Motivic Complexes of Suslin and Voevodsky

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0   Introduction: Connections with K-theory

In this report we sketch some of the insights and consequences of recent work by Andrei Suslin and Vladimir Voevodsky concerning algebraic K-theory and motivic cohomology. We can trace these developments to a lecture at Luminy by Suslin in 1987 and to Voevodsky's Harvard thesis in 1992. What results is a powerful general theory of sheaves with transfers on schemes over a field, a theory developed primarily by Voevodsky with impressive applications by Suslin and Voevodsky.

Criteria for a good motivic cohomology theory originate in topology. This should be a theory which plays some of the same role in algebraic geometry as singular cohomology plays in algebraic topology. One important aspect of singular cohomology is its relationship to (complex, topological) K-theory as formalized by the Atiyah-Hirzebruch spectral sequence for a topological space T [1]

$$E_2^{p,q} = H^p(X, K_{top}^q) \Rightarrow K_{top}^{p+q}(T)$$

where K_{top}^q is the qth coefficient of the generalized cohomology theory given by topological K-theory (equal to $\mathbb{Z}$ if $q \leq 0$ is even and 0 otherwise). Indeed, when tensored with the rational numbers, this spectral sequences collapses to give $K_{top}^n(T) \otimes \mathbb{Q} = \bigoplus_{p+q=n, p \geq 0, q \leq 0} H^p(T, K_{top}^q) \otimes \mathbb{Q}$. This direct sum decomposition can be defined intrinsically in terms of the weight spaces of Adams operations acting upon $K_n^{top}(T)$. This becomes particularly suggestive when compared to the well known results of Alexander Grothendieck [17] concerning algebraic K_0 of a smooth scheme X:

$$K_0(X) \otimes \mathbb{Q} = \bigoplus CH^d(X) \otimes \mathbb{Q}$$

where $CH^d(X)$ is the Chow group of codimension d cycles on X modulo rational equivalence; moreover, this decomposition is once again given in terms of weight spaces for Adams operations.

Working now in the context of schemes (typically of finite type over a field k), William Dwyer and Friedlander [8] developed a topological K-theory for schemes (called etale K-theory) which also has such an Atiyah-Hirzebruch spectral sequence with E_2-term the etale cohomology of the scheme. In [5], Spencer Bloch introduced complexes $Z_*^d(X)$ for X quasi-projective over a field which consist of certain algebraic cycles of codimension d on the product of X and affine spaces of varying dimensions. The homology of $Z_*^d(X)$ is closely related to the (higher Quillen) algebraic K-theory of X. If $CH^d(X,n)$ denotes the $n-th$ homology group of the Bloch complex $Z_*^d(X)$ and if X is a smooth scheme, then

$$K_n(X) \otimes \mathbb{Q} = \bigoplus_d CH^d(X,n) \otimes \mathbb{Q}$$

(see also [20]); this decomposition is presumably given in terms of weight spaces for Adams operations on K-theory. Together with Stephen Lichtenbaum, Bloch has moreover established a spectral sequence [6] converging to algebraic K-theory in the special case that X is the spectrum of a field F

$$E_2^{p,q} = CH^{-q}(SpecF, -p - q) \Rightarrow K_{-p-q}(SpecF).$$

As anticipated many years ago by Alexander Beilinson [2], there should be such a spectral sequence for a quite general smooth scheme

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

converging to algebraic K-theory whose E_2-term is *motivic cohomology*. Moreover, Beilinson [4] and Lichtenbaum [22] anticipated that this motivic cohomology should be the cohomology of *motivic chain complexes*. Although such a spectral sequence still eludes us (except in the case of the spectrum of a field), the complexes $\mathbb{Z}(n)$ of Voevodsky and Suslin (see §4) satisfy so many of the properties required of motivic complexes that we feel comfortable in calling their cohomology motivic cohomology. The first sections of this exposition are dedicated to presenting some of the formalism which leads to such a conclusion. As we see in §5, a theorem of Suslin [27] and duality established by Friedlander and Voevodsky [14] imply that Bloch's higher Chow groups $CH^d(X,n)$ equal motivic cohomology groups of Suslin-Voevodsky for smooth schemes X over a field k "which admits resolution of singularities."

The Beilinson-Lichtenbaum Conjecture (cf [2], [3] [18]) predicts that the conjectural map of spectral sequences from the conjectured spectral sequence converging to algebraic K-theory mod-ℓ to the Atiyah-Hirzebruch spectral sequence converging to etale K-theory mod-ℓ should be an isomorphism on E_2-terms (except for a fringe effect whose extent depends upon the mod-ℓ etale cohomological dimension of X)) for smooth schemes over

a field k in which ℓ is invertible. This would reduce the computation of mod-ℓ K-theory of many smooth schemes to a question of computing "topological invariants" which in many cases has a known solution. In §6, we sketch the proof by Suslin and Voevodsky that the "Bloch-Kato Conjecture" for a field k and a prime ℓ invertible in k implies this Beilinson-Lichtenbaum Conjecture for k and ℓ. As discussed in the seminar by Bruno Kahn, Voevodsky has proved the Bloch-Kato Conjecture for $\ell = 2$ (in which case it was previously conjectured by John Milnor and thus is called the Milnor Conjecture.) Recent work by B. Kahn and separately by Charles Weibel and John Rognes establishes that computations of the 2-primary part of algebraic K-theory for rings of integers in number fields can be derived using special arguments directly from the Beilinson-Lichtenbaum Conjecture and the Bloch-Lichtenbaum spectral sequence.

1 Algebraic Singular Complexes

The elementarily defined *Suslin complexes* $Sus_*(X)$ provide a good introduction to many of the fundamental structures underlying the general theory developed by Voevodsky. Moreover, the relationship between the mod-n cohomology of $Sus_*(X)$ and the etale cohomology mod-n of X stated in Theorem 1.1 suggests the close relationship between etale motivic cohomology mod-ℓ and etale cohomology mod-ℓ.

As motivation, we recall from algebraic topology the following well known theorem of A. Dold and R. Thom [7]. If T is a reasonable topological space (e.g., a C.W. complex) and if $SP^d(T)$ denotes the d-fold symmetric product of T, then the homotopy groups of the group completion $(\coprod_d Sing.(SP^d(T)))^+$ of the simplicial abelian monoid $\coprod_d Sing.(SP^d(T))$ are naturally isomorphic to the (singular) homology of T. Here, $Sing.(SP^d(T))$ is the (topological) singular complex of the space $SP^d(T)$, whose set of n-simplices is the set of continuous maps from the topological n-simplex $\Delta[n]$ to $SP^d(T)$.

Suppose now that X is a scheme of finite type over a field k; each $SP^d(X)$ is similarly a scheme of finite type over k. Let Δ^n denote $Spec\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$ and let Δ^* denote the evident cosimplicial scheme over k which in codimension n is Δ^n. We define the Suslin complex $Sus_*(X)$ of X to be the chain complex associated to the simplicial abelian group $(\coprod_d Hom_{Sch/k}(\Delta^*, SP^d(X)))^+$.

Various aspects of $Sus_*(X)$ play an important role in our context. First, $Sus_*(X)$ equals $c_{equi}(X, 0)(\Delta^*)$, where $c_{equi}(X, 0)$ is a sheaf in the *Nisnevich topology* on the category Sm/k of smooth schemes over the field k. Second, the sheaf $c_{equi}(X, 0)$ is a *presheaf with transfers*. Third, if we denote by $\underline{C}_*(c_{equi}(X, 0))$ the complex of Nisnevich sheaves with transfers (sending a smooth scheme U to $c_{equi}(X, 0)(U \times \Delta^*)$), then this complex of sheaves

has homology presheaves which are *homotopy invariant*: the natural pull-back

$$c_{equi}(X,0)(U \times \Delta^*) \to c_{equi}(X,0)(U \times \mathbb{A}^1 \times \Delta^*)$$

induces an isomorphism on homology groups.

Theorem 1.1. *([28]). Let X be a quasi-projective scheme over an algebraically closed field k and let n be a positive integer relatively prime to the exponential characteristic of k. Then the mod-n cohomology of $Sus_*(X)$ (i.e., the cohomology of the complex $\mathbf{R}Hom(Sus_*(X), \mathbb{Z}/n))$ is given by*

$$H^*(Sus_*(X), \mathbb{Z}/n) \simeq H^*_{et}(X, \mathbb{Z}/n),$$

where the right hand side is the etale cohomology of the scheme X with coefficients in the constant sheaf $\mathbb{Z}/n$.

Quick sketch of proof. This theorem is proved using the rigidity theorem of Suslin and Voevodsky stated below as Theorem 2.5. We apply this to the (graded) homotopy invariant (cf. Lemma 2.4) presheaves with transfers

$$\Phi_i(-) = H_i(c_{equi}(X,0)(- \times \Delta^*)) \otimes \mathbb{Z}[1/p])$$

where p is the exponential characteristic of k. An auxiliary topology, the "qfh topology" is introduced which has the property that the free $\mathbb{Z}[1/p]$ sheaf in this topology represented by X equals $c_{equi}(X,0) \otimes \mathbb{Z}[1/p]$. Since $Sus_*(X) = \Phi_*(Spec k)$, Theorem 2.5 and the comparison of cohomology in the qfh and etale topologies provides the following string of natural isomorphisms:

$$Ext^*_{Ab}(Sus_*(X), \mathbb{Z}/n) = Ext^*_{EtShv/X}((\Phi_*)_{et}, \mathbb{Z}/n) = Ext^*_{qfhShv/X}((\Phi_*)_{qfh}, \mathbb{Z}/n)$$

$$Ext^*_{qfhShv(k)}(\mathbb{Z}[1/p](X), \mathbb{Z}/n) = H^*_{qfh}(X, \mathbb{Z}/n) = H^*_{et}(X, \mathbb{Z}/n).$$

$\square$

These concepts of presheaves with transfers, Nisnevich sheaves, and homotopy invariant presheaves will be explained in the next section. Even before we investigate their definitions, we can appreciate their role from the following theorem of Voevodsky.

Theorem 1.2. *[31, 5.12] Assume that k is a perfect field. Let*

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

be a short exact sequence of Nisnevich sheaves on Sm/k with transfers. Then the resulting triple of chain complexes of abelian groups

$$F_1(\Delta^*) \to F_2(\Delta^*) \to F_3(\Delta^*) \to F_1(\Delta^*)[1]$$

is a distinguished triangle (i.e., determines a long exact sequence in homology groups).

Quick sketch of proof. Let P denote the presheaf cokernel of $F_1 \to F_2$. Then the kernel and cokernel of the natural map $P \to F_3$ have vanishing associated Nisnevich sheaves. The theorem follows from an acyclicity criterion for $Q(\Delta^*)$ in terms of the vanishing of $Ext^*(Q_{Nis}, -)$ for any presheaf with transfers Q on Sm/k (with associated Nisnevich sheaf Q_{Nis}). A closely related acyclicity theorem is stated as Theorem 3.5 below. $\square$

One consequence of Theorem 1.2 (and Proposition 2.2 below) is the following useful property. The resulting long exact sequence in Suslin homology is far from evident if one works directly with the definition of the Suslin complex.

Corollary 1.3. *[31, 5.17] Let k be a perfect field and X a scheme of finite type over k. Then for any open covering $X = U \cup V$ of X*

$$Sus_*(U \cap V) \to Sus_*(U) \oplus Sus_*(V) \to Sus_*(X) \to Sus_*(U \cap V)[1]$$

is a distinguished triangle.

2 Nisnevich sheaves with Transfers

Let Sm/k denote the category of smooth schemes over a field k. (In particular, such a scheme is of finite type over k.) Then the *Nisnevich topology* on Sm/k (cf. [23]) is the Grothendieck topology (finer than the Zariski topology and less fine than the etale topology) whose coverings $\{U_i \to U\}_{i \in I}$ are etale coverings with the property that for each point $u \in U$ there exists some $i \in I$ and some point $\tilde{u} \in U_i$ mapping to u such that the induced map of residue fields $k(u) \to k(\tilde{u})$ is an isomorphism. A key property of this topology is that its points are Hensel local rings.

In order to consider singular schemes which admit resolutions by smooth schemes, we shall also consider the stronger *cdh topology* on the category Sch/k of schemes of finite type over k. This is defined to be the minimal Grothendieck topology for which Nisnevich coverings are coverings as are proper, surjective morphisms of the following type:

$$W \coprod U_1 \xrightarrow{p \coprod i} U$$

where $i : U_1 \to U$ is a closed embedding and $p^{-1}(U - U_1) \to U - U_1$ is an isomorphism.

We shall often have need to assume that the field "admits resolution of singularities" as formulated in the following definition. At this time, this hypothesis is only known to hold for fields of characteristic 0. As one can see, the cdh topology is designed to permit the study of singular schemes over a field which admits resolution of singularities by employing coverings by smooth schemes.

Definition 2.1. *A field k is said to admit resolution of singularities provided that*

1. *For any scheme of finite type X over k there is a proper, birational, surjective morphism $Y \to X$ such that Y is a smooth scheme over k.*

2. *For any smooth scheme X over k and any proper, birational, surjective map $q : X' \to X$, there exists a sequence of blow-ups $p : X_n \to \cdots \to X_1 = X$ with smooth centers such that p factors through q.*

We define the presheaf of abelian groups

$$c_{equi}(X, 0) : (Sm/k)^{op} \to Ab$$

to be the evident functor whose values on a smooth connected scheme U is the free abelian group on the set of integral closed subschemes on $X \times U$ finite and surjective over U. This is a sheaf for the etale topology and hence also for the Nisnevich topology; indeed, as mentioned following the statement of Theorem 1.1, $c_{equi}(X, 0)$ can be constructed as the sheaf in the qfh-topology (stronger than the etale topology) associated to the presheaf sending U to the free abelian group on $Hom_{Sch/k}(U, X)$.

We shall have occasion to consider other Nisnevich sheaves defined as follows:

$$z_{equi}(X, r) : (Sm/k)^{op} \to Ab$$

sends a connected smooth scheme U to the group of cycles on $U \times X$ equidimensional of relative dimension r over U. In particular, if X is proper over k, then $c_{equi}(X, 0) = z_{equi}(X, 0)$.

One major advantage of our Nisnevich and cdh topologies when compared to the Zariski topology is the existence of Mayer-Vietoris, localization, and blow-up exact sequences as stated below.

Proposition 2.2. *(cf. [29, 4.3.7;4.3.1;4.3.2]) For any smooth scheme X over k and any Zariski open covering $X = U \cup V$, the sequence of sheaves in the Nisnevich topology*

$$0 \to c_{equi}(U \cap V, 0) \to c_{equi}(U, 0) \oplus c_{equi}(V, 0) \to c_{equi}(X, 0) \to 0$$

of Mayer-Vietoris type is exact.

For any scheme X of finite type over k, any open covering $X = U \cup V$, and any closed scheme $Y \subset X$, the sequences of sheaves in the cdh topology

$$0 \to c_{equi}(U \cap V, 0)_{cdh} \to c_{equi}(U, 0)_{cdh} \oplus c_{equi}(V, 0)_{cdh} \to c_{equi}(X, 0)_{cdh} \to 0$$

$$0 \to z_{equi}(Y, r)_{cdh} \to z_{equi}(X, r)_{cdh} \to z_{equi}(X - Y, r)_{cdh} \to 0$$

of Mayer-Vietoris and localization type are exact.

For any scheme X of finite type over k, any closed subscheme $Z \subset X$, and any proper morphism $f : X' \to X$ whose restriction $f^{-1}(X - Z) \to X - Z$ is an isomorphism, the sequences of sheaves in the cdh topology

$$0 \to c_{equi}(f^{-1}(Z), 0)_{cdh} \to c_{equi}(X', 0)_{cdh} \oplus c_{equi}(Z, 0)_{cdh} \to c_{equi}(X, 0)_{cdh} \to 0$$

$$0 \to z_{equi}(f^{-1}(Z)), r)_{cdh} \to z_{equi}(X', r)_{cdh} \oplus z_{equi}(Z, r))_{cdh} \to z_{equi}(X, r))_{cdh} \to 0$$

of blow-up type are exact.

Remarks on the proof. The only issue is exactness on the right. We motivate the proof of the exactness of the localization short exact sequences using Chow varieties, assuming that X is quasi-projective. Let W be a smooth connected scheme and $Z \subset (X - Y) \times W$ a closed integral subscheme of relative dimension r over W. Such a Z is associated to a rationally defined map from W to the Chow variety of some projective closure of X. The projection to W of the graph of this rational map determines a cdh-covering $W' \to W$ restricted to which the pull-back of Z on $(X - Y) \times W'$ extends to a cycle on $X \times W'$ equidimensional of relative dimension r over W'. $\qquad\qquad\square$

We next introduce the important notion of transfers (i.e., functoriality with respect to finite correspondences).

Definition 2.3. *The category of smooth correspondences over k, $SmCor(k)$, is the category whose objects are smooth schemes over k and for which*

$$Hom_{SmCor(k)}(U, X) = c_{equi}(X, 0)(U),$$

the free abelian group of finite correspondences from U to X. A presheaf with transfers is a contravariant functor

$$F : (SmCor(k))^{op} \to Ab.$$

The structure of presheaves with transfers on $c_{equi}(X, 0)$ and $z_{equi}(X, r)$ is exhibited using the observation that if Z is an equidimensional cycle over a smooth scheme X and if $W \to X$ is a morphism of schemes of finite type, then the pull-back of Z to W is well defined since the embedding of the graph of $W \to X$ in $W \times X$ is a locally complete intersection morphism [15]. Consequently, if $U \leftarrow W \to X$ is a finite correspondence in $SmCor(k)$, then we obtain transfer maps by first pulling back cycles of X to W and then pushing them forward to U. The reader should be forewarned that earlier papers of Voevodsky, Suslin, and Friedlander use the condition on a presheaf that it be a "pretheory

of homological type" which is shown in [32, 3.1.10] to be implied by the existence of transfers.

One can easily prove the following lemma which reveals the key property of homotopy invariance possessed by the algebraic singular complex used to define Suslin homology. For any presheaf F on Sm/k, we employ the notation $\underline{C}_*(F)$ for the complex of presheaves on Sm/k sending U to the complex $F(U \times \Delta^*)$.

Lemma 2.4. *Let $F : (Sm/k)^{op} \to Ab$ be a presheaf on Sm/k and consider $\underline{h}^{-i}(F) :$ $(Sm/k)^{op} \to Ab$ sending U to the i-th homology of $\underline{C}_*(F)$ (for some non-negative integer i). Then $\underline{h}^{-i}(F)$ is homotopy invariant:*

$$\underline{h}^{-i}(F)(U) = \underline{h}^{-i}(F)(U \times \mathbb{A}^1).$$

As we saw in our sketch of proof of Theorem 1.1, the following rigidity theorem of Suslin and Voevodsky, extending the original rigidity theorem of Suslin [26] is of considerable importance.

Theorem 2.5. *[28, 4.4] Let Φ be a homotopy invariant presheaf with transfers satisfying $n\Phi = 0$ for some integer n prime to the residue characteristic of k. Let S_d be the henselization of $\mathbb{A}^d$ (i.e., affine d-space) at the origin. Then*

$$\Phi(S_d) = \Phi(Spec k).$$

Idea of Proof. In a now familiar manner, the theorem is reduced to an assertion that any two sections of a smooth relative curve $X \to S$ with good compactification which coincide at the closed point of S induce the same map $\Phi(X) \to \Phi(S)$. The difference Z of these sections is a finite correspondence from S to X. Since Φ is a homotopy invariant presheaf with transfers, to show that the map induced by Z is 0 it suffices to show that the difference is 0 in the relative Picard group $Pic(\overline{X}, Y)/n \subset H^2_{et}(\overline{X}, j_!(\mu_n))$, where $\overline{X} \to S$ is a good compactification, $Y = \overline{X} - X$, and $j : X \subset \overline{X}$. The proper base change theorem implies that it suffices to show that the image of Z is 0 upon base change to the closed point of S. This is indeed the case since the two sections were assumed to coincide on the closed point. $\square$

The following theorem summarizes many of the results proved by Voevodsky in [31] and reformulated in [32]. In particular, this theorem enables us to replace consideration of cohomology in the Nisnevich topology by cohomology in the Zariski topology for smooth schemes.

Theorem 2.6. *[32, 3.1.11] If $F : (SmCor(k))^{op} \to Ab$ is a homotopy invariant presheaf with transfers, then its associated Nisnevich sheaf F_{Nis} is also a homotopy invariant*

presheaf with transfers and equals (as a presheaf on Sm/k) the associated Zariski sheaf F_{Zar}.

Moreover, if k is perfect, then

$$H^i_{Zar}(-, F_{Zar}) = H^i_{Nis}(-, F_{Nis})$$

for any $i \geq 0$, and these are homotopy invariant presheaves with transfer.

To complete the picture relating sheaf cohomology for different topologies we mention the following result which tells us that if we consider the cdh topology on schemes of finite type over k then the resulting cohomology equals Nisnevich cohomology whenever the scheme is smooth.

Proposition 2.7. *[14, 5.5] Assume that k is a perfect field admitting resolution of singularities. Let F be a homotopy invariant presheaf on Sm/k with transfers. Then for any smooth scheme of finite type over k*

$$H^*_{cdh}(X, F_{cdh}) = H^*_{Nis}(X, F_{Nis}) = H^*_{Zar}(X, F_{Zar}).$$

Remark on Proof. The proof uses the techniques employed in the proof of Theorem 3.5 below applied to the cone of $\mathbb{Z}(\mathcal{U}) \to \mathbb{Z}(U)$, where $\mathcal{U}$ is an arbitrarily fine hypercovering of U for the cdh topology consisting of smooth schemes. □

3 Formalism of the Triangulated Category DM_k

Voevodsky's approach [32] to motives for smooth schemes and for schemes of finite type over a field admitting resolution of singularities entails a triangulated category $DM^{eff}_{gm}(k)$ of effective geometric motives. Roughly speaking, $DM^{eff}_{gm}(k)$ is obtained by adjoining kernels and cokernels of projectors to the localization (to impose homotopy invariance) of the homotopy category of bounded complexes on the category of smooth schemes and finite correspondences. Voevodsky then inverts the "Tate object" $\mathbb{Z}(1)$ in this category to obtain his triangulated category $DM_{gm}(k)$ of geometric motives. (See [21] for another approach to the triangulated category of mixed motives by Marc Levine.)

In this section, we focus our attention upon another triangulated category introduced by Voevodsky which we denote by DM_k for notational convenience. (Voevodsky's notation is $DM^{eff}_-(k)$.) Voevodsky proves [32, 3.2.6] that his category $DM^{eff}_{gm}(k)$ of effective geometric motives embeds as a full triangulated subcategory of DM_k. Furthermore, as we see in Theorem 5.7 below, under this embedding the Tate motive is quasi-invertible so that DM_{gm} is also a full triangulated subcategory of DM_k.

Definition 3.1. *Let X be a scheme over over a field k. Assume either that X is smooth or that X is of finite type and k admits resolution of singularities. We define the motive of X to be*

$$M(X) \equiv \underline{C}_*(c_{equi}(X,0)) : (Sm/k)^{op} \to C_*(Ab).$$

Similarly, we define the motive of X with compact supports to be

$$M^c(X) \equiv \underline{C}_*(z_{equi}(X,0)) : (Sm/k)^{op} \to C_*(Ab).$$

We shall use the usual (but confusing) conventions when working with complexes. Our complexes will have cohomological indexing, meaning that the differential increases degree by 1. We view this differential of degree $+1$ as shifting 1 position to the right. If K is a complex, then $K[1]$ is the complex obtained from K by shifting 1 position to the left. This has the convenience when working with (hyper-) cohomology that $H^i(X, K[1]) = H^{i+1}(X, K)$.

We now introduce the triangulated category DM_k designed to capture the Nisnevich cohomology of smooth schemes over k and the cdh cohomology of schemes of finite type over k.

Definition 3.2. *Denote by $Shv_{Nis}(SmCor(k))$ the category of Nisnevich sheaves with transfers and let $D_-(Shv_{Nis}(SmCor(k)))$ denote the derived category of complexes of $Shv_{Nis}(SmCor(k))$ which are bounded above. We define*

$$DM_k \subset D_-(Shv_{Nis}(SmCor(k)))$$

to be the full subcategory of those complexes with homotopy invariant cohomology sheaves.

By Lemma 2.4 and Theorem 2.6, $M(X)$ and $M^c(X)$ are objects of the triangulated category DM_k.

We obtain the following relatively formal consequence of our definitions.

Proposition 3.3. *[32, 3.1.8,3.2.6] If X is smooth over k, then for any $K \in DM_k$,*

$$H_{Zar}^n(X, K) = Hom_{DM_k}(M(X), K[n]);$$

in particular, if X is smooth, then

$$Hom_{DM_k}(M(X), M(Y)[i]) = H_{Zar}^i(X, \underline{C}_*(Y)).$$

If X is of finite type over k and k admits resolution of singularities, then

$$H_{cdh}^n(X, K_{cdh}) = Hom_{DM_k}(M(X), K[n]_{cdh}).$$

Taking $X = Spec\,k$, we obtain an interpretation of $Sus_*(Y)$ in terms of DM_k.

Corollary 3.4. *If Y is a scheme of finite type over k, then the homology of $Sus_*(Y)$ is given by $Hom_{DM_k}(\mathbb{Z}[*], M(Y))$.*

The machinery of presheaves with transfers and the formulation of the cdh topology permits the following useful vanishing theorem. This is an extension of an earlier theorem of Voevodsky asserting the equivalence of the conditions on a homotopy invariant presheaf with transfers that the homology sheaves of $\underline{C}_*(F)_{Zar}$ vanish and that $Ext^*_{NisShv}(F_{Nis}, -) = 0$ [31, 5.9].

Theorem 3.5. *[14, 5.5.2] Assume F is a presheaf with transfers on Sm/k where k is a perfect field which admits resolution of singularities. If $F_{cdh} = 0$, then $\underline{C}_*(F)_{Zar}$ is quasi-isomorphic to 0.*

Idea of Proof. If $\underline{C}_*(F)_{Zar}$ is not quasi-isomorphic to 0, let $\underline{h}_n(F)_{Zar}$ be the first non-vanishing cohomology sheaf. Using Theorem 2.6 and techniques of [31], we conclude that a non-zero element of this group determines a non-zero element of

$$Hom_{D(Sm/k)_{Nis}}(\underline{C}_*(F)_{Nis}, \underline{h}_n(F)_{Nis}[n]) = Ext^n_{NisShv}(F_{Nis}, \underline{h}_n(F)_{Nis}).$$

On the other hand, using a resolution of F by Nisnevich sheaves which are the free abelian sheaves associated to smooth schemes, we verify that the vanishing of F_{cdh} together with [31, 5.9] implies that

$$Ext^*_{NisShv}(F_{Nis}, G_{Nis}) = 0$$

for any homotopy invariant presheaf G with transfers. $\square$

In conjunction with Proposition 2.2, Theorem 3.5 leads to the following distinguished triangles for motives and motives with compact support.

Corollary 3.6. *Assume that the field k admits resolution of singularities and that X is a scheme of finite type over k. If $X = U \cup V$ is a Zariski open covering, then we have the following distinguished triangles of Mayer-Vietoris type*

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$$

$$M^c(X) \to M^c(U) \oplus M^c(V) \to M^c(U \cap V) \to M^c(X)[1].$$

If $Y \subset X$ is a closed subscheme with Zariski open complement U, then we have the following distinguished triangle of localization type

$$M^c(Y) \to M^c(X) \to M^c(U) \to M^c(Y)[1].$$

Finally, if $f : X' \to X$ is a proper morphism and $Z \subset X$ is a closed subscheme such that the restriction of f above $X - Z$, $f_| : X' - f^{-1}(Z) \to X - Z$ is an isomorphism, then we have the following distinguished triangles for abstract blow-ups:

$$M(f^{-1}(Z)) \to M(X') \oplus M(Z) \to M(X) \to M(f^{-1}(Z))[1]$$

$$M^c(f^{-1}(Z)) \to M^c(X') \oplus M^c(Z) \to M^c(X) \to M^c(f^{-1}(Z))[1]$$

Armed with these distinguished triangles, one can obtain results similar to those of Henri Gillet and Christophe Soulé in [16].

We next introduce the *Tate motive* $\mathbb{Z}(1)[2]$ in DM_k and define the *Tate twist* of motives.

Definition 3.7. *We define the Tate motive $\mathbb{Z}(1)[2]$ to be the cone of $M(Spec k) \to M(\mathbb{P}^1)$. We define the Tate twist by*

$$M(X)(1) = cone\{M(X) \to M(X \times \mathbb{P}^1)[-2]\},$$

$$M^c(X)(1) = cone\{M^c(X) \to M^c(X \times \mathbb{P}^1)[-2]\}.$$

Thus, if X is projective and k admits resolution of singularities,

$$M(X)(1) = M^c(X \times \mathbb{A}^1)[-2].$$

We briefly introduce the analogous triangulated category for the etale site.

Definition 3.8. *Denote by $Shv_{et}(SmCor(k))$ the category of presheaves with transfers which are sheaves on the etale site of (Sm/k) and let $D_-(Shv_{et}(SmCor(k)))$ denote the derived category of complexes of $Shv_{et}(SmCor(k))$ which are bounded above. We define*

$$DM_{k,et} \subset D_-(Shv_{et}(SmCor(k)))$$

to be the full subcategory of those complexes with homotopy invariant cohomology sheaves.

Observe that the exact functor $\pi^* : Shv_{Nis}(SmCor(k)) \to Shv_{et}(SmCor(k))$ induces a natural map

$$Hom_{DM_k}(K, L) \to Hom_{DM_{k,et}}(\pi^*K, \pi^*L).$$

Voevodsky observes that

$$Hom_{DM_{k,et}}(M(X), K[n]) = H^n_{et}(X, K)$$

for any $K \in DM_{k,et}$.

4   Motivic Cohomology and Homology

Having introduced the triangulated category DM_k, we now proceed to consider the motivic complexes $\mathbb{Z}(n) \in DM_k$ whose cohomology and homology is motivic cohomology and homology. Other authors (e.g., Lichtenbaum and Friedlander-Gabber) have considered similar complexes; the importance of the approach of Suslin and Voevodsky is the context in which these complexes are considered. The many properties established for DM_k enable many good formal properties to be proved.

Definition 4.1. *For a given positive integer n, let F_n be the sum of the images of the n embeddings*

$$c_{equi}((\mathbb{A}^1 - \{0\})^{n-1}, 0) \to c_{equi}((\mathbb{A}^1 - \{0\})^n, 0)$$

determined by the embeddings $(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_{n-1})$. We define

$$\mathbb{Z}(n) = \underline{C}_*(c_{equi}((\mathbb{A}^1 - \{0\})^n, 0)/F_n))[-n].$$

For any positive integer m, we define

$$\mathbb{Z}/m(n) = \underline{C}_*(c_{equi}((\mathbb{A}^1 - \{0\})^n, 0)/F_n) \otimes \mathbb{Z}/m)[-n].$$

Observe that Mayer-Vietoris implies that $\mathbb{Z}(1)$ defined as in Definition 4.1 agrees with (i.e., is quasi-isomorphic to) $\mathbb{Z}(1)$ as given in Definition 3.7; similarly, for any $n > 0$,

$$\mathbb{Z}(n) = \underline{C}_*(c_{equi}(\mathbb{P}^n, 0)/c_{equi}(\mathbb{P}^{n-1}))[-2n].$$

Moreover, if k admits resolution of singularities, then localization implies that

$$\mathbb{Z}(n) = \underline{C}_*(z_{equi}(\mathbb{A}^n, 0))[-2n].$$

We obtain the following determination of $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ which we would require of any proposed definition of motivic complexes.

Proposition 4.2. *[32, 3.4.3]*

(a.) $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$.

(b.) $\mathbf{G}_m \simeq \mathbb{Z}(1)[1]$, where $\mathbf{G}_m$ is viewed as a sheaf of abelian groups.

We now introduce motivic cohomology.

Definition 4.3. *For any scheme of finite type over a field k, we define the motivic cohomology of X by*

$$H^i(X, \mathbb{Z}(j)) = H^i_{cdh}(X, \mathbb{Z}(j)_{cdh}).$$

For any positive integer m, we define the mod-m motivic cohomology of X by

$$H^i(X, \mathbb{Z}/m(j)) = H^i_{cdh}(X, \mathbb{Z}/m(j)_{cdh}).$$

Thus, if X is smooth and k is perfect, then Theorem 2.6 and Proposition 3.3 imply that motivic cohomology is Zariski hypercohomology (where the complex $\mathbb{Z}(j)$ of Nisnevich sheaves is viewed as a complex of Zariski sheaves by restriction):

$$H^i(X, \mathbb{Z}(j)) = H^i_{Zar}(X, \mathbb{Z}(j)) = Hom_{DM_k}(M(X), \mathbb{Z}(j)[i]).$$

Similarly, if k admits resolution of singularities, then for any X of finite type over k

$$H^i(X, \mathbb{Z}(j)) = Hom_{DM_k}(M(X), \mathbb{Z}(j)[i]).$$

If d denotes the dimension of X, then

$$H^i(X, \mathbb{Z}(j)) = 0 \quad \text{whenever } i > d + j.$$

The following theorem relating Milnor K-theory to motivic cohomology appears in various guises in [5] and [24]. The reader is referred to [30] for a direct proof given in our present context.

Theorem 4.4. *For any field k and any non-negative integer n, there is a natural isomorphism*

$$K^M_n(k) \simeq H^n(Spec k, \mathbb{Z}(n))$$

where $K^M_*(k)$ is the Milnor K-theory of k.

So defined, motivic cohomology is cohomology with respect to the Zariski site for smooth schemes (and with respect to the cdh site for more general schemes of finite type) as anticipated by Beilinson. One can also consider the analogous cohomology with respect to the etale site following the lead of Lichtenbaum.

As usual, we let μ_ℓ denote the sheaf of ℓ-th roots of unity on $(Sm/k)_{et}$.

Theorem 4.5. *[32, 3.3] Define the etale motivic cohomology $H^i_{et}(X, \mathbb{Z}(j))$ of a scheme X of finite type over k by*

$$H^i_{et}(X, \mathbb{Z}(j)_{et}) = Hom_{DM_{k,et}}(M(X)_{et}, \mathbb{Z}(j)_{et}[i]);$$

similarly for any positive integer relatively prime to the residue characteristic of k, define

$$H^i_{et}(X, \mathbb{Z}/m(j)) \equiv H^i_{et}(X, \mathbb{Z}/m(j)_{et}) = Hom_{DM_{k,et}}(M(X)_{et}, \mathbb{Z}/m(j)_{et}[i]).$$

Then there is a natural quasi-isomorphism

$$\mu_m^{\otimes j} \to \mathbb{Z}/m(j)_{et}$$

In particular, this gives an isomorphism

$$H^*_{et}(X, \mu_m^{\otimes j}) \xrightarrow{\cong} H^*_{et}(X, \mathbb{Z}/m(j)).$$

Sketch of proof. By Proposition 4.2.b, μ_m is quasi-isomorphic to $\mathbb{Z}/m(1)$. More generally, we construct an explicit map $\mu_m^{\otimes j}(F(\zeta_m)) \to \mathbb{Z}/m(j)(F(\zeta_m))$ where F is a field extension of k and ζ_m is a primitive m-th root of unity and show that this map is $Gal(F(\zeta_m)/F)$-invariant. This determines a map of etale sheaves with transfers $\mu_m^{\otimes j} \to \mathbb{Z}/m(j)_{et}$. By Theorems 1.1 and 2.5, this map is a quasi-isomorphism. $\qquad\square$

Because the etale cohomology of a Hensel local ring is torsion, we readily conclude the following proposition using Proposition 2.7.

Proposition 4.6. *For any smooth scheme,*

$$H^*(X, \mathbb{Z}(j)) \otimes \mathbb{Q} = H^*_{et}(X, \mathbb{Z}(j)) \otimes \mathbb{Q}.$$

As we shall see in the next section, motivic cohomology is dual to motivic locally compact homology for smooth schemes over a field admitting resolution of singularities. This locally compact homology was initially formulated in [14] (essentially following the definition in [10]) using $\underline{C}_*(z_{equi}(X, r))$. To rephrase this in terms of our triangulated category DM_k, we need the following proposition.

Proposition 4.7. *[32, 4.2.8] Let X be a scheme of finite type over a field k and let r be a non-negative integer. Then there is a natural isomorphism in DM_k*

$$\underline{C}_*(z_{equi}(X, r)) \simeq \underline{Hom}_{DM_k}(\mathbb{Z}(r)[2r], M^c(X))$$

where $\underline{Hom}$ denotes internal Hom in the derived category of unbounded complexes of Nisnevich sheaves with transfers.

We now define three other theories: motivic cohomology with compact supports, motivic homology, and motivic homology with locally compact supports. We leave implicit the formulation of these theories with mod-m coefficients.

Definition 4.8. *Let X be a scheme of finite type over a field k which admits resolution of singularities. Then we define*

$$H_c^i(X, \mathbb{Z}(j)) = Hom_{DM_k}(M^c(X), \mathbb{Z}(j)[i])$$

$$H_i^{lc}(X, \mathbb{Z}(j)) = Hom_{DM_k}(\mathbb{Z}(j)[i], M^c(X))$$

$$H_i(X, \mathbb{Z}(j)) = Hom_{DM_k}(\mathbb{Z}(j)[i], M(X)).$$

Since $c_{equi}(-, 0)$ is covariantly functorial (using push-forward of cycles), we conclude that $H^*(X, \mathbb{Z}(j))$ is contravariantly functorial and $H_*(X, \mathbb{Z}(j))$ is covariantly functorial for morphisms of schemes of finite type over k. Similarly, the functoriality of $z_{equi}(-, 0)$ implies that $H_c^*(X, \mathbb{Z}(j)$ (respectively,$H_*^{lc}(X, \mathbb{Z}(j)$ is contravariant (resp. covariant) for proper maps and covariant (resp. contravariant) for flat maps.

We recall the bivariant theory introduced in [14], which is closely related to a construction in [10] and which is an algebraic version of the bivariant morphic cohomology introduced by Friedlander and Lawson in [11]:

$$A_{r,i}(Y, X) \equiv H_{cdh}^{-i}(Y, \underline{C}_*(z_{equi}(X, r))_{cdh}).$$

This bivariant theory is used in §5 when considering the duality relationship between motivic cohomologies and homologies.

We conclude this section with a proposition, proved by Voevodsky, which interprets this bivariant theory in the context of the triangulated category DM_k and the Tate twist of Definition 3.7.

Proposition 4.9. *[32, 4.2.3] Let k be a field admitting resolution of singularities and X, Y schemes of finite type over k. There is a natural isomorphism*

$$A_{r,i}(Y, X) = Hom_{DM_k}(M(Y)(r)[2r + i], M^c(X)).$$

As special cases of $A_{r,i}(Y, X)$, we see that

$$A_{0,i}(Y, \mathbb{A}^j) = H^{2j-i}(Y, \mathbb{Z}(j))$$

(since localization implies that $\mathbb{Z}(j)[2j]$ is quasi-isomorphic to $M^c(\mathbb{A}^j)$) and

$$A_{r,i}(Spec k, X) = H_{2r+i}^{lc}(X, \mathbb{Z}(r))$$

(since $M(Spec k)(r) = \mathbb{Z}(r)$).

5 Duality with Applications

In [13], Friedlander and H.B. Lawson prove a moving lemma for families of cycles on a smooth scheme which enables one to make all effective cycles of degree bounded by some constant to intersect properly all effective cycles of similarly bounded degree. This was used to establish duality isomorphisms [12], [9] between Lawson homology (cf. [18]) and morphic cohomology (cf. [11]), topological analogues of motivic homology with locally compact supports and motivic cohomology.

Theorem 5.3 presents the result of adapting the moving lemma of [13] to our present context of DM_k. As consequences of this moving lemma, we show that a theorem of Suslin implies that Bloch's higher Chow groups of a smooth scheme over a field which admits resolution of singularites equals motivic cohomology as defined in §4. We also prove that applying Tate twists is fully faithful in DM_k.

We first translate the moving lemma of [13] into a statement concerning the presheaves $z_{equi}(X, *)$. The moving lemma enables us to move cycles on $U \times W \times X$ equidimensional over a smooth W to become equidimensional over $U \times W$ provided that U is also smooth. (In other words, cycles are moved to intersect properly each of the fibres of the projection $U \times W \times X \to U \times W$.)

Theorem 5.1. *[14, 7.4] Assume that k admits resolution of singularities, that U is a smooth, quasi-projective, equidimensional scheme of dimension n over k, and that X is a scheme of finite type over k. For any $r \geq 0$, the natural embedding of presheaves on Sm/k*

$$\mathcal{D} : z_{equi}(X, r)(U \times -) \to z_{equi}(X \times U, r + n)$$

induces a quasi-isomorphism of chain complexes

$$\mathcal{D} : z_{equi}(X, r)(U \times \Delta^*) \to z_{equi}(X \times U, r + n)(\Delta^*).$$

As shown in [14, 7.1], the hypothesis that k admits resolution of singularities may be dropped provided that we assume instead that X and Y are both projective and smooth.

Applying Theorem 5.1 to the map of presheaves

$$z_{equi}(X, r)(\Delta^* \times \mathbb{A}^1 \times -) \to z_{equi}(X \times \mathbb{A}^1, r + 1)(\Delta^* \times -)$$

and using Lemma 2.4, we obtain the following homotopy invariance property.

Corollary 5.2. *Assume that k admits resolution of singularites. Then the natural map of presheaves induced by product with $\mathbb{A}^1$*

$$z_{equi}(X, r) \to z_{equi}(X \times \mathbb{A}^1, r + 1)$$

induces a quasi-isomorphism

$$\underline{C}_*(z_{equi}(X,r)) \xrightarrow{\simeq} \underline{C}_*(z_{equi}(X \times \mathbb{A}^1, r+1)).$$

Massaging Theorem 5.1 into the machinery of the previous sections provides the following duality theorem.

Theorem 5.3. *[14, 8.2] Assume that k admits resolution of singularities. Let X, Y be schemes of finite type over k and let U be a smooth scheme of pure dimension n over k. Then there are natural isomorphisms*

$$A_{r,i}(Y \times U, X) \equiv H^{-i}_{cdh}(Y \times U, \underline{C}_*(z_{equi}(X,r))_{cdh}) \xrightarrow{\simeq}$$

$$H^{-i}_{cdh}(Y, \underline{C}_*(z_{equi}(X \times U, r+n))_{cdh}) \equiv A_{r+n,i}(Y, X \times U).$$

Setting $Y = Speck$, $X = \mathbb{A}^j$, and $r = 0$, we obtain the following duality relating motivic cohomology to motivic homology with locally compact supports.

Corollary 5.4. *Assume that k admits resolution of singularities and that U is a smooth scheme of pure dimension n over k. Then there are natural isomorphisms*

$$H^m(U, \mathbb{Z}(j)) \xrightarrow{\simeq} H^{lc}_{2n-m}(U, \mathbb{Z}(n-j))$$

provided $n \geq j$.

Proof. We obtain the following string of equalities provided $n \geq j$:

$$H^m(U, \mathbb{Z}(j)) = H^m(U, \underline{C}_*(z_{equi}(\mathbb{A}^j, 0))[-2j])$$

$$\xrightarrow{\simeq} H^m(Speck, \underline{C}_*(z_{equi}(U \times \mathbb{A}^j, n))[-2j]) = H^m(Speck, \underline{C}_*(z_{equi}(U, n-j))[-2j])$$

$$= Hom_{DM_k}(\mathbb{Z}(n-j)[2n-2j], M^c(U)[m-2j]) = H^{lc}_{2n-m}(X, \mathbb{Z}(j)).$$

$\square$

The following theorem was proved by Suslin in [27] using a different type of moving argument which applies to cycles over affine spaces. The content of this theorem is that Bloch's complex (consisting of cycles over algebraic simplices which meet the pre-images of faces properly) is quasi-isomorphic to complex of cycles equidimensional over simplices.

Theorem 5.5. *Let X be a scheme of finite type of pure dimension n over a field k and assume that either X is affine or that k admits resolution of singularities. Let $Z^j_*(X)$ denote the Bloch complex of codimension j cycles (whose cohomology equals Bloch's higher Chow groups $CH^j(X, *)$. Then whenever $0 \leq j \leq n$, the natural embedding*

$$\underline{C}_*(z_{equi}(X, n-j))(Speck) \to Z^j_*(X)$$

is a quasi-isomorphism.

Combining Corollary 5.4 and Theorem 5.5, we obtain the following comparison of motivic cohomology and Bloch's higher Chow groups.

Corollary 5.6. *Let X be a smooth scheme of finite type of pure dimension n over a field k and assume that k admits resolution of singularities. Then there is a natural isomorphism*

$$H^{2j-i}(X, \mathbb{Z}(j)) \simeq CH^j(X, i).$$

Another important consequence of Theorem 5.3 is the following theorem.

Theorem 5.7. *[32, 4.3.1] Let X, Y be schemes of finite type over a field k which admits resolution of singularities. Then the natural map*

$$Hom_{DM_k}(M(X), M(Y)) \to Hom_{DM_k}(M(X)(1), M(Y)(1))$$

is an isomorphism.

Sketch of proof. We use the following identification (cf. [32, 4.23.])

$$A_{r,i}(X, Y)) = Hom_{DM_k}(\underline{C}_*(c_{equi}(X, 0))(r)_{cdh}[2r + i], \underline{C}_*(z_{equi}(Y, 0))_{cdh}).$$

Using localization, we reduce to the case that X, Y are projective. Then,

$$Hom_{DM_k}(M(X)(1), M(Y)(1)) = Hom_{DM_k}(M(X)(1), M^c(Y \times \mathbb{A}^1)[-2])$$

equals $A_{1,0}(X, Y \times \mathbb{A}^1)$ by Proposition 4.9 which is isomorphic to $A_{0,0}(X, Y) = Hom_{DM_k}(X, Y)$ by Theorem 5.3. $\qquad\square$

6   Conjecture of Beilinson -Lichtenbaum

In this section, we sketch a theorem of Suslin and Voevodsky which permits K-theoretic conclusions provided that one can prove the Bloch-Kato Conjecture. Since this conjecture for the prime 2 is precisely the Milnor Conjecture recently proved by Voevodsky [33], the connection established by Suslin and Voevodsky has important applications to the 2-primary part of algebraic K-theory.

Throughout this section ℓ is a prime invertible in k and k is assumed to admit resolution of singularities. We recall the Bloch-Kato Conjecture.

Conjecture 6.1. *(Bloch-Kato conjecture in weight n over k) For any field F over k, the natural homomorphism*

$$K_n^M(F)/\ell \to H_{et}^n(F, \mu_\ell^{\otimes n})$$

is an isomorphism. In other words,

$$H^n(SpecF, \mathbb{Z}/\ell(n)) \xrightarrow{\simeq} H_{et}^n(SpecF, \mathbb{Z}/\ell(n)).$$

If K is a complex of sheaves on some site, we define $\tau_{\leq n}(K)$ to be the natural sub-complex of sheaves such that

$$\underline{H}^i(\tau_{\leq n}(K))) = \begin{cases} \underline{H}^i(K) & i \leq n \\ 0 & i > n \end{cases}$$

Definition 6.2. *Let $\pi : (Sm/k)_{et} \to (Sm/k)_{Zar}$ be the evident morphism of topologies on smooth schemes over k. Let $\mathbf{R}\pi_*(\mu_\ell^{\otimes n})$ denote the total right derived image of the sheaf $\mu_\ell^{\otimes n}$. We denote by $B/\ell(n)$ the complex of sheaves on $(Sm/k)_{Zar}$ given by*

$$B/\ell(n) = \tau_{\leq n}\mathbf{R}\pi_*(\mu_\ell^{\otimes n}).$$

As shown in [30, 5.1], $B/\ell(n)$ is a complex of presheaves with transfers with homotopy invariant cohomology sheaves. By Propositions 2.7 and 3.3, this implies the natural isomorphism for any smooth scheme X over k

$$H^i_{Zar}(X, B/\ell(n)) \xrightarrow{\simeq} Hom_{DM_k}(M(X), B/\ell(n)[i]),$$

where the cohomology is Zariski hypercohomology.

The following conjecture of Beilinson [2], related to conjectures of Lichtenbaum [22], is an intriguing generalization of the Bloch-Kato conjecture. We use the natural quasi-isomorphism $\mu_\ell^{\otimes n} \simeq \mathbb{Z}/\ell(n)_{et}$ of Theorem 4.5 plus the acyclicity of $\mathbb{Z}/\ell(n)$ in degrees greater than n to conclude that the natural maps

$$\mathbb{Z}/\ell(n) \to \mathbf{R}\pi_*\mathbb{Z}/\ell(n)_{et} \simeq \mathbf{R}\pi_*\mu_\ell^{\otimes n} \leftarrow B/\ell(n)$$

determine a natural map (in the derived category of complexes of sheaves in the Zariski topology)

$$\mathbb{Z}/\ell(n) \to B/\ell(n).$$

Conjecture 6.3. *(Beilinson-Lichtenbaum Conjecture in weight n over k) The natural morphism*

$$\mathbb{Z}/\ell(n) \to B/\ell(n)$$

is a quasi-isomorphism of complexes of sheaves on $(Sm/k)_{Zar}$.

Remark A well known conjecture of Beilinson [2], [3] and Christophe Soulé [25] asserts that $H^i(X, \mathbb{Z}(n))$ vanishes for $i < 0$. Since $H^i_{et}(X, \mu_\ell^{\otimes n}) = 0$ for $i < 0$, Conjecture 6.3 incorporates the mod-ℓ analogue of the Beilinson-Soulé Conjecture.

We now state the theorem of Suslin and Voevodsky. M. Levine provided a forerunner of this theorem in [19].

Theorem 6.4. *[30, 5.9] Let k be a field which admits resolution of singularities and assume that the Bloch-Kato conjecture holds over k in weight n. Then the Beilinson-Lichtenbaum conjecture holds over k in weight n.*

Sketch of Proof. One readily verifies that the validity of the Bloch-Kato Conjecture in weight n implies the validity of this conjecture in weights less than n. Consequently, proceeding by induction, we may assume the validity of the Beilinson-Lichtenbaum Conjecture in weights less than n. Moreover, since both $\mathbb{Z}/\ell(n)$ and $B/\ell(n)$ have cohomology presheaves which are homotopy invariant presheaves with transfers annihilated by multiplication by n, we may apply the rigidity theorem (Theorem 2.5) to conclude that to prove the asserted quasi-isomorphism $\mathbb{Z}/\ell(n) \to B/\ell(n)$ it suffices to prove for all extension fields F over k that the induced map

$$H^*(SpecF, \mathbb{Z}/\ell(n)) \to H^*(SpecF, B/\ell(n))$$

is an isomorphism. By construction, $H^i(SpecF, \mathbb{Z}/\ell(n)) = 0$ for $i > n$, so that it suffices to prove

$$H^i(SpecF, \mathbb{Z}/\ell(n)) \xrightarrow{\simeq} H^i_{et}(SpecF, \mu_\ell^{\otimes n}) \quad i \leq n.$$

Suslin and Voevodsky easily conclude that it suffices to prove that

$$H^i(SpecF, \mathbb{Z}/\ell(n)) \to H^i_{et}(SpecF, \mu_\ell^{\otimes n}) \quad i < n$$

is injective (assuming the validity of the Bloch-Kato Conjecture in weight n). This in turn is implied by the assertion that

$$H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n)_{cdh}) \to H^n(\partial\Delta_F^j, B/\ell(n)_{cdh})$$

is injective for all j, where $\partial\Delta_F^j$ is the (singular) boundary of the j-simplex over F whose cohomology fits in Mayer-Vietoris exact sequence for a covering by two contractible closed subschemes whose intersection is $\partial\Delta_F^{j-1}$.

We denote by S^1 the scheme obtained from $\mathbb{A}^1$ by gluing together $\{0\}, \{1\}$. We have natural embeddings

$$H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n)_{cdh}) \to H^{n+1}(\partial\Delta_F^j \times S^1, \mathbb{Z}/\ell(n)_{cdh})$$

$$H^n(\partial\Delta_F^j, B/\ell(n)_{cdh}) \to H^{n+1}(\partial\Delta_F^j \times S^1, B/\ell(n)_{cdh}).$$

Any cohomology class in $H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n))$ which does not arise from $H^n(SpecF, \mathbb{Z}/\ell(n))$ vanishes on some open subset $U \subset \partial\Delta_F^j \times S^1$ containing all the points of the form $p_i \times \infty$ where $\infty \in S^1$ is the distinguished point. In other words, all such cohomology lies in

the image of $H_{\mathcal{Z}}^{n+1}(\partial \Delta_F^j \times S^1, \mathbb{Z}/\ell(n)_{cdh})$, the direct limit of cohomology with supports in closed subschemes missing each of the points $p_i \times \infty$.

The localization distinguished triangle of Corollary 3.6 gives us long exact sequences in cohomology with coefficients $\mathbb{Z}/\ell(n)_{cdh}$ and $B/\ell(n)_{cdh}$ and a map between these sequences; the terms involve the cohomology of S (the semi-local scheme of the set $\{p_i \times \{\infty\}\}$), of $\Delta_F^j \times S^1$ with supports in $\mathcal{Z}$, and of $\Delta_F^j \times S^1$ itself. Although S is not smooth, one can conclude that our Bloch-Kato hypothesis implies that $H^n(S, \mathbb{Z}/\ell(n)_{cdh}) \to H^n(S, B/\ell(n)_{cdh})$ is surjective. Another application of the localization distinguished triangle plus induction (on n) implies that the map on cohomology with supports in $\mathcal{Z}$ is an isomorphism. The required injectivity now follows by an easy diagram chase. $\qquad \square$

An important consequence of Theorem 6.4 is the following result of Suslin and Voevodsky.

Proposition 6.5. *[30, 7.1] The Bloch-Kato conjecture holds over k in weight n if and only if for any field F of finite type over k the Bockstein homomorphisms*

$$H_{et}^n(F, \mu_{\ell^m}^{\otimes n}) \to H_{et}^{n+1}(F, \mu_\ell^{\otimes n})$$

are zero for all $m > 0$.

Comment about the Proof. If the Bloch-Kato conjecture holds, then $H_{et}^n(F, \mu_{\ell^m}^{\otimes n})$ consists of sums of products of elements of $H_{et}^1(F, \mu_{\ell^m})$. The vanishing of the Bockstein homomorphism on classes of cohomology degree 1 follows immediately from Hilbert's Theorem 90.

The proof of the converse is somewhat less direct.

$\qquad \square$

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