0. Brief overview

Cycles and bundles are intrinsic invariants of algebraic varieties
Close connections going back to Grothendieck
Work with quasi-projective varieties over a field k

1. Algebraic vector bundles

Affine Varieties

We consider finitely generated commutative k-algebras, which are in natural 1-1 correspondence with affine varieties (over k).

Definition 1.1. An A-module P is projective if it is a direct summand of a free module. Equivalently, the A-module p is projective if whenever $M \to N$ is a surjective map of A-modules any map $f : P \to N$ of A-modules lifts to a map $\tilde{f} : P \to M$.

Remark 1.2. If T is a compact Hausdorff topological space and if $C(T)$ is the ring of continuous functions on T, then for any topological vector bundle $E \to T$ there is a natural action of $C(T)$ on the global sections of E, $\Gamma(E)$. This determines a natural bijective correspondence between isomorphism classes of topological vector bundles on T and finitely generated projective $C(T)$-modules.

Example 1.3. Let A be a Dedekind domain with field of fractions K. Recall that a *fractional ideal* $I \subset K$ is an A-submodule of K. The ideal class group is the group of fractional ideals modulo principal ideals, where the group operation is product of ideals.

Fractional ideals are the rank 1 projective A-modules P, where the rank of P is defined to be the dimension of the F vector space $P \otimes_A F$ for any ring homomorphism $A \to F$ from A to a field F.

Quasi-projective Varieties

Recall 1.4. Recall projective space$\mathbb{P}^N$, whose k points are equivalence classes of $N+1$-tuple, $\langle a_0, \dots, a_N \rangle$, some entry of which is non-zero. Two $N+1$-tuples $(a_0, \dots, a_N), (b_0, \dots, b_N)$ are equivalent if there exists some $0 \neq c \in k$ such that $(a_0, \dots, a_N) = (cb_0, \dots, cb_N)$.

If $F(X_0, \dots, X_N)$ is a homogeneous polynomial, then the zero locus $Z(F) \subset \mathbb{P}^N$ is well defined.

Recall that $\mathbb{P}^N$ is covered by standard affine opens $U_i = \mathbb{P}^N \backslash Z(X_i)$.

Recall the Zariski topology on $\mathbb{P}^N$, a base of open sets for which are the subsets of the form $U_G = \mathbb{P}^N \backslash Z(G)$.

Recall 1.5. Recall the notion of a presheaf on a topological space T: a contravariant functor from the category whose objects are open subsets of T and whose morphisms are inclusions.

Recall that a sheaf is a presheaf satisfying the sheaf axiom: for all open subsets U, V,

$$F(U \cup V) = F(U) \times_{F(U \cap V)} F(V).$$

Recall structure sheaf of "regular functions" $\mathcal{O}_{\mathbb{P}^N}$, sections of $\mathcal{O}_{\mathbb{P}^N}(U)$ on any open.

Definition 1.6. A projective variety X is a space with a sheaf of commutative rings $\mathcal{O}_X$ which admits a closed embedding into some $\mathbb{P}^N$, $i : X \subset \mathbb{P}^N$, so that $\mathcal{O}_X$ is the quotient of the sheaf $\mathcal{O}_{\mathbb{P}^N}$ by the ideal sheaf of those regular functions which vanish on X.

A quasi-projective variety U is once again a space with a sheaf of commutative rings $\mathcal{O}_U$ which admits a locally a closed embedding into some $\mathbb{P}^N$, $j : U \subset \mathbb{P}^N$, so that the closure $\overline{U} \subset \mathbb{P}^N$ of U admits the structure of a projective variety and so that $\mathcal{O}_U$ equals the restriction of $\mathcal{O}_{\overline{U}}$ to $U \subset \overline{U}$.

A quasi-projective variety U is said to be affine if U admits a closed embedding into some $\mathbb{A}^N = \mathbb{P}^N \backslash Z(X_0)$ so that $\mathcal{O}_U$ is the quotient of $\mathcal{O}_{\mathbb{A}^N}$ by the sheaf of ideals which vanish on U.

Remark 1.7. Any quasi-projective variety U has a base of (Zariski) open subsets which are affine.

Most quasi-projective varieties are neither projective nor affine.

There is a bijective correspondence between affine varieties and finitely generated commutative k-algebras. If U is an affine variety, then $\Gamma(\mathcal{O}_U)$ is the corresponding finitely generated k-algebra. Conversely, if A is written as a quotient $k[x_1, \ldots, x_N] \to A$, then $SpecA \to Speck[x_1, \ldots, x_N] = \mathbb{A}^N$ is the corresponding closed embedding of the affine variety $SpecA$.

Example 1.8. Let F be a polynomial in variables $X_0, \ldots, X_N$ homogeneous of degree d (i.e., $F(ca_0, \ldots, ca_N) = c^d F(a_0, \ldots, a_N)$. Then the zero locus $Z(F) \subset \mathbb{P}^N$ is called a hypersurface of degree d. For example if $N = 2$, then $Z(F)$ is 1-dimensional (i.e., a curve). If $k = \mathbb{C}$ and if the Jacobian of F does not vanish anywhere on $C = Z(F)$ (i.e., if C is *smooth*), then C is a projective, smooth, algebraic curve of genus $\frac{(d-1)(d-2)}{2}$.

Quasi-coherent Sheaves

Definition 1.9. Let X be a quasi-projective variety. A *quasi-coherent sheaf* $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules (i.e., an abelian sheaf equipped with a pairing $\mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ of sheaves satisfying the condition that for each open $U \subset X$ this pairing gives $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module) with the property that there exists an open covering $\{U_i \subset X; i \in I\}$ by affine open subsets so that $\mathcal{F}_{|U_i}$ is the sheaf associated to an $O_X(U_i)$-module M_i for each i.

If each of the M_i can be chosen to be finitely generated as an $O_X(U_i)$-module, then such a quasi-coherent sheaf is called *coherent.*

Definition 1.10. Let X be a quasi-projective variety. A coherent sheaf $\mathcal{E}$ on X is said to be an algebraic vector bundle if $\mathcal{E}$ is locally free. In other words, if there exists a (Zariski) open covering $\{U_i; i \in I\}$ of X such that $\mathcal{E}_{|U_i} \simeq \mathcal{O}^{e_i}_{X|U_i}$ for each i.

Remark 1.11. If quasi-projective variety is affine, then an algebraic vector bundle on X is equivalent to a projective $\Gamma(\mathcal{O}_X)$-module.

Construction 1.12. If M is a free A-module of rank r, then the symmetric algebra $Sym^\bullet_A(M)$ is a polynomial algebra of r generators over A and the structure map $\pi : \operatorname{Spec} Sym^\bullet_A(M) \to \operatorname{Spec} A$ is just the projection $\mathbb{A}^r \times \operatorname{Spec} A \to \operatorname{Spec} A$. This construction readily globalizes: if $\mathcal{E}$ is an algebraic vector bundle over X, then

$$\pi_{\mathcal{E}} : \mathbb{V}(\mathcal{E}) \equiv \operatorname{Spec} Sym^\bullet_{\mathcal{O}_X}(\mathcal{E})^* \ \to \ X$$

is locally a product projection: if $\{U_i \subset X; i \in\}$ is an open covering restricted to which $\mathcal{E}$ is trivial, then the restriction of $\pi_{\mathcal{E}}$ above each U_i is isomorphic to the product projection $\mathbb{A}^r \times U_i \to U_i$. In the above definition of $\pi_{\mathcal{E}}$ we consider the symmetric algebra on the dual $\mathcal{E}^* = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, so that the association $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E}^*)$ is covariantly functorial.

Thus, we may alternatively think of an algebraic vector bundle in terms of the map $\pi_{\mathcal{E}}$ of varieties.

Remark 1.13. We should be looking at the maximal ideal spectrum of a variety over a field k, rather than simply the k rational points, whenever k is not algebraically closed. We suppress this point, for we will soon switch to prime ideal spectra (i.e., work with schemes of finite type over k). However, we do point out that the reason it suffices to consider the maximal ideal spectrum rather the spectrum of all prime ideals is the validity of the Hilbert Nullstellensatz. One form of this important theorem is that the subset of maximal ideals constitute a dense subset of the space of prime ideals (with the Zariski topology) of a finitely generated commutative k-algebra.

<u>Examples</u>

Example 1.14. Rank 1 vector bundles $O_{\mathbb{P}^N}(k), k \in \mathbb{Z}$ on $\mathbb{P}^N$. The sections of $O_{\mathbb{P}^N}(j)$ on the basic open subset $U_G = \mathbb{P}^N\ Z(G)$ are given by the formula

$$O_{\mathbb{P}^N}(k)(U_G) = k[X_0, \dots, X_N, 1/G]_{(j)}$$

(i.e., ratios of homogeneous polynomials of total degree j).

In terms of the trivialization on the open covering $U_i, 0 \leq i \leq N$, the patching functions are given by $X_i^k / X_{i'}^j$.

$\Gamma(O_{\mathbb{P}^N}(j))$ has dimension $\binom{N+j}{j}$ if $j > 0$, dimension 1 if $j = 0$, and 0 otherwise. Thus, using the fact that $O_{\mathbb{P}^N}(j) \otimes_{\mathcal{O}_X} O_{\mathbb{P}^N}(j') = O_{\mathbb{P}^N}(j + j')$, we conclude that $\Gamma(O_{\mathbb{P}^N}(j))$ is not isomorphic to $\Gamma(O_{\mathbb{P}^N}(j'))$ provided that $j' \neq j$.

Proposition 1.15. *(Grothendieck) Each vector bundle on $\mathbb{P}^1$ has a unique decomposition as a finite direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(k), k \in \mathbb{Z}$.*

Example 1.16. Serre's conjecture (proved by Quillen and Suslin) asserts that every algebraic vector bundle on $\mathbb{A}^N$ (or any affine open subset of $\mathbb{A}^N$) is trivial. In more algebraic terms, every finitely generated projective $k[x_1, \dots, x_n]$-module is free.

Example 1.17. Let $X = Grass_{n,N}$, the Grassmann variety of n-planes in P^N (i.e., $n+1$-dimensional subspaces of k^{N+1}). We can embed $Grass_{n,N}$ as a Zariski closed subset of $\mathbb{P}^{M-1}$, where $M = \binom{N+1}{n+1}$, by sending the subspace $V \subset k^{N+1}$ to its $n+1$-st exterior power $\Lambda^{N+1}V \subset \mathbb{P}^M$. There is a natural rank n algebraic vector bundle $\mathcal{E}$ on X provided with an embedding in the trivial rank $N+1$ dimensional vector bundle $\mathcal{O}_X^{N+1}$ (in the special case $n = 1$, this is $\mathcal{O}_{\mathbb{P}^N}(-1) \subset \mathcal{O}_{\mathbb{P}^N}^{N+1}$) whose fibre above a point in X is the corresponding subspace. Of equal importance is the natural rank $N - n$-dimensional quotient bundle $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^N}^{N+1}/\mathcal{E}$.

This example readily generalizes to flag varieties.

Example 1.18. Let A be a commutative k-algebra and recall the module $\Omega_{A/k}$ of Kähler differentals. These globalize to a quasi-coherent sheaf Ω_X on a quasi-projective variety X over k. If X is *smooth* of dimension r, then Ω_X is an algebraic vector bundle over X of rank r.

Definition 1.19. Let X be a quasi-projective variety. We define $Pic(X)$ to be the abelian group whose elments are isomorphism classes of rank 1 algebraic vector bundles on X (also called "invertible sheaves"). The group structure on $Pic(X)$ is given by tensor product.

So defined, $Pic(X)$ is a generalization of the construction of the Class Group (of fractional ideals modulo principal ideal) for $X = \operatorname{Spec} A$ with A a Dedekind domain.

Example 1.20. By examing patching data, we readily verify that

$$H^1(X, \mathcal{O}_X^*) \ = \ Pic(X)$$

where $\mathcal{O}_X^*$ is the sheaf of abelian groups on X with sections $\Gamma(U, \mathcal{O}_X^*)$ defined to be the invertible elements of $\Gamma(U, \mathcal{O}_X)$ (with group structure given by multiplication).

If $k = \mathbb{C}$, then we have a short exact sequence of *analytic sheaves* of abelian sheaves on the analytic space $X(\mathbb{C})^{an}$,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{exp}{\to} \mathcal{O}_X^* \to 0.$$

We use identification due to Serre of analytic and algebraic vector bundles on a projective variety. If $X = C$ is a smooth curve, this identification enables us to conclude the short exact sequence

$$0 \to \mathbb{C}^g/\mathbb{Z}^{2g} \to Pic(C) \to H^2(C, \mathbb{Z})$$

since $H^1(C, \mathcal{O}_C) \simeq H^0(C, \Omega_C) = \mathbb{C}^g$ (where g is the genus of C). In particular, we conclude that for a curve of positive genus, $Pic(C)$ is very large, having a "continuous part" (which is an abelian variety).

Example 1.21. A $K3$ surface S over the complex numbers is characterized by the conditions that $H^0(S, \Lambda^2(\Omega_S)) = 0 = H^1_{sing}(S, \mathbb{Q})$. Even though the homotopy type of a $K3$ surface is well defined, the rank of $Pic(S)$ can vary from 1 to 20. [The dimension of $H^2_{sing}(S, \mathbb{Q})$ is 22.]

Grothendieck groups K_0

Definition 1.22. Let X be a quasi-projective variety. We define $K_0(X)$ to be the quotient of the free abelian group generated by isomorphism classes $[\mathcal{E}]$ of (algebraic) vector bundles $\mathcal{E}$ on X modulo the equivalence relation generated pairs $([\mathcal{E}], [\mathcal{E}_1] + [\mathcal{E}_2])$ for each short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of vector bundles.

Remark 1.23. Let A be a finitely generated k-algebra. Observe that every short exact sequence of projective A-modules splits. Thus, the equivalence relation defining $K_0(A)$ is generated by pairs $([\mathcal{E}_1 \oplus \mathcal{E}_2], [\mathcal{E}_1] + [\mathcal{E}_2])$. Every element of $K_0(A)$ can be written as $[P] - [m]$ for some non-negative integer m; moreover, projective modules P, Q determine the same class in $K_0(A)$ if and only if there exists some non-negative integer m such that $P \oplus A^m \simeq Q \oplus A^m$.

Proposition 1.24. *$K_0(\mathbb{P}^N)$ is a free abelian group of rank $N+1$. Moreover, for any $k \in \mathbb{Z}$, the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(k), \ldots, \mathcal{O}_{\mathbb{P}^N}(k+N)$ generate $K_0(\mathbb{P}^N)$.*

Proof. One obtains a relation among $N+2$ invertible sheaves from the Koszul complex on $N+1$ dimensional vector space V:

$$0 \to \Lambda^{N+1}V \otimes S^{*-N-1}(V) \to \cdots \to V \otimes S^{*-1}(V) \to S^*(V) \to k \to 0.$$

One shows that the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(j), j \in \mathbb{Z}$ generate $K_0(\mathbb{P}^N)$ using Serre's theorem that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ there exist integers $m, n > 0$ and a surjective map of $\mathcal{O}_{\mathbb{P}^N}$-modules $\mathcal{O}_{\mathbb{P}^N}(m)^n \to \mathcal{F}$.

One way to show that the rank of $K_0(\mathbb{P}^N)$ equals $N+1$ is to use Chern classes. □

Smooth varieties

Remark 1.25. For any $d, N > 0$, we consider the d-fold symmetric product $S^d(P^N)$ of the projective space P^N is a well defined algebraic variety. Unfortunately, even for these relatively simply varieties, the computation of $K_0(S^d(P^N))$ has not been achieved. The problem is that X is not smooth.

Definition 1.26. A quasi-projective variety X is smooth of dimension n at some point $x \in X$ if there exists an open neighborhood $x \in U \subset X$ and k polynomials $f_1, \ldots, f_k$ in $n+k$ variables (viewed as regular functions on $\mathbb{A}^{n+k}$) vanishing at $0 \in \mathbb{A}^{n+k}$ with Jacobian $|\frac{\partial f_i}{\partial x_j}|(0)$ of rank k and an isomorphism of U with $Z(f_1, \ldots, f_k) \subset \mathbb{A}^{n+k}$ sending x to 0.

In more algebraic terms, a point $x \in X$ is smooth if there exists an open neighborhood $x \in U \subset X$ and a map $p : U \to \mathbb{A}^n$ sending x to 0 which is flat and unramified at x.

Definition 1.27. Let X be a quasi-projective variety. Then $K_0'(X)$ is the Grothendieck group of isomorphism classes of coherent sheaves on X, where the equivalence relation is generated pairs $([\mathcal{E}], [\mathcal{E}_1] + [\mathcal{E}_2])$ for short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of $\mathcal{O}_X$-modules.

Example 1.28. Let $A = k[x]/x^2$. Consider the short exact sequence of A-modules

$$0 \to k \to A \to k \to 0$$

where k is an A-module via the agumentation map (i.e., x acts as multiplication by 0), where the first map sends $a \in k$ to $ax \in A$, and the second map sends x to 0. We conclude that the class $[A]$ of the rank 1 free module equals $2[k]$.

On the other hand, because A is a local ring, $K_0(A) = \mathbb{Z}$, generated by the class $[A]$. Thus, the natural map $K_0(\operatorname{Spec} A) \to K_0'(\operatorname{Spec} A)$ is not surjective. The map is, however, injective, as can be seen by observing that $dim_k(-) : K_0'(\operatorname{Spec} A) \to \mathbb{Z}$ is well defined.

Theorem 1.29. *If X is smooth, then the natural map $K_0(X) \to K_0'(X)$ is an isomorphism.*

Proof. Smoothness implies that every coherent sheaf has a finite resolution by vector bundles, This enables us to define a map

$$K_0'(X) \ \to \ K_0(X)$$

by sending a coherent sheaf $\mathcal{F}$ to the alternating sum $\Sigma_{i=1}^{N}(-1)^i\mathcal{E}_i$, where $0 \to \mathcal{E}_N \to \cdots \mathcal{E}_0 \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ by vector bundles.

Injectivity follows from the observation that the composition

$$K_0(X) \;\to\; K_0'(X) \;\to\; K_0(X)$$

is the identity. Surjectivity follows from the observation that $\mathcal{F} = \Sigma_{i=1}^{N}(-1)^i\mathcal{E}_i$ so that the composition

$$K_0'(X) \;\to\; K_0(X) \;\to\; K_0'(X)$$

is also the identity. □

References

H [1] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

W [2] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: eric@math.nwu.edu

2. Algebraic cycles

We use the notation $\operatorname{Spec} A$ to denote the prime ideal spectrum of a finitely generated commutative k-algebra A. As a set, $\operatorname{Spec} A$ is the set of primes ideals. We give $\operatorname{Spec} A$ the *Zariski topology*: a subset $S \subset \operatorname{Spec} A$ is closed if there exists some $I \subset A$ such that S is the set of prime ideals containing I. In particular, a point $x \in \operatorname{Spec} A$ is closed (i.e., equal to its own closure) if and only if the corresponding prime ideal is maximal. We provide $X = \operatorname{Spec} A$ with its sheaf of regular functions $\mathcal{O}_X$, defined as in the first lecture. Namely, a basis of open subsets of X are the affine open subsets $X_f = \operatorname{Spec} A_f \subset \operatorname{Spec} A$, and $\mathcal{O}_X(X_f) = A_f$.

As in the first lecture, this "globalizes" to provide any quasi-projective variety X locally closed (for the Zariski topology) in some $\mathbb{P}^N$ with the structure of a scheme: a topological space equipped with a sheaf $\mathcal{O}_X$ of commutative rings which is locally isomorphic to a ringed space of the form $\operatorname{Spec} A$ as above. For example, we can view the scheme theoretic points of $\mathbb{P}^N$ as the non-tautological homogeneous prime ideals of the commutative graded k-algebra $k[X_0, \ldots, X_N]$. One important property that the sheaf $\mathcal{O}_X$ satisfies for any X is that its it stalk

$$\mathcal{O}_{X,x} \ = \ colim_{x \in U} \ \mathcal{O}_X(U)$$

is a local ring. The residue field of this local ring $\mathcal{O}_{X,x}$ is a finite field extension of k if and only x is a closed point.

We shall adopt the convention that a quasi-projective algebraic variety (over k) is a reduced (but not necessarily irreducible) scheme of finite type over $\operatorname{Spec} k$ which admits a locally closed embedding in some projective space $\mathbb{P}^M$.

Recall that X is said to be *integral* if $\mathcal{O}_X(U)$ is an integral domain for all open subsets $U \subset X$. The field of fractions K of such an integral variety is the field of fractions of $\mathcal{O}_X(U)$ for any affine open subset U. If $\mathcal{O}_X(U)$ is integrally closed in K for every affine open subset U, then the stalk $\mathcal{O}_{X,x}$ at any (scheme-theoretic point) $x \in X$ of codimension 1 is a discrete valuation ring.

Definition 2.1. Let X is an integral variety regular in codimension 1 and let K be its field of fractions. For any $0 \neq f \in K$, we define the *principal divisor* (f) associated to f to be the following formal sum of codimension 1, irreducible subvarieties

$$(f) = \sum_{x \in X^{(1)}} v_x(f)\overline{x}.$$

Here, $X^{(1)} \subset X$ consists of the scheme-theoretic points of codimension 1, $v_x : K^* \to \mathbb{Z}$ is the discrete valuation at $x \in X^{(1)}$, and $\overline{x} \subset X$ is the codimension 1 irreducible subvariety of X given as the closure of x.

A formal sum

$$D = \sum_{x \in X^{(1)}} n_x \overline{x}, \quad n_x \in \mathbb{Z}$$

with all but finitely many n_x equal to 0 is said to be a *locally principal* divisor provided that for every $x \in X^{(1)}$ there exists some Zariski open neighborhood $x \in U_x \subset X$ and some $f_x \in K$ such that $D_{|U_x} = (f_x)_{|U_x}$.

Construction 2.2. Assume that X is integral and regular in codimension 1. Let $\mathcal{L} \in Pic(X)$ be a locally free sheaf of rank 1 (i.e., a "line bundle" or "invertible sheaf") and assume that $\Gamma(\mathcal{L}) \neq 0$. Then any $0 \neq s \in \Gamma(\mathcal{L})$ determines a well defined locally principal divisor on X, $Z(s) \subset X$. Namely, if $\mathcal{L}_{|U} \simeq \mathcal{O}_{X|U}$ is trivial when restricted to some open $U \subset X$, then $s_U \in \mathcal{L}(U)$ determines an element of $\mathcal{O}_X(U)$ well defined up to a unit in $\mathcal{O}_X(U)$ (i.e., an element of $\mathcal{O}_X^*(U)$) so that the valuation $v_x(s)$ is well defined for every $x \in U^{(1)}$. We define $Z(s)$ by the property that $Z(s)_U = (s_U)_{|U}$ for any open $U \subset X$ restricted to which $\mathcal{L}$ is trivial, and where (s_U) denotes the divisor of an element of $\mathcal{O}_X(U)$ corresponding to s_U under any $(\mathcal{O}_X)_{|U}$ isomorphism $\mathcal{L}_{|U} \simeq (\mathcal{O}_X)_{|U}$.

Remark 2.3. If X is a projective, as well as integral and regular in codimension 1, then the *degree* of a principal divisor,

$$deg(f) \equiv \sum_{x \in X^{(1)}} v_x(f),$$

equals 0. On the other hand, the coefficients $v_x(s)$ of the locally principal divisor associated to a global section $s \in \Gamma(\mathcal{L})$ are all non-negative.

Theorem 2.4. *Assume that X is an integral variety regular in codimension 1. Let $\mathcal{D}(X)$ denote the group of locally principal divisors on X modulo principal divisors. Then the above construction determines a well defined isomoprhism*

$$Pic(X) \ \simeq \ \mathcal{D}(X).$$

Moreover, if $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, then $D(X)$ equals the group $Z^1(X)$ of Weil divisors: the free abelian group on closed irreducible subvarieties of X of codimension 1.

Proof. If $s, s' \in \Gamma(\mathcal{L})$ are non-zero global sections, then there exists some $f \in K = frac(\mathcal{O}_X)$ such that with respect to any trivialization of $\mathcal{L}$ on some open covering $\{U_i \subset X\}$ of X the quotient of the images of s, s' in $\mathcal{O}_X(U_i)$ equals f. A line bundle $\mathcal{L}$ is trivial if and only if it is isomorphic to $\mathcal{O}_X$ which is the case if and only if it has a global section $s \in \Gamma(X)$ which never vanishes if and only if $(s) = 0$. If $\mathcal{L}_1, \mathcal{L}_2$ are two such line bundles with non-zero global sections s_1, s_2, then $(s_1 \otimes s_2) = (s_1) + (s_2)$.

Thus, the map is a well defined homomorphism on the monoid of those line bundles with a non-zero global section. By Serre's theorem concerning coherent sheaves generated by global sections, for any line bundle $\mathcal{L}$ there exists a positive integer n such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is generated by global sections (and in particular, has non-zero global sections), where we have implicitly chosen a locally closed embedding $X \subset \mathbb{P}^M$ and taken $\mathcal{O}_X(n)$ to be the pull-back via this embedding of $\mathcal{O}_{\mathbb{P}^M}(n)$. Thus, we can send such an $\mathcal{L} \in Pic(X)$ to $(s) - (w)$, where $s \in \Gamma(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$ and $w \in \Gamma(\mathcal{O}_X(n))$.

The fact that $Pic(X) \to \mathcal{D}(X)$ is an isomorphism is an exercise in unravelling the formulation of the definition of line bundle in terms of local data.

Recall that a domain A is a unique factorization domain if and only every prime of height 1 is principal. Whenever $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, every codimension 1 subvariety $Y \subset X$ is thus locally principal, so that the natural inclusion $D(X) \subset Z^1(X)$ is an equality. □

Remark 2.5. This is a first example of relating bundles to cycles, and moreover a first example of duality. Namely, $Pic(X)$ is the group of rank 1 vector bundles; the group $Z^1(X)$ of Weil divisors is a group of cycles. Moreover, $Pic(X)$ is contravariant with respect X whereas $Z^1(X)$ is covariant with respect to equidimensional maps.

To relate the two as in the above theorem, some smoothness conditions are required
.

Definition 2.6. Let X be a quasi-projective algebraic variety. An *algebraic r-cycle* on X if a formal sum

$$\sum_Y n_Y[Y], \quad Y \text{ irreducible of dimension } r, \quad n_Y \in \mathbb{Z}$$

with all but finitely many n_Y equal to 0.

Equivalently, an algebraic r-cycle is a finite integer combination of points of X of dimension r.

If $Y \subset X$ is a subvariety each of whose irredicuble components $Y_1, \ldots, Y_m$ is r-dimensional, then the algebraic r-cycle

$$Z = \sum_{i=1}^{m} [Y_i]$$

is called the *cycle associated* to Y.

The group of (algebraic) r-cycles on X will be denoted $Z_r(X)$.

For example, if X is an integral variety of dimension d (i.e., the field of fractions of X has transcendence d over k), then a Weil divisor is an algebraic $d-1$-cycle. In the following definition, we extend to r-cycles the equivalence relation we impose on locally principal divisor when we consider these modulo principal divisors. As motivation, observe that if C is a smooth curve and $f \in frac(C)$, then f determines a morphism $f : C \to \mathbb{P}^1$ and

$$(f) = f^{-1}(0) - f^{-1}(\infty),$$

where $f^{-1}(0), f^{-1}(\infty)$ are the scheme-theoretic fibres of f above $0, \infty$.

Definition 2.7. Two r-cycles Z, Z' on a quasi-projective variety X are said to be *rationally equivalent* if there exist algebraic $r+1$-cycles $W_0, \ldots, W_n$ on $X \times \mathbb{P}^1$ for some $n > 0$ with the property that each component of each W_i projects onto an open subvariety of $\mathbb{P}^1$ and that i $Z = W_0[0], Z' = W_n[\infty]$, and $W_i[\infty] = W_{i+1}[0]$ for $0 \leq i < n$. Here, $W_i[0]$ (respecitvely, $W_i[\infty]$ denotes the cycle associated to the scheme theoretic fibre above $0 \in \mathbb{P}^1$ (resp., $\infty \in \mathbb{P}^1$) of the restriction of the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ to (the components of) W_i.

The *Chow group* $CH_r(X)$ is the group of r-cycles modulo rational equivalence.

Observe that in the above definition we can replace the role of $r+1$-cycles on $X \times \mathbb{P}^1$ and their geometric fibres over $0, \infty$ by $r+1$-cycles on $X \times U$ for any non-empty Zaristik open $U \subset X$ and geometric fibres over any two k-rational points $p, q \in U$.

Remark 2.8. Given some $r+1$ dimensional irreducible subvariety $V \subset X$ together with some $f \in k(V)$, we may define $(f) = \sum_S ord_S(f)[S]$ where S runs through the codimension 1 irreducible subvarieties of V. Here, $ord_S(-)$ is the valuation centered on S if V is regular at the codimension 1 point corresponding to S; more generally, $ord_S(f)$ is defined to be the length of the $O_{V,S}$-module $O_{V,S}/(f)$.

We readily check that (f) is rationally equivalent to 0: namely, we associate to (V, f) the closure $W = \Gamma_f \subset X \times \mathbb{P}^1$ of the graph of the rational map $V \dashrightarrow \mathbb{P}^1$ determined by f. Then $(f) = W[0] - W[\infty]$.

Conversely, given an $r+1$-dimensional irreducible subvariety W on $X \times \mathbb{P}^1$ which maps onto $\mathbb{P}^1$, the composition $W \subset X \times \mathbb{P}^1 \stackrel{pr_2}{\to} \mathbb{P}^1$ determines $f \in frac(W)$ such that

$$(f) \ = \ W[0] - W[\infty].$$

Thus, the definition of rational equivalence on r-cycles of X can be given in terms of the equivalence relation generated by

$$\{(f), f \in frac(W); W \text{ irreducible of dimension } r+1\}$$

In particular, we conclude that the subgroup of principal divisors inside the group of all locally principal divisors consists precisely of those locally principal divisors which are rationally equivalent to 0.

One should view $CH_*(X)$ as a homology/cohomology theory. These groups are covariantly functorial for proper maps $f: X \to Y$ and contravariantly functiorial for flat maps $W \to X$, so that they might best be viewed as some sort of "Borel-Moore homology theory.

Example 2.9. Let $X = \mathbb{A}^N$. Then any $N-1$-cycle (i.e., Weil divisor) $Z \in CH_{N-1}(\mathbb{A}^N)$ is principal, so that $CH_{N-1}(\mathbb{A}^N) = 0$.

More generally, consider the map $\mu : \mathbb{A}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ which sends $(x_1, \ldots, x_n), t$ to $\langle t \cdot x_1, \ldots, t \cdot x_n, 1\rangle, t$. Consider an ireducible subvariety $Z \subset \mathbb{A}^N$ of dimension $r > N$ not containing the origin and $\overline{Z} \subset \mathbb{P}^N$ be its closure. Let $W = \mu^{-1}(\overline{Z} \times \mathbb{A}^1)$. Then $W[0] = \emptyset$ whereas $W[1] = Z$. Thus, $CH_r(\mathbb{A}^N) = 0$ for any $r < N$.

Example 2.10. Arguing in a similar geometric fashion, we see that the inclusion of a linear plane $P^{N-1} \subset \mathbb{P}^N$ induces an isomorphism $CH_r(\mathbb{P}^{N-1}) = CH_r(\mathbb{P}^N)$ provided that $r < N$ and thus we conclude by induction that $CH_r(\mathbb{P}^N) = \mathbb{Z}$ if $r \leq N$. Namely, consider $\mu : \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ sending $\langle x_0, \ldots, x_N\rangle, t$ to $\langle x_), \ldots, x_{N-1}, t \cdot x_N\rangle, t$ and set $W = \mu^{-1}(Z \times \mathbb{A}^1)$ for any Z not containing $\langle 0, \ldots, 0, 1\rangle$. Then $W[0] = pr_{N*}(Z), W[1] = Z$.

Example 2.11. Let C be a smooth curve. Then $Pic(C) \simeq CH_0(X)$.

Definition 2.12. If $f : X \to Y$ is a proper map of quasi-projective varieties, then the proper push-forward of cycles determines a well defined homomorphism

$$f_* : CH_r(X) \ \to \ CH_r(Y), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to $d \cdot [f(Z)] \in CH_r(Y)$ where $[k(Z) : k(f(Z))] = d$ if $dim\ Z = dim\ f(Z)$ and is sent to 0 otherwise.

If $g : W \to X$ is a flat map of quasi-projective varieties of relative dimension e, then the flat pull-back of cycles induces a well defined homomorphism

$$g^* : CH_r(X) \ \to \ CH_{r+e}(W), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to the cycle on W associated to $Z \times_X W \subset W$.

Proposition 2.13. *Let Y be a closed subvariety of X and let $U = X \backslash Y$. Let $i : Y \to X, j : U \to X$ be the inclusions. Then the sequence*

$$CH_r(Y) \stackrel{i_*}{\to} CH_r(X) \stackrel{j^*}{\to} CH_r(U) \to 0$$

is exact for any $r \geq 0$.

Proof. If $V \subset U$ is an irreducible subvariety of U of dimension r, then the closure of V in X, $\overline{V} \subset X$, is an irreducible subvariety of X of dimension r with the property that $j^*([\overline{V}]) = [V]$. Thus, we have an exact sequence

$$Z_r(Y) \xrightarrow{i_*} Z_r(X) \xrightarrow{j^*} Z_r(U) \to 0.$$

If $Z = \sum_i n_i[Y_i]$ is a cycle on X with $j^*(Z) = 0 \in CH_r(U)$, then $j^*Z = \sum_{W,f}(f)$ where each $W \subset U$ is an irreducible subvarieties of U of dimension $r+1$ and $f \in k(W)$. Thus, $Z' = \sum_i n_i[\overline{Y}_i] - \sum_{\overline{W},f}(f)$ is an r-cycle on Y with the property that $i_*(Z')$ is rationally equivalent to Z. Exactness of the asserted sequence of Chow groups is now clear.

□

Corollary 2.14. *Let $H \subset \mathbb{P}^N$ be a hypersurface of degree d. Then $CH_{N-1}(\mathbb{P}^N \backslash H) = \mathbb{Z}/d\mathbb{Z}$.*

Example 2.15. Mumford shows that if S is a projective smooth surface with a non-zerol global algebraic 2-form (i.e., $H^0(S, \Lambda^2(\Omega_S)) \neq 0$), then $CH_0(S)$ is *not finite dimensional* (i.e., must be very large).

Bloch's Conjecture predicts that if S is a projective, smooth surface with geometric genus equal to 0 (i.e., $H^0(S, \Lambda^2(\Omega_S)) = 0$), then the natural map from $CH_0(S)$ to the (finite dimensional) Albanese variety is injective.

Theorem 2.16. *Let X be a smooth quasi-projective variety of dimension d. Then there exists a pairing*

$$CH_r(X) \otimes CH_s(X) \xrightarrow{\bullet} CH_{d-r-s}, \quad d \geq r+s,$$

with the property that if $Z = [Y], Z' = [W]$ are irreducible cycles of dimension r, s respectively and if $Y \cap W$ has dimension $\leq d-r-s$, then $Z \bullet Z'$ is a cycle which is a sum with positive coefficients indexed by the irreducible subvarieties of $Y \cap W$ of dimension $d-r-s$.

For notational purposes, we shall often write $CH^s(X)$ for $CH_{d-s}(X)$. With this indexing convention, the intersection pairing has the form

$$CH^s(X) \otimes CH^t(X) \xrightarrow{\bullet} CH^{s+t}(X).$$

Proof. Classically, this was proved by showing the following geometric fact: given a codimension r cycle Z and a codimension s cycle $W = \sum_j m_j R_j$ with $r+s \leq d$, then there is another codimension r cycle $Z' = \sum_i n_i Y_i$ rationally equivalent to Z (i.e., determining the same element in $CH^r(X)$) such that Z' meets W "properly"; in other words, every component $C_{i,j,k}$ of each $Y_i \cap R_j$ has codimension $r+s$. One then defines

$$Z' \bullet W = \sum_{i,j,k} n_i m_j \cdot int(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}$$

where $int(Y_i \cap R_j, C_{i,j,k})$ is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a Z'' rationally equivalent to both Z, Z' and also intersecting W properly, then $Z' \bullet W$ is rationally equivalent to $Z" \bullet W$.

In [2], Blaine Lawson and I showed how one could generalize this, considering any finite dimensional family of codimension r cycles Z_α and any finite dimensional family of codimension s cycles W_β, moving simultaneously each Z_α to a suitable

Z'_α. Bill Fulton has asserted that earlier treatments of the Chow Moving Lemma were inadequate, and this generalized result gives the first complete proof.

A completely different proof is given by William Fulton and Robert MacPherson. (cf [3]). They use a powerful geometric technique discovered by MacPherson called *deformation to the normal cone.* For $Y \subset X$ closed, the deformation space $M_Y(X)$ is a variety mapping to $\mathbb{P}^1$ defined as the complement in the *blow-up* of $X \times \mathbb{P}^1$ along $Y \times \infty$ of the blow-up of $X \times \infty$ along $Y \times \infty$. One readily verifies that $Y \times \mathbb{P}^1 \subset M(X,Y)$ restricts above $\infty \neq p \in \mathbb{P}^1$ to the given embedding $Y \subset X$; and above ∞, restricts to the inclusion of Y into the normal cone $C_Y(X) = Spec(\oplus_{n \geq 0} \mathcal{I}_Y^n / \mathcal{I}_Y^{n_1})$, where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf defining $Y \subset X$. When $Y \subset X$ is a regular closed embedding, then this normal cone is a bundle, the normal bundle $N_Y(X)$.

This enables a regular closed embedding (e.g., the diagonal $\delta : X \to X \times X$ for X smooth) to be deformed into the embedding of the 0-section of the normal bundle $N_{\delta(X)}(X \times X)$. One defines the intersection of Z, W as the intersection of $\delta(X), Z \times W$ and thus one reduces the problem of defining intersection product to the special case of intersection of the 0-section of the normal bundle $N_X(X \times X)$ with the normal cone $N_{(Z \times W) \cap \delta(X)}(Z \times W)$. □

Grothendieck introduced many basic techniques which we now use as a matter of course when working with bundles. The following *splitting principle* is one such technique, a technique which enable one to frequently reduce constructions for arbitrary vector bundles to those which are a sum of line bundles.

Proposition 2.17. *Let $\mathcal{E}$ be a rank $r+1$ vector bundle on a quasi-projective variety X and define $p_1 : \mathbb{P}(\mathcal{E}) = Proj\ (Sym_{O_X} \mathcal{E}) \to X$ to be the projective bundle of lines in $\mathcal{E}$. Then $p_1^* : CH_*(X) \to CH_{*+r}(\mathbb{P}(\mathcal{E}))$ is split injective and $p_1^*(\mathcal{E}) = \mathcal{E}_1$ is a direct sum of a rank r bundle and a line bundle.*

Applying this construction to $\mathcal{E}_1$ on $\mathbb{P}(\mathcal{E})$, we obtain $p_2 : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$; proceeding inductively, we obtain

$$p = p_r \circ \cdots \circ p_1 : \mathbb{F}(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{r-1}) \to X$$

with the property that $p^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$ is split injective and $p^*(\mathcal{E})$ is a direct sum of line bundles.

We now introduce Chern classes and the Chern character, once again following Grothendieck's point of view.

Construction 2.18. Let $\mathcal{E}$ be a rank r vector bundle on a smooth, quasi-projective variety X of dimension d. Then $CH^*(\mathbb{P}(\mathcal{E}))$ is a free module over $CH^*(X)$ with generators $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$, where $\zeta \in CH^1(\mathbb{P}(\mathcal{E}))$ denotes the divisor class associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. .

We define the i-th Chern class $c_i(\mathcal{E}) \in CH^i(X)$ of $\mathcal{E}$ by the formula

$$CH^*(\mathbb{P}(\mathcal{E})) \;=\; CH^*(X)[\zeta] / \sum_{i=0}^{r} (-1)^i \pi^*(c_i(\mathcal{E})) \cdot \zeta^{r-i}.$$

We define the total Chern class $c(\mathcal{E})$ by the formula

$$c(\mathcal{E}) \;=\; \sum_{i=0}^{r} c_i(\mathcal{E})$$

and set $c_t(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E}) t^i$. Then the *Whitney sum* formula asserts that $c_t(\mathcal{E} \oplus \mathcal{F}) = c_t(\mathcal{E}) \cdot c_t(\mathcal{F})$.

We define the *Chern roots*, $\alpha_1, \ldots, \alpha_r$ of $\mathcal{E}$ by the formula

$$c_t(\mathcal{E}) \;=\; \prod_{i=1}^{r}(1+\alpha_i t)$$

where the factorization can be viewed either as purely formal or as occurring in $\mathbb{F}(\mathcal{E})$. Observe that $c_k(\mathcal{E})$ is the k-th elementary symmetric function of these Chern roots.

In other words, the Chern classes of the rank r vector bundle $\mathcal{E}$ are given by the expression for $\zeta^r \in CH^r(\mathbb{P}(\mathcal{E}))$ in terms of the generators $1, \zeta, \ldots, \zeta^{r-1}$. Thus, the Chern classes depend critically on the identification of the first Chern class ζ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the multiplicative structure on $CH^*(X)$. The necessary structure for such a definition of Chern classes is called an *oriented multiplicative cohomology theory.* The splitting principle guarantees that Chern classes are uniquely determined by the assignment of first Chen classes to line bundles.

We refer the interested reader to [3] for the definition of "operational Chern classes" defined for bundles on a non necessarily smooth variety.

Construction 2.19. Let X be a smooth, quasi-projective variety, let $\mathcal{E}$ be a rank r vector bundle over X, and let $\pi : \mathbb{F}(\mathcal{E}) \to X$ be the associated bundle of flags of $\mathcal{E}$. Write $\pi^*(\mathcal{E}) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where each $\mathcal{L}_i$ is a line bundle on $\mathbb{F}(\mathcal{E})$. Then $c_t(\pi^*(\mathcal{E})) = \prod_{i=1}^r (1 \oplus c_1(\mathcal{L}_i))t$.

We define the Chern character of $\mathcal{E}$ as

$$ch(\mathcal{E}) = \sum_{i=1}^{r}\{1 + c_1(\mathcal{L}_i) + \frac{1}{2}c_1(\mathcal{L}_i)^2 + \frac{1}{3!}c_1(\mathcal{L}_i)^3 + \cdots\} = \sum_{i=1}^{r} exp(c_t(\mathcal{L}_i)),$$

where this expression is verified to lie in the image of the injective map $CH^*(X) \otimes \mathbb{Q} \to CH^*(\mathbb{F}(\mathcal{E})) \otimes \mathbb{Q}$. (Namely, one can identify $ch_k(\mathcal{E})$ as the k-th power sum of the Chern roots, and therefore expressible in terms of the Chern classes using Newton polynomials.)

Since $\pi^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E})), \quad \pi^* : CH^*(X) \to CH^*(\mathbb{F}(\mathcal{E}))$ are ring homomorphisms, the splitting principle enables us to immediately verify that ch is also a ring homomoprhism (i.e., sends the direct sum of bundles to the sum in $CH^*(X)$ of Chern characters, sends the tensor product of bundles to the product in $CH^*(X)$ of Chern characters).

Grothendieck's formulation of the Riemann-Roch theorem is an assertion of the behaviour of the Chern character ch with respect to push-forward maps induced by a proper smooth map $f : X \to Y$ of smooth varieties. It is not the case that ch commutes with the these push-forward maps; one must modify the push forward map in K-theory by multiplication by the Todd class.

This modification of the Todd class is necessary even when consideration of the push-forward of a divisor. Indeed, the Todd class

$$td : K_0(X) \to A^*(X)$$

is characterized by the properties that

- i. $td(L) = c_1(L)/(1 - exp(-c_1(L));$
- ii. $td(E_1 \oplus E_2) = td(E_1) \cdot td(E_2)$; and
- iii. $td \circ f^* = f^* \circ td$.

The reader is recommended to consult [1] for a very nice overview of Grothendieck's Riemann-Roch Theorem.

Theorem 2.20. *(Grothendieck's Riemann-Roch Theorem)*
Let $f : X \to Y$ be a projective map of smooth varieties. Then for any $x \in K_0(X)$, we have the equality

$$ch(f_!(x)) \cdot td(T_Y) \;=\; f_*(ch(x) \cdot td(T_X))$$

where T_X, T_Y are the tangent bundles of X, Y and $td(T_X), td(T_Y)$ are their Todd classes.

Here, $f_! : K_0(X) \to K_0(Y)$ is defined by identifying $K_0(X)$ with $K_0'(X)$, $K_0(Y)$ with $K_0'(Y)$, and defining $f_! : K_0'(X) \to K_0'(Y)$ by sending a coherent sheaf $\mathcal{F}$ on X to $\sum_i (-1)^i R^i f_*(F)$. The map $f_* : CH_*(X) \to CH_*(Y)$ is proper push-forward of cycles.

Just to make this more concrete and more familiar, let us consider a very special case in which X is a projective, smooth curve, Y is a point, and $x \in K_0(X)$ is the class of a line bundle $\mathcal{L}$. (Hirzebruch had earlier proved a version of Grothendieck's theorem in which the target Y was a point.)

Example 2.21. Let C be a projective, smooth curve of genus g and let $f : C \to Spec\mathbb{C}$ be the projection to a point. Let $\mathcal{L}$ be a line bundle on C with first Chern class $D \in CH^1(C)$. Then

$$f_!([\mathcal{L}]) = dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) \in \mathbb{Z},$$

and $ch : K_0(Spec\mathbb{C}) = \mathbb{Z} \to A^*(Spec\mathbb{C}) = \mathbb{Z}$ is an isomorphism. Let $K \in CH^1(C)$ denote the "canonical divisor", the first Chern class of the line bundle Ω_C, the dual of T_C. Then

$$td(T_C) = \frac{-K}{1 - (1 + K + \frac{1}{2}K^2)} = 1 - \frac{1}{2}K.$$

Recall that $deg(K) = 2g - 2$. Since $ch([\mathcal{L}]) = 1 + D$, we conclude that

$$f_*(ch([\mathcal{L}]) \cdot td(T_C)) = f_*((1 + D) \cdot (1 - \frac{1}{2}K)) = deg(D) - \frac{1/2}{d}eg(K).$$

Thus, in this case, Riemann-Roch tell us that

$$dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) = deg(D) + 1 - g.$$

For our purpose, Riemann-Roch is especially important because of the following consequence.

Corollary 2.22. *Let X be a smooth quasi-projective variety. Then*

$$ch_* : K_0(X) \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$$

is a ring isomorphism.

Proof. The essential ingredient is the Riemann-Roch theorem. Namely, we have a natural map $CH^*(X) \to K_0'(X)$ sending an irreducible subvariety Y to the $\mathcal{O}_X$-module $\mathcal{O}_Y$. We show that the composition with the Chern character is an isomorphism by applying Riemann-Roch to the map from $\tilde{Y} \to Y \to X$ where $\tilde{Y} \to Y$ is a resolution of singularites. On the other hand, clearly $CH^*(X) \to K_0'(X)$ is surjective. □

References

B [1] R. Bott. Math Review MR0116022 (22 # 6817) of Le théorème de Riemann-Roch by Armand Borel and Jean-Pierre Serre.

FL [2] E. Friedlander and H.B. Lawson, Moving algebraic cycles of bounded degree. Inventiones Math. **132** (1998), 91-119.

Fu [3] W. Fulton, *Intersection Theory*, Springer 1984.

H [4] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

W [5] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: `eric@math.nwu.edu`

3. Topological K-theory

Shortly after Grothendieck introduced $K_0(X)$ for an algebraic variety X, M. Atiyah and F. Hirzebrch introduced the analogous theory for topological spaces [2] based on topological vector bundles. We consider this theory $K^*_{top}(-)$ for two reasons: first, the theory developed by Atiyah and Hirzebruch has been a model for 40 years of effort in algebraic K-theory, effort that has recently produced significant advances; second, topological K-theory of the underlying analytic space of a complex variety X, X^{an}, provides a much more computable theory to which algebraic K-theory maps.

A *(complex) topological vector bundle* of rank r, $p : E \to T$, on a space T is a continuous map with fibers each given the structure of a complex vector space of dimension r such that there is an open covering $\{U_i \subset T\}$ with the property that there exist homeomorphisms $\phi_i : \mathbb{C}^r \times U_i \xrightarrow{\sim} E_{|U_i}$ over U_i which are $\mathbb{C}$-linear on each fiber. One readily verifies that such a topological vector bundle $p : E \to T$, on T of rank r determines and is determined by *patching data*: a collection of continuous, fiber-wise linear homemorphisms for each i, j

$$\phi_j^{-1} \circ \phi_i = \theta_{i,j} : \mathbb{C}^r \times (U_i \cap U_j) \simeq (E_{|U_i})_{U_j} = (E_{|U_j})_{U_i} \simeq \mathbb{C}^r \times (U_i \cap U_j).$$

This in turn is equivalent to a 1-cocycle

$$\{\theta_{i,j} \in Maps_{cont}(U_{i,j}, GL(r, \mathbb{C}))\} \in Z^1(T, GL(r, \mathbb{C})).$$

Indeed, two such 1-cocycles determine isomorphic topological vector bundles if and only if they differ by a co-boundary. Thus, the topological vector bundles on T of rank r are "classified" by $H^1(T, GL(r, \mathbb{C}))$.

It is instructive to recognize the distinction between an algebraic vector bundle on the complex algebraic variety X and a topological vector bundle on the associated analytic space X^{an}. If X is a projective smooth variety with the property that $CH^*(X) \otimes \mathbb{Q} \to H^*(X^{an}, \mathbb{Q})$ is not surjective, then we know that not every topological vector bundle on X admits an algebraic structure. We can view this in terms of patching functions as asserting that a given topological 1-cocycle $\{\theta_{i,j}\} \in Z^1(X^{an}, GL(r, \mathbb{C}))$ is not cohomologous to a cocycle for which each $\theta_{i,j} \in Mor(X, GL_{r,\mathbb{C}})$. If C is a projective, smooth curve of genus $g > 1$, we have seen that $K_0(C)$ contains the uncountable abelian group (complex torus) $Pic^0(C) \simeq \mathbb{C}^g/\Lambda$ every element of which maps to $0 \in K^0_{top}(C^{an})$. This reflects the fact that algebraic vector bundles which are non-isomorphic as algebraic vector bundles can become isomorphic when viewed as topological vector bundles. In terms of patching data, what occurs here is that some algebraic patching data are coboundaries of continuous but not algebraic $0 - cochains$.

Our examples of algebraic vector bundles immediately give us many examples of topological vector bundles as we observe in the following proposition.

Proposition 3.1. *Let X be a complex variety with underlying analytic space X^{an}. Then any algebraic vector bundle $\mathcal{E}$ on X of rank r determines naturally a rank r topological vector bundle on X^{an}.*

Consider the Grassmannian $Grass_n(\mathbb{P}^N)$ of dimension n-planes in projective N-space, or equivalently of n dimensional linear quotients of $\mathbb{C}^{N+1}$. A map (i.e., algebraic morphism) from an algebraic variety X to $Grass_n(\mathbb{P}^N)$ corresponds naturally to a rank n algebraic vector bundle $\mathcal{E}$ over X together a choice of $N+1$ global sections which determine a surjective bundle map $\mathcal{O}_X \to \mathcal{E}$. A map (i.e., continuous function) from a topological space T to $Grass_n(\mathbb{P}^N)^{an}$ corresponds naturally to a rank n topological vector bundle on T together a choice of $N+1$ global sections which determine a surjective bundle map $(\mathbb{C}^{N+1} \times T) \to E$.

We shall use the notation

$$Grass_n(\mathbb{P}^\infty)^{an} \equiv \varinjlim_N Grass_n(\mathbb{P}^N)^{an}.$$

The following statements about topological vector bundles are not valid (in general) for algebraic vector bundles. These properties suggest that topological K-theory is better behaved than algebraic K-theory.

Proposition 3.2. *(cf.* [1]*) Let T be a compact Hausdorff space. If $p: E \to T$ is a topological vector bundle on T, then for some $N > 0$ there is a surjective map of bundles on T, $(\mathbb{C}^{N+1} \times T) \to E$.*

Any surjective map $E \to F$ of topological vector bundles on T admits a splitting over T.

The set of homotopy classes of maps $[T, Grass_\infty(\mathbb{P}^N)^{an}]$ is in natural 1-1 correspondence with the set of isomorphism classes of rank n topological vector bundles on T.

Proof. The first statement is proved using a partition of unity argument.

The proof of the second statement is proved by establishing a Hermetian metric on E (so that $E \simeq F \oplus F^\perp$), which is achieved by once again using a partition of unity argument.

To prove the last statement, one verifies that if $T \times I \to G$ is a homotopy relating continuous maps $f, g : T \to G$ and if E is a topological vector bundle on G, then $f^*E \simeq g^*E$ as topological vector bundles on T. Once again, a partition of unity argument is the key ingredient in the proof. □

Proposition 3.3. *There is a natural homotopy equivalence*

$$BU(n) \stackrel{\simeq}{\to} Grass_n(\mathbb{P}^\infty)^{an},$$

where BU_n is the classifying space of the Lie group U_n of $n \times n$ unitary matrices.

For any space T, the set of homotopy classes of maps

$$[T, BU \times \mathbb{Z}], \quad BU = \varinjlim_n BU_n$$

admits a natural structure of an abelian group induced by block sum of matrices $U_n \times U_m \to U_{n+m}$. We define

$$K^0_{top}(T) \equiv [T, BU \times \mathbb{Z}].$$

For any compact, Hausdorff space T, $K^0_{top}(T)$ is naturally isomorphic to the Grothendieck group of topological vector bundles on T:

$$K^0_{top}(T) \simeq \frac{\mathbb{Z}[\text{iso classes of top vector bundles on } T]}{[E] = [E_1] + [E_2], \text{ whenever } E \simeq E_1 \oplus E_2}.$$

Proof. To verify the first assertion, one observes that the Grassmannian $Grass_\infty(\mathbb{P}^N)$ is a homogeneous space for $GL_{N+1}(\mathbb{C})$, with stabilizer the parabolic subgroup with reductive factors GL_{N+1-n}, GL_n. Equivalently, $Grass_\infty(\mathbb{P}^N)$ is the quotient by a free action of GL_n on the quotient of the Stieffel variety

$$GL_{N+1}(\mathbb{C})/GL_{N+1-n} \rtimes M_{N+1-n,n}.$$

The assertion now follows from the observation that the colimit with respect to N of the analytic spaces of these Stieffel varieties is contractible.

(External) direct sum of matrices gives a monoid structure on $\sqcup_n BU_n$ which determines a (homotopy associative and commutative) H-space structure on $BU \times Z$ which we view as the mapping telescope of the self map

$$\sqcup_n BU_n \to \sqcup_n BU_n, \quad BU_i \times \{\star \in BU_1\} \to BU_{i+1}.$$

The (abelian) group structure on $[T, BU \times \mathbb{Z}]$ is then determined.

To show that this mapping telescope is actually an H-space, one must verify that it has a 2-sided identity up to *pointed* homotopy: one must verify that product on the left with $\star \in BU_1$ gives a self map of $BU \times \mathbb{Z}$ which is related to the identity via a base-point preserving homotopy. (Such a verification is not difficult, but the analogous verification fails if we replace the topological groups U_n by discrete groups $GL_n(A)$ for some unital ring A.)

Finally, the previous proposition implies a natural isomorphism of monoids

$$\text{iso classes top vector bundles} \xrightarrow{\sim} [T, \sqcup_n BU_n]$$

whenever T is a compact Hausdorff space. One readily verifies that taking the colimit of the self-map given by translation $[T, \sqcup_n BU_n] \to [T, \sqcup_n BU]$ has the effect of group completing this monoid. Thus, the isomorphism of monoids induces an isomorphism of group completions. □

Example 3.4. Since the Lie groups U_n are connected, the spaces BU_n are simply connected and thus

$$K^0_{top}(S^1) = \pi_1(BU \times \mathbb{Z}) = 0.$$

It is useful to extend $K^0_{top}(-)$ to a relative theory which applies to pairs (T, A) of spaces (i.e., T is a topological space and $A \subset T$ is a closed subset). In the special case that $A = \emptyset$, then $T/\emptyset = T_+/\star$, the pointed space obtained by taking the disjoint union of T with a point $\star$ which we declare to be the basepoint.

Definition 3.5. If T is a pointed space with basepoint t_0, we define the reduced K-theory of T by

$$\tilde{K}^*_{top}(T) \equiv K^*_{top}(T, t_0).$$

For any pair (T, A), we define

$$K^0_{top}(T, A) \equiv \tilde{K}^0_{top}(T/A)$$

thereby extending our earlier definition of $K^0_{top}(T)$.

For any $n > 0$, we define

$$K^n_{top}(T, A) \equiv \tilde{K}^0_{top}(\Sigma^n(T/A)).$$

In particular, for any $n \geq 0$, we define

$$K^{-n}_{top}(T) \equiv K^{-n}_{top}(T, \emptyset) \equiv \tilde{K}^0_{top}(\Sigma^n(T_+)).$$

Observe that

$$\tilde{K}^0_{top}(S \wedge T) = ker\{K^0_{top}(S \times T) \to K^0_{top}(S) \oplus K^0_{top}(T)\},$$

so that (external) tensor product of bundles induces a natural pairing

$$K^{-i}_{top}(S) \otimes K^{-j}_{top}(T) \to K^{-i-j}_{top}(S \times T).$$

Just to get the notation somewhat straight, let us take T to be a single point $T = \{t\}$. Then $T_+ = \{t, \star\}$, the 2-point space with new point $\star$ as base-point. Then $\Sigma^2(T_+)$ is the 2-sphere S^2, and thus

$$K_{top}^{-2}(\{t\}) = ker\{K_{top}^0(S^2)) \to K_{top}^0(\star)\}.$$

We single out a special element, *the Bott element*

$$\beta = [\mathcal{O}_{\mathbb{P}^1}(1)] - [\mathcal{O}_{\mathbb{P}^1}] \in K_{top}^{-2}(pt)),$$

where we have abused notation by identifying $(\mathbb{P}^1)^{an}$ with S^2 and the images of algebraic vector bundles on $\mathbb{P}^1$ in $K_{top}^0((\mathbb{P}^1)^{an})$ have the same names as in $K_0(\mathbb{P}^1)$.

The most famous, and most important, theorem in topological K-theory is the following formulation by Atiyah of R. Bott's famous theorem that there is a natural homotopy equivalence $\Omega(U) \to BU \times \mathbb{Z}$. Since U is naturally equivalent to $\Omega(BU \times \mathbb{Z})$, we conclude that

$$\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

Atiyah interprets this 2-fold periodicity in terms of K-theory as follows.

Theorem 3.6. *(Bott Periodicity) For any space T and any $i \geq 0$, mutliplication by the Bott element induces a natural isomorphism*

$$\beta : K_{top}^{-i}(T) \to K_{top}^{-i-2}(T).$$

In particular, taking T to be a point, we conclude that $\tilde{K}_{top}^0(S^2) = \mathbb{Z}$, generated by the Bott element.

Example 3.7. Let S^0 denote $\{*, \star\} = *_+$. According to our definitions, the K-theory $K_{top}(*)$, of a point equals the reduced K-theory of S^0. In particular, for $n > 0$,

$$K_{top}^{-n}(*) = \tilde{K}_{top}^{-n}(S^0) = \tilde{K}_{top}^0(S^n) = \pi_n(BU).$$

Thus, we conclude

$$K_{top}^n(*) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

We can reformulate this by writing

$$K_{top}^i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i+n \text{ is even} \\ 0 & \text{if } i+n \text{ is odd} \end{cases}$$

As observed by Atiyah and Hirzebruch, topological K-theory enjoys all of the properties of singular cohomology (other than the fact that its value on a point is somewhat more complicated), and thus can be computed for many familiar spaces. This is formalized in the following theorem.

Theorem 3.8. *The sequence of contravariant functors $K_{top}^n(-,-)$ on pairs satisfies the following properties:*

- *(Homotopy invariance) If $f, g : (S, B) \to (T, A)$ are continuous maps of pairs related by a homotopy $F : S \times I \to T$ restricting to $B \times I \to A$, then $f^* = g^*$.*
- *(Exactness) For every pair (T, A), there is a long exact sequence*

$$\cdots \to K_{top}^n(X, A) \to K_{top}^n(X, \emptyset) \to K_{top}^n(A, \emptyset) \to K_{top}^{n+1}(X, A) \to \cdots$$

- *(Excision) For every $U \subset A$ whose closure is contained in the interior of A, the inclusion $(T-U, A-U) \to (T,A)$ induces an isomorphism*

$$K^*_{top}(T,A) \simeq K^*_{top}(T-U, A-U).$$

Thus, $K^*_{top}(-)$ (respectively, $\tilde{K}^*_{top}(-)$) is an unreduced (resp., reduced) generalized cohomology theory.

Example 3.9. Assume that T is a finite cell complex with only even dimensional cells. Then our previous computation of $K^*_{top}(pt)$ and the preceding theorem imply that $K^0_{top}(T)$ is a free abelian group on the cells of T and that $K^1_{top}(T)$ is 0.

As for algebraic K-theory, we can define Chern classes

$$c_i : K^0_{top}(T) \to H^{2i}(T, \mathbb{Z})$$

where the right hand side is the singular cohomology of the topological space T. Perhaps the simplest way to do this is to identify canonical classes

$$c_i \in H^{2i}(BU(n), \mathbb{Z}), \quad n \geq i$$

which arise from fundamental classes of algebraic cycles on $Grass_n(\mathbb{P}^\infty)$. Then, since an element $\zeta \in K^0_{top}(T)$ corresponds to a homotopy class of maps $f_\zeta : T \to BU \times \mathbb{Z}$ and since

$$H^j(BU, \mathbb{Z}) = H^j(BU(n), \mathbb{Z}), \quad j \leq 2n$$

we can define $c_i(\zeta) = f^*(c_i)$. Alternatively, we could approximate T by compact space and use the splitting principle together with the computation of $H^*(\mathbb{P}(E), \mathbb{Z})$ as a $H^*(T, \mathbb{Z})$ module.

Once we have Chern classes, we can define the Chern character either in terms of Newton polynomials in the Chern classes, in terms of power sums using the splitting principle, or simply by pulling back universal Chern character components

$$ch_i \in H^{2i}(BU, \mathbb{Q}$$

defined universally (using one of these methods) on universal bundles over Grassmannians.

Another important computational tool is the following Atiyah-Hirzebruch spectral sequence which relates $K^*_{top}(-)$ to (singular) cohomology with integer coefficients.

Theorem 3.10. *For any topological space T, there exists a spectral sequence*

$$E_2^{p,q} = H^p(T, \tilde{K}^{p+q}_{top}(S^p)) = H^p(T, K^q_{top}(pt)) \Rightarrow K^{p+q}_{top}(T).$$

Proof. There are two basic approaches to proving this spectral sequence. The first is to assume T is a cell complex, then consider T as a filtered space with $T_n \subset T$ the union of cells of dimension $\leq n$. The properties of $K^*_{top}(-)$ stated in the previous theorem give us an exact couple associated to the long exact sequences

$$\cdots \to \oplus K^q_{top}(S^n) \simeq K^q_{top}(T_n/T_{n-1}) \to K^q_{top}(T_n) \to K^q_{top}(T_{n-1}) \to \oplus K^{q+1}_{top}(S^n) \to \cdots$$

where the direct sum is indexed by the n-cells of T.

The second approach applies to a general space T and uses the Postnikov tower of $BU \times \mathbb{Z}$. This is a tower of fibrations whose fibers are Eilenberg-MacLane spaces for the groups which occur as the homotopy groups of $BU \times \mathbb{Z}$. □

Remark 3.11. The Postnikov tower argument together with a knowledge of the k-invariants of $BU \times \mathbb{Z}$ shows that after tenoring with $\mathbb{Q}$ this Atiyah-Hirzebruch spectral sequence collapses; in other words, that $E_2^{*,*} \otimes \mathbb{Q} = E_\infty^{*,*} \otimes \mathbb{Q}$.

We give one more important formal property of K-theory, thereby asserting that K-theory satisfies the properties necessary to be an "oriented multiplicative generalized cohomology theory."

Let $p : E \to T$ be a rank r, topological vector bundle on T and let $\pi : \mathbb{P}(E \oplus 1) \to T$ denote the associated projectivized bundle; the fiber over a point $t \in T$ is a copy of $\mathbb{P}^r$ with $p^{-1}(t) \simeq A^r \subset \mathbb{P}^r$; equivalently, we view this fiber as consisting of lines in the fiber of $E \oplus 1$ above t. Observe that there is a tautological map of bundles on $\mathbb{P}(E \oplus 1)$, $\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1) \to \pi^*(E \oplus 1)$, whose restriction to the line above a point $\tilde{t} \in \pi^{-1}(t)$ is simply the embedding of that line into $E \oplus 1$. We compose this map with the projection $\pi^*(E \oplus 1) \to \pi^* E$ and consider the corresponding global section $s \in \mathcal{O}_{\mathbb{P}(E\oplus 1)}(1) \otimes \pi^* E$ whose zero locus is precisely $T \simeq \mathbb{P}(1) \subset \mathbb{P}(E \oplus 1)$.

External wedge product with s gives maps

$$s \wedge (-) : \Lambda^i(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(1) \otimes \pi^* E) \to \Lambda^{i+1}(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(1) \otimes \pi^* E),$$

thereby endowing $\Lambda^*(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(1) \otimes \pi^* E)$ with the structure of a cochain complex of vector bundles exact over $\mathbb{P}(E)$ (but not exact over $T \simeq \mathbb{P}(1) \subset \mathbb{P}(E \oplus 1)$). Above any point $t \in T$, the restriction of this complex can be viewed as a class in $K^0_{top}(\mathbb{P}^r, \mathbb{P}^{r-1}) = K^0_{top}(S^{2r}) = \mathbb{Z}$ which is readily seen to be a generator.

Thus, λ_E is a topological K-theory *orientation class* for $\mathbb{P}(E \oplus 1)/\mathbb{P}(E)$, the *Thom space* for the vector bundle $p : E \to T$.

Theorem 3.12. *(Thom isomorphism) Let $p : E \to T$ be a topological vector bundle on T and let $\pi : \mathbb{P}(E \oplus 1) \to T$ denote the associated projectivized bundle.*

Then multiplication by λ_E, the class of

$$\sum_i (-1)^i \Lambda^i(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(1) \otimes \pi^* E) \in K^0_{top}(\mathbb{P}(E \oplus 1), \mathbb{P}(E)),$$

determines an isomorphism

$$K^*_{top}(T) \xrightarrow{\sim} K^*_{top}(\mathbb{P}(E \oplus 1), \mathbb{P}(E)).$$

Remark 3.13. Observe that this construction applies in the context of an algebraic vector bundle $\mathcal{E}$ over an algebraic variety X, giving us a Thom class in $K_0(\mathbb{P}(\mathcal{E} \oplus 1), \mathbb{P}(\mathcal{E}))$. Once again, multiplication by this class is an isomorphism.

There are several reasons why topological K-theory has sometimes proved to be a more useful computational tool than singular cohomology.

- $K^0_{top}(-)$ can be torsion free, even though $H^{ev}(-, \mathbb{Z})$ might have torsion. This is the case, for example, for compact Lie groups.
- $K^*_{top}(-)$ is essentially $\mathbb{Z}/2$-graded rather than graded by the natural numbers.
- $K^*_{top}(-)$ has interesting cohomology operations not seen in cohomology.

Definition 3.14. Let T be a compact Hausforff space and $E \to T$ be a topological vector bundle of rank r. Define

$$\lambda_t(E) = \sum_{i=0}^r [\Lambda^i E] t^i \in K^0_{top}(T)[t],$$

a polynomial with constant term 1 and thus an invertible element in $K^0_{top}[[t]]$. Extend this to a homomorphism

$$\lambda_t : K^0_{top}(T) \to (1 + K^0_{top}(X)[[t]])^*,$$

(using the fact that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$) and define $\lambda^i : K^0_{top}(T) \to K^0_{top}(T)$ to be the coefficient of t^i of λ_t.

More generally, we define these λ operations on $K^0_{top}(T)$ for any T by defining them first on the universal vector bundles over Grassmannians and using the functoriality of $K^0_{top}(-)$.

Definition 3.15. For any topological space T, define

$$\psi_t(x) = \sum_{i \geq 0} \psi^i(X)t^i \equiv rank(x) - t \cdot \frac{d}{dt}(log\lambda_{-t}(x))$$

for any $x \in K^0_{top}(T)$.

The *Adams operations* ψ^k satisfy many good properties, some of which we list below.

Proposition 3.16. *For any topological space T, any $x, y \in K^0_{top}(T)$, any $k > 0$*

- $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$.
- $\psi^k(xy) = \psi^k(x)\psi^k(y)$.
- $\psi^k(\psi^\ell(x) = \psi^{k\ell}(x)$.
- $ch_q(\psi^k(x)) = k^q ch_q(x) \in H^{2q}(T, \mathbb{Q})$.
- *$\psi^p(x)$ is congruent modulo p to x^p if p is a prime number.*
- *$\psi^k(x) = x^k$ whenever x is a line bundle*

In particular, if E is a sum of line bundles $\oplus_i L_i$, then $\psi^k(E) = \oplus((L_i)^k)$, the k-th power sum. By the splitting principle, this property alone uniquely determines ψ^k.

We introduce further operations, the γ-operations on $K_0^{top}(T)$.

Definition 3.17. For any topological space T, define

$$\gamma_t(x) = \sum_{i \geq 0} \gamma^i(X)t^i \equiv \lambda_{t/1-t}(x)$$

for any $x \in K^0_{top}(T)$.

Basic properties of these γ-oerations include the following

1. $\gamma_t(x + y) = \gamma_t(x)\gamma_t(y)$
2. $\gamma([L] - 1) = 1 + t([L] - 1)$.
3. $\lambda_s(x) = \gamma_{s/1+s}(x)$

Using these γ operations, we define the γ filtration on $K^0_{top}(T)$ as follows.

Definition 3.18. For any topological space T, define $K^{\gamma,1}_{top}(T)$ as the kernel of the rank map

$$K^{\gamma,1}_{top}(T) \equiv ker\{\text{rank} : K^0_{top}(T) \to K^0_{top}(\pi_0(T))\}.$$

For $n > 1$, define

$$K^0_{top}(T)^{\gamma,n} \subset K^{\gamma,0}_{top}(T) \equiv K^0_{top}(T)$$

to be the subgroup generated by monomials $\gamma^{i_1}(x_1) \cdots \gamma^{i_k}(x_k)$ with $\sum_j i_j \geq n, x_i \in K^{\gamma,1}_{top}(T)$.

Finally, we use the Adams operations and the γ-filtration to describe in the following theorem the relationship between $K^0_{top}(T)$, a group which has no natural grading, and the graded group $H^{ev}(T, \mathbb{Q})$.

Theorem 3.19. *Let T be a finite cell complex. Then for any $k > 0$, ψ^k restricts to a self-map of each $K^{\gamma,n}_{top}(T)$ and satisfies the property that it induces multiplication by k^n on the quotient*

$$\psi^k(x) = k^n \cdot x, \quad x \in K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)).$$

Furthermore, the Chern character ch induces an isomorphism

$$ch_n : K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)) \otimes Q \simeq H^{2n}(T, \mathbb{Q}).$$

In particular, the preceding theorem gives us a K-theoretic way to define the grading on $K^0_{top}(T) \otimes \mathbb{Q}$ induced by the Chern character isomorphism. The graded piece of (the associated graded of) $K^0_{top}(T) \otimes \mathbb{Q}$ corresponding to $H^{2n}(T, \mathbb{Q})$ consists of those classes x for which $\psi^k(x) = k^n x$ for some (or all) $k > 0$.

References

A [1] M. Atiyah, K-thoery, Benjamin 1967.

AH [2] M. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symp in Pure Math, vol 3, A.M.S. (1961).

B [3] R. Bott. Math Review MR0116022 (22 # 6817) of Le théorème de Riemann-Roch by Armand Borel and Jean-Pierre Serre.

FL [4] E. Friedlander and H.B. Lawson, Moving algebraic cycles of bounded degree. Inventiones Math. **132** (1998), 91-119.

Fu [5] W. Fulton, *Intersection Theory*, Springer 1984.

H [6] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

W [7] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: `eric@math.nwu.edu`

4. Higher Algebraic K-theory

With the "model" of topological K-theory in mind, we proceed to investigate the formulation of the algebraic K-theory of a variety in positive degrees. The reader should be forewarned of an indexing confusion: if X is a quasi-projective complex algebraic variety, then there is a natural map

$$K_i(X) \to K_{top}^{-i}(X^{an}).$$

The explanation of the change of index is that these higher algebraic K-groups were originally defined for affine schemes $X = SpecR$, so that the contravariance of K-theory with respect to maps of schemes is seen as covariance of K-theory with respect to maps of rings.

The strategy is to formulate a suitable space $\mathcal{K}(X)$ with the property that $\pi_0(\mathcal{K}(X)) = K_0(X)$; we then define $K_i(X) = \pi(\mathcal{K}(X))$. Indeed, the space $\mathcal{K}(X)$ which is constructed is an infinite loop space, enabling the definition of $K_i(X, \mathbb{Z}/n) \equiv \pi_i(\mathcal{K}(X), \mathbb{Z}/n)$ for every $i \geq 0, n > 1$. Many procedures have been developed to produce such spaces, some of which give different answers. However, K-theorists accept Quillen's constructions (of which we shall mention three, which give the same K-groups whenever each is defined) as the "correct" ones, for they produce K-groups which are well behaved and somewhat computable.

The construction of these spaces (or spectra) is quite interesting, but we shall be content to try to convey some sense of homotopy-theoretic group completion which plays a key role and to merely sketch various ideas involved in Quillen's constructions. Before we discuss these definitions (and some of the formal properties of $K_*(X)$), we briefly mention the "lower K-groups" $K_1(-), K_2(-)$ since a criterion for a good definition of the space $\mathcal{K}(X)$ is that $\pi_1(\mathcal{K}(X)$ should give K_1, $\pi_2(\mathcal{K}(X))$ should give K_2.

Definition 4.1. Let R be a ring (assumed associative, as always and with unit). We define $K_1(R)$ by the formula

$$K_1(R) \;\equiv\; GL(R)/[GL(R), GL(R)],$$

where $GL(R) = \varinjlim_n GL(n, R)$ and where $[GL(r), GL(R)]$ is the commutator subgroup of the group $GL(R)$. Thus, $K_1(R)$ is the maximal abelian quotient of $GL(R)$,

$$K_1(R) \;=\; H_1(GL(R), \mathbb{Z}).$$

The commutator subgroup $[GL(R), GL(R)]$ equals the subgroup $E(R) \subset GL(R)$ defined as the subgroup generated by elementary matrices $E_{i,j,}(r), r \in R, i \neq j$ (i.e., matrices which differ by the identity matrix by having r in the (i, j) position). This group is readily seen to be *perfect* (i.e., $E(R) = [E(r), E(R)]$); indeed, it is an elementary exercise to verify that $E(n, R) = E(R) \cap GL(n, R)$ is perfect for $n \geq 3$.

Proposition 4.2. *If R is a commutative ring, then the determinant map*

$$det : K_1(R) \to R^\times$$

from $K_1(R)$ to the mulitplicative group of units of R provides a natural splitting of $R^\times = GL(1, R) \to GL(R) \to K_1(R)$. Thus, we can write

$$K_1(R) \;=\; R^\times \times SL(R)$$

where $SL(R) = ker\{det\}$.

If R is a field or more generally a local ring, then $SK_1(R) = 0$. Moreover, $SK_1(\mathcal{O}_F) = 0$ for the ring of integers $\mathcal{O}_F$ in a number field F.

The consideration of $K_1(-)$ played an important role in the "Congruent subgroup problem" by Bass, Milnor, and Tate: for a ring of integers $\mathcal{O}$ in a number field F, is it the case that all subgroups of finite index in $GL(n,\mathcal{O})$ are given as the kernels of maps $GL(n,\mathcal{O}) \to GL(n,\mathcal{O}/\mathcal{I})$ for some ideal $\mathcal{I} \subset \mathcal{O}$)? The answer is yes if the number field F admits a real embedding, and no otherwise.

For a group ring $\mathbb{Z}[G]$ of a discrete group G, the Whitehead group

$$Wh(G) \ = \ K_1(\mathbb{Z}[G])/ < \pm g, g \in G >$$

is an important topological invariant of a connected cell complex with fundamental group G.

One can think of $K_0(R)$ as the "stable group" of projective modules "modulo trivial projective modules" and of $K_1(R)$ of the stabilized group of automorphisms of the trivial projective module modulo "trivial automorphisms" (i.e., the elementary matrices up to isomorphism. This philosophy can be extended to the definition of K_2, but has not been extended to $K_i, i > 2$. Namely, $K_2(R)$ can be viewed as the relations among the trivial automorphisms (i.e., elementary matrices) modulo those relations which hold universally.

Definition 4.3. Let $St(R)$, the Steinberg group of R, denote the group generated by elements $X_{i,j}(r), i \neq j, r \in R$ subject to the following commutator relations:

$$[X_{i,j(r)}, X_{k,\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq \ell \\ X_{i,\ell}(rs) & \text{if } j = k, i \neq \ell \\ X_{k,j}(-rs) & \text{if } j \neq k, i = \ell \end{cases}$$

We define $K_2(R)$ to be the kernel of the map $St(R) \to E(R)$, given by sending $X_{i,j}(r)$ to the elementary matrix $E_{i,j}(r)$, so that we have a short exact sequence

$$1 \to K_2(R) \to St(R) \to E(R) \to 1.$$

Proposition 4.4. *The short exact sequence*

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

is the universal central extension of the perfect group $E(R)$. Thus, $K_2(R) = H_2(E(R),\mathbb{Z})$, the Schur multiplier of $E(r)$.

Proof. Once can show that a universal central extension of a group E exists if and only E is perfect. In this case, a group S mapping onto E is the universal central extension if and only if S is also perfect and $H_2(S,\mathbb{Z}) = 0$. □

Example 4.5. If R is a field, then $K_1(F) = F^\times$, the non-zero elements of the field viewed as an abelian group under multiplication. By a theorem of Matsumoto, $K_2(F)$ is characterized as the target of the "universal Steinberg symbol". Namely, $K_2(F)$ is isomorphic to the free abelian group with generators "Steinberg symbols" $\{a,b\},\ a,b \in F^\times$ and relations

- i. {ac,b} = {a,b} {c,b},
- ii. {a,bd} = {a,b} {a,d},
- iii. $\{a, 1-a\} = 1, \quad a \neq 1 \neq 1-a.$ (*Steinberg relation*)

Observe that for $a \in F^\times,\ -a = \frac{1-a}{1-a^{-1}}$, so that

$$\{a,-a\} = \{a,1-a\}\{a,1-a^{-1}\}^{-1} = \{a,1-a^{-1}\}^{-1} = \{a^{-1},1-a^{-1}\} = 1.$$

Then we conclude the skew symmetry of these symbols:

$$\{a,b\}\{b,a\} = \{a,-a\}\{a,b\}\{b,a\}\{b,-b\} = \{a,-ab\}\{b,-ab\} = \{ab,-ab\} = 1.$$

Milnor used this presentation of $K_2(F)$ as the starting point of his definition of the *Milnor K-theory* of a field F.

Definition 4.6. Let F be a field with multiplicative group of units $F^\times$. The Milnor K-group $K_n^{Milnor}(F)$ is defined to be the n-th graded piece of the graded ring defined as the tensor algebra $\bigoplus_{n\geq 0}(F^\times)^{\otimes n}$ modulo the ideal generated by elements $\{a, 1-a\}, a \neq 1 \neq 1-a$ of homogeneous degree 2.

In particular, $K_1(F) = K_1^{Milnor}(F), K_2(F) = K_2^{Milnor}(F)$ for any field F, and $K_n^{Milnor}(F)$ is a quotient of $\Lambda^n(F^\times)$. For F an infinite field, Suslin proved that there are natural maps

$$K_n^{Milnor}(F) \to K_n(F) \to K_n^{Milnor}(F)$$

whose composition is $(-1)^{n-1}(n-1)!$. This immediately implies, for example, that the higher K-groups of a field of high transcendence degree are very large.

Remark 4.7. It is difficult to even briefly mention K_2 of fields without also mentioning the deep and import theorem of Mekurjev and Suslin: for any field F and positive integer n,

$$K_2(F)/nK_2(F) \ \simeq \ {}_nBr(F)$$

where ${}_nBr(F)$ denotes the subgroup of the Brauer group of F consisting of elements which are n-torsion.

The most famous success of K-theory in recent years is the following theorem of Voevodsky, extending foundational work of Milnor.

Theorem 4.8. *Let F be a field of characteristic $\neq 2$. Let $W(F)$ denote the Witt ring of F, the quotient of the Grothendieck group of symmetric inner product spaces modulo the ideal generated by the hyperbolic space $\langle 1\rangle \oplus \langle -1\rangle$ and let $I = ker\{W(F) \to \mathbb{Z}/2\}$ be given by sending a symmetric inner product space to its rank (modulo 2). Then the map*

$$K_n^{Milnor}(F)/2 \cdot K_n^{Milnor}(F) \to I^n/I^{n+1}, \quad \{a_1, \ldots, a_n\} \mapsto \prod_{i=1}^{n}(\langle a_i\rangle - 1)$$

is an isomorphism for all $n \geq 0$. Here, $\langle a\rangle$ is the 1 dimensional symmetric inner product space with inner product $(-,-)_a$ defined by $(c,d)_a = acd$.

We next turn to the construction of Quillen K-theory spaces, the most readily accessible of which is that of the *Quillen plus construction* for a ring R (e.g., an affine algebraic variety).

Proposition 4.9. *let R be a ring (associative, with unit). There is a unique homotopy class of maps*

$$i : BGL(R) \ \to \ BGL(R)^+$$

of connected spaces (of the homotopy type of C.W. complexes) with the following properties:

- *$i_\# : \pi_1(BGLR) \to \pi_1(BGL(R)^+)$ is the abelianization map $GL(R) \to GL(R)/[GL(R), GL(R)] = K_1(R)$.*

- *For any local coefficient system on $BGL(R)^+$ (i.e., any $K_1(R)$-module M), $i^* : H^*(BGL(R)^+, M) \to H^*(BGL(R), i^*M))$ is an isomorphism.*

Proof. The uniqueness up to homotopy of such a map follows from obstruction theory. For example, the uniquess is equivalent to the uniquess of a homology equivalence from the covering space of $BGL(R)$ associated to the subgroup $[GL(R), GL(R)] \subset \pi_1(BGL(R))$ to a simply connected space which is a homology equivalence.

The existence of this map depends upon the fact that $[GL(R), GL(R)] = E(R)$ is a perfect normal subgroup of $GL(R)$. One adds 2-cells to $BGL(R)$ to kill $E(R) \subset \pi_1(BGL(R))$ and then using the fact that $E(R)$ is perfect and normal one verifies that one can add 3-cells to kill the resulting extra homology in dimension 2 in such a way that the resulting space has the same homology as $BGL(R)$. □

Remark 4.10. By construction, $\pi_1(BGL(R)^+) = K_1(R)$. Moreover, one verifies directly that $\pi_2(BGL(R)^+) = H_2(BGL(R)^+)$ is the kernel of the universal central extension of $E(R)$. Namely, let F denote the homotopy fibre of the acylic map $BGL(R) \to BGL(R)^+$ and recall that the boundary map in the long exact homotopy sequence sends $\pi_2(BGL(R)^+) = H_2(BGL(R)^+)$ to the center of $E(R) = ker\{\pi_1(BGL(R)) \to \pi_1(BGL(R)^+)\}$. Since $H_1(F) = H_2(F) = 0$, we conclude that $\pi_1(F)$ is perfect and thus $\pi_2(BGL(R)^+ \to \pi_1(F) \to E(R)$ is the universal central extension of $E(R)$.

Definition 4.11. Let R be an associative ring with unit. We define the K-theory space of R to be the space

$$\mathcal{K}(R) \equiv K_0(R) \times BGL(R)^+.$$

Although we are now close to a full computation of $K_*(\mathbb{Z})$, finite fields and formal constructions applied to finite fields given us the only examples of rings for which the K-theory is completely known. Since Quillen had the following computation in mind when he introduced his plus construction, this indicates how difficult it is to compute k-groups. One should contrast this calculation with the computation $K_i^{Milnor} = 0, i \geq 2$.

Theorem 4.12. *Let $\mathbb{F}_q$ be a finite field with q elements. Then $BGL(\mathbb{F}_q)^+$ is homotopy equivalent to the homotopy fibre of the map $1 - \Psi^q : BU \to BU$. Consequently,*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if} \quad i = 0 \\ \mathbb{Z}/q^j - 1 & \text{if} \quad i = 2j > 0 \\ 0 & \text{if} \quad i = 2j - 1 > 0 \end{cases}$$

A "cheap" way to define the K-theory space of a quasi-projective variety is to define the K-theory as that of a *Jouanalou resolution* as exhibited in the following proposition. The fact that this agrees with more satisfying definitions is a consequence of a theorem of Quillen asserting that such an affine fibre bundle induces an isomorphism on K-groups.

Proposition 4.13. *Let X be a quasi-projective variety over a field. Then there exists an affine torsor $P \to X$ with P itself affine.*

Proof. Consider the affine variety J_n of rank 1 projectors of k^{n+1} (i.e., linear maps $p : k^{n+1} \to k^{n+1}$ of rank 1 satisfying $p^2 = p$). Then J^n is the 0-locus inside the affine space of all $n+1 \times n+1$ matrices $A = (a_{i,j})$ of the polynomial equations in the coordinates $x_{i,j}$ which impose the conditions that $A{\cdot}A = A, det(A) = 0, det(I-A) = 0$. The natural map $J^n \to \mathbb{P}^n$ sending a projector p to the line $p(k^{n+1})$ has fibre over p the vector group $Hom_k(ker\{p\}, im\{p\})$. If $X \subset \mathbb{P}^n$ is closed, then we can take $P \to X$ to be the restriction of $J^n \to \mathbb{P}^n$, $P = J^n \times_{\mathbb{P}^n} X \to X$. □

An H-space T is a pointed space of the homotopy type of a pointed C.W. complex equipped with a continuous pointed pairing $\mu : T \times T \to T$ with the property that $\mu(t_0, -), \mu(-, t_0) : T \to T$ are both homotopic as pointed maps to the identity. A homotopy associative H-space equipped with a homotopy inverse $i : T \to T$ is called a group-like H-space. It is a very useful observation that if an H-space T has the homotopy type of a C.W. complex and if $pi_0(T)$ is a group, then it is group-like.

Proposition 4.14. *The K-theory space $\mathcal{K}(R)$ admits the structure of a group-like H-space. Moreover, If Y is any group-like H-space, and if $f : BGL(R) \to Y$ is a map, then there exist a unique homotopy class of maps $BGL(R)^+ \to Y$ whose composition with $BGL(R) \to \mathcal{K}(R)$ is homotopic to f.*

Let $\mathcal{P}$ denote the abelian monoid of isomorphism classes of finitely generated projective R-modules. Then the K-theory space $\mathcal{K}(R)$ together with its H-group structure can be characterized as follows: there is a natural homotopy class of maps of H-spaces

$$\coprod_{[P]\in\mathcal{P}} BAut(P) \ \to \ \mathcal{K}(R)$$

which is a homotopy-theoretic group completion in the sense that it satisfies the following two properties.

- The induced map on connected components is the group completion map $\mathcal{P} \to K_0(R)$.
- The induced map on homology can be identified with the localization map
$$H_*(\coprod_{[P]} BAut(P)) \to \mathbb{Z}[K_0(R)] \otimes_{\mathbb{Z}[\mathcal{P}])} H_*(\coprod_{[P]} BAut(P)).$$

Such a homotopy-theoretic group completion of an H-space T is the universal map (in the pointed homotopy category) of H-spaces from T to a group-like H-space.

Our definition of the K-theory space is not satisfactory for several reasons. The H-space structure is not very natural, the extension to non-affine varieties is also unnatural, and we have as yet no way to define $K_1(-, \mathbb{Z}/n)$. The accepted definition which escapes all of these difficulties involves the Quillen Q-construction. Quillen's comparison for X affine of the Q-construction and Quillen plus construction is via a third construction, the $S^{-1}S$-construction of Quillen. This is presented in [8].

Recall that a *symmetric monoidal category* S is a (small) category with a unit object $e \in S$ and a functor $\square : S \times S \to S$ which is associative and commutative up to coherent natural isomorphisms. For example, if we consider the category $\mathcal{P}$ of finitely generated projective R-modules, then the direct sum $\oplus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative but only commutative up to natural isomorphism. The symmetric monoidal category relevant for the K-theory of a ring R is the category $Iso(\mathcal{P})$ whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms between projective R-modules.

Quillen's construction of $S^{-1}S$ for a symmetric monoidal category S is appealing, modelling one way we would introduce inverses to form the group completion of an abelian monoid.

Definition 4.15. Let S be a symmetric monoidal category. The category $S^{-1}S$ is the category whose objects are pairs $\{a,b\}$ of objects of S and whose maps from $\{a,b\}$ to $\{c,d\}$ are equivalence classes of compositions of the following form:

$$\{a,b\} \stackrel{s\Box-}{\to} \{s\Box a, s\Box b) \stackrel{(f,g)}{\to} \{c,d\}$$

where s is some object of S, f,g are morphisms in S. Another such composition

$$\{a,b\} \stackrel{s'\Box-}{\to} \{s'\Box a, s'\Box b) \stackrel{(f',g')}{\to} \{c,d\}$$

is declared to be the same map in $S^{-1}S$ from $\{a,b\}$ to $\{c,d\}$ if and only if there exists some isomorphism $\theta : s \to s'$ such that $f = f' \circ (\theta\Box a), g = g' \circ (\theta\Box b)$.

Heuristically, we view $\{a,b\} \in S^{-1}S$ as representing $a-b$, so that $\{s\Box a, s\Box b\}$ also represents $a-b$. Moreover, we are forcing morphisms in S to be invertible in $S^{-1}S$. If we were to apply this construction to the natural numbers $\mathbb{N}$ viewed as a discrete category with addition as the operation, then we get $\mathbb{N}^{-1}\mathbb{N} = \mathbb{Z}$.

We briefly recall the construction of the "classifying space" of a (small) category $\mathcal{C}$. Namely, we associate to $\mathcal{C}$ its "nerve" $N\mathcal{C}$, the *simplicial set* whose set of 0-simplices is the set of objects of $\mathcal{C}$ and whose set of n-simplices for $n > 0$ is the set of sequences of maps $X_0 \to \cdots \to X_n$ in $\mathcal{C}$; face maps are given by either dropping an end object or composing adjacent maps; degeneracies are given by inserting an identity map. We then define $B\mathcal{C}$ to be the *geometric realization* of the nerve of $\mathcal{C}$,

$$B\mathcal{C} \ \equiv \ |N\mathcal{C}|.$$

Since the geometric realization functor $|-| : \{\textit{simplicial sets}\} \to \{\textit{spaces}\}$ takes values in C.W. complexes, $B\mathcal{C}$ is a C.W. complex.

An informative example is the category $\mathcal{C}(P)$ of simplices of a polyhedron (objects are simplices, maps are inclusions). Then $B\mathcal{C}(P)$ can be identified with the first barycentric subdivision of P.

Theorem 4.16. *(Quillen) Let S be a symmetric monoidal category with the property that for all $s,t \in S$ the map $s\Box- : Aut(t) \to Aut(s\Box t)$ is injective. Then the natural map $BS \to B(S^{-1}S)$ is a homotopy-theoretic group completion.*

In particular, if S denotes the category whose objects are finite dimensional projective R-modules and whose maps are isomorphisms (so that $BS = \coprod_{[P]} BAut(P)$), then $\mathcal{K}(R)$ is homotopy equivalent to $B(S^{-1}S)$.

Quillen proves this theorem using ingenious techniques of recognizing when a functor between categories induces a homotopy equivalence on classifying spaces and when a triple $\mathcal{D} \to \mathcal{D} \to \mathcal{E}$ of categories determines a fibration seuqence of spaces (i.e., yields a long exact sequence of homotopy groups). This leads us to Quillen's Q-construction which applies to any small exact category (e.g., the category $\mathcal{P}(X)$ of algebraic vector bundles on a variety X, or the category $\mathcal{M}(X)$ of coherent sheaves on X).

Definition 4.17. An exact category $\mathcal{P}$ is an additive category equipped with a family of "exact sequences" $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{P}$ which satisfies the following: there exists an embedding of $\mathcal{P}$ in an abelian category $\mathcal{A}$ such that

- A sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ is $\mathcal{P}$ is exact if and only if it is exact in $\mathcal{A}$.
- If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence in $\mathcal{A}$ with $A_1, A_3 \in \mathcal{P}$, then this is an exact sequence of $\mathcal{P}$ (in particular, $A_2 \in \mathcal{P}$).

If a map $i : X_1 \to X_2$ in $\mathcal{P}$ fits in an exact sequence of the form $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{P}$ then we say that i is an admissible monomorphism and write it as $X_1 \overset{i}{\rightarrowtail} X_2$; if $j : X_2 \to X_3$ in $\mathcal{P}$ fits in an exact sequence of the form $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{P}$, then we say that j is an admissible epimorphism and write it as $X_2 \overset{j}{\twoheadrightarrow} X_3$.

If S is a symmetric monoidal category, then S is an exact category when provided with the class of split exact sequences (i.e., those sequences isomorphic to $e \to X_1 \to X_1 \Box X_2 \to X_2 \to e$).

Definition 4.18. Let $\mathcal{P}$ is an exact category, we define $Q\mathcal{P}$ to be the category whose objects are the same as the objects of $\mathcal{P}$ and whose maps $X \to Y$ are equivalence classes of diagrams $X \twoheadleftarrow Z \rightarrowtail Y$. Two such diagram are equivalent (and thus determine the same morphism in $Q\mathcal{P}$ provided that there is an isomorphism between the two diagrams which is the identity on both X and Y (i.e., an isomorphism $f : Z \to Z'$ such that $i = i' \circ f, j = j' \circ f$).

If $Y \twoheadleftarrow W \rightarrowtail S$ is another morphism in $Q\mathcal{P}$ then the composition is defined to be the naturally induced $X \twoheadleftarrow Z \times_Y W \rightarrowtail S$.

Remark 4.19. We can identify morphisms $X \to Y$ in $Q\mathcal{P}$ with an isomorphism of $Y_2/Y_1 \overset{\sim}{\to} X$ where $Y_1 \twoheadrightarrow Y_2 \to Y$ is an "admissible layer" of Y.

Theorem 4.20. *Let R be a ring and let $\mathcal{P}(R)$ denote the exact category of finitely generated projective R-modules. Let S denotes the category whose objects are those of $\mathcal{P}(R)$ and whose maps are the isomorphisms of $\mathcal{P}(R)$. Then there is a natural homotopy equivalence*

$$B(S^{-1}S) \simeq \Omega BQ\mathcal{P}(R).$$

In particular,

$$K_i(R) \equiv \pi_i \mathcal{K}(R)) \simeq \pi B(S^{-1}S) \simeq \pi_{i+1} BQ\mathcal{P}(R).$$

Definition 4.21. For any variety (or scheme) X, define the K-theory space $\mathcal{K}(X)$ to be $\Omega BQ(\mathcal{P}(X))$, the loop space on the classifying space of the Quillen construction applied to the exact category of algebraic vector bundles over X. Moreover, define $K_i(X)$ by

$$K_i(X) \equiv \pi_{i+1}\mathcal{K}(X)$$

and define $K_i(X, \mathbb{Z}/n)$ by

$$K_i(X, \mathbb{Z}/n) \equiv \pi_{i+1}(\mathcal{K}(X), \mathbb{Z}/n), n > 0; \quad K_0(X, \mathbb{Z}/n) = K_0(X) \otimes \mathbb{Z}/n).$$

We conclude this section with the following theorem of Quillen, extending work of Bloch. This gives another hint of the close connection of algebraic K-theory and algebraic cycles.

Theorem 4.22. *Let X be a smooth algebraic variety over a field. For any $i \geq 0$, let $\mathcal{K}_i$ denote the sheaf on X (for the Zariski topology) sending an open subset $U \subset X$ to $K_i(U)$. Then there is a natural isomorphism*

$$CH^i(X) = H^i(X, \mathcal{K}_i)$$

relating the Chow group of codimension i cycles on X and the Zariski cohomology of the sheaf $\mathcal{K}_i$. In particular, for $i = 1$, this becomes the familiar identification $Pic(X) = H^1(X, \mathcal{O}_X^*)$.

References

A [1] M. Atiyah, K-thoery, Benjamin 1967.

AH [2] M. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symp in Pure Math, vol 3, A.M.S. (1961).

BMS [3] H. Bass, J. Milnor, and J.-P. Serre, Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$, Publ. Math. I.H.E.S. [bf 33, 1967.

Ber [4] J. Berrick, An Approach to Algebraic K-theory, Pitman 1982.

B [5] R. Bott. Math Review MR0116022 (22 # 6817) of Le théorème de Riemann-Roch by Armand Borel and Jean-Pierre Serre.

FL [6] E. Friedlander and H.B. Lawson, Moving algebraic cycles of bounded degree. Inventiones Math. **132** (1998), 91-119.

Fu [7] W. Fulton, *Intersection Theory*, Springer 1984.

G [8] D. Grayson, *Higher algebraic K-theory. II (after Daniel Quillen)* L.N.M. # 551, pp 217-240.

H [9] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

M [10] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math Studies 72, Princeton University press 1971.

Q [11] D. Quillen, Higher algebraic K-theory, I. L.N.M # 341, pp. 85 – 147.

W [12] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: `eric@math.nwu.edu`

5. Higher Chow Groups and Beilinson's Conjectures

5.1. **Bloch's formula.** One interesting application of Quillen's techniques involving the Q construction is the following theorem of Quillen, extending work of Bloch. This gives another hint of the close connection of algebraic K-theory and algebraic cycles.

Theorem 5.1. *Let X be a smooth algebraic variety over a field. For any $i \geq 0$, let $\mathcal{K}_i$ denote the sheaf on X (for the Zariski topology) sending an open subset $U \subset X$ to $K_i(U)$. Then there is a natural spectral sequence*

$$E_2^{p,q}(X) = \ H^p(X, \mathcal{K}_q) \Rightarrow K_{p+q}(X).$$

Moreover, there is a natural isomorphism (Bloch's formula)

$$CH^q(X) \simeq H^q(X, \mathcal{K}_q)$$

relating the Chow group of codimension q cycles on X and the Zariski cohomology of the sheaf $\mathcal{K}_q$.

In particular, for $q = 1$, Bloch's formula becomes the familiar isomorphism relating the Chow group of divisors on a smooth variety to $H^1(X, \mathcal{O}_X^*)$.

Proof. Quillen's techniques apply to $K'_*(X) = \pi_{*+1}(BQ\mathcal{M}(X))$, the K-theory of coherent $\mathcal{O}_X$-modules. When X is smooth, Quillen verifies that the natural map $K_*(X) \to K'_*(X)$ is an isomorphism.

The key result needed for the existence of a spectral sequence is Quillen's localization theorem for a not necessarily smooth scheme X: if $Y \subset X$ is closed with Zariski open complement $U = X - Y$, then there is a natural long exact sequence

$$\cdots \to K'_{q+1}(U) \to K'_q(Y) \to K'_q(X) \to K'_q(U) \to \cdots .$$

Applying this localization sequence (slightly generalized) to the filtration of the category $\mathcal{M}(X)$ of coherent $\mathcal{O}_X$-modules

$$\mathcal{M}(X) = \mathcal{M}_0(X) \supset \mathcal{M}_1(X) \supset \mathcal{M}_2(X) \supset \cdots$$

where $\mathcal{M}_p(X)$ denotes the subcategory of those coherent sheaves whose support has codimension $\geq p$, Quillen obtains long exact sequences

$$\cdots K_i(\mathcal{M}_{p+1}) \to K_i(\mathcal{M}_p(X)) \to \coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(\mathcal{M}_{p+1}) \to \cdots$$

Here, X^p denotes the set of points of X of codimension p.

These exact sequences (for varying p) determine an exact couple and thus a spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X).$$

Following Gersten, Quillen considers the sequences

gersten (5.1.1)
$$0 \to K_q \to \coprod_{x \in X^0} K_q(k(x)) \to \coprod_{x \in X^1} K_{q-1}(k(x)) \to \cdots$$

where the differential in this sequence is d_1 of the above spectral sequence, given as the composition

$$\coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(\mathcal{M}_{p+1}) \to \coprod_{x \in X^{p+1}} K_{i-1}(k(x)).$$

Gersten conjectured that this sequence should be exact if X is the spectrum of a regular local ring; Quillen proved this for local rings of the form $O_{X,x}$ where X is a smooth variety.

We now consider the sequences (5.1.1) for varying open subsets $U \subset X$ in place of X, thereby getting an exact sequence of *sheaves* on X

Gersten (5.1.2)
$$0 \to \mathcal{K}_q \to \coprod_{x \in X^0} i_{x*}K_q(k(x)) \to \coprod_{x \in X^1} i_{x*}K_{q-1}(k(x)) \to \cdots$$

Since the sheaves $i_{x*}K_q(k(x))$ are flasque, we conclude that the p-th cohomology of the sequence (5.1.2) equals $H^p(X, \mathcal{K}_q)$ which is the E_2-term of our spectral sequence.

Finally, Quillen verifies that the cokernel of

$$\coprod_{x \in X^{q-1}} K_1(k(x)) \to \coprod_{x \in X^q} K_0(k(x))$$

is canonically isomorphic to $CH^q(X)$ provided that X is smooth. □

5.2. **Derived categories.** In order to formulate motivic cohomology, we need to introduce the language of derived categories. Let $\mathcal{A}$ be an abelian category (e.g., the category of modules over a fixed ring) and consider the category of chain complexes $CH^\bullet(\mathcal{A})$. We shall index our chain complexes so that the differential has degree $+1$. We assume that $\mathcal{A}$ has enough injectives and projectives, so that we can construct the usual derived functors of left exact and right exact functors from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For example, if $F : \mathcal{A} \to \mathcal{B}$ is right exact, then we define $L_iF(A)$ to be the i-th homology of the chain complex $F(P_\bullet)$ obtained by applying F to a projective resolution $P_\bullet \to A$ of A; similarly, if $G : \mathcal{A} \to \mathcal{B}$ is left exact, then $R^jG(A) = H^j(I^\bullet)$ where $A \to I^\bullet$ is an injective resolution of A.

The usual verification that these derived functors are well defined up to canonical isomorphism actually proves a bit more. Namely, rather take the homology of the complexes $F(P_\bullet), G(I^\bullet)$, we consider these complexes themselves and observe that they are independent *up to quasi-isomorphism* of the choice of resolutions. Recall, that a map $C^\bullet \to D^\bullet$ is a quasi-isomorphism if it induces an isomorphism on homology; only in special cases is a complex $C^\bullet$ quasi-isomorphic to its homology $H^\bullet(C^\bullet)$ viewed as a complex with trivial differential.

We define the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ as the category obtained from the category of $CH^\bullet(\mathcal{A})$ of chain complexes of $\mathcal{A}$ by inverting quasi-isomorphisms. Of course, some care must be taken to insure that such a localization of $CH^\bullet(\mathcal{A})$ is well defined. Let $Hot(CH^\bullet(\mathcal{A}))$ denote the homotopy category of chain complexes of $\mathcal{A}$: maps from the chain complex $C^\bullet$ to the chain complex $D^\bullet$ in $\mathcal{H}(CH^\bullet(\mathcal{A}))$ are chain homotopy equivalence classes of chain maps. Since chain homotopic maps induce the same map on homology, we see that $\mathcal{D}(\mathcal{A})$ can also be defined as the category obtained from $Hot(CH^\bullet(\mathcal{A}))$ by inverting quasi-isomorphisms.

The derived category $\mathcal{D}(\mathcal{A})$ of the abelian category $CH^\bullet(\mathcal{A})$ is a *triangulated category*. Namely, we have a shift operator $(-)[n]$ defined by

$$(A^\bullet[n])^j \equiv A^{n+j}.$$

This indexing is very confusing (as would be any other); we can view $A^\bullet[n]$ as $A^\bullet$ shifted "down" or "to the left". We also have *distinguished triangles*

$$A^\bullet \to B\bullet \to C^\bullet \to A^\bullet[1]$$

defined to be those "triangles" quasi-isomorphic to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of chain complexes.

This notation enables us to express *Ext*-groups quite neatly as

$$Ext^i_{\mathcal{A}}(A,B) = H^i(Hom_{\mathcal{A}}(P_\bullet, B) = Hom_{\mathcal{D}(\mathcal{A})}(A[-i], B)$$
$$= Hom_{\mathcal{D}(\mathcal{A})}(A, B[i]) = H^i(Hom_{\mathcal{A}}(A, P^\bullet)).$$

5.3. **Bloch's Higher Chow Groups.** From our point of view, motivic cohomology should be a "cohomology theory" which bears a relationship to $K_*(X)$ analogous to the role Chow groups $CH^*(X)$ bear to $K_0(X)$ (and analogous to the relationship of $H^*_{sing}(T)$ to $K^*_{top}(T)$). In particular, motivic cohomology will be doubly indexed.

We now discuss a relatively naive construction by Spencer Bloch of "higher Chow groups" which satisfies this criterion. We shall then consider a more sophisticated version of motivic cohomology due to Suslin and Voevodsky.

We work over a field k and define Δ^n to be $\operatorname{Spec} k[x_0, \ldots, x_n]/(\sum_i x_i - 1)$, the algebraic n-simplex. As in topology, we have face maps $\partial_i : \Delta^{n-1} \to \Delta^n$ (sending the coordinate function $x_i \in k[\Delta^n]$ to 0) and degeneracy maps $\sigma_j : \Delta^{n+1} \to \Delta^n$ (sending the coordinate function $x_j \in k[\Delta^n]$ to $x_j + x_{j+1} \in k[\Delta^{n+1}]$). More generally, a composition of face maps determines a face $F \simeq \Delta^i \to \Delta^n$. Of course, $\Delta^n \simeq \mathbb{A}^n$.

Bloch's idea is to construct a chain complex for each q which in degree n would be the codimension q-cycles on $X \times \Delta^n$. In particular, the 0-th homology of this chain complex should be the usual Chow group $CH^q(X)$ of codimension q cycles on X modulo rational equivalence. This can not be done in a completely straightforward manner, since one has no good way in general to restrict a general cycle on $X \times \Delta[n]$ via a face map ∂_i to $X \times \Delta^{n-1}$. Thus, Bloch only considers codimension q cycles on $X \times \Delta^n$ which restrict properly to all faces (i.e., to codimension q cycles on $X \times F$).

Definition 5.2. Let X be a variety over a field k. For each $p \geq 0$, we define a complex $z_p(X, *)$ which in degree n is the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$dim_k(Z \cap (X \times F) \leq dim_k(F) + p.$$

The differential of $z_p(X, *)$ is the alternating sum of the maps induced by restricting cycles to codimension 1 faces. Define the *higher Chow homology groups* by

$$CH_p(X, n) = H_n(z_p(X, *)), \quad n, p \geq 0.$$

If X is locally equi-dimensional over k (e.g., X is smooth), let $z^q(X, n)$ be the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$codim_{X \times F}(Z \cap (X \times F)) \geq q.$$

Define the *higher Chow cohomology groups* by

$$CH^q(X, n) = H_n(z^q(X, *), \quad n, q \geq 0,$$

where the differential of $z^q(X, *)$ is defined exactly as for $z_p(X, *)$.

Bloch, with the aid of Marc Levine, has proved many remarkable properties of these higher Chow groups.

Theorem 5.3. *Let X be a quasi-projective variety over a field. Bloch's higher Chow groups satisfy the following properties:*

- $CH_p(-,*)$ *is covariantly functorial with respect to proper maps.*
- $CH^q(-,*)$ *is contravariantly functorial on Sm_k, the category of smooth quasi-projective varieties over k.*
- $CH_p(X,0) = CH_p(X)$*, the Chow group of p-cycles modulo rational equivalence.*
- *(Homotopy invariance)* $\pi^* : CH_p(X,*) \xrightarrow{\sim} CH_{p+1}(X \times \mathbb{A}^1)$.
- *(Localization) Let $i : Y \to X$ be a closed subvariety with $j : U = X - Y \subset X$ the complement of Y. Then there is a distinguished triangle*

$$z_p(Y,*) \xrightarrow{i_*} z_p(X,*) \xrightarrow{j^*} z_p(U,*) \to z_p(Y,*)[1]$$

- *(Projective bundle formula) Let $\mathcal{E}$ be a rank n vector bundle over X. Then $CH^*(\mathbb{P}(\mathcal{E})*)$ is a free $CH^*(X,*)$-module on generators $1, \zeta, \ldots, \zeta^{n-1} \in CH^1(\mathbb{P}(\mathcal{E}), 0)$.*
- *For X smooth, $K_i(X) \otimes \mathbb{Q} \simeq \oplus_q CH^q(X,i) \otimes \mathbb{Q}$ for any $i \geq 0$. Moreover, for any $q \geq 0$,*

$$(K_i(X) \otimes \mathbb{Q})^{(q)} \simeq CH^q(X,i) \otimes \mathbb{Q}.$$

- *If F is a field, the $K_n^M(F) \simeq CH^n(\mathit{Spec}\,F, n)$.*

The most difficult of these properties, and perhaps the most important, is localization. The proof requires a very subtle technique of moving cycles. Observe that $z_p(X,*) \to z_p(U,*)$ is not surjective because the conditions of proper intersection on an element of $z_p(U,n)$ (i.e, a cycle on $U \times \Delta^n$) might not continue to hold for the closure of that cycle in $X \times \Delta^n$.

5.4. **Sheaves and Grothendieck topolologies.** The motivic cohomology groups of Suslin-Voevodsky are sheaf cohomology groups, and this formulation in terms of sheaves for a suitable topology provides much more flexibility. Moreover, this fits the spirt of the Beilinson conjectures which have motivated many of the developments which relate K-theory to algebraic cycles.

Before discussing Beilinson's conjectures, let us briefly consider Grothendieck's approach to sheaf theory and introduce both the etale and Nisnevich topologies.

Grothendieck had the insight to realize that one could formulate sheaves and sheaf cohomology in a setting more general than that of topological spaces. What is essential in sheaf theory is the notion of a covering, but such a covering need not consist of open subsets.

Definition 5.4. A (Grothendieck) site is the data of a category $\mathcal{C}/X$ of schemes over a given scheme X which is closed under fiber products and a distinguished class of morphisms (e.g., Zariski open embeddings; or etale morphisms) closed under composition, base change and including all isomorphisms. A covering of an object $Y \in \mathcal{C}/X$ for this site is a family of distinguished morphisms $\{g_i : U_i \to Y\}$ with the property that $Y = \cup_i g_i(U_i)$.

The data of the site $\mathcal{C}/X$ together with its associated family of coverings is called a Grothendieck topology on X.

Example 5.5. Recall that a map $f : U \to X$ of schemes is said to be *etale* if it is flat, unramified, and locally of finite type. Thus, open immersions and covering space maps are examples of etale morphisms. If $f : U \to X$ is etale, then for each point $u \in U$ there exist affine open neighborhoods $SpecA \subset U$ of u and $SpecR \subset X$ of $f(u)$ so that A is isomorphic to $(R[t]/g(t))_h$ for some monic polynomial $g(t)$ and some h so that $g'(t) \in (R[t]/g(t))_h$ is invertible.

The (small) etale site X_{et} has objects which are etale morphisms $Y \to X$ and coverings $\{U_i \to Y\}$ consist of families of etale maps the union of whose images equals Y. The big etale site X_{ET} has objects $Y \to X$ which are locally of finite type over X and coverings $\{U_i \to Y\}$ defined as for X_{et} consisting of families of etale maps the union of whose images equals Y. If k is a field, we shall also consider the site $(Sm/k)_{et}$ which is the full subcategory of $(\mathrm{Spec}\, k)_{ET}$ consisting of smooth, quasi-projective varieties Y over k.

An instructive example is that of $X = SpecF$ for some field F. Then an etale map $Y \to X$ with Y connected is of the form $SpecE \to SpecF$, where E/F is a finite separable field extension.

Definition 5.6. A presheaf sets (respectively, groups, abelian groups, rings, etc) on a site $\mathcal{C}/X$ is a contravariant functor from $\mathcal{C}/X$ to $(sets)$ (resp., to groups, abelian groups, rings, etc). A presheaf $P : (\mathcal{C}/X)^{op} \to (sets)$ is said to be a sheaf if for every covering $\{U_i \to Y\}$ in $\mathcal{C}/X$ the following sequence is exact:

$$P(Y) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_X U_j).$$

(Similarly, for presheaves of groups, abelian presheaves, etc.) In other words, if for every Y, the data of a section $s \in P(Y)$ is equivalent to the data of sections $s_i \in P(U_i)$ which are compatible in the sense that the restrictions of s_i, s_j to $U_i \times_X U_j$ are equal.

The category of abelian sheaves on a Grothendieck site $\mathcal{C}/X$ is an abelian category with enough injectives, so that we can define sheaf cohomology in the usual way. If $F : \mathcal{C}/X)^{op} \to (Ab)$ is an abelian sheaf, then we define

$$H^i(X_{\mathcal{C}/X}, F) = R^i\Gamma(X, F).$$

Etale cohomology has various important properties. We mention two in the following theorem.

Theorem 5.7. *Let X be a quasi-projective, complex variety. Then the etale cohomology of X with coefficients in (constant) sheaf $\mathbb{Z}/n$, $H^*(X_{et}, \mathbb{Z}/n)$, is naturally isomorphic to the singular cohomology of X^{an},*

$$H^*(X_{et}, \mathbb{Z}/n) \;\simeq\; H^*_{sing}(X^{an}, \mathbb{Z}/n).$$

Let $X = Speck$, the spectrum of a field. Then an abelian sheaf on X for the etale topology is in natural 1-1 correspondence with a (continuous) Galois module for the Galois group $Gal(\overline{k}/k)$. Moreover, the etale cohomology of X with coefficients in such a sheaf F is equivalent to the Galois cohomology of the associated Galois module,

$$H^*(k_{et}, F) \;\simeq\; H^*(Gal(\overline{F}/F), F(k)).$$

From the point of view of sheaf theory, the essence of a continuous map $g : S \to T$ of topological spaces is a mapping from the category of open subsets of T to the open subsets of S. We shall consider a map of sites $g : \mathcal{C}/X \to \mathcal{D}/Y$ to a functor from

$\mathcal{C}/Y$ to cC/X which takes distinguished morphisms to distinguished morphisms. For example, one of Beilinson's conjectures involves the map of sites

$$\pi : X_{et} \to X_{Zar}, \quad (U \subset X) \mapsto U \to X.$$

Such a map of sites induces a map on sheaf cohomology: if $F : (\mathcal{D}/Y)^{op} \to (Ab)$ is an abelian sheaf on $\mathcal{C}/Y$, then we obtain a map

$$H^*(Y_{\mathcal{D}/Y}, F) \to H^*(X_{\mathcal{C}/X}, g^*F).$$

5.5. **Beilinson's Conjectures.** We give below a list of conjectures due to Beilinson which relate motivic cohomology and K-theory. Bloch's higher Chow groups go some way toward providing a theory which satisfies these conjectures. Namely, Beilinson conjectures the existence of complexes of sheaves $\Gamma_{Zar}(r)$ whose cohomology (in the Zariski topology) $H^p(X, \Gamma_{Zar}(r))$ one could call "motivic cohomology". If we set

$$H^p(X, \Gamma_{Zar}(r)) = CH^r(X, 2r-p),$$

then many of the cohomological conjectures Beilinson makes for his conjectured complexes are satisfied by Bloch's higher Chow groups $CH^{\bullet}(X, *)$.

Conjecture 5.8. *Let X be a smooth variety over a field k. Then there should exist complexes of sheaves $\Gamma_{Zar}(r)$ of abelian groups on X with the Zariski topology, well defined in $\mathcal{D}(AbSh(X_{Zar}))$, functorial in X, and equipped with a graded product, which satisfy the following properties:*

(1) $\Gamma_{Zar}(1) = \mathbb{Z}$; $\Gamma_{Zar}(1) \simeq G_m[-1]$.
(2) $H^{2n}(X, \Gamma_{zar}(n)) = A^n(X)$.
(3) $H^i(Spec\, k, \Gamma_{Zar}(i)) = \mathcal{K}_i^M k$, *Milnor K-theory.*
(4) *(Motivic spectral sequence) There is a spectral sequence of the form*

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q)) \Rightarrow K_{-p-q}(X)$$

which degenerates after tensoring with $\mathbb{Q}$. Moreover, for each prime ℓ, there is a mod-ℓ version of this spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q) \otimes^L \mathbb{Z}/\ell) \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell)$$

(5) $gr_\gamma^r(K_j(X) \otimes \mathbb{Q} \simeq \mathbb{H}^{2r-i}(X_{Zar}, \Gamma_{Zar}(r))_{\mathbb{Q}}$.
(6) *(Beilinson-Lichtenbaum Conjecture)* $\Gamma_{Zar} \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq r}\mathbf{R}\pi_*(\mu_\ell^{\otimes} r)$ *in the derived category* $\mathcal{D}(AbSh(X_{Zar}))$ *provided that ℓ is invertible in $\mathcal{O}_X$, where $\pi : X_{et} \to X_{Zar}$ is the change of topology morphism.*
(7) *(Vanishing Conjecture)* $\Gamma_{Zar}(r)$ *is acyclic outside* $[1, r]$ *for* $r \geq 1$.

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (4) using "motivic complexes" $\Gamma_{Zar}(r)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X, \mathbf{\Gamma}_{Zar}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\Gamma_{Zar}(-q/2) = 0$ if $-q$ is not an even non-positive integer and $\Gamma_{Zar}(-q/2) = \Gamma_{Zar}(i)$ is $-q = 2i \geq 0$.

(1) and (2) just "normalize" our complexes, assuring us that they extend usual Chow groups and what is known in codimensions 0 and 1. Note that (1) and (2) are compatible in the sense that

$$H^2(X, \Gamma_{Zar}(1)) = H^2(X, \mathcal{O}_X^*[-1]) = H^1(X, \mathcal{O}_X^*) = Pic(X).$$

(3) asserts that for a field k, the n-th cohomology of $\Gamma_{Zar}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory. This has been verified for Bloch's higher Chow groups by Suslin-Nesternko and Totaro.

The (integral) spectral sequence of (4) has been established thanks to the work of many authors. This spectral sequence "collapses" at the E_2-level when tensored with $\mathbb{Q}$, so that $E_2 \otimes \mathbb{Q} = E_\infty \otimes \mathbb{Q}$. (5) asserts that this collapsing can be verified by using Adams operations, interpreted using the γ-filtration.

The vanishing conjecture of (7) is the most problematic, and there is no consensus on whether it is likely to be valid. However, (6) incorporates the mod-ℓ version of the vanishing conjecture.

(6) asserts that if we consider the complexes $\Gamma_{Zar}(r)$ modulo ℓ (in the sense of the derived category), then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes r}$ coefficients, where μ_ℓ is the etale sheaf of ℓ-th roots of unity (isomorphic to $\mathbb{Z}/\ell$ if all ℓ-th roots of unity are in k. If the terms in the mod-ℓ spectral sequence were simply etale cohomology, then we would get etale K-theory which would violate the vanishing conjectured in (7) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo ℓ should be the cohomology of complexes which involve a truncation.

More precisely, $\mathbf{R}\pi_* F$ is a complex of sheaves for the Zariski topology (given by applying π_* to an injective resolution $F \to I^\bullet$) with the property that $H^*_{Zar}(X, \mathbf{R}\pi_* F) = H^*_{et}(X, F)$. Now, the n-th truncation of $\mathbf{R}\pi_* F$, $\tau_{\leq n}\mathbf{R}\pi_* F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $\mathbf{R}\pi_* F$ in degrees $\leq n$ and are 0 in degrees greater than n. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Speck$, then $H^p(Speck, \tau_{\leq n}\mathbf{R}\pi_* \mu_\ell^{\otimes n})$ equals $H^p_{et}(Speck, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

Now, the Suslin-Voevodsky motivic complexes $\mathbb{Z}(n)$ are complexes of sheaves for the *Nisnevich topology*, a Grothendieck topology with finer than the Zariski topology but less fine than the etale topology.

Definition 5.9. A *Nisnevich covering* $\{U_i \to X\}$ of a scheme X is an etale covering with the property that every for every point $x \in X$ there exists some $u_x \in U_i$ mapping to x inducing an isomorphism $k(x) \to k(u_x)$ of residue fields.

We define $(Sm/k)_{Nis}$ to be the Grothendieck site whose objects are smooth quasi-projective varieties over a field k and whose distinguished class of morphisms consists of etale morphisms $g : U \to X$ with the property that for every point $x = g(u)$ there exists some point $u_x \in U$ with $k(x) \simeq k(u_x)$.

The Nisnevich topology is particularly useful for the study of blow-ups a smooth variety along a smooth subvariety. Although many of the proofs of Suslin-Voevodsky require use of this topology rather than the Zariski topology, there are various theorems to the effect that cohomology with respect to the Nisnevich topology agrees with that with respect to the Zariski topology for special types of coefficient sheaves.

It is instructive to observe that the "localization" of a scheme X at a point $x \in X$ is the spectrum of the local ring $\mathcal{O}_{X,x}$ if we consider the Zariski topology, is the spectrum of the *henselization* of $\mathcal{O}_{X,x}$ if we consider the Nisnevich topology, and is the the spectrum the *strict henselization* of $\mathcal{O}_{X,x}$ if we consider the etale topology.

References

A [1] M. Atiyah, K-thoery, Benjamin 1967.

AH [2] M. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symp in Pure Math, vol 3, A.M.S. (1961).

BMS [3] H. Bass, J. Milnor, and J.-P. Serre, Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n} (n \geq 2)$, Publ. Math. I.H.E.S. [bf 33, 1967.

Bei [4] A. Beilinson, Height Paring between algebraic cycles. K-theory , arithmetic, and geometry. Lecture Notes in Mathematics #1289, springer (1986).

Ber [5] J. Berrick, An Approach to Algebraic K-theory, Pitman 1982.

Bl [6] S. Bloch, Algebraic cycles and higher K-theory. Advances in Math **61** (1986), 267-304.

B [7] R. Bott. Math Review MR0116022 (22 # 6817) of Le théorème de Riemann-Roch by Armand Borel and Jean-Pierre Serre.

FL [8] E. Friedlander and H.B. Lawson, Moving algebraic cycles of bounded degree. Inventiones Math. **132** (1998), 91-119.

Fu [9] W. Fulton, *Intersection Theory*, Springer 1984.

G [10] D. Grayson, *Higher algebraic K-theory. II (after Daniel Quillen)* L.N.M. # 551, pp 217-240.

H [11] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

M [12] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math Studies 72, Princeton University press 1971.

Q [13] D. Quillen, Higher algebraic K-theory, I. L.N.M # 341, pp. 85 – 147.

W [14] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: eric@math.nwu.edu