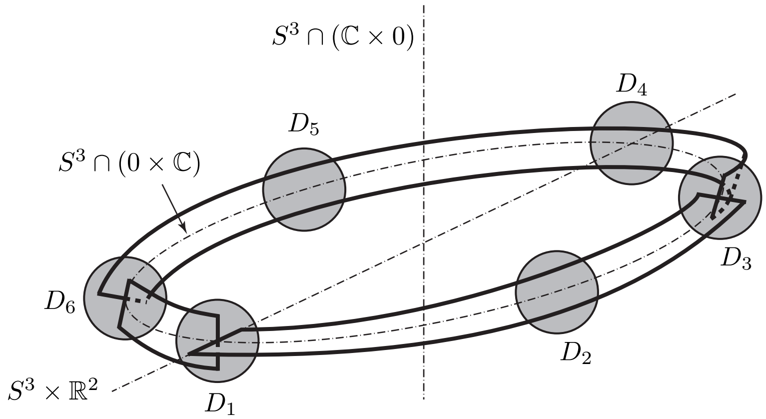


FIGURE 18.6.

When $B^{(n)}$ is already defined, $B^{(n+1)}$ depends on the type of the component S_n of ∂M_i. If S_n is not adjacent to an active plumbing block, then $B^{(n+1)} = R_{\frac{1}{d}}(B^{(n)})$, so that $B^{(n)}$ and $B^{(n+1)}$ fit together to form a smooth ribbon in D_n. Otherwise $B^{(n+1)} = \xi^{\frac{1}{2}} R_{\frac{1}{d}}(B^{(n)})$, where $\xi^{\frac{1}{2}}$ is the $\frac{\pi}{2}$–rotation $(z, z') \mapsto (z, z'\exp(i\frac{\pi}{2}))$ around $S^3 \cap (\mathbb{C} \times 0)$, so that $B^{(n)}$ and $B^{(n+1)}$ abut orthogonally at the point $(\omega^n, 0)$. Figure 18.6 presents an example where $e = 2$, $d = 6$ and $k = 3$.

As a first approximation to the definitive mode, the band B_1^* will then be taken to be $B \cap M_i^*$. The knot K_i^* (first approximation) will of course be the closure of $\partial B_i^* - \partial M_i^*$ plus, in case the ring number is 1, the circle $S^3 \cap (0 \times \mathbb{C}) = $ (thering).

We next want to define the markings C_i^{+*} and C_i^{-*} (first approximations). If, at the points where $B^{(n)}$ and $B^{(n+1)}$ do not fit, we add to B two quarter of discs as indicated in Figure 18.7(a), we get a piecewise smooth ribbon. There are two ways to do so, inducing respectively a right-handed or left-handed quarter twist on the ribbon. Let B^+ (resp.B^-) denote the ribbon obtained by this process, using the right-handed (resp. left-handed) choice everywhere; by construction, the ribbon B^+ (resp. B^-) has unknotted central thread and has e (resp. $e - k$) right-handed half-twists.

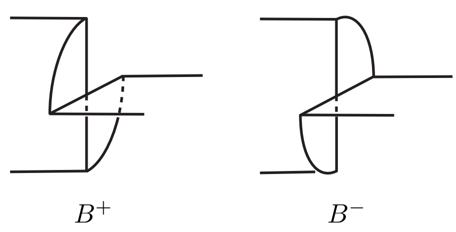


FIGURE 18.7.

For each ball D_n, the pair $(D_n, \partial B^- \cap D_n)$ is a rational tangle pair. Then, the marking C_i^{+*} is defined on the component ∂D_n of ∂M_i^* by the slope of this rational tangle, which is the slope gotten by isotoping the arcs $\partial B^+ \cap D_n$ into ∂D_n fixing their end points. The marking C_i^{-*} is defined similarly, using the rational tangle pairs $(D_n, \partial B^- \cap D_n)$.

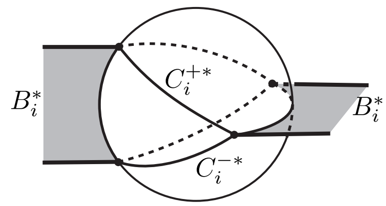


FIGURE 18.8.

Note that, when the squares $B^{(n)}$ and $B^{(n+1)}$ fit together, the two markings C_i^{+*} and C_i^{-*} coincide on ∂D_n and form a square with the trace of the band B_i^*. On the other hand, when $B^{(n)}$ and $B^{(n+1)}$ do not fit, the three markings induced by C^{+*}, C_i^{+*}, C_i^{-*} and B_i^* describe a tetrahedral graph on ∂D_n, as shown on Figure 18.7(b). This completes our construction of the model (M_i^*, K_i^*) in its approximate form.

The above construction has been designed so that there exists a degree $+1$ isomorphism $\sigma_i\colon (M_i, K_i) \to (M_i^*, K_i^*)$ mapping B_i, C_i^+, C_i^- respectively to B_i^*, C_i^{+*}, C_i^{-*}.

The model (M_i^*, K_i^*) clearly enjoys a group $\mathcal{G}_i$ of conformal symmetries which is a subgroup of the group $\mathcal{H}_i \subset \mathrm{O}(4)$ generated by the glide rotation $T_{\frac{1}{d}}$, the $\frac{\pi}{2}$–rotation $\xi^{\frac{1}{2}}$ around $S^3 \cap (\mathbb{C} \times 0)$, the π–rotation T and $R_{\frac{1}{2d}} T R_{\frac{1}{2d}}^{-1}$ around, respectively, $S^3 \cap \mathbb{R}^2$ and $R_{\frac{1}{2d}}(S^3 \cap \mathbb{R}^2)$ and, when $e = \frac{k}{2}$ (namely $a = 0$), the reflection ρ through the plane $S^3 \cap (\mathbb{C} \times \mathbb{R})$. More precisely, one gets generators for $\mathcal{G}_i$ by replacing, in the definition of $\mathcal{H}_i$ above, the integer d by a certain divisor $\frac{d}{\delta}$ of d determined as follows: δ is the least integer > 0 such that, for all n, the boundary component $S_{n+\delta}$ leads to an active block precisely if S_n does, where subscripts are read modulo d.

Using Theorems 16.4 and 16.8, one readily checks that every admissible automorphism of (M_i^*, K_i^*) is pairwise isotopic to a unique element of $\mathcal{G}_i$.

Thus, our preliminary model (M_i^*, K_i^*) almost satisfies the properties required. The only failure is that the markings induced by B_i^*, C_i^{+*} and C_i^{-*} on each (round sphere) component of ∂M_i^* do not describe a regular square or regular tetrahedral graph (up to conformal equivalence); thus our Gluing Condition is not satisfied. We consequently modify K_i^*, the band, and the markings, by an ambient $\mathcal{G}_i$–equivariant isotopy supported on a small neighbourhood of ∂M_i^*, so that, for each component S of ∂M_i^*, the square or tetrahedron drawn on S by B_i^*, C_i^{+*}, C_i^{-*} is regular for the standard metric of S^3. This $\mathcal{G}_i$–equivariant isotopy is easily constructed by consideration of the quotient $U/\mathcal{G}_i$, with U a small $\pm$–cyclic $\mathcal{G}_i$–invariant collar neighbourhood of ∂M_i^*. This change preserves the properties previously enjoyed by our model; it produces the final model (M_i^*, K_i^*).

Note that, when (M_i, K_i) is adjacent to no active block, this final conformal model is essentially the one we used in the proof of Theorem 18.5.

Thus, we have now defined for absolutely every block (M_i, K_i) a model (M_i^*, K_i^*), equipped with two bands B_i^{+*} and B_i^{-*} (possibly equal) and two boundary markings C_i^{+*} and C_i^{-*} (possibly equal). We can ensure inductively that whenever the pairs (M_i, K_i) and (M_j, K_j) (equipped with preferred bands and markings) are admissibly isomorphic, the corresponding models M_i^* and M_j^* are admissibly isomorphic by a *conformal* (degree ± 1) automorphism of S^3. Then, these models clearly satisfy the Naturality Condition and Gluing Condition. The models (M_i^*, K_i^*) consequently fit together to give (as for Theorem 18.5) the embedding of (M, K) asserted by Theorem 18.1.

This completes the proof of Theorem 18.1 in its full generality. □

We now indicate how Theorem 18.1 and its proof lead in practice to a calculation of $\mathrm{Sym}(M, K; C)$ as a group; this calculation was left incomplete in Chapter 16 because of unresolved group extension questions. Here the marked arborescent pair $(M, K; C)$ is as for Theorem 18.1 and we exclude the cases already studied adequately, namely $\Gamma = \overset{e}{\textcircled{r}}$ (where the knot is Seifert fibred) or $\Gamma = \overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\bullet}$ (where the knot is the Borromean rings).

By Fact 18.9, some vertex v or edge e of Γ is fixed by $\mathrm{Sym}(M, K; C)$.

(a) If a vertex v is fixed, the geometrised symmetry group $\mathcal{G} \subset \mathrm{O}(4)$ respects the corresponding block (M_v, K_v) of (M, K), where M_v has been embedded in S^3 by the proof of Theorem 18.1. Then $\mathcal{G}$ is a subgroup, readily determined using Chapter 16, of the isometry group $\mathcal{G}_v^*$ of the model (M_v^*, K_v^*). Note that, algebraically, $\mathcal{G}_v^*$ is described as an extension of a cyclic or dihedral group by $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and that the extension may or may not be split according to the parity of the weight of the central vertex v.

(b) If no vertex of Γ is fixed, let e be an edge that is fixed. Then the geometrised symmetry group $\mathcal{G}$ (from the proof of Theorem 18.1) respects the round 2–sphere S in $M \subset S^3$ corresponding to e. Further S is equatorial (geodesic) in S^3, the intersection $K \cap S$ is the four vertices of an equatorial square in S, and some element of $\mathcal{G}$ exchanges the two sides of S. Thus $\mathcal{G}$ is a subgroup of the order 32 isometry group $\mathcal{H}$ of S^3 respecting S and the "square" $K \cap S$. Better, $\mathcal{G}$ lies in the order 16 subgroup $\mathcal{H}'$ of $\mathcal{H}$ consisting of these elements that exchange the two complementary components of S in S^3 precisely if they exchange the two pairs of opposite edges of the equatorial square of S containing $K \cap S$ (indeed $\mathcal{G} \subset \mathcal{H}'$ because $\mathcal{G}$ respects the bands of the plumbing blocks adjacent to S in (M^3, K)). Again Chapter 16 lets us determine which subgroup of $\mathcal{H}'$ corresponds to the group $\mathcal{G}$.

For drawing pictures, the models that we have used in the proof of Theorem 18.1 may become unnecessarily cumbersome. In practice, it is often convenient to relax somewhat the Naturality Condition imposed on these models, restricting it to those admissible isomorphisms $M_i^*, K_i^*) \to (M_j^*, K_j^*)$ that are realised (via σ_i and σ_j) by degree ± 1 automorphisms of $(M, K; C)$ which respect the plumbing blocks.

This often enables us to use more primitive models, (as in the proof of Theorem 18.5 for example), enjoying less symmetries than the ones we have just given, but easier to draw (see examples below).

In particular, the use of spheres with regular tetrahedral graphs can sometimes be eliminated. For instance, assume that no degree -1 symmetry of Γ fixes an active vertex. Then, performing suitable moves (2.1) and (2.2) along one half of the active vertices gives a new graph Γ' whose symmetry group $\mathrm{Sym}(\Gamma')$, defined in the obvious way, is naturally isomorphic to $\mathrm{Sym}(\Gamma)$ and every degree ± 1 symmetry of Γ' sends a vertex of weight e to a vertex of weight $\pm e$. Plumbing atomic tangles according to Γ' again yields $(M, K; C)$, but the preferred bands and markings of the plumbing blocks are now characteristic. Then, the models for these blocks constructed in the proof of Theorem 18.5 (with just conformal squares on boundary spheres) suffice to realise $\mathrm{Sym}(M, K; C)$ as a finite subgroup of O(4).

We now give a few examples illustrating Theorem 18.1, taken from the Examples 16.19–16.22 that we had considered at the end of Chapter 16.

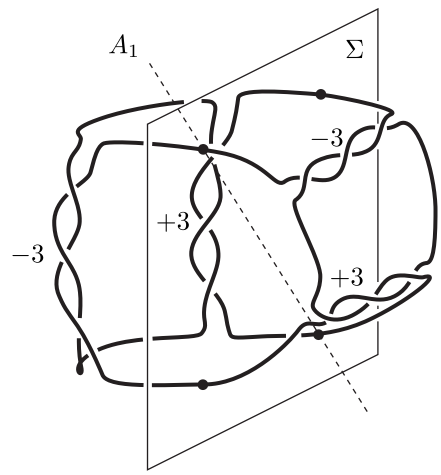
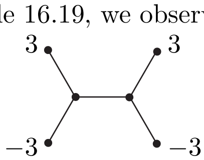


FIGURE 18.9.

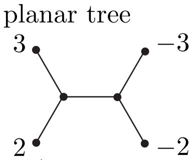
EXAMPLE 18.10. In Example 16.19, we observed that for the graph



$\mathrm{Sym}(S^3, K) \cong \mathrm{Sym}(\Gamma) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with half of the symmetries of degree -1.

Noting that the centre of Γ (Fact 18.9) is an edge corresponding to a Conway sphere Σ in (S^3, K), the symmetry group of (S^3, K) is then realised as in Figure 18.9 by the group generated the π–rotation around the axis $A_1 \subset \Sigma$ and by reflection in the two points of $K \cap \Sigma$ not in A_1. This group of conformal automorphisms lies in O(4) when Σ is an equatorial in S^3 and $K \cap \Sigma$ is equatorial square in Σ.

EXAMPLE 18.11. The knot 8_{17} of Example 16.20 is an example where our full machinery with tetrahedral spheres is unnecessarily complicated. Indeed, the knot is also described by the weighted planar tree



where the non-trivial (degree -1) is much more apparent. We can then realise $\mathrm{Sym}(S^3, K) \cong \mathbb{Z}_2$ by a configuration very similar to that of Figure 18.9. This is represented in Figure 18.10, where the non-trivial element of $\mathrm{Sym}(S^3, K)$ is the reflection in the two antipodal points P_1 and P_2 indicated.

EXAMPLE 18.12. Next, consider the graph

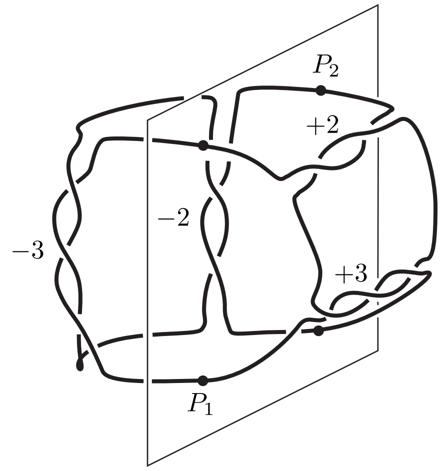
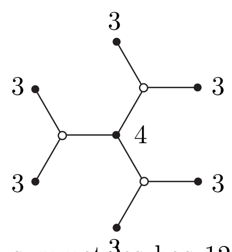


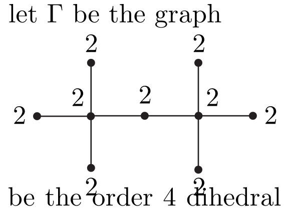
FIGURE 18.10.



The group SymΓ of graph symmetries has 12 elements, and is generated by the dihedral group D_3 of rigid motions of $\mathbb{R}^2$ respecting the embedding of Γ in the plane, and by the involution moving precisely the vertices of weight 3. Also no symmetry has degree −1 and $\mathcal{K}(\Gamma) = 0$.

Then, after isotopy in $S^3 \subset \mathbb{C}^2$, the knot classified by Γ is, as in Figure 18.11, respected by the π–rotations ξ: $(z, z') \mapsto (z, -z')$ and T: $(z, z') \mapsto (\overline{z}, \overline{z}')$ and by the glide-rotation R: $(z, z') \mapsto (z\omega^2, z'\omega)$ with $\omega = \exp(2i\frac{\pi}{6})$; and every symmetry of the knot is realised by a product of these isometries.

EXAMPLE 18.13. Lastly, let Γ be the graph



Sym(Γ) is easily seen to be the order 4 dihedral group D_2, while $\mathcal{K}(\Gamma) \cong \mathbb{Z}_2$. Moreover, note that the identity has both degree +1 and −1, and therefore that the corresponding knots admits 16 distinct symmetries. Thus when $\mathrm{Sym}(S^3, K)$ is geometrically realised by Theorem 18.1, the full group of 16 symmetries of the central active block will be induced.

The knot (S^3, K) associated to K is represented by Figure 18.12, where $\mathrm{Sym}(M, K; C)$ is realised by the subgroup of O(4) generated by the π–rotations around the three axes A_1, A_2 and A_3 and by the reflection through a plane (containing components of K) obtained by π–rotation of the plane of the paper, around the axis A_1 (compare Figure 18.11).

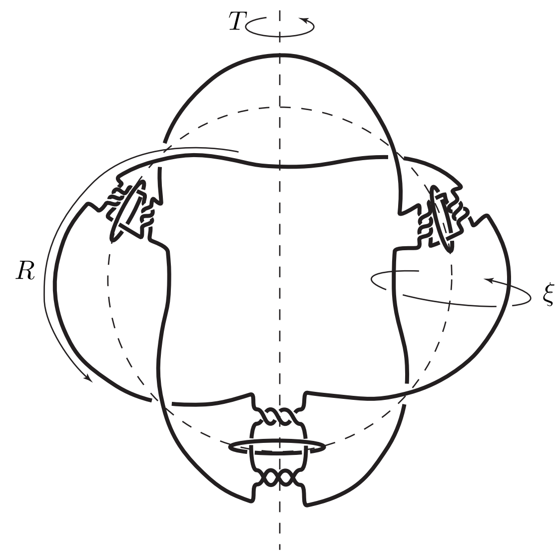


FIGURE 18.11.

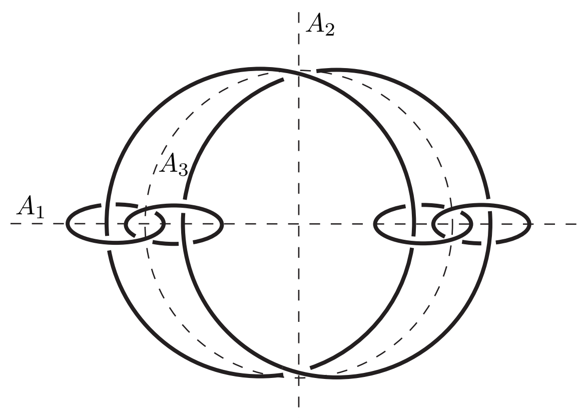


FIGURE 18.12.

APPENDIX A

Branched covers of arborescent pairs

A.1. Branched covers of knot pairs

Consider a knot pair. A ***branched cover*** of (M, K) consists of a 3–manifold N and of a map $p : N \to M$ such that:

(a) The restriction $p_{|p^{-1}(M-K)}\colon p^{-1}(M-K) \to M-K$ is a covering map.

(b) Near $p^{-1}(K)$, the map p is locally modelled by a map $q_n : \mathbb{C}\times\mathbb{R} \to \mathbb{C}\times\mathbb{R}$ defined by $q(z,t) = (z^n, t)$, for some integer $n \geqslant 1$. Namely, for every $x \in p^{-1}(K)$, there are charts $\widetilde{\varphi}\colon \widetilde{U} \to \mathbb{C}\times\mathbb{R}$ and $\varphi : U \to \mathbb{C}\times\mathbb{R}$, respectively identifying a neighbourhood $\widetilde{U}$ of x and a neighbourhood U of $p(x)$ with open subsets of $\mathbb{C}\times\mathbb{R}$, such that $q\widetilde{\varphi} = \varphi p$ and $\widetilde{\varphi}(x) \in 0\times\mathbb{R}$. (Observe that $\varphi^{-1}(0\times\mathbb{R}) = U\cap K$ whenever $n > 1$.)

Sometimes, a branched cover of (M, K) is also called a ***cover of*** M ***branched along*** K.

In this Appendix, we are mostly concerned with 2–***fold branched covers*** or ***double branched covers***, namely with those for which $p^{-1}(y)$ consists of 2 points for every $y \in M-K$ and of one point for every $y \in K$. For such a 2–fold branched cover, observe that necessarily $n = 2$ in Condition (b) above.

For a cover $p\colon N \to M$ branched along K, the restriction $p_{|p^{-1}(M-K)}\colon p^{-1}(M-K) \to M-K$ is a nontrivial covering map above each neighbourhood of a point of K. From the classification of coverings, it follows that, up to the obvious notion of isomorphism, the 2–fold branched covers of (M, K) are in one-to-one correspondence with the homomorphisms $H_1(M-K) \to \mathbb{Z}_2$ which are non-trivial on each meridian of K. It follows that (M, K) admits a unique 2–fold branched cover when $H_1(M;\mathbb{Z}_2) = 0$ and each component of ∂M meets K in an even number of points.

In the cases we are interested in, M is embedded in S^3 and bounded by spheres each meeting K in an even number of points (with possibly $M = S^3$), so that the above property always holds. We can therefore talk of *the* 2–fold branched cover of (M, K).

For an arborescent knot, it turns out that its 2–fold branched cover belongs to a special class of 3–manifolds, called graph manifolds. By definition, a graph manifold is a 3–manifold N containing a family T of disjoint 2–tori such that each closed-up component of $N-T$ admits a (locally trivial) S^1–bundle structure over a surface. These graph manifolds were studied and classified by F. Waldhausen in [**Wal**$_2$].

PROPOSITION A.1. *The* 2*–fold branched cover* N *of an arborescent pair is a graph manifold.*

PROOF. Recall from Chapters 3 and 7 that an arborescent pair is one which can be split along disjoint Conway spheres to get a family of rational tangle pairs (see Figure A.1(a)) and of hollow elementary pairs (see Figure A.1(b)). Thus, it suffices to prove that the 2–fold branched covers of a rational tangle pair and of a hollow elementary pair is an S^1–bundle.

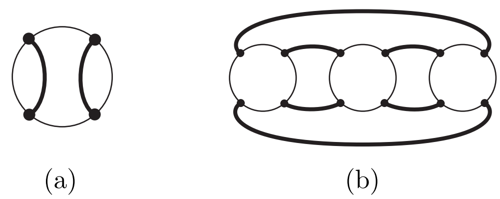


Figure A.1.

On the solid torus $V = S^1 \times B^2$, consider the involution τ defined by $\tau(u, v) = (\overline{u}, \overline{v})$, where S^1 and B^2 are viewed as the unit circle and disk in the complex plane $\mathbb{C}$. Then τ can be seen geometrically as the π–rotation shown on Figure A.2(a). By consideration of a fundamental domain for the action of τ, observe that the quotient $M = V/\tau$ is a 3–ball and that, if $K \subset M$ is the image of the fixed point set of τ, the pair (M, K) is the standard rational tangle pair of see Figure A.2(b). The quotient map $V \to M$ exhibits V as the 2–fold branched cover of the rational tangle pair (M, K).

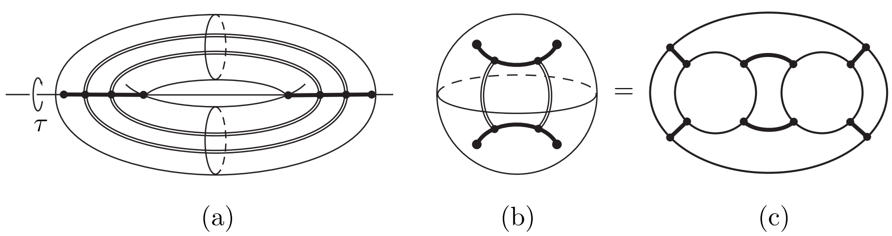


Figure A.2.

In $V = S^1 \times B^2$, consider two fibres f_1, $f_2 = S^1 \times *$ which are invariant under τ. If we remove from V two τ–invariant tubular neighbourhoods U_1, U_2 of f_1, f_2, observe that the knot pair made of the quotient space $V - \mathrm{int}(U_1 \cup U_2)/\tau$ and of the image of the fixed point set of τ is isomorphic to the hollow elementary pair of Figure A.1(b); see Figure A.2(b–c). This shows that the 2–fold branched cover of the hollow elementary pair is isomorphic to the S^1–bundle $V - \mathrm{int}(U_1 \cup U_2) \cong S^1 \times (B^2 - \text{two discs})$.

This concludes the proof of Proposition A.1. □

A.2. Seifert manifolds

A ***Seifert fibration*** of a compact 3–manifold M is a foliation of M by leaves that are circles, and which is locally of the following type: Near the boundary ∂M, the foliation is a locally trivial S^1–bundle; and any leaf in the interior of M has a foliated neighbourhood $U \cong S^1 \times B^2$ where the leaves of the foliation are, for two coprime integers p, q, the images of $z \in S^1 \mapsto (z^p, z_0 z^q)$ where z_0 ranges over B^2, considering S^1 and B^2 as the unit circle and disk in the complex plane $\mathbb{C}$. Viewing the solid torus $U \cong S^1 \times B^2$ standardly positioned in S^3, the fibres are $\{p,q\}$ torus knots wrapping around the core of U. Observe that a Seifert fibration is locally trivial outside of finitely many ***singular*** leaves where $p = \pm 1$. A leaf which is not singular is ***regular*** or ***generic***. The leaves of a Seifert fibration are usually called its ***fibres***. (In this context, the word *foliation* is an anachronism, and can be replaced by *partition* or *decomposition* which is what Seifert had to do in [**Sei$_1$**], before the introduction of foliations.)

Sometimes, the notion of Seifert fibration is extended so as to include a third local type for a decomposition of M into circles. A fibre of this third type admits a neighbourhood $U \cong S^1 \times B^2$ where the fibres of the fibration consist of the core $S^1 \times 0$ and of the images of the maps $z \in S^1 \mapsto (z_1, z_2 z)$, namely meridian circles linking the core. Observe that these fibres do not form a foliation near the core of U. We have encountered such fibres in Chapter 2, when extending the Seifert fibration of a knot complement to all of S^2. We will for this appendix call such a decomposition of M into circles a ***generalised Seifert fibration***. In a generalised Seifert fibration, the fibres that are not of foliation type (namely where the fibration is not a Seifert fibration in the original sense) are said to be ***infinitely singular***.

A ***Seifert manifold*** is a 3–manifold which admits a Seifert fibration. Similarly a ***generalised Seifert manifold*** is one which admits a generalised Seifert fibration.

In his seminal paper [**Sei$_1$**], H. Seifert classified Seifert fibrations of closed manifolds, up to fibred isomorphism, namely up to isomorphism of fibred manifolds that maps fibres to fibres. We now present his results, extended to generalised Seifert fibrations of manifolds with boundary, *but restricting to oriented manifolds and with a slightly different terminology.*

The idea is to extract certain invariants from a generalised Seifert fibration of an oriented 3–manifold M. A first invariant is the space B of the fibres of the fibration, also called its ***base***. In view of the local models of the fibration, this base B is a surface. Although we will not make use of this fact, we should mention that B actually has a richer structure, as a 2–dimensional orbifold structure(see [**Thu$_1$, BonS$_1$, BonS$_2$, Bon$_3$**]).

When M is connected, the surface B is connected and is characterised by its number d of boundary components and by its ***genus*** defined as follows (following [**Wal$_2$**]): $g = 1 - (\chi + d)/2 \geqslant 0$ when B is orientable and $g = \chi + d - 1 \leqslant 0$ otherwise, where χ is the Euler characteristic of B. This genus g puts isomorphism types of closed connected 2–manifolds into one-to-one correspondence with the integers $\mathbb{Z}$; for instance $g = 1$, 0, -1, -2 respectively for the torus, the sphere, the projective plane and the Klein bottle, respectively. (It should be emphasised that the orientability of M does not entail orientability of B.)

Another invariant is associated to each fibred tubular neighbourhood U of a fibre of M, *provided we choose a section Q of the fibration on ∂U*, namely Q is a circle meeting each fibre of ∂U in exactly one point. In $H_1(\partial U) \cong \mathbb{Z}^2$, complete the class of Q to a basis $\{Q, H\}$ where H is represented by a fibre of the fibration, oriented so that $Q \cdot H = +1$ for the boundary orientation on ∂U. Then, the kernel of $H_1(\partial U) \to H_1(U)$ is generated by $\alpha Q + \beta H$ for certain $\alpha, \beta \in \mathbb{Z}$. The number $\frac{\beta}{\alpha} \in \mathbb{Q} \cup \infty$ is the ***Seifert invariant of U with respect to the section Q***. Observe that $\frac{\beta}{\alpha} \in \mathbb{Z}$ when the fibration of U has no singular fibre, and that $\frac{\beta}{\alpha} = \infty$

if and only if U is not of foliation type (namely the fibration of U is not a Seifert fibration, but a generalised Seifert fibration).

Now consider a generalised Seifert fibration of a connected manifold M. Choose a union U of disjoint fibred tubular neighbourhoods of fibres of M, such that the induced fibration on the closure M_0 of $M - U$ is a locally trivial S^1–bundle. In particular, U must contain all singular fibres and we also require that $\partial M_0 \neq \varnothing$, so that this S^1–bundle admits a global section (since its base has the homotopy type of a bouquet of circles). Fix such a section Q. The data consisting of U and Q will be called a ***presentation*** of the generalised Seifert fibration.

Imitating the terminology and notation of Chapter 11, we define the ***raw data vector*** associated to M, U and Q to be $(g; m_1, m_2, \ldots, m_k)$ where these data are the following.

(1) The number $g \in \mathbb{Z}$ is the genus of the base B of the fibration.
(2) The invariants $m_1, \ldots, m_k \in \mathbb{Q} \cup \{\infty, \varnothing\}$ are respectively associated to the components $T_1, \ldots, T_k$ of ∂M_0; m_i is the symbol $\varnothing$ when T_k is also a boundary component of M, and otherwise $m_i \in \mathbb{Q} \cup \infty$ is the Seifert invariant of the component U_i of U that is adjacent to T_i, with respect to the restriction of the section Q to ∂U_i.

This raw data vector clearly depends on the choices of U and Q. To neutralise the effect of these choices, consider the ***fractional data vector*** $(g; e_0; \overline{m}_1, \ldots, \overline{m}_p)$ defined as follows.

(1) $g \in \mathbb{Z}$ is the genus of the base of the fibration.
(2) Each $\overline{m}_j$ is in $\mathbb{Q}/\mathbb{Z} \cup \{\infty, \varnothing\}$, and the collection $\{\overline{m}_1, \ldots, \overline{m}_p\}$ is obtained from the collection $\{m_1, \ldots, m_k\}$ of elements of $\mathbb{Q} \cup \{\infty, \varnothing\}$ by removing all $m_i \in \mathbb{Z}$ and by replacing each $m_j \in \mathbb{Q} - \mathbb{Z}$ by its reduction in $\mathbb{Q}/\mathbb{Z}$.
(3) The invariant e_0 is the symbol $\varnothing$ if some m_i is $\varnothing$ or ∞, and $e_0 = -\sum_{i=1}^{p} m_i$ when all m_i are in $\mathbb{Q}$.

To stay closer to Seifert's original approach, we can also consider the ***Seifert data vector*** $(g; b; \widetilde{m}_1, \ldots, \widetilde{m}_p)$ where:

(1) $g \in \mathbb{Z}$ is the genus of the base of the fibration.
(2) $\widetilde{m}_i = \overline{m}_i$ if $\overline{m}_i = \infty$ or $\varnothing$, and otherwise $\widetilde{m}_i$ is the unique number in $]0, 1[$ representing the class $\overline{m}_i \in \mathbb{Q}/\mathbb{Z}$.
(3) $b = \varnothing$ if $e_0 = \varnothing$, and $b = -e_0 - \sum_{i=1}^{p} \widetilde{m}_i \in \mathbb{Z}$ if $e_0 \in \mathbb{Q}$ (so that all $\widetilde{m}_i$ are in $\mathbb{Q}$).

Clearly, the fractional data vector is completely determined by the Seifert data vector, and conversely.

In his paper [**Sei**$_1$], Seifert used a slightly different form for this list of invariants. First of all, he considered only Seifert fibrations (without any singular fibre of Seifert invariant ∞) of *closed* manifolds. In this case, each $\widetilde{m}_i$ is in $\mathbb{Q} \cap (0, 1)$, and can be written as $\widetilde{m}_i = \frac{\beta_i}{\alpha_i}$ for a unique pair of coprime integers α_i, β_i with $0 < \beta_i < \alpha_i$. Then, Seifert's form of our Seifert data vector $(g; b; \widetilde{m}_i, \ldots, \widetilde{m}_i)$ is $(\mathrm{O}, \mathrm{o}; g; b; (\alpha_1 \beta_1), \ldots, (\alpha_p, \beta_p))$ if $g \geqslant 0$, and $(\mathrm{O}, \mathrm{n}; -g; b; (\alpha_1, \beta_1), \ldots, (\alpha_p, \beta_p))$ if $g < 0$. Here, the symbol O recalls that the 3–manifold is oriented (Seifert also considered non-orientable manifolds), and the symbols o and n indicate whether the base B of the Seifert fibration is orientable or non-orientable.

As in Chapter 11, we can also consider an alternate normalisation, by defining the ***normalised data vector*** $(g; e; \widehat{m}_1, \ldots, \widehat{m}_p)$. Here:

(1) $g \in \mathbb{Z}$ is the genus of the base of the fibration.
(2) $\widehat{m}_i = \overline{m}_i$ if $\overline{m}_i = \infty$ or $\varnothing$, and otherwise $\widehat{m}_i$ is the unique number in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right]$ representing the class $\overline{m}_i \in \mathbb{Q}/\mathbb{Z}$.
(3) $e = \varnothing$ if $e_0 = \varnothing$, and otherwise $e = e_0 + \sum \widehat{m}_i \in \mathbb{Z}$ (when all $\widehat{m}_i$ are in $\mathbb{Q} \cap \left(-\frac{1}{2}, \frac{1}{2}\right]$).

The fundamental result of Seifert in [**Sei**$_1$] is that these invariants completely determine the 3–manifold and its Seifert fibration. The extension to generalised Seifert fibrations of 3–manifolds with boundary is straightforward; see also [**Orl**]. Thus:

Theorem A.2. *Consider two connected oriented* 3*–manifolds M and M', each equipped with a generalised Seifert fibration. There exists a degree* $+1$ *isomorphism $\varphi\colon M \to M'$ sending fibration to fibration if and only if any one of the following three (obviously equivalent) conditions holds:*

(i) *The fractional data vectors $(g; e_0; \overline{m}_1, \ldots, \overline{m}_p)$ associated to M and M' coincide modulo permutation of the $\overline{m}_i$.*
(ii) *Their Seifert data vectors $(g; b; \widetilde{m}_1, \ldots, \widetilde{m}_p)$ coincide modulo permutation of the $\widetilde{m}_i$.*
(iii) *Their normalised data vectors $(g; e; \widehat{m}_1, \ldots, \widehat{m}_p)$ coincide modulo permutation of the $\widehat{m}_i$.* □

It turns out that the topological classification of Seifert manifolds "almost" coincides with the fibred classification of Theorem A.2. Indeed, this topological classification was completed by F. Waldhausen [**Wal**$_2$], P. Orlik, E. Vogt and H. Zieschang [**OVZ**]. Rephrased in the terminology of fractional data vectors, this classification reads:

Theorem A.3. *Two connected Seifert fibred oriented 3–manifolds are (degree* $+1$*) isomorphic if and only if, modulo permutation of the $\overline{m}_i \in \mathbb{Q}/\mathbb{Z} \cup \{\varnothing\}$, their fractional data vectors $(g; e_0; \overline{m}_1, \ldots, \overline{m}_p)$ differ only by a composition of the following exchanges:*

$(0; \varnothing; \overline{m}, \varnothing) \leftrightarrow (0; \varnothing;\ )$
$(-1; \varnothing; \varnothing) \leftrightarrow \left(0; \varnothing; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}, \varnothing\right)$
$(-2; 0;\ ) \leftrightarrow \left(0; 0; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}\right)$
$\left(-1; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}\right) \leftrightarrow \left(0; \tfrac{\alpha}{\beta}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}, -\overline{\tfrac{\alpha}{\beta}}\right)$
$(-1; e;\ ) \leftrightarrow \left(0; -\tfrac{1}{e}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{e}}\right)$ *if* $e \neq 0$
$(-1; 0;\ ) \leftrightarrow \left(0; 0; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}\right)$
$\left(0; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}\right) \leftrightarrow \left(0; -\tfrac{\beta}{\alpha'}; -\overline{\tfrac{\beta}{\alpha'}}\right)$ *when* $\alpha' = \alpha^{\pm 1} \bmod \beta$
$\left(0; -\tfrac{\beta_1}{\alpha_1} - \tfrac{\beta_2}{\alpha_2}; \overline{\tfrac{\beta_1}{\alpha_1}}, \overline{\tfrac{\beta_2}{\alpha_2}}\right) \leftrightarrow \left(0; -\tfrac{\beta}{\alpha}; \overline{\tfrac{\beta}{\alpha}}\right)$ *when* $\frac{\beta}{\alpha} = \frac{\alpha_1\beta_2 + \beta_1\alpha_2}{\alpha_1'\beta_2 + \beta_1'\alpha_2}$ *and when* α_1' *and* β_1' *are such that* $\alpha_1\beta_1' - \alpha_1'\beta_1 = 1$. □

Here, as in Chapter 11, $\overline{\frac{\beta}{\alpha}}$ denotes the class of $\frac{\beta}{\alpha}$ in $\mathbb{Q}/\mathbb{Z}$.

The non-fibred isomorphisms corresponding to the above exchanges are probably best described by the corresponding isomorphisms between Montesinos knots, which we encountered in Chapter 10 and §11.2. Indeed, we will show in §A.3 (see Theorem A.6) that the 2–fold branched cover of a Montesinos pair admits a Seifert fibration whose fractional data vector is obtained from the one of the pair by reversing the sign of the first entry. And any isomorphism between knot pairs certainly lifts to an isomorphism between their 2–fold branched covers.

We will not give in detail the topological classification of generalised Seifert manifold. However, we should probably mention that it is easily deducted from Theorem A.3. Indeed, if a generalised Seifert fibration of M is not a Seifert fibration, namely if it has a singular fibre of Seifert invariant ∞, consider a simple closed curve in the base B running through the point corresponding to this singular fibre. The preimage of this curve under the projection $M \to B$ is a 2–sphere, which decomposes M into a connected sum of two 3–manifolds or into a self-connected sum of one manifold; however, the one or two manifolds of this decomposition admit a generalised Seifert fibration with one fibre of type ∞. (Compare Figure 8.31.)

Starting from this observation, one easily decomposes M into a connected sum of lens spaces, including copies of $S^1 \times S^2$. Since on the other hand, a Seifert manifold is prime (see [**Wal**$_2$] for instance), the classification of generalised Seifert manifolds easily follows from Theorem A.3 and from the unique factorisation of 3–manifolds into prime summands that we have encountered in Chapter 2 [**Kne, Hak**$_2$**, Mil**].

A.3. Double branched covers of Montesinos pairs

The main achievement of this section is to show that the double branched cover of a Montesinos pair is a generalised Seifert manifold, and to identify the data vectors of this Seifert manifold from those of the Montesinos pair, as defined in Chapter 11.

First, consider the hollow Montesinos tangle (M, K) of Figure A.3. The same argument as in the proof of Proposition A.1 shows that the double branched cover of (M, K) is isomorphic to $S^1 \times (S^2 - n \text{ discs})$ where $n \geqslant 0$ is the number of boundary components of (M, K). Also, each fibre $S^1 \times *$ projects to an arc or a circle in (M, K), according to whether it is invariant under the covering involution or not.

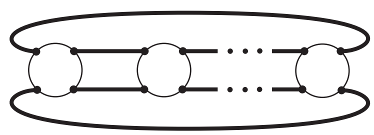


FIGURE A.3.

Observe that the images of the fibres have slope ∞ on the boundary of (M, K), whose boundary curves have slope 0, which lifts to a section of the S^1–bundle in the double branched cover.

We next consider the knot pair underlying the ring tangle.

LEMMA A.4. *The double branched cover of the ring pair is the (unique) S^1–bundle N over the Möbius strip that has orientable total space.*

PROOF. Consider the Möbius strip Σ as the quotient of the square $[0, 1] \times [0, 1]$ by the equivalence relation that glues each point $[0, y]$ of the left hand side to the point $[1, 1 - y]$ on the right hand side. Then, the bundle N is obtained from $[0, 1] \times [0, 1] \times S^1$ by gluing each point $(0, y, u)$ to $(1, 1 - y, \bar{u})$, considering S^1 as the unit circle in $\mathbb{C}$ and letting $\bar{u}$ denote the complex conjugate of $u \in S^1$.

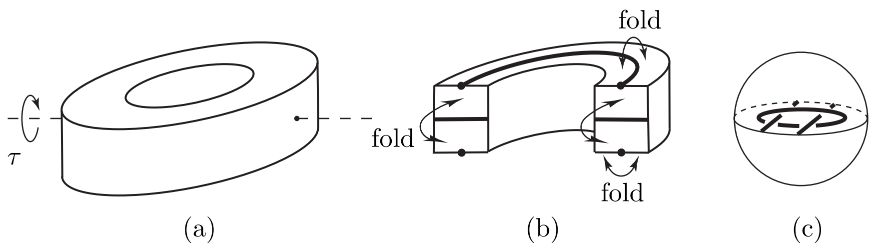


FIGURE A.4.

Consider the involution τ of N that comes from the involution τ' of $[0,1] \times [0,1] \times S^1$ defined by $\tau'(x,y,u) = (1-x, y, \bar{u})$. See Figure A.4(a).

The fixed point set of $\tau\colon N \to N$ consists of the images of all the fixed points $(\frac{1}{2}, y, \pm 1)$ of τ', but also of the images of the points $(0, \frac{1}{2}, u)$ and $(1, \frac{1}{2}, u)$.

To analyse the quotient space N/τ, consider the fundamental domain $Q = [0,1] \times [0,1] \times J$ where $J \subset S^1$ is the upper semi-circle consisting of all $u \in S^1$ with nonnegative imaginary part, as in Figure A.4(b). The quotient space N/τ is then obtained from Q by gluing each point $(0,y,u)$ to $(0,1-y,u)$, each $(1,y,u)$ to $(1,1-y,u)$, each $(x,y,1)$ to $(1-x,y,1)$, and each $(x,y,-1)$ to $(1-x,y,-1)$. See Figure A.4(b).

Folding faces of Q according to these gluing instructions, one readily sees that the quotient space N/τ is isomorphic to a ball, in such a way that the image of the fixed point set of τ is as indicated in Figure A.4(c). As a consequence, the quotient pair $(N, \mathrm{Fix}\,(\tau))/\tau$ is isomorphic to the ring pair (M,K).

In other words, N is the double branched cover of the ring pair (M,K). □

Observe the following consequence of our proof of Lemma A.4: On the boundary of the double branched cover N of the ring tangle (M,K), the fibres project to arcs and curves of slope ∞. Also, the S^1–bundle N admits a section Q, image of the square $[0,1] \times [0,1] \times *$, such that ∂Q projects to an arc of slope 0 on $\partial(M,K)$.

Combining our analysis of the case of the hollow Montesinos pair and of the ring tangle, we get:

LEMMA A.5. *Let (M,K) be a presented Montesinos pair with r ring tangles, d boundary components, and no rational tangles. Then, the double branched cover N of (M,K) admits an S^1–bundle structure, with base the d times punctured surface of genus $-r$. Moreover, the fibres project to arcs and curves of slope ∞ on the boundary $\partial(M,K)$, and the boundary of a certain section Q of the bundle projects to curves of slope* 0.

PROOF. The pair (M,K) can be decomposed into a hollow Montesinos pair (M_0,K_0) and ring tangles $(M_1,K_1), \dots, (M_r,K_r)$. We already saw that the preimage $N_i \subset N$ of $M_i \subset M$ admits an S^1–bundle structure. By our observation on the slopes, these fibrations can be modified so as to fit along each ∂N_i, $i \geqslant 1$, to give an S^1–bundle on N. The base of this fibration is obtained by capping off with a Möbius strip r boundary components of the base of N_0, which is a $(d+r)$ times punctured sphere. Therefore, the base of the fibration of N is a d times punctured surface of genus $-r$. Similarly, the sections considered on the N_i can be modified to give a section Q of N as desired. □

THEOREM A.6. *Let (M,K) be a presented Montesinos pair, with raw data vector $(r; m_1, \dots, m_k)$ as defined in Chapter 11. Then, the double branched cover N of (M,K) admits a generalised Seifert fibration which has raw data vector $(-r; m_1, \dots, m_k)$, for some presentation of this fibration.*

PROOF. Let $(M_1, K_1), \dots, (M_n, K_n)$ be the rational tangles of the presentation of (M,K), and let (M_0, K_0) be the closure of their complement. Let $N_i \subset N$ denote the preimage of $M_i \subset M$.

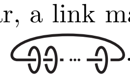
By Lemma A.5, N_0 admits an S^1–bundle structure with base a punctured surface of genus $-r$. For each $i \geqslant 1$, N_i is a solid torus. It is then easy to extend the fibration of ∂N_i induced by the fibration of N_0 to a generalised Seifert fibration of N_i. (Hint: On the boundary of the solid torus $S^1 \times B^2$, any S^1–bundle structure is isotopic to a "linear" one, and therefore extends to a generalised Seifert fibration.) Thus, the fibration of N_0 extends to a generalised Seifert fibration of N.

Consider the presentation of this fibration of N which consists of the union U of the N_i with $i \geqslant 1$ and of the section Q of N_0 provided by Lemma A.5. In view of the definition of the slope of rational tangles in §1.2 and of the Seifert invariants in §A.2, it is immediate that the Seifert invariant of each N_i, $i \geqslant 1$, with respect to Q is equal to the slope of the rational tangle (M_i, K_i). It follows that the Seifert fibration of N has raw data vector $(-r; m_1, \dots, m_k)$ for the presentation defined by U and Q. □

If we apply a non-dihedral permutation to the rational tangle slopes of the fractional data vector of a Montesinos pair (M,K), the results of Chapter 11 show we usually obtains a Montesinos pair (M', K') which is not isomorphic to (M,K). However, Theorems A.3 and A.6 show that (M,K) and $M', K')$ have isomorphic double branched covers. This provides many examples of different knots that have isomorphic double branched cover.

A.4. Montesinos knots that are Seifert fibred

In Chapter 16, we needed to know for which pairwise irreducible Montesinos knots (S^3, K) it is possible to put a Seifert fibration on the complement $S^3 - U(K)$ of a tubular neighbourhood $U(K)$ of K. Observe that such a fibration of $S^3 - U(K)$ extends to a generalised Seifert fibration of S^3 for which K consists of fibres. Thus, the question is equivalent to asking which Montesinos knots consist of fibres of a generalised Seifert fibration of S^3.

First of all, an easy argument (based for instance on a computation of the first homology group) shows that there is only one generalised Seifert fibration of S^3 with a singular fibre of infinite slope. The fractional data vector of this fibration is $(0, \varnothing; \varnothing)$. In particular, a link made up of fibres of this fibration is either a trivial link or a keyring link . If, in addition, the link is pairwise irreducible, it must be either the trivial knot or the Hopf link  ; observe that these are Montesinos knots.

Having solved the problem for generalised Seifert fibrations of S^3 with a singular fibre of infinite slope, we can now restrict attention to Seifert fibrations (with all slopes finite).

The main idea for our analysis is the observation that, if K consists of fibres of a Seifert fibration of S^3, this fibration lifts to a Seifert fibration of the double branched cover N of (S^3, K). A local analysis easily proves this property. Each fibre of S^3 lifts to 1 or 2 fibres in N. It easily follows that the base of the fibration of N either is equal to the base B of the fibration of S^3 (when a generic fibre of S^3 lifts to a single fibre in N), or is a 2–fold branched cover of B, branched at points of B corresponding to some singular fibres or to components of K. See the proof of Theorem A.8 below for more details.

LEMMA A.7. *Let K be a union of fibres of a Seifert fibration of S^3. If (S^3, K) is a Montesinos knot, then the Seifert fibration of S^3 lifts to a Seifert fibration of the double branched cover N of (S^3, K) whose fractional data vector is of type $\left(0; e_0; \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_2}{\alpha_2}}, \dots, \overline{\frac{\beta_n}{\alpha_n}}\right)$, where $n \leqslant 3$ and where, in addition, $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$ if $n = 3$.*

PROOF. By the classification Theorem A.3, the base B of the Seifert fibration of S^3 is necessarily a 2–sphere.

The Seifert fibration of N lifts to a Seifert fibration of N whose base either is the base B of S^3 or is a double branched cover of B. In particular, the base of the fibration of N is orientable. Therefore, the fractional data vector of this fibration is of type $\left(g; e_0; \overline{\frac{\beta_1}{\alpha_1}}, \overline{\frac{\beta_2}{\alpha_2}}, \dots, \overline{\frac{\beta_n}{\alpha_n}}\right)$ with $g \geqslant 0$.

First consider the case where $g = 0$. Assume, in search for a contradiction, that the list of the α_i is not one of those occurring in the statement of the lemma. Under these conditions, it is proved in [**OVZ, Orl**] that the centre of the fundamental group of N is infinite cyclic, generated by the image of a generic fibre.

The covering involution τ of N sends fibre to fibre. Also, if we coherently orient these fibres, it respects this orientation. It therefore follows that τ induces the identity on the centre of $\pi_1(N)$.

On the other hand, if (S^3, K) is a Montesinos knot, we constructed in §A.3 another Seifert fibration of N, for which τ reverses the orientation of a generic fibre. By [**OVZ, Orl**], the square of this generic fibre is a non-trivial element of the centre of $\pi_1(N)$ (we need to take the square in case the base of the fibration is non-orientable). We then have an element of the centre of $\pi_1(N)$ which is not fixed by τ, contradicting our first conclusion.

This proves that, if $g = 0$, the Seifert invariants must be of the type given in the statement of the lemma.

To conclude the proof, we need to exclude the case where $g \geqslant 1$. This easily follows from the classification of Seifert manifolds. Indeed, if $g \geqslant 1$, Theorem A.3 shows that N cannot admit a Seifert fibration whose base has non-positive genus. On the other hand, if (S^3, K) is a Montesinos knot, §A.3 proves that N does have such a fibration, giving the contradiction required. Therefore, $g = 0$.

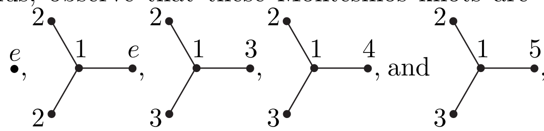
This concludes the proof of Lemma A.7. □

THEOREM A.8. *The pairwise irreducible Montesinos knot (S^3, K) is Seifert fibred if and only if its fractional data vector, as defined in Chapter 11, is one of*

$$(0; e;\ ) \text{ with } e \neq 0$$
$$(1; e;\ ) \cong \left(0; -\tfrac{1}{e}; \overline{\tfrac{1}{e}}, \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}}\right) \text{ with } e \neq 0$$
$$\pm\left(0; -\tfrac{1}{6}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{3}}\right)$$
$$\pm\left(0; -\tfrac{1}{12}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{4}}\right)$$
$$\pm\left(0; -\tfrac{1}{30}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{5}}\right).$$

The corresponding Seifert knots are, in this order, the torus knot (or link) $\{2, e\}$, the union of the torus knot $\{2, e\}$ and of the axis linking this torus knot twice, the *torus knot* $\{3,4\}$*, the union of the torus knot* $\{2,3\}$ *and of the axis linking this torus knot 3 times, and the torus knot* $\{3,5\}$*.*

In terms of our plumbing calculus, observe that these Montesinos knots are those associated to the weighted trees



modulo reversal of the signs of all weights.

Proof. Assume that K consists of fibres of a Seifert fibration of S^3. By Theorem A.3, the fractional data vector of this fibration is $(0;\pm1;\ )$, or $\left(0;-\frac{1}{\alpha};\overline{\frac{1}{\alpha}}\right)$, or $\left(0;-\frac{\beta_1}{\alpha_1}-\frac{\beta_2}{\alpha_2};\overline{\frac{\beta_1}{\alpha_1}},\overline{\frac{\beta_2}{\alpha_2}}\right)$ with $\alpha_1\beta_2+\alpha_2\beta_1=\pm1$.

Consider the double branched cover N of (S^3,K). The fibration of S^3 lifts to a Seifert fibration of N. To analyse this fibration, whose invariants are easily computed (following [**Mon**$_1$]). We have to consider two cases.

Case 1 *The preimage in* N *of a generic fibre of* S^3 *is disconnected*

In other words, the restriction of the branched cover above a generic fibre of S^3 is a trivial 2–fold cover.

Let us analyse what the local behavior of this branched cover can be near a fibre f of S. Choose a fibred tubular neighbourhood U of f, as well as a section Q of the fibration on ∂U. In the case we are considering, the preimage $\widetilde{Q}$ of Q gives a section of the fibration on the boundary of the preimage $\widetilde{U}$ of U in N. An easy computation shows that, if $\frac{\beta}{\alpha}\in\mathbb{Q}$ is the Seifert invariant of U with respect to Q, then either $\widetilde{U}$ consists of two solid tori each of Seifert invariant $\frac{\beta}{\alpha}$ with respect to $\widetilde{Q}$, or $\widetilde{U}$ is a single solid torus of Seifert invariant $\frac{2\beta}{\alpha}$ with respect to $\widetilde{Q}$. In the second case, the projection from the (disc) base of $\widetilde{U}$ to the (disc) base of U is a 2–fold cover branched at the image of f, and the covering map $\widetilde{U}\to U$ is branched along f if and only if α is odd.

In particular, the base $\widetilde{B}$ of the fibration of N is a 2–fold branched cover of the base B of S^3. Because (S^3,K) is a Montesinos knot, Lemma A.7 implies that $\widetilde{B}$ is a sphere. It follows that the covering map $\widetilde{B}\to B$ is branched at exactly 2 points, and therefore that K consists of only 1 or 2 fibres of S^3.

If the Seifert fibration of S^3 has no singular fibre, namely is the Hopf fibration up to degree ±1 isomorphism, then K consists of 2 fibres of this fibration. (If K was connected, a generic fibre would have linking number ±1 with it and would have connected preimage in N.) Therefore, K is the Hopf link, which is a Montesinos knot of fractional data vector $(0;\pm2;\ )$.

If the Seifert fibration of S^3 has exactly one singular fibre, its fractional data vector is $\left(0;-\frac{1}{\alpha};\overline{\frac{1}{\alpha}}\right)$. Since K consists of one or two fibres of this fibration, (S^3,K) is one of the trivial knot, the $\{2,2\alpha\}$ torus link, or the $\{2,4\alpha\}$ torus link. These are Montesinos knots, with fractional data vectors $(0;1;\ )$, $(0;2\alpha;\ )$, $(0;4\alpha;\ )$, and certainly are on the list of Theorem A.8.

Having analysed these two easy special cases, we now turn to the more general case, where the fibration of S^3 has two singular fibres. Choose two fibred tubular neighbourhoods U_1 and U_2 of these fibres, and a section Q of the fibration on

$S^3 - U_1 \cup U_2$. This gives a presentation of the Seifert fibration of S^3, whose raw data vector is $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}\big)$ with $\alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$ and $\alpha_1, \alpha_2 \geqslant 2$.

In N, the preimage $\widetilde{Q}$ of Q is a section of the fibration on the preimage of $S^3 - U_1 \cup U_2$. (This requires a little verification when a component of K is in $S^3 - U_1 \cup U_2$.) By our local analysis of the branched cover, the corresponding presentation of the Seifert fibration of N has raw data vector $\big(0; \frac{2\beta_1}{\alpha_1}, \frac{2\beta_2}{\alpha_2}\big)$, $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_1}{\alpha_1}, \frac{2\beta_2}{\alpha_2}\big)$ or $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_2}{\alpha_2}\big)$.

If the fibration of N has raw data vector $\big(0; \frac{2\beta_1}{\alpha_1}, \frac{2\beta_2}{\alpha_2}\big)$, then K consists of one or two of the singular fibres of S^3, and therefore is the unknot or the Hopf link.

The case where the fibration of N has raw data vector $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_2}{\alpha_2}\big)$ is excluded by Lemma A.7, since (S^3, K) is also a Montesinos knot.

When the fibration has raw data vector $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_1}{\alpha_1}, \frac{2\beta_2}{\alpha_2}\big)$ with $\alpha_2 = 2$, then K consists of a non-singular fibre of the fibration of S^3. This non singular fibre is a $\{2, \alpha_1\}$ torus knot in S^3, and therefore is the Montesinos knot of fractional data vector $(0; \alpha_1;\ )$.

The last case is when the fibration of N has raw data vector $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_1}{\alpha_1}, \frac{2\beta_2}{\alpha_2}\big)$ with $\alpha_2 \neq 2$. Combining Lemma A.7 with the condition that $\alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$, a straightforward computation shows that the fractional data vector of the fibration of N must be one of $\big(0; -\frac{1}{\alpha_2}; \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{\alpha_2}}\big)$ with α_2 odd, or $\big(0; -\frac{1}{6}; \overline{\frac{1}{2}}, \overline{\frac{1}{3}}, \overline{\frac{1}{3}}\big)$ modulo reversal of all signs. Combining Theorem A.6 with the topological classification of Seifert manifolds given in Theorem A.3, it follows that, as a Montesinos knot, (S^3, K) has fractional data vector $\pm\big(0; -\frac{1}{\alpha_2}; \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{\alpha_2}}\big)$ with α_2 odd, or $\pm\big(0; -\frac{1}{6}; \overline{\frac{1}{2}}, \overline{\frac{1}{3}}, \overline{\frac{1}{3}}\big)$.

CASE 2 *The preimage in N of each fibre of S^3 is connected*

In this case, the fibrations of N and S^3 have the same base B.

To analyse the fibration locally, consider a fibred tubular neighbourhood U of a fibre f of S^3, and its preimage $\widetilde{U}$ in N. In this case, not every section on ∂U lifts to a section on $\widetilde{\partial}U$. However, considering the homomorphism $H_1(\partial U) \to \mathbb{Z}/2$ classifying the covering map $\partial\widetilde{U} \to \partial U$, it is always possible to find a section Q on ∂U whose preimage on $\partial\widetilde{U}$ is disconnected; any component $\widetilde{Q}$ of the preimage of Q then is a section on $\partial\widetilde{U}$. If U has Seifert invariant $\frac{\beta}{\alpha}$ with respect to Q, an easy computation then shows that $\widetilde{U}$ has Seifert invariant $\frac{\beta}{2\alpha}$ with respect to $\widetilde{Q}$. Moreover, the covering map is branched along the fibre f if and only if β is odd.

Let us start with the case where the Seifert fibration of S^3 has two singular fibres. Let U_1, U_2 be fibred tubular neighbourhoods of these singular fibres, and let $U_3, \ldots, U_n$ be fibred tubular neighbourhoods of the components of K which are distinct from these fibres (if any). For $i = 1, \ldots, n-1$, choose on ∂U_i a section Q_i whose preimage in N is disconnected. Standard obstruction theory enable us to extend the Q_i to a section Q of the closure of $S^3 - \bigcup_{i=1}^n U_i$. Because the fundamental group of the planar surface Q is generated by $Q_1, \ldots, Q_{n-1}$, the preimage of Q is disconnected, and a component $\widetilde{Q}$ of this preimage gives a section of the fibration of the preimage of $S^3 - \bigcup_{i=1}^n U_i$. The section Q gives a presentation of the fibration of S^3 whose raw data vector is $\big(0; \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \beta_3, \beta_4, \ldots, \beta_n\big)$.

By our local analysis, the section $\widetilde{Q}$ gives a presentation of N whose raw data vector is $\big(0; \frac{\beta_1}{2\alpha_1}, \frac{\beta_2}{2\alpha_2}, \frac{\beta_3}{2}, \frac{\beta_4}{2}, \ldots, \frac{\beta_n}{2}\big)$.

If $n = 2$, K must be the unknot on the Hopf link.

If $n \geqslant 3$, Lemma A.7 puts some restrictions on what the fractional data vector of N can be. Combining them with the above formulas, a straightforward computation shows that this fractional data vector is, up to reversal of all signs, one of $\bigl(0; -\frac{1}{12}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{4}}\bigr)$ or $\bigl(0; -\frac{1}{30}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{5}}\bigr)$.

By Theorems A.6 and A.3, it follows that the Montesinos knot (S^3, K) must have fractional data vectors $\bigl(0; -\frac{1}{12}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{4}}\bigr)$ or $\bigl(0; -\frac{1}{30}; \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{3}}, \overline{\tfrac{1}{5}}\bigr)$.

It remains to consider the case where the fibration of S^3 has 0 or 1 singular fibres. The same arguments then shows that the Montesinos knot (S^3, K) must be the unknot, the unlink, or have fractional data vector $(0; -\frac{1}{2\alpha}; \overline{\tfrac{1}{2\alpha}}, \overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}})$.

This concludes our proof that, if the Montesinos knot (S^3, K) is Seifert fibred, its fractional data vector must be one of those listed in Theorem A.8.

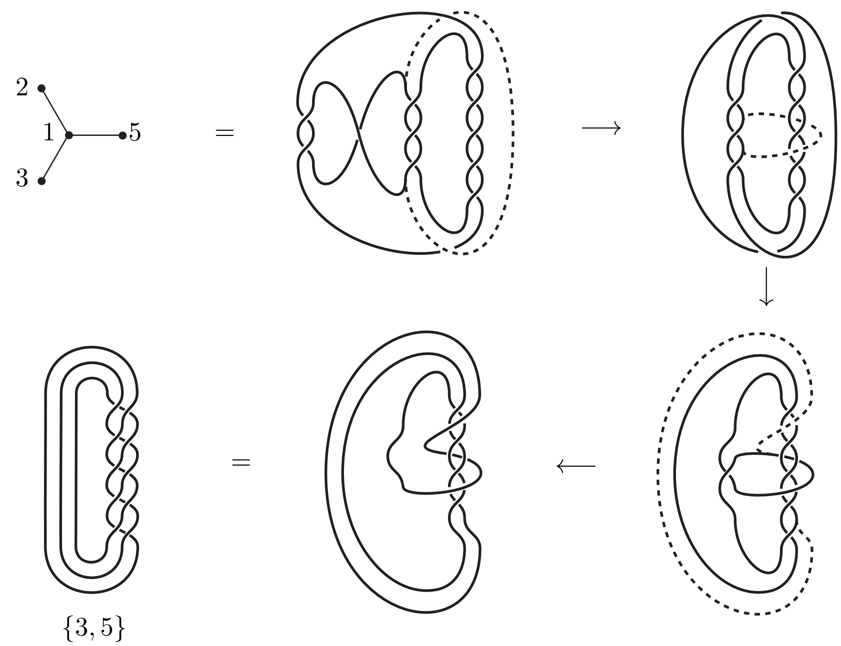


Figure A.5.

Conversely, we have to show that, if the fractional data vector of (S^3, K) is one of those listed, then (S^3, K) is Seifert fibred. This is (relatively) easily done by a string manipulation identifying this knot to the corresponding Seifert fibred knot given in the statement of Theorem A.8. For instance, Figure A.5 gives an example, corresponding to the last exception.

This concludes the proof of Theorem A.8. □

References

[Ago] I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. 4 (2000), 431–449.

[Ale] J.W. Alexander, *On the subdivision of* 3*–space by a polyhedron*, Proc. Nat. Acad. Sci. U.S.A. 10 (1924), 6–8.

[AleB] J. W. Alexander, G. B. Briggs, *On types of knotted curves*, Annals of Math. 28 (1926), 562–586.

[And_1] E.M. Andreev, *Convex polyhedra in Lobačevskiĭ spaces*, Mat. Sb. 81 (1970), 445–478.

[And_2] E.M. Andreev, *Convex polyhedra of finite volume in Lobačevskiĭ space*, Mat. Sb. 83 (1970), 256–260.

[Ban] C. Bankwitz, *Über die Torsionszahlen der alternierenden Knoten*, Math. Ann. 103 (1930), 145–161.

[BanS] C. Bankwitz, H.G. Schumann, *Über Viergeflechte*, Abh. Math. Sem. Univ. Hamburg 10 (1934), 263–284.

[BenP] R. Benedetti, C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.

[Boi_1] M. Boileau, *Inversibilité des nœuds de Montesinos*, Dissertation, Publications Mathématiques d'Orsay 81-3, Université de Paris-Sud, 1981.

[Boi_2] M. Boileau, *Groupe des symétries des noeuds de bretzel et de Montesinos*, Publications Mathématiques d'Orsay, Université de Paris-Sud, 1982.

[BoiLP] M. Boileau, B. Leeb, J. Porti, *Geometrization of* 3*–dimensional orbifolds*, Ann. of Math. 162 (2005), 195–290.

[BoiMP] M. Boileau, S. Maillot, J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses 15, Société Mathématique de France, 2003.

[BoiP] M. Boileau, J. Porti, *Geometrization of* 3*–orbifolds of cyclic type*, Astérisque 272, Société Mathématique de France, 2001.

[BoiS] M. Boileau, L.C. Siebenmann, *A planar classification of pretzel knots and Montesinos knots*, Publications Mathématiques d'Orsay, Université de Paris-Sud, 1980.

[BoiZ] M. Boileau, B. Zimmermann, *Symmetries of nonelliptic Montesinos links*, Math. Ann. 277 (1987), 563–584.

[Bon_1] F. Bonahon, *Involutions et fibrés de Seifert dans les variétés de dimension* 3, Thèse de 3ème cycle, University of Paris XI, Orsay, 1979.

[Bon_2] F. Bonahon, *Cobordism of automorphisms of surfaces*, Ann. Sci. École Norm. Sup. 16 (1983), 237–270.

[Bon_3] F. Bonahon, *Geometric structures on* 3*-manifolds*, in: *Handbook of geometric topology* (R.J. Daverman, R.B. Sher eds.), 93–164, North-Holland, 2002.

[Bon_4] F. Bonahon, *Low-dimensional geometry: from euclidean surfaces to hyperbolic knots*, Student Math. Library 49, American Mathematical Society, 2009.

[$BonS_1$] F. Bonahon, L.C. Siebenmann, *The characteristic toric splitting of irreducible compact 3-orbifolds*, Math. Ann. 278 (1987), 441–479.

[$BonS_2$] F. Bonahon, L.C. Siebenmann, *The classification of Seifert fibred 3-orbifolds*, in: *Low-dimensional topology (Chelwood Gate, 1982)*, 19–85, London Math. Soc. Lecture Note 95, Cambridge University Press, 1985.

[$BonS_3$] F. Bonahon, L.C. Siebenmann, *Seifert 3–orbifolds and their role as natural crystalline parts of arbitrary compact irreducible 3–orbifolds*, Publications Mathématiques d'Orsay, Université de Paris-Sud, 1983.

[Bra] K. Brauner, *Zur Geometrie der Functionen zweier Veränderlichen: II-IV*, Abh. Math. Sem. Hamburg 6 (1928), 1–54.

[BroT] R. Brown, T.L. Thickstun, *Low-dimensional topology (Bangor, 1979)*, London Math. Soc. Lecture Note Series 48, Cambridge University Press, 1982.

[Bud_1] R. Budney, *JSJ-decompositions of knot and link complements in the 3–sphere*, Enseign. Math. 52 (2006), 319–359.

[Bud_2] R. Budney, *Little cubes and long knots*, Topology 46 (2007), 1–27.

[Bur_1] W. Burau, *Kennzeichnung des Schlauchknoten*, Abh. Math. Sem. Hamburg 9 (1932) 125–133.

[Bur_2] W. Burau, *Kennzeichnung des Schlauchverkettungen*, Abh. Math. Sem. Hamburg 10 (1934) 285–297.

[BurZ] G. Burde, H. Zieschang, *Knots*, de Gruyter Studies in Math. 5, Walter de Gruyter & Co., Berlin, 1985.

[$Burk_1$] J.J. Burkhardt, *Bemerkungen zur arithmetischen Berechnung der Bewegungsgruppen*, Commentarii Math. Helvetici 2 (1930), 91–98.

[$Burk_2$] J.J. Burkhardt, *Die Bewegungsgruppen der Kristallographie*, Birkhäuser, 1946.

[BusK] P. Buser, H. Karcher, *Gromov's almost flat manifolds*, Astérisque 81, Société Mathématique de France, Paris, 1981.

[Car] M.P. do Carmo, *Riemannian geometry* (Transl. Francis Flaherty), Mathematics: Theory & Applications, Birkhäuser Boston Inc., 1992.

[CaoZ] H.-D. Cao, X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. 10 (2006), 165–492; *Erratum*, Asian J. Math. 10 (2006), 663.

[Cau] A. Caudron, *Classification des nœuds et des enlacements*, Publ. Math. Orsay 82-4, University of Paris XI, Orsay, 1982.

[Cer] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Math. 53, Springer-Verlag, 1968.

[CheE] J. Cheeger, D.G. Ebin, *Comparison theorems in Riemannian geometry* North-Holland Mathematical Library 9, North-Holland Publishing Co., 1975.

[ChoK] B. Chow, D. Knopf, *The Ricci flow: an introduction*, Math. Surveys and Monographs 110, American Math. Soc., 2005.

[ChoLN] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics 77, American Mathematical Society; Science Press, 2006.

[ChoEtAl] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *The Ricci flow: techniques and applications, Part I. Geometric aspects*, Math. Surveys and Monographs 135, American Math. Soc., 2007; *Part II. Analytic aspects*, Math. Surveys and Monographs 144, American Math. Soc., 2008.

[ConM] P.E. Conner, D. Montgomery, *Transformation groups on a* $K(\pi, 1)$, I, Michigan Math. J. 6 (1959), 405–412.

[ConR] P.E. Conner, F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, in: *Topology of manifolds. Proceedings of the University of Georgia Topology of Manifolds Institute, 1969*, (J. C. Cantrell and C. H. Edwards eds.), 227–264, Markham Publishing Co., 1970.

[Conw] J.H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, in: 1970 *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, 329–358, Pergamon Press, 1970.

[CooHK] D. Cooper, C.D. Hodgson, S.P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs 5, Mathematical Society of Japan, 2000.

[CuGLS] M. Culler, C. McA. Gordon, J. Luecke, P.B. Shalen, *Dehn surgery on knots*, Ann. of Math. 125 (1987), 237–300.

[Dre] A. Dress, *Newman's theorems on transformation groups*, Topology 8 (1969), 203–207.

[Dun_1] W.D. Dunbar, *Fibered orbifolds and crystallographic groups*, doctoral dissertation, Princeton University, 1981.

[Dun_2] W.D. Dunbar, *Geometric orbifolds*, Rev. Mat. Univ. Complut. Madrid 1 (1988), 67–99.

[EisN] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. Math. Studies 110, Princeton Univ. Press, 1985.

[Feu_1] C.D. Feustel, *On the torus theorem and its applications*, Trans. Amer. Math. Soc. 217 (1976), 1–43.

[Feu_2] C.D. Feustel, *On the torus theorem for closed* 3*-manifolds*, Trans. Amer. Math. Soc. 217 (1976), 45–57.

[FloO] W. Floyd, U. Oertel, *Incompressible surfaces via branched surfaces*, Topology 23 (1984), 117–125.

[Gab_1] D. Gabai, *The Murasugi sum is a natural geometric operation I*, in *Low-dimensional topology (San Francisco, Calif., 1981)*, 131–143, Contemp. Math. 20, Amer. Math. Soc., 1983.

[Gab_2] D. Gabai, *The Murasugi sum is a natural geometric operation II*, in *Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982)*, 93–100, Contemp. Math. 44, Amer. Math. Soc., 1985.

[Gif] C.H. Giffen, *On transformations of the* 3*-sphere fixing a knot*, Bull. Amer. Math. Soc. 73 (1967), 913–914.

[Gor] C.McA. Gordon, *Dehn surgery and* 3*–manifolds*, in: *Low dimensional topology*, 21–71, IAS/Park City Math. Ser., 15, Amer. Math. Soc., 2009.

[Gro] M. Gromov, *Structures métriques pour les variétés riemanniennes*, (J. Lafontaine, P. Pansu eds.), Textes Mathématiques 1, CEDIC, Paris, 1981.

[Hak_1] W. Haken, *Theorie der Normalflächen*, Acta Math. 105 (1961), 245–375.

[Hak_2] W. Haken, *Ein Verfahren zur Aufspaltung einer* 3*–Mannigfaltigkeit in irreduzible* 3*–Mannigfaltigkeiten*, Math. Z. 76 (1961), 427–467.

[Hak_3] W. Haken, *Über das Homöomorphieproblem der* 3*–Mannigfaltigkeiten. I*, Math. Z. 80 (1962), 89–120.

[HarW] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, 1938.

[Has_1] M.G. Haseman, *On knots, with a census of the amphicheirals with twelve crossings*, Trans. Roy. Soc. Edinburgh 52 (1917-8), 235–255.

[Has_2] M.G. Haseman, *Amphicheiral knots*, Trans. Roy. Soc. Edinburgh 52 (1919-20), 597–602.

[Hat] A.E. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. 117 (1983), 553–607.

[HatT] A.E. Hatcher, W.P. Thurston, *Incompressible surfaces in* 2*–bridge knot complements*, Invent. Math. 79 (1985), 225–246.

[Hemi] G. Hemion, *On the classification of homeomorphisms of* 2*–manifolds and the classification of* 3*–manifolds*, Acta Math. 142 (1979), 123–155.

[Hemp] J. Hempel, 3*–Manifolds*, Ann. of Math. Studies 86, Princeton University Press, 1976.

[HendL] H. Hendriks, F. Laudenbach, *Difféomorphismes des sommes connexes en dimension trois*, Topology 23 (1984), 423–443.

[HendM] H. Hendriks, D. McCullough, *On the diffeomorphism group of a reducible* 3*-manifold*, Topology Appl. 26 (1987), 25–31.

[Henr] M. Henry, *Classe d'homotopie d'une sphère plongée dans* $\#_q S^2 \times S^1$, C. R. Acad. Sci. Paris Ser. I Math. 297 (1983), 425–428.

[HirM] F. Hirzebruch, K.H. Mayer, $\mathrm{O}(n)$*-Mannigfaltigkeiten, exotische Sphären und Singularitäten*, Lecture Notes in Mathematics 57, Springer-Verlag, 1968.

[HirNK] F. Hirzebruch, W.D. Neumann, S.S. Koh, *Differentiable manifolds and quadratic forms*, Lect. Notes in Pure and Applied Math., Marcel Dekker, 1971.

[Hos] J. Hoste, *The enumeration and classification of knots and links*, in: *Handbook of knot theory*, 209–232, Elsevier B. V., Amsterdam, 2005.

[HTW] J. Hoste, M. Thistlethwaite, J. Weeks, *The first* 1,701,936 *knots*, Math. Intelligencer 20 (1998), 33–48.

[ITC] *International tables for X-ray crystallography* (N.F.M. Henry, L. Lonsdale eds.), Kynoch Press, 1969.

[JacO] W.H. Jaco, U. Oertel, *An algorithm to decide if a* 3*-manifold is a Haken manifold*, Topology 23 (1984), 195–209.

[$JacS_1$] W.H. Jaco, P.B. Shalen, *Seifert fibered spaces in irreducible, sufficiently-large* 3*-manifolds*, Bull. Amer. Math. Soc. 82 (1976), 765–767.

[$JacS_2$] W.H. Jaco, P.B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. 21, American Mathematical Society, 1979.

[Joh_1] K. Johannson, *Équivalences d'homotopie des variétés de dimension* 3, C. R. Acad. Sci. Paris Sér. A-B 281 (1975), A1009–A1010.

[Joh_2] K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundaries*, Lecture Notes in Mathematics 761, Springer-Verlag, 1979.

[Joh_3] K. Johannson, *On the mapping class group of simple* 3*-manifolds*, in *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, 48–66, Lecture Notes in Math. 722, Springer, 1979.

[Jør] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. 98 (1976), 739–749.

[Kap] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics 183, Birkhäuser, 2001.

[Kas] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics 155, Springer-Verlag, 1995.

[Kaw] A. Kawauchi, *The invertibility problem on amphicheiral excellent knots*, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), 399–402.

[Kau] L. Kauffman, *State models and the Jones polynomial*, Topology 26 (1987), 395–407.

[KimT] P.K. Kim, J.L. Tollefson, *Splitting the PL involutions of nonprime* 3–*manifolds*, Michigan Math. J. 27 (1980), 259–274.

[Kirb] R.C. Kirby, *Problems in low dimensional manifold theory*, in: *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976)*, Part 2, 273–312, Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., 1978.

[KirbS] R.C. Kirby, L.C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Mathematics Studies 88, Princeton University Press, 1977.

[Kirk_1] T.P. Kirkman, *The enumeration, description, and construction of knots of fewer than* 10 *crossings*, Trans. Roy. Soc. Edinburgh 32 (1883–84), 281–309.

[Kirk_2] T.P. Kirkman, *The* 364 *unifilar knots of ten crossings, enumerated and described*, Trans. Roy. Soc. Edinburgh 32 (1884–85), 483–491.

[Kirk_3] T.P. Kirkman, *On the twists of Listing and Tait*, Proc. Royal Soc. Edinburgh 13 (1884–85), 363–367.

[Kirk_4] T.P. Kirkman, *Examples upon the reading of the circle or circles of a knot*, Proc. Royal Soc. Edinburgh 13 (1885–86), 693–698.

[KleiL] B. Kleiner, J. Lott, *Notes on Perelman's papers*, Geometry & Topology 12 (2008), 2587–2855.

[Kne] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutsch. Math.-Verein. 38 (1929), 248-260.

[Kob] T. Kobayashi, *Structures of full Haken manifolds*, Osaka J. Math. 24 (1987), 173–215.

[Lac_1] J. Lacan, *Le séminaire* XXI, 1973–74: *Les non-dupes errent*, transcripts accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_2] J. Lacan, *Le séminaire* XXII, 1974–75: *R.S.I.*, transcripts accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_3] J. Lacan, J.-A. Miller, *Le séminaire* XXIII, 1975–76: *Le sinthome*, Le Seuil, 2005; transcripts also accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_4] J. Lacan, *Le séminaire* XXIV, 1976–77: *L'insu que sait de l'une bévue s'aile à mourre*, transcripts accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_5] J. Lacan, *Le séminaire* XXV, 1977–78: *Le moment de conclure*, transcripts accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_6] J. Lacan, *Le séminaire* XXVI, 1978–79: *La topologie et le temps*, transcripts accessible at `http://gaogoa.free.fr/SeminaireS.htm`.

[Lac_7] J. Lacan, *Le séminaire* XXVII, 1979–80: *Dissolution*, transcripts accessible at
`http://gaogoa.free.fr/SeminaireS.htm`.

[Lack] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. 140 (2000), 243–282.

[LacM] M. Lackenby, R. Meyerhoff, *The maximal number of exceptional Dehn surgeries*, preprint, `arXiv:0808.1176`.

[Lis] J.B. Listing, *Vorstudien zur Topologie*, Göttinger Studien (Abtheilung 1) 1 (1847), 811–875.

[Lit_1] C.N. Little, *On knots, with a census for order* 10, Trans. Connecticut Academy Sci. 18, Vol. 7 (1885), 27–43.

[Lit_2] C.N. Little, *Non-alternate* ± *knots of order eight and nine*, Trans. Royal. Soc. Edinburgh 35 (1890), 253–255.

[Lit_3] C.N. Little, *Alternate* ± *knots of order eleven*, Trans. Royal. Soc. Edinburgh 36 (1890-91), 663-664

[Lit_4] C.N. Little, *Non-alternate* ± *knots*, Trans. Royal. Soc. Edinburgh 39 (1898-99), 771–778.

[Luo] F. Luo, *On a problem of Fenchel*, Geom. Dedicata 64 (1997), 277–282.

[MeeS] W.H. Meeks, P. Scott, *Finite group actions on* 3*–manifolds*, Invent. Math. 86 (1986), 287–346.

[$MeeY_1$] W.H. Meeks, S.-T. Yau, *The equivariant Dehn's lemma and loop theorem*, Comment. Math. Helv. 56 (1981), 225–239.

[$MeeY_2$] W.H. Meeks, S.-T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. 112 (1980), 441–484.

[$MeeY_3$] W.H. Meeks, S.-T. Yau, *The equivariant loop theorem for three-dimensional manifolds and a review of the existence theorems for minimal surfaces*, in: *The Smith conjecture (New York, 1979)*, 153–163, Pure Appl. Math. 112, Academic Press, 1984.

[Men_1] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984), 37–44.

[Men_2] W. Menasco, *Determining incompressibility of surfaces in alternating knot and link complements*, Pacific J. Math. 117 (1985), 353–370.

[MenT] W. Menasco, M. Thistlethwaite, *The classification of alternating links*, Ann. of Math. 138 (1993), 113–171.

[Mey] R. Meyerhoff, *A lower bound for the volume of hyperbolic* 3*–manifolds*, Canadian J. Math. 39 (1987), 1038–1056.

[Mil] J. Milnor, *A unique decomposition theorem for* 3*-manifolds*, Amer. J. Math. 84 (1962), 1–7.

[Moi] E.E. Moise, *Geometric topology in dimensions* 2 *and* 3, Graduate Texts in Mathematics 47, Springer-Verlag, 1977.

[Mon_1] J.M. Montesinos, *Variedades de Seifert que són recubridores cíclicos de dos hojas*, Bol. Soc. Mat. Mexicana 18 (1973), 1–32.

[Mon_2] J.M. Montesinos, *Revêtements ramifiés de noeuds, espaces fibrés de Seifert et scindements de Heegaard*, Publications Mathématiques d'Orsay, Université de Paris-Sud, 1979.

[Mor_1] J.W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, in: *The Smith conjecture (New York, 1979)*, 37–125, Pure Appl. Math., 112, Academic Press, 1984.

[$MorT_1$] J.W. Morgan, G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs 3, American Math. Soc.; Clay Mathematics Institute, 2007.

[$MorT_2$] J.W. Morgan, G. Tian, *Completion of the proof of the geometrization conjecture*, preprint, 2008, `arxiv.org/abs/0809.4040` .

[Mos_1] G.D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Institut des Hautes Études Scientifiques, Publications Mathématiques 34 (1968), 53–104.

[Mos_2] G.D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies 78, Princeton University Press, 1973.

[Mun] J. Munkres, *Differentiable isotopies on the* 2*-sphere*, Michigan Math. J. 7 (1960), 193–197.

[Mur] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology 26 (1987), 187–194.

[NeuZ] W.D. Neumann, D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985), 307–332.

[New] M.H.A. Newman, *A theorem on periodic transformations of spaces*, Quart. J. Math. 2 (1931), 1–9.

[Oer] U. Oertel, *Incompressible branched surfaces*, Invent. Math. 76 (1984), 385–410.

[Orl] P. Orlik, *Seifert manifolds*, Lect. Notes in Math. 291, Springer-Verlag, 1972.

[OVZ] P. Orlik, E. Vogt, H. Zieschang, *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten*, Topology 6 (1967), 49–64.

[Ota_1] J.-P. Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension* 3, Astérisque 235, Société Mathématique de France, 1996.

[Ota_2] J.-P. Otal, *Thurston's hyperbolization of Haken manifolds*, in: *Surveys in differential geometry, Vol. III (Cambridge, MA, 1996)*, 77–194, International Press, 1998.

[$Pere_1$] G.Y. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arxiv.org/abs/math/0211159`.

[$Pere_2$] G.Y. Perelman, *Ricci flow with surgery on three-manifolds*, `arxiv.org/abs/math/0303109`.

[$Pere_3$] G.Y. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, `arxiv.org/abs/math/0307245`.

[$Perk_1$] K.A. Perko, *On covering spaces of knots*, Glasnik Mat. Ser. III 9(29) (1974), 141–145.

[$Perk_2$] K.A. Perko, *On the classification of knots*, Proc. Amer. Math. Soc. 45 (1974), 262–266.

[$Perk_3$] K.A. Perko, *On* 10*-crossing knots*, Portugal. Math. 38 (1979), 5–9.

[$Perk_4$] K.A. Perko, *Invariants of* 11*–crossing knots*, Prépublications Math. d'Orsay 80, Université de Paris-Sud, 1980.

[Ran] R. von Randow, *Zur Topologie von dreidimensionalen Baummannigfaltigkeiten*, Bonn. Math. Schr. 14, 1962.

[Rat] J.G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[Ril_1] R.F. Riley, *An elliptical path from parabolic representations to hyperbolic structures*, in: *Topology of low-dimensional manifolds (Proceedings of the Second Sussex Conference, Chelwood Gate, 1977)*, 99–133, Lecture Notes in Math. 722, Springer, 1979.

[Ril_2] R.F. Riley, *Seven excellent knots*, in: *Low-dimensional topology (Bangor, 1979)*, 81–151, London Math. Soc. Lecture Note Series 48, Cambridge University Press, 1982.

[Rol] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, Publish or Perish, Inc., 1976.

[Sak] M. Sakuma, *The geometries of spherical Montesinos links*, Kobe J. Math. 7 (1990), 167–190.

[SchaS] M. Scharlemann, J. Schultens, *Comparing Heegaard and JSJ structures of orientable* 3*–manifolds*, Trans. Amer. Math. Soc. 353 (2001), 557–584.

[Sch_1] H. Schubert, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, Sitz.-Ber. Heidelberger Akad. Wiss. Math.-Nat. (1949), 57–104.

[Sch_2] H. Schubert, *Knoten und Vollringe*, Acta Math. 90 (1953), 131–286.

[Sch_3] H. Schubert, *Knoten mit zwei Brücken*, Math. Z. 65 (1956), 133–170.

[Sch_4] H. Schubert, *Bestimmung der Primfaktorzerlegung von Verkettungen*, Math. Z. 76 (1961), 116–148.

[SchuS] H. Schubert, K. Soltsien, *Isotopie von Flächen in einfachen Knoten*, Abh. Math. Sem. Univ. Hamburg 27 (1964), 116–123.

[Sco_1] G.P. Scott, *A new proof of the annulus and torus theorems*, Amer. J. Math. 102 (1980), 241–277.

[Sco_2] G.P. Scott, *Strong annulus and torus theorems and the enclosing property of characteristic submanifolds of* 3*-manifolds*, Quart. J. Math. Oxford 35 (1984), 485–506.

[Sco_3] G.P. Scott, *The geometries of* 3*–manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.

[Sei_1] H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. 60 (1933), 147–238.

[Sei_2] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. 110 (1935), 571.

[Sel] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, in: *Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960)*, 147–164, Tata Institute of Fundamental Research, Bombay, 1960.

[Sie_1] L.C. Siebenmann, *Exercices sur les nœuds rationnels*, lecture notes, Université de Paris-Sud, 1977.

[Sie_2] L.C. Siebenmann, *On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology* 3*-spheres*, in: *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, 172–222, Lecture Notes in Math. 788, Springer, 1980.

[Sie_3] L.C. Siebenmann, *Sweet-TEX*, `http://www.ctan.org/tex-archive /systems/mac/sweet-tex/`

[Sin] I.M. Singer, *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math. 13 (1960), 685–697.

[Sma] S. Smale, *Diffeomorphisms of the* 2*-sphere* Proc. Amer. Math. Soc. 10 (1959), 621–626.

[Smi] P. A. Smith , *Transformations of finite period III: Newman's theorem*, Annals of Math. 42 (1941), 446–458.

[Sou_1] P. Soury, *Chaînes, nœuds, surfaces: la topologie de Lacan* (J. Lafont ed.), École de la Cause Freudienne, 1982.

[Sou_2] P. Soury, *Chaînes et Nœuds: 1ère partie* (M. Thomé, P. Soury eds.), École de la Cause Freudienne, 1988.

[Sou_3] P. Soury, *Chaînes et Nœuds: 3ème partie* (M. Thomé, P. Soury eds.), École de la Cause Freudienne, 1986.

[Stoi] A. Stoimenow, *On the number of links and link polynomials*, Q. J. Math. 55 (2004), 87–98.

[SunT] C. Sundberg, M. Thistlethwaite, *The rate of growth of the number of prime alternating links and tangles*, Pacific J. Math. 182 (1998), 329–358.

[Tai_1] P.G. Tait, *On knots I*, Trans. Roy. Soc. Edinburgh 28 (1876-7), 145–190.

[Tai_2] P.G. Tait, *On knots II*, Trans. Roy. Soc. Edinburgh 32 (1883-4), 327–342.

[Tai_3] P.G. Tait, *On knots III*, Trans. Roy. Soc. Edinburgh 32 (1884-5),493–506.

[Tho] W.H. Thomson, *On vortex motion*, Trans. Roy. Soc. Edinburgh 25 (1867), 217–260.

[Thi] M. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology 26 (1987), 297–309.

[Thu_1] W.P. Thurston, *The geometry and topology of three-manifolds*, lecture notes, Princeton University, 1976-80.

[Thu_2] W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[Thu_3] W.P. Thurston, *Hyperbolic structures on* 3*–manifolds, I: Deformation of acylindrical manifolds*, Ann. of Math. 124 (1986), 203–246

[Thu_4] W.P. Thurston, *Hyperbolic structures on* 3*–manifolds, II: Surface groups and* 3*–manifolds which fiber over the circle*, unpublished preprint, *ca.* 1980, `arXiv:math/9801045`.

[Thu_5] W.P. Thurston, *Hyperbolic structures on* 3*–manifolds, III: Deformations of 3-manifolds with incompressible boundary*, unpublished preprint, *ca.* 1980, `arXiv:math/9801058v1`.

[Thu_6] W.P. Thurston, *Three-dimensional geometry and topology, Vol. 1* (Edited by Silvio Levy), Princeton Math. Series 35, Princeton University Press, 1997.

[Tol_1] J.L. Tollefson, *Periodic homeomorphisms of* 3*-manifolds fibered over* S^1, Trans. Amer. Math. Soc. 223 (1976), 223–234.

[Tol_2] J.L. Tollefson, *Involutions of Seifert fiber spaces*, Pacific J. Math. 74 (1978), 519–529.

[Tol_3] J.L. Tollefson, *Involutions of sufficiently large* 3*-manifolds*, Topology 20 (1981), 323–352.

[Tur] V.G. Turaev, *Classification of oriented Montesinos links via spin structures*, in *Topology and geometry—Rohlin Seminar*, 271–289, Lecture Notes in Math., 1346, Springer, 1988.

[Wal_1] F. Waldhausen, *Gruppen mit Zentrum und* 3*-dimensionale Mannigfaltigkeiten*, Topology 6 (1967), 505–517.

[Wal$_2$] F. Waldhausen, *Eine Klasse von* 3*-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. 3 (1967), 308–333; Invent. Math. 4 (1967), 87–117.

[Wal$_3$] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. 87 (1968), 56–88.

[Wal$_4$] F. Waldhausen, *On the determination of some bounded* 3*–manifolds by their fundamental group alone*, in: *Proceedings of the International Symposium on Topology and its Applications*, Herceg-Novi 1968, Beograd (1969), 331–332.

[Wal$_5$] F. Waldhausen, *Recent results on sufficiently large* 3*-manifolds*, in: *Algebraic and geometric topology* (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, 21–38, Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., 1978.

[Wic] B. Wicha-Krause, *On embedded* 2*-spheres in* 3*-manifolds*, Math. Z. 184 (1983), 235–243.

[Wol] J.A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Co., 1967.

[Zar] O. Zariski, *On the topology of algebraic singularities*, Amer. J. Math. 54 (1932), 453–465.

[Zie] H. Zieschang, *Classification of Montesinos knots*, in *Topology (Leningrad, 1982)*, 378–389, Lecture Notes in Math., 1060, Springer, Berlin, 1984.

Index