LECTURES ON THE COHOMOLOGY OF FINITE GROUP SCHEMES

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Contents

0. Introduction

This paper is a revised version of five lectures given in Nantes in December 2001. We have revised the lectures themselves so that they might provide an introduction to some of the techniques and computations of cohomology of finite group schemes which have been developed since the publication of J. Jantzen's book [14].

The goal of those Nantes lectures was to provide an introduction to the cohomology of finite group schemes over a field k of characteristic $p > 0$ and to explain the important role played by the cohomology of (strict polynomial) functors. The focal point of these lectures was a theorem of E. Friedlander and A. Suslin asserting that the cohomology of finite group schemes is finitely generated (see Theorem 4.7 below). The somewhat innovative proof of this theorem has led to numerous further results; in these lectures we have restricted attention to those results bearing on the qualitative description of the cohomology algebra of a finite group scheme.

The reader can obtain a quick guide to these edited lectures by glancing at the table of contents. In the first lecture, we introduce the concepts and terminology which underline our subject. In particular, we recall the definition of the Frobenius kernels of an algebraic group and the Frobenius twists of a module. The second lecture summarizes some of the techniques which one can find for example in [14] which are used to compute cohomology. The relationship of this subject with the theme of the Nantes meeting, cohomology in categories of functors, is explained in the third lecture. Strict polynomial functors are introduced and their relationship with polynomial representations is explained. The fourth lecture is dedicated to an outline of the proof of finite generation of the cohomology of finite group schemes. Here, computations of cohomology in the category of strict polynomial functors

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plays a central role in the construction of certain universal classes; these computations follow closely the computations of V. Franjou, J. Lannes, and L. Schwartz [9] of ordinary functor cohomology. Finally, in Lecture 5 we describe how the techniques introduced to prove finite generation lead to a qualitative description of the cohomology algebra $H^*(G, k)$ of a finite group scheme. This follows work of D. Quillen [15] who determined the maximal ideal spectrum of the cohomology of a finite group.

We thank the organizers of the Nantes conference for the opportunity to visit Nantes and participate in that very successful conference. We also take this opportunity to thank Andrei Suslin for sharing with us many fundamental ideas he has contributed to the study of the cohomology of finite group schemes. Finally, we are especially grateful to Julia Pevtsova who corrected many errors in a preliminary draft of these notes.

1. Affine group schemes

Let k be a field of characteristic $p > 0$, fixed throughout this paper. We begin our discussion by defining an affine group scheme (implicitly assumed to be over k) and considering a few interesting examples.

Definition 1.1. An affine group scheme is a representable functor

$$G : (\text{fin.gen.comm.}k - alg) \to (grps)$$

We denote by $k[G]$ the representing finitely generated commutative k-algebra (the *coordinate algebra*) of G. To give such a representable functor is equivalent to giving a finitely generated commutative Hopf algebra (over k).

Example 1.2. $G = \mathbb{G}_a$, the additive group. This is the functor which takes a commutative k-algebra A to the underlying abelian group (which we might denote A^+). The coordinate algebra of $\mathbb{G}_a$ is $k[\mathbb{G}_a] = k[t]$, with coproduct $\nabla(t) = t \otimes 1 + 1 \otimes t$.

Example 1.3. $G = GL_n$, the general linear group, sends a commutative k-algebra A to the group of $n \times n$ invertible matrices $\{a_{i,j}\}$ with coefficients in A. The coordinate algebra of GL_n is given by

$$k[GL_n] = k[x_{i,j}, t]_{1 \leq i,j \leq n}/det(x_{i,j})t - 1$$

with coproduct

$$\nabla(x_{i,j}) = \Sigma x_{i,k} \otimes x_{k,j}.$$

Example 1.4. Let π be a (discrete) group. We view π as an affine group scheme by letting π also denote "the constant functor with value π." In other words, this functor sends a commutative k-algebra A to the group $\pi^{|\pi_0(A)|}$, where $\pi_0(A)$ is the set of indecomposable non-trivial idempotents in A and $|\pi_0(A)|$ denotes the cardinality of $\pi_0(A)$.

Example 1.5. For any positive integer r, we consider the "r-th Frobenius kernel" of GL_n which is denoted $GL_{n(r)}$. This is the functor which sends a commutative k-algeba A to the group of $n \times n$ invertible matrices $(a_{i,j})$ with coefficients in A which satisfy the property that $a_{i,j}^{p^r} = \delta_{i,j}$ (i.e., equal to 1 if $i = j$ and 0 otherwise). The coordinate algebra $k[GL_{n(r)}]$ is the quotient of $k[GL_n]$ by the (Hopf) ideal generated by $x_{i,j}^{p^r} - \delta_{i,j}$. More explicitly, we can write $k[GL_{n(r)}] = k[x_{i,j}]/(x_{i,j}^{p^r} - \delta_{i,j})$.

Similarly, the r-th Frobenius kernel of $\mathbb{G}_a$ sends A to the group of elements of A whose p^r-th power is 0. The coordinate algebra of $\mathbb{G}_{a(r)}$ is given by $k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r}$, whereas the dual algebra is given by $k\mathbb{G}_{a(r)} = k[X_1, \ldots, X_r]/(X_i^p)$ where one can view the dual generator X_i as the operator $\frac{1}{p^{i-1}!}\frac{d^{p^{i-1}}}{dt^{p^{i-1}}}$ on $k[t]$.

Example 1.6. Let g be a finite dimensional p-restricted Lie algebra of k and let $V(g)$ denote its restricted enveloping algebra, the quotient of the universal enveloping algebra $U(g)$ of g by the ideal generated by $\{X^p - X^{[p]}, X \in g\}$ (where $(-)^{[p]} : g \to g$ is the p-th power operation of g), Then the k-linear dual of $V(g)$, which we denote by $V(g)^\#$, is a finite dimensional commutative Hopf algebra over k and thus corresponds to an affine group scheme over k.

Remark 1.7. An affine group scheme G is said to be *finite* if $k[G]$ is finite dimensional. For example, if G corresponds to a finite group π as in Example 1.4 or if G is a group scheme as in Example 1.5 or G is associated to a finite dimensional p-restricted Lie algebra as in Example 1.6, then G is a finite group scheme. The linear dual is called the *group algebra* of G, denoted kG, consistent with the usual terminology of the group algebra of a discrete group π. In Example 1.6, the group algebra kG of the group scheme G associated to the p-restricted Lie algebra g is $V(g)$, the restricted enveloping algebra of g.

One usually refers to an affine group scheme G whose coordinate algebra is integral (i.e., reduced and irreducible) as an (affine) algebraic group. For example, both $\mathbb{G}_a$ of Example 1.2 and GL_n of Example 1.3 are algebraic groups.

Remark 1.8. A finite group scheme G is said to be *infinitesimal* if the coordinate algebra $k[G]$ is local. An infinitesimal group G is said to be of height $\leq r$ if G admits a closed embedding $G \hookrightarrow GL_{n(r)}$ (i.e., if $a^{p^r} = 0$ for every element a in the augmentation ideal of $k[G]$). For any infinitesimal group scheme G of height 1 we have an isomorphism of algebras:

$$kG \simeq V(LieG).$$

Conversely, if g is a finite dimensional p-restricted Lie algebra, then $V(g)^\#$ is the coordinate algebra of an infinitesimal group scheme G of height 1. This establishes an equivalence of categories between finite dimensional p-restricted Lie algebras and infinitesimal group schemes of height 1.

We next introduce the concept of a G-module for an affine group scheme (sometimes called a rational G-module).

Definition 1.9. Let G be an affine group scheme over k. Then a G-module M is a k-vector space provided with an A-linear group action

$$(1.10) \qquad\qquad G(A) \times (M \otimes A) \to M \otimes A$$

for all finitely generated commutative k-algebras A, functorial with respect to A. (Here, and below, the tensor product is over k.)

Equivalently, such a G-module M is a k-vector space provided with the structure of a comodule for $k[G]$; namely, a k-linear map

$$(1.11) \qquad\qquad \nabla_M : M \to M \otimes k[G].$$

To verify this equivalence, observe that the pairing (1.10) in the special case $A = k[G]$ is written

$$Hom_{k-alg}(k[G], k[G]) \times (M \otimes k[G]) \to M \otimes k[G].$$

This determines a comodule structure of the form (1.11) by restricting to $id_{k[G]} \in Hom_{k-alg}(k[G], k[G])$. Conversely, given a comodule structure ∇_M, we get a pairing of the form (1.10) as the following composition

$$Hom_{k-alg}(k[G], A) \times (M \otimes A) \to Hom_{k-alg}(k[G], A) \times (M \otimes k[G] \otimes A) \to$$

$$M \otimes A \otimes A \to M \otimes A$$

where the first map is given by ∇_M, the second by the natural pairing, and the third by the ring structure on A.

If the G-module M is finite dimensional (as a k vector space), we may give another useful formulation of the concept of a G-module. Namely, suppose that M is n-dimensional and identify the affine group scheme of k-automorphisms of M with GL_n. Then to give M the structure of a G-module is equivalent to giving a homomorphism $\rho_M : G \to GL_n$ of affine group schemes.

An important example of a G-module is the coordinate algebra itself. We readily check that the coproduct on $k[G]$, $\nabla : k[G] \to k[G] \otimes k[G]$, corresponds to the right regular representation of G on the functions of G: $(g \in G, f(-) \in k[G]) \mapsto f(-\cdot g) \in k[G]$.

Suppose that $H \subset G$ is a closed subgroup scheme of the affine group scheme G (i.e., $k[G] \to k[H]$ is surjective). Then for any H-module N, we consider the H-fixed points of $k[G] \otimes N$, where H acts on $k[G]$ via the right regular representation. We use the notation

$$Ind_H^G N = (k[G] \otimes N)^H$$

to denote the G-module with G action given by the left regular representation of G on $k[G]$.

One very useful aspect of this induction functor is given by the following theorem which is often called *Frobenius reciprocity*.

Theorem 1.12. *(cf. [14, 3.4]) If $H \subset G$ is a closed subgroup of the affine group scheme G, then $Ind_H^G(-)$ is right adjoint to the restriction functor. In other words, for every H-module N and every G-module M, there is a natural isomorphism*

$$Hom_H(M, N) \simeq Hom_G(M, Ind_H^G N).$$

In particular, if N is an injective H-module, then $Ind_H^G N$ is an injective G-module. For example, $k[G] = Ind_e^G k$ is an injective G-module.

Observe that sending $m \in M$ to $m \otimes \epsilon \in M \otimes k[G]$ determines a homomorphism $M \to M \otimes k[G]$ of G-modules, where $\epsilon : G \to k$ is evaluation at the identity (i.e., the co-unit of the Hopf algebra $k[G]$). A direct calculation shows that the map $M \otimes k[G] \to M_{tr} \otimes k[G]$ defined by $m \otimes f \mapsto (1 \otimes f)\nabla_M(m)$ is an isomorphism of G-modules, where M_{tr} is a trivial G-module isomorphic to M as a k-vector space. Since $k[G]$ is an injective G-module, this verifies that any G-module can be embedded into an injective module.

Consequently, the category of G-modules is an abelian category with enough injectives, so that we may use standard homological algebra to define

$$Ext_G^i(M, N) = R^i Hom_G(M, -)(N)$$

for any pair of G-modules M,N. As usual, we denote $Ext^*_G(k, M)$ by $H^*(G, M)$, so that

$$H^i(G, M) = R^i Hom_G(k, -)(M) = R^i(-)^G(M)$$

where the G-fixed point functor sends a G-module M to $M^G \subset M$, the maximal subspace of M on which G acts trivially. We readily verify that Theorem 1.12 implies that

$$H^*(H, N) \simeq H^*(G, Ind^G_H N)$$

whenever $H \subset G$ is a closed subgroup scheme and N is a H-module.

Let $\phi : k \to k$ denote the p-th power map which sends $\alpha \in k$ to $\alpha^p \in k$. (ϕ is often called the *arithmetic Frobenius map*.) Given a k vector space V, we obtain a new k-vector space $V^{(1)}$ defined as the base change of V via ϕ,

$$V^{(1)} = k \otimes_\phi V.$$

If k is perfect (i.e., if ϕ is an isomorphism), then

$$V^{(1)} \simeq V, \quad \alpha \otimes v \mapsto \alpha^{1/p} v$$

identifies $V^{(1)}$ via a semi-linear map with V, so that we may view $V^{(1)}$ as the vector space V with the modified k-action given by $(\alpha, v) \mapsto \alpha^{1/p} v$. $V^{(1)}$ is called the (first) *Frobenius twist* of V.

Definition 1.13. If G is an affine group scheme, we denote by $G^{(r)}$ the affine group scheme whose coordinate algebra is $k[G]^{(r)}$, the rth Frobenius twist of $k[G]$. Moreover, we denote by $G_{(r)}$ the affine group scheme defined as the kernel of the natural map

$$G_{(r)} = ker\{\Phi^r : G \to G^{(r)}\},$$

where $\Phi^{r*} : k[G^{(r)}] \to k[G]$ is the k-linear map sending $f \in k[G]^{(r)}$ to $f^{p^r} \in k[G]$.

If G is defined over the finite field $\mathbb{F}_{p^r}$ so that $G = G_{\mathbb{F}_{p^r}} \times_{\text{Spec}\,\mathbb{F}_{p^r}} \text{Spec}\,k$, then

$$\Phi^{r*} = F^{r*} \circ \phi^r : k[G]^{(r)} \simeq k[G] \to k[G].$$

Here, F^r is the so-called *geometric Frobenius* of G, defined as the base change from $\mathbb{F}_{p^r}$ to k of the p^r-th power map on $\mathbb{F}_{p^r}[G_{\mathbb{F}_{p^r}}]$. Thus, for such G we can identify $G_{(r)}$ with the kernel of F^r,

$$G_{(r)} = ker\{F^r : G \to G\}.$$

In the special case $G = GL_n$, we readily verify that $(GL_n)_{(r)}$ so defined equals $GL_{n(r)}$ as discussed in Example 1.5.

We conclude that whenever G is defined over $\mathbb{F}_{p^r}$, a G-module M determines a new G-module $M^{(r)}$, the *r-th Frobenius twist* of M. If $\rho_M : G \to GL_n$ is the representation associated to the G-module M, then

$$\rho_{M^{(r)}} = F^r \circ \rho : G \to GL_n \to GL_n$$

is the representation associated to $M^{(r)}$. Observe that $M^{(r)}$ is trivial as a $G_{(r)}$-module, so that

$$H^0(G_{(r)}, M) \neq H^0(G_{(r)}, M^{(r)})$$

whenever M is non-trivial as a $G_{(r)}$-module. Similarly, the cohomology $H^*(GL_n, M)$ can be quite different from $H^*(GL_n, M^{(1)})$. Indeed, this difference plays an important role in our techniques for computation.

2. Cohomological techniques

Much of the second lecture of this series was dedicated to explaining weights associated to the action of a torus with the goal of giving some insight into the effect that Frobenius twist plays in cohomology. This written version adds to the original lecture by giving a brief introduction to some of the techniques used in the computation of cohomology. The reader is referred to the book of J. Jantzen [14] for a much more complete exposition of these techniques.

The algebraic group GL_1 is typically denoted $\mathbb{G}_m$ and called the multiplicative group. The coordinate algebra $k[\mathbb{G}_m]$ is given by

$$k[\mathbb{G}_m] = k[t, t^{-1}] = k[u, v]/(uv - 1)$$

with coproduct $t \mapsto t \otimes t$. A split torus of rank n is an algebraic group isomorphic to $\mathbb{G}_m^{\times n}$. The subgroup $T_n \subset GL_n$ of diagonal matrices is the usual model for such a split torus of rank n.

The representation theory of a split torus is particularly easy to describe as the following proposition recalls.

Proposition 2.1. *Every T_n-module splits as a direct sum of 1-dimensional irreducible T_n-mod ules. An irreducible T_n-module is given by its weight $\underline{\lambda} = (\lambda_1, \ldots \lambda_n) \in \mathbb{Z}^{\oplus n}$, where for a given finitely generated k-algebra A the diagonal matrix*

$$\begin{pmatrix} x_1 & & \\ & \ldots & \\ & & x_n \end{pmatrix} \in (A^*)^n$$

acts on the rank 1 A-module via multiplication by $x_1^{\lambda_1} \ldots x_n^{\lambda_n}$.

Similarly, every $T_{n(r)}$-module splits as a direct sum of 1-dimensional $T_{n(r)}$-modules, where $T_n(r)$ is the r-th Frobenius kernel of T_n. The weights $\underline{\lambda} = (\lambda_1, \ldots \lambda_n)$ of $T_{n(r)}$ can be viewed as taking values in $\{0, 1, \ldots p^r - 1\}^n$ since any $(x_1, \ldots, x_n) \in T_{n(r)}(A)$ satisfies $x_i^{p^r} = 1$.

Example 2.2. The most basic example is the action of T_n on an n-dimensional vector space given by multiplication; in this case, the weights of this action are all of the form $(0, \ldots, 0, 1, 0, \ldots 0)$. We view this action as given by the pairing of algebraic groups $\mu : T_n \times \mathbb{G}_a^{\times n} \to \mathbb{G}_a^{\times n}$, which is equivalent to the data of a compatible collection of pairings $\mu : (A^*)^{\times n} \times A^{\oplus n} \to A^{\oplus n}$ for every finitely generated k-algebra A.

A second basic example is the action of T_n on the dual vector space, given by

$$\mu : Hom_{\text{grp sch}}(\mathbb{G}_a^{\times n}, \mathbb{G}_a) \to Hom_{\text{grp sch}}(\mathbb{G}_a^{\times n}, \mathbb{G}_a).$$

Since we define $Hom_{\text{grp sch}}(\mathbb{G}_a^{\times n}, \mathbb{G}_a)$ as a $\mathbb{G}_m$-module so that the evaluation pairing $Hom_{\text{grp sch}}(\mathbb{G}_a^{\times n}, \mathbb{G}_a) \times \mathbb{G}_a^{\times n} \to \mathbb{G}_a$ is $\mathbb{G}_m$ equivariant with $\mathbb{G}_m$ acting trivially on the right hand side, the resulting weights of $Hom_{\text{grp sch}}(\mathbb{G}_a^{\times n}, \mathbb{G}_a)$ are all of the form $(0, \ldots, 0, -1, 0, \ldots 0)$. . Observe that under this action

$$(\alpha_1, \ldots, \alpha_n), (\psi_1(-), \ldots \psi_n(-)) \mapsto (\psi_1(\alpha_1^{-1} \cdot -), \ldots, \psi_n(\alpha_n^{-1} \cdot -)),$$

which is the usual contragredient action.

Rather than discuss maximal tori and weights for general reductive groups, we describe the situation for the example of primary interest, that of the algebraic group GL_n.

Proposition 2.3. *Let M be a GL_n-module.*

(1) *As a T_n-module, $M \simeq \oplus M_{\underline{\lambda}}$. The T_n-submodule $M_{\underline{\lambda}} \subset M$ is called the $\underline{\lambda}$-weight subspace.*

(2) *The Frobenius twist $M^{(r)}$ of M has weight decomposition $M^{(r)} = \oplus M_{p^r\underline{\lambda}}$.*

(3) *Let $\mathbb{G}_m \subset T_n$ denote the subgroup of scalar multiples of the identity. As a GL_n-module, M splits as a direct sum $M = \oplus M_d$, where $M_d \subset M$ is the weight subspace of weight d with respect to the action of $\mathbb{G}_m$.*

Observe that $H^i(T_n, M) = 0, i > 0$ since T_n is semisimple. On the other hand, the cohomology of $\mathbb{G}_a$ is quite interesting. We recall its computation, including its weight structure where the action of $\mathbb{G}_m$ on $H^*(\mathbb{G}_a, k)$ is that induced by the multiplication action of $\mathbb{G}_m$ on $\mathbb{G}_a$.

Theorem 2.4. *(cf. [5]).*

(1) $H^*(\mathbb{G}_a, k) = \Lambda^*(y_1, y_2, \dots) \otimes k[x_1, x_2 \dots], \quad p \neq 2.$

$$H^*(\mathbb{G}_a, k) = k[y_1, y_2, \dots], \quad p = 2$$

where each $y_i \in H^1(\mathbb{G}_a, k), x_i \in H^2(\mathbb{G}_a, k)$.

(2) *Let $F : \mathbb{G}_a \to \mathbb{G}_a$ be the (geometric) Frobenius endomorphism. Then*

$$F^*(x_i) = x_{i+1}, \quad F^*(y_i) = y_{i+1}.$$

(3) *The weight of x_i is $-p^i$ and of y_i is $-p^{i-1}$.*

(4) *If $(\alpha \cdot -) : \mathbb{G}_a \to \mathbb{G}_a$ denotes multiplication by $\alpha \in k$, then*

$$(\alpha \cdot -)^*(x_i) = \alpha^{p^i} x_i, \quad (\alpha \cdot -)^*(y_i) = \alpha^{p^{i-1}} y_i,$$

(5) $H^*(\mathbb{G}_{a(r)}, k) = \Lambda^*(y_1, \dots, y_r) \otimes k[x_1, \dots x_r], \quad p \neq 2.$

$$H^*(\mathbb{G}_{a(r)}, k) = k[y_1, \dots, y_r], \quad p = 2.$$

The reader puzzled about the fact that the generator $y_1 \in H^1(\mathbb{G}_{a(1)}, k) = Hom_{grpsch}(\mathbb{G}_{a(1)}, \mathbb{G}_a)$ has weight -1 whereas the generator $x_1 \in H^2(\mathbb{G}_{a(1)}, k)$ has weight $-p$ might find it helpful to know that x_1 is the Bockstein of y_1. Thus, if y_1 is represented by some function $f \in k[\mathbb{G}_{a(1)}]$, then x_1 is represented by $\delta(f) \in k[\mathbb{G}_{a(1)}^2]$ defined by

$$\delta(f)(g_1, g_2) = \frac{f(g_1^p) + f(g_2^p) - f(g_1^p + g_2^p)}{p}.$$

A very useful technique for computations is the Lyndon-Hochschild-Serre (L-H-S) (first quadrant, cohomological) spectral sequence

(2.5) $$E_2^{p,q} = H^p(G/N, H^q(N, M)) \Longrightarrow H^{p+q}(G, M)$$

relating the cohomology of G with coefficients in the G-module M to the cohomology of G/N with coefficients in the G/N-module $H^*(N, M)$, the cohomology of the normal subgroup scheme N with coefficients in the restriction of M to N.

Example 2.6. Let $B_n \subset GL_n$ denote the subgroup of upper triangular matrices, and let $U_n \subset B_n$ denote the subgroup of strictly upper triangular matrices. We utilize the short exact sequence

$$1 \to U_n \to B_n \to T_n \to 1$$

and the semi-simplicity of T_n to conclude that

$$H^*(B_n, M) \simeq (H^*(U_n, M))^{T_n}.$$

Similarly, for any $r \geq 1$, we conclude

$$H^*(B_{n(r)}, M) \simeq (H^*(U_{n(r)}, M))^{T_{n(r)}}.$$

In the special case of trivial coefficients (i.e., $M = k$), we may make further progress in the computation of $H^*(B_n, k)$ by using a central series for U_n to express U_n as a succession of central extensions of products of root subgroups (i.e., subgroups isomorphic to $\mathbb{G}_a$ stabilized by T_n). Then the action of T_n stabilizes each of these extensions and thus induces a T_n-action on their associated L-H-S spectral sequences.

Indeed, if we pass to the first Frobenius kernel $B_{n(1)}$ of B_n and assume that $p > n$, then this strategy gives a complete calculation of $H^*(B_{n(1)}, k)$ as a $T_{n(1)}$-module. Namely, we consider the height 1 central extensions

$$1 \to \mathbb{G}_{a(1)} \to U_{(1)} \to \overline{U}_{(1)} \to 1,$$

associated to this central series for U_n. Applying the exact functor $(-)^{T_{n(1)}}$ to each of the associated L-H-S spectral sequences, we obtain spectral sequences of the form

$$(2.7) \qquad E_2^{p,q} = (H^p(\overline{U}_{(1)}, k) \otimes H^q(\mathbb{G}_{a(1)}, k))^{T_{n(1)}} \implies H^{p+q}(U_{(1)}, k)^{T_{n(1)}}.$$

If $p > n$, then the computation of Theorem 2.4 together with the multiplicative structure of (2.5) implies that all of the differentials of (2.7) are 0. Thus, we obtain an isomorphism of $T_{(n)}$-modules

$$H^*(U_{n(1)}, k) \simeq H^*(gr(U_n)_{(1)}, k), \quad gr(U_n) \simeq \mathbb{G}_a^{\times N}, N = \frac{n(n-1)}{2}.$$

Assuming $p > n$, this enables one to fully compute $H^*(B_{n(1)}, k) = (H^*(U_{n(1)}, k))^{T_{n(1)}}$ by taking the $T_{n(1)}$ invariants of

$$H^*(gr(U_n)_{(1)}, k) = \otimes_{i=1}^{N} H^*(\mathbb{G}_{a(1)}, k).$$

Finally, a weight argument (for $p > n$) implies that $H^*(B_{(1)}, k) \simeq S^*(u_n^{(\#(1)}$ where $u - n = Lie(U_n)$. (See [14, 12.12] for details of this weight argument; the earlier part of the above argument using the L-H-S spectral sequence is replaced in [14] by a different spectral sequence argument.)

As we recall in the following corollary of "Kempf's Vanishing Theorem", the cohomology $H^*(GL_n, M)$ is isomorphic to $H^*(B_n, M)$. We state this theorem more generally for an arbitrary affine algebraic group G; we remind the reader that a Borel subgroup $B \subset G$ is a maximal closed, connected, reduced, solvable subgroup scheme.

Theorem 2.8. *(cf. [5],[18, 3.1]) Let G be an affine algebraic group and $B \subset G$ be a Borel subgroup. Then for any G-module M, the natural restriction map*

$$H^*(G, M) \to H^*(B, M)$$

is an isomorphism.

Example 2.9. A construction of G. Hochschild provides a natural map

$$g^{\#(1)} \to H^2(V(g), k),$$

where $g^{\#}$ is the linear dual of the p-restricted Lie algebra g. Namely, $H^2(V(g), k)$ can be naturally identified with isomorphism classes of extensions of p-restricted Lie algebras of the form

$$1 \to k \to \tilde{g} \to g \to 1,$$

where k is equipped with the trivial p-restriction as well as trivial Lie bracket. For any linear map $\psi : g \to k$, we define following Hochschild the p-restricted Lie algebra $\tilde{g}$ with Lie algebra structure the direct sum $k \oplus g$ and with p-restriction given by $(\alpha, X)^{[p]} = (\psi(X)^p, X^{[p]})$.

Let G be an affine group scheme and let I denote the augmentation ideal of $k[G]$, the maximal ideal at the identity $e \in G$. Then we set

$$gr(k[G]) = \oplus_{n \geq 0} I^n / I^{n-1},$$

and readily verify that the commutative Hopf algebra structure on $k[G]$ determines a commutative Hopf algebra structure on $gr(k[G])$. We denote the associated affine group scheme by $gr(G)$. If M is a G-module, then the standard "Hochschild complex" $C^*(G, M)$ admits an associated filtration whose associated graded complex is the Hochschild complex $C^*(gr(G), M)$ where M is viewed as a trivial $gr(G)$-module. This leads to the following general form of the "May spectral sequence."

Theorem 2.10. *(cf. [14, 9.13] For any affine group scheme G and G-module M, there is a natural first quadrant spectral sequence of cohomological type*

$$E_1^{s,t}(M) = H^{s+t}(gr(G)_{(s)} \otimes M) \Longrightarrow H^{s+t}(G, M).$$

For $G = GL_{n(r)}$, this specializes to

$$E_1^{*,*}(M) = \bigotimes_{i=1}^{r} S^*(gl_n^{\#(i)}[2]) \otimes \Lambda^*(gl_n^{\#(i-1)}[1]) \otimes M \Longrightarrow H^*(GL_{n(r)}, M),$$

where $S^*(gl_n^{\#(i)}[2])$ denotes the symmetric algebra generated by the vector space $gl_n^{\#(i)}$ in degree 2, $\Lambda^*(gl_n^{\#(i-1)}[1])$ the exterior algebra generated by $gl_n^{\#(i-1)}$ in degree 1, and the notation specifies the structure of the spectral sequence with its GL_n-action.

Example 2.11. We apply Example 2.9 and Theorem 2.10 to sketch a computation of $H^*(V(gl_n), k) = H^*(GL_{n(1)}, k)$ for $p \geq n$. Even though this sketch omits several somewhat difficult arguments, it can serve to suggest the manner in which computations can be made.

The May spectral sequence for $GL_{n(1)}$ and $M = k$ has the form

$$E_1^{2s,t}(k) = S^s(gl_n^{\#(1)}[2]) \otimes \Lambda^t(gl_n, k) \Longrightarrow H^{2s+t}(GL_{n(1)}, k).$$

where $H^t(gl_n, k) = H^t(\Lambda^*(gl_n^\#))$ is the Lie algebra cohomology of gl_n. The Hochschild construction of Example 2.9 implies that $E_1^{2,0}(k) = S^1(gl_n^{\#(1)}[2])$ consists of permanent cycles; by multiplicativity of the spectral sequence, we conclude that $E_1^{*,0}(k) = S^*(gl_n^{\#(1)}[2])$ consists of permanent cycles. A direct computation of $d_1^{0,*}$ implies that $E_2^{0,*}(k) = H^*(gl_n, k)$, the cohomology of the universal enveloping algebra of the Lie algebra gl_n. Thus, the E_2-page of the May spectal sequence has the form

$$E_2^{2s,t}(k) = S^s(gl_n^{\#(1)}[2]) \otimes H^t(gl_n, k) \Longrightarrow H^{2s+t}(GL_{n(1)}, k).$$

As verified in [10, 1.1] if $p > n$ then

$$H^*(gl_n, k) = (\Lambda^*(gl_n^\#))^{(GL_n)_1}$$

is an exterior algebra on generators in degrees $1, 3, \ldots, 2n-1$, whereas the latter is shown in [2] (cf. [14, 12.10]) to be isomorphic to $(\Lambda^*(gl_n^\#))^{GL_n}$. We assume inductively that the first i generators of $H^*(gl_n, k)$ transgress to some non-zero element

of $E_{2i}^{*,0}(k)$. An argument of Borel enables us to conclude that on the E_{2i+1}-page we have $E_{2i+1}^{*,j}(k) = 0, 0 < j \leq i$, so that $E_{2i+1}^{*,0}(k)$ is the quotient of $S^*(gl_n^{\#(1)}[2])$ by the ideal generated by the transgressions of elements of $\oplus_{j=1}^{i} H^j(gl_n, k)$.

On the other hand, $H^{*>0}(GL_n, k) = 0$; this is a special case of the usual Kempf vanishing theorem, but could be rederived using Theorem 2.8 and a weight argument showing $H^{*>0}(U_n, k)^{T_n} = 0$. The GL_n invariance of $H^*(gl_n, k)$ implies that $i + 1$-st generator of $H^*(gl_n, k)$ must transgress to some non-zero element of $E_{2i+2}^{2i,0}(k)$. We conclude that $H^*(GL_{n(1)}, k)$ is isomorphic to $S^*(gl_n^{(1)\#}[2])$ modulo the ideal generated by the transgressions of $H^{*>0}(gl_n, k)$ (which necessarily equals the ideal generated by the $GL_n^{(1)}$-invariant elements of positive degree). (See [11] for details.)

We conclude this lecture by examining the fundamental class

(2.12) $e_1 \in H^2(GL_n, gl_n^{(1)}) \simeq Ext^2_{GL_n}(V_n^{(1)}, V_n^{(1)})$

where gl_n denotes the adjoint module (i.e., n^2-dimensional vector space of $n \times n$ matrices with GL_n acting via conjugation) and V_n is the natural GL_n-module associated to the identity representation $GL_n \to Aut_k(V_n)$. This fundamental class enables a straight-forward proof of the finite generation of $H^*(GL_{n(1)}, k)$ (cf. Theorem 4.1). The role of strict polynomial functors and their cohomology in Lecture 4 will be to establish suitable higher order fundamental classes e_r which will enable the proof of finite generation of $H^*((GL_{n(r)}, k)$ and thus the cohomology of any finite group scheme.

First, observe that

$$H^2(GL_n, gl_n) = H^2(B_n, gl_n) = 0 = H^2(U_n, gl_n)^{T_n} = 0$$

because no weight of $H^2(U_n, k)$ is the negative of a weight of gl_n provided that $p > n$. (This can be verified as in Example 2.6 using the computation of Theorem 2.4 or more directly using the May spectral sequence.) This emphasizes the role of Frobenius twists. The possibility that $H^2(GL_n, gl_n^{(1)})$ is non-zero can be seen from our knowledge that the weights of $gl_n^{(1)}$ are p times the weights of gl_n.

Example 2.13. Let $W_2(k)$ denote the Witt vectors of length 2 over k, so that $W_2(k)$ is the Artinian k-algebra whose underlying additive structure is as a non-trivial extension of k by k. (Thus, $W_2(\mathbb{F}_p) = \mathbb{Z}/p^2\mathbb{Z}$.) Then we have an extension of affine group schemes over k,

$$1 \to gl_n^{(1)} \to GL_{n,W_2(k)} \to GL_n \to 1$$

which corresponds to a class in $H^2(GL_n, gl_n^{(1)})$; since this extension does not split, this class is non-trivial and is one representation of our fundamental class e_1.

Another representation of the class e_1 uses the May spectral sequence of Theorem 2.10 for $G = GL_{n(1)}$ and $M = gl_n^{(1)}$. There is a canonical $GL_n^{(1)}$-invariant "identity element" $id \in gl_n^{(1)} \otimes gl_n^{\#(1)} \simeq (E_2^{2,0})(gl_n^{(1)})$ which determines a class in $H^2(GL_n, gl_n^{(1)})$ using the L-H-S spectral sequence for the short exact sequence

$$1 \to GL_{n(1)} \to GL_n \to GL_n^{(1)} \to 1.$$

As yet another representation of e_1, we consider the exact sequence of GL_n-modules

(2.14) $0 \to V_n^{(1)} \to S^p(V_n) \to \Gamma^p(V_n) \to V_n^{(1)} \to 0,$

where $S^p(V_n) = (V_n^{\otimes p})/\Sigma_p$ is the p-th symmetric power of V_n represented concretely by the vector space of polynomials in n variables homogeneous of degree p, and $\Gamma^p(V_n) = (V_n^{\otimes p})^{\Sigma_p} = (S^p(V_n))^{\#}$. The map $V_n^{(1)} \to S^p(V_n)$ is given by $v \mapsto v^p$, the map $\Gamma^p(V_n) \to V_n^{(1)}$ is the dual of this map, and the map $S^p(V_n) \to \Gamma^p(V_n)$ is the symmetrization map. The extension (2.14) corresponds to the class

$$e_1 \in Ext^2_{GL_n}(V_n^{(1)}, V_n^{(1)}) \simeq H^2(GL_n, gl_n^{(1)}).$$

3. Polynomial modules and functors

In this lecture, we restrict our attention to GL_n. A GL_n-module is frequently called a *rational representation*, for the data necessary to provide an N-dimensional vector space with the structure of a GL_n-module consists of N^2-matrix coefficients viewed as regular functions on GL_n. Regular functions on GL_n can in turn be viewed as rational functions in the n^2 matrix coordinates of GL_n. Should these N^2 rational functions all be polynomial functions of the matrix coordinates of GL_n, namely lie in

(3.1) $\qquad k[M_n] = k[x_{i,j}]_{1 \le i,j \le n} \subset k[x_{i,j}; t]_{1 \le i,j \le n}/det(x_{i,j})t - 1 = k[GL_n],$

then the GL_n-module is said to be a *polynomial module* (or a polynomial representation of GL_n).

In this lecture, we shall see how to interpret such polynomial modules and their cohomology in terms of "strict polynomial functors" and we shall see how this functor point of view affords computational advantages. The formulation of strict polynomial functors is at first somewhat daunting, but the reader should keep in mind the fact that these functors are so defined in order to play the same role in connection with polynomial representations of GL_n as the role played by more familiar polynomial functors in connection with representations of the discrete group $GL_n(k)$.

We begin with the definition of the *Schur algebra*.

Definition 3.2. Let n, d be position integers, consider the Hopf algebra $k[M_n]$ of (3.1), and let $k[M_n]_d \subset k[M_n]$ denote the subspace of homogeneous polynomials of degree d. Then $k[M_n]_d$ is closed under the coproduct of $k[M_n]$ and its linear dual (which is a finite dimensional k-algebra)

$$S(n, d) = (k[M_n]_d)^{\#}$$

is called the Schur algebra (of rank n and degree d).

A module for $S(n, d)$ is called a polynomial module for GL_n homogeneous of degree d.

Thus, a module for $S(n, d)$ is a comodule for $k[M_n]_d \subset k[GL_n]$, a GL_n-module whose matrix coefficients are homogenous polynomial functions of the matrix coordinates of GL_n. For future reference, we recall that

(3.3) $\qquad S(n, d) = (S^d(End_k(k^n))^{\#} = \Gamma^d(End_k(k^n))$

$$= ((End_k(k^n))^{\otimes d})^{\Sigma_d} = End_{k\Sigma_d}((k^n)^{\otimes d})$$

Let $\mathcal{P}ol_{n,d} \subset (Mod_{GL_n})$ denote the full-subcategory of polynomial modules for GL_n homogeneous of degree d. Then essentially by definition we have an equivalence of categories

(3.4) $\qquad\qquad\qquad \mathcal{P}ol_{n,d} \simeq (Mod_{S(n,d)})$

between this category and the category of modules for the Schur algebra $S(n, d)$.

The following theorem tells us that GL_n-cohomology of polynomial modules can be computed as the cohomology of Schur algebras.

Theorem 3.5. *(cf. [6], [13, 3.12.1]) Let*

$$\mathcal{P}ol_n = \oplus_{d \geq 0} \mathcal{P}ol_{n,d} \subset (Mod_{GL_n})$$

denote the category of polynomial modules for GL_n.

(1) *A GL_n-module is polynomial if and only if all of its T_n-weights are non-negative.*

(2) *Any polynomial module M for GL_n is canonically a direct sum of polynomial modules for GL_n homogeneous of degree d, $M = \oplus_{d \geq 0} M_d$. Moreover, the homogenous summand of degree d is the weight space of degree d of the GL_n-module with respect to the scalar matrices $\mathbb{G}_m \subset T_n$.*

(3) *If M, N are polynomial modules for GL_n, then*

$$Ext^*_{GL_n}(M, N) \simeq Ext^*_{\mathcal{P}ol_n}(M, N).$$

Let $\mathcal{V}_k$ denote the category of vector spaces over k and let $(\mathcal{V}_k)^f \subset \mathcal{V}_k$ denote the full subcategory of finite dimensional k-vector spaces. As defined below, a strict polynomial functor is a collection of polynomial modules M_n for GL_n for each $n \geq 1$ together with compatibility of actions as n varies.

Definition 3.6. A strict polynomial functor $T : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ is the data of an association

$$T(V) \in (\mathcal{V}_k)^f, \quad \forall V \in (\mathcal{V}_k)^f$$

together with maps of affine schemes

$$T_{V,W} : Hom_k(V, W) \to Hom_k(T(V), T(W)), \quad \forall V, W \in (\mathcal{V}_k)^f$$

satisfying the following:

- $T_{V,V}(id_V) = id_{T(V)}, \forall V \in (\mathcal{V}_k)^f$.
- $\forall U, V, W \in (\mathcal{V}_k)^f$,

$$
\begin{array}{ccc}
Hom_k(U, V) \times Hom_k(V, W) & \longrightarrow & Hom_k(U, W) \\
\scriptstyle{T_{U,V} \times T_{V,W}} \downarrow & & \downarrow \scriptstyle{T_{U,W}} \\
Hom_k(T(U), T(V)) \times Hom_k(T(V), T(W)) & \longrightarrow & Hom_k(T(U), T(W))
\end{array}
$$

commutes, where the horizontal maps are given by composition.

If T is a strict polynomial functor with the property that $T_{V,W}$ has degree bounded by some integer which can be chosen independent of V, W, then we say that T has bounded degree; if each $T_{V,W}$ is homogeneous of degree d, then we say that T is homogeneous of degree d.

We denote by $\mathcal{P}_d$ the category of strict polynomial functors homogeneous of degree d, and $\mathcal{P}$ the category of strict polynomial functors of bounded degree.

Remark 3.7. If the field k is infinite, then a strict polynomial functor T can be described more simply as a functor $T : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ with the property that each $T_{V,W} : Hom_k(V, W) \to Hom_k(T(V), T(W))$ is a polynomial function (i.e., a map of sets having the property that with respect to a choice of bases for $Hom_k(V, W), Hom_k(T(V), T(W))$ the coordinates of $T_{V,W}(f)$ are polynomial in the coordinates of $f \in Hom_k(V, W)$).

Observe that a map of affine schemes

$$F : Hom_k(V, W) \to Hom_k(T(V), T(W))$$

is equivalent to a map of coordinate algebras

$$F^* : S^*(Hom_k(T(V), T(W))^\#) \to S^*(Hom_k(V, W)^\#)$$

which is equivalent to a linear map of k-vector spaces

$$Hom_k(T(V), T(W))^\# \to S^*(Hom_k(V, W)^\#).$$

To say that F is homogeneous of degree d is to say that this last map has image in $S^d(Hom_k(V, W)^\#) \subset S^*(Hom_k(V, W)^\#)$, so that the data associated to this map is equivalent to a linear map of the form $\Gamma^d(Hom_k(V, W)) \to Hom_k(T(V)), T(W))$.

Thus, if T is a strict polynomial functor homogeneous of degree d, we may replace the structure maps $T_{V,W}$ by equivalent linear maps which we shall continue to denote by $T_{V,W}$,

$$T_{V,W} : \Gamma^d(Hom_k(V, W)) \to Hom_k(T(V)), T(W)).$$

Proposition 3.8. *Let T be a strict polynomial functor homogenous of degree d. Then $T(k^n)$ has the natural structure of a polynomial module for GL_n for each $n \geq 0$,*

Proof. If A is any finitely generated commutative k-algebra, then we may define $T(A \otimes V)$ to be $A \otimes T(V)$ and we may consider the base-change of $T_{V,W}$ to obtain

$$T_{V,W} \otimes A : Hom_A(A \otimes V, A \otimes W) \to Hom_A(A \otimes T(V), A \otimes T(W)).$$

In particular, the composition

$$GL_n(A) \subset Hom_A(A \otimes k^n, A \otimes k^n) \to Hom_A(A \otimes T(k^n), A \otimes T(k^n))$$

for varying A determines a GL_n-module structure on $T(k^n)$ which by construction is polynomial. $\square$

Example 3.9. We give some common examples of strict polynomial functors.
   (1) The identity $I : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ is a strict polynomial functor homogeneous of degree 1.
   (2) For any $r \geq 1$, $I^{(r)} : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ given by $V \mapsto V^{(r)}$ is a strict polynomial functor homogeneous of degree p^r.
   (3) For any $d > 0$, $\otimes^d : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ given by $V \mapsto V^{\otimes^d}$ is a strict polynomial functor homogeneous of degree d.
   (4) For any $d > 0$, $\Lambda^d : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ given by $V \mapsto \Lambda^d(V)$ is a strict polynomial functor homogeneous of degree d.
   (5) For any $d > 0$, $S^d : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ given by $V \mapsto S^d(V)$ is a strict polynomial functor homogeneous of degree d.
   (6) For any $d > 0$, $\Gamma^d : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ given by $V \mapsto \Gamma^d(V)$ is a strict polynomial functor homogeneous of degree d. More generally, for any $n, d > 0$, $\Gamma^d(Hom_k(k^n, -))$ is a strict polynomial functor of degree d.
   (7) If $T : (\mathcal{V}_k)^f \to (\mathcal{V}_k)^f$ is a strict polynomial of degree d, then $T^\#$ given by $V \mapsto T(V^\#)^\#$ is also a strict polynomial functor of degree d. Moreover, T is a projective object of the category $\mathcal{P}$ of strict polynomial functors of bounded degree if and only if $T^\#$ is an injective object of $\mathcal{P}$.

The following proposition makes more explicit various homological algebra constructions in $\mathcal{P}$, our category of strict polynomial functors of bounded degree. Observe that if T is a strict polynomial functor homogeneous of degree d then there is a natural map (i.e., natural transformation of functors)

$$(3.10) \qquad\qquad T(k^n) \otimes \Gamma^d(Hom_k(k^n, -)) \to T$$

given for each $V \in (\mathcal{V}_k)^f$ as the adjoint of $T_{k^n, V} : \Gamma^d(Hom_k(k^n, V)) \to Hom_k(T(k^n), T(V))$.

Proposition 3.11. *(cf. [13, 2.10] The category $\mathcal{P}$ of strict polynomial functors of bounded degree is isomorphic to the direct sum of categories strict polynomial functors homogeneous of degree d for $d > 0$,*

$$\mathcal{P} \simeq \oplus_d \mathcal{P}_d.$$

Moreover, for any $n > 0$, the functor $\Gamma^d(Hom_k(k^n, -)) \in \mathcal{P}_d$ is a projective object.

If T is a strict polynomial functor homogeneous of degree d, then the natural map (3.10) is surjective provided that $n \geq d$. Thus, for $n \geq d$, $\Gamma^d(Hom_k(k^n, -))$ is a projective generator of $\mathcal{P}_d$.

We now formulate the theorem that tells us that we can compute Ext-groups of polynomial GL_n-modules in terms of Ext-groups of strict polynomial functors.

Theorem 3.12. [13, 3.2] *For positive integers $n \geq d$, there are natural equivalences of abelian categories (with enough injective and projective objects)*

$$\mathcal{P}_d \simeq (Mod_{S(n,d)}) \simeq \mathcal{P}ol_{n,d}.$$

Consequently, for any pair of strict polynomial functors S, T homogeneous of degree d, there are natural isomorphisms of graded groups

$$(3.13) \qquad\qquad Ext^*_{\mathcal{P}_d}(S, T) \simeq Ext^*_{\mathcal{P}ol_{n,d}}(S(k^n), T(k^n)) \simeq$$

$$Ext^*_{S(n,d)}(S(k^n), T(k^n)) \simeq Ext^*_{GL_n}(S(k^n), T(k^n)).$$

Proof. (Outline of proof.) The map $\mathcal{P}_d \to (Mod_{S(n,d)})$ is given by $T \mapsto T(k^n)$. The action of $S(n, d) = \Gamma^d(End_k(k^n))$ (cf. (3.3)) on $T(k^n)$ is given by (3.10). The proof that this is an equivalence of categories is more or less a direct computation using the explicit inverse sending $(Mod_{S(n,d)}) \to \mathcal{P}_d$ given by $M \mapsto \Gamma^d(Hom(k^n, -)) \otimes M$. The equivalence $(Mod_{S(n,d)}) \simeq \mathcal{P}ol_{n,d}$ is that of 3.4.

The first three isomorphisms of (3.13) follow from the equivalences of categories. The last is given by (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude this lecture by mentioning a few of the computational advantages one has when computing $Ext_{\mathcal{P}}$-groups. One is the existence of *complexes of functors* (discussed in some detail in other lectures). For example, one has the (exact) Koszul complex

$$(3.14) \qquad 0 \to \Lambda^d \to \Lambda^{d-1} \otimes S^1 \to \cdots \to \Lambda^1 \otimes S^{d-1} \to S^d \to 0$$

and the not necessarily exact DeRahm complex

$$(3.15) \qquad 0 \to S^d \to \Lambda^1 \otimes S^{d-1} \to \cdots \to \Lambda^{d-1} \otimes S^1 \to \Lambda^d \to 0$$

A second advantage is the very concrete nature of injectives and projectives. For example, the functors S^d are injective and thus cohomologically acyclic.

A third useful computational tool, especially in conjunction with the above complexes, is the following acyclicity result. This result appears in much strengthened form in [8, 1.7].

Theorem 3.16. *(cf.* [9], [13, 2.13] *Let T, T' be homogenous strict polynomial functors of positive degree and let A be an additive functor (e.g., a Frobenius twist of a strict polynomial functor homogeneous of degree 1). Then*

$$Ext_{\mathcal{P}}^*(A, T \otimes T') = 0.$$

4. Finite generation of cohomology

In this lecture, we outline the proof of the finite generation of the cohomology of finite group schemes, Theorem 4.7. We first sketch the proof of finite generation for infinitesimal group schemes of height 1, for this special case introduces the general method of proving finite generation. We then discuss the existence and basic properties of the fundamental classes

$$e_r \in Ext_{\mathcal{P}}^{2p^{r-1}}(I^{(r)}, I^{(r)}).$$

Using these classes, we then sketch the proof of finite generation of $H^*(GL_{n(r)}, k)$. Finally, we discuss the relatively straight-forward manner in which finite generation of $H^*(GL_{n(r)}, k)$ implies the finite generation of $H^*(G, k)$ for any finite group scheme G.

The following theorem was first formulated and proved in [11] although the result might well have been known previously.

Theorem 4.1. [11] *Let G be an infinitesimal group scheme of height 1 (i.e., $k[G]$ is a finite connected algebra whose maximal ideal consists of elements whose p-th power is 0) and let M be a finite dimensional G-module. Then $H^*(G, k)$ is a finitely generated algebra and $H^*(G, M)$ is a finite module over $H^*(G, k)$.*

Proof. (Sketch of proof.) As in Example (2.11), the Hochschild construction of Example 2.9 implies that the May spectral sequence of Theorem 2.10 has the form

$$E_2^{2s,t}(M) = S^s(g^{(1)\#}) \otimes H^t(g, M) \Longrightarrow H^{2s+t}(G, M).$$

Here $H^*(g, M)$ is the Lie algebra cohomology of $g = Lie(G)$ (i.e., the cohomology of the universal enveloping algebra $U(g)$ of g). The "shape" of this spectral sequence implies that $S^*(g^{(1)\#}) = E_2^{*,0}(k)$ consists of "permanent cycles" (i.e., the differentials d_r vanish on $E_r^{*,0}(k)$). This implies that $E_r^{*,*}(M)$ is a module over $S^*(g^{(1)\#})$ and that d_r is a homomorphism of $S^*(g^{(1)\#})$-modules.

Now assume that M is finite dimensional. Then $E_2^{*,*}(M)$ is a finite $S^*(g^{(1)\#})$-module. Since $E_r^{*,*}(M)$ is a subquotient of $E_{r-1}^{*,*}(M)$, we conclude that each $E_r^{*,*}(M)$ and thus also $E_\infty^{*,*}(M)$ are finite $S^*(g^{(1)\#})$-modules. In particular, $E_\infty^{*,*}(k)$ is a finite $S^*(g^{(1)\#})$-modules which implies that $E_\infty^{*,*}(k)$ is finitely generated which implies by a result of L. Evens [7, 2.1] that $H^*(G, k)$ is finitely generated. Moreover, the spectral sequence $\{E_r^{*,*}(M)\}$ is a module over the spectral sequence (of algebras) $\{E_r^{*,*}(k)\}$, so that the action of $S^*(g^{(1)\#})$ on $E_r^{*,*}(M)$ factors through $E_r^{*,*}(k)$ and thus the action of $S^*(g^{(1)\#})$ on $H^*(G, M)$ factors through $H^*(G, k)$. We therefore conclude that $H^*(G, M)$ is a finite $H^*(G, k)$-module. $\square$

To extend this argument to more general finite group schemes G, we require a finitely generated subalgebra of $E_2^{*,*}(k)$ for the May spectral sequence consisting of

permanent cycles and with respect to which the $E_r^{*,*}(M)$ is a finite module. We can no longer argue that the shape of May spectral sequence guarantees the existence of such an algebra. Instead, we construct explicit generators of such an algebra, the fundamental classes $\{e_i\}$, whose basic properties suffice to guarantee that they generate such an algebra.

The following complete calculation of $Ext_{\mathcal{P}}^*(I^{(r)}, I^{(r)})$ is the heart of the proof of finite generation. The proof follows closely the arguments of [9].

Theorem 4.2. [13, 4.10] *The Ext-algebra $Ext_{\mathcal{P}}^*(I^{(r)}, I^{(r)})$ is a commutative k-algebra generated by elements*

$$e_i^{(r-i)} \in Ext_{\mathcal{P}}^{2p^{i-1}}(I^{(r)}, I^{(r)})$$

subject only to the relations $(e_i^{(r-i)})^p = 0$.

Proof. (Comments on proof.) We use all of the computational tools mentioned at the end of Lecture 3. Namely, we proceed by induction first on r and then for a given r by induction on j to compute $Ext_{\mathcal{P}}^*(I^{(r)}, S^{p^{r-j}(j)})$. Inputs to this computation include the vanishing of $Ext_{\mathcal{P}}^{*>0}(-, S^d)$ for any $d \geq 0$ because of the injectivity of S^d, the vanishing of $Ext_{\mathcal{P}}^*(I^{(r)}, \Lambda^i \otimes S^j)$ for $i, j > 0$ by Theorem 3.16, and the exactness of the Koszul complex (3.14). One additional input which enables this computation is a theorem of P. Cartier [4] which determines the cohomology of the DeRham complex (3.15); namely, the DeRham complex is acyclic if $(p, d) = 1$ and equals the first Frobenius twist of the DeRham complex relating $S^{\frac{d}{p}}$ to $\Lambda^{\frac{d}{p}}$ if $p|d$. $\square$

Further work with Ext-groups in the category $\mathcal{P}$ of strict polynomial functors of bounded degree verifies that e_r is related i n a natural way to a power of e_1.

Theorem 4.3. [13, 5.7] *The image of*

$$(e_1^{p-1})^{\otimes p^{r-1}} \in (Ext_{\mathcal{P}}^{2(p-1)}(I^{(1)}, I^{(1)}))^{\otimes p^{r-1}}$$

is a scalar multiple of the image of

$$e_r^{p-1} \in Ext_{\mathcal{P}}^{2(p-1)p^{r-1}}(I^{(r)}, I^{(r)})$$

in $Ext_{\mathcal{P}}^{2(p-1)p^{r-1}}(\Gamma^{p^{r-1}(1)}, S^{p^{r-1}(1)})$.

Theorem 4.3 enables us to conclude the existence of non-zero classes in the cohomology of GL_n which restrict non-trivially to the cohomology of $GL_{n(1)}$.

Theorem 4.4. [13, 6.2] *For any $n \geq 2, r \geq 1$, the image of e_r under the composition*

$$Ext_{\mathcal{P}}^{2p^{r-1}}(I^{(r)}, I^{(r)}) \to Ext_{GL_n}^{2p^{r-1}}(V_n^{(r)}, V_n^{(r)}) =$$
$$H^{2p^{r-1}}(GL_n, gl_n^{(r)}) \to H^{2p^{r-1}}(GL_{n(1)}, k) \otimes gl_n^{(r)}$$

is non-zero.

Theorem 4.4 together with a bit more work implies the following corollary.

Corollary 4.5. *The class $e_r \in H^{2p^{r-1}}(GL_n, gl_n^{(r))})$ restricts to a non-trivial class in $H^{2p^{r-1}}(GL_{n(r)}, k) \otimes gl_n^{(r)}$. We view this restriction as a non-zero map*

$$e_r : gl_n^{(r)\#} \to H^{2p^{r-1}}(GL_{n(r)}, k).$$

This map annihilates the 1-dimensional GL_n-invariant subspace of $gl_n^{(r)\#}$. Moreover, its composition with the restriction map to $H^{2p^{r-1}}(GL_{n(1)}, k)$ is given up to non-zero scalar multiple as the composition

$$gl_n^{(r)\#} \to S^{p^{r-1}}(gl_n^{(1)\#}[2]) \to H^{2p^{r-1}}(GL_{n(1)}, k)$$

where the first map is the p^{r-1}-st power map and the second is the edge homomorphism in the May spectral sequence of Theorem 2.10.

Corollary 4.5 enables us to adapt the proof of Theorem 4.1 to provide a proof of finite generation of cohomology of $GL_{n(r)}$.

Theorem 4.6. *Let $n \geq 2, r \geq 1$ and let M be a finite dimensional $GL_{n(r)}$-module. Then $H^*(GL_{n(r)}, k)$ is a finitely generated algebra and $H^*(GL_{n(r)}, M)$ is a finite module over $H^*(GL_{n(r)}, k)$.*

Proof. (Sketch of proof.) As in the proof of Theorem 4.1, we analyze the May spectral sequences of Theorem 2.10, $\{E_r^{*,*}(k)\}$ for $H^*(GL_{n(r)}, k)$ and $\{E_r^{*,*}(M)\}$ for $H^*(GL_{n(r)}, M)$.

Let

$$S^*(gl_n^{(r)\#}[2p^{r-i}]) \subset S^*(gl_n^{(i)\#}[2])$$

denote the poynomial subalgebra generated by the subspace $gl_n^{(r)} \subset S^{p^{r-i}}(gl_n^{(i)\#}[2])$ of p^{r-i}-th powers of $gl_n^{(i)\#}[2]$. Then Corollary 4.5 together with the evident naturality of our constructions with respect to $GL_{n(r)} \to GL_{n(r)}/GL_{n(r-1)} \simeq GL_{n(1)}$ implies that

$$\bigotimes_{i=1}^{r} S^*(gl_n^{(r)\#}[2p^{r-i}]) \otimes M \ \subset \ E_1^{*,*}(M)$$

consists of permanent cycles.

Clearly, $E_1^{*,*}(M)$ is a finite $\otimes_{i=1}^{r} S^*(gl_n^{(r)\#}[2p^{r-i}])$ module. As argued in the proof of Theorem 4.1, this implies that $E_\infty^{*,*}(k)$ is a finitely generated algebra and thus also $H^*(GL_{n(r)}, k)$ is also finitely generated. Since the action of $\otimes_{i=1}^{r} S^*(gl_n^{(r)\#}[2p^{r-i}])$ on $H^*(GL_{n(r)}, M)$ factors through $H^*(GL_{n(r)}, k)$, we conclude that $H^*(GL_{n(r)}, M)$ is a finite $H^*(GL_{n(r)}, k)$ module. $\square$

We are now in a position to outline the remainder of the proof of finite generation of $H^*(G, k)$ for an arbitrary finite group scheme. This proof relies on earlier work of L. Evens [7] who, together with B. Venkov [19], proved the finite generation of the cohomology algebra of a finite group.

Theorem 4.7. [13, 1.1] *Let G be a finite group scheme and let M be a finite dimensional G-module. Then $H^*(G, k)$ is a finitely generated algebra and $H^*(G, M)$ is a finite module over $H^*(G, k)$.*

Proof. (Outline of proof.) If G an an infinitesimal group scheme of height $\leq r$, then G admits an embedding as a closed subgroup scheme of $GL_{n(r)}$. Shapiro's Lemma,

$$H^*(G, M) \simeq H^*(GL_{n(r)}, Ind_G^{GL_{n(r)}} M),$$

in conjunction with Theorem 4.6 easily implies the assertions of the theorem for such infinitesimal group schemes G.

For applications considered in the next lecture, we utilize a different proof of finite generation for G infinitesimal. Namely, a closed embedding $G \subset GL_{n(r)}$ induces a map of spectral sequences

$$E_1^{*,*}(GL_{n(r)}, k) = \bigotimes_1^r S^*(gl_n^{(i)\#}[2]) \otimes \bigotimes_1^r \Lambda^*(gl_n^{(i-1)\#}[1])$$

(4.8)

$$\downarrow$$

$$E_1^{*,*}(G, k) = \bigotimes_1^r S^*(g^{(i)\#}[2]) \otimes \bigotimes_1^r \Lambda^*(g^{(i-1)\#}[1])$$

which is surjective on $E_1^{*,*}$. Thus, the argument given in the proof of Theorem 4.6 applies, since it suffices to show that $H^*(G, k)$ is a finite module over $\otimes_{i=1}^r S^*(gl_n^{(r)\#}[2p^{r-i}])$ and is thus finitely generated. Similarly, $H^*(G, M)$ is finite as a $\otimes_{i=1}^r S^*(gl_n^{(r)\#}[2p^{r-i}])$ module and thus also as a $H^*(G, k)$ module.

Since $H^*(G, M) \otimes_k K = H^*(G_K, M \otimes_k K)$ for any field extension K/k, to prove finite generation for an arbitrary finite group scheme we may assume that k is algebraically closed. In this case, the split extension

$$1 \to G^o \to G \to \pi_0(G) \to 1$$

is necessarily a semi-direct product of an infinitesimal group scheme and a finite group. Then, one readily adapts results of [7] to conclude finite generation for G knowing finite generation for G^o and using the Evens-Venkov theorem asserting finite generation for $\pi_0(G)$. (see [13, 1.9,1.10] for details). $\square$

5. Qualitative description of $H^{ev}(G, k)$

In [15], D. Quillen described the maximal ideal spectrum $|\pi|$ of the commutative algebra $H^{ev}(\pi, k)$ for a finite group π in terms of the elementary abelian p-subgroups of π. This remains a remarkable work, both for introducing the possibility of identifying the maximal ideal spectrum as well as for the completeness of the result. For example, Quillen gives us an explicit description of the maximal ideal spectrum of $H^{ev}(GL_n(\mathbb{F}_q), k), q = p^d$, even though we know very little about the individual cohomology groups $H^i(GL_n(\mathbb{F}_q, k)$. (For example, we do not even know what is the smallest positive degree such that $H^i(GL_n(\mathbb{F}_q), k) \neq 0$.) It is interesting to note that Quillen also observed that $H^i(GL_\infty(\mathbb{F}_q), k) = 0$ for $i > 0$, a fact which is closely related to the fact that $k = S^0 \in \mathcal{P}$ is acyclic.

Theorem 5.1. [15] *Let π be a finite group, assume that k is algebraically closed, and let $|\pi|$ denote the maximal ideal spectrum of $H^{ev}(\pi, k)$. Then the natural map*

$$\varinjlim_{\{E \to \pi\}} |E| \to |\pi|.$$

is a homeomorphism, where the indexing category for the colimit is the category whose objects are elementary abelian subgroups of π and whose maps are compositions of group inclusions and maps induced by conjugations by elements of π.

Recall that if E is an elementary abelian p-group of rank n, then $H^*(E, k) \simeq k[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n)$ where each $x_i \in H^2(E, k), y_i \in H^1(E, k)$ for $p \neq 2$ (for $p = 2$, $H^*(E, k) \simeq k[y_1, \ldots, y_n]$ with each $y_i \in H^1(E, k)$). Thus, $|E|$ is an affine space of dimension n. Theorem 5.1 tells us that the Krull dimension of the

commutative ring $H^{ev}(\pi, k)$ equals the maximal rank among elementary abelian p-subgroups of π. We can restate Theorem 5.1 as asserting that $|\pi|$ is the identification space of the following projection

$$(5.2) \qquad \coprod_{E \ max} |E|/W_E \to |\pi|,$$

where the coproduct is indexed by conjugacy classes of maximal elementary abelian p-subgroups of π and where W_E denotes the normalizer of E modulo its centralizer as a subgroup of π. Moreover, points of $e \in |E|/W_E, e' \in |E'|/W_{E'}$ are mapped via (5.2) to the same point of $|\pi|$ if and only if there exist conjugates $\tilde{E}, \tilde{E}'$ of E, E' and a point $e'' \in |\tilde{E} \cap \tilde{E}'|/W_{\tilde{E} \cap \tilde{E}'}$ mapping to e, e'.

To prove Theorem 5.1, Quillen proves i.) that the map from the coproduct is surjective by showing that any cohomology class $\zeta \in H^*(\pi, k)$ which restricts to $0 \in H^*(E, k)$ for every elementary abelian subgroup $E \subset \pi$ is nilpotent; ii.) that any point of $|E|/W_E$ not in the image of $|E'|/W_{E'}$ with E' a proper subgroup of E maps injectively into $|\pi|$ by showing that any class in a certain localization of $H^{ev}(E, k)^{W_E}$ admits a p-th power in the image of $H^{ev}(\pi, k)$.

As first observed by Friedlander-Parshall, the maximal ideal spectrum of the even dimensional cohomology of a finite dimensional restricted Lie algebra also has an explicit description. Conditions on the prime p required by Friedlander-Parshall were relaxed by Andersen-Janzten and eliminated altogether by Suslin-Friedlander-Bendel.

Theorem 5.3. *(cf. [11],[2],[18]) Let G be an infinitesimal group scheme of height 1, let $g = Lie G$, and assume that k is algebraically closed. Denote by $N_p(g) \subset g$ the p-nilpotent cone of g, the set of elements $x \in g$ satisfying $x^{[p]} = 0$. Then there is a natural homeomorphism*

$$\Psi : N_p(G) \xrightarrow{\sim} |G|$$

where $|G|$ denotes the maximal ideal spectrum of $H^{ev}(G, k)$.

Theorem 5.3 was generalized to arbitrary infinitesimal group schemes in two papers by Suslin-Friedlander-Bendel [17], [18] in a form which is more precise even in the height 1 case. (Namely, these papers deal with schemes rather than maximal ideal spectra. Among other advantages, this permits them to consider an arbitrary field k.) The schemes that generalize the variety $N_p(g)$ of Theorem 5.3 are introduced in the next proposition.

Proposition 5.4. [17, 1.5] *Let G be an affine group scheme. Then the functor on commutative k-algebras*

$$A \mapsto Hom_{Grps/A}(\mathbb{G}_{a(r)} \otimes A \to G \otimes A)$$

is representable by an affine scheme $V_r(G)$.
   For $G = GL_n$,

$$V_r(GL_n)(k) = \{(\alpha_1, \ldots, \alpha_r) \in M_n(k)^r \big| \alpha_i^p = 0 = [\alpha_i, \alpha_j]\}.$$

In the case $r = 1$, $V_1(G)$ is the scheme whose underlying variety is the p-nilpotent cone $N_p(Lie(G))$ considered in Theorem 5.3.

We call a homomorphism $\alpha : \mathbb{G}_{a(r)} \otimes A \to G \times A$ a *1-parameter subgroup* of height r defined over A.

Recall from Example 1.5 that $k\mathbb{G}_{a(r)} = k[X_1, \ldots, X_r]/(X_i^p)$ where X_i is the operator $\frac{1}{p^{i-1}!}\frac{d^{p^{i-1}}}{dt^{p^{i-1}}}$ on $k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r}$. We consider the map

$$\epsilon : k\mathbb{G}_{a(1)} \to \mathbb{G}_{a(r)}, \quad u \mapsto X_r,$$

where $u \in k\mathbb{G}_{a(1)}$ is the dual of $t \in k[t]/t^p = k[\mathbb{G}_{a(1)}]$. So defined, ϵ is not a map of Hopf algebras (i.e., does not commute with the coproduct), but does induce a map on cohomology

$$\epsilon^* : H^*(\mathbb{G}_{a(r)}, k) \to H^*(\mathbb{G}_{a(1)}).$$

Observe that a 1-parameter subgroup $\alpha : \mathbb{G}_{a(r)} \to G$ determines a homomorphism of graded algebras

$$\epsilon^* \circ \alpha^* : H^{ev}(G, k) \to H^{ev}(\mathbb{G}_{a(r)}, k) \to H^{ev}(\mathbb{G}_{a(1)}, k) \simeq k[t].$$

This determines a natural set-theoretic map

$$V_r(G) \to Proj(H^{ev}(G, k)), \quad \alpha \mapsto ker\{\epsilon^* \circ \alpha^*\},$$

where $Proj(H^{ev}(G, k))$ denotes the maximal non-trival homogeneous prime ideals of the graded algebra $H^{ev}(G, k)$. The following proposition asserts that this set-theoretic map admits a natural refinement as a map of schemes.

Proposition 5.5. [17, 1.14] *For any affine group scheme G, there is a natural homomorphism of graded commutative k-algebras*

(5.6) $$\psi : H^{ev}(G, k) \to k[V_r(G)]$$

which multiplies degrees by $p^r/2$.

In the case $r = 1$ and k algebraically closed, the map on affine varieties induced by ψ is the homeomorphism Ψ of Theorem 5.3.

Proof. The construction of this map is of sufficient independent interest that we sketch it here. Let

$$u : \mathbb{G}_{a(r)} \otimes k[V_r(G)] \to G \otimes k[V_r(G)]$$

correspond to

$$id_{k[V_r(G)]} \in V_r(G)(k[V_r(G)]) = Hom_{Grps/k[V_r(G)]}(\mathbb{G}_{a(r)} \otimes k[V_r(G)], G \otimes k[V_r(G)]).$$

Consider

$$u^* : H^*(G, k) \to H^*(G, k) \otimes k[V_r(G)] = H^*(G \otimes k[V_r(G)], k[V_r(G)])$$
$$\to H^*(\mathbb{G}_{a(r)} \otimes k[V_r(G)], k[V_r(G)]) = H^*(\mathbb{G}_{a(r)}, k) \otimes k[V_r(G)].$$

For any $\zeta \in H^{2j}(G, k)$, we define $\psi(\zeta)$ to be the coefficient of x_r^j, where $x_r = \epsilon^*(x) \in H^2(\mathbb{G}_{a(r)}, k)$ with $x \in H^2(\mathbb{G}_{a(1)}, k)$ the chosen polynomial generator. $\square$

We proceed to outline how Suslin-Friedlander-Bendel construct an "inverse modulo p-nilpotents" of ψ, thereby verifying that ψ determines a homeomorphism $\Psi : V_r(G) \to Spec\, H^{ev}(G, k)$ of prime ideal spectra. The following theorem explicitly exhibits such an "inverse" for ψ in the special case of $G = GL_{n(r)}$.

Theorem 5.7. [17, 5.2] *The fundamental classes $e_i \in H^{2p^{i-1}}(GL_n, gl_n^{(i)})$ determine a map of algebras*

$$\otimes_{i=1}^r e_i^{(r-i)} : \bigotimes_{i=1}^r S^*(gl_n^{(r)\#}[2p^{i-1}]) \to H^{ev}(GL_{n(r)}, k)$$

which factors through the quotient map $\otimes_{i=1}^{r} S^*(gl_n^{(r)\#}[2p^{i-1}]) \to k[V_r(G)]$ associated to the embedding of $V_r(GL_n) \subset (M_n)^r$. Thus, $\otimes_{i=1}^{r} e_i^{(r-i)}$ determines a map

$$\phi : k[V_r(GL_n)] \to H^{ev}(GL_{n(r)}, k).$$

Moreover, the composition

$$\psi \circ \phi : k[V_r(GL_n)] \to H^*(GL_{n(r)}, k) \to k[V_r(GL_n)],$$

equals F^r, the r-th iterate of the (geometric) Frobenius sending generators of the k-algebra $k[V_r(GL_n)]$ to their p^r-th power.

As in the proof of finite generation, establishing a qualitative description for $H^{ev}(GL_{n(r)}, k)$ goes a long way toward establishing a similar description of infinitesimal groups of height $\leq r$. In particular, Theorem 5.7 together with the naturality of the May spectral sequence (see (4.8)) easily implies the following corollary.

Corollary 5.8. *For any infinitesimal group scheme G of height $\leq r$, $\psi : H^{ev}(G, k) \to k[V_r(G)]$ has image containing $F^r(k[V_r(G)]) \subset k[V_r(G)]$. In particular, ψ is surjective modulo p-th powers.*

To complete the qualitative description of $H^{ev}(G, k)$ for G infinitesimal we must show that ψ is "injective modulo nilpotents". This is verified by showing that a class $\zeta \in H^{ev}(G, k)$ which restricts to 0 via every 1-parameter subgroup is nilpotent, a result analoguous to Quillen's result asserting the cohomology of $H^*(\pi, k)$ is detected modulo nilpotents by restrictions to elementary abelian subgroups of a finite group π.

Thus, Suslin-Friedlander-Bendel conclude the following.

Theorem 5.9. [18, 5.2] *Let G be an infinitesimal group of height $\leq r$. Then the map of affine schemes associated to (5.6),*

$$\Psi : V_r(G) \to Spec\, H^{ev}(G, k),$$

is a homeomorphism.

Quite recently, Friedlander and J. Pevtsova have introduced a qualitative description of $H^*(G, k)$ for any finite group scheme which encompasses the case of finite groups presented in Theorem 5.1 and that of infinitesimal group schemes presented in Theorem 5.9. This generalization loses the scheme-theoretic information of Theorem 5.9 and requires the assumption that k be algebraically closed.

Definition 5.10. Let G be a finite group scheme over the algebraically closed field k. An abelian p-point of G is a flat map of algebras $\alpha : k[u]/u^p \to kG$ which factors through some abelian subgroup scheme of G. Two such abelian p-points α, β are said to be equivalent provided that they satisfy the following condition: for every finite dimensional G-module M, $\alpha^*(M)$ is projective (as a $k[u]/u^p$-module) if and only $\beta^*(M)$ is projective.

The set of equivalence classes of abelian p-points of G is denoted $P(G)$. This set is given a topology by defining a subset $Y \subset P(G)$ to be closed if and only if there exists a finite dimensional module M such that Y consists of those equivalence classes of abelian p-points α for which $\alpha^*(M)$ is not projective.

The primary motivation for the above definition is the consideration of "support varieties" for G-modules, a topic which we do not consider for lack of time but which is a natural extension to the subject matter of this lecture. However, Definition

5.10 does enable us to formulate the following qualitative description of $H^{ev}(G, k)$ for an arbitrary finite group scheme G.

Theorem 5.11. [12, 4.8] *Let G be a finite group scheme over an algebraically closed field k. Then there is a natural homeomorphism*

$$P(G) \overset{\sim}{\longrightarrow} Proj(|G|),$$

sending an abelian p-point $\alpha : k[u]/u^p \to H^{ev}(G, k)$ to the homogeneous ideal $ker\{\alpha^*\}$.

To prove Theorem 5.11, Friedlander and Pevtsova use the following theorem recently proved by A. Suslin extending a result by C. Bendel [3] which itself extended results of Suslin-Friedlander-Bendel. We say that a finite group scheme is *quasi-elementary* if it is isomorphic to a product of the form $\mathbb{G}_{a(r)} \times \mathbb{Z}/p^s$ for some $r, s \geq 0$.

Theorem 5.12. [16] *(A. Suslin) Let G be a finite group scheme, Λ be a unital associative G-algebra, and $\zeta \in H^{ev}(G, \Lambda)$ be a homogeneous cohomology class of even degree. Then ζ is nilpotent if and only if ζ_K restricts to a nilpotent class in $H^{ev}(\mathcal{E}_K, \Lambda_K)$ for every field extension K/k and every quasi-elementary subgroup scheme $\mathcal{E}_K$ of G_K.*

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