INTERSECTION PRODUCTS FOR SPACES OF ALGEBRAIC CYCLES

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This is a slightly expanded version of three lectures given by the author in Bologna in December 1997. Our purpose is to introduce to algebraic geometers some constructions developed in recent years for spaces of cycles that relate to classical intersection theory. The work discussed appears primarily in joint papers of the author and H. Blaine Lawson as well as a joint paper of the author and Ofer Gabber. We recommend the reader consult the expository article [L2] which not only provides further details concerning spaces of algebraic cycles but also provides a somewhat different point of view.

Unless mention is made to the contrary, all varieties considered will be quasi-projective varieties over the complex field $\mathbf{C}$. By a closed subvariety of such a variety X, we shall mean (implicitly) closed in the Zariski topology on X, but otherwise when we speak of the topology on X we shall mean the analytic topology. As mentioned in the lectures, certain aspects of what we discuss hold over an arbitrary field (e.g., moving lemmas as in [F-L3]) or over a field which admits resolution of singularities (e.g., duality theorems as in [FV]).

§1 Topological abelian groups of algebraic cycles

In this lecture, we survey some of the basic definitions and properties of Lawson homology, a homology theory defined in terms of homotopy groups of topological abelian groups of algebraic cycles.

Let X be a (complex, quasi-projective) variety. Then the effective 0-cycles on X of a given degree d (i.e., formal sums of points of X with non-negative integer coefficients whose sum equals d) admit the natural structure of an algebraic variety $SP^d(X)$, the d-fold symmetric power of X. The formal sum of 0-cycles determines a topological monoid structure on

$$\mathcal{C}_0(X) \;\equiv\; \coprod_{d\geq 0} SP^d(X),$$

where we set $SP^0(X)$ equal to a point (the set of effective 0-cycles contains only the empty – or zero – cycle). The naïve group completion of this topological monoid is the topological abelian group

$$Z_0(X) \equiv \mathbf{Z}[X] \;=\; C_0(X)^{\times 2}/\sim$$

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of all 0-cycles on X (or, equivalently, the free abelian group on X), where the equivalence relation $\sim$ consists of pairs of pairs $(\sum x_i, \sum y_j) \sim (\sum z_k, \sum w_\ell)$ such that $\sum x_i + \sum w_\ell = \sum y_j + \sum z_k$ and where $Z_0(X)$ is provided with the quotient topology with respect to $\mathcal{C}_0(X)^{\times 2} \to Z_0(X)$.

Theorem 1.1 (Dold-Thom [D-T]). *The integral singular homology of the complex projective variety X can be computed as*

$$H_*(X) \ = \ \pi_*(\mathbf{Z}[X]) \ = \ \pi_*(\Omega\, \mathrm{B}\, \mathcal{C}_0(X))$$

where $\mathrm{B}\,\mathcal{C}_0(X)$ is the classifying space of the abelian topological monoid $\mathcal{C}_0(X)$ and $\Omega\,\mathrm{B}\,\mathcal{C}_0(X)$ is its loop space.

Now, we proceed with a very similar construction for positive dimensional cycles on algebraic varieties. We begin by assuming that our variety is projective so that we can use Chow varieties as recalled below.

Theorem 1.2 ([C-W]). *Let X be a closed subvariety of some projective space $\mathbf{P}^N$. Then the set of effective, degree d dimension r-cycles on X admits a natural structure of a projective algebraic variety $C_{r,d}(X)$.*

Moreover, the Chow monoid

$$\mathcal{C}_r(X) \ \equiv \ \coprod_{d \geq 0} C_{r,d}(X)$$

is an abelian topological monoid which is independent up to algebraic isomorphism of the chosen embedding $X \subset \mathbf{P}^N$.

The independence of projective embedding of $\mathcal{C}_r(X)$ is proved in [B].

Definition 1.3. *Let X be a projective variety. The topological abelian group of algebraic r-cycles is defined to be the naïve group completion of the Chow monoid $\mathcal{C}_r(X)$*

$$Z_r(X) \ \equiv \ \mathcal{C}_r(X)^{\times 2}/ \sim,$$

where the equivalence relation $\sim$ consists of pairs of pairs $(Z_1, Z_2) \sim (W_1, W_2)$ such that $Z_1 + W_2 = Z_2 + W_1$ and where $Z_r(X)$ is provided with the quotient topology.

The (bi-indexed) Lawson homology groups are defined to be the homotopy groups of these topological abelian groups (pointed at 0)

$$L_rH_n(X) \ \equiv \ \pi_{n-2r}(Z_r(X)), \quad n \geq 2r.$$

(cf. Remark 1.11 for an explanation of this indexing.)

The reader consulting the literature should be forewarned that early papers on Lawson homology used a somewhat different (less convenient, but essentially equivalent) definition. From the point of view of getting sensible algebraic invariants, it seemed evident that one wanted to use a homotopy theoretic group completion of $\mathcal{C}_r(X)$. Thus, the first formulations used $\Omega\,\mathrm{B}\,\mathcal{C}_r(X)$ or the group completion of the simplicial monoid $Sing.(\mathcal{C}_r(X))$. However, the underlying discrete group of $Z_r(X)$ is precisely the group of all r-cycles on X, so that this more concrete object has definite advantages. It was proved by Paulo Lima-Filho [Li2] and by Ofer Gabber and the author [F-G] that the natural map $\mathcal{C}_r(X) \to Z_r(X)$ is a homotopy theoretic group completion: the induced map in integral homology can be identified with the following localization

$$H_*(\mathcal{C}_r(X)) \ \to \ \mathbf{Z}[\pi_0^+] \otimes_{\mathbf{Z}[\pi_0]} H_*(\mathcal{C}_r(X))$$

where $\pi_0 = \pi_0(\mathcal{C}_r(X))$ is the discrete abelian monoid of connected components of $\mathcal{C}_r(X)$. Another reassuring property of the topological abelian groups $Z_r(X)$ is that they are "nice" as topological spaces; namely, $Z_r(X)$ has the natural structure of a (countable) C.W. complex (cf. [F4]).

We shall require constructions on these topological abelian groups which require working modulo homotopy equivalences which are continuous group homomorphisms. We recall that this context is a familiar one in algebraic geometry.

Formalism. *The category whose objects are topological abelian groups of the homotopy type of C.W. complexes and whose maps are homotopy equivalence classes of continuous group homomorphisms is equivalent to the derived category of abelian groups $\mathcal{D}^+(Ab)$ defined as the the triangulated category of chain complexes of abelian groups (with differential of degree $+1$) which are bounded above localized with respect to quasi-isomorphisms.*

This equivalence can be realized by sending the topological abelian group Z to the normalized chain complex $\widetilde{Z}$ associated to the simplical abelian group $Sing.(Z)$. Under this correspondence, $H_i(Z)$ is naturally isomorphic to $\pi_i(\widetilde{Z})$.

When working with the cycle spaces, we sometimes consider these as topological abelian groups $Z_r(X)$ and sometimes as chain complexes $\widetilde{Z}_r(X)$.

The following theorem of Blaine Lawson is the foundation for all that follows. We recall that $\mathbf{P}^{N+1}$ can be viewed as the union of all lines from a fixed hyperplane $\mathbf{P}^N \subset \mathbf{P}^{N+1}$ to a fixed point $x_\infty \in \mathbf{P}^{N+1}$ not on the hyperplane, $\mathbf{P}^{N+1} = \mathbf{P}^N \# x_\infty$. The algebraic suspension of a closed subvariety $X \subset \mathbf{P}^N$ is defined to be the subvariety $\Sigma\, X \subset \mathbf{P}^{N+1}$ given as the union of all lines from X to x_∞, where once again $\mathbf{P}^{N+1} = \mathbf{P}^N \# x_\infty$. Stated more algebraically, if the subvariety $X \subset \mathbf{P}^N$ is given by homogeneous equations $F_i(T_0, \dots, T_N)$, then $\Sigma \subset \mathbf{P}^{N+1}$ is given by the same set of homogeneous equations now viewed as equations in the variables $T_0, \dots, T_{N+1}$.

Theorem 1.4 Lawson Suspension Theorem [L1]**.** *Let $X \subset \mathbf{P}^N$ be a closed subvariety. Then sending an irreducible subvariety $Y \subset X$ of dimension r to $\Sigma\, Y \subset \mathbf{P}^{N+1}$ determines a continuous group homomorphism*

$$\Sigma : Z_r(X) \ \to \ Z_{r+1}(\Sigma\, X)$$

which is a homotopy equivalence. In other words,

$$\Sigma : \widetilde{Z}_r(X) \ \to \ \widetilde{Z}_{r+1}(\Sigma\, X)$$

is a quasi-isomorphism in $\mathcal{D}^+(Ab)$.

Remarks. The essence of this theorem is a moving lemma. Namely, let

$$\mathcal{C}_{r+1,d}(\Sigma\, X, X) \subset \mathcal{C}_{r+1,d}(\Sigma\, X)$$

denote the submonoid of those effective $r+1$-cycles on ΣX which intersect $X \subset \Sigma X$ properly (i.e., in pure dimension r). Then it is not difficult to see that

$$\Sigma : \mathcal{C}_{r,d}(X) \to \mathcal{C}_{r+1,d}(\Sigma X, X)$$

is a deformation retract. The content of the theorem is the assertion that

$$C_{r+1,d}(\Sigma X, X) \ \subset \ C_{r+1}(\Sigma X)$$

induces a homotopy equivalence of homtopy theoretic group completions. One can view this as the assertion that after group completion any "bounded family" of r-cycles on ΣX can be moved so that each member of the family meets X properly.

As Lawson observed, this theorem plus Theorem 1.1 immediately gives the following computation (since $\mathbf{P}^N = \Sigma^r \equiv \underbrace{\Sigma \circ \cdots \circ \Sigma}_{r\ times} P^{N-r}$).

Corollary 1.5. *For any $N \geq r \geq 0$,*

$$L_r H_n(\mathbf{P}^N) \ = \ H_n(\mathbf{P}^{N-r})$$

where $H_n(\mathbf{P}^{N-r})$ denotes the integral singular homology of the analytic space $\mathbf{P}^{N-r}$.

Indeed, the possibility of such a result was a major motivation of Lawson who sought to find an analogue of Theorem 1.1 for higher dimensional cycles along the lines of a theorem of F. Almgren which asserted that integral homology of X could be computed as the homotopy groups of topological abelian groups of "integral cycles" (i.e., rectifiable currents on X with trivial boundary topologized with the flat norm topology) [A].

From the outset, we have taken a different point of view: Lawson homology groups are often algebraic invariants which have no classical algebraic topology representation. For example, we have the following elementary computation.

Proposition 1.6 [F1]**.** *Let X be a projective algebraic variety. Then $L_r H_{2r}(X)$ equals the group of algebraic r-cycles modulo algebraic equivalence.*

We now extend our consideration to quasi-projective varieties, following the lead of Lima-Filho [Li1].

Definition 1.7. *Let $X \subset \mathbf{P}^N$ be a closed subvariety, let $X_\infty \subset X$ be a closed embedding, and let $U = X - X_\infty$. Then we define*

$$Z_r(U) \ \equiv \ Z_r(X)/Z_r(X_\infty), \quad L_r H_n(U) = \pi_{n-2r}(Z_r(U)).$$

Justification of this definition is a result of Lima-Filho that $Z_r(U)$ is independent (i.e., the isomorphism class of $\widetilde{Sing}.(Z_r(U))$) of the choice of projective closure X of U and of the embedding $X \subset \mathbf{P}^N$ (cf. [Li1], [F-G;2.2]).

The following localization theorem (due to Lima-Filho and Friedlander-Gabber) provides a useful computational tool.

Theorem 1.8 [Li-1], [F-G]. *Let X be a projective algebraic variety with closed subvariety $X_\infty \subset X$ and Let $U = X - X_\infty$. Then*

$$Z_r(X_\infty) \to Z_r(X) \to Z_r(U)$$

is a fibration sequence; in other words, we have the following natural distinguished triangle in $\mathcal{D}^+(Ab)$

$$\widetilde{Z}_r(X_\infty) \to \widetilde{Z}_r(X) \to \widetilde{Z}_r(U) \to \widetilde{Z}_r(X_\infty)[1].$$

We next consider operations on Lawson homology introduced by the author and Barry Mazur in [F-M1]. These operations are appropriate to discuss in a meeting on intersection theory, for their definition is essentially intersection-theoretic.

Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$ linearly embedded in $\mathbf{P}^{m+n+1}$ with disjont images. Then we may view $\mathbf{P}^{m+n+1}$ as the union of all lines from points on $\mathbf{P}^m$ to ponts on $\mathbf{P}^n$ and we write

$$\mathbf{P}^{n+m+1} \ = \ \mathbf{P}^m \# \mathbf{P}^n.$$

For any closed subvarieties $X \subset \mathbf{P}^m, Y \subset \mathbf{P}^n$, we define the algebraic join

$$X \# Y \ \subset \ \mathbf{P}^m \# \mathbf{P}^n = \mathbf{P}^{n+m+1}$$

as the union of all lines from points of X to points of Y. Taking $n = 0$ and $Y = x_\infty$, we obtain the algebraic suspension discussed earlier. If X is given by the set of homogeneous equations $\{F_i(S_0, \dots, S_m)\}$ and Y by the set of homogeneous equations $\{G_j(T_0, \dots, T_n)\}$, then $X \# Y$ is given by the union of these two sets of homogeneous equations viewed as functions in $S_0, \dots, S_m, T_0, \dots, T_n$. Observe that

$$X \# P^n \ = \ \Sigma^{n+1} X \ \subset \mathbf{P}^{m+n+1}.$$

This join construction extends by linearity to a continuous bilinear pairing of topological groups

$$\# : Z_r(X) \times Z_s(Y) \ \to \ Z_{r+s}(X \# Y).$$

Definition 1.9 [F-M1]. *Let $X \subset \mathbf{P}^N$ be a closed subvariety and let $r > 0$. We consider the chain of continuous group homomorphisms*

$$Z_r(X) \times Z_0(\mathbf{P}^1) \xrightarrow{\#} Z_{r+1}(X \# \mathbf{P}^1) \xleftarrow{\Sigma^2} Z_{r-1}(X), \tag{1.9.1}$$

where the first is given by join of cycles and the second is two-fold algebraic suspension (or join with $x_\infty \in \mathbf{P}^1$). We define

$$h : Z_r(X) \to Z_{r-1}(X), \quad h : \widetilde{Z}_r(X) \to \widetilde{Z}_{r-1}(X)$$

to be the map (i.e., isomorphism class of maps in $\mathcal{D}^+(Ab)$) obtained by restricting the first map of (1.9.1) to $Z_r(X) \times \{x_\infty\}$ and composing this restriction with the homotopy inverse of Σ^2.

Consider the map $\mathbf{P}^1 \to Z_0(\mathbf{P}^1)$ given by sending a point x to $x - x_\infty$. We define

$$s : Z_r(X) \to \Omega^2 Z_{r-1}(X), \quad s : \widetilde{Z}_r(X) \to \widetilde{Z}_{r-1}(X)[-2]$$

as the map induced by adjunction by the composition of

$$Z_r(X) \times \mathbf{P}^1 \to Z_r(X) \times Z_0(\mathbf{P}^1) \xrightarrow{\#} Z_r(X \times \mathbf{P}^1)$$

and the homotopy inverse of Σ^2.

We view the operation h on $Z_r(X)$ as taking an r-cycle on X and intersecting it with a fixed hyperplane. Indeed, this operation does "cover" the operation in homology given by sending the homology class of a cycle (as defined below) to its intersection with the class of a hyperplane. Thus, h evidently depends on the choice of embedding $X \subset \mathbf{P}^N$.

What is at first disturbing is that the operation h enables us to "continuously intersect" all r-cycles on X with a fixed hyperplane, whereas if r is less than the dimension of X then any hyperplane contains some r-cycles of X so that h can not possibly be represented on such cycles as the intersection with the given hyperplane. Since the construction of h requires a choice of homotopy inverse for Σ, we see that h is not canonically defined. Indeed, if one fixes a given hyperplane L, then a representative for the homotopy class of h can be chosen so that on those effective r-cycles intersecting L properly this representative takes such an effective cycle Z to $Z \bullet L$.

The operation s is more interesting. We view this geometrically as associating to any r-cycle on X the family of $r-1$-cycles on X parametrized by $\mathbf{P}^1$ given by intersecting the cycle with a fixed Lefschetz pencil of hyperplanes. Once again, we see that this can only be literally correct if we fix the pencil of hyperplanes and consider effective cycles which intersect properly each member of the pencil.

The following proposition mentions some of the good properties of the s operation.

Proposition 1.10 [F-G], [F2]. *Let $X \subset \mathbf{P}^N$ be a closed subvariety and let $X_\infty \subset X$ be a closed subvariety.*

(a.) The s operation on $Z_*(X)$ restricts to an operation on $Z_*(X_\infty)$ and thereby determines operations

$$s : Z_r(U) \to \Omega\, Z_{r-1}(U), \quad \widetilde{Z}_r(U) \ \to \ \widetilde{Z}_{r-1}(U)[-2].$$

(b.) The s operation can be represented as the composition of the map $Z_r(U) \times Z_0(\mathbf{P}^1) \to Z_r(U \times \mathbf{P}^1)$ sending an r-cycle ζ on U and a 0-cycle τ on $\mathbf{P}^1$ to $\zeta \times \tau$ and the map $Z_r(U \times \mathbf{P}^1) \to Z_{r-1}(U)$ given by intersection with $U \times x_\infty$. In particular, s is independent of the choice of projective closure $U \subset X$ and projective embedding $X \subset \mathbf{P}^N$.

(c.) The map

$$s^r : Z_r(U) \to \pi_0(Z_r(U)) \to \pi_{2r}(Z_0(U))$$

sends an algebraic equivalence class of r-cycles to its fundamental homology class in integral Borel-Moore homology $H_{2r}^{BM}(U) = H_{2r}(X, X_\infty)$.

Remark 1.11. *Proposition 1.10 enables us to conclude that $s^r : Z_r(U) \to \Omega^r\, Z_0(U)$ determines a natural map (independent of projective closure $U \subset X$ and projective embedding $X \subset \mathbf{P}^N$)*

$$L_r H_{2r+i}(U) \ \to \ H_{2r+i}^{BM}(U).$$

Using the s operation, Barry Mazur and the author introduced filtrations on singular homology and on algebraic cycles. The "topological filtration on homology" is defined by

$$T_iH_n(X) \equiv im\{s^i : L_iH_n(X) \to H_n(X)\}$$

and investigated in [F-M1], [F-M2]. In particular, this topological filtration was shown to equal the "correspondence filtration" which is formulated in purely algebro-geometric terms in terms of images of homological correspondences. The S-filtration on cycles considered in [F4], [F5] is given by

$$S^jZ_r(X) \equiv ker\{s^j : Z_r(X) \to \pi_0(Z_r(X)) \to L_{r-j}H_{2r}(X)\}.$$

This filtration can also be described in purely algebro-geometric terms in terms of images under correspondences of cycles homologically equivalent to 0 on smaller dimensional subvarieties. Questions about these filtrations are directly related to the (generalized) Hodge Conjecture and Grothendieck's standard conjectures. For abelian varieties, the topological filtration on homology is investigated in a recent paper by Salman Abdulali [Ab]. A recent paper of C. Peters [P] investigates the stability of the S-filtration on cycles in certain cases.

§2 Intersection pairings

In this lecture, we discuss various aspects of the our joint paper with Ofer Gabber [F-G]. The reader familiar with intersection theory as developed in William Fulton's book [Fu] will find that we are presenting some aspects of that theory in a way that we treat all cycles of a given dimension at the same time. In particular, an intersection pairing

$$Z_r(U) \times Z_s(U) \to Z_{r+s-n}(U), \quad n = dim(U)$$

is constructed for any smooth quasi-projective variety by defining Gysin maps for regular immersions and considering the Gysin map $\Delta^! : Z_{r+s}(U \times U) \to Z_{r+-n}(U)$. We also discuss another approach to intersecting spaces of cycles based on the Moving Lemma developed by the author and Blaine Lawson [FL-3]. This approach has the advantage that it applies over an arbitrary field, providing interesting structure to motivic cohomology (cf. [F-V]).

As usual, all varieties discussed are quasi-projective complex algebraic varieties unless explicit mention to the contrary.

We begin with the following "homotopy invariance" property.

Proposition 2.1 [F-G;2.3]. *Let E be a locally free rank e sheaf on a quasi-projective variety U and let*

$$\pi : \mathbf{V}(E) = \mathrm{Spec}\, Sym_{\mathcal{O}_X} E^* \;\to\; U$$

be the associated algebraic vector bundle. Then

$$\pi^* : Z_r(U) \;\to\; Z_{r+e}(\mathbf{V}(E))$$

is a homotopy equivalence, where π^* denotes flat pull-back of cycles.

The proof of this important property is surprisingly easy. Using the localization property (Theorem 1.8), we reduce to the case in which the bundle is trivial. We can similarly localize the context of the Lawson suspension theorem (trivializing the line bundle $\mathcal{O}_X(1)$ determining $\Sigma\, X$), thereby obtaining the desired homotopy equivalence.

Proposition 2.1 enables the following natural definition of the operational first chern class of a line bundle.

Definition 2.2. *Let L be a line bundle on the quasi-projective variety U. For any $r \geq 0$, we define*

$$c_1(L) : Z_{r+1}(U) \ \to \ Z_r(U)$$

as the composition of $Z_{r+1}(U) \to Z_{r+1}(\mathbf{V}(L))$ induced by the 0-section $0 : X \to \mathbf{V}(L)$ followed by a choice of homotopy inverse for $\pi^* : Z_r(U) \to Z_{r+1}(\mathbf{V}(L))$.

Of course, $c_1(L)$ is only well defined up to homotopy; in other words, the isomorphism class of

$$c_1(L) : \widetilde{Z}_{r+1}(X) \ \to \ \widetilde{Z}_r(X)$$

is well defined in $\mathcal{D}^+(Ab)$. Observe that $c_1(L)$ is a generalization of the h operation: take U to be the suspension $\Sigma\, W$ of some variety W and take L to be $\mathcal{O}_U(1)$.

From our perspective of looking at the entire topological group of algebraic cycles on a variety, it is natural to formulate intersection with the group of all divisors (not simply one divisor at a time). We state such a formulation in the next theorem. Only in the case of dvisiors (as in Theorems 2.3 and 2.4) can we successfully define intersection product on cycle spaces of a variety which is not necessarily smooth.

Theorem 2.3 [F-G;3.1]**.** *Let X be an irreducible projective variety. Let*

$$Div(X) = \coprod_{\alpha \in NS(X)^+} Div_\alpha(X)$$

be the disjoint union of projective varieties whose points are effective Cartier divisors on X. Let $Div(X)^+$ denote the homotopy-theoretic group completion of $Div(X)$, constructed as the infinite mapping telescope of the self-map given by addition with a chosen very ample divisor. Then there is a natural (in $\mathcal{D}^+(Ab)$) pairing

$$Z_{r+1}(X) \times Div(X)^+ \ \to \ Z_r(X)$$

satisfying the following properties:

(a.) For any effective Cartier divisor L, $c_1(L)$ is represented by restricting this pairing to $Z_{r+1}(X) \times \{L\}$. In particular, the homotopy class of $c_1(L)$ depends only upon the equivalence class of L in $\pi_0 Div(X)^+ = NS(X)$, the Neron-Severi group of X.

(b.) There is a fibration sequence

$$\mathbf{P}^\infty \to Div(X)^+ \to Pic(X);$$

in particular, $\pi_2 Div(X) = \mathbf{Z}$.

(c.) Restricting this pairing to $Z_{r+1}(X) \times S^2$ (where $S^2 = \mathbf{P}^1 \subset \mathbf{P}^\infty \subset Div(X)^+$) determines the s operation

$$Z_{r+1}(X) \wedge S^2 \ \to \ Z_r(X).$$

We next observe that $c_1(L)$ can be refined to take values in spaces of cycles on the support of the Weil divisor associated to the Cartier divisor L. As one might expect, we do not need to assume we are working with smooth varieties when intersecting with (Cartier) divisors.

Theorem 2.4 [F-G;2.5]**.** *Let D, D' be effective Cartier divisors on a quasi-projective variety U with support the closed subvarieties $i_D : |D| \to U,\ i_{D'} : |D'| \to U$. There exists a Gysin map (well defined up to homotopy, or up to isomorphism in $\mathcal{D}^+(Ab)$) of*

$$i_D^! : Z_{r+1}(U) \to Z_r(|D|), \quad i_D^! : \widetilde{Z}_{r+1}(U) \to \widetilde{Z}_r(|D|)$$

satisfying the following properties:

(a.) A representative of $i_D^!$ can be chosen whose restriction to effective cycles which meet $|D|$ properly is the usual intersection with a divisor.

(b.) The composition of $i_D^!$ with the map induced by i_D equals $c_1(\mathcal{O}(D)) : Z_{r+1}(U) \to Z_r(U)$.

(c.) $i_D! + i_{D'}^! = i_{D+D'}^! : \widetilde{Z}_{r+1}(U) \to \widetilde{Z}_r(|D| \cup |D'|)$.

(d.) $i_D^! \circ i_{D'}^! = i_{D'}^! \circ i_D^! : \widetilde{Z}_{r+1}(U) \to \widetilde{Z}_{r-1}(|D| \cap |D'|)$.

(e.) $i_D^!$ depends naturally as a map in $\mathcal{D}^+(Ab)$ upon the pair (U, D).

The proof of Theorem 2.4 utilizes the trivialization of $\mathbf{V}(\mathcal{O}(-D)) \to U$ when restricted to $U - |D|$ plus the localization sequence to construct a lifting of $Z_{r+1}(U) \to Z_{r+1}(\mathbf{V}(\mathcal{O}(-D))$ to $Z_r(|D|) \simeq Z_{r+1}(\mathbf{V}(i_D^*\mathcal{O}(-D))$.

Once one has a good formulation of intersection with a divisor, one can easily conclude the following projective bundle computation. As shown to us by Grothendieck, this enables us to define Chern classes of vector bundles in this theory.

We denote by $p_E : \mathbf{P}(E) = Proj(Sym_{\mathcal{O}_U} E^*) \to U$ the projectivization of $\mathbf{V}(E) \to U$; thus if E has rank e, then the fibres of p_E are projective spaces of dimension $e - 1$.

Corollary 2.5 [F-G;2.5]**.** *Let E be a rank e vector bundle over a quasi-projective variety U. Then the map*

$$\sum_{0/leqj<e} c_1(\mathcal{O}_{\mathbf{P}(E)}(1))^j \circ p_E^* : \prod_{0 \leq j < e} Z_{r+j}(U) \to Z_{r+e-1}(\mathbf{P}(E))$$

is a homotopy equivalence.

Our next task is to extend the definition of the Gysin maps of Theorem 2.4 to a regular closed embedding $W \subset U$ of codimension greater than 1. As one might expect, the construction of this Gysin map uses the technique of deformation to the normal cone of a regular embedding.

Theorem 2.6 [F-G;3.4]**.** *Let U be a quasi-projective variety and let $i_W : W \to U$ be a regular closed embedding of codimension c. Then there exists a Gysin map (well defined up to homotopy)*

$$i_W^! : Z_{r+c}(U) \ \to \ Z_r(W)$$

satisfying the following properties:

(a.) A representative of $i_W^!$ can be chosen whose restriction to effective $r + c$-cycles Z on U which meet W properly is given by

$$i_W^!(Z) \ = \ W \bullet Z$$

where $- \bullet -$ denotes the intersection product on the smooth variety U.

(b.) If W is of codimension 1 in U, then $i_W^!$ equals the Gysin map of Theorem 2.4.
(c.) If $i_V : V \to W$ is a regular embedding of codimension c', then

$$(i_W \circ i_V)^! = i_V^! \circ i_W^! : Z_{r+c+c'}(U) \to Z_r(V).$$

(d.) If $g : U' \to U$ is flat, then

$$i^!_{W\times_X X'} \circ g^* = g'^* \circ i^!_W;$$

if $f : X^{\sim} \to X$ is proper, then

$$i^!_W \circ f_* = f'_* \circ i^!_{W\times_X X'}.$$

If U is smooth, then the diagonal $\Delta : U \to U \times U$ is a regular closed embedding. Thus, Theorem 2.6 immediately provides the following intersection pairing on cycle spaces of a smooth variety.

Corollary 2.7. *Let U be a smooth variety of pure dimension n. We define the intersection product on cycle spaces (well defined up to homotopy) as the map*

$$\Delta^! \circ (- \times -) : Z_r(U) \times Z_s(U) \to Z_{r+s}(U \times U) \to Z_{r+s-n}(U)$$

sending a pair (ζ_1, ζ_2) to $\Delta^!(\zeta_1 \times \zeta_2)$. A representative of this map can be chosen so that it sends a pair of effective cycles (ζ_1, ζ_2) of dimensions r, s on U which meet properly to their intesection product $\zeta_1 \bullet \zeta_2$.

We conclude this lecture with a sketch of a different approach to intersection of cycle spaces following joint work of the author and Blaine Lawson. Here is a formulation of the Moving Lemma of [F-3]. We remind the reader that a "continuous algebraic map" from X to Y is a morphism from the weak normalization of X to Y.

Theorem 2.8. *Let X be a projective variety of pure dimension n, let r, s, be non-negative integers with $r + s \geq n$, and let e be a positive integer. Then there exists a Zariski open neighborhood $\mathcal{O} \subset \mathbf{A}^1$ of $0 \in \mathbf{A}^1$ and a continuous algebraic map*

$$\Psi : \mathcal{C}_s(X) \times \mathcal{O} \ \to \ \mathcal{C}_s(X)^2$$

satisfying the following conditions:
(a.) $\psi_0^+ - \psi_0^- : \mathcal{C}_s(X) \to Z_s(X)$ is the natural inclusion.
(b.) If Z_1, Z_2 are effective cycles on X of degree $\leq e$ of dimension r, s respectively, then for any $0 \neq t \in \mathcal{O}$ Z_1 intersects $(\psi_t^+ - \psi_t^-)(Z_2)$ properly outside of the singular locus of X.
Here, we have denoted by $\psi_t^+ \times \psi_t^-$ the restriction of Ψ to $\mathcal{C}_s(X) \times \{t\}$ for any $t \in \mathcal{O}$.

In other words, once one bounds the degree of the effective cycles under consideration, Ψ gives a means of uniformly moving via rational moves all effective s-cycles of degree $\leq e$ so that each intersects properly all effective r-cycles of degree $\leq e$ outside of the singular locus of X. For X smooth, this implies that given compact subsets $R \subset Z_r(X), S \subset Z_s(X)$, we can find a rational move of all of $Z_s(X)$ which moves each s-cycle in S to a rationally equivalent cycle meeting properly every r-cycle in R.

In particular, we conclude that for X projective and smooth the intersection product on homotopy groups induced by the pairing of Corollary 2.7 can be realized more geometrically as follows.

Corollary 2.9. *Let X be projective and smooth of pure dimension n, and let r, s, be non-negative integers with $r+s \geq n$. Then the bilinear map on homotopy groups*

$$L_rH_m(X) \otimes L_sH_p(X) \;\to\; L_{r+s-n}H_{m+p-2n}(X)$$

induced by the pairing $\Delta^!_X \circ (- \times -)$ of Corollary 2.8 can be realized as follows: represent a class $\alpha \in L_rH_m(X)$ by a continuous map $a : S^{m-2r} \to Z_r(X)$ and a class $\beta \in L_sH_p(X)$ by a continuous map $b : S^{p-2s} \to Z_s(X)$; using an appropriate Ψ as in Theorem 2.8, move each s-cycle in the image of b so that it meets properly each r-cycle in the image of a; then choose some $0 \neq t \in \mathcal{O}$ and take the homotopy class of the map sending a point $x \wedge y \in S^{m+p-2r-2s} = S^{m-2r} \wedge S^{p-2s}$ to $a(x) \bullet (\psi^=_t - \psi^-_t)(y) \in Z_{r+s-n}(X)$.

The reason that Theorem 2.8 does not provide intersection of cycle spaces on a smooth variety U which is not projective is that the moving achieved in Theorem 2.8 on a projective closure X of U might move certain effective s-cycles of degree $\leq e$ supported on $X_\infty = X - U$ to cycles with components meeting U. In other words, what Theorem 2.8 does not permit us to do is move cycles on X with a specified behaviour on $X_\infty \subset X$. A different approach to moving cycles which does permit such conditions on the moving but whoch applies in a much restricted situation has been given by Andrei Suslin [S].

§3 Cocycles and Duality

As we saw in the last lecture, we have a suitable intersection product on cycle spaces when our underlying quasi-projective variety is smooth. We now ask how one might proceed for singular varieties. Although we have no "magic answers", we do see that cocycle spaces admit a cup product pairing without hypotheses of smoothness. With this in mind, we can speculate how one might be able to refine intersection products using the stratification of a variety by singular loci.

In this lecture we survey "morphic cohomology" as introduced by the author and Blaine Lawson. As we shall see, morphic cohomology has a "join product" which is compatible with cup product in singular cohomology. One interesting feature of morphic cohomology is that it naturally maps (via a cap product with a fundamental class) to Lawson homology and this map is an isomorphism for smooth varieties. Moreover, this duality isomorphism for smooth varieties sends the join product in morphic cohomology to the intersection product in Lawson homology. Thus, in some sense, we may view join product in morphic homology as providing some sort of "intersection theory" on singular varieties.

We briefly mention a possible refinement of this theory which might more fully reflect the geometry of singular varieties. We conclude by mentioning motivic analogues of the theory we have discussed.

As mentioned in Remark 3.6, a somewhat more sophisticated theory which we call "topological cycle cohomology" has been developed (partly in response to a question of Robert Lateveer following this lecture) which has particularly pleasing properties [F6].

Definition 3.1. *An algebraic s-cocycle on a variety U with values in an equidimensional variety Y is a cycle ζ on $U \times Y$ each irreducible component of which meets each $\{u\} \times Y$ in codimension s.*

The following proposition immediately implies that such cocycles are contravariant in U for U normal, covariant in Y for Y projective (with a shift in codimension).

Proposition 3.2. [FL-1;1.5] *If U is normal variety and Y is a projective variety of pure dimension n, then the abelian monoid of effective s-cocycles on U with values in Y can be naturally identified with the monoid of morphisms $Hom(U, \mathcal{C}_{n-s}(Y))$.*

To find a suitable topology on $Hom(U, \mathcal{C}_{n-s}(Y))$ is somewhat subtle, for we want the topology to be as "algebraic" as possible yet to provide invariants which have good formal properties (cf. [F4] for a discussion of relative merits of other possible topologies).

Definition 3.3. *Assume that U is a normal variety of pure dimension m and that Y is a projective variety of pure dimension n. Sending $f : U \to \mathcal{C}_{n-s}(Y)$ to its graph Γ_f determines a natural monomorphism*

$$Hom(U, \mathcal{C}_{n-s}(Y)) \ \to \ \mathcal{C}_{m+n-s}(U \times Y).$$

We provide $Hom(X, \mathcal{C}_{n-s}(Y))$ with the subspace topology for this embedding and define $Z^s(X, Y)$ to be the topological abelian group given as the the naïve group completion of the topological abelian monoid $Hom(U, \mathcal{C}_{n-s}(Y))$.

Similarly, if U is a normal variety of pure dimension m, then this graph construction determines a natural monomorphism

$$\mathcal{C}^s(U) \ \equiv \ Hom(U, \mathcal{C}_0(\mathbf{P}^s))/Hom(U, \mathcal{C}_0(\mathbf{P}^{s-1})) \ \to \ \mathcal{C}_m(U \times \mathbf{P}^s)/\mathcal{C}(U \times \mathbf{P}^{s-1}).$$

Once again, we provide $\mathcal{C}^s(U)$ with the subspace topology for this embedding and define $Z^s(X)$ to be the topological abelian group given as the the naïve group completion of the topological abelian monoid $\mathcal{C}^s(U)$.

The following proposition assures us that these topological abelian groups are reasonably well behaved from a topologist's point of view.

Proposition 3.4. [F-L2,C.3; F4,1.5,1.7] *Let U be a normal equidimensional variety and Y a projective equidimensional variety. The topology on $Hom(U, \mathcal{C}_{n-s}(Y))$ is characterizedd by the property that a sequence of maps $\{f_i; i \in \mathbf{N}\}$ converges if and only if both a.) the sequence converges in the compact open topology and b.) for some Zariski locally closed embedding $U \times Y \subset \mathbf{P}^N$, the graphs of the maps f_i have bounded degree. In particular, if $X = U$ is compact, then $Hom(X, \mathcal{C}_{n-s}(Y))$ has the compact-open topology.*

Furthermore, the topological abelian groups $Z^s(U, Y)$ and $Z^s(U)$ admit the structure of C-W complexes and are homotopy-theoretic group completions of the topological abelian monoids $\mathcal{C}^s(U, Y), \mathcal{C}^s(U)$.

The algebraic join $\# : C_{r,d}(Y) \times C_{r',d'}(Y') \to C_{r+r'+1,dd'}(Y \# Y')$ considered in §1 induces an external pairing on cocycles

$$Hom(U, \mathcal{C}_{n-s}(Y)) \times Hom(U', \mathcal{C}_{n'-s'}(Y')) \to Hom(U \times U', \mathcal{C}_{n-s+n'-s'+1}(Y \# Y')).$$

Taking U equal to U', $Y = \mathbf{P}^s$, and $Y' = \mathbf{P}^{s'}$, we obtain pairings

$$Z^s(U) \otimes Z^{s'}(U) \to [Hom(U, \mathcal{C}_1(\mathbf{P}^{s+s'+1}))/Hom(U, \mathcal{C}_1(\mathbf{P}^{s+s'}))]^+ \ \simeq \ Z^{s+s'}(U)$$

where the indicated equivalence is proven using the observation that the equivalence $Z_0(\mathbf{P}^{s+s'}) \ \to \ Z_1(\mathbf{P}^{s+s'+1})$ in the Lawson suspension theorem (Theorem 1.4) is determined by retractions which are algebraic.

Definition 3.5. *Let U be a normal equidimensional variety. Then the morphic cohomology groups of U are the homotopy groups of $Z^s(U)$, denoted as follows:*

$$L^sH^{2s-j}(U) \;\equiv\; \pi_j(Z^s(U)).$$

As above, the join pairing induces an associative pairing

$$\# : L^sH^{2s-j}(U) \otimes L^{s'}H^{2s'-j'}(U) \to L^{s+s'}H^{2s+2s'-j-j'}(U).$$

Remark 3.6. *In* [F6]*, we introduce a more sophisticated formulation of such a cohomology theory (which we call topological cycle cohomology) which agrees with morphic cohomology for smooth varieties. Topological cycle cohomology is defined for all varieties. The important property that it possesses which fails for the morphic cohomology of normal but not smooth varieties is excision. Topological cycle cohomology in conjunction with Lawson homology constitute a "Poincaré duality theory with supports" as axiomitized by S. Bloch and A. Ogus* [B-O].

In Remark 1.11, we observed that there is a natural map from Lawson homology to Borel-Moore homology. We have a similarly natural map from morphic cohomology to (singular) cohomology. Recall that $Z_0(\mathbf{A}^s)$ has the homotopy type of the Eilenberg-MacLane space $K(\mathbf{Z}, 2s)$ and thereby represents singular cohomology.

Proposition 3.7. [F-L1;6.3] *Let U be a normal equidimensional variety. The composition of natural homomorphisms of topological monoids*

$$Hom(U, \mathcal{C}_0(\mathbf{P}^s)) \to Hom_{cont}(U, Z_0(\mathbf{P}^s)) \to Hom_{cont}(U, Z_0(\mathbf{A}^s))$$

induces a continuous group homomorphism

$$\Phi : Z^s(U) \to Hom_{cont}(U, Z_0(\mathbf{A}^s))$$

whose map on homotopy groups in degree j is denoted

$$\Phi^{s,2s-j} : L^sH^{2s-j}(U) \to H^{2s-j}(U).$$

So defined, $\Phi^{*,*}$ is a ring homomorphism, where the product on morphic cohomology is the join pairing of (3.5.) and the product on integral singular cohomology is cup product.

As to be expected, algebraic vector bundles have chern classes with values in morphic cohomology. Namely, if E is a rank e algebraic vector bundle on U generated by $N+1$ global sections, then E determines

$$U \to Grass^e(\mathbf{P}^N) \to \mathcal{C}_{N-e}(\mathbf{P}^N).$$

If we define the total Segre class (equal to the inverse of the total chern class) of E to be the class of this map

$$s(E) \;\in\; \pi_0 Z^e(X, \mathbf{P}^N) \simeq \oplus_{i=0}^N L^iH^{2i}(U).$$

Mark Walker and the author have recently proved the desired Whitney sum formula for these Segre classes [F-W], sharpening the original verification of the author and Blaine Lawson in [F-L1].

We proceed to relate cocycle spaces to cycle spaces.

Definition 3.8. *Let X be a normal variety of pure dimension m and Y a projective variety of pure dimension n. Then the duality maps are the continuous homomorphisms*

$$\mathcal{D} : Z^s(U,Y) \ \to \ Z_{m+n-s}(U \times Y),$$
$$\mathcal{D} : Z^s(U) \ \to \ Z_m(X \times A^s) \ \simeq \ Z_{m-s}(X).$$

determined by sending $f : U \to \mathcal{C}_{n-s}(Y)$ to its graph Γ_f

The author and Blaine Lawson prove that in the case $U = X$ this duality map is compatible with cap product with the fundamental homology class $[X]$ of X in the sense that the following square commutes

$$\begin{array}{ccc} L^sH^{2s-j}(X) & \xrightarrow{\mathcal{D}_*} & L_{m-s}H_{2m-2s+j}(X) \\ \Phi\downarrow & & \downarrow\Phi \\ H^{2s-j}(X) & \xrightarrow{\cap[X]} & H_{2m-2s+j}(X) \end{array}$$

A proof of this can be found in the original duality paper [F-L2], whereas a more "geometric" proof is presented in [F-L4].

The following duality theorem was proved by Blaine Lawson and the author for smooth projective varieties in [F-L2] and for general smooth varieties by the author in [F-4]. The proof is an application of Theorem 2.8 which shows that a family of cycles parametrized by a sphere or the product of a sphere and an interval can be moved to a new family consisting of cocycles (i.e, each member of the new family intersects properly each fibre of the projection $X \times Y \to X$).

Theorem 3.9. *Let U be a smooth variety of dimension m and let Y be a smooth projective variety of dimension n. Then the duality maps*

$$\mathcal{D} : Z^s(U,Y) \ \to \ Z_{m+n-s}(U \times Y),$$
$$\mathcal{D} : Z^s(U) \ \to \ Z_m(X \times \mathbf{A}^s) \ \simeq \ Z_{m-s}(X)$$

are homotopy equivalences, inducing isomorphisms from morphic cohomology to Lawson homology groups

$$L^sH^{2s-j}(U,Y) \ \simeq \ L_{2m+2n-2s+j}(U \times Y)$$
$$L^sH^{2s-j}(U) \ \simeq \ L_{2m-2s+j}(U).$$

To complete the compatibility picture of join product in morphic cohomology and intersection product in Lawson homology, we state the following (whose proof should also ppear in [F-W]).

Proposition 3.10. *If U is a smooth connected variety of dimension m, then the follow square of topological abelian groups commutes up to homotopy for any s,t with $s+t \leq m$:*

$$\begin{array}{ccc} Z^s(U) \times Z^t(U) & \xrightarrow{\#} & Z^{s+t}(U) \\ \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ Z_{m-s}(U) \times Z_{m-t}(U) & \xrightarrow{\bullet} & Z_{m-s-t}(U). \end{array}$$

Let us briefly speculate how one might find intermediate theories analogous to intersection homology groups in our context. A different approach by Pawel Gajer can be found in [G]. Our idea is to relax the condition on a cocycle on X with values in Y that it meet each fibre $\{x\} \times Y$ properly. Of course, if we impose no condition on this intersection, we get arbitrary cycles which determine our (Lawson) homology theory.

Challenge. *Define a "middle perversity Lawson homology theory" for normal varieties X for which there is an intersection pairing.*

For example, one might try to proceed as follows. Let X be a normal variety of dimension m and let

$$X \supset X^{(1)} \supset X^{(2)} \supset \cdots$$

be a stratification of X with $X^{(i)} - X^{(i+1)}$ smooth of codimension i in X. Consider the subgroup

$$P^s(X) \ \subset \ Z_{2m-s}(X \times \mathbf{A}^m)$$

consisting of $(2m-s)$-cycles ζ on $X \times \mathbf{A}^m$ with the property that $|\zeta| \cap (X^{(i)} \times \mathbf{A}^m) \ \subset \ X^{(i)} \times \mathbf{A}^m$ has codimension greater or equal to $s + i/2$, where $|\zeta|$ denotes the support of the cycle ζ. The challenge is to provide a singular moving lemma which would enable us to define a pairing from $P^s(X) \times P^t(X)$ to $P^{s+t}(X)$.

We conclude our lectures by mentioning that the constructions we have presented have "motivic" analogues. For these analogues, the role of algebraic equivalence of cycles is replaced by rational equivalence. One important feature of these analogues is that they can be formulated for varieties over an arbitrary field k.

Once one is familiar with Lawson homology and morphic cohomology, then the "naïve" motivic versions are not difficult to formulate. We utilize Chow varieties as before. Rather than impose the analytic topology and view cycle spaces as topological abelian groups, we consider the "Suslin complex" of the algebraically defined monoid of effective cycles and group complete the resulting simplicial abelian monoid. This approach was briefly mentioned by Ofer Gabber and the author in [F-G], then taken somewhat more seriously in [F2]. For example, a naïve formulation of motivic cohomology $H^*(U, \mathbf{Z}(s))$ of a normal variety U over a field k is given as the cohomology of the chain complexes associated to the simplicial abelian groups

$$A^s(U) \ = \ [\frac{Hom(\Delta^\bullet \times U, \mathcal{C}_0(\mathbf{P}^s))}{Hom(\Delta^\bullet \times U, \mathcal{C}_0(\mathbf{P}^{s-1}))}]^+$$

where $\Delta^\bullet = \{n \mapsto \Delta^n\}$ is the cosimplicial variety with $\Delta^n = \operatorname{Spec} k[x_0, \ldots, x_n]/\Sigma x_i = 1$.

The reader interested in such constructions is invited to consult [F-V] where Vladimir Voevodsky and the author formulate motivic homology and cohomology using this point of view and prove a duality theorem analogous to Theorem 3.9. The usefulness of this latter duality theorem can be seen by the proof given in [F-V] that for smooth varieties motivic cohomology groups are isomorphic to the higher Chow groups of Spencer Bloch.

To prove various properties of motivic cohomology and homology, one needs to follow the sophisticated point of view of Andrei Suslin and Vladimir Voevodsky (cf. [F7]). The reader somewhat familiar with the constructions in these notes might find that [F6] serves as an introduction to some of the derived category formalism used by Suslin and Voevodsky.

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