A GLIMPSE OF ALGEBRAIC K-THEORY:

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During the first three days of September, 1997, I had the privilege of giving a series of five lectures at the beginning of the "School on Algebraic K-Theory and Applications" at the International Center for Theoretical Physics in Trieste. What follows are the written notes of my lectures, essentially in the form distributed to the audience. I am especially grateful to Professor Adremi Kuku for the opportunity to participate in this workshop and whose encouragement motivated me to prepare these informal notes.

Lecture I: $K_0(-)$ and $K_1(-)$

Perhaps the first new concept that arises in the study of K-theory, and one which recurs frequently, is that of the group completion of an abelian monoid. The basic example: the group completion of the monoid $\mathbf{N}$ of natural numbers is the group $\mathbf{Z}$ of integers. Recall that an abelian monoid M is a set together with a binary, associative, commutative operation $+ : M \times M \to M$ and a distinguished element $0 \in M$ which serves as an identify (i.e., $0 + m = m$ for all $m \in M$). Then we define the group completion $\gamma : M \to M^+$ by setting M^+ equal to the quotient of the free abelian group with generators $[m], m \in M$ modulo the subgroup generated by elements of the form $[m]+[n]-[m+n]$ and define $\gamma : M \to M^+$ by sending $m \in M$ to $[m]$. We frequently refer to M^+ as the *Grothendieck group* of M.

Universal property. *Let M be an abelian monoid and $\gamma : M \to M^+$ denote its group completion. Then for any homomorphism $\phi : M \to A$ from M to a group A, there exists a unique homomorphism $\phi^+ : M^+ \to A$ such that $\phi^+ \circ \gamma = \phi : M \to A$.*

This leads almost immediately to K-theory. Let R be a ring (always assumed associative with unit, but not necessarily commutative). Recall that an (always assumed left) R-module P is said to be projective if there exists another R-module Q such that $P \oplus Q$ is a free R-module.

Definition I.1. *Let $\mathcal{P}(R)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of finitely generated projective R-modules. Then we define $K_0(R)$ to be $\mathcal{P}(R)^+$.*

Warning: The group completion map $\gamma : \mathcal{P}(R) \to K_0(R)$ is frequently not injective.

One of the exercises asks you to verify that if $j : R \to S$ is a ring homomorphism and if P is a finitely generated projective R-module, then $S \otimes_R P$ is a finitely

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generated projective S-module. Using the universal property of the Grothendieck group, you should also check that this construction determines $j_* : K_0(R) \to K_0(S)$. Indeed, we see that $K_0(-)$ is a (covariant) functor from rings to abelian groups.

Example If $R = F$ is a field, then a finitely generated F-module is just a finite dimensional F-vector space. Two such vector spaces are isomorphic if and only if they have the same dimension. Thus, $\mathcal{P}(F) \simeq \mathbf{N}$ and $K_0(F) = \mathbf{Z}$.

Example. Let $K/\mathbf{Q}$ be a finite field extension of the rational numbers (K is said to be a *number field*) and let $\mathcal{O}_K \subset K$ be the ring of algebraic integers in K. Thus, $\mathcal{O}$ is the subring of those elements $\alpha \in K$ which satisfy a monic polynomial $p_\alpha(x) \in \mathbf{Z}[x]$. Recall that $\mathcal{O}_K$ is a Dedekind domain. The theory of Dedekind domains permits us to conclude that

$$K_0(\mathcal{O}_K) = \mathbf{Z} \oplus Cl(K)$$

where $Cl(K)$ is the ideal class group of K.

A well known theorem of Minkowski asserts that $Cl(K)$ is finite for any number field K (cf. [Rosenberg]). Computing class groups is devilishly difficult. We do know that there only finitely many cyclotomic fields (i.e., of the form $\mathbf{Q}(\zeta_n)$ obtained by adjoining a primitive n-th root of unity to $\mathbf{Q}$) with class group $\{1\}$. The smallest n with non-trivial class group is $n = 23$ for which $Cl(Q(\zeta_{23})) = \mathbf{Z}/3$. A check of tables shows, for example, that $Cl(\mathbf{Q}(\zeta_{100})) = \mathbf{Z}/65$.

The K-theory of integral group rings has several important applications in topology. For a group π, the integral group ring $\mathbf{Z}[\pi]$ is defined to be the ring whose underlying abelian group is the free group on the set $[g], g \in \pi$ and whose ring structure is defined by setting $[g] \cdot [h] = [g \cdot h]$. Thus, if π is not abelian, then $\mathbf{Z}[\pi]$ is not a commutative ring.

Application. *Let X be a path-connected space with the homotopy type of a C.W. complex and with fundamental group π. Suppose that X is a retract of a finite C.W. complex. Then the Wall finiteness obstruction is an element of $K_0(\mathbf{Z}[\pi])$ which vanishes if and only if X is homotopy equivalent to a finite C.W. complex.*

We now consider topological K-theory for a topological space X. This is also constructed as a Grothendieck group and is typically easier to compute than algebraic K-theory of a ring R. Moreover, results first proved for topological K-theory have both motivated and helped to prove important theorems in algebraic K-theory.

Definition I.2. *Let $\mathbf{F}$ denote either the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. An $\mathbf{F}$-vector bundle on a topological space X is a continuous open surjective map $p : E \to X$ satisfying*
(a.) For all $x \in X$, $p^{-1}(x)$ is a finite dimensional $\mathbf{F}$-vector space.
(b.) There are continuous maps $E \times E \to E, \mathbf{F} \times E \to E$ which provide the vector space structure on $p^{-1}(x)$, all $x \in X$.
(c.) For all $x \in X$, there exists an open neighborhood $U_x \subset X$, an $\mathbf{F}$-vector space V, and a homeomorphism $\psi_x : V \times U_x \to p^{-1}(U_x)$ over U_x (i.e., $pr_2 = p \circ \psi_x : V \times U_x \to U_x$) compatible with the structure in (b.).

Examples. Let $X = S^1$, the circle. The projection of the Möbius band M to its equator $p : M \to S^1$ is a rank 1, real vector bundle over S^1. Let $X = S^2$, the 2-sphere. Then the projection $p : T_{S^2} \to S^2$ of the tangent bundle is a non-trivial vector bundle. Let $X = S^2$, but now view X as the complex projective line, so that points of X can be viewed as complex lines through the origin in $\mathbf{C}^2$ (i.e., complex subspaces of $\mathbf{C}^2$ of dimension 1). Then there is a natural rank 1, complex line bundle $E \to X$ whose fibre above $x \in X$ is the complex line parametrized by x; if $E - o(X) \to X$ denotes the result of removing the origin of each fibre, then we can identify $E - o(X)$ with $\mathbf{C}^2 - \{0\}$.

Definition I.3. *Let $Vect_{\mathbf{F}}(X)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of $\mathbf{F}$-vector bundles of X. We define*

$$K^0_{top}(X) = Vect_{\mathbf{C}}(X)^+, \quad KO^0_{top}(X) = Vect_{\mathbf{R}}(X)^+.$$

(This definition will agree with our more sophisticated definition of topological K-theory given in Definition III.2 provided that the X has the homotopy type of a finite dimensional C.W. complex.)

The reason we use a superscript 0 rather than a subscript 0 for topological K-theory is that it determines a contravariant functor. Namely, if $f : X \to Y$ is a continuous map of topological spaces and if $p : E \to Y$ is an $\mathbf{F}$-vector bundle on Y, then $pr_2 : E \times_Y X \to X$ is an $\mathbf{F}$-vector bundle on X. This determines

$$f^* : K^0_{top}(Y) \to K^0_{top}(X).$$

Example. Let n_{S^2} denote the "trivial" rank n, real vector bundle over S^2 (i.e., $pr_2 : \mathbf{R}^n \times S^2 \to S^2$). Then $T_{S^2} \oplus 1_{S^2} \simeq 3_{S^2}$. We conclude that $Vect_{\mathbf{R}}(S^2) \to K\mathcal{O}^0_{top}(S^2)$ is not 1-1.

Here is an early theorem of Richard Swan relating algebraic and topological K-theory. You can find a full proof, for example, in [Rosenberg].

Swan's Theorem. *Let $\mathbf{F} = \mathbf{R}$ (respectively, $= \mathbf{C}$), let X be a compact Hausdorff space, and let $\mathcal{C}(X, \mathbf{F})$ denote the ring of continuous functions $X \to \mathbf{F}$. For any $E \in Vect_{\mathbf{F}}(X)$, define the $\mathbf{F}$-vector space of global sections $\Gamma(X, E)$ to be*

$$\Gamma(X, E) = \{s : X \to E \ \mathit{continuous}; p \circ s = id_X\}.$$

Then sending E to $\Gamma(X, E)$ determines isomorphisms

$$KO^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbf{R})), \quad K^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbf{C})).$$

So far, we have only considered degree 0 algebraic and topological K-theory. Before we consider $K_n(R), n \in \mathbf{N}, K^n_{top}(X), n \in \mathbf{Z}$, we look explicitly at $K_1(R)$.

Denote by $GL_n(R)$ the group of invertible $n \times n$ matrices in R (i.e., an element of $GL_n(R)$ is an $n \times n$ matrix with entries in R and which admits a two-sided inverse under matrix multiplication). Denote by $GL(R)$ the union over n of $GL_n(R)$, where the inclusion $GL_n(R) \subset GL_{n+1}(R)$ sends an $n \times n$ matrix $(a_{i,j})$ to the $(n+1) \times (n+1)$ matrix whose (i, j)-th entry equals $a_{i,j}$ if both i, j are $\leq n$, whose $(n+1, n+1)$-entry is 1, and whose other entries are 0.

Definition I.4. *For any ring R, we define $K_1(R)$ by*

$$K_1(R) = GL(R)/[GL(R), GL(R)],$$

where $[GL(R), GL(R)]$ denotes the commutator subgroup of $GL(R)$.

Thus, $K_1(R)$ is the maximal abelian quotient group of $GL(R)$.

The following plays an important role in further developments in algebraic K-theory.

Whitehead Lemma. $[GL(R), GL(R)] \subset GL(R)$ *is the normal subgroup generated by elementary matrices (i.e., those matrices with at most one non-zero diagonal element and with diagonal elements all equal to 1).*

Example If R is a commutative ring, then sending an invertible $n \times n$ matrix to its determinant determines a well defined surjective homomorphism

$$det : K_1(R) \to R^* = \{\text{invertible elements in } R\}.$$

The kernel of det is denoted $SK_1(R)$. If $R = \mathbf{C}[x_0, x_1, x_2]/x_1 + x_2 + x_3 - 1$, then $SK_1(R) = \mathbf{Z}$.

The following theorem is not at all easy, but it does tell us that nothing surprising happens for rings of integers in number fields.

Theorem of Bass-Milnor-Serre. *If $\mathcal{O}_K$ is the ring of integers in a number field K, then $SK_1(\mathcal{O}_K) = 0$.*

Application The work of Bass-Milnor-Serre was dedicated to solving the following question: is every subgroup $H \subset SL(\mathcal{O}_K)$ of finite index a "congruent subgroup" (i.e., of the form $ker\{SL(\mathcal{O}_K) \to SL(O/p^n)\}$ for some prime ideal $p \subset \mathcal{O}_K$.

The preceding theorem is complemented by the following classical result due to Dirichlet (cf. [Rosenberg]).

Dirichlet's Theorem. *Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then*

$$O_K^* = \mu(K) \oplus \mathbf{Z}^{r_1+r_2-1}$$

where $\mu(K) \subset K$ denotes the finite subgroup of roots of unity and where r_1 (respectively, r_2) denotes the number of embeddings of K into $\mathbf{R}$ (resp., number of conjugate pairs of embeddings of K into $\mathbf{C}$).

Application Let π be a finitely generated group and consider the Whitehead group

$$Wh(\pi) = K_1(R)/\{\pm g; g \in \pi\}.$$

A homotopy equivalence of finite complexes with fundamental group π has an invariant (its "Whitehead torsion") in $Wh(\pi)$ which determines whether or not this is a simple homotopy equivalence (given by a chain of "elementary expansions" and "elementary collapses").

Much of our discussion in future lectures will require the language and concepts of category theory. Indeed, working with categories will give us a method to consider various kinds of K-theories simultaneously.

I shall assume that you are familiar with the notion of an abelian category, the standard example of which is the category Mod_R of R-modules for some ring R or the category Mod_R^{fg} of finitely generated R-modules (in which case we must take R to be Noetherian). Recall that in an abelian category $\mathcal{A}$, the set of morphisms $Hom_{\mathcal{A}}(B, C)$ for any $A, B \in Obj\ \mathcal{A}$ has the natural structure of an abelian group; moreover, for each $A, B \in Obj\ \mathcal{A}$, there is an object $B \oplus C$ which is both a product and a coproduct; moreover, any $f : A \to B$ in $Hom_{\mathcal{A}}(A, B)$ has both a kernel and a cokernel. In an abelian category, we can work with exact sequences just as we do in the category of abelian groups.

Warning. $\mathcal{P}(R)$ is not an abelian category. For example, if $R = \mathbf{Z}$, then $n : \mathbf{Z} \to \mathbf{Z}$ is a homomorphism of projective R-modules whose kernel is not projective and thus is not in $\mathcal{P}(\mathbf{Z})$.

Definition I.5. *An exact category $\mathcal{P}$ is a full additive subcategory of some abelian category $\mathcal{A}$ such that*

(a.) There exists some set $S \subset Obj\ \mathcal{A}$ such that every $A \in Obj\ \mathcal{A}$ is isomorphic to some element of S.

(b.) If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence in $\mathcal{A}$ with both $A_1, A_3 \in Obj\ \mathcal{P}$, then $A_2 \in Obj\ \mathcal{P}$.

An admissible monomorphism (respectively, epimorphism) in $\mathcal{P}$ is a monomorphism $A_1 \to A_2$ (resp., $A_2 \to A_3$) in $\mathcal{P}$ which fits in an exact sequence of the form of (b.).

Definition I.6. *If $\mathcal{P}$ is an exact category, we define $K_0(\mathcal{P})$ to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of objects of $\mathcal{P}$ (with respect to $\oplus$) modulo the equivalence relation $[A_2] - [A_1] - [A_3]$ for every exact sequence of the form (I.5.b).*

One of the exercises asks you to show that $K_0(R)$ equals $K_0(\mathcal{P}_R)$, where $\mathcal{P}_R$ is the exact category of finitely generated projective R-modules.

Important example Let $Mod_R^{fg} \subset Mod_R$ be the exact category of finitely generated R-modules. Then we give a special name to the 0-th K-group of this category:

$$G_0(R) \equiv K_0(Mod_R^{fg}).$$

Definition I.7. *Let $\mathcal{P}$ be an exact category in which all exact sequences split. Consider pairs (A, α) where $A \in Obj\ \mathcal{P}$ and α is an automorphism of A. Direct sums and exact sequences of such pairs are defined in the obvious way. Then $K_1(\mathcal{P})$ is defined to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of such pairs modulo the relations given by short exact sequences.*

You can find a proof in [Rosenberg] that $K_1(\mathcal{P}_R)$ equals $K_1(R)$.

Lecture II. Classifying spaces and higher K-theory

Much of our discussions will require some basics of homotopy theory. Recall that two continuous maps $f, g : X \to Y$ between topological spaces are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ with $F_{|X\times\{0\}} = f, F_{|X\times\{1\}} = g$ (where I denotes the unit interval $[0,1]$). If $x \in X, y \in Y$ are chosen ("base points"), then two ("pointed") maps $f, g : (X, \{x\}) \to (Y, \{y\})$ are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ such that $F_{|X\{\times 0\}} = f, F_{|X\times\{1\}} = g$, and $F_{|\{x\}\times I} = \{y\}$ (i.e., F must project $\{x\} \times I$ to $\{y\}$. We use the notation $[(X, x), (Y, y)]$ to denote the pointed homotopy classes of maps from (X, x) (previously denoted $(X, \{x\})$) to $(Y, \{y\})$.

Recall that for any $n \geq 0$ and any pointed space (X, x),

$$\pi_n(X, x) \equiv [(S^n, \infty), (X, x)].$$

For $n = 0$, $\pi_n(X, x)$ is a pointed set; for $n \geq 1$, a group; for $n \geq 2$, an abelian group. If (X, x) is "nice", then $\pi_n(X, x) \simeq [S^n, X]$; moreover, if X is path connected, then the isomorphism class of $\pi_n(X, x)$ is independent of $x \in X$.

A **relative C.W. complex** is a topological pair (X, A) (i.e., A is a subspace of X) such that there exists a sequence of subspaces $A = X_{-1} \subset X_0 \subset \cdots \subset X_n \subset \cdots$ of X with union equal to X such that X_n is obtained from X_{n-1} by "attaching" n-cells (i.e., possibly infinitely many copies of the closed unit disk in $\mathbf{R}^n$, where "attachment" means that the boundary of the disk is identified with its image under a continous map $S^{n-1} \to X_{n-1}$) and such that a subset $F \subset X$ is closed if and only if $X \cap X_n \subset X_n$ is closed for all n. A space X is a C.W. complex if $(X, \emptyset)$ is a relative C.W. complex. A pointed C.W. complex (X, x) is a relative C.W. complex for $(X, \{x\})$.

C.W. complexes have many good properties. For example, the *Whitehead theorem* tell us that if $f : X \to Y$ is a continuous map of connected C.W. complexes such that $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence.

Moreover, C.W. complexes are quite general: If (T, t) is a pointed topological space, then there exists a pointed C.W. complex (X, x) and a continuous map $g : (X, x) \to (T, t)$ such that $g_* : \pi_*(X, x) \to \pi_*(T, t)$ is an isomorphism.

Recall that a continuous map $f : X \to Y$ is said to be a fibration if it has the homotopy lifting property: given any commutative square of continuous maps

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \cap\big\downarrow & & \big\downarrow \\ A \times I & \longrightarrow & Y \end{array}$$

then there exits a map $A \times I \to X$ whose restriction to $A \times \{0\}$ is the upper horizontal map and whose composition with the right vertical map equals the lower horizontal map. A very important property of fibrations is that if $f : X \to Y$ is a fibration, then there is a long exact sequence of homotopy groups for any $x_o \in X, y \in Y$:

$$\cdots \to \pi_n(f^{-1}(y), x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \pi_{n-1}(f^{-1}(y), x_0) \to \cdots$$

If $f : (X, x) \to (Y, y)$ is any pointed map of spaces, we can naturally construct a fibration $\tilde{f} : \tilde{X} \to Y$ together with a homotopy equivalence $X \to \tilde{X}$ over Y. We denote by $htyfib(f)$ the fibre $\tilde{f}^{-1}(y)$ of $\tilde{f}$.

Definition II.1. *Let G be a topological group and X a topological space. Then a* **G-torsor** *over X (or principal G-bundle) is a continuous map $p : E \to X$ together with a continuous action of G on E over X such that there exists an open covering $\{U_i\}$ of X homeomorphisms $G \times U_i \to E_{|U_i}$ for each i respecting G-actions (where G acts on $G \times U_i$ by left multiplication on G).*

Example Assume that G is a discrete group. Then a G-torsor $p : E \to X$ is a normal covering space with covering group G.

Theorem (Milnor). *Let G be a topological group with the homotopy type of a C.W. complex. There there exists a connected C.W. complex BG and a G-torsor $\pi : EG \to BG$ such that sending a continuous function $X \to BG$ to the G-torsor $X \times_{BG} EG \to X$ over X determines a 1-1 correspondence*

$$[X, BG] \xrightarrow{\simeq} \{\textit{isom classes of } G\textit{-torsors over } X\}$$

Moreover, the homotopy type of BG is thereby determined; furthermore, EG is contractible.

Corollary. *If G is discrete, then $\pi_1(BG, *) = G$ and $\pi_n(BG, *) = 0$ for all $n > 0$ (where $*$ is some choice of base point). Moreover, these properties characterize the C.W. complex BG up to homotopy type.*

Sketch of proof. If $n > 0$, then the facts that $\pi_1(S^n) = 0$ and EG is contractible imply that $[S^n, BG] = \{0\}$. The fact that $\pi_1(BG, *) = G$ is classical covering space theory.

The proof of the following proposition is fairly elementary, using a standard projection resolution of $\mathbf{Z}$ as a $\mathbf{Z}[\pi]$-module.

Proposition. *Let π be a discrete group and let A be a $\mathbf{Z}[\pi]$-module. Then*

$$H^*(B\pi, A) = Ext^*_{\mathbf{Z}[\pi]}(\mathbf{Z}, A) \equiv H^*(\pi, A)$$

$$H_*(B\pi, A) = Tor^{\mathbf{Z}[\pi]}_*(\mathbf{Z}, A) \equiv H_*(\pi, A).$$

Now, vector bundles are not G-torsors but rather fibre bundles for the topological groups $O(n)$ (respectively, $U(n)$) in the case of a real (resp., complex) vector bundle of rank n. Nevertheless, because $O(n)$ (resp., $U(n)$) acts faithfully and transitively on $\mathbf{R}^n$ (resp., $\mathbf{C}^n$), we can readily conclude using the above proposition that

$$[X, BO(n)] = \{\text{isom classes of real rank n vector bundles over X}\}$$

$$[X, BU(n)] = \{\text{isom classes of complex rank n vector bundles over X}\}$$

We now introduce the "Quillen plus construction" which immediately gives us Daniel Quillen's first definition of $K_i(R), i > 0$.

Theorem: Plus construction. *Let G be a discrete group and $H \subset G$ be a perfect normal subgroup. Then there exists a C.W. complex BG^+ and a continuous map*

$$\gamma : BG \to BG^+$$

such that $ker\{\pi_1(BG) \to \pi_1(BG^+)\} = H$ and such that $\tilde{H}_*(htyfib(\gamma), \mathbf{Z}) = 0$. Moreover, γ is unique up to homotopy.

Using the Whitehead Lemma, we can easily prove that commutator subgroup $[GL(R), GL(R)]$ is perfect. (One verifies that an $n \times n$ elementary matrix is itself a commutator of elementary matrices provided that $n \geq 4$.)

Definition II.2. *For any ring R, let*

$$\gamma : BGL(R) \to BGL(R)^+$$

denote the Quillen plus construction with respect to $E(R) \subset GL(R)$. We define

$$K_i(R) \equiv \pi_i(BGL(R)^+), \quad i > 0.$$

This construction is closely connected to the group completions of our first lecture. In some sense, $\coprod_n BGL(n,R)$ is "up to homotopy a commutative topological monoid" and $BGL(R) \times \mathbf{Z}$ is a group completion in an appropriate sense. There are several technologies which have been introduced in part to justify this informal description, but we shall not discuss them here.

Remark Essentially by definition, $K_1(R)$ as defined in the first lecture agrees with that given in Definition II.2. Moreover, for any $K_1(R)$-module A,

$$H^*(BGL(R)^+, A) = H^*(BGL(R), A).$$

When Quillen formulated his definition of $K_*(R)$, he also made the following fundamental computation. Indeed, this computation was a motivating factor for Quillen's definition.

Theorem: Quillen's computation for finite fields. *Let $\mathbf{F}_q$ be a finite field. Then the space $BGL(\mathbf{F}_q)^+$ can be described as the homotopy fibre of a computable map. This leads to the following computation for $i > 0$:*

$$K_i(\mathbf{F}_q) = \mathbf{Z}/q^j - 1 \qquad \text{if } i = 2j - 1$$

$$K_i(\mathbf{F}_q) = 0 \qquad \text{if } i = 2j.$$

As you probably know, homotopy groups are notoriously hard to compute. So Quillen has played a nasty trick on us, giving us very interesting invariants with which we struggle to make the most basic calculations. For example, a fundamental problem (and one which my later lectures will address) is to compute $K_i(\mathbf{Z})$. Quite recently, there has been dramatic progress in making this computation.

Rather than try to compute the homotopy groups directly, another approach which many mathematicians have used in past years is to try to study the homology of the group $GL(R)$ since there is somewhat close relationship between homology and homotopy groups. Here is an elementary such relationship.

Proposition: Schur multipliers. *For any ring R,*

$$K_2(R) = H_2(E(R), \mathbf{Z})$$

where $E(R) \subset GL(R)$ is the normal subgroup generated by elementary matrices (equal to $[GL(R), GL(R)]$ by the Whitehead Lemma).

Sketch of proof. We have a map of fibration sequences

$$\begin{array}{ccccc} htyfib(\tilde{\gamma}) & \longrightarrow & BE(R) & \longrightarrow & BE(R)^+ \\ \simeq\downarrow & & \downarrow & & \downarrow \\ htyfib(\gamma) & \longrightarrow & BGL(R) & \longrightarrow & BGL(R)^+ \end{array}$$

where $\tilde{\gamma}$ is the plus construction for $BE(R)$ with respect to $E(R) = E(R)$. Since $\pi_1(BE(R)^+) = 0$,

$$\pi_2(BE(R)^+) = H_2(BE(R)^+, \mathbf{Z}) = H_2(E(R), \mathbf{Z}).$$

Since $BE(R) \to BGL(R)$ is a normal covering space with group $K_1(R)$, so is $BE(R)^+ \to BGL(R)^+$. Thus,

$$K_2(R) = \pi_2(BGL(R)^+) = \pi_2(BE(R)^+).$$

The original definition of $K_2(R)$ was given by John Milnor [Milnor]. Milnor defined $K_2(R)$ as the kernel of the universal central extension

$$0 \to K_2(R) \to St(R) \to E(R) \to 1$$

of the perfect subgroup $E(R)$. The total group of this central extension, called the **Steinberg group** $St(R)$ by Milnor in honor of John Steinberg's related work, is also of considerable interest. It is quite elementary to show that Quillen's definition of $K_2(R)$ agrees with Milnor's.

What are the higher K-groups of an exact category? These are defined in terms of another construction by Quillen, the "Quillen Q-construction."

Definition II.3. *Let $\mathcal{P}$ be an exact category and let $Q\mathcal{P}$ be the category obtained from $\mathcal{P}$ by applying the Quillen Q-construction (as discussed below). Then*

$$K_i(\mathcal{P}) = \pi_{i+1}(Q\mathcal{P}), \quad i \geq 0.$$

Observe that this definition includes K_0. To make sense of this definition , we need to define the category $Q\mathcal{P}$ and then explain what are the homotopy groups of a category.

Definition II.4. *Let $\mathcal{P}$ be an exact category. We define the category $Q\mathcal{P}$ as follows. We set Obj $Q\mathcal{P}$ equal to Obj $\mathcal{P}$. For any $A, B \in$ Obj $Q\mathcal{P}$, we define*

$$Hom_{Q\mathcal{P}}(A,B) = \{A \overset{j}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B; j \textit{ (resp } i\textit{) admis epi (resp. mono)}/\sim\}$$

where the equivalence relation is generated by pairs $A \twoheadleftarrow X \rightarrowtail B, A \twoheadleftarrow X' \rightarrowtail B$ which fit in a commutative diagram

$$\begin{tikzcd} A & X \arrow[l, "j"'] \arrow[r, "I"] \arrow[d, "\simeq"'] & B \arrow[d, "="] \\ A \arrow[u, phantom] & X' \arrow[l, "j'"'] \arrow[r, "i'"] & B \end{tikzcd}$$

It is helpful in understanding this definition to view an element of $Hom_{Q\mathcal{P}}(A,B)$ as a "layer" of B: namely, the data of such an element is equivalent to two admissible monomorphisms $B_1 \rightarrowtail B_2 \rightarrowtail B$ together with an isomorphism $B_1/B_2 \simeq A$.

To compose $A \twoheadleftarrow X \rightarrowtail B$ and $B \twoheadleftarrow Y \rightarrowtail C$ in $Q\mathcal{P}$, we define $Z = X \times_B Y$ and observe that $A \twoheadleftarrow X \twoheadleftarrow Z$ is an admissible epimorphism and $Z \rightarrowtail Y \rightarrowtail C$ is an admissible monomorphism.

To define the homotopy groups of $Q\mathcal{P}$, we shall introduce the concept of a simplicial set. Let Δ denote the category whose objects we denote by $\underline{n} = \langle 0, 1, \ldots, n \rangle$ indexed by $n \in \mathbf{N}$ and whose morphisms are given by

$$Hom_\Delta(\underline{m}, \underline{n}) = \{\text{non-decreasing maps } \langle 0, 1, \ldots, n\rangle \to \langle 0, 1, \ldots, m\rangle\}.$$

We have special names for certain morphisms in Δ (which generate under composition all the morphisms of Δ):

$$\partial_i : \underline{n-1} \to \underline{n} \ (\textit{skip } i); \quad \sigma_j : \underline{n+1} \to \underline{n} \ (\text{repeat j}).$$

These satisfy certain standard relations which many topologists know by heart.

Definition II.5. *A simplicial set $S_.$ is a functor $\Delta^{op} \to (sets)$.*

In other words, $S_.$ consists of a set S_n for each $n \geq 0$ and maps $d_i : S_n \to S_{n-1}, s_j : S_n \to S_{n+1}$ satisfying the relations given by the relations satisfied by $\partial_i, \sigma_j \in \Delta$.

Important example Let T be a topological space. Then the **singular complex** $Sing_.T$ is a simplicial set. Recall that $Sing_nT$ is the set of continuous maps $\Delta^n \to T$, where $\Delta^n \subset \mathbf{R}^{n+1}$ is the standard n-simplex consisting of those points $\underline{x} = (x_0, \ldots, x_n)$ with each $x_i \geq 0$ and $\sum x_i = 1$. Since any map $\mu : \underline{n} \to \underline{m}$ determines a (linear) map $\Delta^n \to \Delta^m$, it also determines $\mu : Sing_mT \to Sing_nT$, so that we may easily verify that $Sing_. : \Delta^{op} \to (sets)$ is a well defined functor.

Definition II.6 (Milnor's geometric realization functor). *For any simplicial set $X_.$, we define its geometric realization as the topological space $|X_.|$ given as follows:*

$$|X_.| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is given by $(x, \mu \circ t) \simeq (\mu \circ x, t)$ whenever $x \in X_m, t \in \Delta^n, \mu : \underline{n} \to \underline{m}$ a map of Δ. This quotient is given the quotient topology, where each $X_n \times \Delta^n$ is topologized as a disjoint union indexed by $x \in X_n$ of copies of $\Delta^n \subset \mathbf{R}^{n+1}$.

Now, simplicial sets are a very good combinatorial model for homotopy theory as the next theorem reveals.

Theorem: Homotopy category.

(a.) Milnor's geometric realization functor is left adjoint to the singular functor; in other words, for every simplicial set $X_.$ and every topological space T,

$$Hom_{(s.sets)}(X_., Sing_.T) = Hom_{(spaces)}(|X_.|, T).$$

(b) For any simplicial set $X_.$, $|X_.|$ is a C.W. complex; moreover, for any topological space T, $Sing.(T)$ is a particularly well behaved type of simplicial set called a Kan complex.

(c.) *For any topological space T and any point $t \in T$, the adjunction morphism*

$$(|Sing_{.}T|, t) \to (T, t)$$

induces an isomorphism on homotopy groups.

(d.) *The adjunction morphisms of (a.) induces an equivalence of categories*

$$(Kan\ cxes)/\sim hom.equiv \quad \simeq (C.W.\ cxes)/\sim hom.equiv$$

We now return to the definition of the homotopy groups of a (small) category.

Definition II.7. *Let $\mathcal{C}$ be a small category. We define the* **nerve** $N\mathcal{C} \in (s.sets)$ *to be the simplicial set whose set of n-simplices is the set of composable n-tuples of morphisms in $\mathcal{C}$:*

$$N\mathcal{C}_n = \{C_n \xrightarrow{\gamma_n} C_{n-1} \to \cdots \xrightarrow{\gamma_1} C_0\}.$$

For $\partial_i : \underline{n-1} \to \underline{n}$, we define $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ to send the n-tuple $C_n \to \cdots \to C_0$ to that $n-1$-tuple given by composing γ_{i+1} and γ_i whenever $0 < i < n$, by dropping $\stackrel{\gamma_1}{\to} C_0$ if $i = 0$ and by dropping $C_n \stackrel{\gamma_n}{\to}$ if $i = n$. For $\sigma_j : \underline{n} \to \underline{n+1}$, we define $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ by repeating C_j and inserting the identity map.

We define the **classifying space** $B\mathcal{C}$ *of the category $\mathcal{C}$ to be $|N\mathcal{C}|$, the geometric realization of the nerve of $\mathcal{C}$.*

Example Let G be a (discrete) group and let $\mathcal{G}$ denote the category with a single object (denoted $*$) and with $Hom_{\mathcal{G}}(*, *) = G$. Then $B\mathcal{G}$ is a model for BG (i.e., $B\mathcal{G}$ is a connected C.W. complex with $\pi_1(B\mathcal{G}, *) = G$ and all higher homotopy groups 0).

Definition II.8. *For any small category $\mathcal{C}$, we define*

$$\pi_i(\mathcal{C}) \equiv \pi_i(B\mathcal{C}), \quad i \geq 0.$$

Furthermore, for any exact category $\mathcal{P}$, we define

$$K_i(\mathcal{P}) \equiv \pi_{i+1}(BQ\mathcal{P}), \quad i \geq 0.$$

Theorem (Quillen). *Let R be a ring and let $\mathcal{P}_R$ denote the exact category of finitely generated projective R-modules. Then*

$$K_i(R) = K_i(\mathcal{P}_R), \quad i \geq 0.$$

Lecture III. Topological K-theory and Algebraic Geometry

In this lecture, we will develop some of the machinery which makes topological K-theory both useful and computable. Not only is this of considerable interest in its own right, but also much of current research by algebraic K-theorists centers around developing similar results for algebraic K-theory.

The topological group $O(n) \subset M_n(\mathbf{R}) \simeq \mathbf{R}^{n^2}$ consists of those real matrices A with $A \cdot A^t = 1$; the topological group $U(n) \subset M_n(\mathbf{C}) \simeq \mathbf{C}^{n^2}$ consists of those complex matrices A with $A \cdot \overline{A}^t = 1$. Stabilizing with respect to n in the usual way, we obtain

$$O = \bigcup_{n \geq 1} O(n), \quad U = \bigcup_{n \geq 1} U(n).$$

Our analysis of classifying spaces for G-torsors with $G = O(n)$ or $U(n)$ is the major ingredient in the proof of the following theorem.

Theorem. *Let X be a finite dimensional C.W. complex. Then*

$$KO^0_{top}(X) = [X, BO \times \mathbf{Z}], \quad K^0_{top}(X) = [X, BU \times \mathbf{Z}].$$

Of fundamental importance in the study of topological K-theory is the following theorem of Raoul Bott. Recall that if (X, x) is pointed space, then the **loop space** ΩX is the function complex (with the compact-open topology) of continuous maps from (S^1, ∞) to (X, x).

Theorem: Bott Periodicity.

(a.) There is a homotopy equivalence

$$BO \times \mathbf{Z} \simeq \Omega^8(BO \times \mathbf{Z})$$

from $BO \times \mathbf{Z}$ to its 8-fold loop space. Moreover, the homotopy groups $\pi_i(BO \times \mathbf{Z})$ are given by $\mathbf{Z}, \mathbf{Z}/2, \mathbf{Z}/2, 0, \mathbf{Z}, 0, 0, 0$ depending upon whether i is congruent to $0, 1, 2, 3, 4, 5, 6, 7$ modulo 8.

(b.) There is a homotopy equivalence

$$BU \times \mathbf{Z} \simeq \Omega^2(BU \times \mathbf{Z})$$

from $BU \times \mathbf{Z}$ to its 2-fold loop space. Moreover, $\pi_i(BU \times \mathbf{Z})$ is $\mathbf{Z}$ if i is even and equals 0 if i is odd.

Thus, both $BO \times \mathbf{Z}$ and $BU \times \mathbf{Z}$ are Ω-spectra as defined as follows.

Definition III.1. *An Ω-spectrum $\underline{E}$ is a sequence of pointed spaces $\{E^0, E^1, \dots\}$, each of which has the homotopy type of a pointed C.W. complex, together with homotopy equivalences relating each E^n with the loop space of E^{n+1}; in other words, a sequence of pointed homotopy equivalences*

$$E^0 \xrightarrow{\simeq} \Omega E^1 \xrightarrow{\simeq} \Omega^2 E^2 \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \Omega^n E^n \to \cdots$$

Theorem. *(cf. [Spanier]) Let $\underline{E}$ be an Ω-spectrum. Then for any topological space X with closed subspace $A \subset X$, set*

$$h^n_{\underline{E}}(X,A) = [(X,A), E^n], \quad n \geq 0$$

Then $(X,a) \mapsto h^*_{\underline{E}}(X,A)$ is a generalized cohomology theory which satisfies all of the Eilenberg-Steenrod axioms except that its value at a point (i.e., $(*, \emptyset)$) may not be that of ordinary cohomology:

(a.) $h^*_{\underline{E}}(-)$ is a functor from the category of pairs of spaces to graded abelian groups.
(b.) for each $n \geq 0$ and each pair of spaces (X,A), there is a functorial connecting homomorphism $\partial : h^n_{\underline{E}}(A) \to h^{n+1}_{\underline{E}}(X,A)$.
(c.) the connecting homomorphisms of (b.) determine long exact sequences for every pair (X,A).
(d.) $h^*_{\underline{E}}(-)$ satisfies excision: i.e., for every pair (X,A) and every subspace $U \subset A$ whose closure lies in the interior of A, $h^*_{\underline{E}}(X,A) \simeq h^*_{\underline{E}}(X-U, A-U)$.

Observe that in the above definition we use the notation $h^*_{\underline{E}}(X)$ for $h^*_{\underline{E}}(X,\emptyset) = h^*_{\underline{E}}(X_+, *)$, where X_+ is the disjoint union of X and a point $*$.

Definition III.2. *The (periodic) topological K-theories $KO^*_{top}(-), K^*_{top}(-)$ are the generalized cohomology theories associated to the Ω-spectra given by $BO \times \mathbf{Z}$ and $BU \times \mathbf{Z}$ with their deloopings given by Bott periodicity. In particular,*

$$K^{2j}_{top}(X) = [X, BU \times \mathbf{Z}], \quad K^{2j-1}_{top}(X) = [X, U],$$

so that we recover our definition of $K^0_{top}(X)$ (and similarly $KO^0_{top}(X)$) whenever X is a finite dimensional C.W. complex.

Tensor product of vector bundles induces a multiplication

$$K^0_{top}(X) \otimes K^0_{top}(X) \to K^0_{top}(X)$$

for any finite dimensional C.W. complex X. This can be generalized by observing that tensor product induces group homomorphisms $U(m) \times U(n) \to U(n+m)$ and thereby maps of classifying spaces

$$BU(m) \times BU(n) \to BU(n+m).$$

With a little effort, one can show that these multiplication maps are compatible up to homotopy with the standard embeddings $U(m) \subset U(m+1), U(n) \subset U(n+1)$ and thereby give us a pairing

$$(BU \times \mathbf{Z}) \times (BU \times \mathbf{Z}) \to BU \times \mathbf{Z}$$

(factoring through the smash product). In this way, $BU \times \mathbf{Z}$ has the structure of an H-space which induces a pairing of spectra and thus a multiplication for the generalized cohomology theory $K^*(-)$. (A completely similar argument applies to $KO^*(-)$).

As an example of how topological K-theory inspired even the early effort in algebraic K-theory we mention the following classical theorem of Hyman Bass. The analogous result in topological K-theory for rank e vector bundles over a finite dimension C.W. complex of dimension $d < e$ can be readily proved using the standard method of "obstruction theory".

Theorem: Bass stability theorem. *Let A be a commutative, noetherian ring of Krull dimension d. Then for any two projective A-modules P, P' of rank $e > d$, if $[P] = [P'] \in K_0(A)$ then P must be isomorphic to P'.*

We next describe the close relationship between $K^*_{top}(X)$ and the integral cohomology $H^*(X) = H^*(X, \mathbf{Z})$ of X. Indeed, we give below the general form of the spectral sequence for any generalized cohomology theory, and observe that this is particularly nice for $K^*_{top}(-)$.

Theorem: Atiyah-Hirzebruch spectral sequence. *For any generalized cohomology theory $h^*_{\underline{E}}(-)$ and any topological space X, there exists a right half-plane spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(X, h^q(*)) \Rightarrow h^{p+q}_{\underline{E}}(X).$$

In the special case of $K^*_{top}(-)$, this takes the following form

$$E_2^{p,q} = H^p(X, \mathbf{Z}(q/2)) \Rightarrow K^{p+q}_{top}(X)$$

where $\mathbf{Z}(q/2) = \mathbf{Z}$ if q is an even non-negative integer and 0 otherwise.

What is a spectral sequence of cohomological type? This is the data of a 2-dimensional array $E_r^{p,q}$ of abelian groups for each $r \geq r_0$ (typically, r_0 equals 0, or 1 or 2; in our case $r_0 = 2$) and homomorphisms

$$d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$$

such that the next array $E_{r+1}^{p,q}$ is given by the cohomology of these homomorphisms:

$$E_{r+1}^{p,q} = ker\{d_r^{p,q}\}/im\{d_r^{p-r,q+r-1}\}.$$

To say that the spectral sequence is "right half plane" is to say $E_r^{p,q} = 0$ whenever $p < 0$. We say that the spectral sequence **converges to the abutment** E^*_∞ (in our case $h^*_{\underline{E}}(X)$) if at each spot (p, q) there are only finitely many non-zero homomorphisms going in and going out and if there exists a decreasing filtration $\{F^p E^n_\infty\}$ on each E^n_∞ so that

$$E^n_\infty \;=\; \bigcup_p F^p E^n_\infty, \;\; 0 = \bigcap_p F^p E^n_\infty,$$

$$F^p E^n_\infty / F^{p+1} E^n_\infty = E_R^{p,n-p} \quad R >> 0.$$

If X is a C.W. complex then we can define its **p-skeleton** $sk_p(X)$ for each $p \geq 0$ as the subspace of X consisting of the union of those cells of dimension $\leq p$. Then the filtration on $h^*_{\underline{E}}(X)$ for the Atiyah-Hirzebruch spectral sequence is given by

$$F^p E^*_\infty = ker\{h^*_{\underline{E}}(X) \rightarrow h^*_{\underline{E}}(sk_p(X)\}.$$

While we are discussing spectral sequences, we should mention the following:

Theorem: Serre spectral sequence. *Let (B, b) be a connected, pointed C.W. complex. For any fibration $p : E \to B$ of topological spaces with fibre $F = p^{-1}(b)$ and for any abelian group A, there exists a convergent first quadrant spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(B, H^q(F, A)) \Rightarrow H^{p+q}(E, A)$$

provided that $\pi_1(B, b)$ acts trivially on $H^*(F, A)$.

The following theorem tells us that topological K-theory tensor the rational numbers is essentially rational cohomology $H^*(-, \mathbf{Q})$.

Theorem (Atiyah-Hirzebruch). *Let X be a C.W. complex. Then there exists a homomorphism of graded rings*

$$ch : K_{top}^*(X) \otimes \mathbf{Q} \to H^*(X, \mathbf{Q})$$

which restricts to isomorphisms

$$K_{top}^0(X) \otimes \mathbf{Q} \simeq H^{ev}(X, \mathbf{Q}), \quad K_{top}^1(X) \otimes Q \simeq H^{odd}(X, \mathbf{Q}).$$

In Lecture V, we shall discuss operations on both topological and algebraic K-theory. These operations originate from the observation the the exterior products $\Lambda^i(P)$ of a projective module P are likewise projective modules and the exterior products $\Lambda^i(E)$ of a vector bundle E are likewise vector bundles. The other simple fact that we use is the following relationship

$$\Lambda^n(V \oplus W) = \oplus_{i+j=n} \Lambda^i(V) \otimes \Lambda^j(W)$$

which one can first check for vector spaces, then extend to either projective modules or vector bundles.

In particular, J. Frank Adams introduced operations

$$\psi^k(-) : K_{top}^0(-) \to K_{top}^0(-), \quad k > 0$$

(called **Adams operations**) which have many applications. We shall use these Adams operations in algebraic K-theory, but here is a short list of famous theorems of Adams using topological K-theory and Adams operations:

Applications (Adams)

(1.) Determination of the number of linearly independent vector fields on the n-sphere S^n for all $n > 1$.

(b.) Determination of the only dimensions (namely, $n = 1, 2, 4, 8$) for which $\mathbf{R}^n$ admits the structure of a division algebra. (The examples of the real numbers $\mathbf{R}$, the complex numbers $\mathbf{C}$, the quaternions, and the Cayley numbers gives us structures in these dimensions.)

(c.) Determination of those (now well understood) elements of the homotopy groups of spheres associated with $KO_{top}^0(S^n)$.

In the remainder of this lecture, I shall review some of the basic concepts of algebraic geometry.

Definition III.3. *A* **sheaf** *of sets F on a topological space X is a contravariant functor*

$$F : (\textit{open sets of } X, \textit{ inclusions}) \to (\textit{sets})$$

satisfying the sheaf axiom: for any open set U, any open covering $\{U_i \to U\}_{i \in I}$,

$$F(U) = \textit{equalizer } \{\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)\}.$$

In other words, an element $s \in F(U)$ (called a "section of F on U") is uniquely determined by elements $s_i \in F(U_i)$ which agree on intersections $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all $i, j \in I$, where $s_{j|U_i \cap U_j}$ denotes the image of $s_i \in F(U_i)$ under the functorially given map $F(U_i) \to F(U_i \cap U_j)$.

For any $x \in X$, the **stalk** *of F_x at x is the following set: colimit$_{x \in U} F(U)$.*

If the functor F is actually a functor to (groups)(respectively, (abelian groups); resp., (rings);etc.), then we say that F is a sheaf of groups (resp., sheaf of abelian groups; resp., sheaf of rings;etc).

Basic Example Let $p : E \to X$ be a continuous map. Define an associated sheaf F by sending an open subset $U \subset X$ to the set of continuous functions $s : U \to E$ with the property that $p \circ s : U \to X$ is the inclusion $U \subset X$.

Recall that if A is a commutative ring we denote by $SpecA$ the set of prime ideals of A. The set $X = SpecA$ is provided with a topology, the **Zariski topology** defined as follows: a subset $Y \subset X$ is closed if and only if there exists some ideal $I \subset A$ such that $Y = \{p \in X; I \subset p\}$. We define the **structure sheaf** $\mathcal{O}_X$ of commutative rings on $X = SpecA$ by specifying its value on the basic open set $X_f = \{p \in SpecA, f \notin p\}$ for some $f \in A$ to be the ring A_f obtained from A by adjoining the inverse to f. (Recall that $A \to A_f$ sends to 0 any element $a \in A$ such that $f^n \cdot a = 0$ for some n). We now use the sheaf axiom to determine the value of $\mathcal{O}_X$ on any arbitrary open set $U \subset X$, for any such U is a finite union of basic open subsets. The stalk $\mathcal{O}_{X,p}$ of the structure sheaf at a prime ideal $p \subset A$ is easily computed to be the local ring $A_p = \{f \notin p\}^{-1}A$.

Thus, $(X = SpecA, \mathcal{O}_X)$ has the structure of a **local ringed space**: a topological space with a sheaf of commutative rings each of whose stalks is a local ring. A map of local ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \to Y$ of topological spaces and a map of sheaves $O_Y \to f_* O_X$ on Y, where $f_* O_X(V) = O_X(f^{-1}(V))$ for any open $V \subset Y$.

If M is an A-module for a commutative ring A, then M defines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X = SpecA$. Namely, for each basic open subset $X_f \subset X$, we define $\tilde{M}(X_f) \equiv A_f \otimes_A M$. This is easily seen to determine a sheaf of abelian groups on X with the additional property that for every open $U \subset X$, $\tilde{M}(U)$ is a sheaf of $\mathcal{O}_X(U)$-modules with structure compatible with restriction to smaller open subsets $U' \subset U$.

Definition III.4. *A local ringed space $(X, \mathcal{O}_X)$ is said to be an* **affine scheme** *if it is isomorphic (as local ringed spaces) to $(X = SpecA, \mathcal{O}_X)$ as defined above. A* **scheme** *$(X, \mathcal{O}_X)$ is a local ringed space for which there exists a finite open covering $\{U_i\}_{i \in I}$ of X such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine scheme.*

*If k is a field, a k-***variety** *is a scheme $(X, \mathcal{O}_X)$ with the property there is a finite open covering $\{U_i\}_{i \in I}$ by affine schemes with the property that each $(U_i, \mathcal{O}_{X|U_i}) \simeq (SpecA_i, \mathcal{O}_{SpecA_i})$ with A_i a finitely generated k-algebra without nilpotents.*

Example The scheme $\mathbf{P}^1_{\mathbf{Z}}$ is a non-affine scheme defined by patching together two copies of the affine scheme $Spec\mathbf{Z}[t]$. So $\mathbf{P}^1_{\mathbf{Z}}$ has a covering $\{U_1, U_2\}$ corresponding to rings $A_1 = \mathbf{Z}[u], A_2 = \mathbf{Z}[v]$. These are "patched together" by identifying the open subschemes $Spec(A_1)_u \subset SpecA_1,\ Spec(A_2)_v \subset SpecA_2$ via the isomorphism of rings $(A_1)_u \simeq (A_2)_v$ which sends u to v^{-1}.

Note that we have used $SpecR$ to denote the local ringed space $(SpecR, \mathcal{O}_{SpecR})$; we will continue to use this abbreviated notation.

Definition. *Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $\mathcal{P}_X$ the exact category of sheaves F of $\mathcal{O}_X$-modules with the property that there exists a finite open covering $\{U_i\}$ of X by affine schemes $U_i = SpecA_i$ and free, finitely generated A_i-modules M_i such that the restriction $F_{|U_i}$ of F to U_i is isomorphic to the sheaf $\tilde{M}_i$ on $SpecA_i$.*

We define the algebraic K-theory of the scheme X by setting

$$K_*(X) = K_*(\mathcal{P}_X).$$

In our last two lectures, we will need a generalization of the concept of a topology on a set, a generalization introduced by Grothendieck for which sheaf theory is still feasible.

Definition III.5. *A* **Grothendieck site** *on a scheme $(X, \mathcal{O}_X)$ consists of a class $\mathcal{C}/X$ of schemes over X (i.e., each provided with a specified morphism to X) closed under fibre products (i.e., if $Y_1 \to X, Y_2 \to X \in \mathcal{C}/X$, then $Y_1 \times_X Y_2 \to X \in \mathcal{C}/X$) together with a class of morphisms $\mathcal{E}$ such that*

(a.) all isomorphisms of $\mathcal{C}/X$ are in $\mathcal{E}$
(b.) $\mathcal{E}$ is closed under composition
(c.) if $V \to U, V' \to U \in \mathcal{E}$, then $V \times_U V' \to V' \in \mathcal{E}$.
(d.) if $Y \to X \in \mathcal{C}/X, U \to Y \in \mathcal{E}$, then the composition $U \to X \in \mathcal{C}/X$.

An $\mathcal{E}$-covering $\{U_i \to U\}_{i\in I}$ consists of morphisms in $\mathcal{E}$ with the property that the union of their images equals U.

A sheaf on such a site is a contravariant functor $F : \mathcal{C}/X \to (sets)$ satisfying the sheaf axiom: for all $\mathcal{E}$-coverings $\{U_i \to U\}_{i\in I}$,

$$F(U) = \textit{equalizer}\ \{\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)\}.$$

The **(Grothendieck) $\mathcal{E}$-topology** on the site $(\mathcal{C}/X, \mathcal{E})$ is the collection of all $\mathcal{E}$-coverings for this site. In our next lecture, we will be particularly interested in the **etale topology** in which $\mathcal{E}$ consists of all etale maps between schemes.

Lecture IV. Some hard open problems

The aims of this lecture are to make abundantly clear that algebraic K-theory is still in the early stages of its development with a great deal of work left to be done and to introduce some of the current work in this subject.

The first conjecture we discuss is one of the oldest. As far as I know, little or no progress has been made on this conjecture in the past 25 years.

Conjecture: Bass finiteness. *Let A be a commutative ring which is finitely generated as an algebra over $\mathbf{Z}$. Is $K_n(Mod_A^{fg})$ (which we also denote by $G_n(A)$) finitely generated for all n?*

Remarks

(a.) Even for $n = 0$, Bass' finiteness conjecture appears to be extremely difficult.

(b.) As seen in the previous lecture, $G_n(A) = K_n(A)$ whenever A is regular. We could try to restrict this conjecture to regular rings A and algebraic K-theory $K_*(A)$. However, considering the more general conjecture might yield a useful approach to proving the conjecture, even if we are only interested in regular rings.

(c.) Here is an example of Bass showing that we must assume A is regular or consider $G_*(A)$. Let $A = \mathbf{Z}[x,y]/x^2$. Then the ideal (x) is infinitely additively generated by $x, xy, xy^2, \dots$. On the other hand, if $t \in (x)$, then $1 + t \in A^*$, so that we see that $K_1(A)$ is not finitely generated.

(d.) Perhaps the best positive result is the following theorem of Quillen: If $\mathcal{O}_K$ is the ring of integers in a number field, the $K_n(\mathcal{O}_K)$ is finitely generated for every $n \geq 0$.

(e.) As pointed out by Bass, it is elementary to show (using general theorems of Quillen and Quillen's computation of the K-theory of finite fields) that if A is finite, then $G_n(A) \simeq G_n(A/radA)$ is finite for every $n \geq 0$. Subsequently, Andremi Kuku proved that $K_n(A)$ is also finite whenever A is finite.

Consider now a complex algebraic variety $X/\mathbf{C}$. For convenience, we assume that S is affine but this is not really necessary for what follows. In this case, $X = SpecA$, where $A = \mathbf{C}[X]$ is a finitely generated ring over $\mathbf{C}$ without nilpotents.

There is a natural map

$$BGL(A) \to Hom_{cont}(X, BU)$$

which induces $K_*(A) \to K_{top}^{-*}(X)$, where

$$K_{top}^{-n}(X) \equiv [S^n \wedge (X, \emptyset), BU \times \mathbf{Z}].$$

We shall find it convenient to write

$$K_n^{top}(X) \equiv K_{top}^{-n}(X),$$

so that we may write our map as

$$K_*(A) \to K_*^{top}(X).$$

We will be interested in studying various versions of this map. Perhaps we should first explain how this map is defined. We first observe that

$$GL_n(A) = Hom_{(schemes)}(X, GL_n), \quad BGL_n(A) = Hom_{(s.schemes)}(X, BGL_n)$$

where GL_n is the affine (group) scheme with

$$\mathbf{C}[GL_n] = \mathbf{C}[x_{1,1}, \dots , x_{n,n}, t]/det(X_{i,j}) \cdot t - 1$$

and BGL_n is the simplicial scheme constructed following the classical "bar construction" (so that BGL_n in degree m equals GL_n^m). Thus, the observation that algebraic morphisms of complex varieties determine continuous maps provides a natural map

$$BGL_n(A) \to Hom_{cont}(X, BGL_n\mathbf{C}) \xleftarrow{\simeq} Hom_{cont}(X, BU_n).$$

The homotopy equivalence on the right is established using the fact that $U_n \subset GL_n\mathbf{C}$ is a homotopy equivalence and the fact that the bar construction used to construct BGL_n leads also to a model for the classifying space for $GL_n\mathbf{C}$ as a topological group. The map we seek is constructed by first taking the limit over n to obtain $BGL(A) \to Hom_{cont}(X, BU)$ and then using the (homotopy) universal property of the Quillen-plus construction to obtain

$$BGL(A)^+ \to Hom_{cont}(X, BU).$$

An interesting case is $X = Spec\mathbf{C}$ (in other words, a point). Milnor showed for any uncountable field F (e.g., $F = \mathbf{C}$) that $K_2(F)$ is also uncountable. On the other hand, $K_2^{top}(pt) = \mathbf{Z}$. These are totally different, but not as different as they might first appear!

To explain this cryptic remark, we introduce K-theory mod-n.

Definition IV.1. *For positive integers $i, n > 1$, let $M(i, \mathbf{Z}/n)$ denote the C.W. complex obtained by attaching an i-cell D^i to S^{i-1} via the map $\partial(D^i) = S^{i-1} \to S^{i-1}$ given by multiplication by n.*

For any connected C.W. complex, we define

$$\pi_i(X, \mathbf{Z}/n) \equiv [M(i, \mathbf{Z}/n), X], \quad i, n > 1.$$

If $X = \Omega^2 Y$, we define

$$\pi_i(X, \mathbf{Z}/n) \equiv [M(i+2, \mathbf{Z}/n), Y], \quad i \geq 0, n > 1.$$

Since $S^{i-1} \to M(i, \mathbf{Z}/n)$ is the cone on the multiplication by n map $S^{i-1} \xrightarrow{n} S^{i-1}$, we have long exact sequences

$$\cdots \to \pi_i(X) \xrightarrow{n} \pi_i(X) \to \pi_i(X/\mathbf{Z}/n) \to \pi_{i-1}(X) \to \cdots$$

Now, we can justify our previous remark. The following theorem, proved by Andrei Suslin, follows a well known conjecture by Stephen Lichtenbaum and Daniel Quillen. (We implicitly use the fact that $BGL(R)^+$ admits two (indeed, infinitely many) deloopings, as does $BQ\mathcal{P}_X$).

Theorem: Q-L Conjecture for algebraically closed fields. *The above map $BL(\mathbf{C})^+ \to BU$ induces an isomorphism*

$$K_*(\mathbf{C}, \mathbf{Z}/n) \xrightarrow{\simeq} K_*^{top}(pt, \mathbf{Z}/n), \quad n \geq 0.$$

More generally, if k is an algebraically closed field of characteristic $p \geq 0$, then there is a natural isomorphism

$$K_*(k, \mathbf{Z}/n) \xrightarrow{\simeq} K_*^{top}(pt, \mathbf{Z}/n), \quad (n,p) = 1.$$

Moreover, if the characteristic of k is a positive integer p, then $K_i(k, \mathbf{Z}/p) = 0$, for all $i > 0$.

Perhaps this is sufficient to motivate our next conjecture, which we might call the Quillen-Lichtenbaum Conjecture for smooth complex algebraic varieties.

Conjecture: Q-L for smooth C varieties. *If X is a smooth complex variety of dimension d, then is the natural map*

$$K_i(X, \mathbf{Z}/n) \to K_i^{top}(X, \mathbf{Z}/n)$$

an isomorphism provided that $i \geq d$?

Remark In "low" degrees, $K_*(X, \mathbf{Z}/n)$ should be more interesting and will not be periodic. For example, $K_{ev}^{top}(X, \mathbf{Z}/n)$ has a contribution from the Brauer group of X whereas $K_0(X, \mathbf{Z}/n)$ does not.

Perhaps of more interest than finitely generated algebras over $\mathbf{C}$ are rings of integers in number fields (especially, $\mathbf{Z}$).

Known computations. *Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then Armand Borel has shown that $K_i(\mathcal{O}_K) \otimes \mathbf{Q}$ has the following dimensions: $dim = 1,\ i = 0$; $dim = r_1 + r_2 - 1,\ i = 1$; $dim = 0,\ i = 2j > 0$; $dim = r_1 + r_2,\ i = 4k+1 > 1$; $dim = r_2,\ i = 4k+3$. Here, r_1 (respectively, r_2) is the number of real embeddings (resp., pairs of complex embeddings) of K into $\mathbf{C}$. Since Quillen has shown that $K_i(\mathcal{O}_K)$ is finitely generated, it therefore suffices in order to completely determine these groups to compute $K_*(\mathcal{O}_K, \mathbf{Z}/n)$ for all $n > 1$.*

Topological considerations have led to explicit conjectures of what these K-groups should be. For example, Stephen Mitchell derived the following explicit conjecture from the work of William Dwyer and myself.

Conjecture: $K_*(\mathbf{Z})$. *The K-theory of $\mathbf{Z}$ should be periodic of period 8, given by $K_i(\mathbf{Z}) = 0,\ 0 < i \equiv 0$; $\mathbf{Z} \oplus \mathbf{Z}/2,\ 1 < i \equiv 1$; $\mathbf{Z}/c_k \oplus \mathbf{Z}/2,\ i \equiv 2$, $\mathbf{Z}/8d_k,\ i \equiv 3$; $0,\ i \equiv 4$; $\mathbf{Z},\ i \equiv 5$; $\mathbf{Z}/c_k,\ i \equiv 6$; $Z/4d_k,\ i \equiv 7$. Here, c_k/d_k is defined to be the reduced expression for B_k/k, where B_k is the k-th Bernoulli number (defined by*

$$\frac{t}{e^t - 1} = 1 + \sum_{m=1}^{\infty} \frac{B_m}{(2m)!} t^{2m} \ )$$

and $i = 4k-1$ or $4k-2$).

The motivation for these conjectures comes from the etale topology, a Grothendieck topology associated to the etale site. For this site, the distinguished morphisms $\mathcal{E}$ are etale morphisms of schemes. A map of schemes $f : U \to V$ is said to be etale if there exist affine open coverings $\{U_i\}$ of U, $\{V_j\}$ of V such that the restriction to U_i of f lies in some V_j and such that the corresponding map of commutative rings $A_i \leftarrow R_j$ is unramified (i.e., for all homomorphisms from R to a field k, $A \otimes_R k \leftarrow k$ is a finite separable k algebra) and flat.

The etale topology was introduced by Grothendieck partly to reinterpret Galois cohomology of fields and partly to algebraically realize singular cohomology of complex algebraic varieties. The following "comparison theorem" proved by Michael Artin and Alexander Grothendieck is an important property of the etale topology.

Theorem (Artin, Grothendieck). *If X is a complex algebraic variety, then*

$$H^*_{et}(X, \mathbf{Z}/n) \simeq H^*_{sing}(X, \mathbf{Z}/n).$$

Here, $H^*_{et}(X, \mathbf{Z}/n)$ denotes the derived functors of the global section functor applied to the constant sheaf $\mathbf{Z}/n$ on the etale site and $H^*_{sing}(X, \mathbf{Z}/n)$ is the (usual) singular cohomology with $\mathbf{Z}/n$ coefficients of X with its analytic topology.

Bill Dwyer and I have constructed a theory which we call **etale K-theory** which is topological K-theory for schemes equipped with the etale site. For this theory, there is an Atiyah-Hirzeburch spectral sequence

$$E_2^{p,q} = H^p_{et}(X, K^q_{et}(pt)) \Rightarrow K^{p+q}_{et}(X, \mathbf{Z}/n)$$

provided that $\mathcal{O}_X$ is a sheaf of $\mathbf{Z}[1/n]$-modules. If we let μ_n denote the etale sheaf of n-th roots of unity and let $\mu_n^{\otimes q/2}$ denote $\mu_n^{\otimes j}$ if $q = 2j$ and 0 if j is odd, then this spectral sequence can be rewritten

$$E_2^{p,q} = H^p_{et}(X, \mu^{\otimes q/2}) \Rightarrow K^{et}_{q-p}(X, \mathbf{Z}/n).$$

Using etale K-theory, we can reformulate the Quillen-Lichtenbaum Conjecture (for number fields), putting it in a quite general context.

Conjecture: Quillen-Lichtenbaum. *Let X be a smooth scheme of finite type over a field k, and assume that n is a positive integer with $1/n$ in k or A. Then the natural map*

$$K_i(X, \mathbf{Z}/n) \to K_i^{et}(X, \mathbf{Z}/n)$$

is an isomorphism for i greater or equal to the mod-n etale cohomological dimension of X.

Much work has been done on this conjecture, as you will undoubtedly hear in other lectures. I should mention the work of Robert Thomason who showed that if you force $K_i(X, \mathbf{Z}/n)$ to satisfy the same periodicity as is satisfied by $K_*^{et}(X, \mathbf{Z}/n)$ (i.e., if you invert the "Bott element"), then the map in the above conjecture is an isomorphism. Moreover, Bill Dwyer and I proved that this map is surjective for $X = Spec\mathcal{O}_K$, the ring of integers in a number field.

In order to understand conjectural spectral sequences converging to algebraic K-theory, we should briefly consider operations in K-theory. If A is a commutative ring and M is an A-module, then we define $\Lambda^i M$ to be the quotient of the i-fold tensor power of M (over A) modulo the relations

$$m_1 \otimes \cdots \otimes m_i - (-1)^{sgn(\sigma)} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(i)}, \ \forall \sigma \in \Sigma_i.$$

We define

$$\lambda_t : \mathcal{P}(A) \to \{1 + tK_0(A)[[t]]\}$$

to send a projective P to the power series $\sum_{i=0}^{\infty} t^i[\Lambda^i(P)]$. Since $\lambda_t(P \oplus Q) = \lambda_t(P) \cdot \lambda_t(Q)$, we obtain a unique extension

$$\lambda_t : K_0(A) \to \{1 + tK_0(A)[[t]]\}$$

The Adams operations $\psi^k : K_0(A) \to K_0(A)$ to which we referred in Lecture III can be slickly defined by

$$\Psi_t(x) = \sum_{k=0}^{\infty} \psi^k(x)t^k = \epsilon(x) - t \cdot \frac{d}{dt} \cdot log\lambda_{-t}(x)$$

where $\epsilon : K_0(A) \to H^0(SpecA, \mathbf{Z})$ is the rank function. We also require the gamma operations on $K_0(A)$, defined equally slickly by

$$\gamma_t(x) = \sum_{k=0}^{\infty} \gamma^k t^k = \lambda_{t/1-t}(x).$$

Alternatively, $\gamma^k(x) = \lambda(x + k - 1)$.

These operations need not be as formally presented as I have done (for the sake of time). The **splitting principle** tells us that for any finitely generated projective A-module (or more generally, locally free coherent sheaf on a scheme X), there exists a map of schemes $X \to \tilde{X}$ such that $K_0(X) \to K_0(\tilde{X})$ is injective and $[P]$ has image the direct sum of locally free sheaves of rank 1. Now, $\psi^k(-)$ is a ring homomorphism and $\psi^k([L]) = [L^{\otimes k}]$ if L is a projective of rank 1. Moreover, $\gamma_t(x+y) = \gamma_t(x) \cdot \gamma_t(y)$, and $\gamma_t([L]-1) = 1 + ([L]-1)t$ if L is projective of rank 1.

Important properties of these operations are that the Adams operations ψ^k are ring homomorphisms and the gamma operations γ^k satisfy $\gamma^k([P]-n) = 0$ if P is a rank n projective and $k > n$.

These operations can be extended to all K-groups.

Definition IV.2. *The γ filtration $\{F^s_\gamma K_*(X);\ s \geq 0\}$ on $K_*(X)$ is defined by setting $F^s_\gamma K_*(X) \subset K_i(X)$ to be the subgroup generated by products $\gamma^{r_1}(x_1) \cdots \gamma^{r_j}(x_j)$ such that $\sum_j r_j \geq s$ and $\epsilon(x_j) = 0$ (some mathematicians also require the additional condition that at most one $x_i \notin K_0(X)$, but this does not affect the filtration rationally).*

The following relationship between the Adams operations and the gamma filtration was presumably known to Adams when he defined his operations.

Proposition. *For any $k > 0$, the Adams operation ψ^k acts on*

$$K_*(X)^{(s)}_{\mathbf{Q}} \equiv (F^s_\gamma K_*(X)/F^{s+1}_\gamma K_*(X)) \otimes \mathbf{Q}$$

by multiplication by k^s. We refer to this quotient as the (rational) weight s part of $K_*(X)$.

As we mentioned earlier, in low degrees algebraic K-theory fails to satisfy the periodicity satisfied by topological K-theory. This means that there can not be a spectral sequence converging to algebraic K-theory of exactly the same form as the Atiyah-Hirzebruch spectral sequence. The following conjecture by Alexander Beilinson and Christophe Soulé suggests how the spectral sequence should be modified.

Conjecture: Beilinson-Soulé Vanishing. *If R is a regular commutative ring, then*

$$K_{2i+1}(R)^{(s)}_{\mathbf{Q}} = 0 = K_{2i}(R)^{(s)}_{\mathbf{Q}}, \quad s \leq i.$$

This conjecture is complemented by the following theorem of Soulé. Indeed, Soulé proves a vanishing theorem for more general rings R with a range depending upon the "stable range" of R.

Theorem (Soulé). *For any field F,*

$$K_n(F)^{(s)}_{\mathbf{Q}} = 0, \quad s > n.$$

My typing is not sufficiently good to draw for you the conjectural form of a spectral sequence converging to algebraic K-theory. Such a spectral sequence for fields has been constructed by Bloch and Lichtenbaum, though the conjectural Beilinson-Soulé vanishing remains to be proved (except at the prime 2, by a recent theorem of Vladimir Voevodsky).

I conclude this lecture by remarking that a different approach using other topological machinery has enabled Lars Hasselholt and Ib Madsen to compute the K-theory of the p-adic integers $\mathbf{Z}_p$. Here is my recollection of the their result, stated for the p-primary part of the K-theory, for the torsion which is prime to p was already known to equal that of $K_*(\mathbf{F}_p)$.

Theorem (Hasselholt-Madsen). *The p-primary part of the algebraic K-theory of the p-adic integers $\mathbf{Z}_p$ is given by*

$$K_*(\mathbf{Z})_{(p)} = \pi_*((U \times BImJ \times ImJ)_{(p)})$$

where the spaces ImJ and $BImJ$ are spaces related to $BGL(\mathbf{F}_\ell)$ for a prime ℓ which satisfies the condition that p is a generator of the units in $\mathbf{Z}_\ell$.

Lecture V. Beilinson's vision

In this lecture, we will consider Alexander Beilinson's vision of what algebraic K-theory should be for smooth schemes and related regular rings. I will also discuss some of the ingredients in the recent work of my colleagues Andrei Suslin and Vladimir Voevodsky.

Although our goal is to describe conjectures which would begin to "explain" algebraic K-theory, let me start by giving one (of many) informal explanation of why algebraic K-theory is so interesting to algebraic geometers (and algebraic number theorists). It has been known for some time that there can not be an algebraic theory whose values on complex algebraic varieties is integral (or even rational) singular homology. Indeed, Jean-Pierre Serre observed that this is not possible even for smooth projective algebraic curves because some such curves have automorphism groups which do not admit a representation which would be implied by functoriality. On the other hand, algebraic K-theory is in some sense integral – we define it without inverting residue characteristics or considering only mod-n coefficients. Thus, if we can formulate a sensible Atiyah-Hirzebruch type spectral sequence converging to algebraic K-theory, then the E_2-term offers an algebraic formulation of integral cohomology.

Now, Spencer Bloch has told us for many years that algebraic K-theory is related to algebraic cycles. Before discussing Beilinson's conjectures, we briefly consider this relationship. If X is an algebraic variety over a field k, then the abelian group of algebraic r-cycles on X is the free abelian group on the set of irreducible, r-dimensional subvarieties of X; equivalently, the free abelian group on the subset of those points in X of height r. Many of the most difficult and famous conjectures of algebraic geometry are questions about algebraic cycles.

Algebraic cycles are typically studied by imposing one of several equivalence relations. The equivalence relation most relevant for algebraic K-theory is *rational equivalence.*

Definition V.1. *Let $Z = \sum_i n_i Y_i$ be an algebraic r-cycle on a k-variety X. Then Z is said to be rationally equivalent to 0 if there exists a finite number of $(r+1)$-dimensional irreducible subvarieties W_j and rational functions $r_j \in k[W_j]^*$ such that*

$$Z = \sum_j [div(r_j)].$$

(Recall that

$$[div(r_j)] = \sum_{V \subset W_j, codim 1} (ord_V(a_j) - ord_V(b_j))[V]$$

where $r_j = a_j/b_j$ with $a_j, b_j \in \mathcal{O}_{W_j,V}$.)

Two algebraic r-cycles Z, Z' on X are said to be rationally equivalent if $Z - Z'$ is rationally equivalent to 0. The abelian group of rational equivalence classes of algebraic r-cycles on X is denoted $A_r(X)$ and called the Chow group of r-cycles. If X has pure dimension d, we also denote this group by $A^{d-r}(X)$, the Chow group of codimension $d-r$-cycles on X.

The first connection between Chow groups and algebraic K-theory occurred at the birth of K-theory is Alexander Grothendieck's proof of the Riemann-Roch Theorem. An important consequence of that theorem is the following result of Grothendieck.

Theorem: Chern character. *If X is a smooth k-variety of dimension d, then there exists a natural isomorphism*

$$ch : K_0(X) \otimes \mathbf{Q} \to A^*(X) \otimes \mathbf{Q}$$

where $A^*(X) = \oplus A^i(X)$. Moreover,

$$K_0(X)^{(i)}_{\mathbf{Q}} = A^i(X) \otimes \mathbf{Q}.$$

At this point, it would be natural to discuss Spencer Bloch's higher Chow groups which give an extension of the preceding theorem to higher K-groups. Since Prof. Weibel has assumed responsibility for the exposition of this theory, I will resist my temptation to give a discussion of higher Chow groups.

Codimension 1 cycles are invariably easier to study than general r-cycles. Before we state Beilinson's conjectures, let us briefly consider this special case.

Definition V.2. *Let X be a k-variety. The Picard group is the group of isomorphism classes of locally free, rank 1 coherent sheaves on X.*

Now, if X is smooth, then such "line bundles" naturally correspond to codimension 1 cycles. (One associates to the line bundle the codimension 1 cycle given locally by the divisor of a non-zero section.) It is well known that

$$Pic(X) = H^1_{Zar}(X, \mathcal{O}_X^*).$$

In modern terminology, $O_X^* = \mathbf{Z}(1)[1]$, where $\mathbf{Z}(1)$ is a complex of sheaves (a complex in this case of length 1) and the shift sending F to $F[n]$ is defined so that $H^i(X, F[n]) = H^{i+n}(X, F)$. Beilinson proposed that we seek complexes of sheaves (in the Zariski topology) $\mathbf{Z}(n)$ which should provide much of the known behaviour of algebraic K-theory.

Here are Beilinson's conjectures. Beilinson seeks complexes of sheaves $\mathbf{Z}(i)$ whose cohomology has good properties. I should explain what we mean by the cohomology of a complex

$$\cdots \to C^n \to C^{n+1} \to \cdots$$

of sheaves. What we do is find a complex of sheaves, each of which is injective,

$$\cdots \to I^n \to I^{n+1} \to \cdots$$

which is "quasi-isomorphic" to our original complex (i.e., has the same cohomology sheaves) and then one takes the cohomology of the complex (of abelian groups) obtained by taking global sections:

$$H^*(X, C^\bullet) \equiv H^*(\Gamma(X, I^\bullet)).$$

Beilinson's Conjectures. *For each $n \geq 0$ there should be a complex of sheaves on the site $(Sm/k, Zar)$ whose objects are smooth schemes of finite type over a given field k and whose coverings are Zariski open coverings. These complexes of sheaves should satisfy the following hypotheses:*

(a.) $\mathbf{Z}(0) = \mathbf{Z}$, $\quad \mathbf{Z}(1) \simeq \mathcal{O}^*[-1]$.

(b.) $H^n(Speck, \mathbf{Z}(n)) = K_n^{Milnor}(k)$.

(c.) $H^{2n}(X, \mathbf{Z}(n)) = A^n(X)$ *whenever* X *is smooth over* k.
(d) Vanishing Conjecture: $\mathbf{Z}(n)$ *is acyclic outside of* $[0, n]$.
(e.) Motivic spectral sequences for X *smooth over* k*:*

$$E_2^{p,q} = H^{p-q}(X, \mathbf{Z}(-q)) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^{p-q}(X, \mathbf{Z}/\ell(-q)) \Rightarrow K_{-p-q}(X, \mathbf{Z}/\ell), \quad if\ 1/\ell \in k.$$

(f.) Beilinson-Lichtenbaum Conjecture:

$$\mathbf{Z}(n) \otimes^L \mathbf{Z}/\ell \simeq \tau_{\leq n} \mathbf{R}\pi_* \mu_\ell^{\otimes n}, \quad if\ 1/\ell \in k.$$

(g.) $H^i(X, \mathbf{Z}(n)) \otimes \mathbf{Q} \simeq K_{2n-i}(X)_{\mathbf{Q}}^{(n)}$.

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (part (e.)) using "motivic complexes" $\mathbf{Z}(n)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. By the way, I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X, \mathbf{Z}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\mathbf{Z}(-q/2) = 0$ if q is not an even non-positive integer and $\mathbf{Z}(-q/2) = \mathbf{Z}(i)$ is $-q = 2i \geq 0$.

Part (a.) just asserts that the first motivic complexes should be what they must be. Part (b.) asserts that for a field k, the n-th cohomology of $\mathbf{Z}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory, the subject of much of one of Prof. Weibel's later lectures. In view of the (integral) spectral sequence of part (e.), part (c.) refines Grothendieck's Theorem by asserting that those terms which contribute to $K_0(X)$ are exactly the Chow groups of X.

The (integral) spectral sequence of part (e.) should "collapse" when tensored with $\mathbf{Q}$, in the sense that there should be no differentials after tensoring with $\mathbf{Q}$. This is the content of part (g.), which refines this statement of "collapsing" by asserting that the contributions from the various terms in the spectral sequence tensor $\mathbf{Q}$ can be identified as pieces of the gamma filtration on $K_*(X)_{\mathbf{Q}}$. Part (d.) incorporates of the Soulé-Beilinson Conjecture. It says that the complex $\mathbf{Z}(n)$ has no sheaf cohomology below degree 0 and above degree n, which tells us that $H^i(X, \mathbf{Z}(n))$ vanishes for $i < 0$ or i greater than n plus the cohomological dimension of X.

The most subtle part of this conjecture, and the part that connects with the Quillen-Lichtenbaum conjecture of the previous lecture, is part (f.). This asserts that if we consider the motivic complexes modulo ℓ, then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes n}$ coefficients, where μ_ℓ is the etale sheaf of ℓ-th roots of unity (isomorphic to $\mathbf{Z}/\ell$ if all ℓ-th roots of unity are in k. If the terms in the modulo ℓ spectral sequence were simply etale cohomology, then we would get etale K-theory which would violate the vanishing of part (d.) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo n should be the cohomology of complexes which involve a truncation.

More precisely, we consider

$$\pi : \text{etale site} \to \text{Zariski site}$$

which is the "continuous map" arising from the fact that every Zariski open inclusion is an etale map. Then $\mathbf{R}\pi_* F$ is a complex of sheaves for the Zariski topology (given by applying π_* to an injective resolution $F \to I^\bullet$) with the property that $H^*_{Zar}(X, \mathbf{R}\pi_* F) = H^*_{et}(X, F)$. Now, the n-th truncation of $\mathbf{R}\pi_* F$, $\tau_{\leq n}\mathbf{R}\pi_* F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $\mathbf{R}\pi_* F$ in degrees $\leq n$ and are 0 in degrees greater than n. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Speck$, then $H^p(Speck, \tau_{\leq n}\mathbf{R}\pi_*\mu_\ell^{\otimes n})$ equals $H^p_{et}(Speck, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

My last topic is a brief motivation for recent work of my colleagues Andrei Suslin and Vladimir Voevodsky. The result of their work is the construction of complexes of sheaves which appear to satisfy Beilinson's conjectures in that we already know that they satisfy many of the expected properties. (This work is discussed in some detail in my June 1997 Bourbaki lecture, as well in other lectures in this workshop.) Moreover, these complexes and related constructions have led Voevodsky to make great progress toward proving the Quillen-Lichtenbaum conjectures. I believe this will be a focus of Prof. Weibel's last lecture.

One motivation for the construction of the complexes of Suslin-Voevodsky is the following famous theorem of Albrecht Dold and René Thom.

Dold-Thom Theorem. *Let X be a C.W. complex and let $SP^d(X)$ denote the d-th symmetric power of X, $SP^d(X) \equiv X^d/\Sigma_d$. Then $\coprod_{d>0} SP^d(X)$ is a topological abelian monoid whose group completion*

$$\mathbf{Z}[X] = \{\coprod_{d\geq 0} SP^d(X)\}^+$$

is a topological abelian group whose homotopy groups are the homology of X,

$$\pi_i(\mathbf{Z}[X], 0) = H_i(X).$$

Andrei Suslin had the idea of using this theorem to define "algebraic homology". To explain this, we first need to define the algebraic singular complex of a variety. Using a technique which was first introduced by Karoubi-Villamayor, we define the "standard algebraic m-simples" $\Delta[m]$ over a field k to be the affine variety

$$\Delta[m] \equiv Speck[x_0, \dots, x_m]/\sum_i x_i - 1.$$

Then, we have standard "skip" and "repeat" morphisms

$$\partial_i : \Delta[m-1] \to \Delta[m], \quad \sigma_j : \Delta[m+1] \to \Delta[m]$$

which enable us to imitate singular complexes of algebraic topology.

Definition V.3. *If Y is a variety, then $Sing^{alg}_{.}(Y)$ is the simplicial set whose set of m-simplices consists of all morphisms (of varieties) $\Delta[m] \to Y$.*

We define the Suslin complex $Sus_*(X)$ of a variety to be the following simplicial abelian group (or its associated chain complex)

$$Sing_*(X) = \{\coprod_{d \geq 0} Sing^{alg}_{.}(SP^d(X)\}^+.$$

The relevance of the Suslin complex $Sus_*(X)$ to the "motivic complexes" $\mathbf{Z}(i)$ is that the construction is similar. More importantly, Suslin and Voevodsky proved the following beautiful theorem using techniques which then evolved to enable them to show many good properties of their complexes.

Suslin-Voevodsky Theorem. *If X is a complex algebraic variety, then*

$$\pi_i(Sus_*(X), \mathbf{Z}/\ell) \simeq H_i(X, \mathbf{Z}/\ell)$$

where the homology on the right is singular homology of X (with its analytic topology).

Thus, the Suslin-Voevodsky theorem tells us that the Suslin complex can achieve one of the most important accomplishments of etale cohomology: an algebraic representation of singular homology with finite coefficients.