1. Lecture 1

1.1. **Overview.** The goal of these lectures is to present some foundations of cohomologiy theories based on algebraic cycles and to provide some insight into the relationship between these cohomology theories and K-theory. The reader is encouraged to consult the freely available manuscript [16] by C. Weibel for many details of algebraic K-theory; a much less detailed reference on the Web can be found at my web page [6], the notes of my lectures given recently in Zurich.

In order to motivate our considerations, I shall begin with a brief discussion of some of the important theorems which relate K-theory to cohomology theory. Several of these theorems we shall investigate in more detail in subsequent lectures.

The first result is due to A. Grothendieck, established at the same time that Grothendieck first introduced K-theory. Grothendieck's K-group of a scheme X is what we now denote $K_0(X)$. I remind you that $K_0(X)$ is the "Grothendieck group" of algebraic vector bundles on X, defined as the abelian group with set of generators $[\mathcal{E}]$ indexed by the isomorphism classes of algebraic vector bundles $\mathcal{E}$ over X and with a relation $[\mathcal{E}] - [\mathcal{E}_1] - [\mathcal{E}_2]$ for each short exact seqquence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to -0$. Although we often think of an algebraic vector bundle $\mathcal{E}$ over X as a scheme over X whose fibres are vector spaces and which is locally in the Zariski topology a product projection, the simplest precise formulation is that $\mathcal{E}$ is a locally free, coherent $\mathcal{O}_X$-module.

Grothendieck's theorem relates $K_0(X)$ to the graded group $CH^*(X)$ of algebraic cycles on X modulo rational equivalence. Grothendieck assigns to a vector bundle $\mathcal{E}$ Chern classes $c_q(X) \in CH^q(X)$ and formulates a graded ring homomorphism $ch_*(-)$, the Chern character, using these Chern classes.

Theorem 1.1. *(cf. [4], [5]) Let X be a smooth variety over a field. Then there is a natural ring isomorphism*

$$ch_* : K_0(X) \otimes \mathbb{Q} \stackrel{\sim}{\to} CH^*(X) \otimes \mathbb{Q}$$

where $CH^q(X)$ is the Chow group of algebraic cycles of codimension q on X modulo rational equivalence. The ring structure on $K_0(X)$ is that determined by tensor product of algebraic vector bundles, the ring structure on $CH^*(X)$ is that of intersection of cycles.

For example, if $\mathcal{L}$ is a line bundle (i.e., a localy free, coherent sheaf of rank 1) with a non-zero global section $s \in \mathcal{L}(X)$, then the first Chern class $c_1([\mathcal{L}])$ of $\mathcal{L}$ is the equivalence class of the zero locus $z = Z(s) \subset X$ and $ch_*([\mathcal{L}]) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$.

Inspired by Grothendieck's introduction of K-theory into algebraic geometry M. Atiyah and F. Hirzebruch established a very analogous theory for topological spaces. This "topological K-theory" of a given space T is a sequence of abelian groups $K^i_{top}(T)$, with $K^0_{top}(T)$ the Grothendieck group of topological vector bundles over T (at least if T is a finite dimensional cell complex). Atiyah and Hirzebruch proved many good properties of their theory, properties which algebraic K-theorists have been trying ever since to establish for algebraic K-theory.

In particular, they established the "Atiyah-Hirzebruch spectral sequence" which establishes clearly the relationship of $K^*_{top}(T)$ to the singular cohomology $H^*_{sing}(T, \mathbb{Z})$ of T.

Theorem 1.2. *(cf.* [1]*) Let T be a finite dimensional cell complex. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(T, K^q(S^0)) \Rightarrow K^{p+q}(T)$$

whose differentials are torsion (so that the spectral sequence degenerates when tensored with $\mathbb{Q}$).

After many years, the analogue of the Atiyah-Hirzebruch spectral sequence has been finally established for algebraic K-theory. The cohomology theory which plays the role of singular cohomology in this analogue is *motivic cohomology*, a theory based on algebraic cycles and one which is bigraded.

Theorem 1.3. *(cf.* [3], [7], [8]*) Let X be a smooth variety over a field. There there exists a convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

It was well known from the beginnings of sheaf cohomology that $H^1_{Zar}(X, \mathcal{O}_X^*)$ can be identified with $Pic(X)$, the group (under tensor product) of isomorphism classes of line bundles. S. Bloch had the insight that there should be a similar relationship involving higher algebraic K-theory. Namely, from Bloch's point of view, $\mathcal{O}_X^*$ is the sheaf $\underline{\mathrm{K}}_1$ associated to the presheaf sending a Zariski open U to $K_1(U)$. The following theorem, Bloch's formula, was proved by Bloch for $q = 2$ and for general q by D. Quillen.

Theorem 1.4. *(cf.* [2], [12]*) Let X be a smooth scheme over a field and denote by $\underline{K}_q$ the sheaf associated to the presheaf sending a Zariski open subset $U \subset X$ to $K_q(X)$. Then there is a natural isomorphism*

$$H^q_{Zar}(X, \underline{K}_q) \simeq CH^q(X).$$

It has been recognized for some time that the algebraic K-theory with mod-n coefficients is more accessible than algebraic K-theory itself. Indeed, later in this lecture, we shall investigate Suslin's determination of the mod-n K-theory of algebraically closed fields. An important example of this principle is the following theorem of A. Merkurjev and A. Suslin which relates the mod-n K_2 of a field to Galois cohomology.

Theorem 1.5. *(cf.* [9]*) Let F be a field and n a positive integer invertible in F. Then*

$$K_2(F) \otimes \mathbb{Z}/n \xrightarrow{\sim} H^2_{et}(Spec\, F, \mu_n^{\otimes n}) = H^2_{Gal}(F, \mu_n^{\otimes 2}).$$

The approach used by Merkurjev and Suslin has been exploited by V. Voevodsky to prove the following celebrated conjecture by J. Milnor. Indeed, Voevodsky in collaboration with M. Rost has apparently established this result not simply for mod-2 coefficients but for mod-n coefficients for all n.

Theorem 1.6. *(cf.* [10], [15]*) Let F be a field of characteristic different from 2. Then*

$$K_i^{Milnor}(F) \otimes \mathbb{Z}/2 \xrightarrow{\sim} H^i_{Gal}(F, \mathbb{Z}/2).$$

This theorem and its mod-n analogue essentially establish the following important conjecture about the mod-n K-theory of varieties.

Conjecture 1.7. *(Quillen-Lichtenbaum Conjecture) Let X be a smooth variety of dimension d over a field F with $1/n \in F$. Then for $i \geq d-1$, $K_i(X, \mathbb{Z}/n)$ is determined by the etale cohomology of X*

Since we have moved into the realm of conjectures, I conclude this overview with the famous Hodge Conjecture formulated in K-theoretic terms.

Conjecture 1.8. *Let X be a projective, smooth variety with associated analytic space X^{an} and let E be a topological vector bundle. Then the class of E in $K^0_{top}(X^{an}) \otimes \mathbb{Q}$ lies in the image of $K_0(X) \otimes \mathbb{Q}$ if and only if the Chern classes of E are of type (q,q) with respect to the Hodge decomposition of $H^*_{sing}(X^{an}, \mathbb{C})$.*

1.2. **Mod-n K-theory of algebraically closed fields.** The first topic we consider is a theorem of A. Suslin which asserts that the algebraic K-theory modulo n is invariant under a base change of algebraically closed fields. Not only is this a beautiful result with a short, elegant proof, it introduces the technique of "Suslin rigidity" which plays a key role in the development of motivic cohomology.

We remind the reader that the algebraic K-theory of a variety X is given as the homotopy groups of an infinite loop space $\mathcal{K}(X)$, $K_i(X) = \pi_i(\mathcal{K}(X))$, associated to algebraic vector bundles. For X smooth, this space is homotopy equivalent to another infinite loop space $\mathcal{K}'(X)$ associated to coherent sheaves on X, but for non-smooth varieties the homotopy groups $K'_i(X) = \pi_i(\mathcal{K}'(X))$ satisfy better formal properties. We also remind the reader that if T is a pointed topological space, then the homotopy group $\pi_i(T, \mathbb{Z}/n)$ for $i > 1$ is the group of pointed homotopy classes from the mod-n Moore space $e^i \cup_{S^{i-1}} S^{i-1}$, the identification space of the boundary of the i-cell e^i via multiplication by n on S^{i-1}. Then

$$K_i(X, \mathbb{Z}/n) \equiv \pi_i(\mathcal{K}(X), \mathbb{Z}/n), \quad K'_i(X, \mathbb{Z}/n) \equiv \pi_i(\mathcal{K}'(X), \mathbb{Z}/n)$$

where we use the infinite loop space structure to define these groups for $i = 0, 1$.

One of the first theorems in algebraic K-theory was D. Quillen's computation of the K-theory of finite fields [11] and thus of the algebraic closure of finite fields. Thus, the following theorem of Suslin computes the mod-n algebraic K-theory of any algebraically closed field of positive characteristic (not dividing n). In subsequent work, Suslin also established the mod-n K-theory of algebraically closed fields of characteristic 0.

Theorem 1.9. *(cf.* [13]*) Let F/F_0 be an extension of algebraically closed fields, let n be a positive integer invertible in F_0, and let X_0 be a quasi-projective variety over F_0. Then*

$$K'_*(X_0, \mathbb{Z}/n) \xrightarrow{\sim} K'_*(X, \mathbb{Z}/n)$$

where $X = \mathit{Spec}\, F \times_{\mathit{Spec}\, F_0} X_0$ and $K'_*(-, \mathbb{Z}/n)$ is the mod-n Quillen K-theory associated to coherent sheaves.

In particular,

$$K_*(F_0, \mathbb{Z}/n) \xrightarrow{\sim} K_*(F, \mathbb{Z}/n)$$

Proof. Write F as the colimit $\varinjlim A$ of finitely generated smooth F_0-algebras A. Since $K_*(X_0 \times_{\mathrm{Spec}\, F_0} \mathrm{Spec}\,(-), \mathbb{Z}/n)$ commutes with colimits and since each F_0-algebra A admits an F_0-section $A \to F_0$ by the Hilbert Nullstellensatz, we conclude that $K'_*(X_0, \mathbb{Z}/n) \subset K'_*(X, \mathbb{Z}/n)$.

To prove surjectivity, Suslin proves that the map $K'_*(X_0 \times_{\mathrm{Spec}\, F_0} \mathrm{Spec}\, A, \mathbb{Z}/n) \to K'_*(X, \mathbb{Z}/n)$ factors as the composition of a retraction $K'_*(X_0 \times_{\mathrm{Spec}\, F_0} \mathrm{Spec}\, A, \mathbb{Z}/n) \to K'_*(X_0)$ given by such a section followed by the map $K'_*(X_0, \mathbb{Z}/n) \to K'_*(X, \mathbb{Z}/n)$ induced by $F_0 \to F$. To do this, we view these two maps as induced by two F-valued points of Spec A, and we observe that any two such points are the image of two rational points of a smooth, connected curve C over Spec F and a map $C \to \operatorname{Spec} A$ over Spec F_0.

Thus, we are reduced to proving that if C is a projective, smooth, connected curve over Spec F and if $p_1, p_2 \in C$ are two rational points, then the induced maps

$$p_1^*, p_2^* : K'_*(X \times C, \mathbb{Z}/n) \to K'_*(X, \mathbb{Z}/n)$$

are equal. Suslin first observes that the maps p^* extend to an action

$$K'_*(X \times C, \mathbb{Z}/n) \otimes C_0(C) \to K'_*(X, \mathbb{Z}/n)$$

where $C_0(C)$ denotes the group of 0-cycles on C.

Now, the critical insight ("Suslin rigidity") is Suslin's proof that this action factors through the quotient $C_0(C) \to Pic(C)$ (i.e., rationally equivalent 0-cycles induce the same pairing). For the special case in which $C = \mathbb{P}^1$, the projective line, this follows from the property of *homotopy invariance* of $K'_*(X \times -, \mathbb{Z}/n)$ (i.e., $\pi^* : K'_*(X, \mathbb{Z}/n) \xrightarrow{\sim} K'_*(X \times \mathbb{A}^1, \mathbb{Z}/n)$). The critical observation of Suslin is that the functor $K'_*(X \times -, \mathbb{Z}/n)$ behaves well with respect to "transfer" with respect to a finite, surjective map from a projective smooth curve C to $\mathbb{P}^1$. Since 0-cycles on C rationally equivalent to 0 are the pre-image via some such finite surjective map $C \to \mathbb{P}^1$ of the difference $\{0\} - \{\infty\}$, we conclude using formal properties of this transfer that rationally equivalent points p_1, p_2 determine equal pairings.

Finally, Suslin uses the well-known fact that $Pic(C) \otimes \mathbb{Z}/n = 0$ to conclude that any two points $p_1, p_2 \in C$ determine the same pairing from $K'_*(X \times C, \mathbb{Z}/n) \to K'_*(X, \mathbb{Z}/n)$. Note that at this point, we use strongly the fact that we are dealing with functors with values in n-torsion abelian groups. $\square$

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2. Lecture 2

The primary aim of this lecture is to present many of the foundations used by Suslin and Voevodsky in their discussion of algebraic singular homology [4]. We view this as an excellent introduction to the roles played by *rigidity* and *homotopy invariance*. Throughout this lecture, k will be a fixed field and (Sch/k) will denote the category of schemes of finite type over an algebraically closed field k.

2.1. **Presheaves with transfer.** The concept of presheaf with transfers is an abstraction of the technique Suslin introduced in [3] to determine the mod-n K-theory of algebraically closed fields as discussed in the previous lecture. There are various slightly different definitions of this concept and the one we present here is the one introduced by Suslin and Voevodsky in [4].

Definition 2.1. A contravariant functor on (Sch/k),

$$\mathcal{F} : (Sch/k)^{op} \to Ab,$$

is said to be a presheaf with transfers if $\mathcal{F}$ is provided with a homomorphism for every finite surjective $p : Y \to S$ with Y irreducible, reduced and S irreducible, regular

$$Tr_{Y/S} : \mathcal{F}(Y) \to \mathcal{F}(S)$$

satisfying the following conditions:

- If p is an isomorphism, then $Tr_{Y/S} \circ p^* = id$.
- If $V \subset S$ is a closed, irreducible, regular subscheme and if $p^{-1}(V) = \sum_i n_i W_i$ (where the multiplicity n_i is the usual intersection multiplicity of $p^{-1}(Y)$ along the component W_i), then the following diagram commutes

pre (2.1.1)

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "Tr_{Y/S}"] \arrow[d] & \mathcal{F}(S) \arrow[d] \\ \oplus_i \mathcal{F}(W_i) \arrow[r, "\sum n_i Tr_{W_i/V}"] & \mathcal{F}(V) \end{tikzcd}$$

Observe that we do not require Y to be regular. Indeed, the way these transfers are employed is as follows. We are given a scheme X over S and some closed, irreducible subscheme $Y \subset X$ which is finite and surjective over S. Then the contravariant functoriality of $\mathcal{F}$ together with the transfer $Tr_{Y/S}$ determines a transfer map

$$Tr_{Y/S} : \mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(S).$$

In other words, if $C_0(X/S)$ denotes the free abelian group on the closed irreducible subschemes $Y \subset X$ which are finite and surjective over S, then $\mathcal{F}$ is provided with a pairing

1 (2.1.2) $$\mathcal{F}(X) \otimes C_0(X/S) \to \mathcal{F}(S).$$

Let $C_1(X/S)$ denote the free abelian group on the closed irreducible subschemes $W \subset X \times \mathbb{A}^1$ which are finite and surjective over $S \times \mathbb{A}^1$, and let $H_0(X/S)$ denote the cokernel of the difference of the two maps $C_1(X/S) \to C_0(X/S)$ given by evaluating at 0,1.

We now see the role of *homotopy invariance*. We can view the following proposition as a generalization of the rigidity used in the previous lecture (by using an awkward notational shift, so that $\mathcal{F} = K'_*(X \times -, \mathbb{Z}/n)$, X in the proposition below is replaced by a smooth projective curve C, and S by $\operatorname{Spec} k$.

Proposition 2.2. *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers. Then the pairing (2.1.2) factors as*

2 (2.2.1) $$\mathcal{F}(X) \otimes H_0(X/S) \to \mathcal{F}(S).$$

We say that X/S is a smooth relative curve with good compactification if X is a smooth, affine, irreducible scheme of relative dimension 1 over S which embeds as an open subset of some $\overline{X}/S$ with $\overline{X}$ normal, $\overline{X} \to S$ proper, and $\overline{X} - X$ admits an affine open neighborhood in $\overline{X}$.

We remind the reader that if $W \subset X$ is a closed subscheme, then the relative Picard group $Pic(X, W)$ is the abelian group of isomorphism classes of pairs $\langle L, \phi \rangle$, where $\mathcal{L}$ is a line bundle on X and $\phi : \mathcal{L}_{|W} \stackrel{sim}{\to} \mathcal{O}_Y$ is a trivialization of $\mathcal{L}$ restricted to W.

pic **Proposition 2.3.** *Assume that S is a normal, affine scheme and that X/S is a smooth relative curve with good compactification. Then $H_0(X/S)$ equals the relative Picard group $Pic(\overline{X}, W)$, where $X \subset \overline{X}$ is a good compactification with complement $W = \overline{X} - X$.*

Proposition 2.3 enables us to investigate the behaviour of $H_0(X/S) \otimes \mathbb{Z}/n$ using etale cohomology. In particular, the proper base change theorem in etale cohomology enables us to conclude with the hypotheses of the preceding theorem that

$$H_0(X/S) \otimes \mathbb{Z}/n \subset H_0(X_0, S_0)$$

where $S_0 \subset S$ is a closed regular subscheme and $X_0 = X \times_S S_0$.

The preceding propositions easily enable Suslin and Voevodsky to prove the following rigidity theorem.

Theorem 2.4. *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers satisfying the condition that $n\mathcal{F} = 0$ for some positive integer n invertible in k, let X denote the henselization of a smooth variety over k at some closed point, and let X/S denote a smooth relative curve with a good compactification.*

If $g_1, g_2 : S \to X$ are two sections of X/S which agree on the closed point of S, then $g_1^* = g_2^* : \mathcal{F}(X) \to \mathcal{F}(S)$.

This enables Suslin and Voevodsky to prove the following theorem which essentially tells us that a presheaf with transfers as in the preceding theorem is essentially locally constant for the etale topology.

loc-triv **Theorem 2.5.** *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers satisfying the condition that $n\mathcal{F} = 0$ for some positive integer n invertible in k, and let S denote the henselization of a smooth variety at a closed point. Then the restriction map*

$$\mathcal{F}(S) \to \mathcal{F}(Spec\, k)$$

is an isomorphism.

2.2. **Grothendieck topologies.** In this section, we shall give a brief introduction to the etale topology and to two other Grothendieck topologies introduced by Voevodsky and used by Suslin and Voevodsky.

Definition 2.6. A site $\mathcal{C}/X$ is a subcategory of the category of schemes over a fixed scheme closed under fiber products and is equipped with a distinguished class of morphisms which is required to be closed under composition, base change and which includes all ismorphisms. One selects as coverings of an object $Y \in \mathcal{C}/X$ families of distinguished morphism $\{g_i : V_i \to Y\}$ with the property that $Y = \cup_i g_i(V)$; one requires various closure properties of coverings: all isomorphisms are coverings, the pull-back of a covering of Y by a morphism $Y' \to Y$ in $\mathcal{C}/X$ should be a covering of Y'.

The data of a site together with coverings consisting of distinguished morphisms is a Grothendieck topology on X.

The most "classical" Grothendieck topology (other than the Zariski topology, which of course is a topology in the usual sense as well as a Grothendieck topology) is the etale topology. The reader is refereed to the book by J. Milne [2] for considerable foundational detail on this important construction. The etale topology plays a significant role in our understanding of K-theory mod-n, and the Suslin-Voevodsky theorem provides a more naive means of determining the etale cohomology with $\mathbb{Z}/n$ coefficients of varieties over k.

Definition 2.7. A map $U \to X$ is said to be etale if it is flat, unramified, and locally of finite type. The small etale site X_{et} is the category of schemes etale over X and whose distinguished morphisms are etale morphisms and whose coverings are all collections $\{g_i : V_i \to Y\}$ of etale morphisms with the property that $Y = \cup_i g_i(V_i)$. The big etale site X_{ET} is the category all schemes locally of finite type over X with distinguished morphisms and coverings as in X_{et}

Example 2.8. The following morphisms are examples of etale morphisms.

- $U \to X$ a Zariski open immersion.
- $\tilde{X} \to X$ a finite covering space
- $\operatorname{Spec} R \to \operatorname{Spec} F$, where F is a field and R is a finite separable F-algebra (i.e., $\overline{F} \otimes_F R$ splits as a product of copies of $\overline{F}$, the algebraic closure of F).
- $V \to X$ a morphism of complex algebraic varieties with the property that $V^{an} \to X^{an}$ is a local homeomorphism.
- If R is a domain, $g(t), h(t) \in R[t]$, then the map $\operatorname{Spec} R[t]/g(t))[\overline{h}(t)^{-1}] \to \operatorname{Spec} R$ is etale provided that $g'(t)$ is invertible in $R[t]/g(t))[\overline{h}(t)^{-1}]$ (i.e., the zero locus of $h(t)$ contains the common zeros of $g(t), g'(t)$.

One of the many insights of Grothendieck is that one can formulate sheaf theory and sheaf cohomology on a site with a Grothendieck topology with essentially no change from the usual sheaf theory for sheaves on a topological space.

Definition 2.9. Let $\mathcal{C}/X$ be a site provided with a Grothendieck topology and let $\mathcal{A}$ denote the category of sets, groups, abelian groups, rings, or modules over a given ring. Then a presheaf on $\mathcal{C}/X$ (i.e., a contravariant functor $\mathcal{F} : (\mathcal{C}/X)^{op} \to \mathcal{A}$) is said to be a sheaf if for all $Y \in \mathcal{C}/X$ and all coverings $\{V_i \to Y\}$ the following sequence is exact:

$$\mathcal{F}(Y) \to \prod_i \mathcal{F}(V_i) \to \prod_{i,j} \mathcal{F}(V_i \times_Y V_j).$$

If $\mathcal{A}$ is an abelian category with enough injectives, then the category (topos) of sheaves for the Grothendieck topology on $\mathcal{C}/X$ with values in $\mathcal{A}$ is an abelian category with enough injectives, permitting us to apply standard homological algebra in defining the sheaf cohomology

$$H^*(\mathcal{C}/X, \mathcal{F}) \equiv R^*\Gamma(X, -)(\mathcal{F}).$$

One important property of etale cohomology is that $H^*(X_{et}, \mathbb{Z}/n) \simeq H^*_{sing}(X^{an}, \mathbb{Z}/n)$ for any quasi-projective complex algebraic variety. We mention another in the following example.

Example 2.10. Let F be a field with separable closure $\overline{F}$. A sheaf of sets $\mathcal{F}$ on F_{et} consists of a set S together with a "continuous" action of $Gal(\overline{F}/F)$ on this set. In other words, an action of the discrete group $\varprojlim_{\{L/F\ Galois\}} Gal(L/F)$ on S with the property that for each element $t \in S$ there exists some (finite) Galois L_t/F with the property that the kernel of $\varprojlim_{L/F} Gal(L/F) \to Gal(L_t/F)$ fixes t.

In particular, if $\mathcal{F}$ is an abelian sheaf on F_{et}, then $H^*(F_{et}, \mathcal{F})$ is the Galois cohomology $H^*_{Gal}(F, \mathcal{F})$.

Recall that Theorem 2.5 involved the henslization S of a smooth variety at a closed point. Although such a henzelization can be defined more algebraically, let us formulate this in terms of the etale topology.

Definition 2.11. Let $X = \operatorname{Spec} A$ be an affine variety and $y \in \operatorname{Spec} A$ be a closed point with associated local ring $\mathcal{O}_{X,y} = R$. Then the *henselization* R^h of R is the colimit of local etale morphisms $R \to B$, $R^h = \varinjlim_{R \to B} B$, where the map of local rings $R \to B$ is required to induce an isomorphism on residue fields. The *strict henselization* R^{sh} of R is the colimit of all local etale morphisms $R \to B$ without the condition that the induced map on residue fields is an isomorphism; thus, the residue field of R^{sh} is the separable closure of the residue field of R.l

If $X = \operatorname{Spec} R$ is the spectrum of a strict hensel local ring (i.e., $R = R^{sh}$), then the global section functor $\mathcal{F} \mapsto \mathcal{F}(X)$ is an equivalence from the category of sheaves on X_{et} with values in $\mathcal{A}$ to the category $\mathcal{A}$.

For a variety X over the algebraically closed field k, a sequence of sheaves on X_{et} with values in an abelian category $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact if and only if its restriction to each of the (strict) henselizations at closed points of X is exact.

Two other Grothendieck topologies play a role in the proof of the Suslin-Voevodsky theorem.

Definition 2.12. An h-covering of a scheme Y is a finite family of morphisms of finite type $g_i : V_i \to Y$ with the property that the induced map $p : \coprod_i V_i \to Y$ is a universal topological epimorphism (i.e., for any $Y' \to Y$, the pull-back $p' : \coprod_i V_i \times_Y Y' \to Y'$ satisfies the property that Y' as a topological space with the Zariski topology is the quotient of of $\coprod_i V_i$). If each of thee morphisms $g_i : V_i \to Y$ is quasi-finite, then the h-covering $\{g_i : V_i \to Y\}$ is called a qfh-covering.

The h (respectively, qfh) topology on X is the site of schemes of finite type over X whose

2.3. **Dold-Thom Theorem and the Suslin complex.** A well known theorem of A. Dold and R. Thom enable one to express the singular homology of a cell complex in terms of the homotopy groups of its symmetric powers.

Theorem 2.13. *(cf. [1]) Let T be a C.W. complex and let $\coprod_{d \geq 0} S^d T$ denote the free abelian monoid on the points of T, where $S^d T$ is the given the quotient topology* *with respect to the natural surjection $T^{\times d} \to S^d$. Then*

$$H_i^{sing}(T) = \pi_i(Sing.(\coprod_{d\geq 0} S^dT)^+)$$

where $Sing.(\coprod_{d\geq 0} S^dT)^+$ is the simplicial abelian group given in degree n as the group completion of the abelian monoid $\coprod_{d\geq 0} Sing_n(S^dT)$. Alternatively, if $\mathbb{Z}(T)$ denotes the topological abelian group on the points of T (topologized as a quotient of $(\coprod_{d\geq 0} S^dT)^{\times 2}$), then

$$H_i^{sing}(T) = \pi_i(\mathbb{Z}(T)).$$

Two observations help to make this homotopy theoretic construction applicable to algebraic geometry. First, if X is an algebraic variety over k, then for each $d > 0$ the d-fold symmentric power S^dX of X is also an algebraic variety. Second, the homotopy groups of a simplicial abelian group $A_\bullet$ are naturally identified with the homology groups of the chain complex given in dimension n by A_n and with diferential the alternating sum of the face maps $d_i : A_n \to A_{n-1}$. This chain complex is easily seen to be quasi-isomorphic to the *normalized chain complex* associated to $A_\bullet$ which is given in dimension n as the intersection of the kernels of the face maps $d_i : A_n \to A_{n-1}$ for $i > 0$ and whose differential is the restriction of d_0 to this kernel.

Recall the conventional notation of $\Delta^n \equiv \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$, the "$n$-simplex over k. The natural face and degeneracy maps determine a cosimplicial object $\Delta^\bullet$ in (Sch/k). Suslin's observation is that it is profitable to consider the alebraic-geometric analogue of the topological construction $T \mapsto Sing.(\coprod_{d\geq 0} S^dT)^+)$.

Definition 2.14. Let $X \in (Sch/k)$. We define

$$Sus_*(X) \equiv \mathcal{N}((\coprod_{d\geq 0} Hom_{(Sch/k)}(\Delta^\bullet, S^dX)^+),$$

the normalized chain complex associated to the simplicial abelian group given in dimension n by $Hom_{(Sch/k)}(\Delta^n, \coprod_{d\geq 0} S^dX)$.

2.4. **Statement of Suslin-Voevodsky theorem.** We conclude this lecture with the statement of the remarkable theorem of Suslin-Voevodsky.

Theorem 2.15. *For $X \in (Sch/k)$ and positive integer n invertible in the algebraically closed field k,*

$$H^*(X_{et}, \mathbb{Z}/n) = H^*(Sus_*(X), \mathbb{Z}/n).$$

In particular, if $k = \mathbb{C}$, then

$$H^*_{sing}(X^{an}, \mathbb{Z}/n) = H^*(Sus_*(X), \mathbb{Z}/n).$$

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3. Lecture 3

3.1. **Freely generate qfh-sheaves.** We recall that if $\mathcal{F}$ is a homotopy invariant presheaf with transfers in the sense of the last lecture, then we have a well defined pairing

$$\mathcal{F}(X) \otimes H_0(X/S) \to \mathcal{F}(S)$$

given by associating to any irreducible $i : Y \subset X$ finite, surjective over S the mapping $Tr_{Y/S} \circ i^* : \mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(S)$. Moreover, recall that any qfh-sheaf admits transfers.

Consider the following presheaf, where $\mathbb{Z}[1/p]$ denotes localization of the residue characteristic of k (so that if k is of characteristic 0, then $\mathbb{Z}[1/p]$ should be read as $\mathbb{Z}$):

$$(Sch/k)^{op} \to Ab, \quad Y \mapsto \mathbb{Z}[1/p]Hom_{(Sch/k)}(Y, X).$$

We denote by $\mathcal{F}_X$ the associated qfh-sheaf.

The following result of Suslin-Voevodsky identifies this sheaf when applied to normal varieties and relates this to the Suslin complex $Sus_*(X)$ of X. One can interpret this theorem as saying that cycles in $X \times Y$ each component of which is finite and surjective over a normal variety Y are locally in the qfh-topology a sum of graphs of morphisms from Y to X.

Theorem 3.1. *If $Y \in (Sch/k)$ is normal, then*

$$\mathcal{F}_X(Y) = C_0(X \times Y/Y) \otimes \mathbb{Z}[1/p].$$

In particular,

$$\mathcal{F}_X(\Delta^\bullet) = Sus_*(X) \otimes \mathbb{Z}[1/p].$$

Sheafifying in the qfh-topology gives us transfers. The following proposition enables us to obtain presheaves which are homotopy invariant.

Proposition 3.2. *Assume that $\mathcal{F}$ is a presheaf with transfers. Let $(\mathcal{F}_*)\tilde{}$ denote the complex of sheaves given in degree q as the qfh- sheaf associated to the presheaf $Y \mapsto \mathcal{F}(Y \times \Delta^q)$. For any $q \geq 0$, consider the presheaf*

$$\mathcal{H}_q((\mathcal{F}_*)\tilde{}\ ) : (Sch/k)^{op} \to Ab$$

defined as the homology sheaves of the complex $(\mathcal{F}_*)\tilde{}$ (or, equivalently, the sheaf associated to the presheaf $Y \mapsto H_q(\mathcal{F}(Y \times \Delta^\bullet))$). Then $\mathcal{H}_q((\mathcal{F}_*)\tilde{}\ )$ is a homotopy invariant presheaf with transfers.

Proof. One first shows that evaluation at $0, 1 \in \Delta^1$ determine chain homotopy maps

$$\mathcal{F}(Y \times \Delta^\bullet \times \Delta^1) \to \mathcal{F}(Y \times \Delta^\bullet).$$

This enables one to show that upon taking qfh-sheaves that $\mathcal{H}_q((\mathcal{F}_*)\tilde{}\ )$ is homotopy invariant. Moreover, the naturality properties of transfers on $\mathcal{F}$ imply that $Tr_{Y/S}$ gives us a transfer map on complexes $\mathcal{F}(X \times \Delta^\bullet) \to \mathcal{F}(S \times \Delta^\bullet)$, so that taking associated qfh-sheaves gives us transfers on $\mathcal{H}_q((\mathcal{F}_*)\tilde{}\ )$. □

3.2. **Proof of Suslin-Voevodsky theorem.** We shall sketch a proof of the following theorem.

Theorem 3.3. *(Suslin Voevodsky* [2]*) Let $\mathcal{F}_X$ denote the qfh-sheaf associated to the prsheaf*

$$(Sch/k)^{op} \to Ab, \quad Y \mapsto \mathbb{Z}[1/p]Hom_{(Sch/k)}(Y, X).$$

Then for any positive integer n invertible in k, the natural maps of complexes of qfh-sheaves

$$\mathcal{F}_X(\Delta^\bullet) \to (\mathcal{F}_{X*})\tilde{} \leftarrow \mathcal{F}_X$$

induce isomorphisms of Ext-groups

ext

$$H^*(Sus_*(X), \mathbb{Z}/n) \xrightarrow{\sim} Ext^*_{qfh}((\mathcal{F}_{X*})\tilde{}, \mathbb{Z}/n) \xleftarrow{\sim} Ext^*_{qfh}(\mathcal{F}_X, \mathbb{Z}/n). \tag{3.3.1}$$

Moreover, $Ext^*_{qfh}(\mathcal{F}_X, \mathbb{Z}/n) \simeq Ext^*(X_{et}, \mathbb{Z}/n)$.

Proof. The two isomorphisms of (3.3.1) are proved considering the two hypercohomology spectral sequences for $Ext^*_{qfh}((\mathcal{F}_{X*})\tilde{}, \mathbb{Z}/n)$. The first isomorphism follows by comparing the homology at each level of the map of complexes of presheaves with transfers

$$\mathcal{F}_X(\Delta^\bullet) \otimes \mathbb{Z}/n \to (\mathcal{F}_{X*})\tilde{} \otimes \mathbb{Z}/n,$$

where the left hand complex is viewed as a complex of constant presheaves. Since the homology presheaves of these complexes are presheaves with transfer which are annihilated by multiplication by n, we can apply the Suslin-Voevodsky theorem to conclude that the induced map on homology sheaves is an isomorphism.

The second isomorphism does not use the fact that $\mathcal{F}_X$ is a presheaf with transfers, but is a general fact that $\mathcal{F}_X \to (\mathcal{F}_X(- \times \Delta^q))\tilde{}$ induces an isomorphism in $Ext^*_{qfh}(-, \mathbb{Z}/n)$.

The right had side is almost by definition $H^*(X_{qfh}, \mathbb{Z}/n)$. The comparison with the etale cohomology of X is achieved by realizing explicitly sufficiently fine qfh coverings together with "resolution of singularities" (in characteristic $p > 0$, one uses de Jong's modifications rather than resolutions which are not known to exist. □

3.3. **Discussion of Chow groups.** Recall that X is said to be *integral* if $\mathcal{O}_X(U)$ is an integral domain for all open subsets $U \subset X$. The field of fractions K of such an integral variety is the field of fractions of $\mathcal{O}_X(U)$ for any affine open subset U. If $\mathcal{O}_X(U)$ is integrally closed in K for every affine open subset U, then the stalk $\mathcal{O}_{X,x}$ at any (scheme-theoretic point) $x \in X$ of codimension 1 is a discrete valuation ring.

Definition 3.4. Let X is an integral variety regular in codimension 1 and let K be its field of fractions. For any $0 \neq f \in K$, we define the *principal divisor* (f) associated to f to be the following formal sum of codimension 1, irreducible subvarieties

$$(f) = \sum_{x \in X^{(1)}} v_x(f)\overline{x}.$$

Here, $X^{(1)} \subset X$ consists of the scheme-theoretic points of codimension 1, $v_x : K^* \to \mathbb{Z}$ is the discrete valuation at $x \in X^{(1)}$, and $\overline{x} \subset X$ is the codimension 1 irreducible subvariety of X given as the closure of x.

A formal sum

$$D = \sum_{x \in X^{(1)}} n_x\overline{x}, \quad n_x \in \mathbb{Z}$$

with all but finitely many n_x equal to 0 is said to be a *locally principal* divisor provided that for every $x \in X^{(1)}$ there exists some Zariski open neighborhood $x \in U_x \subset X$ and some $f_x \in K$ such that $D_{|U_x} = (f_x)_{|U_x}$.

Definition 3.5. Let X be a quasi-projective algebraic variety. An *algebraic r-cycle* on X if a formal sum

$$\sum_Y n_Y[Y], \quad Y \text{ irreducible of dimension } r, \quad n_Y \in \mathbb{Z}$$

with all but finitely many n_Y equal to 0.

Equivalently, an algebraic r-cycle is a finite integer combination of points of X of dimension r.

If $Y \subset X$ is a subvariety each of whose irreducible components $Y_1, \ldots, Y_m$ is r-dimensional, then the algebraic r-cycle

$$Z = \sum_{i=1}^{m} [Y_i]$$

is called the *cycle associated* to Y.

The group of (algebraic) r-cycles on X will be denoted $Z_r(X)$.

Two r-cycles Z, Z' on a quasi-projective variety X if their difference lies in the subgroup $Z_{r,rat}(X) \subset Z_r(X)$ generated by cycles of the form $W_{|X\times\{p\}} - W_{|X\times\{q\}}$, where $U \subset \mathbb{P}^1$ is a Zariski open set containing points $p, q \in U$ and $W \subset X \times U$ is a cycle each of whose irreducible components maps surjectively onto U.

The *Chow group* $CH_r(X) = Z_r(X)/Z_{r,rat}(X)$ is the group of r-cycles modulo rational equivalence.

Theorem 3.6. *(cf.* [1]*) Assume that X is an integral variety regular in codimension 1. Let $\mathcal{D}(X)$ denote the group of locally principal divisors on X modulo principal divisors. Then there is a natural isomoprhism*

$$Pic(X) \stackrel{\sim}{\to} \mathcal{D}(X).$$

If $\mathcal{L} \in Pic(X)$ has a non-zero global section $s \in \mathcal{L}(X) = \Gamma(X, \mathcal{L})$, then this isomorphism sends $\mathcal{L}$ to $\sum_{x\in X^{(1)}} v_x(s)\overline{x}$.

Moreover, if $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, then

$$\mathcal{D}(X) \stackrel{\sim}{\to} CH^1(X),$$

the Chow group of codimension 1 cycles on X modulo rational equivalence.

Remark 3.7. Not only is this an example of relating bundles to cycles, but it is also an example of duality. Namely, $Pic(X)$ is contravariant, whereas $CH_r(-)$ is covariant for proper maps. This suggests that for $Pic(X)$ to be isomorphic to $CH^1(X)$, some smoothness condition on X is required .

Observe that in the above definition we can replace the role of $r + 1$-cycles on $X \times \mathbb{P}^1$ and their geometric fibres over $0, \infty$ by $r + 1$-cycles on $X \times U$ for any non-empty Zaristik open $U \subset X$ and geometric fibres over any two k-rational points $p, q \in U$.

Remark 3.8. Given some $r+1$ dimensional irreducible subvariety $V \subset X$ together with some $f \in k(V)$, we may define $(f) = \sum_S ord_S(f)[S]$ where S runs through the codimension 1 irreducible subvarieties of V. Here, $ord_S(-)$ is the valuation centered on S if V is regular at the codimension 1 point corresponding to S; more generally, $ord_S(f)$ is defined to be the length of the $O_{V,S}$-module $O_{V,S}/(f)$.

We readily check that (f) is rationally equivalent to 0: namely, we associate to (V, f) the closure $W = \Gamma_f \subset X \times \mathbb{P}^1$ of the graph of the rational map $V \dashrightarrow \mathbb{P}^1$ determined by f. Then $(f) = W_{|X\times\{0\}} - W_{X\times\{\infty\}}$.

Conversely, given an $r+1$-dimensional irreducible subvariety W on $X \times \mathbb{P}^1$ which maps onto $\mathbb{P}^1$, the composition $W \subset X \times \mathbb{P}^1 \stackrel{pr_2}{\to} \mathbb{P}^1$ determines $f \in frac(W)$ such that

$$(f) \ = \ W_{|X\times\{0\}} - W_{X\times\{\infty\}}.$$

Thus, the definition of rational equivalence on r-cycles of X can be given in terms of the equivalence relation generated by

$$\{(f), f \in frac(W); W \text{ irreducible of dimension } r+1\}$$

In particular, we conclude that the subgroup of principal divisors inside the group of all locally principal divisors consists precisely of those locally principal divisors which are rationally equivalent to 0.

Example 3.9. For essentially formal reasons, $Pic(X) \simeq H^1(X, \mathcal{O}_X^*)$. If k is the complex field, we can use the exponential sequence of sheaves in the analytic topology

$$0 \to \mathbb{Z} \to \mathcal{O}_X \stackrel{exp}{\to} \mathcal{O}_X^* \to 0$$

to conclude that the kernel of $H^1(X^{an}, \mathcal{O}_X^*) \to H^2(X^{an}, \mathbb{Z})$ is the complex vector space $H^1(X^{an}, \mathcal{O}_X)$ modulo the discrete subgroup $H^1(X^{an}, \mathbb{Z})$. For example, if $X = C$ is a smooth, projective curve of genus g, then $CH^1(C)$ fits in a short exact sequence

$$0 \to \mathcal{C}^g/\mathbb{Z}^{2g} = H^1(C, \mathcal{O}_C^*)/H^1(C^{an}, \mathbb{Z}) \to CH^1(C) \to \mathbb{Z} = H^2(C^{an}, \mathbb{Z}) \to 0.$$

Example 3.10. Let $X = \mathbb{A}^N$. Then any $N-1$-cycle (i.e., Weil divisor) $Z \in CH_{N-1}(\mathbb{A}^N)$ is principal, so that $CH_{N-1}(\mathbb{A}^N) = 0$.

More generally, consider the map $\mu : \mathbb{A}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ which sends $(x_1, \ldots, x_n), t$ to $\langle t \cdot x_1, \ldots, t \cdot x_n, 1\rangle, t$. Consider an ireducible subvariety $Z \subset \mathbb{A}^N$ of dimension $r > N$ not containing the origin and $\overline{Z} \subset \mathbb{P}^N$ be its closure. Let $W = \mu^{-1}(\overline{Z} \times \mathbb{A}^1)$. Then $W[0] = \emptyset$ whereas $W[1] = Z$. Thus, $CH_r(\mathbb{A}^N) = 0$ for any $r < N$.

Example 3.11. Arguing in a similar geometric fashion, we see that the inclusion of a linear plane $P^{N-1} \subset \mathbb{P}^N$ induces an isomorphism $CH_r(\mathbb{P}^{N-1}) = CH_r(\mathbb{P}^N)$ provided that $r < N$ and thus we conclude by induction that $CH_r(\mathbb{P}^N) = \mathbb{Z}$ if $r \leq N$. Namely, consider $\mu : \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ sending $\langle x_0, \ldots, x_N\rangle, t$ to $\langle x_), \ldots, x_{N-1}, t \cdot x_N\rangle, t$ and set $W = \mu^{-1}(Z \times \mathbb{A}^1)$ for any Z not containing $\langle 0, \ldots, 0, 1\rangle$. Then $W[0] = pr_{N*}(Z), W[1] = Z$.

Example 3.12. Let C be a smooth curve. Then $Pic(C) \simeq CH_0(X)$.

Definition 3.13. If $f : X \to Y$ is a proper map of quasi-projective varieties, then the proper push-forward of cycles determines a well defined homomorphism

$$f_* : CH_r(X) \ \to \ CH_r(Y), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to $d \cdot [f(Z)] \in CH_r(Y)$ where $[k(Z) : k(f(Z))] = d$ if $dim\ Z = dim\ f(Z)$ and is sent to 0 otherwise.

If $g : W \to X$ is a flat map of quasi-projective varieties of relative dimension e, then the flat pull-back of cycles induces a well defined homomorphism

$$g^* : CH_r(X) \ \to \ CH_{r+e}(W), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to the cycle on W associated to $Z \times_X W \subset W$.

Proposition 3.14. *Let Y be a closed subvariety of X and let $U = X \backslash Y$. Let $i : Y \to X, j : U \to X$ be the inclusions. Then the sequence*

$$CH_r(Y) \xrightarrow{i_*} CH_r(X) \xrightarrow{j^*} CH_r(U) \to 0$$

is exact for any $r \geq 0$.

Proof. If $V \subset U$ is an irreducible subvariety of U of dimension r, then the closure of V in X, $\overline{V} \subset X$, is an irreducible subvariety of X of dimension r with the property that $j^*([\overline{V}]) = [V]$. Thus, we have an exact sequence

$$Z_r(Y) \xrightarrow{i_*} Z_r(X) \xrightarrow{j^*} Z_r(U) \to 0.$$

If $Z = \sum_i n_i[Y_i]$ is a cycle on X with $j^*(Z) = 0 \in CH_r(U)$, then $j^*Z = \sum_{W,f}(f)$ where each $W \subset U$ is an irreducible subvarieties of U of dimension $r+1$ and $f \in k(W)$. Thus, $Z' = \sum_i n_i[\overline{Y}_i] - \sum_{\overline{W},f}(f)$ is an r-cycle on Y with the property that $i_*(Z')$ is rationally equivalent to Z. Exactness of the asserted sequence of Chow groups is now clear.

□

Corollary 3.15. *Let $H \subset \mathbb{P}^N$ be a hypersurface of degree d. Then $CH_{N-1}(\mathbb{P}^N \backslash H) = \mathbb{Z}/d\mathbb{Z}$.*

Example 3.16. Mumford shows that if S is a projective smooth surface with a non-zerol global algebraic 2-form (i.e., $H^0(S, \Lambda^2(\Omega_S)) \neq 0$), then $CH_0(S)$ is *not finite dimensional* (i.e., must be very large).

Bloch's Conjecture predicts that if S is a projective, smooth surface with geometric genus equal to 0 (i.e., $H^0(S, \Lambda^2(\Omega_S)) = 0$), then the natural map from $CH_0(S)$ to the (finite dimensional) Albanese variety is injective.

3.4. **Intersection product.**

Theorem 3.17. *Let X be a smooth quasi-projective variety of dimension d. Then there exists a pairing*

$$CH_r(X) \otimes CH_s(X) \xrightarrow{\bullet} CH_{d-r-s}, \quad d \geq r+s,$$

with the property that if $Z = [Y], Z' = [W]$ are irreducible cycles of dimension r, s respectively and if $Y \cap W$ has dimension $\leq d-r-s$, then $Z \bullet Z'$ is a cycle which is a sum with positive coefficients indexed by the irreducible subvarieties of $Y \cap W$ of dimension $d-r-s$.

For notational purposes, we shall often write $CH^s(X)$ for $CH_{d-s}(X)$. With this indexing convention, the intersection pairing has the form

$$CH^s(X) \otimes CH^t(X) \ \xrightarrow{\bullet} \ CH^{s+t}(X).$$

Proof. Classically, this was proved by showing the following geometric fact: given a codimension r cycle Z and a codimension s cycle $W = \sum_j m_j R_j$ with $r + s \leq d$, then there is another codimension r cycle $Z' = \sum_i n_i Y_i$ rationally equivalent to Z (i.e., determining the same element in $CH^r(X)$) such that Z' meets W "properly"; in other words, every component $C_{i,j,k}$ of each $Y_i \cap R_j$ has codimension $r + s$. One then defines

$$Z' \bullet W = \sum_{i,j,k} n_i m_j \cdot int(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}$$

where $int(Y_i \cap R_j, C_{i,j,k})$ is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a Z'' rationally equivalent to both Z, Z' and also intersecting W properly, then $Z' \bullet W$ is rationally equivalent to $Z'' \bullet W$.

To be continued next lecture ...

□

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4. Lecture 4

4.1. **Intersection product.** Let us briefly return to the ring structure on $CH^*(X)$ for a smooth variety X. This ring structure arises from intersection of cycles, as described in the following theorem.

Theorem 4.1. *Let X be a smooth quasi-projective variety of dimension d. Then there exists a pairing*

$$CH_r(X) \otimes CH_s(X) \xrightarrow{\bullet} CH_{d-r-s}, \quad d \geq r+s,$$

with the property that if $Z = [Y], Z' = [W]$ are irreducible cycles of dimension r, s respectively and if $Y \cap W$ has dimension $\leq d-r-s$, then $Z \bullet Z'$ is a cycle which is a sum with positive coefficients indexed by the irreducible subvarieties of $Y \cap W$ of dimension $d-r-s$.

Thus, the intersection product determine an intersection pairing of the form

$$CH^s(X) \otimes CH^t(X) \ \xrightarrow{\bullet}\ CH^{s+t}(X).$$

Proof. Classically, this was proved by showing the following geometric fact: given a codimension r cycle Z and a codimension s cycle $W = \sum_j m_j R_j$ with $r+s \leq d$, then there is another codimension r cycle $Z' = \sum_i n_i Y_i$ rationally equivalent to Z (i.e., determining the same element in $CH^r(X)$) such that Z' meets W "properly"; in other words, every component $C_{i,j,k}$ of each $Y_i \cap R_j$ has codimension $r+s$. One then defines

$$Z' \bullet W = \sum_{i,j,k} n_i m_j \cdot int(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}$$

where $int(Y_i \cap R_j, C_{i,j,k})$ is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a Z'' rationally equivalent to both Z, Z' and also intersecting W properly, then $Z' \bullet W$ is rationally equivalent to $Z'' \bullet W$.

In [3], Blaine Lawson and I showed how one could generalize this, considering any finite dimensional family of codimension r cycles Z_α and any finite dimensional family of codimension s cycles W_β, moving simultaneously each Z_α to a suitable Z'_α so that each Z'_α meets properly each W_β. This is achieved by showing that the classical argument admits parametrizations by large dimensional parameter spaces. This classical argument consists of two steps, in order to move Z into better position with respect to W. The first step involves a choice of a finite projection $\pi_L : X \subset \mathbb{P}^N$ onto $\mathbb{P}^n$, $n = dim(X)$, which determines the *projectiving cone* $C_L(Z) = \pi_L^*(\pi_{L*}(Z)) \subset \mathbb{P}^N$. The first parametrized family is the family of "linear centers" L parametrizing these projections. The second parametrized family is the family of moves in $\mathbb{P}^N$ which enables us to move the resulting projecting cone $C_L(Z)$ in order that $C_L(Z) \bullet X$ meets Y properly on X. The variation in the family of projections π_L enables us to arrange that the *residual cycle* $R_L(Z) = C_L(Z) \bullet X - Z$ has improved intersection with W that Z.

A completely different proof is given by William Fulton and Robert MacPherson. (cf [4]). They use a powerful geometric technique discovered by MacPherson called *deformation to the normal cone.* For $Y \subset X$ closed, the deformation space $M_Y(X)$ is a variety mapping to $\mathbb{P}^1$ defined as the complement in the *blow-up* of $X \times \mathbb{P}^1$ along $Y \times \infty$ of the blow-up of $X \times \infty$ along $Y \times \infty$. One readily verifies that $Y \times \mathbb{P}^1 \subset M(X,Y)$ restricts above $\infty \neq p \in \mathbb{P}^1$ to the given embedding $Y \subset X$; and above ∞, restricts to the inclusion of Y into the normal cone $C_Y(X) = Spec(\oplus_{n\geq 0}\mathcal{I}_Y^n/\mathcal{I}_Y^{n_1})$, where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf defining $Y \subset X$. When $Y \subset X$ is a regular closed embedding, then this normal cone is a bundle, the normal bundle $N_Y(X)$.

This enables a regular closed embedding (e.g., the diagonal $\delta : X \to X \times X$ for X smooth) to be deformed into the embedding of the 0-section of the normal bundle $N_{\delta(X)}(X \times X)$. One defines the intersection of Z, W as the intersection of $\delta(X), Z \times W$ and thus one reduces the problem of defining intersection product to the special case of intersection of the 0-section of the normal bundle $N_X(X \times X)$ with the normal cone $N_{(Z\times W)\cap\delta(X)}(Z \times W)$. □

4.2. **Chern classes and the Chern character.** Grothendieck introduced many basic techniques which we now use as a matter of course when working with bundles. The following *splitting principle* is one such technique, a technique which enable one to frequently reduce constructions for arbitrary vector bundles to those which are a sum of line bundles.

Proposition 4.2. *Let $\mathcal{E}$ be a rank $r+1$ vector bundle on a quasi-projective variety X and define $p_1 : \mathbb{P}(\mathcal{E}) = Proj\ (Sym_{\mathcal{O}_X}\mathcal{E}) \to X$ to be the projective bundle of lines in $\mathcal{E}$. Then $p_1^* : K_*(X) \to K_{*+r}(\mathbb{P}(\mathcal{E}))$ is split injective and $p_1^*(\mathcal{E}) = \mathcal{E}_1$ is a direct sum of a rank r bundle and a line bundle.*

Applying this construction to $\mathcal{E}_1$ on $\mathbb{P}(\mathcal{E})$, we obtain $p_2 : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$; proceeding inductively, we obtain

$$p = p_r \circ \cdots \circ p_1 : \mathbb{F}(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{r-1}) \to X$$

with the property that $p^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$ is split injective and $p^*(\mathcal{E})$ is a direct sum of line bundles.

We now introduce Chern classes and the Chern character, once again following Grothendieck's point of view.

Construction 4.3. Let $\mathcal{E}$ be a rank r vector bundle on a smooth, quasi-projective variety X of dimension d. Then $CH^*(\mathbb{P}(\mathcal{E}))$ is a free module over $CH^*(X)$ with generators $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$, where $\zeta \in CH^1(\mathbb{P}(\mathcal{E}))$ denotes the divisor class associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We define the i-th Chern class $c_i(\mathcal{E}) \in CH^i(X)$ of $\mathcal{E}$ by the formula

$$CH^*(\mathbb{P}(\mathcal{E})) \ = \ CH^*(X)[\zeta]/\sum_{i=0}^{r}(-1)^i\pi^*(c_i(\mathcal{E}))\cdot\zeta^{r-i}.$$

We define the total Chern class $c(\mathcal{E})$ by the formula

$$c(\mathcal{E}) \ = \ \sum_{i=0}^{r} c_i(\mathcal{E})$$

and set $c_t(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E})t^i$. Then the *Whitney sum* formula asserts that $c_t(\mathcal{E} \oplus \mathcal{F}) = c_t(\mathcal{E}) \cdot c_t(\mathcal{F})$.

We define the *Chern roots*, $\alpha_1, \ldots, \alpha_r$ of $\mathcal{E}$ by the formula

$$c_t(\mathcal{E}) \ = \ \prod_{i=1}^{r}(1 + \alpha_i t)$$

where the factorization can be viewed either as purely formal or as occurring in $\mathbb{F}(\mathcal{E})$. Observe that $c_k(\mathcal{E})$ is the k-th elementary symmetric function of these Chern roots.

In other words, the Chern classes of the rank r vector bundle $\mathcal{E}$ are given by the expression for $\zeta^r \in CH^r(\mathbb{P}(\mathcal{E}))$ in terms of the generators $1, \zeta, \ldots, \zeta^{r-1}$. Thus, the Chern classes depend critically on the identification of the first Chern class ζ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the multiplicative structure on $CH^*(X)$. The necessary structure for such a definition of Chern classes is called an *oriented multiplicative cohomology theory.* The splitting principle guarantees that Chern classes are uniquely determined by the assignment of first Chen classes to line bundles.

We refer the interested reader to [4] for the definition of "operational Chern classes" defined for bundles on a non necessarily smooth variety.

Construction 4.4. Let X be a smooth, quasi-projective variety, let $\mathcal{E}$ be a rank r vector bundle over X, and let $\pi : \mathbb{F}(\mathcal{E}) \to X$ be the associated bundle of flags of $\mathcal{E}$. Write $\pi^*(\mathcal{E}) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where each $\mathcal{L}_i$ is a line bundle on $\mathbb{F}(\mathcal{E})$. Then $c_t(\pi^*(\mathcal{E})) = \prod_{i=1}^r (1 \oplus c_1(\mathcal{L}_i))t$.

We define the Chern character of $\mathcal{E}$ as

$$ch(\mathcal{E}) = \sum_{i=1}^r \{1 + c_1(\mathcal{L}_i) + \frac{1}{2} c_1(\mathcal{L}_i)^2 + \frac{1}{3!} c_1(\mathcal{L}_i)^3 + \cdots\} = \sum_{i=1}^r exp(c_t(\mathcal{L}_i)),$$

where this expression is verified to lie in the image of the injective map $CH^*(X) \otimes \mathbb{Q} \to CH^*(\mathbb{F}(\mathcal{E})) \otimes \mathbb{Q}$. (Namely, one can identify $ch_k(\mathcal{E})$ as the k-th power sum of the Chern roots, and therefore expressible in terms of the Chern classes using Newton polynomials.)

Since $\pi^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E})), \quad \pi^* : CH^*(X) \to CH^*(\mathbb{F}(\mathcal{E}))$ are ring homomorphisms, the splitting principle enables us to immediately verify that ch is also a ring homomoprhism (i.e., sends the direct sum of bundles to the sum in $CH^*(X)$ of Chern characters, sends the tensor product of bundles to the product in $CH^*(X)$ of Chern characters).

Grothendieck's formulation of the Riemann-Roch theorem is an assertion of the behaviour of the Chern character ch with respect to push-forward maps induced by a proper smooth map $f : X \to Y$ of smooth varieties. It is not the case that ch commutes with the these push-forward maps; one must modify the push forward map in K-theory by multiplication by the Todd class.

This modification of the Todd class is necessary even when consideration of the push-forward of a divisor. Indeed, the Todd class

$$td : K_0(X) \to A^*(X)$$

(given explicitly for a vector bundle E in terms of the Charn roots α_i of E as $\prod_i \frac{\alpha_i}{1-e^{\alpha_i}}$) is characterized by the properties that

- i. $td(L) = c_1(L)/(1 - exp(-c_1(L));$
- ii. $td(E_1 \oplus E_2) = td(E_1) \cdot td(E_2)$; and
- iii. $td \circ f^* = f^* \circ td$.

The reader is recommended to consult [2] for a very nice overview of Grothendieck's Riemann-Roch Theorem.

Theorem 4.5. *(Grothendieck's Riemann-Roch Theorem)*

Let $f : X \to Y$ be a projective map of smooth varieties. Then for any $x \in K_0(X)$, we have the equality

$$ch(f_!(x)) \cdot td(T_Y) \quad = \quad f_*(ch(x) \cdot td(T_X))$$

where T_X, T_Y are the tangent bundles of X, Y and $td(T_X), td(T_Y)$ are their Todd classes.

Here, $f_! : K_0(X) \to K_0(Y)$ is defined by identifying $K_0(X)$ with $K_0'(X)$, $K_0(Y)$ with $K_0'(Y)$, and defining $f_! : K_0'(X) \to K_0'(Y)$ by sending a coherent sheaf $\mathcal{F}$ on X to $\sum_i (-1)^i R^i f_*(F)$. The map $f_* : CH_*(X) \to CH_*(Y)$ is proper push-forward of cycles.

Just to make this more concrete and more familiar, let us consider a very special case in which X is a projective, smooth curve, Y is a point, and $x \in K_0(X)$ is the class of a line bundle $\mathcal{L}$. (Hirzebruch had earlier proved a version of Grothendieck's theorem in which the target Y was a point.)

Example 4.6. Let C be a projective, smooth curve of genus g and let $f : C \to \operatorname{Spec} \mathbb{C}$ be the projection to a point. Let $\mathcal{L}$ be a line bundle on C with first Chern class $D \in CH^1(C)$. Then

$$f_!([\mathcal{L}]) = dimH^0(C, \mathcal{L}) - dimH^1(C, \mathcal{L}) \in \mathbb{Z},$$

and $ch : K_0(\operatorname{Spec} \mathbb{C}) = \mathbb{Z} \to A^*(\operatorname{Spec} \mathbb{C}) = \mathbb{Z}$ is an isomorphism. Let $K \in CH^1(C)$ denote the "canonical divisor", the first Chern class of the line bundle Ω_C, the dual of T_C. Then

$$td(T_C) = \frac{-K}{1 - (1 + K + \frac{1}{2}K^2)} = 1 - \frac{1}{2}K.$$

Recall that $deg(K) = 2g - 2$. Since $ch([\mathcal{L}]) = 1 + D$, we conclude that

$$f_*(ch([\mathcal{L}]) \cdot td(T_C)) = f_*((1 + D) \cdot (1 - \frac{1}{2}K)) = deg(D) - \frac{1}{2}deg(K).$$

(Note that $1 \in CH^0(X)$ is the fundamental class of X; that $f_* : CH^*(X) \to CH^*(\operatorname{Spec} \mathbb{C})$ simply takes the 0-cycle component.) Thus, in this case, Riemann-Roch tell us that

$$dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) = deg(D) + 1 - g.$$

For our purpose, Riemann-Roch is especially important because of the following consequence.

Theorem 4.7. *Let X be a smooth quasi-projective variety. Then*

$$ch_* : K_0(X) \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$$

is a ring isomorphism.

Proof. The essential ingredient is the Riemann-Roch theorem. Namely, we have a natural map $CH^*(X) \to K_0'(X)$ sending an irreducible subvariety Y to the class $[\mathcal{O}_Y]$ of the $\mathcal{O}_X$-module $\mathcal{O}_Y$. We put a filtration on $K_0'(X)$ using the dimension of support of a coherent sheaf $\mathcal{F} \in K_0'(X)$ and conclude using "localization" and "devissage" (see Lecture 5) that this natural map induces a surjection from $CH^*(X)$ to the associated graded group of $K_0'(X)$.

We show that the composition with the Chern character is an isomorphism on $CH^*(X) \otimes \mathbb{Q}$ by applying Grothendieck's Riemann-Roch theorem to each closed immersion $Y \subset X$, for Y an irreducible smooth subvariety of X. Namely, Riemann Roch implies that $ch_*([\mathcal{O}_Y]) \in CH^*(X)$ has the form the sum of Y and terms of lower dimension. Indeed, this argument applies to irreducible subvarieties Y of X with singularities, by observing that the contribution of singularities is also of higher codimension using a localization sequence and induction.

Thus, the associated graded map of ch_* is an isomorphism, which imples that ch_* is also an isomorphism.

□

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5. Lecture 5

5.1. **Quillen's localization theorem and Bloch's formula.** Our next topic is a sketch of Quillen's proof of Bloch's formula, which is also a a brief discussion of aspects of Quillen's remarkable paper [4]. For K_2, this formula was proved by Bloch in [2].

We remind the reader that in this paper Quillen introduces the "Quillen Q-construction" $Q\mathcal{E}$ of an exact category $\mathcal{E}$ and defines the K-groups of $\mathcal{E}$ to be the homotopy groups of the classifying space of $Q\mathcal{E}$. Of most importance to us are the abelian category $\mathcal{M}_X$ of coherent sheaves on a Noetherian scheme $\mathcal{O}_X$ and the exact subcategory $\mathcal{P}_X \subset \mathcal{M}_X$, yielding

$$\mathcal{K}_*(X) = \pi_*(BQ\mathcal{P}_X), \quad K'_*(X) = \pi_*(BQ\mathcal{M}_X).$$

A key ingredient in Quillen's proof of Bloch's formula is the localization sequence for K'_*, extending known localization sequences for low K-groups. Quillen formulates his results in an abstract, categorical setting.

Theorem 5.1. *(Localization Theorem of Quillen, cf.* [4]*) Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory with quotient category $\mathcal{A}/\mathcal{B}$. Then there is a long exact sequence of Quillen K-groups*

$$\cdots K_1(\mathcal{A}) \to K_1(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

In conjunction with Quillen's "devissage theorem", this localization theorem implies the following:

loc **Theorem 5.2.** *Consider $X \in (Sch/k)$ and let $\mathcal{M}_r(X)$ denote the Serre subcategory of the category $\mathcal{M}_X$ consisting of coherent sheaves whose support has codimension $\geq r$. Then there is a natural long exact sequence*

$$\cdots \to \coprod_{x \in X_r} K_{i+1}k(x) \to K_i(\mathcal{M}_{r+1}(X)) \to K_i(\mathcal{M}_r(X)) \to \coprod_{x \in X_r} K_ik(x) \to \cdots$$

Here, X_r denotes the set of points of X of codimension r.

Consequently, there is a spectral sequence

$$E_1^{p,q}(X) = \coprod_{x \in X_p} K_{-p-q}k(x) \Rightarrow K'_{-n}(X)$$

relating the K-theory of the residue fields of points of X to the K' theory of X.

Proof. Quillen's devissage theorem tells us that $K_i(\mathcal{M}_r(X)/\mathcal{M}_{r+1}(X))$ is naturally isomorphic to $\coprod_{X_r} K_ik(x)$. The asserted exact sequences patch together to give an exact couple, with the indexing of the spectral sequence determined by this exact couple. □

Definition 5.3. The Gersten complex for K'_n is the complex

$$0 \to K'_nX \to \coprod_{x \in X_0} K_nk(x) \to \coprod_{x \in X_1} K_{n-1}k(x) \to \cdots \to \coprod_{x \in X_n} K_0k(x) \to 0$$

determined by the exact sequences of Theorem 5.2.

Essentially by inspection, we have the following thereom concerning the relationship of the spectral sequence of Theorem 5.2 and the exactness of the Gersten complex.

Proposition 5.4. *Let $X \in (Sch/k)$. Then the following conditions are equivalent:*

1.) For every $r \geq 0$, the inclusion $\mathcal{M}_{r+1}(X) \to \mathcal{M}_r(X)$ induces the zero map on K-groups.

2.) In the spectral sequence of Theorem 5.2, for all $q, E_2^{p,q} = 0$ for $p > 0$ and the edge homomorphism $K'_{-q}X \to E_2^{0,q}X$ is an isomorphism.

3.) The Gersten complex for X is exact.

Here is Quillen's theorem establishing the validity of Bloch's formula.

Theorem 5.5. *(Bloch's formula by Quillen* [4]*) Let $X \in Sch(k)$ be regular. Then there is a cannoical isomorphism*

$$H^q(X, \mathcal{K}_q) \ \simeq CH^q(X)$$

where $\mathcal{K}_q$ is the sheaf on X (for the Zariski topology) associated to the presheaf $U \mapsto K_q(U)$.

Proof. Granted the above analysis of the Quillen spectral sequence, there are two additional ingredients in the proof.

The first is Quillen's theorem that the Gersten resolution is exact for $\operatorname{Spec} O_{X,x}$ whenever $X \in Sch(k)$ and $x \in X$ is a regular point. This tells us that the Gersten complex for $K'_n(X)$ becomes a resolution of resolution of $K_n(X)$ by flasque sheaves

$$0 \to K_nX \to \coprod_{x \in X_0} i_{x*}K_nk(x) \to \coprod_{x \in X_1} i_{x*}K_{n-1}k(x) \to \cdots$$

Consequently, the E_2-term of the Quillen spectral sequence has the form

$$E_2^{p,q}(X) = H^p(X, \mathcal{K}_{-q}) \Rightarrow K_{-p-q}(X).$$

The second is Quillen's determination of the last differential in the Gersten complex

$$\coprod_{x \in X_{q-1}} K_1k(x) \xrightarrow{d_1} \coprod_{x \in X_q} K_0k(x) = Z^q(X).$$

Quillen concludes that the image of this map is precisely the codimension q cycles rationally equivalent to 0. □

5.2. **Derived categories.** In order to formulate motivic cohomology, we need to introduce the language of derived categories. Let $\mathcal{A}$ be an abelian category (e.g., the category of modules over a fixed ring) and consider the category of chain complexes $CH^\bullet(\mathcal{A})$. We shall index our chain complexes so that the differential has degree $+1$. We assume that $\mathcal{A}$ has enough injectives and projectives, so that we can construct the usual derived functors of left exact and right exact functors from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For example, if $F : \mathcal{A} \to \mathcal{B}$ is right exact, then we define $L_iF(A)$ to be the i-th homology of the chain complex $F(P_\bullet)$ obtained by applying F to a projective resolution $P_\bullet \to A$ of A; similarly, if $G : \mathcal{A} \to \mathcal{B}$ is left exact, then $R^jG(A) = H^j(I^\bullet)$ where $A \to I^\bullet$ is an injective resolution of A.

The usual verification that these derived functors are well defined up to canonical isomorphism actually proves a bit more. Namely, rather take the homology of the complexes $F(P_\bullet), G(I^\bullet)$, we consider these complexes themselves and observe that they are independent *up to quasi-isomorphism* of the choice of resolutions. Recall, that a map $C^\bullet \to D^\bullet$ is a quasi-isomorphism if it induces an isomorphism on homology; only in special cases is a complex $C^\bullet$ quasi-isomorphic to its homology $H^\bullet(C^\bullet)$ viewed as a complex with trivial differential.

We define the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ as the category obtained from the category of $CH^\bullet(\mathcal{A})$ of chain complexes of $\mathcal{A}$ by inverting quasi-isomorphisms. Of course, some care must be taken to insure that such a localization of $CH^\bullet(\mathcal{A})$ is well defined. Let $Hot(CH^\bullet(\mathcal{A}))$ denote the homotopy category of chain complexes of $\mathcal{A}$: maps from the chain complex $C^\bullet$ to the chain complex $D^\bullet$ in $\mathcal{H}(CH^\bullet(\mathcal{A}))$ are chain homotopy equivalence classes of chain maps. Since chain homotopic maps induce the same map on homology, we see that $\mathcal{D}(\mathcal{A})$ can also be defined as the category obtained from $Hot(CH^\bullet(\mathcal{A}))$ by inverting quasi-isomorphisms.

The derived category $\mathcal{D}(\mathcal{A})$ of the abelian category $CH^\bullet(\mathcal{A})$ is a *triangulated category.* Namely, we have a shift operator $(-)[n]$ defined by

$$(A^\bullet[n])^j \;\equiv\; A^{n+j}.$$

This indexing is very confusing (as would be any other); we can view $A^\bullet[n]$ as $A^\bullet$ shifted "down" or "to the left". We also have *distinguished triangles*

$$A^\bullet \to B\bullet \to C^\bullet \to A^\bullet[1]$$

defined to be those "triangles" quasi-isomorphic to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of chain complexes.

This notation enables us to express Ext-groups quite neatly as

$$\begin{aligned} Ext^i_{\mathcal{A}}(A,B) &= H^i(Hom_{\mathcal{A}}(P_\bullet, B)) = Hom_{\mathcal{D}(\mathcal{A})}(A[-i], B) \\ &= Hom_{\mathcal{D}(\mathcal{A})}(A, B[i]) = H^i(Hom_{\mathcal{A}}(A, P^\bullet)). \end{aligned}$$

5.3. **Bloch's Higher Chow Groups.** From our point of view, motivic cohomology should be a "cohomology theory" which bears a relationship to $K_*(X)$ analogous to the role Chow groups $CH^*(X)$ bear to $K_0(X)$ (and analogous to the relationship of $H^*_{sing}(T)$ to $K^*_{top}(T)$). In particular, motivic cohomology will be doubly indexed.

We now discuss a relatively naive construction by Spencer Bloch of "higher Chow groups" which satisfies this criterion. We shall then consider a more sophisticated version of motivic cohomology due to Suslin and Voevodsky.

We work over a field k and define Δ^n to be $\operatorname{Spec} k[x_0,\ldots,x_n]/(\sum_i x_i - 1)$, the algebraic n-simplex. As in topology, we have face maps $\partial_i : \Delta^{n-1} \to \Delta^n$ (sending the coordinate function $x_i \in k[\Delta^n]$ to 0) and degeneracy maps $\sigma_j : \Delta^{n+1} \to \Delta^n$ (sending the coordinate function $x_j \in k[\Delta^n]$ to $x_j + x_{j+1} \in k[\Delta^{n+1}]$). More generally, a composition of face maps determines a face $F \simeq \Delta^i \to \Delta^n$. Of course, $\Delta^n \simeq \mathbb{A}^n$.

Bloch's idea is to construct a chain complex for each q which in degree n would be the codimension q-cycles on $X \times \Delta^n$. In particular, the 0-th homology of this chain complex should be the usual Chow group $CH^q(X)$ of codimension q cycles on X modulo rational equivalence. This can not be done in a completely straightforward manner, since one has no good way in general to restrict a general cycle on $X \times \Delta[n]$ via a face map ∂_i to $X \times \Delta^{n-1}$. Thus, Bloch only considers codimension q cycles on $X \times \Delta^n$ which restrict properly to all faces (i.e., to codimension q cycles on $X \times F$).

Definition 5.6. Let X be a variety over a field k. For each $p \geq 0$, we define a complex $z_p(X,*)$ which in degree n is the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$dim_k(Z \cap (X \times F)) \leq dim_k(F) + p.$$

The differential of $z_p(X,*)$ is the alternating sum of the maps induced by restricting cycles to codimension 1 faces. Define the *higher Chow homology groups* by

$$CH_p(X,n) = H_n(z_p(X,*)), \quad n,p \geq 0.$$

If X is locally equi-dimensional over k (e.g., X is smooth), let $z^q(X,n)$ be the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$codim_{X \times F}(Z \cap (X \times F)) \geq q.$$

Define the *higher Chow cohomology groups* by

$$CH^q(X,n) = H_n(z^q(X,*)), \quad n,q \geq 0,$$

where the differential of $z^q(X,*)$ is defined exactly as for $z_p(X,*)$.

Bloch, with the aid of Marc Levine, has proved many remarkable properties of these higher Chow groups.

Theorem 5.7. *Let X be a quasi-projective variety over a field. Bloch's higher Chow groups satisfy the following properties:*

- $CH_p(-,*)$ *is covariantly functorial with respect to proper maps.*
- $CH^q(-,*)$ *is contravariantly functorial on Sm_k, the category of smooth quasi-projective varieties over k.*
- $CH_p(X,0) = CH_p(X)$*, the Chow group of p-cycles modulo rational equivalence.*
- *(Homotopy invariance)* $\pi^* : CH_p(X,*) \xrightarrow{\sim} CH_{p+1}(X \times \mathbb{A}^1)$.
- *(Localization) Let $i : Y \to X$ be a closed subvariety with $j : U = X - Y \subset X$ the complement of Y. Then there is a distinguished triangle*

$$z_p(Y,*) \xrightarrow{i_*} z_p(X,*) \xrightarrow{j^*} z_p(U,*) \to z_p(Y,*)[1]$$

- *(Projective bundle formula) Let $\mathcal{E}$ be a rank n vector bundle over X. Then $CH^*(\mathbb{P}(\mathcal{E})*)$ is a free $CH^*(X,*)$-module on generators $1, \zeta, \ldots, \zeta^{n-1} \in CH^1(\mathbb{P}(\mathcal{E}), 0)$.*
- *For X smooth, $K_i(X) \otimes \mathbb{Q} \simeq \oplus_q CH^q(X,i) \otimes \mathbb{Q}$ for any $i \geq 0$. Moreover, for any $q \geq 0$,*

$$(K_i(X) \otimes \mathbb{Q})^{(q)} \simeq CH^q(X,i) \otimes \mathbb{Q}.$$

- *If F is a field, the $K_n^M(F) \simeq CH^n(Spec\, F, n)$.*

The most difficult of these properties, and perhaps the most important, is localization. The proof requires a very subtle technique of moving cycles. Observe that $z_p(X,*) \to z_p(U,*)$ is not surjective because the conditions of proper intersection on an element of $z_p(U,n)$ (i.e, a cycle on $U \times \Delta^n$) might not continue to hold for the closure of that cycle in $X \times \Delta^n$.

5.4. **Beilinson's Conjectures.** We give below a list of conjectures due to Beilinson which relate motivic cohomology and K-theory. Bloch's higher Chow groups go some way toward providing a theory which satisfies these conjectures. Namely, Beilinson conjectures the existence of complexes of sheaves $\Gamma_{Zar}(r)$ whose cohomology (in the Zariski topology) $H^p(X, \Gamma_{Zar}(r))$ one could call "motivic cohomology". If we set

$$H^p(X, \Gamma_{Zar}(r)) = CH^r(X, 2r-p),$$

then many of the cohomological conjectures Beilinson makes for his conjectured complexes are satisfied by Bloch's higher Chow groups $CH^\bullet(X,*)$.

Conjecture 5.8. *(Beilinson [1]) Let X be a smooth variety over a field k. Then there should exist complexes of sheaves $\Gamma_{Zar}(r)$ of abelian groups on X with the Zariski topology, well defined in $\mathcal{D}(AbSh(X_{Zar}))$, functorial in X, and equipped with a graded product, which satisfy the following properties:*

(1) $\Gamma_{Zar}(1) = \mathbb{Z}$; $\Gamma_{Zar}(1) \simeq G_m[-1]$.
(2) $H^{2n}(X,\Gamma_{zar}(n)) = CH^n(X)$.
(3) $H^i(Spec\,k,\Gamma_{Zar}(i)) = \mathcal{K}_i^M k$, *Milnor K-theory.*
(4) *(Motivic spectral sequence) There is a spectral sequence of the form*

$$E_2^{p,q} = H^{p-q}(X,\Gamma_{Zar}(q)) \Rightarrow K_{-p-q}(X)$$

which degenerates after tensoring with $\mathbb{Q}$. Moreover, for each prime ℓ, there is a mod-ℓ version of this spectral sequence

$$E_2^{p,q} = H^{p-q}(X,\Gamma_{Zar}(q) \otimes^L \mathbb{Z}/\ell) \Rightarrow K_{-p-q}(X,\mathbb{Z}/\ell)$$

(5) $gr^r_\gamma(K_j(X) \otimes \mathbb{Q} \simeq \mathbb{H}^{2r-j}(X_{Zar},\Gamma_{Zar}(r))_{\mathbb{Q}}$.
(6) *(Beilinson-Lichtenbaum Conjecture) $\Gamma_{Zar} \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq r}\mathbb{R}\pi_*(\mu_\ell^{\otimes r})$ in the derived category $\mathcal{D}(AbSh(X_{Zar}))$ provided that ℓ is invertible in $\mathcal{O}_X$, where $\pi : X_{et} \to X_{Zar}$ is the change of topology morphism.*
(7) *(Vanishing Conjecture) $\Gamma_{Zar}(r)$ is acyclic outside $[1,r]$ for $r \geq 1$.*

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (4) using "motivic complexes" $\Gamma_{Zar}(r)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X,\mathbf{\Gamma}_{Zar}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\Gamma_{Zar}(-q/2) = 0$ if $-q$ is not an even non-positive integer and $\Gamma_{Zar}(-q/2) = \Gamma_{Zar}(i)$ is $-q = 2i \geq 0$.

(1) and (2) just "normalize" our complexes, assuring us that they extend usual Chow groups and what is known in codimensions 0 and 1. Note that (1) and (2) are compatible in the sense that

$$H^2(X,\Gamma_{Zar}(1)) = H^2(X,\mathcal{O}_X^*[-1]) = H^1(X,\mathcal{O}_X^*) = Pic(X).$$

(3) asserts that for a field k, the n-th cohomology of $\Gamma_{Zar}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory. This has been verified for Bloch's higher Chow groups by Suslin-Nesterenko and Totaro.

The (integral) spectral sequence of (4) has been established thanks to the work of many authors. This spectral sequence "collapses" at the E_2-level when tensored with $\mathbb{Q}$, so that $E_2 \otimes \mathbb{Q} = E_\infty \otimes \mathbb{Q}$. (5) asserts that this collapsing can be verified by using Adams operations, interpreted using the γ-filtration.

The vanishing conjecture of (7) is the most problematic, and there is no consensus on whether it is likely to be valid. However, (6) incorporates the mod-ℓ version of the vanishing conjecture and has apparently been proved by Rost and Voevodsky.

(6) asserts that if we consider the complexes $\Gamma_{Zar}(r)$ modulo ℓ (in the sense of the derived category), then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes r}$ coefficients, where μ_ℓ is the etale sheaf of ℓ-th roots of unity (isomorphic to $\mathbb{Z}/\ell$ if all ℓ-th roots of unity are in k. If the terms in the mod-ℓ spectral sequence were simply etale cohomology, then we would get etale K-theory which would violate the vanishing conjectured in (7) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo ℓ should be the cohomology of complexes which involve a truncation.

More precisely, $\mathbb{R}\pi_*F$ is a complex of sheaves for the Zariski topology (given by applying π_* to an injective resolution $F \to I^\bullet$ of etale sheaves) with the property that $H^*_{Zar}(X, \mathbb{R}\pi_*F) = H^*_{et}(X, F)$. Now, the n-th truncation of $\mathbb{R}\pi_*F$, $\tau_{\leq n}\mathbb{R}\pi_*F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $\mathbb{R}\pi_*F$ in degrees $\leq n$ and are 0 in degrees greater than n. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Speck$, then $H^p(Speck, \tau_{\leq n}\mathbb{R}\pi_*\mu_\ell^{\otimes n})$ equals $H^p_{et}(Speck, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

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5. Lecture 5

5.1. **Quillen's localization theorem and Bloch's formula.** Our next topic is a sketch of Quillen's proof of Bloch's formula, which is also a a brief discussion of aspects of Quillen's remarkable paper [4]. For K_2, this formula was proved by Bloch in [2].

We remind the reader that in this paper Quillen introduces the "Quillen Q-construction" $Q\mathcal{E}$ of an exact category $\mathcal{E}$ and defines the K-groups of $\mathcal{E}$ to be the homotopy groups of the classifying space of $Q\mathcal{E}$. Of most importance to us are the abelian category $\mathcal{M}_X$ of coherent sheaves on a Noetherian scheme $\mathcal{O}_X$ and the exact subcategory $\mathcal{P}_X \subset \mathcal{M}_X$, yielding

$$\mathcal{K}_*(X) = \pi_*(BQ\mathcal{P}_X), \quad K'_*(X) = \pi_*(BQ\mathcal{M}_X).$$

A key ingredient in Quillen's proof of Bloch's formula is the localization sequence for K'_*, extending known localization sequences for low K-groups. Quillen formulates his results in an abstract, categorical setting.

Theorem 5.1. *(Localization Theorem of Quillen, cf.* [4]*) Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory with quotient category $\mathcal{A}/\mathcal{B}$. Then there is a long exact sequence of Quillen K-groups*

$$\cdots K_1(\mathcal{A}) \to K_1(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

In conjunction with Quillen's "devissage theorem", this localization theorem implies the following:

loc **Theorem 5.2.** *Consider $X \in (Sch/k)$ and let $\mathcal{M}_r(X)$ denote the Serre subcategory of the category $\mathcal{M}_X$ consisting of coherent sheaves whose support has codimension $\geq r$. Then there is a natural long exact sequence*

$$\cdots \to \coprod_{x \in X_r} K_{i+1}k(x) \to K_i(\mathcal{M}_{r+1}(X)) \to K_i(\mathcal{M}_r(X)) \to \coprod_{x \in X_r} K_i k(x) \to \cdots$$

Here, X_r denotes the set of points of X of codimension r.

Consequently, there is a spectral sequence

$$E_1^{p,q}(X) = \coprod_{x \in X_p} K_{-p-q}k(x) \Rightarrow K'_{-n}(X)$$

relating the K-theory of the residue fields of points of X to the K' theory of X.

Proof. Quillen's devissage theorem tells us that $K_i(\mathcal{M}_r(X)/\mathcal{M}_{r+1}(X))$ is naturally isomorphic to $\coprod_{X_r} K_i k(x)$. The asserted exact sequences patch together to give an exact couple, with the indexing of the spectral sequence determined by this exact couple. □

Definition 5.3. The Gersten complex for K'_n is the complex

$$0 \to K'_n X \to \coprod_{x \in X_0} K_n k(x) \to \coprod_{x \in X_1} K_{n-1}k(x) \to \cdots \to \coprod_{x \in X_n} K_0 k(x) \to 0$$

determined by the exact sequences of Theorem 5.2.

Essentially by inspection, we have the following thereom concerning the relationship of the spectral sequence of Theorem 5.2 and the exactness of the Gersten complex.

Proposition 5.4. *Let $X \in (Sch/k)$. Then the following conditions are equivalent:*

1.) For every $r \geq 0$, the inclusion $\mathcal{M}_{r+1}(X) \to \mathcal{M}_r(X)$ induces the zero map on K-groups.

2.) In the spectral sequence of Theorem 5.2, for all $q, E_2^{p,q} = 0$ for $p > 0$ and the edge homomorphism $K'_{-q}X \to E_2^{0,q}X$ is an isomorphism.

3.) The Gersten complex for X is exact.

Here is Quillen's theorem establishing the validity of Bloch's formula.

Theorem 5.5. *(Bloch's formula by Quillen* [4]*) Let $X \in Sch(k)$ be regular. Then there is a cannoical isomorphism*

$$H^q(X, \mathcal{K}_q) \ \simeq CH^q(X)$$

where $\mathcal{K}_q$ is the sheaf on X (for the Zariski topology) associated to the presheaf $U \mapsto K_q(U)$.

Proof. Granted the above analysis of the Quillen spectral sequence, there are two additional ingredients in the proof.

The first is Quillen's theorem that the Gersten resolution is exact for $\operatorname{Spec} O_{X,x}$ whenever $X \in Sch(k)$ and $x \in X$ is a regular point. This tells us that the Gersten complex for $K'_n(X)$ becomes a resolution of resolution of $K_n(X)$ by flasque sheaves

$$0 \to K_nX \to \coprod_{x \in X_0} i_{x*}K_nk(x) \to \coprod_{x \in X_1} i_{x*}K_{n-1}k(x) \to \cdots$$

Consequently, the E_2-term of the Quillen spectral sequence has the form

$$E_2^{p,q}(X) = H^p(X, \mathcal{K}_{-q}) \Rightarrow K_{-p-q}(X).$$

The second is Quillen's determination of the last differential in the Gersten complex

$$\coprod_{x \in X_{q-1}} K_1k(x) \xrightarrow{d_1} \coprod_{x \in X_q} K_0k(x) = Z^q(X).$$

Quillen concludes that the image of this map is precisely the codimension q cycles rationally equivalent to 0. □

5.2. **Derived categories.** In order to formulate motivic cohomology, we need to introduce the language of derived categories. Let $\mathcal{A}$ be an abelian category (e.g., the category of modules over a fixed ring) and consider the category of chain complexes $CH^\bullet(\mathcal{A})$. We shall index our chain complexes so that the differential has degree $+1$. We assume that $\mathcal{A}$ has enough injectives and projectives, so that we can construct the usual derived functors of left exact and right exact functors from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For example, if $F : \mathcal{A} \to \mathcal{B}$ is right exact, then we define $L_iF(A)$ to be the i-th homology of the chain complex $F(P_\bullet)$ obtained by applying F to a projective resolution $P_\bullet \to A$ of A; similarly, if $G : \mathcal{A} \to \mathcal{B}$ is left exact, then $R^jG(A) = H^j(I^\bullet)$ where $A \to I^\bullet$ is an injective resolution of A.

The usual verification that these derived functors are well defined up to canonical isomorphism actually proves a bit more. Namely, rather take the homology of the complexes $F(P_\bullet), G(I^\bullet)$, we consider these complexes themselves and observe that they are independent *up to quasi-isomorphism* of the choice of resolutions. Recall, that a map $C^\bullet \to D^\bullet$ is a quasi-isomorphism if it induces an isomorphism on homology; only in special cases is a complex $C^\bullet$ quasi-isomorphic to its homology $H^\bullet(C^\bullet)$ viewed as a complex with trivial differential.

We define the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ as the category obtained from the category of $CH^\bullet(\mathcal{A})$ of chain complexes of $\mathcal{A}$ by inverting quasi-isomorphisms. Of course, some care must be taken to insure that such a localization of $CH^\bullet(\mathcal{A})$ is well defined. Let $Hot(CH^\bullet(\mathcal{A}))$ denote the homotopy category of chain complexes of $\mathcal{A}$: maps from the chain complex $C^\bullet$ to the chain complex $D^\bullet$ in $\mathcal{H}(CH^\bullet(\mathcal{A}))$ are chain homotopy equivalence classes of chain maps. Since chain homotopic maps induce the same map on homology, we see that $\mathcal{D}(\mathcal{A})$ can also be defined as the category obtained from $Hot(CH^\bullet(\mathcal{A}))$ by inverting quasi-isomorphisms.

The derived category $\mathcal{D}(\mathcal{A})$ of the abelian category $CH^\bullet(\mathcal{A})$ is a *triangulated category*. Namely, we have a shift operator $(-)[n]$ defined by

$$(A^\bullet[n])^j \;\equiv\; A^{n+j}.$$

This indexing is very confusing (as would be any other); we can view $A^\bullet[n]$ as $A^\bullet$ shifted "down" or "to the left". We also have *distinguished triangles*

$$A^\bullet \to B\bullet \to C^\bullet \to A^\bullet[1]$$

defined to be those "triangles" quasi-isomorphic to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of chain complexes.

This notation enables us to express Ext-groups quite neatly as

$$Ext^i_{\mathcal{A}}(A,B) = H^i(Hom_{\mathcal{A}}(P_\bullet, B)) = Hom_{\mathcal{D}(\mathcal{A})}(A[-i], B)$$
$$= Hom_{\mathcal{D}(\mathcal{A})}(A, B[i]) = H^i(Hom_{\mathcal{A}}(A, P^\bullet)).$$

5.3. **Bloch's Higher Chow Groups.** From our point of view, motivic cohomology should be a "cohomology theory" which bears a relationship to $K_*(X)$ analogous to the role Chow groups $CH^*(X)$ bear to $K_0(X)$ (and analogous to the relationship of $H^*_{sing}(T)$ to $K^*_{top}(T)$). In particular, motivic cohomology will be doubly indexed.

We now discuss a relatively naive construction by Spencer Bloch of "higher Chow groups" which satisfies this criterion. We shall then consider a more sophisticated version of motivic cohomology due to Suslin and Voevodsky.

We work over a field k and define Δ^n to be $\operatorname{Spec} k[x_0, \dots, x_n]/(\sum_i x_i - 1)$, the algebraic n-simplex. As in topology, we have face maps $\partial_i : \Delta^{n-1} \to \Delta^n$ (sending the coordinate function $x_i \in k[\Delta^n]$ to 0) and degeneracy maps $\sigma_j : \Delta^{n+1} \to \Delta^n$ (sending the coordinate function $x_j \in k[\Delta^n]$ to $x_j + x_{j+1} \in k[\Delta^{n+1}]$). More generally, a composition of face maps determines a face $F \simeq \Delta^i \to \Delta^n$. Of course, $\Delta^n \simeq \mathbb{A}^n$.

Bloch's idea is to construct a chain complex for each q which in degree n would be the codimension q-cycles on $X \times \Delta^n$. In particular, the 0-th homology of this chain complex should be the usual Chow group $CH^q(X)$ of codimension q cycles on X modulo rational equivalence. This can not be done in a completely straightforward manner, since one has no good way in general to restrict a general cycle on $X \times \Delta[n]$ via a face map ∂_i to $X \times \Delta^{n-1}$. Thus, Bloch only considers codimension q cycles on $X \times \Delta^n$ which restrict properly to all faces (i.e., to codimension q cycles on $X \times F$).

Definition 5.6. Let X be a variety over a field k. For each $p \geq 0$, we define a complex $z_p(X, *)$ which in degree n is the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$dim_k(Z \cap (X \times F)) \leq dim_k(F) + p.$$

The differential of $z_p(X,*)$ is the alternating sum of the maps induced by restricting cycles to codimension 1 faces. Define the *higher Chow homology groups* by

$$CH_p(X,n) = H_n(z_p(X,*)), \quad n,p \geq 0.$$

If X is locally equi-dimensional over k (e.g., X is smooth), let $z^q(X,n)$ be the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$codim_{X\times F}(Z \cap (X \times F)) \geq q.$$

Define the *higher Chow cohomology groups* by

$$CH^q(X,n) = H_n(z^q(X,*)), \quad n,q \geq 0,$$

where the differential of $z^q(X,*)$ is defined exactly as for $z_p(X,*)$.

Bloch, with the aid of Marc Levine, has proved many remarkable properties of these higher Chow groups.

Theorem 5.7. *Let X be a quasi-projective variety over a field. Bloch's higher Chow groups satisfy the following properties:*

- $CH_p(-,*)$ *is covariantly functorial with respect to proper maps.*
- $CH^q(-,*)$ *is contravariantly functorial on Sm_k, the category of smooth quasi-projective varieties over k.*
- $CH_p(X,0) = CH_p(X)$*, the Chow group of p-cycles modulo rational equivalence.*
- *(Homotopy invariance)* $\pi^* : CH_p(X,*) \xrightarrow{\sim} CH_{p+1}(X \times \mathbb{A}^1)$.
- *(Localization) Let $i : Y \to X$ be a closed subvariety with $j : U = X - Y \subset X$ the complement of Y. Then there is a distinguished triangle*

$$z_p(Y,*) \xrightarrow{i_*} z_p(X,*) \xrightarrow{j^*} z_p(U,*) \to z_p(Y,*)[1]$$

- *(Projective bundle formula) Let $\mathcal{E}$ be a rank n vector bundle over X. Then $CH^*(\mathbb{P}(\mathcal{E})*)$ is a free $CH^*(X,*)$-module on generators $1,\zeta,\ldots,\zeta^{n-1} \in CH^1(\mathbb{P}(\mathcal{E}),0)$.*
- *For X smooth, $K_i(X) \otimes \mathbb{Q} \simeq \oplus_q CH^q(X,i) \otimes \mathbb{Q}$ for any $i \geq 0$. Moreover, for any $q \geq 0$,*

$$(K_i(X) \otimes \mathbb{Q})^{(q)} \simeq CH^q(X,i) \otimes \mathbb{Q}.$$

- *If F is a field, the $K_n^M(F) \simeq CH^n(Spec\, F, n)$.*

The most difficult of these properties, and perhaps the most important, is localization. The proof requires a very subtle technique of moving cycles. Observe that $z_p(X,*) \to z_p(U,*)$ is not surjective because the conditions of proper intersection on an element of $z_p(U,n)$ (i.e, a cycle on $U \times \Delta^n$) might not continue to hold for the closure of that cycle in $X \times \Delta^n$.

5.4. **Beilinson's Conjectures.** We give below a list of conjectures due to Beilinson which relate motivic cohomology and K-theory. Bloch's higher Chow groups go some way toward providing a theory which satisfies these conjectures. Namely, Beilinson conjectures the existence of complexes of sheaves $\Gamma_{Zar}(r)$ whose cohomology (in the Zariski topology) $H^p(X,\Gamma_{Zar}(r))$ one could call "motivic cohomology". If we set

$$H^p(X,\Gamma_{Zar}(r)) = CH^r(X, 2r-p),$$

then many of the cohomological conjectures Beilinson makes for his conjectured complexes are satisfied by Bloch's higher Chow groups $CH^\bullet(X,*)$.

Conjecture 5.8. *(Beilinson [1]) Let X be a smooth variety over a field k. Then there should exist complexes of sheaves $\Gamma_{Zar}(r)$ of abelian groups on X with the Zariski topology, well defined in $\mathcal{D}(AbSh(X_{Zar}))$, functorial in X, and equipped with a graded product, which satisfy the following properties:*

(1) $\Gamma_{Zar}(1) = \mathbb{Z}$; $\Gamma_{Zar}(1) \simeq G_m[-1]$.
(2) $H^{2n}(X, \Gamma_{zar}(n)) = CH^n(X)$.
(3) $H^i(Spec\, k, \Gamma_{Zar}(i)) = \mathcal{K}_i^M k$, *Milnor K-theory.*
(4) *(Motivic spectral sequence) There is a spectral sequence of the form*

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q)) \Rightarrow K_{-p-q}(X)$$

which degenerates after tensoring with $\mathbb{Q}$. Moreover, for each prime ℓ, there is a mod-ℓ version of this spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q) \otimes^L \mathbb{Z}/\ell) \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell)$$

(5) $gr_\gamma^r(K_j(X) \otimes \mathbb{Q} \simeq \mathbb{H}^{2r-j}(X_{Zar}, \Gamma_{Zar}(r))_{\mathbb{Q}}$.
(6) *(Beilinson-Lichtenbaum Conjecture) $\Gamma_{Zar} \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq r}\mathbb{R}\pi_*(\mu_\ell^{\otimes r})$ in the derived category $\mathcal{D}(AbSh(X_{Zar}))$ provided that ℓ is invertible in $\mathcal{O}_X$, where $\pi : X_{et} \to X_{Zar}$ is the change of topology morphism.*
(7) *(Vanishing Conjecture) $\Gamma_{Zar}(r)$ is acyclic outside $[1,r]$ for $r \geq 1$.*

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (4) using "motivic complexes" $\Gamma_{Zar}(r)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X, \mathbf{\Gamma}_{Zar}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\Gamma_{Zar}(-q/2) = 0$ if $-q$ is not an even non-positive integer and $\Gamma_{Zar}(-q/2) = \Gamma_{Zar}(i)$ is $-q = 2i \geq 0$.

(1) and (2) just "normalize" our complexes, assuring us that they extend usual Chow groups and what is known in codimensions 0 and 1. Note that (1) and (2) are compatible in the sense that

$$H^2(X, \Gamma_{Zar}(1)) = H^2(X, \mathcal{O}_X^*[-1]) = H^1(X, \mathcal{O}_X^*) = Pic(X).$$

(3) asserts that for a field k, the n-th cohomology of $\Gamma_{Zar}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory. This has been verified for Bloch's higher Chow groups by Suslin-Nesterenko and Totaro.

The (integral) spectral sequence of (4) has been established thanks to the work of many authors. This spectral sequence "collapses" at the E_2-level when tensored with $\mathbb{Q}$, so that $E_2 \otimes \mathbb{Q} = E_\infty \otimes \mathbb{Q}$. (5) asserts that this collapsing can be verified by using Adams operations, interpreted using the γ-filtration.

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