Filtrations on Algebraic Cycles and Homology

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In recent years, there has been a renewed interest in obtaining invariants for an algebraic variety X using the Chow monoid $\mathcal{C}_r(X)$ of effective r-cycles on X. This began with the fundamental paper of Blaine Lawson [L] which introduced in the context of complex projective algebraic varieties the study of the homotopy groups of the group completion $Z_r(X)$ of $\mathcal{C}_r(X)$. The resultant *Lawson homology* has numerous good properties, most notably that reflected in the "Lawson suspension theorem." An algebraic version of Lawson's analytic approach was developed by the author in [F], permitting a study of projective varieties over arbitrary algebraically closed fields. Subsequent work has focussed on complex varieties: the author and Barry Mazur introduced operations in Lawson homology which led to interesting filtrations in (singular) homology [F-Mazur]; the author and Blaine Lawson introduced a bivariant theory with the purpose of constructing a cohomology theory associated to cycle spaces [F-Lawson]; Paulo Lima-Filho [Lima-Filho] (see also [F-Gabber]) extended Lawson homology to quasi-projective varieties; and the author and Ofer Gabber established an intersection theory in Lawson homology [F-Gabber]. A related theory, the "algebraic bivariant cycle complex" introduced in [F-Gabber], is applicable to quasi-projective varieties over an arbitrary field and bears some resemblance to certain candidates for motivic cohomology.

In this paper, we return to the study of filtrations for complex projective algebraic varieties begun in [F-Mazur]. We consider the filtration on algebraic cycles given by kernels of iterates of an operation (the so-called s-operation) in Lawson homology introduced in [F-Mazur]. Theorem 3.2 gives an alternate description of this "S-filtration" in terms of correspondences. We also consider the "topological filtration" on homology given by images of iterates of the s-operation. For example, Proposition 4.2 demonstrates that the equality of this filtration with a geometric filtration considered by A. Grothendieck is implied by one of Grothendieck's "Standard Conjectures", Grothendieck's Conjecture B.

In order to pursue our analysis of these filtrations, we begin in section 1 with a detailed investigation of alternate formulations of the s-operation. This discussion relies heavily on the foundational work of [F-Gabber]. We continue this analysis in section 2 with the introduction of a "graph mapping" on cycles associated to a "Chow correspondence". The application of this mapping in later sections as well as its occurence in [F-Mazur2] suggests that this construction codifies a fundamental aspect of the functoriality of algebraic cycles. We anticipate that these somewhat foundational sections will prove useful in future developments of Lawson homology.

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Section 3 compares the S-filtration on cycles to filtrations considered by Madhav Nori in [Nori] and by Spencer Bloch and Arthur Ogus in [Bloch-Ogus]. The S-filtration is subordinate to the Bloch-Ogus filtration and dominates Nori's filtration. In fact, we show that the S-filtration has a description in terms of correspondences exactly parallel to that of Nori's, except that our correspondences are permitted to have singular domain.

In [F-Mazur], the topological filtration associated to images of the s-operation was introduced, shown to be subordinate to Grothendieck's geometric filtration (for smooth projective varieites), and equality of these filtrations was conjectured. In section 4, we reconsider this conjecture. In particular, we present a proof of an unpublished result of R. Hain [Hain] asserting the equality of these filtrations for "sufficiently general" abelian varieties. Our proof, somewhat different from Hain's original proof, fits in the general context of a study of the inverse of the Lefschetz operator whose algebraicity is the content of Grothendieck's Conjecture B.

We conclude this paper by presenting in section 5 a spectral sequence which codifies the relationship between algebraic cycles and homology as seen from the point of view of iterates of the s-operation. In particular, both the S-filtration on cycles and the topological filtration on homology appear in this spectral sequence.

Thoughout this paper, we restrict our attention to complex, quasi-projective algebraic varieties.

This work is an outgrowth of numerous discussions. The influence of Ofer Gabber is evident throughout. The example of abelian varieties is due to Dick Hain. Most importantly, our understanding of operations and filtrations evolved through many discussions with Barry Mazur. We gratefully thank the interest and support of each of these friends.

1. The s-operation revisited

In this section, we recall the s-operation in Lawson homology introduced in [F-Mazur] and further considered in [F-Gabber]. As in the latter paper, we view this operation as the map in homology associated to the "s-map", a map in the derived category of chain complexes of abelian groups. The central result of this section is Theorem 1.3 which establishes three alternate formulations of this operation. We point out in Proposition 1.6 that the cycle map is factorized by the s-operation not just for projective varieties but also for quasi-projective varieties. Proposition 1.7 verifies expected naturality properties of our s-operation.

We begin by recalling the cycle spaces $Z_r(U)$ and cycle complexes $\tilde{Z}_r(U)$ of r-cycles on a quasi-projective variety U. The independence of $Z_r(U), \tilde{Z}_r(U)$ of the choice of projective closure $U \subset X \subset \mathbf{P}^N$ is verified in [Lima-Filho] and also [F-Gabber].

Definition 1.1. Let X be a complex projective variety and r a non-negative integer. The *Chow monoid* $\mathcal{C}_r(X)$ is the disjoint union of the Chow varieties $C_{r,d}(X)$ of effective r-cycles on X of degree d for some non-negative integer d. The cycle space $Z_r(X)$ is defined to be the topological abelian group given as the group completion of the abelian monoid $\mathcal{C}_r(X)$ provided with the quotient topology associated to the surjective map $\mathcal{C}_r(X)^2 \to Z_r(X)$, where $\mathcal{C}_r(X)^2$ is given the analytic topology. If $Y \subset X$ is a closed subvariety, we define

$$Z_r(U) \equiv Z_r(X)/Z_r(Y) \quad , \quad U = X - Y.$$

The normalized chain complex of the simplicial abelian group $\mathrm{Sing.}(Z_r(U))$ of singular chains on the topological group $Z_r(U)$ will be denoted by $\tilde{Z}_r(U)$. Finally, we define the *Lawson homology groups* of U to be

$$L_rH_n(U) \ \equiv \ \pi_{n-2r}(Z_r(U)) \ \simeq \ H_{n-2r}(\tilde{Z}_r(U)).$$

The above definition of Lawson homology groups is that given in [F-Gabber]. In [F], Lawson homology groups were defined for complex projective varieties as the homotopy groups of the homotopy theoretic group completion $\Omega B\mathcal{C}_r(X)$ of the Chow monoid $\mathcal{C}_r(X)$ viewed as a topological monoid. This was shown in [Lima-Filho], [F-Gabber] to be naturally homotopy equivalent to $Z_r(X)$. In [F-Mazur], the Lawson homology groups were viewed (using [F;2.6]) as the homotopy groups of the direct limit $\mathrm{LimSing.}\mathcal{C}_r(X)$ of copies of the simplicial abelian monoid $\mathrm{Sing.}\mathcal{C}_r(X)$ of singular simplices of the Chow monoid, where the direct limit is indexed by a "base system" associated to $\pi_O(\mathcal{C}_r(X))$. We shall frequently reference [F], [F-Mazur] for properties of $Z_r(X)$, $\tilde{Z}_r(X)$ which have been proved in [F], [F-Mazur] for either $\Omega B\mathcal{C}_r(X)$ or $\mathrm{LimSing.}\mathcal{C}_r(X)$.

The localization theorem of [Lima-Filho], [F-Gabber] asserts that

$$\tilde{Z}_r(Y) \to \tilde{Z}_r(X) \to \tilde{Z}_r(U)$$

is a distinguished triangle whenever Y is a closed subvariety of X with complement $U = X - Y$. In other words, the short exact sequence of topological abelian groups $Z_r(Y) \to Z_r(X) \to Z_r(X)/Z_r(Y)$ yields a long exact seqeunce in homotopy groups.

In [F-Mazur], operations were introduced on the Lawson homology groups using the geometric construction of the "join" of two cycles. Namely, if $V \subset \mathbf{P}^M$ and $W \subset \mathbf{P}^N$ are closed subvarieties of disjoint projective spaces, then we may view $\mathbf{P}^{M+N+1}$ as consisting of all points on (projective) lines from points on $\mathbf{P}^M$ to points on $\mathbf{P}^N$ and we define the join $V\#W \subset \mathbf{P}^{M+N+1}$ to be the subvariety of those points lying on lines between points of V and points of W. The initial formulation of the operations for a projective variety X was in terms of the join pairings of effective algebraic cycles $C_{r,d}(X) \times C_{j,e}(\mathbf{P}^t) \to C_{r+j+1,de}(X\#\mathbf{P}^t)$ inducing

$$Z_r(X) \times Z_j(\mathbf{P}^t) \to Z_{r+j+1}(X\#\mathbf{P}^t) \simeq Z_{r-t+j}(X) \quad , \quad r-t+j \geq 0$$

where the right-hand equivalence is that given by the Lawson suspension theorem. In particular, the s-operation was defined to be the map in homotopy groups obtained from the induced pairing on homotopy groups

$$\pi_{n-2r}(Z_r(X)) \otimes \pi_2(Z_0(\mathbf{P}^1)) \to \pi_{n-2r+2}(Z_{r-1}(X)) \quad , \quad r > 0$$

by restricting to the canonical generator of $\pi_2(Z_0(\mathbf{P}^1))$:

$$s : L_rH_n(X) \to L_{r-1}H_n(X).$$

Mapping $\mathbf{P}^1$ to $Z_0(\mathbf{P}^1)$ by sending a point $p \in \mathbf{P}^1$ to $p - \{\infty\}$, we obtain an "s-map"

$$Z_r(X) \wedge \mathbf{P}^1 \to Z_{r-1}(X)$$

well defined up to homotopy which determines the s-operation.

Observe that $Z_0(\mathbf{P}^1)_{deg0}$ is quasi-isomorphic to $\mathbf{Z}[2]$, the chain complex whose only non-zero term is a $\mathbf{Z}$ in in degree 2. Consequently, the above map $\mathbf{P}^1 \to Z_0(\mathbf{P}^1)_{deg0} \subset Z_0(\mathbf{P}^1)$ determines a map in the derived category $\mathbf{Z}[2] \to Z_0(\mathbf{P}^1)$ which depends only upon the choice of quasi-isomorphism $\mathbf{Z}[2] \simeq Z_0(\mathbf{P}^1)_{deg0}$. Thus, with somewhat more precision, we may view the s-map as a map (well defined in the derived category)

$$s : \tilde{Z}_r(X)[2] = \tilde{Z}_r(X) \otimes \mathbf{Z}[2] \to \tilde{Z}_{r-1}(X)$$

obtained by restricting the join pairing

$$\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^1) \to \tilde{Z}_{r+1}(X \# \mathbf{P}^1) \simeq \tilde{Z}_{r-1}(X)$$

via $\mathbf{Z}[2] \to Z_0(\mathbf{P}^1)$.

The naturality of this construction permits one to extend the definition of the s-operation to the Lawson homology of quasi-projective varieties. Namely, if $Y \subset X$ is a closed subvariety of the projective variety X, then the join operation determines pairings

$$Z_r(X)/Z_r(Y) \times Z_j(\mathbf{P}^t) \to Z_{r+j+1}(X \# \mathbf{P}^t)/Z_{r+j+1}(Y \# \mathbf{P}^t).$$

So defined, the s-map determines a map of distinguished triangles

$$\begin{array}{ccccc} \tilde{Z}_r(Y)[2] & \to & \tilde{Z}_r(X)[2] & \to & \tilde{Z}_r(U)[2] \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Z}_{r-1}(Y) & \to & \tilde{Z}_{r-1}(X) & \to & \tilde{Z}_{r-1}(U) \end{array}$$

Proposition 1.2 *If X is connected and smooth of dimension $n > 0$, then $H_2(\tilde{Z}_{n-1}(X))$ is naturally isomorphic to $\mathbf{Z}$ with canonical generator determined by any pencil of divisors coming from a 2-dimensional space of sections of a line bundle on some smooth projective closure of X. Similarly, if X is an irreducible, projective variety of dimension $n > 0$, then $\pi_2(Div(X)^+)$ is naturally isomorphic to $\mathbf{Z}$, where $Div(X)^+$ denotes the homotopy theoretic group completion of the abelian topological monoid of effective Cartier divisors.*

Consequently, a choice of quasi-isomorphism $\mathbf{Z}[2] \simeq Z_0(\mathbf{P}^1)_{deg0}$ determines a natural map $\mathbf{Z}[2] \to \tilde{Z}_{n-1}(X)$ which induces an isomorphism in H_2 in the first case. In the second case, there is a natural homotopy class of maps $\mathbf{P}^1 \to Div(X)^+$ inducing an isomorphism in π_2 which is independent of the choice of pencil of divisiors.

Proof. Assume X is connected and smooth of dimension $n > 0$. We choose a projective closure $X \subset \overline{X} \subset \mathbf{P}^N$ such that $\overline{X}$ is smooth. Since $Z_{n-1}(Y)$ is discrete where $Y = \overline{X} - X$, we conclude that $Z_{n-1}(\overline{X}) \to Z_{n-1}(X)$ induces an isomorphism

$$H_2(Z_{n-1}(\overline{X})) \simeq \pi_2(Z_{n-1}(\overline{X})) \to \pi_2(Z_{n-1}(X)) \simeq H_2(Z_{n-1}(X)).$$

By [F;4.5], if L is any line bundle on $\overline{X}$ and $\mathbf{P}^1 \subset Proj(\Gamma(L))$ is any pencil of divisors, then

$$\mathbf{P}^1 \to Proj(\Gamma(L)) \to Z_{n-1}(\overline{X})$$

determines a quasi-isomorphism $\mathbf{Z} \simeq \pi_2(Z_{n-1}(\overline{X}))$. We conclude that

$$\mathbf{Z}[2] \simeq \tilde{Z}_0(\mathbf{P}^1)_{deg0} \to \tilde{Z}_{n-1}(\overline{X}) \to \tilde{Z}_{n-1}(X)$$

induces an isomorphism in H_2; as a map in the derived category, this map is independent of the choice of pencil of divisors.

If X projective and irreducible, the proof of [F;4.5] shows that $Div(X)^+$ fits in a fibration sequence

$$\mathbf{P}^\infty \to Div(X)^+ \to Pic(X)$$

so that $\pi_2(Div(X)^+) \simeq \mathbf{Z}$. Once again, the generator of $\pi_2(Div(X)^+) \simeq \pi_2(\mathbf{P}^\infty)$ is determined by any pencil of divisors.

In the following theorem, we present other formulations of the s-map involving intersection products introduced in [F-Gabber]. As remarked in [F-Gabber], these alternate formulations establish the non-obvious property that $s : \tilde{Z}_r(X)[2] \to \tilde{Z}_{r-1}(X)$ (as a map in the derived category) is independent of the projective embedding of X. We gratefully thank O. Gabber for suggesting the proof of 1.3.b) presented below, a simplification of our original proof.

Theorem 1.3. *Let X be a complex, quasi-projective variety and r a postive integer.*
a.) *The s-map equals (in the derived category) the map defined by restricting the following pairing*

$$i_X^! \circ \times : \tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^1) \to \tilde{Z}_{r-1}(X)$$

to $\tilde{Z}_0(\mathbf{P}^1)_{deg0} \simeq \mathbf{Z}$, where $i_X : X \subset X \times \mathbf{P}^1$ embeds X as the divisor $X \times \infty$, where $i_X^!$ is the Gysin map associated to this divisor, and where $\times : Z_r(X) \times Z_0(\mathbf{P}^1) \to Z_r(X \times \mathbf{P}^1)$ sends (Z,p) to $Z \times \{p\}$. In particular, as a map in the derived category, the s-map is independent of choice of projective closure $X \subset \overline{X}$ and of projective embedding $\overline{X} \subset \mathbf{P}^n$.
b.) *If X is connected and smooth of dimension $n > 0$, then the s-map equals (in the derived category) the restriction of the intersection pairing*

$$\tilde{Z}_r(X) \otimes \tilde{Z}_{n-1}(X) \to \tilde{Z}_{r-1}(X)$$

via the map $\mathbf{Z}[2] \to \tilde{Z}_{n-1}(X))$ of (1.2).
c.) *If X is an irreducible, projective variety of dimension $n > 0$, then the homotopy class of the s-map is determined by the restriction of the intersection pairing*

$$Z_r(X)) \wedge Div(X)^+ \to Z_r(X)$$

via the map $\mathbf{P}^1 \to Div(X)^+$ of (1.2).

Proof. For X projective, the equality of the s-operation with $i_X^! \circ \tilde{\omega}$ is proved in [F-Gabber;2.6], which immediately implies for X projective that the s-map is independent of projective embedding. The proof given applies equally well to X quasi-projective, once one replaces cycles spaces $Z_*(X)$ by the appropriate quotient spaces $Z_*(\overline{X})/Z_*(Y)$, where $X \subset \overline{X}$ is a projective closure with complement Y. Moreover, the proof that $\tilde{Z}_r(X)$ is independent of a choice of compactification $X \subset \overline{X}$ (up to natural isomorphism in the derived category) is achieved by dominating any two compactifications by a third; the naturality of $i^!_{\overline{X}} \circ \times$ easily enables one to extend this argument to show that the s-map is likewise independent of a choice of compactification.

To prove b.), we consider the following diagram

$$\begin{tikzcd}
\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^1) \arrow[r, "\times"] \arrow[d, "i_* \times 1"] & \tilde{Z}_r(X \times \mathbf{P}^1) \arrow[r, "i^!"] \arrow[d, "(i \times 1)_*"] & \tilde{Z}_{r-1}(X) \arrow[d, "i_*"] \\
\tilde{Z}_r(X \times \mathbf{P}^1) \otimes \tilde{Z}_0(\mathbf{P}^1) \arrow[r, "\times"] \arrow[d, "1 \times pr_2^*"] & \tilde{Z}_r((X \times \mathbf{P}^1) \times \mathbf{P}^1) \arrow[r, "\delta^!_{\mathbf{P}}"] \arrow[d, "(1 \times pr_2)^*"] & \tilde{Z}_{r-1}(X \times \mathbf{P}^1) \arrow[d, "="] \\
\tilde{Z}_r(X \times \mathbf{P}^1) \otimes \tilde{Z}_n(X \times \mathbf{P}^1) \arrow[r, "\times"] & \tilde{Z}_{n+r}((X \times \mathbf{P}^1) \times (X \times \mathbf{P}^1)) \arrow[r, "\Delta^!_{X \times P}"] & \tilde{Z}_{r-1}(X \times \mathbf{P}^1) \\
\tilde{Z}_r(X \times \mathbf{P}^1) \otimes \tilde{Z}_{n-1}(X) \arrow[r, "\times"] \arrow[u, "1 \times pr_1^*"] & \tilde{Z}_{n+r-1}((X \times \mathbf{P}^1) \times X) \arrow[r, "\delta^!_X"] \arrow[u, "(1 \times pr_1)^*"] & \tilde{Z}_{r-1}(X \times \mathbf{P}^1) \arrow[u, "="] \\
\tilde{Z}_r(X) \otimes \tilde{Z}_{n-1}(X) \arrow[r, "\times"] \arrow[u, "i_* \times 1"] & \tilde{Z}_{n+r-1}(X \times X) \arrow[r, "\Delta^!_X"] \arrow[u, "(i \times 1)_*"] & \tilde{Z}_{r-1}(X) \arrow[u, "i_*"]
\end{tikzcd}$$

Since $i_* : Z_r(X) \to Z_r(X \times \mathbf{P}^1)$ admits a right inverse (namely, pr_{1*}), we conclude using the naturality of the isomorphism $H_2(\tilde{Z}_{n-1}(X))$ that it suffices to verify the commutativity of this diagram (in the derived category). The commutativity of the left squares follow from the naturality of the push-forward (for proper maps) and pull-back (for flat maps) functoriality of $\tilde{Z}_r(X)$. The commutativity of the top and bottom right squares follows from [F-Gabber;3.4.d] applied to the proper maps $\delta_P : X \times \mathbf{P}^1 \to X \times \mathbf{P}^1 \times \mathbf{P}^1$ and $\Delta_X : X \to X \times X$. Finally, the commutativity of the middle right squares follows from [F-Gabber;3.4.d] applied to the flat maps $1 \times pr_2 : (X \times \mathbf{P}^1) \times (X \times \mathbf{P}^1) \to X \times \mathbf{P}^1 \times \mathbf{P}^1$ and $1 \times pr_1 : (X \times \mathbf{P}^1) \times (X \times \mathbf{P}^1) \to X \times \mathbf{P}^1 \times X$.

Part c.) is proved in [F-Gabber;3.1]

The first part of the following corollary is a consequence of (1.3.b), the second of [F;3.5]

Corollary 1.4. *Let $\mathcal{C}_{r,\leq d}(X)$ denote the submonoid of $\mathcal{C}_r(X)$ of effective r-cycles on X of degree $\leq d$ with respect to some locally closed embedding $X \subset \mathbf{P}^N$.*

a.) *Assume X is smooth and connected of dimension $n > 0$. If $\mathbf{P}^1 \simeq P \subset Proj(\Gamma(X, O(e))$ is a sufficiently general pencil of effective divisors of degree $e >> d$ with chosen base point $E \in P$, then*

$$\mathcal{C}_{r,\leq d}(X) \wedge P \to Z_{r-1}(X)$$

sending (Z,D) to $Z\cdot D - Z\cdot E$ is homotopic to the restriction of the s-map via $\mathcal{C}_{r,\le d}(X)\subset Z_{r-1}(X)$.

b.) *Assume X is projective of dimension $n>0$. Then for any $d>0$ and all $e>>d$, there exists a continuous algebraic map*

$$\mathcal{C}_{r,\le d}(X)\times \mathbf{P}^1 \to \mathcal{C}_{r-1,\le de}(X)$$

which fits in a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathcal{C}_{r,\le d}(X)\times \mathbf{P}^1 & & \to & & \mathcal{C}_{r-1,\le de}(X) \\
\downarrow & & & & \downarrow \\
Z_r(X)\wedge Z_0(\mathbf{P}^1) & \xrightarrow{s} & Z_{r-1}(X) & \xrightarrow{e} & Z_{r-1}(X)
\end{array}$$

whose vertical arrows are induced by the natural inclusions.

Proof. A proof that the generic divisor of degree $e>>0$ meets every cycle of $\mathcal{C}_{r,\le d}(X)$ properly is given in [Lawson;5.11]. Consequently, a.) follows from Theorem 1.3.b and [F-Gabber;3.5.a] (which asserts that the restriction of the intersection pairing to cycles which intersect properly is homotopic to the ususal intersection product).

As defined in [F-Mazur] (for X projective), the s-map is induced by the composition $Z_r(X)\times\mathbf{P}^1\to Z_{r+1}(X\#\mathbf{P}^1)\to Z_{r-1}(X)$. By its very definition, the first map when restricted to $C_{r,d}(X)$ is given by a continuous algebraic map $C_{r,d}(X)\times\mathbf{P}^1\to C_{r+1,d}(X\#\mathbf{P}^1)$. As shown in [F;3.5] for any projective variety Y, every sufficiently large multiple M of the inverse of the Lawson suspension isomorphism $\pi_*(Z_s(Y))\to\pi_*(Z_{s+1}(\Sigma Y))$ when restricted to $C_{s+1,d}(\Sigma Y)$ is represented by a continuous algebraic map $C_{s+1,d}(\Sigma Y)\to C_{s,dM}(Y)$ in the sense that when composed with the inclusion $C_{s,dM}(Y)\to Z_s(Y)$ the map is homotopic to the restriction to $C_{s+1,d}(\Sigma Y)$ of $M\cdot\Sigma^{-1}: Z_{s+1}(X)\to Z_s(X)\to Z_s(X)$. This implies the second assertion of the corollary.

In each of the maps of Corollary 1.5 below, one obtains s^j by additively extending the domain of the indicated composition from $Z_r(X)\times P$ to $Z_r(X)\times Z_0(P)$ and then restricting to the appropriate factor of the Eilenberg-MacLane space $Z_0(P)$.

Corollary 1.5. *Let X be a complex, quasi-projective variety of positive dimension and let $r\ge j>0$. Then the s-map is induced by each of the following compositions:*
i.) $Z_r(X)\times(\mathbf{P}^1)^{\times j}\to Z_r(X\times(\mathbf{P}^1)^{\times j})\to Z_{r-j}(X)$.
ii.) $Z_r(X)\times\mathbf{P}^j\to Z_r(X\times\mathbf{P}^j)\to Z_{r-j}(X)$.
iii.) $Z_r(X)\times\mathbf{P}^j\to Z_{r+1}(X\#\mathbf{P}^j)\to Z_{r-j}(X)$ *provided that X is projective.*
iv.) $Z_r(X)\times(\mathbf{P}^1)^{\times j}\to Z_{r+j}(X\#(\mathbf{P}^1)^{\#j})\to Z_{r-j}(X)$ *provided that X is projective.*

In i.) and ii.), the left maps are given by product and the right maps are Gysin maps for the appropriate regular immersion; in iii.) and iv.), the left maps are given by the algebraic join and the right maps by the Lawson suspension theorem.

Proof. The fact that s^j is given by iv.) follows directly from its (original) definition of the s-map in terms of the join pairing and the inverse of Lawson suspension. That i.) also determines s^j follows from (1.3.a.).

To prove ii.), we proceed as follows. Let $P \subset \mathbf{P}^j \times (\mathbf{P}^1)^{\times j}$ be the closure of the graph of the birational map relating $\mathbf{P}^j$ to $(\mathbf{P}^1)^{\times j}$. We employ the following diagram, commutative in the derived category, to equate the maps given by i.) and ii.):

$$\begin{tikzcd}
\tilde{Z}_r(X) \otimes \tilde{Z}_0((\mathbf{P}^1)^{\times j}) \arrow[r] & \tilde{Z}_r(X \times (\mathbf{P}^1)^{\times j}) \arrow[r] & \tilde{Z}_{r-j}(X) \\
\tilde{Z}_r(X) \otimes \tilde{Z}_0(P) \arrow[r] \arrow[u] \arrow[d] & \tilde{Z}_r(X \times P) \arrow[r] \arrow[u] \arrow[d] & \tilde{Z}_{r-j}(X) \arrow[u] \arrow[d] \\
\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j) \arrow[r] & \tilde{Z}_r(X \times \mathbf{P}^j) \arrow[r] & \tilde{Z}_{r-j}(X)
\end{tikzcd}$$

Finally, we show that iii.) also determines the s-map. Iterating the argument of [F-Gabber;2.6] j times, we see that the composition of the maps

$$\tilde{Z}_r(X) \otimes \tilde{Z}_0((\mathbf{P}^1)^{\times j}) \to \tilde{Z}_r(X \times (\mathbf{P}^1)^{\times j}) \to \tilde{Z}_{r-i}(X \times (\mathbf{P}^1)^{\times j-i}) \to \tilde{Z}_{r-i}(X)$$

is trivial for $i < j$ when restricted to $\tilde{Z}_r(X) \otimes \tilde{Z}_0((\mathbf{P}^1)_{deg 0})^{\otimes j}$. Using common blow-ups $P_i \to (\mathbf{P}^1)^{\times j-i}, P_i \to \mathbf{P}^{j-i}$ as for ii.), we conclude that the composition

$$\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j) \to \tilde{Z}_r(X \times \mathbf{P}^j) \to \tilde{Z}_{r-i}(X \times \mathbf{P}^{j-i}) \to \tilde{Z}_{r-i}(X)$$

is also trivial for $i < j$ when restricted to the summand of $\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j)$ given by the natural splitting of the projection $\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j) \to \tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j)/\tilde{Z}_0(\mathbf{P}^{j-1})$. In other words, if we abuse notation and denote this summand by $\tilde{Z}_r(X) \otimes \tilde{Z}_0(\mathbf{P}^j)/\tilde{Z}_0(\mathbf{P}^{j-1})$, then we conclude that $\tilde{Z}_r(X) \times \tilde{Z}_0(\mathbf{P}^j)/\tilde{Z}_0(\mathbf{P}^{j-1}) \to \tilde{Z}_r(X \times \mathbf{P}^j)$ is homotopic to its composition with $pr_1^* \circ i^! : \tilde{Z}_r(X \times \mathbf{P}^j) \to \tilde{Z}_{r-j}(X) \to \tilde{Z}_r(X \times \mathbf{P}^j)$. Now consider the following diagram

$$\begin{tikzcd}
Z_r(X) \times Z_0(\mathbf{P}^j) \arrow[r, "\#"] \arrow[d, "\times"'] & Z_{r+1}(X \# \mathbf{P}^j) & Z_{r-j}(X) \arrow[l, "\Sigma^{j+1}"'] \\
Z_r(X \times \mathbf{P}^j) \arrow[r, "\pi^*"] \arrow[d, shift right, "i^!"'] & Z_{r+1}(W) \arrow[u, "p_*"'] & \\
Z_{r-j}(X) \arrow[u, shift right, "pr_1^*"'] & &
\end{tikzcd}$$

where $X \subset \mathbf{P}^N$ is a projective embedding and W is the closed subset of $X \times \mathbf{P}^j \times \mathbf{P}^{N+j+1}$ consisting of triples (x, y, z) with z lying on the line from x to y. The maps $\pi : W \to X \times \mathbf{P}^j$ and $p : W \to X \# \mathbf{P}^j \subset \mathbf{P}^{N+j+1}$ are the projections; the square is easily seen to commute. We easily verify that

$$p_* \circ \pi^* \circ pr_1^* = \Sigma^{j+1} : Z_{r-j}(X) \to Z_{r+1}(X \# \mathbf{P}^j).$$

This, together with the preceding verification, implies the equality of the compositions in ii.) and iii.) when restricted to $\tilde{Z}_r(X) \times \tilde{Z}_0(\mathbf{P}^j)/\tilde{Z}_0(\mathbf{P}^{j-1})$.

One important property of the s-map is that it factors the cycle map to homology. For X projective, this is one of the basic properties proved in [F-Mazur]; for X quasi-projective, this is a conseqeunce of naturality as we make explicit in the following proposition.

Proposition 1.6. *Let X be a quasi-projective variety. Then the cycle map*

$$\gamma : Z_r(X) \to H_{2r}^{BM}(X)$$

sending an algebraic r-cycle Z to its class $\gamma(Z)$ in the 2r-th Borel-Moore homology group of X is given by the following composition

$$Z_r(X) \to \pi_0(Z_r(X)) \to \pi_{2r}(Z_0(X)) \simeq H_{2r}^{BM}(X)$$

induced by the adjoint $Z_r(X) \to \Omega^{2r} Z_0(X)$ of $s^r : Z_r(X) \wedge (\mathbf{P}^1)^{\wedge r} \to Z_0(X)$.

Proof. Let $X \subset \overline{X}$ be a projective closure with complement Y. Then $\overline{X}/Y$ is a one-point compactification of X and $Z_0(\overline{X})/Z_0(\overline{Y}) \simeq Z_0(\overline{X}/Y)/Z_0(\{\infty\})$. Consequently, the Dold-Thom theorem applied to $\overline{X}/Y$ implies the natural isomorphisms

$$\pi_0(\Omega^{2r} Z_0(X)) \simeq \pi_{2r}(Z_0(X)) \simeq \tilde{\pi}_{2r}(Z_0(\overline{X}/Y)) \simeq \tilde{H}_{2r}(\overline{X}/Y) \simeq H_{2r}^{BM}(X).$$

Thus, the proposition follows from the commutative square

$$\begin{array}{ccc} Z_r(\overline{X}) \wedge (\mathbf{P}^1)^{\wedge r} & \xrightarrow{s^r} & Z_0(\overline{X}) \\ \downarrow & & \downarrow \\ Z_r(X) \wedge (\mathbf{P}^1)^{\wedge r} & \xrightarrow{s^r} & Z_0(X) \end{array}$$

and the surjectivity of $Z_r(\overline{X}) \to Z_r(X)$.

We conclude this section by verifying some basic naturality properties of the s-map.

Proposition 1.7. *Let X be a quasi-projective complex variety and r a positive integer.*
a.) *If $f : X \to Y$ is a proper map of varieties, then the following square commutes (in the derived category):*

$$\begin{array}{ccc} \tilde{Z}_r(X) & \xrightarrow{s} & \tilde{Z}_{r-1}(X)[-2] \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \tilde{Z}_r(Y) & \xrightarrow{s} & \tilde{Z}_{r-1}(Y)[-2] \end{array}$$

b.) *If $g : X' \to X$ is a flat map of varieties of pure relative dimension c, then the following square commutes (in the derived category):*

$$\begin{array}{ccc} \tilde{Z}_r(X) & \xrightarrow{s} & \tilde{Z}_{r-1}(X)[-2] \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \tilde{Z}_{r+c}(X') & \xrightarrow{s} & \tilde{Z}_{r+c-1}(X')[-2] \end{array}$$

c.) *If X is smooth of pure dimension n and if r' is a positive integer with $r + r' > n$, then the following square commutes (in the derived category):*

$$\begin{array}{ccc} \tilde{Z}_r(X) \otimes \tilde{Z}_{r'}(X) & \stackrel{1\otimes s}{\rightarrow} & \tilde{Z}_r(X) \otimes \tilde{Z}_{r'-1}(X) \\ \bullet \downarrow & & \downarrow \bullet \\ \tilde{Z}_{r+r'-n}(X) & \stackrel{s}{\rightarrow} & \tilde{Z}_{r+r'-n-1}(X) \end{array}$$

where $\bullet$ denotes the intersection product of [F-Gabber].

Proof. To prove (a.), it suffices to observe that f_* induces a commutative diagram of cycle spaces

$$\begin{array}{ccccc} Z_r(X) \times Z_0(\mathbf{P}^1) & \rightarrow & Z_r(X \times \mathbf{P}^1) & \leftarrow & Z_{r-1}(X) \\ f_* \times 1 \downarrow & & (f \times 1)_* \downarrow & & \downarrow f_* \\ Z_r(Y) \times Z_0(\mathbf{P}^1) & \rightarrow & Z_r(Y \times \mathbf{P}^1) & \leftarrow & Z_{r-1}(Y) \end{array} .$$

The proof of (b.) is similar. By [F-Gabber;3.5], the diagram

$$\begin{array}{ccc} \tilde{Z}_r(X) \otimes \tilde{Z}_{r'}(X) \otimes \tilde{Z}_{n-1}(X) & \stackrel{1\times\bullet}{\rightarrow} & \tilde{Z}_r(X) \otimes \tilde{Z}_{r'-1}(X) \\ \bullet \otimes 1 \downarrow & & \downarrow \bullet \\ \tilde{Z}_{r+r'-n}(X) \otimes \tilde{Z}_{n-1}(X) & \stackrel{\bullet}{\rightarrow} & \tilde{Z}_{r+r'-n-1}(X) \end{array}$$

commutes in the derived category. Thus, (c.) follows by applying Theorem 1.3.b).

2. Graph mappings associated to Chow correspondences

A "Chow correspondence" from Y to X of relative dimension r is a continuous algebraic map $f : Y \to \mathcal{C}_r(X)$. Such a map determines a cycle Z_f on $Y \times X$ equidimensional over Y of relative dimension r (cf. [F-Mazur2] for an extensive discussion). In this section, we investigate the "graph mapping"

$$\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$$

induced by a Chow correspondence$f : Y \to \mathcal{C}_r(X)$. The key ingredient in the definition of Γ_f is the "trace map"

$$tr : \mathcal{C}_k(\mathcal{C}_r(X)) \to \mathcal{C}_{r+k}(X)$$

introduced in [F-Lawson;7.1]. This is defined to send an irreducible subvariety $W \subset \mathcal{C}_r(X)$ of dimension k to $pr_{X*}(Z_W)$, where Z_W is the cycle on $W \times X$ given as the correspondence equidimensional over W associated to the inclusion morphism $W \subset \mathcal{C}_r(X)$. We also consider composition of Chow correspondences, associating to continuous algebraic maps $f : Y \to \mathcal{C}_r(X)$, $g : X \to \mathcal{C}_s(T)$ a continuous algebraic map $g \cdot f : Y \to \mathcal{C}_{r+s}(T)$.

In more detail, the section begins with a definition of the graph mapping and presents an intersection-theoretic interpretation for smooth varieties. This interpretation encompasses intersection with correspondences not necessarily equidimensional over their domain. The result of most interest in this section is Theorem 2.4, a corollary of which exhibits for a given cycle a cycle which is algebraically equivalent to a multiple of the original cycle and which is equidimensional over a projective space. The section ends with a verification that the graph mapping construction commutes with compositions.

Definition 2.1. Let Y, X be projective algebraic varieties and let $f : Y \to \mathcal{C}_r(X)$ be a Chow correspondence. We define the *graph mapping* associated to f

$$\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$$

to be the group completion of the composition

$$tr \circ f_* : \mathcal{C}_k(Y) \to \mathcal{C}_k(\mathcal{C}_r(X)) \to \mathcal{C}_{r+k}(X)$$

where f_* is the map functorially induced by f (cf. [F;2.9]) and where tr is the trace map of [F-Lawson;7.1] described above.

Let $V_f \subset X$ denote $pr_{X*}(| Z_f |)$, the projection to X of the support of Z_f on $Y \times X$, the cycle associated to f. Then $tr \circ f_*$ factors through a map

$$(tr \circ f_*)\breve{} : \mathcal{C}_r(Y) \to \mathcal{C}_{r+k}(V_f)$$

whose group completion

$$\tilde{\Gamma}_f : Z_k(Y) \to Z_{r+k}(V_f)$$

we call the *refined graph mapping.*

An explicit description of $\Gamma_f(W)$ for W irreducible of dimension k on Y is as follows. Let ω denote the generic point of W. If $f(\omega)$ has dimension $< k$ as a scheme-theoretic point of $C_r(X)$, then $\Gamma_f(W) = 0$. Otherwise, let $\sum A_i$ denote the cycle with Chow point $f(\omega)$, where each irreducible A_i is a subvariety of $X_{k(f(\omega))}$. If the generic point of A_i maps to a scheme-theoretic point of X of dimension $k+r$, let B_i denote the closure of this point in X; otherwise, take B_i to be empty. Then $\Gamma_f(W) = \sum B_i$. Using this description, we immediately conclude that

$$\Gamma_f(W) = pr_{X*}(Z_{f\circ i}) = \Gamma_{f\circ i}(W) \tag{2.1.1}$$

where $i : W \to Y$ is the closed immersion of W in Y.

Our first proposition concerning Γ_f provides an intersection-theoretic interpretation in the special case in which both Y and X are smooth.

We are much indebted to Ofer Gabber for pointing out an error in an earlier version of the second assertion of Proposition 2.2 and guiding us to the following formulation.

Proposition 2.2. *Consider a Chow correspondence $f : Y \to \mathcal{C}_r(X)$ with both X and Y projective and smooth. The graph mapping*

$$\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$$

sends an irreducible subvariety W of dimension k on Y to

$$\Gamma_f(W) = pr_{X*}(pr_Y^*(W) \cdot Z_f),$$

where Z_f is the cycle on $Y \times X$ associated to f and where $pr_Y^*(W) \cdot Z_f$ denotes intersection of cycles (meeting properly) on the smooth variety $Y \times X$.

Conversely, consider some $m+r$-cycle Z on $Y \times X$, where m denotes the dimension of Y. There exist smooth projective varieties Y_i of dimension $m - c_i$, maps $g_i : Y_i \to Y$, and Chow correspondences $f_i : Y_i \to \mathcal{C}_{r+c_i}(X)$ such that for any irreducible subvariety W of Y of dimension k

$$\sum \Gamma_{f_i}(W_i) \quad , \quad pr_X*(pr_Y^*(W) \cdot Z)$$

are rationally equivalent, where $W_i = g_i^!(W)$ is a $(k-c_i)$-cycle on Y_i representing the Gysin pullback of W.

Proof. Let $i : W \subset Y$ be an irreducible subvariety of dimension k and let $j : V \to Y$ be a (Zariski) open immersion with the property that $i' : T \equiv W \cap V \to V$ is a regular immersion. Since $\Gamma_f(W) = pr_{X*}(Z_{f \circ i})$, to prove the first assertion it suffices to prove that $Z_{f \circ j \circ i'}$ (which equals the restriction of $Z_{f \circ i}$ to $T \times X \subset W \times X$) equals $pr_V^*(T) \cdot Z_{f \circ j}$. The Gysin pullback $(i' \times 1)^!(Z_{f \circ j})$ equals (essentially by definition) the intersection $pr_V^*(T) \cdot Z_{f \circ j}$. Thus, the equality $\Gamma_f(W) = pr_{X*}(pr_Y^*(W) \cdot Z_f)$ follows from the fact that

$$Z_{f \circ j \circ i'} = (i' \times 1)^!(Z_{f \circ j})$$

(cf. [F-Mazur;3.1]).

To prove the converse, we immediately reduce to the case that Z is irreducible, so that $pr_1 : Z \to Y$ has image some subvariety $V \subset Y$ of dimension $m - c$. Then $Z \to V$ is generically of relative dimension $r + c$, thereby determining a rational map $g_V : V - - > \mathcal{C}_{r+c}(X)$. Let $V' \subset V \times \mathcal{C}_{r+c}(X)$ be the graph of this rational map; thus, V' is the closure of the graph of a morphism $g_U : U \to \mathcal{C}_{r+c}(X)$ with domain some dense open subset of V. Let $g : Y' \to Y$ denote the composition of some smooth resolution $h : Y' \to V'$ (i.e., a proper, birational map with Y' smooth) and the projection $pr_1 : V' \to V$ and let $f' : Y' \to \mathcal{C}_{r+c}(X)$ denote the composition $pr_2 \circ h$. If $U' \subset Y'$ is an open subset lying in V', then $Z_{f'}$ restricted to $U' \times X$ maps via $g \times 1$ isomorphically onto some dense open subset of Z, so that $(g \times 1)_*(Z_{f'}) = Z$.

By the first half of the proposition,

$$\Gamma_{f'}(g^! W) = pr_{X*}(pr_{Y'}^*(g^! W) \cdot Z_{f'}).$$

On the other hand, since $pr_X : Y' \times X \to X$ equals $pr_X \circ (g \times 1) : Y' \times X \to Y \times X \to X$,

$$pr_{X*}(pr_{Y'}^*(g^! W) \cdot Z_{f'}) = pr_{X*} \circ (g \times 1)_*(pr_Y^*(g^! W) \cdot Z_{f'}).$$

Applying the projection formula ([Fulton;8.1.1.c]) and the equality $pr_Y^*(g^! W) = (g \times 1)^!(pr_Y^* W)$, we conclude that

$$(g \times 1)_*(pr_Y^*(g^! W) \cdot Z_{f'}) \quad , \quad pr_Y^*(W) \cdot (g \times 1)_*(Z_f')$$

are rationally equivalent. Thus, the proof is completed by applying the equality $(g \times 1)_*(Z_{f'}) = Z$ verified above.

In the next proposition, we verify that the graph mapping commutes with the s-operation considered in detail in section 1.

Proposition 2.3. *Let Y, X be projective algebraic varieties and consider a continuous algebraic map $f : Y \to \mathcal{C}_r(X)$. Let $V_f \subset X$ denote $pr_{X*}(| Z_f |)$, the projection to X of the support of the cycle Z_f on $Y \times X$ associated to f. For any pair of positive integers r, k, the following diagram commutes (in the derived category)*

$$\begin{array}{ccc} \tilde{Z}_k(Y) & \stackrel{s}{\to} & \tilde{Z}_{k-1}(Y)[-2] \\ \tilde{\Gamma}_f \downarrow & & \downarrow \tilde{\Gamma}_f \\ \tilde{Z}_r(V_f) & \stackrel{s}{\to} & \tilde{Z}_{r-1}(V_f)[-2] \end{array}$$

where $\tilde{\Gamma}_f : Z_*(Y) \to Z_{*+r}(V_f)$ is the refined graph mapping.

Proof. Let $i_X : X \to X \times \mathbf{P}^1$, $i_C : \mathcal{C}_r(X) \to \mathcal{C}_r(X) \times \mathbf{P}^1$ denote the fibre inclusions above $\infty \in \mathbf{P}^1$. We consider the following diagram of cycle spaces

$$\begin{array}{ccccc} Z_k(Y) \times Z_0(\mathbf{P}^1) & \stackrel{\times}{\to} & Z_k(Y \times \mathbf{P}^1) & \stackrel{i_Y^!}{\to} & Z_{k-1}(Y) \\ f_* \times 1 \downarrow & & f_* \times 1 \downarrow & & \downarrow f_* \\ Z_k(\mathcal{C}_r(X)) \times Z_0(\mathbf{P}^1) & \stackrel{\times}{\to} & Z_k(\mathcal{C}_r(X) \times \mathbf{P}^1) & \stackrel{i_C^!}{\to} & Z_{k-1}(\mathcal{C}_r(X)) \\ tr \times 1 \downarrow & & (tr \times 1) \circ \times \downarrow & & tr \downarrow \\ Z_{r+k}(X) \times Z_0(\mathbf{P}^1) & \stackrel{\times}{\to} & Z_{r+k}(X \times \mathbf{P}^1) & \stackrel{i_X^!}{\to} & Z_{r+k-1}(X) \end{array}$$

where $\times : Z_j(V) \times Z_0(W) \to Z_j(V \times W)$ sends (Z, w) to $Z \times w$. By (1.3.a), the horizontal rows induce the s-map, whereas the left and right columns induce Γ_f. Consequently, to prove the weak form of the proposition with the refined graph mapping $\tilde{\Gamma}_f$ replaced by the graph mapping Γ_f, it suffices to prove the commutativity (up to homotopies through group homomorphisms) of the above diagram. The upper and lower left squares commute as can be seen by inspection; the upper right square commutes by [F-Gabber;3.4.d].

To verify the (homotopy) commutativity of the lower right square, we employ the projective bundle theorem of [F-Gabber;2.5] which implies that

$$i_{C*} \oplus pr_1^* : \tilde{Z}_k(\mathcal{C}_r(X)) \oplus \tilde{Z}_{k-1}(\mathcal{C}_r(X)) \to \tilde{Z}_k(\mathcal{C}_r(X) \times \mathbf{P}^1)$$

is a quasi-isomorphism with quasi-inverse $pr_{C*} \times i_C^!$. Observe that $i_C^!$ vanishes on the summand $i_{C*}(\tilde{Z}_k(\mathcal{C}_r(X)))$ and that this summand maps via $(tr \times 1) \circ \times$ to the summand $i_{X*}(\tilde{Z}_k(X))$ of $\tilde{Z}_k(X \times \mathbf{P}^1)$ on which $i_X^!$ vanishes. Since pr_C^* is left inverse to $i_C^!$ and pr_X^* is left inverse to $i_X^!$ in the derived category, we may verify the homotopy commutativity of the lower right square by showing the commutativity of the square obtain by replacing $i^!_C$, $i^!_X$ by pr^*_C, pr^*_X. The commutativity of this latter square is easily seen by inspection.

We now consider the corresponding diagram for the refined graph mapping (where V denotes V_f):

$$\begin{array}{ccccc}
Z_k(Y) \times Z_0(\mathbf{P}^1) & \stackrel{\times}{\to} & Z_k(Y \times \mathbf{P}^1) & \stackrel{i^!_Y}{\to} & Z_{k-1}(Y) \\
\tilde{\Gamma}_f \times 1 \Big\downarrow & & \Big\downarrow & & \Big\downarrow \tilde{\Gamma}_f \\
Z_{r+k}(V) \times Z_0(\mathbf{P}^1) & \stackrel{\times}{\to} & Z_{r+k}(V \times \mathbf{P}^1) & \stackrel{i^!_V}{\to} & Z_{r+k-1}(V)
\end{array}$$

whose middle row is induced by the middle column of the preceding diagram. The commutativity of the left square of the above diagram follows from the commutativity of the left squares of the preceding diagram. To prove the homotopy commutativity of the right square, we proceed as above using the projective bundle theorem to verify homotopy commutativity on each summand of $\tilde{Z}_k(Y \times \mathbf{P}^1)$, where

$$i_{Y*} \oplus pr_1^* : \tilde{Z}_k(Y) \oplus \tilde{Z}_{k-1}(Y) \to \tilde{Z}_k(Y \times \mathbf{P}^1)$$

is a quasi-isomorphism. On the summand $i_{Y*}(\tilde{Z}_k(Y))$, $i^!_Y$ vanishes as does $i^!_V \circ \tilde{\Gamma}_f$. On the summand $pr_1^*(\tilde{Z}_{k-1}(Y))$, the required commutativity follows by replacing the maps $i^!_Y$, $i^!_V$ by their left inverses pr^*_Y, pr^*_V (in the derived category) and verifying commutativity by inspection.

For a given Chow variety $C_{r+1,d}(X\#\mathbf{P}^j)$ of some suspension $X\#\mathbf{P}^j$ of a projective variety X, there exists some positive integer E (depending upon X, r, j) such that for all $e > E$ there exists some continuous algebraic map

$$\nu_e : C_{r+1,d}(X\#\mathbf{P}^j) \to C_{r-j,de}(X)$$

with the property that $\nu_e \circ \Sigma^{j+1}$ is algebraically homotopic to multiplication by e. Hence, (1.5.iii) implies that the following diagram commutes up to homotopy

$$\begin{array}{ccccc}
C_{r,d}(X) \times \mathbf{P}^j & \stackrel{\#}{\to} & C_{r+1,d}(X\#\mathbf{P}^j) & \stackrel{\nu_e}{\to} & C_{r-j,de}(X) \\
\Big\downarrow & & & & \Big\downarrow \\
Z_r(X) \wedge Z_0(\mathbf{P}^j) & \stackrel{s^j}{\to} & Z_{r-j}(X) & \stackrel{e}{\to} & Z_{r-j}(X)
\end{array}$$

thereby generalizing (1.4.b).

Theorem 2.4. *Let X be a projective algebraic variety, $f : Y \to C_{r,d}(X)$ be a Chow correspondence, and $i : W \subset Y$ an irreducible subvariety of Y of dimension k. For any j with $0 < j \leq r$, let $\#(f) : Y \times \mathbf{P}^j \to C_{r+1,d}(X\#\mathbf{P}^j)$ denote the Chow correspondence given by the composition the composition*

$$\# \circ f \times 1 : Y \times \mathbf{P}^j \to C_{r,d}(X) \times \mathbf{P}^j \to C_{r+1,d}(X\#\mathbf{P}^j).$$

We consider the graph mappings $\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$ and $\Gamma_{\#(f)} : Z_{k+j}(Y \times \mathbf{P}^j) \to Z_{r+k+j+1}(X\#\mathbf{P}^j)$. These are related as follows.
a.) $\Gamma_{\#(f)}(W \times \mathbf{P}^j) = \Sigma^{j+1}(\Gamma_f(W))$.
b.) $e \cdot \Sigma^{j+1}\Gamma_f(W)$, $\Sigma^{j+1}\Gamma_{\nu_e \circ \#(f)}(W \times \mathbf{P}^j)$ are effectively rationally equivalent, where $\nu_e : C_{r+1,d}(X\#\mathbf{P}^j) \to C_{r-j,de}(X)$ is as discussed above.
c.) $e \cdot \Gamma_f(W) = pr_{X*}(e \cdot Z_f)$, $\Gamma_{\nu_e \circ \#(f)}(W \times \mathbf{P}^j) = pr_{X*}(Z_{\nu_e \circ \#(f)})$ are rationally equivalent.

Proof. Using (2.1.1), we immediately reduce to the case that $W = Y$. Assertion a.) follows from the observation that the generic point of $W \times \mathbf{P}^j$ is mapped via $f \times 1$ to the Chow point of the cycle on $X\#\mathbf{P}^j$ whose closure is $\Sigma^{j+1}(\Gamma_f(W))$, since $\Gamma_f(W)$ is the cycle on X which is the closure of the cycle whose Chow point is the image under f of the generic point of W.

For any Chow corespondence $g : Y \to C_{s,c}(X)$ the composition of g with $\Sigma : C_{s,c}(X) \to C_{s+1,c}(\Sigma X)$ has the effect of sending the cycle Z_g on $Y \times X$ equidimensional over Y to its fibre-wise suspension $\Sigma_Y(Z_g)$ on $Y \times \Sigma(X)$. These cycles satisfy $\Sigma(pr_{X*}(Z_g)) = pr_{\Sigma X*}(Z_{\Sigma \circ g})$. Moreover, an algebraic homotopy $F : W \times \mathbf{P}^j \times C \to C_{r+1,d}(X\#\mathbf{P}^j)$ relating two Chow correspondences $f_1, f_2 : W \times \mathbf{P}^j \to C_{r+1,d}(X\#\mathbf{P}^j)$ has associated cycle Z_F which provides an effective rational equivalence between the associated cycles Z_{f_1}, Z_{f_2} (as verified, for example, in [F-Mazur2]). Since $\Sigma^{j+1} \circ \nu_e : C_{r+1,d}(X\#\mathbf{P}^j) \to C_{r+1,de}(X\#\mathbf{P}^j)$ is algebraically homotopy equivalent to multiplication by e, we conclude that

$$\Sigma^{j+1}\Gamma_{\nu_e \circ \#(f)}(W \times \mathbf{P}^j) \ , \ e \cdot \Gamma_{\#(f)}(W \times \mathbf{P}^j) = e \cdot \Sigma^{j+1}(\Gamma_f(W))$$

are rationally equivalent.

The Lawson suspension theorem remains valid for algebraic bivariant cycle complexes [F-Gabber;4.6.c], so that the rational equivalence classes of $r+k$-cycles on X (i.e., $\pi_0(A_{r+k}(*, X))$) map isomorphically via Σ^{j+1} to rational equivalence classes of $r+k+j+1$-cycles on $\Sigma^{j+1}(X)$. Thus, c.) follows from b.).

We specialize Theorem 2.4 to the special case in which W is simply a point. One can interpret the assertion of Corollary 2.5 as providing a method of moving a cycle Z to a rationally equivalent cycle which is equidimensional over a projective space.

Corollary 2.5. *Let Z be an effective r-cycle of degree d on a projective variety X and let $\zeta : \mathbf{P}^j \to C_{r+1,d}(X\#\mathbf{P}^j)$ send $t \in \mathbf{P}^j$ to $Z\#t$ for some j with $0 < j \leq r$.*
a.) $\Gamma_\zeta(\mathbf{P}^j) = \Sigma^{j+1}(Z)$.
b.) $e \cdot \Sigma^{j+1}(Z)$, $\Sigma^{j+1}\Gamma_{\nu_e \circ \zeta}(\mathbf{P}^j)$ are effectively rationally equivalent.
c.) $e \cdot Z$, $\Gamma_{\nu_e \circ \zeta}(\mathbf{P}^j)$ are rationally equivalent.
d.) The image of $(\nu_e \circ \zeta)_*([\mathbf{P}^j]) \in H_{2j}(C_{r-j,de}(X))$ in $H_{2j}(Z_{r-j}(X))$ equals the Hurewicz image of $s^j(e \cdot \{Z\}) \in \pi_{2j}(Z_{r-j}(X))$.

Proof. Specializing Theorem 2.4 to the case in which $Y = W$ is a point, we obtain the first three assertions. To determine the Hurewicz image of $s^j(e \cdot \{Z\})$, we use (1.5.ii) and observe that the Hurewicz image of $\{Z\} \wedge S^{2j} \in \pi_{2j}(Z_r(X) \wedge Z_0(\mathbf{P}^j))$ is the image of

$\{Z\} \otimes [\mathbf{P}^j] \in H_{2j}(C_{r,d}(X) \times \mathbf{P}^j)$. Applying H_{2j} to the diagram preceding Theorem 2.4, we conclude the last assertion.

We now consider the composition of Chow correspondences.

Definition 2.6. Let Y,X,T be projective varieties and consider continuous algebraic maps

$$f : Y \to \mathcal{C}_r(X) \quad , \quad g : X \to \mathcal{C}_s(T).$$

Then the composition product

$$g \cdot f : Y \to \mathcal{C}_{r+s}(T)$$

is defined as the composition

$$tr \circ g_* \circ f : Y \to \mathcal{C}_r(X) \to \mathcal{C}_r(\mathcal{C}_s(T)) \to \mathcal{C}_{r+s}(T).$$

One application of the following proposition is a proof (in [F-Mazur2]) that "correspondence homomorphisms" behave well with respect to the correspondence product.

Proposition 2.7. *Let Y,X,T be projective varieties and consider Chow correspondences*

$$f : Y \to \mathcal{C}_r(X) \quad , \quad g : X \to \mathcal{C}_s(T).$$

Then the graph mapping associated to the composition product is given as the composition of graph mappings:

$$\Gamma_{g \cdot f} = \Gamma_g \circ \Gamma_f : Z_*(Y) \to Z_{*+r+s}(T).$$

Proof. We consider an irreducible subvariety W of Y of dimension k and proceed to prove that $\Gamma_{g \cdot f}(W) = \Gamma_g(\Gamma_f(W))$. We interpret $\Gamma_f(W)$ in terms of generic points as follows. Let $\omega \in W \subset Y$ be the generic point of W and let $\chi = f(\omega) \in \mathcal{C}_r(X)$. Then χ is the Chow point of an effective cycle $\sum A_i$ with each A_i an irreducible subvariety of $X_{k(\chi)}$. ($k(\chi)$ denotes the residue field of the scheme-theoretic point $\chi \in \mathcal{C}_r(X)$). Let $\chi_i \in X$ be the scheme-theoretic point defined as the image of the generic point of A_i under the composition $A_i \subset X_{k(\chi)} \to X$. Then $\Gamma_f(W)$ is the sum of those subvarieties $\{\chi_i\}^- \subset X$ which are of dimension $k + r$.

Consider now $\gamma_i = g(\chi_i)$, a scheme-theoretic point of $C_s(T)$. Then γ_i is the Chow point of a cycle $\sum C_{i,j}$, where each $C_{i,j}$ is an irreducible subvariety of $T_{k(\gamma_i)}$. Let $\gamma_{i,j} \in T$ be the scheme-theoretic point defined as the image of the generic point of $C_{i,j}$ under the composition $C_{i,j} \subset T_{k(\gamma_i)} \to T$. Then $\Gamma_g(\Gamma_f(W))$ is the sum of those subvarieties $\{\gamma_{i,j}\}^- \subset T$ which are of dimension k+r+s.

On the other hand, let $\tau = g_*(\chi)$, a scheme-theoretic point of $C_r(C_s(T))$. Then τ is the Chow point of an effective r-cycle $\sum R_i$ with each R_i an irreducible subvariety of $C_s(T)_{k(\tau)}$. (In fact, $R_i = g_*(A_i)$, $A_i \subset X_{k(\chi)}$). Let $\rho_i \in C_s(T)$ denote the generic point of

R_i. Then ρ_i is the Chow point of a cycle $\sum D_{i,j}$ with each $D_{i,j}$ an irreducible subvariety of $T_{k(\rho_i)}$. Let $\delta_{i,j} \in T$ (respectively, $\delta'_{i,j} \in T_{k(\omega)}$) be the scheme-theoretic point defined as the image of the generic point of $D_{i,j}$ under the composition $D_{i,j} \subset T_{k(\rho_i)} \to T$ (resp., $D_{i,j} \subset T_{k(\rho_i)} \to T_{k(\omega)}$). Essentially by definition, $g \cdot f(\omega) = tr(\tau) \in C_{r+s}(T)_{k(\omega)}$ is the Chow point of the cycle defined as the sum of those subvarieties $\{\delta'_{i,j}\}^- \subset T_{k(\omega)}$ which are of dimension r+s. Thus, $\Gamma_{g\cdot f}(W)$ is the sum of those subvarieties $\{\delta_{i,j}\}^- \subset T$ which are of dimension k+r+s.

Finally, we verify by inspection the equality of the set (with possibly repeated elements) of those $\gamma_{i,j} \in T$ of dimension k+r+s and the set of those $\delta_{i,j} \in T$ of dimension k+r+s.

3. Filtrations on Cycles

As observed in [F-Mazur;1.4], considering kernels of iterates of the s-operation on $\pi_0(Z_r(X))$ provides an increasing filtration on algebraic r-cycles beginning with the subgroup of those cycles algebraically equivalent to 0 and ending with those homologically equivalent to 0. In Theorem 3.2, we identify this "S-filtration" in terms of images under graph mappings of cycles homologically equivalent to 0. This identification is closely related to a filtration introduced by Nori [Nori] and is readily verified to dominate Nori's filtration whenever the latter is defined. In fact, we show that one can view the S-filtration as merely the extension of Nori's filtration to include possible contribution from singular varieties. We also show that our S-filtration is dominated by a filtration considered by Bloch and Ogus [Bloch-Ogus], thereby strengthening an observation of Nori's that his filtration is dominated by that of Bloch and Ogus.

Definition 3.1. (cf. [F-Mazur;1.4]) Let X be a quasi-projective algebraic variety and r a non-negative integer. Two r-cycles Z_1, Z_2 are said to be τ_k equivalent for some k with $0 \leq k \leq r$ if $Z_1 - Z_2 \in Z_r(X)$ lies in the kernel of

$$Z_r(X) \xrightarrow{\pi} \pi_0(Z_r(X)) \xrightarrow{s^k} \pi_{2k}(Z_{r-k}(X)).$$

We call the resulting filtration

$$\{S_k Z_r(X)\} = \{Z \in Z_r(X) : Z \ \tau_k\text{−equivalent to } 0\}$$

the S-filtration.

In particular, two algebraic r-cycles are τ_0 -equivalent if and only if they are algebraically equivalent and are τ_r-equivalent if and only if they are homologically equivalent (with respect to singular Borel-Moore homology).

In what follows, we shall often abuse notation by applying the s-operation to elements of $Z_r(X)$ (viewed as a discrete group) rather than to their equivalence classes in $\pi_0(Z_r(X))$.

In the following theorem, we use the properties of the s-operation developed in section 1 to provide an interpretation of τ_k equivalence inspired by an equivalence relation introduced by Nori in [Nori].

Theorem 3.2. *For any projective algebraic variety X, $S_k Z_r(X) \subset Z_r(X)$ is the subgroup generated by cycles Z of the following form: there exists a projective variety Y of dimension $2k+1$, a Chow correspondence $f : Y \to \mathcal{C}_{r-k}(X)$, and a k-cycle W on Y homologically equivalent to 0 such that Z is rationally equivalent to $\Gamma_f(W)$.*

Proof. To verify the theorem in the special case $k = 0$, we recall (cf. [Fulton]) that the subgroup of r-cycles algebraically equivalent to 0 is generated by cycles $Z = Z' - Z''$ of the following form: there exists a smooth connected curve C, an effective cycle V on $X \times C$, and points w', w'' on C such that Z', Z'' are the fibres of V over w', w''. We readily verify that this is equivalent to the assertion that $Z = \Gamma_f(W)$, where $f : C \to C_{r,d}(X)$ sends w', w'' to the Chow points of Z', Z'' and $W = [w'] - [w'']$.

We now assume that $k > 0$. Consider an algebraic r-cycle Z given as a difference of two effective r-cycles $Z = Z' - Z''$ and assume that $s^k(Z) = 0$ (which is equivalent to $s^k(Z') = s^k(Z'')$). Consider

$$\zeta', \zeta'' : \mathbf{P}^k \to C_{r+1,d}(X \# \mathbf{P}^k)$$

sending $t \in \mathbf{P}^k$ to $Z' \# t$, $Z'' \# t$. By (1.5.iii.), the square

$$\begin{array}{ccc} C_{r,d}(X) \times \mathbf{P}^k & \overset{\#}{\to} & C_{r+1,d}(X \# \mathbf{P}^k) \\ \downarrow & & \downarrow \\ Z_r(X) \wedge Z_0(\mathbf{P}^k) & \overset{\Sigma^{k+1} \circ s^k}{\to} & Z_{r+1}(X \# \mathbf{P}^k) \end{array}$$

commutes up to homotopy. Since

$$\{Z\} \otimes [\mathbf{P}^k] \quad , \quad \{Z'\} \otimes [\mathbf{P}^k] \quad \in H_{2k}(C_{r,d}(X) \times \mathbf{P}^k)$$

map to the Hurewicz images of

$$\{Z\} \wedge S^{2k} \quad , \quad \{Z'\} \wedge S^{2k} \quad \in \pi_{2k}(Z_r(X) \wedge Z_0(\mathbf{P}^k))$$

we conclude that

$$\zeta'_*([\mathbf{P}^k]), \zeta''_*([\mathbf{P}^k]) \in H_{2k}(C_{r+1,d}(X \# \mathbf{P}^k)$$

have images in $H_{2k}(Z_{r+1}(X \# \mathbf{P}^k)) \simeq H_{2k}(Z_{r-k}(X))$ equal to the Hurewicz images of $s^k(Z')$, $s^k(Z'')$ and thus are equal. Since the homology of $Z_{r+1}(X \# \mathbf{P}^k)$ is the direct limit (with respect to translation by elements in $\pi_0(\mathcal{C}_{r+1}(X \# \mathbf{P}^k))$) of the homology of $\mathcal{C}_{r+1}(X \# \mathbf{P}^k)$, we conclude that

$$\zeta'_{A*}([\mathbf{P}^k]) = \zeta''_{A*}([\mathbf{P}^k]) \in H_{2k}(C_{r+1,d+a}(X \# \mathbf{P}^k))$$

where ζ'_A, ζ''_A denote the compositions of ζ', ζ'' with translation by some $A \in C_{r+1,a}(X \# \mathbf{P}^k)$ of sufficiently high degree a.

Choose E sufficiently large that there exists for all $e' \geq E$ some continuous algebraic map $\nu_{e'} : C_{r+1,d+a}(X \# \mathbf{P}^k) \to C_{r-k,(d+a)e'}(X)$ representing e' times a homotopy inverse to

Lawson suspension $\Sigma^{k+1} : Z_{r-k}(X) \to Z_{r+k+1}(X)$ and let $\zeta'_{e'}, \zeta''_{e'}$ denote the compositions $\nu_{e'} \circ \zeta'_A, \nu_{e'} \circ \zeta''_A$. Then

$$\zeta'_{e',*}([\mathbf{P}^k]) = \zeta''_{e',*}([\mathbf{P}^k]) \in H_{2k}(C_{r-k,(d+a)e'}(X)).$$

By taking successive hyperplane sections which contain the images of $\zeta'_{e'}, \zeta''_{e'}$ and the singular locus of the preceding hyperplane section, we may apply the Lefschetz hyperplane theorem for singular varieties [Andreotti-Frankel] to obtain some closed subvariety $Y_{e'} \subset C_{r-k,(d+a)e'}(X)$ of dimension $2k+1$ such that $\zeta'_{e'}, \zeta''_{e'}$ factor through $g'_{e'}, g''_{e'} : \mathbf{P}^k \to Y_{e'}$ and

$$g'_{e'*}([\mathbf{P}^k]) = g''_{e'*}([\mathbf{P}^k]) \ \in \ H_{2k}(Y'_e).$$

(A similar application of [Andreotti-Frankel] is presented in detail in [F-Mazur2;3.2].) We define $f_{e'} : Y_{e'} \to \mathcal{C}_{r-k}(X)$ to be the inclusion and we define the cycle $W_{e'} \in Z_k(Y_{e'})$ as

$$W_{e'} \equiv g'_{e'*}([\mathbf{P}^k]) - g''_{e'*}([\mathbf{P}^k])$$

so that $W_{e'}$ is homologically equivalent to 0 on $Y_{e'}$.

We claim that Z is rationally equivalent to $\Gamma_{f_{e+1}}(W_{e+1}) - \Gamma_{f_e}(W_e)$. Namely, $\Sigma^{k+1}Z'$ is rationally equivalent to $\Gamma_{\zeta'}([\mathbf{P}^k])$ by (2.5.a), whereas the latter equals $\Gamma_{\zeta'_A}([\mathbf{P}^k])$ because the graph mapping is unaffected by the addition of a constant family. Consequently, (2.5.c) implies that $\Gamma_{\zeta'_{e'}}([\mathbf{P}^k])$ is rationally equivalent to $e'Z'$. On the other hand, $\Gamma_{\zeta'_{e'}}([\mathbf{P}^k]) = \Gamma_{f_{e'}}(g'_{e'*}([\mathbf{P}^k]))$. Similarly, $e'Z''$ is rationally equivalent to $\Gamma_{f_{e'}}(g''_{e'*}([\mathbf{P}^k]))$, thereby proving that $e'Z$ is rationally equivalent to $\Gamma_{f_{e'}}(W_{e'})$.

To prove the converse statement of the theorem, suppose Z is rationally equivalent to $\Gamma_f(W)$, for some projective variety Y of dimension $2k+1$, continuous algebraic map $f : Y \to \mathcal{C}_{r-k}(X)$, and k-cycle W on Y homologically equivalent to 0. We must show $s^k(Z) = 0$. Clearly, we may assume $Z = \Gamma_f(W)$. The hypothesis that W is homologically equivalent to 0 is equivalent to the condition that $s^k(W) = 0$. Consequently, $s^k(Z) = 0$ by Proposition 2.3.

In the proof above of the converse statement, we did not require any constraint on the dimension of Y. Thus, Theorem 3.2 remains valid if the assertion is changed by dropping the condition that Y be of dimension $2k+1$.

Nori's filtration $\{A_kCH_r(X)\}$ on the (discrete) group of algebraic r-cycles on a projective, smooth variety X is defined as follows: $A_kCH_r(X) \subset Z_r(X)$ is the subgroup generated by those cycles rationally equivalent to cycles of the form

$$pr_X*((pr_Y^*W \cdot Z), \quad W \in Z_k(Y), \quad Z \in Z_{r+c-k}(Y \times X)$$

where Y is a projective smooth variety of some dimension c, $W \in Z_k(Y)$ is homologically equivalent to 0, and $pr_Y : Y \times X \to Y, pr_X : Y \times X \to X$ are the projections.

Using Proposition 2.2, we re-interpret Nori's filtration using our graph mapping in terms exactly parallel to the condition of Thereom 3.2. We see for X smooth that the τ_k-filtration differs from Nori's only in that one permits not necessarily smooth domains

Y for the graph mapping Γ_f associated to a Chow correspondence $f : Y \to \mathcal{C}_{r-k}(X)$. One can view the graph mapping as a useful formalism which permits consideration of singular varieties (which do not readily fit into a formalism involving intersection pairings).

Corollary 3.3. *Let X be a smooth, projective algebraic variety. Then*

$$A_k CH_r(X) \subset S_k Z_r(X).$$

Moreover, $A_k CH_r(X)$ is the subgroup of $Z_r(X)$ generated by those r-cycles Z rationally equivalent to $\Gamma_f(W)$ for some smooth projective variety Y, continuous algebraic map $f : Y \to \mathcal{C}_{r-k}(X)$, and k-cycle W on Y homologically equivalent to 0.

Proof. To prove the containment $A_k CH_r(X) \subset S_k Z_r(X)$, we consider an element $\xi \equiv \pi_{1*}((\pi_2^* u \cdot v))$ of $A_k CH_r(X)$, with $u \in Z_k(Y)$ homologically equivalent to 0 so that $s^k(u) = 0$. By (1.7.b), $s^k(\pi_2^* u) = 0$; by (1.7.c), $s^k((\pi_2^* u \cdot v)) = 0$; by (1.7.a), $s^k(\xi) = 0$. Hence, $\xi \in S_k Z_r(X)$.

By Proposition 2.2, $\Gamma_f(W) = pr_{X*}(pr_Y^* W \cdot Z_f)$ whenever X, Y are smooth, so that if the k-cycle W on Y is homologically equivalent to 0 then $\Gamma_f(W)$ lies in $A_k CH_r(X)$ for any $f : Y \to \mathcal{C}_{r-k}(X)$. (This also follows from Theorem 3.2.) Let $A'_k CH_r(X) \subset A_k CH_r(X)$ denote the subgroup generated by cycles rationally equivalent to such cycles $\Gamma_f(W)$ as f varies. We proceed to show that this inclusion is the identity.

Consider an arbitrary generator of $A_k CH_r(X)$

$$pr_X * (pr_Y^* W \cdot Z), \quad W \in Z_k(Y), \quad Z \in Z_{r+c-k}(Y \times X)$$

as in the definition of Nori's filtration. Applying Proposition 2.2 once again, we conclude that $pr_X * (pr_Y^* W \cdot Z)$ is rationally equivalent to $\sum \Gamma_{f_i}(W_i)$ where $g_i : Y_i \to Y$ is a map from a smooth projective variety Y_i of dimension dim(Y)$-c_i$ for some $c_i \geq 0$, $f_i : Y_i \to \mathcal{C}_{r-k+c_i}(X)$ is a Chow correspondence, and $W_i = g_i^!(W)$ is a $k - c_i$-cycle on Y_i. Let $\nu_e : \mathcal{C}_{r-k+c_i+1}(X \# \mathbf{P}^{c_i}) \to \mathcal{C}_{r-k}(X)$ be as in Theorem 2.4 so that $\nu_e \circ \Sigma^{c_i}$ is homotopic to multiplication by e. By part c.) of Theorem 2.4, $\Gamma_{f_i}(W_i)$ is rationally equivalent to

$$\Gamma_{\nu_{e+1} \circ \#(f)}(W_i \times \mathbf{P}^{c_i}) - \Gamma_{\nu_e \circ \#(f)}(W_i \times \mathbf{P}^{c_i}).$$

The fact that W is homologically equivalent to 0 on Y implies that $W_i = g_i^!(W)$ is homologically equivalent to 0 on Y_i (cf. [Fulton;19.2]). Thus, each $W_i \times \mathbf{P}^{c_i}$ is homologically equivalent to 0 on $Y_i \times \mathbf{P}^{c_i}$, so that each $\Gamma_{f_i}(W_i)$ and thus also $pr_X * ((pr_Y^* W \cdot Z)$ is in $A'_k CH_r(X)$.

Nori constructs examples of algebraic r-cycles homologically equivalent to 0 but not in $A_{r-1} CH_r(X)$. His examples are of the form of the restriction $i^!(W)$ of some $W \in Z_{r+h}(V)$ with $cl_V(W) \neq 0 \in H_{2r+2h}(V)$ to a sufficiently general complete intersection $i : X = V \cap D_1 \cap ... \cap D_h \subset V$ of a projective, smooth variety V. Nori shows that these cycles can not be in the (r-1)st stage of his filtration, whereas they can indeed be homologically equivalent to 0. It seems likely that Nori's examples are examples of cycles $Z = i^!(W) \in Z_r(X)$ with $s^r(Z) = 0, s^{r-1}(Z) \neq 0$.

We next turn to the filtration of Bloch and Ogus [Bloch-Ogus]. The k-th stage of their filtration,

$$B_k CH_r(X) \subset Z_r(X)$$

is the subgroup generated by those algebraic r-cycles Z for which their exists an $r+k+1$-dimensional subvariety V of X suppporting Z such that Z is homologically equivalent to 0 on V. $B_0CH_r(X)$ is the subgroup of r-cycles algebraically equivalent to 0 [B-O;7.3], hence equal to $A_0CH_r(X) = S_0CH_r(X)$.

In the following proposition, we prove that the S-filtration is dominated by that of Bloch-Ogus. This result was first proved by O. Gabber by different methods.

Proposition 3.4. *Let X be a complex projective algebraic variety and r a non-negative integer. Then for all $k \leq r$,*

$$S_k Z_r(X) \subset B_k CH_r(X).$$

Proof. By Theorem 3.2, it suffices to consider a cycle $Z \in Z_r(X)$ of the form $\Gamma_f(W)$, for some projective variety Y of dimension $2k+1$, continuous algebraic map $f : Y \to \mathcal{C}_{r-k}(X)$, and k-cycle W on Y homologically equivalent to 0. To prove the proposition, it suffices to exhibit some $g : V \to X$ with V a projective variety of dimension $r+k+1$ and a cycle $Z' \in Z_r(V)$ with $g_*(Z') = Z$ and with $\gamma(Z') = 0 \in H_{2r}(V)$.

We take V equal to V_f, where $\tilde{\Gamma}_f : Z_*(Y) \to Z_{*+r-k}(V_f)$ is the refined graph mapping, and take Z' equal to $\tilde{\Gamma}_f(W)$. Since W is homologically equivalent to 0,

$$s^k(W) = 0 \in H_{2k}(\tilde{Z}_0(Y));$$

by Proposition 2.3, this implies that

$$s^k(\tilde{\Gamma}_f(W)) = 0 \in H_{2k}(\tilde{Z}_{r-k}(V))$$

which implies that

$$\gamma(\tilde{\Gamma}_f(W)) = s^r(\tilde{\Gamma}_f(W)) = 0 \in H_{2r}(\tilde{Z}_0(V)) = H_{2r}(V).$$

4. Homology filtrations and the Grothendieck's Conjecture B

If X is a projective, smooth variety of dimension n, then the Strong Lefschetz Theorem asserts that

$$h^{n-i} : H_{2n-i}(X, \mathbf{Q}) \to H_i(X, \mathbf{Q})$$

is an isomorphism, where h denotes intersection with the homology class of a hyperplane section. A. Grothendieck has conjectured (in a conjecture referred to as Grothendieck's Conjecture B; cf. [Grothendieck] and [Kleiman]) that the inverse of this isomorphism

$$\Lambda_X^{n-i} : H_i(X, \mathbf{Q}) \to H_{2n-i}(X, \mathbf{Q})$$

is an “algebraic correspondence.” In other words, there exists some homology class $\ell_X^{n-i} \in H_{4n-2i}(X \times X)$ in the linear span of the set of fundamental classes of $2n-i$-dimensional subvarieties of $X \times X$ such that for any $u \in H_i(X, \mathbf{Q})$ with Poincaré dual $\tilde{u} \in H^{2n-i}(X, \mathbf{Q})$

$$\Lambda_X^{n-i}(u) = pr_{2*}(pr_1^* \tilde{u} \cap \ell_X^{n-i}) = \tilde{u} \backslash \ell_X^{n-i}.$$

As we see below, this conjecture is closely related to a conjecture of [F-Mazur] that the “topological filtration” on $H_m(X, \mathbf{Q})$ with r-th term

$$T_r H_m(X, \mathbf{Q}) = \text{image}\{s^r : \pi_{m-2r}(Z_r(X)) \otimes Q \to \pi_m(Z_0(X)) \otimes \mathbf{Q}\}$$

equals the “geometric” (or “niveau”) filtration on $H_m(X, \mathbf{Q})$ whose r-th term is

$$G_r H_m(X, \mathbf{Q}) = \text{span}\{i_*(H_m(Y)); i : Y \subset X \text{ with } \dim(Y) \leq m-r\}.$$

More specifically, we prove in Proposition 4.2 that if a resolution of singularities $\tilde{Y}$ of each subvariety $Y \subset X$ satisfies Grothendieck’s Conjecture B then we do indeed have the equality of topological and geometric filtrations on $H_*(X, \mathbf{Q})$. More generally, in Proposition 4.3 we verify that if X satisfies Grothendieck’s Conjecture B then the “primitive filtration” is subordinate to the topological filtration. As a corollary, we conclude a result of R. Hain [Hain] that the topological and geometric filtrations are equal for a sufficiently general abelian variety.

In [F-Mazur2], a Chow correspondence $f : Y \to C_r(X)$ with Y,X projective is shown to determine a *correspondence homomorphism*

$$\Phi_f : H_*(Y) \to H_{*+2r}(X)$$

which can be described as the following composition:

$$H_*(Y) \simeq \pi_*(Z_0(Y)) \overset{\Gamma_{f*}}{\to} \pi_*(Z_r(X)) \overset{s_*^r}{\to} \pi_{*+2r}(Z_0(X)) \simeq H_{*+2r}(X).$$

The following proposition is the homological analogue of the second half of Proposition 2.2.

Proposition 4.1. *Let X be a projective, smooth variety of dimension n and let Z be an $n+r$-cycle on $X \times X$ for some $r \geq 0$. Then there exist projective smooth varieties X_i of dimension $n-c_i$, maps $g_i : X_i \to X$, and Chow correspondences $f_i : X_i \to \mathcal{C}_{r+c_i}(X)$ such that for any $\alpha \in H_m(X, \mathbf{Q})$ with Poincarè dual $\tilde{\alpha} \in H^{2n-m}(X, \mathbf{Q})$*

$$\sum \Phi_{f_i}(\alpha_i) = pr_{2*}(pr_1^* \tilde{\alpha} \cap [Z]) = \tilde{\alpha} \backslash [Z]$$

where Φ_{f_i} is the correspondence homomorphism associated to the Chow correspondence f_i and $\alpha_i = f_i^*(\tilde{\alpha})\tilde{}$, the Gysin pullback of α via f_i.

Proof. Clearly, we may assume that Z is irreducible. As argued in the proof of Proposition 2.2, $pr_1 : Z \subset X \times X \to X$ has image some irreducible subvariety V of X of dimension

$n-c$ and thus determines a rational map $V -- > \mathcal{C}_{r+c}(X)$. This in turn determines a Chow correspondence $f'' : V' \to \mathcal{C}_{r+c}(X)$ where V' is the graph of this rational map. We define $g = pr_1 \circ h : X' \to X$, given by a resolution of singularities $h : X' \to V'$; we define $f' = f'' \circ h : X' \to \mathcal{C}_{r+c}(X)$. As seen in the proof of Proposition 2.2, $(g \times 1)_*(Z_{f'}) = Z$.

Let $\alpha' = f'^*(\tilde{\alpha})\breve{}$. Applying the projection formula, we conclude that

$$\tilde{\alpha}' \backslash [Z_{f'}] = f'^*(\tilde{\alpha}) \backslash [Z_{f'}] = \tilde{\alpha} \backslash (f' \times 1)_*([Z_{f'}]) = \tilde{\alpha} \backslash [Z].$$

By [F-Mazur2], the left-hand side of the above equality equals the image of α' under the correspondence homomorphism $\Phi_{f'}$ associated to the Chow correspondence $f' : X' \to \mathcal{C}_{r+c}(X)$.

Proposition 4.1 enables us to easily conclude that Grothendieck's Conjecture B implies the equality of the topological and geometric filtrations.

Proposition 4.2. *Let X be a projective, smooth variety of dimension n. Assume that Grothendieck's Conjecture B is valid for a resolution of singularities of each irreducible subvariety $Y \subset X$ of dimension $m-r$. Then*

$$T_r H_m(X, \mathbf{Q}) = G_r H_m(X, \mathbf{Q}).$$

Proof. As shown in [F-Mazur], $T_r H_m(X, \mathbf{Q}) \subset G_r H_m(X, \mathbf{Q})$ for any projective, smooth X. To prove the reverse inclusion, consider a class $\alpha \in H_m(X, \mathbf{Q})$ lying in the image of $H_m(Y, \mathbf{Q})$ with $Y \subset X$ of dimension $m-r$. Let $\tilde{Y} \to Y$ be a resolution of singularities (i.e., a proper birational map with $\tilde{Y}$ smooth) satisfying Grothendieck's Conjecture B. We recall that the theory of weights of Mixed Hodge Structures developed by P. Deligne implies that there exists some $\gamma \in H_m(\tilde{Y}, \mathbf{Q})$ mapping to α (cf. [F-Mazur;A.1]). Since the connected components of $\tilde{Y}$ are resolutions of the irreducible components of Y, we may assume that Y is irreducible and thus $\tilde{Y}$ is connected.

The Strong Lefschetz Theorem for $\tilde{Y}$ implies that there exists some $\delta \in H_{m-2r}(\tilde{Y}, \mathbf{Q})$ with $\Lambda^r_{\tilde{Y}}(\delta) = \gamma$. By hypothesis, there exists an $m+r$-cycle Z on $\tilde{Y} \times \tilde{Y}$ such that

$$pr_{2*}(pr_1^*(\delta) \bullet [Z]) = c \cdot \gamma \in H_m(\tilde{Y}, \mathbf{Q}) \ , \ c \neq 0 \in \mathbf{Q}.$$

By Proposition 4.1, there exist projective smooth varieties Y_i of dimension $m-r-c_i$, maps $g_i : Y_i \to \tilde{Y}$, and Chow correspondences $f_i : Y_i \to \mathcal{C}_{r+c_i}(\tilde{Y})$ such that

$$\sum \Phi_{f_i}(\delta_i) = pr_{2*}(pr_1^*(\delta) \cdot [Z])$$

where $\delta_i = g_i^*(\tilde{\delta})\breve{} \in H_{i-2r}(Y_i, \mathbf{Q})$ is the Gysin pullback of δ. Let $q_i : Y_i \to \mathcal{C}_{r+c_i}(X)$ be the composition of f_i and the map $\mathcal{C}_{r+c_i}(\tilde{Y}) \to \mathcal{C}_{r+c_i}(X)$ induced by $\tilde{Y} \to Y \to X$. Then

$$\sum \Phi_{q_i}(\delta_i) = c \cdot \alpha \in H_m(X, \mathbf{Q})$$

thereby showing that α lies in $C_rH_m(X,\mathbf{Q})$, the r-th stage of the "correspondence filtration" on $H_m(X,\mathbf{Q})$ (which contains $C_{r+c}H_m(X,\mathbf{Q})$ for any $c \geq 0$). Since $C_rH_m(X,Q)$ has been shown in [F-Mazur;7.3] to equal $T_rH_m(X,\mathbf{Q})$, we conclude that

$$G_rH_m(X,\mathbf{Q}) \subset T_rH_m(X,\mathbf{Q})$$

as required.

If X is a projective, smooth variety of dimension n, we define the "primitive filtration" on $H_m(X,\mathbf{Q})$ as follows. For $i \leq n$, the primitive subspace $\mathrm{Prim}(H_i(X,\mathbf{Q})) \subset H_i(X,\mathbf{Q})$ is the kernel of $h : H_i(X,\mathbf{Q}) \to H_{i-2}(X,\mathbf{Q})$ whereas $\mathrm{Prim}(H_{2n-i}(X,\mathbf{Q})) = \Lambda^{n-i}(\mathrm{Prim}(H_i(X,\mathbf{Q}))$. For $i \leq n$, we define

$$P_rH_{2n-i}(X,\mathbf{Q}) = \sum_{j\geq r} h^j(\mathrm{Prim}(H_{2n+2j-i}(X,\mathbf{Q})))$$

and

$$P_rH_i(X,\mathbf{Q}) = \sum_{j\geq r} h^{n+j-i} \circ \Lambda^{n+2j-i}(\mathrm{Prim}(H_{i-2j}(X,\mathbf{Q}))).$$

The following proposition provides a useful lower bound for the topological filtration of a variety satisfying Grothendieck's Conjecture B.

Proposition 4.3. *Let X be a projective, smooth variety of dimension n satisfying Grothendieck's Conjecture B. Then*

$$P_rH_m(X,\mathbf{Q}) \subset T_rH_m(X,\mathbf{Q}).$$

Proof. Observe that $h : H_*(X,\mathbf{Q}) \to H_{*-2}(X,\mathbf{Q})$ is an algebraic correspondence, for $h(u) = pr_{2*}(pr_1^*(u)\cdot\gamma(\Delta_H))$, where H is a hyperplane section of X and Δ_H is its image in $X \times X$ under the diagonal map. Since composition of algebraic correspondences are again algebraic [Kleiman], we conclude that if X satisfies Grothendieck's Conjecture B, then h^j and Λ^{n+j-i} are also algebraic correspondences for any j.

Consequently, Proposition 4.1 implies that

$$P_rH_m(X,\mathbf{Q}) \subset C_rH_m(X,\mathbf{Q})$$

where $C_rH_m(X,\mathbf{Q})$ ($= T_rH_m(X,\mathbf{Q})$ by [F-Mazur;7.3]) is the r-th stage of the correspondence filtration on $H_m(X,\mathbf{Q})$.

As proved by D. Lieberman [Lieberman], an abelian variety satisfies Grothendieck's Conjecture B. Thus, Proposition 4.3 implies the following result, first proved by R. Hain by more explicit means.

Proposition 4.4. (cf. [Hain]) *If X is a sufficiently general abelian variety, then*

$$T_r H_m(X, \mathbf{Q}) = G_r H_m(X, \mathbf{Q}).$$

For example, the latter equality is valid whenever the special Mumford-Tate group of X equals the full symplectic group on $H_1(X, \mathbf{Q})$.

Proof. We observe that the primitive filtration is the filtration by irreducible summands of the symplectic group $Sp(H_1(X, \mathbf{Q}))$ acting on $H_*(X, \mathbf{Q}) = \Lambda^*(H_1(X, \mathbf{Q}))$. If X is "sufficiently general", the special Mumford-Tate group (cf. [Mumford]) equals $Sp(H_1(X, \mathbf{Q}))$. Since the filtration of $H_*(X, C)$ by sub-Hodge structures is stabilized by the special Mumford-Tate group, the filtration by sub-Hodge structures is also a filtration of symplectic modules. Since the associated quotients of this filtration are non-trivial at those stages for which the associated quotients of the primitive filtration are non-trivial, we conclude that the primitive filtration (complexified) equals the filtration by sub-Hodge structures whenever the special Mumford-Tate group equals the symplectic group. On the other hand, the topological filtration contains the primitive filtration and is subordinate to this Hodge filtration. Thus, all three filtrations must be equal whenever the special Mumford-Tate group equals the symplectic group.

5. The spectral sequence

The purpose of this final section is to present a spectral sequence incorporating both the S-filtration (in the guise of τ_k-equivalence) and the topological filtration. The reader inclined towards a motivic point of view could envision the various terms of the spectral sequence as candidates for new "motives."

We recall that the join operation determines a pairing of abelian topological groups

$$Z_r(X) \times Z_0(\mathbf{P}^1) \to Z_{r+1}(X \# \mathbf{P}^1)$$

and thus a map of normalized chain complexes

$$\tilde{Z}_r(X)[2] \to \tilde{Z}_{r+1}(X \# \mathbf{P}^1) \simeq \tilde{Z}_{r-1}(X).$$

Our spectral sequence arises from consideration of the sequence of chain complexes

$$\tilde{Z}_n(X)[2n] \to \tilde{Z}_{n-1}(X)[2n-2] \to \ldots \to \tilde{Z}_1(X)[2] \to \tilde{Z}_0(X).$$

Proposition 5.1. *For any projective variety X, there exists a (strongly convergent) second quadrant spectral sequence of homological type*

$$E^2_{s,t} = H_{t-s}(Q_{-s/2}) \Rightarrow H_{s+t}(X)$$

whose differentials $d^k_{s,t}$ have bidegree $(-k, k-1)$, where $Q_{s/2} = 0$ unless s is an even integer with $0 \leq s \leq 2n$.

Moreover, the abutment $\sum_{s+t=m} E^{\infty}_{s,t}$ is the associated graded group of $H_m(X)$ with respect to the topological filtration (as considered in section 4). Furthermore, $E^{2k+2}_{-2r,0}$ is naturally isomorphic to the group of algebraic r-cycles on X modulo τ_k equivalence.

Proof. For $0 \leq r \leq n$, we define Q_r to be the mapping cone of the following composition

$$\tilde{Z}_{r+1}(X)[2r+2] \simeq \tilde{Z}_{r+1}(X)[2r] \otimes \tilde{Z}_0(\mathbf{P}^1)_{deg0} \to \tilde{Z}_{r+2}(X\#\mathbf{P}^1)[2r]$$

whose first map is induced by the quasi-isomorphism $\mathbf{Z}[2] \simeq \tilde{Z}_0(\mathbf{P}^1)_{deg0}$ and whose second is induced by the join pairing. Since $\tilde{Z}_{r+2}(X\#\mathbf{P}^1)[2r]$ is quasi-isomorphic to $\tilde{Z}_r(X)[2r]$, we have a family of distinguished triangles

$$\tilde{Z}_{r+1}(X)[2r+2] \to \tilde{Z}_r(X)[2r] \to Q_r.$$

Using the above sequence of chain complexes, we obtain an exact couple in homology

$$... \to \bigoplus_{r,s} H_{2r+s}(\tilde{Z}_{r+1}(X)[2r+2]) \to \bigoplus_{r,s} H_{2r+s}(\tilde{Z}_r(X)[2r]) \to \bigoplus_{r,s} H_{2r+s}(Q_r) \to ...$$

determining our spectral sequence. The convergence follows from the fact that $Q_r = 0$ unless $0 \leq r \leq n$.

To identify the filtration on the abutment $H_*(\tilde{Z}_0(X))$, we observe that a class in $H_*(Q_r)$ is a permanent cycle if it lifts to a class in $H_*(\tilde{Z}_r(X)[2r])$; such a permanent cycle in $H_*(Q_r)$ modulo boundaries is the image of $H_*(\tilde{Z}_r(X)[2r])$ in $H_*(\tilde{Z}_0(X))$ modulo the image of $H_*(\tilde{Z}_{r+1}(X)[2r+2])$ in $H_*(\tilde{Z}_0(X))$.

Finally, to identify $E^k_{-2r,0}$, we observe that all classes in $E^2_{-2r,0} = H_{2r}(Q_r)$ are permanent cycles, thus lifting to classes in $H_{2r}(\tilde{Z}_r(X)[2r]) = \pi_0(Z_r(X))$. The image in $H_{2r}(Q_r)$ of d^2 is the image of

$$\ker\{H_{2r}(\tilde{Z}_r(X)[2r]) \to H_{2r}(\tilde{Z}_{r-1}(X)[2r-2])\},$$

which is the group of algebraic r-cycles τ_1 -equivalent to 0 modulo algebraic equivalence. Thus, the quotient $E^4_{-2r,0}$ is the group of algebraic r-cycles modulo τ_1-equivalence. We argue similarly for any $k \leq r$: the image in $E^{2k}_{-2r,0}$ of d^{2k} is the image of

$$\ker(H_{2r}(\tilde{Z}_r(X)[2r]) \to H_{2r}(\tilde{Z}_{r-k}(X)[2r-2k])),$$

which is the group of algebraic r-cycles τ_k -equivalent to 0 modulo algebraic equivalence.

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