

RATIONAL COHOMOLOGY AND SUPPORTS FOR LINEAR ALGEBRAIC GROUPS

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Dedicated to David J. Benson

0. INTRODUCTION

What follows are rough “notes” based upon four lectures given by the author at PIMS in Vancouver over the period June 27 to June 30, 2016.

- Lecture I. Affine groups schemes over k .
- Lecture II. Algebraic representations.
- Lecture III. Cohomological support varieties.
- Lecture IV. Support varieties for linear algebraic groups.

The primary goal of these lectures was to publicize the author’s recent efforts to extend to representations of linear algebraic groups the “theory of support varieties” which has proved successful in the study of representations of finite group schemes. The first two lectures offer a quick review of relevant background for the study of affine group schemes and their representations. The third lecture discusses cohomological support varieties of finite group schemes and mentions challenges to extending this theory to linear algebraic groups (also discussed in the last paragraph of this introduction). In the fourth and final lecture, we provide an introduction to the author’s theory of support varieties using 1-parameter subgroups following work of A. Suslin, C. Bendel, and the author [57], [58]. The text contains a few improvements on results in the literature (see, for example, Remark 4.14).

We encourage others to follow the work discussed here by sharpening the formulations, extending general theory, providing much better computations, and working out many interesting examples. Towards the end of Lecture IV, we give a list of various explicit problems which might be of interest to some readers. We conclude this text by introducing “formal 1-parameter subgroups” leading to “formal support varieties” in Proposition 4.28, a promising but still unexplored structure.

The reader will find undue emphasis on the work of the author together with collaborators Chris Bendel, Jon Carlson, Brian Parshall, Julia Pevtsova, and Andrei Suslin. A quick look at references given will see that numerous other mathematicians have played seminal roles in developing support varieties, including Daniel Quillen who launched this entire subject.

We conclude this introduction with a brief sketch of the evolution of support varieties for a “group-like object” G and introduce our perspective on their role in the study of their representations (which we usually refer to as G -modules).

Support varieties emerged from D. Quillen’s work [47], [48] on the cohomology of finite groups. The reader attracted by homological computations might become

distracted (as we have been, at times) from our goal of illuminating representation theory by the numerous puzzles and questions concerning cohomology which arise from the geometric perspective of support varieties.

In the late 1960's, and even now, few complete calculations of the cohomology of finite groups were known. Quillen developed foundations for the equivariant cohomology theory introduced by A. Borel [10], a key tool in his determination of the prime ideal spectrum of the cohomology algebra $H^*(G, k)$ for *any finite group* [47], [48]. This enabled Quillen to answer a question of Atiyah and Swan on the growth of projective resolutions of k as a kG -module [59]. This is just one example of Quillen's genius: proving a difficult conjecture by creating a new context, establishing foundations, and proving a geometric, refined result which implies the conjecture.

A decade later, Jon Alperin and Len Evens considered the growth of projective resolutions of finite dimensional kG -modules [2], [3]. They recognized that Quillen's geometric description of the "complexity" of the trivial module k for kG had an extension to arbitrary finite dimensional modules. Following this, Jon Carlson formulated in [11] the (cohomological) support variety $|G|_M$ of a finite dimensional kG -module, a closed subvariety of "Quillen's variety" $|G|$. At first glance, one might think this construction is unhelpful: one starts with a kG -module M and one obtains invariants of M by considering the structure of the Ext -algebra $Ext_G^*(M, M)$ as a module over the cohomology algebra $H^*(G, k)$. Yet in the hands of Carlson and others, this has proved valuable in studying the representation theory of G and more general "group-like" structures.

One early development in the study of support varieties was an alternative construction proposed by J. Carlson for an elementary abelian p -group $E \simeq \mathbb{Z}/p^{\times s}$ and proved equivalent to the cohomological construction by G. Avrunin and L. Scott [4]. J. Carlson's fundamental insight was to reformulate the cohomological variety $|E|$ as a geometric object $V(E)$ whose points are related to kE without reference to cohomology; Carlson then reformulated the support variety $|E|_M$ of a finite dimensional kE -module in "local" terms without reference to homological constructions such as the Ext algebra $Ext_E^*(M, M)$. Only much later was this extended by J. Pevtsova and the author [28], [29] to apply not just to elementary abelian p -groups but to all finite groups; indeed, in doing so, Friedlander and Pevtsova formulated this comparison for all finite group schemes.

This leads us to other "group-like" objects. Such a consideration was foreshadowed by the work of Avrunin-Scott who solved Carlson's conjecture by considering a different Hopf algebra (the restricted enveloping algebra of an abelian Lie algebra with trivial restriction) whose underlying algebra is isomorphic to kE . B. Parshall and the author wrote a series of papers (see, for example, [24],[25],[26],[27]) introducing and exploring a support theory for the p -restricted representations of an arbitrary finite dimensional restricted Lie algebra. This entailed the consideration of the cohomology algebra $H^*(U^{[p]}(\mathfrak{g}), k)$ of the restricted enveloping algebra $U^{[p]}(\mathfrak{g})$; modules for $U^{[p]}(\mathfrak{g})$ are p -restricted representations of \mathfrak{g} .

Subject to restrictions on the prime p , work of Parshall and the author together with work of J. Jantzen [38] showed that Carlson's conjecture for elementary abelian p -groups generalized to any finite dimensional restricted Lie algebra \mathfrak{g} , comparing $|\mathfrak{g}|_M$ to the generalization $V(\mathfrak{g})_M$ of Carlson's rank variety defined in "local" representation-theoretic terms rather than using homological constructions.

(Subsequently, A. Suslin, C. Bendel and the author formulated and proved such a comparison for all primes p [58].) This comparison enables proofs of properties for the support variety construction $M \mapsto V(\mathfrak{g})_M$, some of which are achieved using homological methods and some using more geometric, representation theoretic methods.

In this paper, “linear algebraic group over k ” refers to a reduced, irreducible, affine group scheme of finite type over k , always assumed to be of characteristic $p > 0$ for some prime p . For such a linear algebraic group G , there is a Frobenius morphism $F : G \rightarrow G^{(1)}$; if G is defined over the prime field \mathbb{F}_p , the Frobenius morphism is an endomorphism $F : G \rightarrow G$. The kernel of F is a height 1 “infinitesimal group scheme” of finite type over k denoted $G_{(1)}$. For a finite group scheme of the form $G_{(r)} = \ker\{F^r : G \rightarrow G^{(r)}\}$ (the r -th Frobenius kernel of the linear algebraic group G) and a finite dimensional $G_{(r)}$ -module M , A. Suslin, C. Bendel, and the author give in [57] and [58] a representation-theoretic formulation, denoted $V(G_{(r)})_M$, of the cohomological support variety $|G_{(r)}|_M$. This alternate description is formulated in terms of the restriction of M along infinitesimal 1-parameter subgroups $\psi : \mathbb{G}_{a(r)} \rightarrow G$.

Finite groups and Frobenius kernels of linear algebraic groups are examples of finite group schemes. In [28], [29], J. Pevtsova and the author extended the theory of support varieties to arbitrary finite group schemes, generalizing “cyclic shifted subgroups” considered by J. Carlson in the case of elementary abelian p -groups and reinterpreting infinitesimal 1-parameter subgroups considered by A. Suslin, C. Bendel, and the author in the case of infinitesimal group schemes over k . The finite dimensionality of the cohomology algebra $H^*(G, k)$ of a finite group scheme proved by A. Suslin and the author [4] plays a crucial role in these theories of cohomological support varieties. In these extensions of the original theory for finite groups, one requires a suitable criterion of the detection modulo nilpotence of elements of $H^*(G, k)$; for finite groups, such a detection result is one of D. Quillen’s key theorems.

Although many of the basic techniques used in establishing properties for cohomological support varieties for finite group schemes do not apply to linear algebraic groups, we have continued to seek a suitable theory of support varieties for linear algebraic groups. After all, a major justification for the consideration of Frobenius kernels is that the collection $\{G_{(r)}, r > 0\}$ has representation theory that of G whenever G is a simply connected, simple linear algebraic group as shown by J. Sullivan [56] (see also [17]).

However, the rational cohomology of a simple algebraic group vanishes in positive degree, so that cohomological methods do not appear possible. Furthermore, if the rational cohomology is non-trivial, it is typically not finitely generated. Finally, there are typically no non-trivial projective G -modules for a linear algebraic group as shown by S. Donkin [19]. Despite these difficulties, we present in Lecture IV a theory of support varieties for linear algebraic groups of “exponential type”.

Throughout these lectures, we use the simpler term “ G -module” rather than the usual “rational G -module” when referring to a “rational representation” of an affine group scheme G . We shall abbreviate $V \otimes_k W$ by $V \otimes W$ for the tensor product of k -vector spaces V, W .

1. LECTURE I: AFFINE GROUP SCHEMES OVER k

This lecture is a “recollection” of some elementary aspects of affine algebraic varieties over a field k and a discussion of group schemes over k . We recommend R. Hartshorne’s book “Algebraic Geometry” [35] and W. Waterhouse’s book “An introduction to affine group schemes” [61] for further reading.

We choose a prime p and consider algebraic varieties over an algebraically closed field k of characteristic $p > 0$. The assumption that k is algebraically closed both simplifies the algebraic geometry (through appeals to the Hilbert Nullstellensatz) and simplifies the form of various affine group schemes. Our hypothesis that k is not of characteristic 0 is necessary both for the existence of various finite group schemes and for the non-triviality of various structures. In Subsection I.3, we discuss some of the special features of working over such a field rather than working over a field of characteristic 0. In Subsection I.4, we discuss restricted Lie algebras and their p -restricted representations.

Here is the outline provided to those attending this lecture.

I.A Affine varieties over k .

- i.) k a field, alg closed;
- ii.) \mathbb{A}^n – affine space over k ;
- iii.) zero loci $Z(\{f_1, \dots, f_m\}) \subset \mathbb{A}^n$;
- iv.) Algebra $k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ of algebraic functions;
- v.) Hilbert nullstellensatz: $X \subset \mathbb{A}^n$ versus $k[x_1, \dots, x_n] \rightarrow k[X]$.

I.B Affine group schemes over k .

- i.) Examples: $\mathbb{G}_a, \mathbb{G}_m, GL_N, U_N$;
- ii.) Product on G gives coproduct on $k[G]$;
- iii.) Group objects in the category of affine algebraic varieties over k ;
- iv.) As representable functors from (comm k -alg) to (grps);
- v.) Hopf algebras.

I.C Characteristic $p > 0$.

- i.) Examples of k with $\text{char}(k) = p$;
- ii.) Geometric Frobenius on affine varieties/ k ;
- iii.) Lang map: $1/F : G \rightarrow G$;
- iv.) Frobenius kernels $G_{(r)} = \ker\{F^r : G \rightarrow G\}$;
- v.) Example of $GL_{N(r)}$.

I.D Lie algebra of G

- i.) $\text{Lie}(G)$, tangent space at identity as derivations on $k[G]$.
- ii.) Lie bracket $[-, -]$ and p -th power $(-)^p$;
- iii.) Examples of $\mathbb{G}_a, \mathbb{G}_m, GL_N$;
- iv.) Relationship to $G_{(1)}$.

Supplementary topics:

I.A Extending consideration to k not algebraically closed. Projective varieties.

I.B Simple algebraic groups and their classification. Working with categories and functors.

I.C Frobenius twists, $F : G \rightarrow G^{(1)}$. Arithmetic and absolute Frobenius maps.

I.D Complex, simple Lie algebras. Root systems.

1.1. Affine group schemes over k . We begin with a cursory introduction to affine algebraic geometry over an algebraically closed field k . For any $n > 0$, we denote by \mathbb{A}^n the set of n -tuples of elements of k , by $\underline{a} \in \mathbb{A}^n$ a typical n -tuple. What distinguishes algebraic geometry from other types of geometries is the role of algebraic functions on an algebraic variety. The ring of algebraic (i.e., polynomial) functions on \mathbb{A}^n is by definition the k -algebra $k[x_1, \dots, x_n]$ of polynomials in n variables with coefficients in k , $p(\underline{x}) = \sum_{\underline{d}} c_{\underline{d}} \underline{x}^{\underline{d}}$, where \underline{d} ranges over non-negative n -tuples $(d_1, \dots, d_n) \in \mathbb{N}^{\times n}$. We refer to $k[x_1, \dots, x_n]$ as the *coordinate algebra* of \mathbb{A}^n . For any $\underline{a} \in \mathbb{A}^n$, the value of $p(\underline{x})$ on \underline{a} is $\sum_{\underline{d}} c_{\underline{d}} \underline{a}^{\underline{d}}$.

The **Hilbert Nullstellensatz** tells us that $p(\underline{x})$ is the 0 polynomial (i.e., equals $0 \in k[x_1, \dots, x_n]$) if and only if $p(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}^n$. This has a general formulation which applies to any quotient $A = k[x_1, \dots, x_n]/I$ with no nonzero nilpotent elements: $\bar{p}(\underline{x}) \in A$ is 0 if and only if $p(\underline{a}) = 0$ for all $\underline{a} \in \mathbb{A}^n$ which satisfy $q(\underline{a}) = 0$ for all $q \in I$.

A closed subvariety of \mathbb{A}^n is the zero locus of a set S of polynomials, $Z(S) \subset \mathbb{A}^n$. Let $\langle S \rangle$ denote the ideal generated by S and let I_S denote the radical ideal of all $g \in k[x_1, \dots, x_n]$ for which some power of g lies in $\langle S \rangle$. Then $Z(S) = Z(I_S)$ and $k[x_1, \dots, x_n]/I_S$ is the ring of equivalence classes of polynomials $p(\underline{x})$ for the equivalence relation $p(\underline{x}) \sim q(\underline{x})$ if and only if $p(\underline{x}) - q(\underline{x})$ vanishes on every $\underline{a} \in Z(S)$. We say that $A = k[x_1, \dots, x_n]/I_S$ is the coordinate algebra of the affine algebraic variety $\text{Spec } A$ with underlying space $Z(S)$; the closed subsets of $Z(S)$ are the subsets $Z(T)$ with $T \supset S$. Thus, there is a natural bijection between the closed subsets of \mathbb{A}^n (i.e., the zero loci $Z(S) = Z(I_S)$) and their coordinate rings $k[x_1, \dots, x_n]/I_S$ of algebraic functions.

More generally, the data of an affine k -scheme (of finite type over k) is a commutative, finitely generated k algebra given non-uniquely as the quotient for some $n > 0$ of $k[x_1, \dots, x_n]$ by some ideal J . An affine scheme determines a functor from the category of commutative, finitely generated k -algebras to sets. For $A = k[x_1, \dots, x_n]/J$, this functor sends R to $\text{Hom}_{k\text{-alg}}(A, R)$; in other words, $\text{Hom}_{k\text{-alg}}(A, R)$ equals the set of all n -tuples $\underline{r} \in R^n$ with the property that $p(\underline{r}) = 0$ for all $p(\underline{x}) \in J$.

Since the Hilbert Nullstellensatz does not apply to an affine k -scheme A containing nilpotent elements, we use the Yoneda Lemma to conclude the identification of an affine scheme with its associated functor; thus, we may abstractly define an affine scheme as a representable functor from finitely generated commutative k -algebras to sets.

1.2. Affine group schemes. As made explicit in Definition 1.1, a linear algebraic group over k is, in particular, an algebraic variety over k . We introduced affine schemes in Subsection I.1 whose coordinate algebra might have nilpotent elements in order to consider Frobenius kernels $G_{(r)}$ of linear algebraic groups (see Definition 1.5).

Definition 1.1. A linear algebraic group G over k is an affine scheme over k whose associated functor is a functor from finitely generated commutative k -algebras to groups such that the coordinate algebra $k[G]$ of G is an integral domain..

For example, the linear algebraic group GL_N is the affine variety of $N \times N$ invertible matrices, with associated coordinate algebra is $k[x_{1,1}, \dots, x_{n,n}, z]/(\det(x_{i,j} \cdot z - 1))$. As a functor, GL_N sends a commutative k -algebra R to the group of $N \times N$ matrices with entries in R whose determinant is invertible in R (with group structure given by multiplication of matrices).

We denote GL_1 by \mathbb{G}_m , the multiplicative group. The coordinate algebra $k[\mathbb{G}_m]$ of \mathbb{G}_m is the polynomial algebra $k[x, x^{-1}] \simeq k[x, y]/(xy - 1)$. The associated functor sends R to the set of invertible elements R^\times of R with group structure given by multiplication.

An even simpler, and for that reason more confusing, example is \mathbb{G}_a , the additive group. The coordinate algebra $k[\mathbb{G}_a]$ of \mathbb{G}_a is $k[T]$. The associated functor sends R to itself, viewed as an abelian group (forgetting the multiplicative structure).

Definition 1.2. An affine group scheme over k is an affine k -scheme whose associated functor is a functor from finitely generated commutative k -algebras to groups.

We shall use an alternate formulation of affine group schemes, in addition to the formulation as a representable functor with values in groups. This formulation can be phrased geometrically as follows: an affine group scheme is a group object in the category of schemes.

To be more precise, we state this formally.

Definition 1.3. An affine group scheme G (over k) is the spectrum associated to a finitely generated, commutative k -algebra $k[G]$ (the coordinate algebra of G) equipped with a coproduct $\Delta_G : k[G] \rightarrow k[G] \otimes k[G]$ such that $(k[G], \Delta_G)$ is a Hopf algebra.

This coproduct gives the functorial group structure on the R -points $G(R) \equiv \text{Hom}_{k\text{-alg}}(k[G], R)$ of G for any finitely generated commutative k -algebra R : namely, composition with Δ_G determines

$$\begin{aligned} \text{Hom}_{k\text{-alg}}(k[G], R) \times \text{Hom}_{k\text{-alg}}(k[G], R) &\simeq \text{Hom}_{k\text{-alg}}(k[G] \otimes k[G], R) \\ &\rightarrow \text{Hom}_{k\text{-alg}}(k[G], R). \end{aligned}$$

For example, the coproduct on the coordinate algebra of GL_N is defined on the matrix function $X_{i,j} \in k[GL_N]$ by $\Delta_{GL_N}(X_{i,j}) = \sum_{\ell} X_{i,\ell} \otimes X_{\ell,j}$.

1.3. Characteristic $p > 0$. In this subsection, we convey some of the idiosyncrasies of characteristic p algebraic geometry. We have already mentioned one: the Frobenius kernels $G_{(r)}$ of a linear algebraic group G are defined only if the ground field k has positive characteristic. Unlike the remainder of the text, in this subsection we allow k to denote an arbitrary field of characteristic p (e.g., a finite field).

Let's begin by mentioning a few examples of fields of characteristic p , where p is a fixed prime number. For any power $q = p^d$ of p , there is a field (unique up to isomorphism) with exactly q elements, denoted \mathbb{F}_q . For any set of "variables" S and any k , there is the field (again, unique up to isomorphism) of all quotients $p(\underline{s})/q(\underline{s})$ of polynomials in the variables in S and coefficients in k such that $q(\underline{s})$ is not the 0 polynomial. Typically, we consider a finite set $\{x_1, \dots, x_n\}$ of variables; in this case, we denote the field $k(x, \dots, x_n)$. If $I \subset k[x_1, \dots, x_n]$ is a prime ideal, then $k[x_1, \dots, x_n]/I$ is an integral domain with field of fractions $\text{frac}(k[x_1, \dots, x_n]/I)$ of

transcendence degree over k equal to the dimension of the affine algebraic variety associated to $k[x_1, \dots, x_n]/I$.

The key property of a field k of characteristic p , and more generally of a commutative k -algebra A , is that $(a + b)^p = a^p + b^p$ for all $a, b \in A$. The p -th power map $(-)^p : A \rightarrow A$, $a \mapsto a^p$ is thus a ring homomorphism. However, if $a \in k$ does not lie in \mathbb{F}_p and if b is such that $b^p \neq 0$, then $(-)^p(a \cdot b) \neq a \cdot (-)^p(b)$ as would be required by k -linearity.

The (geometric) Frobenius map $F : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ is a map of k -algebras (i.e., it is a k -linear ring homomorphism) defined by sending an element $a \in k$ to itself, sending any x_i to x_i^p . Thus $F(\sum_{\underline{d}} c_{\underline{d}} \cdot \underline{x}^{\underline{d}}) = \sum_{\underline{d}} c_{\underline{d}} \cdot x^{p \cdot \underline{d}}$. Viewed as a self-map of affine space \mathbb{A}^n , $F : \mathbb{A}^n \rightarrow \mathbb{A}^n$ sends the n -tuple (a_1, \dots, a_n) to the n -tuple (a_1^p, \dots, a_n^p) (in other words, the inverse image of the maximal ideal $(x_1 - a_1, \dots, x_n - a_n)$ is the maximal ideal $(x_1 - a_1^p, \dots, x_n - a_n^p)$).

Definition 1.4. Let A be a finitely generated commutative k -algebra and express A in terms of generators and relations by $k[x_1, \dots, x_n]/(p_1, \dots, p_m)$. For any $p(\underline{x}) = \sum_{\underline{d}} c_{\underline{d}} \cdot \underline{x}^{\underline{d}} \in k[x_1, \dots, x_n]$, set $\phi(p(\underline{x})) = \sum_{\underline{d}} c_{\underline{d}}^p \cdot \underline{x}^{\underline{d}} \in k[x_1, \dots, x_n]$; thus $\phi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ is an isomorphism of algebras which is semi-linear over k .

We define $A^{(1)} = k[x_1, \dots, x_n]/(\phi(p_1), \dots, \phi(p_m))$.

We define the Frobenius map to be the k -linear map given by

$$F : A^{(1)} \rightarrow A, \quad \bar{x}_i \mapsto (\bar{x}_i)^p,$$

where \bar{x}_i is the image of x_i under either the projection $k[x_1, \dots, x_n] \twoheadrightarrow A^{(1)}$ or the projection $k[x_1, \dots, x_n] \twoheadrightarrow A$. Hence, if the ideal (p_1, \dots, p_m) is generated by elements in $\mathbb{F}_p[x_1, \dots, x_n]$ (i.e., if A is defined over \mathbb{F}_p), then the Frobenius map is an endomorphism $F : A \rightarrow A$.

To verify that $F : A^{(1)} \rightarrow A$ is well defined, we observe that $F(\phi(p(\underline{x}))) = (p(\underline{x}))^p$ for any $p(\underline{x}) \in k[x_1, \dots, x_n]$, so that $F((\phi(p_1), \dots, \phi(p_m))) \subset (p_1, \dots, p_m)$.

An intrinsic way to define $A^{(1)}$ is given in [33], which gives a quick way to show that the definition of $A^{(1)}$ does not depend upon generators and relations. Namely, $A^{(1)}$ is isomorphic as a k -algebra to the base change of A via the map $\phi : k \rightarrow k$ sending $a \in k$ to a^p .

One of the author's favorite constructions is the following construction of Serge Lang ([42]) using the Frobenius. Namely, if G is an affine group scheme over k which is defined over \mathbb{F}_p , then we have a morphism of affine k -schemes (but not of affine group schemes)

$$1/F : G \xrightarrow{id \times F} G \times G \xrightarrow{id \times inv} G \times G \xrightarrow{\mu} G.$$

If G is a simple algebraic group over k , then G is defined over \mathbb{F}_p and $1/F$ is a covering space map of G over itself (i.e., $1/F$ is finite, etale), a phenomenon which is not possible for Lie groups or linear algebraic groups over a field of characteristic 0.

We conclude this subsection with the example most relevant for our purposes, namely the example of Frobenius kernels.

Definition 1.5. Let G be a linear algebraic group over k . Then for any positive integer r , we define the r -th Frobenius kernel of G to be the affine group scheme defined as the kernel of the r -th iterate of Frobenius, $F^r : G \rightarrow G^{(r)}$. The functor

associated to $G_{(r)}$ sends a finitely generated commutative k -algebra R to the kernel of the r -th iterate of the Frobenius, $F^r : G(R) \rightarrow G^{(r)}(R)$.

So defined, the coordinate algebra $k[G_{(r)}]$ of $G_{(r)}$ is the quotient of $k[G]$ by the p^r -th power of the maximal ideal of the identity of G . (This quotient is well defined for any field, but is a Hopf algebra if and only if k is of characteristic p .)

A good example is the r -th Frobenius kernel of GL_N . We identify the functor $R \mapsto GL_{N(r)}(R)$ as sending R to the group (under multiplication) of $N \times N$ matrices with coefficients in R whose p^r -th power is the identity matrix. In characteristic p , if A, B are two such $N \times N$ matrices, then $(A \cdot B)^{p^r} = (A^{p^r}) \cdot (B^{p^r})$, so that $GL_{N(r)}(R)$ is indeed a group.

1.4. Restricted Lie algebras. In his revolutionary work on continuous actions (of Lie groups on real vector spaces), Sophus Lie showed that the continuous action of a Lie group is faithfully reflected by its “linearization”, the associated action of its Lie algebra. We may view the Lie algebra of a Lie group as the tangent space at the identity equipped with a Lie bracket on pairs of tangent vectors which is a “first order infinitesimal approximation” of the commutator of pairs of elements of the group. Exponentiation sends a Lie algebra map to a neighborhood of the identity in the Lie group.

This property of the Lie algebra to faithfully reflect the action of the Lie group fails completely in our context of representation theory of affine group schemes over a field of characteristic p . Instead, one should consider all “infinitesimal neighborhoods” $G_{(r)}$ of the identity of G . Nevertheless, the Lie algebra \mathfrak{g} of G and its representations play a central role in our considerations.

Definition 1.6. A Lie algebra \mathfrak{g} over k is a vector space equipped with a binary operation $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $[x, x] = 0$ for all $x \in \mathfrak{g}$ and the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

A Lie algebra is p -restricted if it has an additional “ p -operation” $[-]^{[p]} : \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies conditions (see [37]) satisfied by the p -th power of matrices in $\mathfrak{gl}_N = \text{Lie}(GL_N)$ and by the p -th power of derivations of algebras (over a field of characteristic p).

Any finite dimensional restricted Lie algebra \mathfrak{g} admits an embedding as a Lie algebra into some \mathfrak{gl}_N such that the p -operation of \mathfrak{g} is the restriction of the p -th power in \mathfrak{gl}_N . The subtlety here is that a Lie algebra is not equipped with an associative multiplication (except, accidentally, for \mathfrak{gl}_N). If $\mathfrak{g} \subset \mathfrak{gl}_N$ is an embedding of p -restricted Lie algebras, then the p -th power in \mathfrak{gl}_N of an element $X \in \mathfrak{g}$ is again in \mathfrak{g} and equals $X^{[p]} \in \mathfrak{g}$.

Given an affine group scheme G over k , the Lie algebra \mathfrak{g} of G can be defined as the space of G -invariant derivations of $k[G]$, a Lie subalgebra of the Lie algebra of all k -derivations of $k[G]$. Alternatively, \mathfrak{g} can be identified with the vector space of k derivations $X : k[G] \rightarrow k[G]$ based at the identity $e \in G$; in other words, elements of \mathfrak{g} can be viewed as k -linear functionals on $k[G]$ satisfying $X(f \cdot h) = f(e)X(h) + h(e)X(f)$, with bracket $[X, Y]$ defined to be the commutator $X \circ Y - Y \circ X$. Because we are working over a field of characteristic p , the p -fold composition of such a derivation X with itself is again a derivation based at e ; sending X to this p -fold composition, $X \mapsto X \circ \cdots \circ X$, equips \mathfrak{g} with a p -operation.

In other words, $Lie(G)$ is a p -restricted Lie algebra. For example, for $G = \mathbb{G}_a$, the associated p -restricted Lie algebra \mathfrak{g}_a is the 1-dimensional vector space k (whose bracket necessarily is 0) and the p -operation sends any $c \in k$ to 0. For $G = \mathbb{G}_m$, the associated Lie algebra is again the 1-dimensional vector space with trivial bracket, but the p -operation sends $a \in k$ to a^p .

As a lead-in to Lecture II, we recall the definition of a p -restricted representation of a restricted Lie algebra \mathfrak{g} . The “differential” of a representation of a group scheme over k is a p -restricted representation of $\mathfrak{g} = Lie(G)$

Definition 1.7. Let \mathfrak{g} be a restricted Lie algebra over k . A p -restricted representation of \mathfrak{g} is a k -vector space V together with a k -bilinear pairing

$$\mathfrak{g} \otimes V \rightarrow V, \quad (X, v) \mapsto X(v)$$

such that $[X, Y](v) = X(Y(v)) - Y(X(v))$ and $X^{[p]}(v)$ equals the result of iterating the action of X p -times, $X(X(\cdots X(v)) \cdots)$.

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of \mathfrak{g} , defined as the quotient of the tensor algebra $T^*(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ by the ideal generated by the relations $X \otimes Y - Y \otimes X - [X, Y]$ for all pairs $X, Y \in \mathfrak{g}$. Then the restricted enveloping algebra of \mathfrak{g} , denoted here as in [39] by $U^{[p]}(\mathfrak{g})$, is the quotient of $U(\mathfrak{g})$ by the ideal generated by the relations $X^{\otimes p} - X^{[p]}$ for all $x \in \mathfrak{g}$. If \mathfrak{g} has dimension n over k , then $U^{[p]}(\mathfrak{g})$ is a finite dimensional k -algebra of dimension n^p .

A structure of a p -restricted representation of \mathfrak{g} on a k -vector space V is naturally equivalent to a $U^{[p]}(\mathfrak{g})$ -module structure on V .

A good example is the “adjoint representation” of a restricted Lie algebra \mathfrak{g} . Namely, we define $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ sending (X, Y) to $X(Y) \equiv [X, Y]$. The Jacobi identity of \mathfrak{g} implies the condition that $[X_1, X_2](Y) = X_1(X_2(Y)) - X_2(X_1(Y))$ and the axioms of a p -operation imply that $X^{[p]}(Y) = [X, [X, \dots [X, Y] \dots]]$.

2. LECTURE II: ALGEBRAIC REPRESENTATIONS

Following Lecture I which discussed finite groups, restricted Lie algebras, Frobenius kernels, and algebraic groups (all of which we would include under the rubric of “group-like structures”), this lecture discusses what are the **algebraic** representations of these objects. Our basic reference for this lecture is the excellent book “Representations of Algebraic Groups” by J. Jantzen [39].

Here is the outline provided to participants attending this second lecture.

II.A Equivalent formulations of rational G -modules.

- i.) For M finite dimensional, matrix coefficients;
- ii.) Functorial actions;
- iii.) Comodules for coalgebra;
- iv.) Locally finite modules for hyperalgebra.

II.B Examples.

- i.) \mathbb{G}_a -modules, \mathbb{G}_m -modules;

- ii.) Modules arising from (strict polynomial) functors;
- iii.) Induced modules;
- iv.) Abelian category.

II.C Weights arising from action of a torus;

- i.) Borel's theorem about stable vector for B solvable;
- ii.) Highest weight of an irreducible;
- iii.) $H^0(\lambda)$ and Weyl character formula;
- iv.) Lusztig's Conjecture.

II.D Representations of Frobenius kernels.

- i.) General theory of representations of Artin algebras (e.g., Wedderburn theorem; injective = projective);
- ii.) Special case of $G_{a(r)}$;
- iii.) G -modules and $\{G_{(r)}\}$ -modules.

Topics for discussion/project:

II.A Working out diagrams for checking properties of coaction. Examples of GL_N -actions which are not algebraic Investigating the action of the Lie algebra on M associated to a rational action of G on M .

II.B Working out properties for the categories of finite dimensional and all rational G -modules. Frobenius reciprocity.

II.C Discussion of roots for a simple algebraic group. Understanding of Weyl's character formula (for complex reps).

II.D Expanded investigation of Artin algebras. Discussion of representations of kE , E elementary abelian. Lie algebra actions.

2.1. Algebraic actions. We are interested in the **algebraic** actions of the “group-like” structures G discussed in the previous lecture on vector spaces V over our chosen field k (which we take to be algebraically closed of characteristic p for some prime p). A group action of G on V is a pairing

$$\mu : G \times V \rightarrow V, \quad (g, v) \mapsto \mu(g, v)$$

whose “adjoint” is the corresponding group homomorphism $\rho_\mu : G \rightarrow \text{Aut}_k(V)$. For simplicity, we first assume that V is of some finite dimension N , so that ρ_μ takes the form

$$\rho_\mu : G \rightarrow \text{Aut}_k(V) \simeq GL_N.$$

A discrete action is one for which no further requirement on ρ_μ is imposed other than it be group homomorphism (on the k -points of G and GL_N , thus of the form $G(k) \rightarrow GL_N(k)$). A continuous action has the additional condition that composition with each matrix function

$$X_{i,j} \circ \rho_\mu : G(k) \simeq GL_N(k) \rightarrow k, \quad 1 \leq i, j \leq N$$

is continuous; for this to be meaningful, k must have a topology (e.g., for the fields \mathbb{R} , \mathbb{C} which are of course of characteristic 0).

Recall that the data of an affine scheme X (e.g., an affine group scheme) is equivalent to that of its coordinate algebra $k[X]$, often called its algebra of “regular

functions”. We prefer to view elements of $k[X]$ as “algebraic functions” from X to k (which formally means a function functorial with respect to maps of finitely generated k -algebras; namely, $f \in k[X]$ is equivalent to the following data: for any finitely generated commutative k -algebra A , a map of sets $\text{Hom}_{k\text{-alg}}(k[X], A) \rightarrow A$ (i.e., a function from the A points of X to A) which is functorial with respect to A . (Observe that $f \in k[X]$ is recovered from this data as the image of the identity $\text{Hom}_{k\text{-alg}}(k[X], k[X]) \rightarrow k[X]$; for any f and any A , we send $\phi \in \text{Hom}_{k\text{-alg}}(k[X], A)$ to the $\phi(f) \in A$.)

Before we formulate the definition of an algebraic action of a general affine algebraic group G on a k -vector space, we first consider algebraic actions of a linear algebraic group. The definition below implicitly uses the Hilbert Nullstellensatz.

Definition 2.1. Let G be a linear algebraic group over k and V a finite dimensional k -vector space of dimension N . Then an action $\mu : G \times V \rightarrow V$ of G on V is defined to be algebraic (usually called “rational”) if each matrix coefficient of μ ,

$$X_{i,j} \circ \rho_\mu : G(k) \rightarrow GL_N(k) \rightarrow k, \quad 1 \leq i, j \leq N,$$

is an element of $k[G]$.

Example 2.2. We give a first example of an algebraic action. Further examples will easily follow from alternative formulations in Proposition 2.3 of the algebraicity condition of Definition 2.1.

Let $G = GL_n$ and let V be the elements of degree d in the polynomial algebra $k[x_1, \dots, x_n]$. We define the group action

$$\mu : GL_n(k) \times V \rightarrow V, \quad g \cdot (x_1^{d_1} \cdots x_n^{d_n}) = (g \cdot x_1)^{d_1} \cdots (g \cdot x_n)^{d_n},$$

where $\sum_i d_i = d$ and where $g \cdot x_j = \sum_i X_{i,j}(g)x_i$. Thus, V is the d -fold symmetric power $S^d(k^n)$ of the “defining representation” of GL_n on k^n .

It is a good (elementary) exercise to verify that each matrix coefficient of μ is an element of $k[GL_n]$.

We can argue similarly for exterior powers $\Lambda^d(k^n)$. For example, $\Lambda^n(k^n)$ is a 1-dimensional representation of GL_n given by

$$\mu : GL_n(k) \times k \rightarrow k, \quad g \cdot v = \det(g)v;$$

the algebraicity condition is simply that $\rho_\mu : GL_n(k) \rightarrow GL_1(k) = k^\times \subset k$ is an element of $k[GL_n]$. Observe that this representation is “invertible”, in the sense that $\mu^{-1} : GL_n(k) \times k \rightarrow k, \quad g \cdot v = \det(g)^{-1}v$ is also algebraic.

We extend the definition of an algebraic action to encompass an affine group scheme over k acting on an arbitrary k -vector space.

Proposition 2.3. *Let G be an affine group scheme over k and V a k -vector space. Then the following two conditions on a group action $\mu : G(k) \times V \rightarrow V$ are equivalent.*

- (1) *There exists a k -linear map $\Delta_V : V \rightarrow V \otimes k[G]$ which provides V with the structure of a $k[G]$ -comodule; the pairing $\mu : G(k) \times V \rightarrow V$ is given by sending (g, v) to $((1 \otimes \text{ev}_g) \circ \Delta_V)(v)$.*
- (2) *There exists a pairing of functors $\underline{\mu} : G(-) \times ((-) \otimes V) \rightarrow (-) \otimes V$ on commutative k -algebras such that $\underline{\mu}(A) \times (A \otimes V) \rightarrow A \otimes V$ is an A linear action of $G(A)$ for any commutative k -algebra A ; the pairing $\mu : G(k) \times V \rightarrow V$ is given by taking $A = k$.*

Moreover, if V is finite dimensional, say of dimension N , then the first two conditions are equivalent to the condition:

(3) The adjoint of μ is a map $G \rightarrow GL_N$ of group schemes over k .

Furthermore, if G is a linear algebraic group and V has dimension N , then these equivalent conditions are equivalent to the algebraicity condition of Definition 2.1.

We define an algebraic action of G on an arbitrary vector space V over k to be one that satisfies the equivalent conditions (1) and (2).

Remark 2.4. We say that a group scheme G over k is a finite group scheme over k if $k[G]$ is finite dimensional (over k). For any finite group scheme G over k and any k -vector space V , there is a natural bijection between comodule structures $\Delta_V : V \rightarrow V \otimes k[G]$ and module structures $(k[G])^\# \otimes V \rightarrow V$. Namely, we associate to Δ_V the pairing

$$(k[G])^\# \otimes V \xrightarrow{1 \otimes \Delta_V} (k[G])^\# \otimes V \otimes k[G] \rightarrow V,$$

where the second map is given by the evident evaluation $(k[G])^\# \otimes k[G] \rightarrow k$.

Notation 2.5. If G is a finite group scheme over k , we denote by kG the algebra $(k[G])^\#$ and refer to kG as the group algebra of G . In [39], kG is called the distribution algebra of G (of k -distributions at the identity) whenever G is an infinitesimal group scheme (i.e., whenever G is a connected, finite group scheme).

If G is a linear algebraic group over k we denote by kG the colimit $\varinjlim_r kG_{(r)}$ and refer to this algebra as the group algebra of G ; once again, this is called the distribution algebra of G by Jantzen in [39]; it also is called the hyperalgebra of G by many authors (e.g., [56]).

2.2. Examples. Now, for some more examples.

Example 2.6. (1) Take G to be any affine group scheme. Then the coproduct $\Delta_G : k[G] \rightarrow k[G] \otimes k[G]$ determines the right regular action $\mu : G \times G \rightarrow G$ (where the first factor of $G \times G$ is the object acted upon and the second factor is the group acting).

(2) Take $G = \mathbb{G}_{a(r)}$ for some $r > 0$. Then $k[\mathbb{G}_{a(r)}]$ equals $k[T]/T^{p^r}$ with linear dual $kG_{a(r)} = k[u_0, \dots, u_{r-1}]/(\{u_i^p\})$; we identify u_i as the k -linear map sending T^n to 0 if $n \neq p^i$ and sending T^{p^i} to 1. Since $k[u_0, \dots, u_{r-1}]/(\{u_i^p\})$ can be identified with the group algebra of the elementary abelian p -group $(\mathbb{Z}/p\mathbb{Z})^{\times r}$, we conclude an equivalence of categories between the category of $\mathbb{G}_{a(r)}$ -representations and the category of representations of $(\mathbb{Z}/p\mathbb{Z})^{\times r}$ on k -vector spaces.

(3) Take $G = \mathbb{G}_a$, with $k[\mathbb{G}_a] = k[T]$ and consider

$$k\mathbb{G}_a \equiv \varinjlim_r (k[G_{a(r)}])^\# = k[u_0, \dots, u_n, \dots]/(\{u_i^p, i \geq 0\}).$$

Then an algebraic action of \mathbb{G}_a on V is equivalent to the data of infinitely many p -nilpotent operators $u_i : V \rightarrow V$ which pair-wise commute such that for any $v \in V$ there exist only finitely many u_i 's with $u_i(v) \neq 0$.

(4) Take $G = \mathbb{G}_m$, with coordinate algebra $k[\mathbb{G}_m] \simeq k[T, T^{-1}]$. A $k[\mathbb{G}_m]$ -comodule structure on V has the form

$$\Delta_V : V \rightarrow V \otimes k[\mathbb{G}_m], \quad v \mapsto \sum_{n \in \mathbb{Z}} p_n(v) \otimes T^n.$$

where each $p_n : V \rightarrow V$ is a k -endomorphism of V . One checks that $\sum_n p_n = id_V$, $p_m \circ p_n = \delta_{m,n} p_n$ which implies that $V = \bigoplus_{n \in \mathbb{Z}} V_n$ where $V_n = \{v \in V : \Delta_V(v) = v \otimes T^n\}$. For $v \in V_n$, $a \in \mathbb{G}_m(k) = k^\times$ acts by sending v to $T^n(a) \cdot v = a^n \cdot v$. In particular, V_n is a direct sum of 1-dimensional irreducible \mathbb{G}_m -modules whose isomorphism class is characterized by $n \in \mathbb{Z}$, the power through which k^\times acts.

It is useful to view the action of \mathbb{G}_m on some 1-dimensional irreducible \mathbb{G}_m -module as the composition of a homomorphism $\lambda : \mathbb{G}_m \rightarrow \mathbb{G}_m$ with the defining action of \mathbb{G}_m on k . Such a homomorphism (or character) is given by a choice of $n \in \mathbb{Z}$ (corresponding to the map on coordinate algebras $k[T, T^{-1}] \rightarrow k[T, T^{-1}]$ sending T to T^n). See Definition 2.7 below.

- (5) Take $G = GL_n$ and fix some $d > 0$. Consider $\rho : GL_n \rightarrow GL_N$ (corresponding to an action of GL_n on a vector space of dimension N) with the property that $X_{i,j} \circ \rho : GL_n \rightarrow k$ extends to a function $GL_n \subset \mathbb{A}^{n^2} \rightarrow k$ which is a homogeneous polynomial of degree d in the n^2 variables of \mathbb{A}^{n^2} for some $d > 0$ independent of (i, j) . Such an action is said to be a polynomial representation homogeneous of degree d of GL_n (of rank N). This generalizes the examples of Example 2.2.

We next recall the definition of the character group $X(G)$ of G , extending the discussion of Example 2.6.4. For our purposes, the diagonalizable affine group schemes of most interest are (split) tori T (isomorphic to some product of \mathbb{G}_m 's) and their Frobenius kernels.

Definition 2.7. Let G be an affine group scheme over k . A character of G is a homomorphism of group schemes over k , $\lambda : G \rightarrow \mathbb{G}_m$. Using the abelian group structure of \mathbb{G}_m , the set of characters of G inherits an abelian group structure which is denoted by $X(G)$.

An affine group scheme G is said to be diagonalizable if its coordinate algebra $k[G]$ is isomorphic as a Hopf algebra to the group algebra $k\Lambda$, where $\Lambda = X(G)$ is the character group of G . (Here, the coproduct on $k\Lambda$ is given by $\lambda \mapsto \lambda \otimes \lambda$.)

For example, \mathbb{G}_m is a diagonalizable group scheme over k with coordinate algebra $k[\mathbb{G}_m] \simeq k\mathbb{Z}$.

Proposition 2.8. *Let G be a diagonalizable group scheme with character group Λ . Then an algebraic representation of G on a k -vector space V has a natural decomposition as a direct sum, $V \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda$, where $V_\lambda = \{v \in V : g \cdot v = \lambda(g) \cdot v, \forall g \in G\}$.*

One important construction which produces algebraic representations is ‘‘induction to G from a closed subgroup $H \subset G$ ’’. This is sometimes called ‘‘co-induction’’ by ring theorists.

Definition 2.9. Let G be an affine group scheme and $H \subset G$ a closed subgroup scheme (i.e., the coordinate algebra of H is the quotient of $k[G]$ by a Hopf ideal). Let $H \times W \rightarrow W$ be an algebraic representation of H . Then the induced representation $ind_H^G(W)$ has underlying vector space given by $(k[G] \otimes W)^H$, the elements of $k[G] \otimes W$ fixed under the diagonal action of H acting on $k[G]$ through the right regular representation and on W as given; the G action $G \times (k[G] \otimes W)^H$ is given by the left regular representation on G .

2.3. Weights for G -modules. If G is a linear algebraic group over k , then a Borel subgroup of G is a maximal solvable, closed, connected algebraic subgroup. With our standing hypothesis that k is algebraically closed, all such Borel subgroups $B \subset G$ are conjugate in G . Any maximal torus T of G (i.e., a product of \mathbb{G}_m 's of maximal rank) is contained in some Borel $B \subset G$ and maps isomorphically onto the quotient of B by its unipotent radical U ; thus $B \simeq U \rtimes T$.

Definition 2.10. Let G be a linear algebraic group, $T \subset G$ a maximal torus, ℓ the rank of T (so that $T \simeq \mathbb{G}_m^{\times \ell}$). Let V be a G -module (i.e., an algebraic representation of G on the k -vector space V). Then the set of weights of V are those characters $\lambda \in X(T)$ with the property that the decomposition of V as a T -module has non-zero λ -eigenspace (i.e., $V_\lambda \neq 0$).

If G is unipotent (for example, the algebraic subgroup of GL_N of upper triangular matrices with 1's on the diagonal), then its maximal torus is simply the identity group. However, for G simple (or, more generally for G reductive), this concept of the weights of a representation is the key to parametrizing the irreducible representations of G as stated in Proposition 2.12.

Definition 2.11. Let G be an affine group scheme over k . A non-zero G -module V (given by an algebraic action $\mu : G \times V \rightarrow V$) is said to be **irreducible** if V contains no non-trivial G submodule; in other words, the only $k[G]$ -comodules contained in V are 0 and V itself.

A non-zero G -module V is said to be indecomposable if there do not exist two non-zero G submodules V', V'' of V such that $V \simeq V' \oplus V''$.

We remind the reader that a reductive algebraic group over k is a linear algebraic group whose maximal connected, normal, unipotent subgroup is trivial. Every reductive algebraic group over k is defined over \mathbb{F}_p .

Proposition 2.12. *Let G be a reductive algebraic group over k , $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. There is a 1-1 correspondence between the dominant weights $X(T)_+ \subset X(T)$ and (isomorphism classes of) irreducible G -modules. Namely, to a dominant weight λ , one associates the irreducible G -module*

$$(2.12.1) \quad L_\lambda \equiv \text{soc}_G(\text{ind}_B^G(k_\lambda))$$

(where the socle of a G -module is the direct sum of all irreducible G -submodules). Here, k_λ is the 1-dimensional B -module with algebraic action $B \times k_\lambda \rightarrow k_\lambda$ sending (b, a) to $\lambda(\bar{b})a$, where $\bar{b} \in T$ is the image of b in the quotient $B \twoheadrightarrow T$ and $T \times k_\lambda \rightarrow k_\lambda$ has adjoint $\lambda : T \rightarrow \mathbb{G}_m$. Moreover, the canonical map $k_\lambda \rightarrow \text{ind}_B^G(k_\lambda)$ identifies k_λ with the (1-dimensional) λ -weight space of $\text{ind}_B^G(k_\lambda)$, and λ is the unique highest weight of $\text{ind}_B^G(k_\lambda)$ and of L_λ .

Although $\text{ind}_B^G(k_\lambda)$ as in Proposition 2.12 is indecomposable, the inclusion $L_\lambda \subset \text{ind}_B^G(k_\lambda)$ is an equality only for "small" λ .

2.4. Representations of Frobenius kernels. In this subsection, G will denote a linear algebraic group over k . We briefly investigate the algebraic representations of the group scheme $G_{(r)} \equiv \ker\{F^r : G \rightarrow G^{(r)}\}$.

Proposition 2.13. *For any $r > 0$, the coordinate algebra $k[G_{(r)}]$ of $G_{(r)}$ is a finite dimensional, local (commutative) k -algebra. Moreover, as a $G_{(r)}$ representation,*

$k[G_{(r)}]$ is isomorphic to its k -linear dual $kG_{(r)}$. Consequently, in the category of $k[G_{(r)}]$ -comodules (naturally isomorphic to the category of $kG_{(r)}$ -modules), an object is injective if and only if it is projective.

For $r = 1$, $kG_{(1)}$ is isomorphic as an algebra to the restricted enveloping algebra of $Lie(G)$ (see Definition 1.7).

We remark that since $kG_{(r)}$ is a finite dimensional k -algebra which is injective as an algebra over itself, many standard techniques for studying the representation theory of Artin algebras over k apply. Of course, $kG_{(r)}$ has more structure: it is a cocommutative Hopf algebra.

For simplicity, assume G is defined over the prime field \mathbb{F}_p (which means that the Hopf algebra $k[G]$ arises as the base change of a Hopf algebra over \mathbb{F}_p , $k[G] = k \otimes_{\mathbb{F}_p} \mathbb{F}_p[G]$). This assumption enables us view the Frobenius map as an endomorphism of G , $F : G \rightarrow G$.

Definition 2.14. Given an algebraic action $\mu : G \times V \rightarrow V$ of an affine group scheme G on a k -vector space V , we define the (first) Frobenius twist $V^{(1)}$ of V to be the k -vector space whose underlying abelian group equals that of V and whose k -linear action is given by $c \cdot v^{(1)} = (c^p v^{(1)})^{(1)}$, for $c \in k, v^{(1)} \in V$; the algebraic action of G on $V^{(1)}$ is defined as

$$\mu^{(1)} \circ (F \times 1) : G \times V^{(1)} \rightarrow G^{(1)} \times V^{(1)} \rightarrow V^{(1)}.$$

We inductively define $V^{(r+1)}$ to be $(V^{(r)})^{(1)}$ for any $r \geq 0$.

Since $G_{(r)}$ is the kernel of F^r , we immediately conclude that the action of $G_{(r)}$ on $V^{(r)}$ is trivial.

Example 2.15. Let G be a simply connected, semi-simple algebraic group over k and consider the irreducible G -module L_λ of highest weight λ . The Steinberg tensor product theorem [55] asserts that

$$(2.15.1) \quad L(\lambda) \simeq L(\lambda_1) \otimes L(\lambda_2)^{(1)} \otimes \cdots \otimes L(\lambda_s)^{(s)}$$

where $\lambda = \sum_{i=0}^s p^i \lambda_i$ and each λ_i is a p -restricted dominant weight. The condition that λ be restricted is defined combinatorially, but is equivalent to the condition that L_λ restricts to an irreducible $G_{(1)}$ -module.

In other words, each irreducible G -module is a tensor product of Frobenius twists of G -modules which arise as irreducible restricted representations of $U^{[p]}(\mathfrak{g})$.

Example 2.15 emphasizes that restricting a G -module V to $kG_{(1)}$ (we view this as taking the first order approximation of the G -action) loses enormous amount of information: for example, irreducible G -modules $L_\lambda, L_{\lambda'}$ have isomorphic restrictions to $kG_{(1)}$ if and only if $\lambda - \lambda'$ can be written as a difference of p -multiples of dominant weights.

The following theorem of J. Sullivan in [56] reveals the close connection of the representation theory of the family $\{G_{(r)}, r > 0\}$ of algebras with the rational representations of G . Recall that if A is a k -algebra and M is an A -module, then M is said to be locally finite if each finite dimensional subspace of M is contained in some finite dimensional A -submodule of M .

Theorem 2.16. *Let G be a simply connected, simple algebraic group over k . Then there is an equivalence of categories between the category of G -modules and locally finite modules for the k -algebra $\varinjlim_r kG_{(r)}$.*

3. LECTURE III: COHOMOLOGICAL SUPPORT VARIETIES

In this lecture we provide a quick overview of the theory of cohomological support varieties for finite groups, p -restricted Lie algebras, and finite group schemes. In the lecture, the author discussed a comparison between one formulation of cohomological support varieties for linear algebraic groups and the theory discussed in the final lecture (i.e., Lecture IV) using 1-parameter subgroups. In the text below, we briefly discuss very recent computations for unipotent linear algebraic groups.

We begin with the outline prepared in advance of the lectures, an outline which does not well summarize the text which follows.

III.A Indecomposable versus irreducible.

- i.) examples of semi-simplicity;
- ii.) examples of $\mathbb{Z}/p^{\times n}$;
- iii.) concept of wild representation type.

III.B Derived functors.

- i.) left exact functors, $(-)^G = \text{Hom}_{G\text{-mod}}(k, -)$;
- ii.) injective resolutions and right derived functors;
- iii.) $\text{Ext}_G^1(k, M)$;
- iv.) representation of $\text{Ext}_G^i(k, M)$ as equivalence classes of extensions.

III.C Commutative algebras and affine varieties.

- i.) $\text{Spec } A$, the prime ideal spectrum;
- ii.) elementary examples;
- iii.) $\text{Spec } H^\bullet(G, k)$;
- iv.) (Krull) dimension and growth;
- v.) $\text{Spec } H^\bullet(G, k)/\text{ann}(\text{Ext}_G^*(M, M))$;
- vi.) Quillen's stratification theorem.
- vii.) Carlson's conjecture for $G = \mathbb{Z}/p^{\times n}$.

III.D Linear algebraic groups.

- i.) $H^*(\mathbb{G}_a, k)$;
- ii.) $H^\bullet(U_3, k)_{\text{red}}$;
- iii.) Definitions of $V^{\text{coh}}(G), V^{\text{coh}}(G)_M$.

Topics for discussion/projects:

III.A Presentation of finite/tame/wild representation type. Presentation of families of indecomposable $\mathbb{Z}/p^{\times 2}$ -modules.

III.B Exposition of representation of $\text{Ext}_G^i(N, M)$ by extension classes. Discussion of other derived functors. Project on spectral sequences.

III.C Discussion of algebraic curves over k . Hilbert Nullstellensatz. Computation of $H^*(\mathbb{Z}/p^{\times n}, k)$.

III.D Open questions about detection modulo nilpotents and finite generation.

3.1. Indecomposable versus irreducible. We revisit the distinction between irreducible and indecomposable as defined in Definition 2.11.

Let R be a (unital associative) ring and consider two left R -modules M, N . Then an extension of M by N is a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ of left R -modules. We utilize the equivalence relation on such extensions for fixed R -modules M, N as the equivalence relation generated by commutative diagrams of R -modules of the form

$$(3.0.1) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

relating the upper extension to the lower extension. The set of such extensions of M by N form an abelian group denoted $Ext_R^1(M, N)$. Cohomology groups (i.e., Ext -groups) at their most basic level are invariants devoted to detecting inequivalent extensions. Rather than give information about basic building blocks (i.e., irreducible R -modules), cohomology can be used to show that a pair of indecomposable R -modules with the same irreducible “constituents” are not isomorphic.

For some purposes, one “kills” such extensions by considering the Grothendieck group $K'_0(R)$ defined as the free abelian group on the set of isomorphism classes of left R -modules modulo the equivalence relation $E \sim M \oplus N$ whenever E is an extension of M by N . This construction eliminates the role of cohomology. Said differently, if R satisfies the condition that every R -module splits as a direct sum of irreducible modules, then (positive degree) cohomology groups $Ext_R^i(M, N)$ vanish.

Rather than consider an abelian category of R -modules, we shall consider the abelian category $Mod_k(G)$ of G -modules for an affine group scheme G over k . If $k[G]$ is finite dimensional over k , then $Mod_k(G)$ is isomorphic to the category $Mod(R)$ of left R -modules, where $R = kG$; for any affine group scheme, $Mod_k(G)$ is equivalent to the abelian category of $k[G]$ -comodules.

The representation theory of G is said to be **semi-simple** if every indecomposable G -module is irreducible.

- Example 3.1.** (1) Let G be a diagonalizable affine group scheme as in Definition 2.7. Then the representation theory of G is semi-simple.
- (2) Let G be the finite group \mathbb{Z}/p ; the coordinate algebra of \mathbb{Z}/p equals $Hom_{sets}(\mathbb{Z}/p, k)$ whose dual algebra is the group algebra $k\mathbb{Z}/p = k[x]/(x^p - 1) \simeq k[t]/t^p$. There are p distinct isomorphism classes of indecomposable \mathbb{Z}/p -modules, represented (as modules for $k[t]/t^p$) by the quotients $k[t]/t^i$, $1 \leq i \leq p$ of $k[t]/t^p$. Only the 1-dimensional “trivial” kG -module k is irreducible.
- (3) Let $G = GL_n(1)$, so that $kG \simeq U^{[p]}(\mathfrak{gl}_n)$ and let $V = S^p(k^n) \simeq k[x_1, \dots, x_n]_p$ denote the p -fold symmetric power of the defining representation k^n of GL_n (see Example 2.2). Consider the subspace $W \subset V$ spanned by $\{x_1^p, \dots, x_n^p\}$. Then W is a G -submodule of V , but there does not exist another G -submodule $V' \subset V$ such that $V \simeq W \oplus V'$.

In some sense, the “ultimate goal” of the representation theory of G is the description of all isomorphism classes of indecomposable G -modules (as for $G = \mathbb{Z}/p$ in Example 3.1.2.) However, this goal is far too optimistic. Even for $G = \mathbb{Z}/p^{\times r}$ (for $r \geq 3$; for $p > 2$, we need only that $r \geq 2$), the representation theory of G is “wild”, a condition which can be formulated as the condition that the abelian category of the finite dimensional representations of *any* finite dimensional k -algebra Λ can be

embedded in the abelian category $\text{mod}_k(G)$ of finite dimensional G -modules. (See, for example, [9].)

3.2. Derived functors. We assume that the reader is familiar with the basics of homological algebra. We refer the reader to C. Weibel’s book “An Introduction to Homological Algebra” [62] for background. In our context, the role of cohomology is to give information about the structure of indecomposable G -modules, structure that arises by successive extensions of irreducible G -modules.

The following proposition (see [39]) insures that the abelian category $\text{Mod}_k(G)$ has enough injectives, thereby enabling the formulation of the $\text{Ext}_G^i(M, N)$ groups as right derived functors of the functor

$$\text{Hom}_G(M, -) : \text{Mod}_k(G) \rightarrow (\text{Ab})$$

from the abelian category of G -modules to the abelian category of abelian groups. (Indeed, this functor takes values in the abelian category of k -vector spaces.) As mentioned in the introduction, for “most” linear algebraic groups G , $\text{Mod}_k(G)$ has no non-trivial projectives [19] so that we can not define $\text{Ext}_G^*(-, -)$ -groups by using a projective resolution of the contravariant variable .

Proposition 3.2. *Let G be an affine group scheme over k . Then $k[G]$ (with G -action given as the left regular representation) is an injective G -module. Moreover, if M is any G -module, then M admits a natural embedding $M \hookrightarrow M \otimes k[G]$ and $M \otimes k[G]$ is an injective G -module.*

Definition 3.3. Let G be an affine group scheme over k . For any pair of G -modules M, N and any $i \geq 0$, we define

$$\text{Ext}_G^i(M, N) \equiv (R^i(\text{Hom}_G(M, -)))(N),$$

the value of the i -th right derived functor of $\text{Hom}_G(M, -)$ applied to N .

In particular, one has the graded commutative algebra $H^*(G, k) \equiv \text{Ext}_G^*(k, k)$. For $p = 2$, $H^*(G, k)$ is commutative. For $p > 2$, we consider the commutative subalgebra $H^\bullet(G, k) \subset H^*(G, k)$ generated by cohomology classes of even degree is a commutative k -algebra; for $p = 2$, we set $H^\bullet(G, k)$ equal to the commutative k -algebra $H^*(G, k)$. An important theorem of B. Venkov [60] and L. Evens [20] asserts that $H^*(G, k)$ is finitely generated for any finite group G ; this was generalized to arbitrary finite group schemes by A. Suslin and the author [33].

Remark 3.4. One can describe $\text{Ext}_G^n(M, N)$ as the abelian group of equivalence classes of n -extensions of M by N (cf. [43, III.5]), where the equivalence relation arises by writing an n -extension as a composition of 1-extensions and using pushing forward and pulling-back of 1-extensions.

3.3. The Quillen variety $|G|$ and the cohomological support variety $|G|_M$.

In what follows, if A is a finitely generated commutative k -algebra (such as $H^\bullet(G, k)$ with grading ignored), then we denote by $\text{Spec } A$ the affine scheme whose set of points is the set of prime ideals of A equipped with the Zariski topology and whose structure sheaf $\mathcal{O}_{\text{Spec } A}$ is a sheaf of commutative k -algebras whose value on $\text{Spec } A$ is A itself. For $A = H^\bullet(G, k)$, we denote by $|G|$ the topological space underlying $\text{Spec } H^\bullet(G, k)$; in other words, we ignore the structure sheaf $\mathcal{O}_{\text{Spec } A}$ on G .

The Atiyah-Swan conjecture for a finite group G states that the *growth* of a minimal projective resolution of k as a G -module should be one less than the largest rank of elementary p -subgroup $E \simeq \mathbb{Z}/p^{\times r} \subset G$. This growth can be seen to equal the Krull dimension of $H^\bullet(G, k)$.

Daniel Quillen proved this conjecture and much more by introducing geometry into the study of $H^*(G, k)$. A simplified version of Quillen's main theorem is the following. Following Quillen, we let $\mathcal{E}(G)$ the category of elementary abelian p -groups of G whose Hom -sets $\text{Hom}_{\mathcal{E}}(E, E')$ consist of group homomorphisms $E \rightarrow E'$ which can be written as a composition of an inclusion followed by conjugation by an element of G .

Theorem 3.5. *Let G be a finite group. If $\zeta \in H^*(G, k)$ is not nilpotent, then there exists some elementary abelian p -subgroup $E \simeq \mathbb{Z}/p^{\times r} \subset G$ such that ζ restricted to $H^*(E, k)$ is non-zero.*

Furthermore, the morphisms $\text{Spec } H^\bullet(E, k) \rightarrow \text{Spec } H^\bullet(G, k)$ are natural with respect to $E \in \mathcal{E}(G)$ and determine a homeomorphism

$$\varinjlim_{E < G} |E| \xrightarrow{\sim} |G|.$$

This is a fantastic theorem. Before Quillen's work, we knew very little about computations of group cohomology and this theorem applies to all finite groups. However, it actually does not compute any of the groups $H^i(G, k)$ for $i > 0$. For example, $H^i(GL_{2n}(\mathbb{F}_{p^d}), k) = 0, 1 \leq i \leq f(n, d)$ with $\varinjlim_d f(n, d) = \infty$. On the other hand, Theorem 3.5 tells us that the Krull dimension of $H^\bullet(GL_{2n}(\mathbb{F}_{p^d}), k)$ equals $d \cdot n^2$, for this is the rank of the largest elementary abelian p -group inside $GL_{2n}(\mathbb{F}_{p^d})$.

J. Alperin and L. Evens initiated in [2] the study of the growth of projective resolutions for an arbitrary finite dimensional kG -module for a finite group G (extending Quillen's theorem for the trivial k -module k). This led Jon Carlson in [11] to introduce the following notion of the support variety of a finite group.

Definition 3.6. Let G be a finite group and denote by $|G|$ the space (with the Zariski topology) underlying $\text{Spec } H^\bullet(G, k)$. For any finite dimensional kG -module M , denote by $I(M) \subset H^\bullet(G, k)$ the ideal of those elements α such that α acts as 0 on $\text{Ext}_G^*(M, M)$. The cohomological support variety $|G|_M$ is the closed subset of $|G|$ defined as the "zero locus" of $I(M)$. In other words,

$$|G|_M = \text{Spec } H^\bullet(G, k)/I(M) \subset |G|.$$

We remark that the ideal $I(M)$ of Definition 3.6 is equal to the kernel of the natural map of graded k -algebras $H^*(G, k) \rightarrow \text{Ext}_G^*(M, M)$ given in degree n by tensoring an n -extension of k by k by M to obtain an n -extension of M by M .

The following theorem states two of Carlson's early results concerning support varieties, both of which have subsequently been show to generalize to all finite group schemes. The second result is especially important (as well as elegant).

Theorem 3.7. (*J. Carlson, [12]*) *Let G be a finite group.*

- (1) *If M is a finite dimensional indecomposable G -module, then the projectivization of $|G|_M$ is connected.*
- (2) *Let $C \subset |G|$ be a (Zariski) closed, conical subvariety of $|G|$. Once given a choice of generators for the ideal in $H^\bullet(G, k)$ defining C , one can explicitly construct a finite dimensional kG -module M_C such that $|G|_{M_C} \simeq C$.*

Following the development of the theory of support varieties for finite groups, various mathematicians considered the generalization of the theory to other “group-like” structures as mentioned in the introduction.

Definition 3.6 can be repeated verbatim for an arbitrary finite group scheme. More interesting, a “representation theoretic model” for $|G|_M$ has been developed for any finite group scheme. This began with the model $\mathcal{N}_p(\mathfrak{g})_M$ of $|G|_M$ in terms of the p -nilpotent cone $\mathcal{N}_p(\mathfrak{g})$ for a finite dimensional $U^{[p]}(\mathfrak{g})$ -module M , where \mathfrak{g} is an arbitrary finite dimensional p -restricted Lie \mathfrak{g} . This was extended to the model $V(G_{(r)})_M$ in terms of infinitesimal 1-parameter subgroups of G for any infinitesimal group scheme $G_{(r)}$ in the work of A. Suslin, C. Bendel, and the author. Finally, in the work of the author and J. Pevtsova, a model $\Pi(G)$ isomorphic to $|G|_M$ was formulated in terms of equivalence classes of π -points. (See Definition 4.7 in the next lecture.)

These geometric models for $|G|_M$ play an important role in proving the following properties of support varieties for these various “group-like” structures.

Theorem 3.8. *Let G be a finite group scheme over k , and let M, N be finite dimensional G -modules.*

- (1) $|G|_M = 0$ if and only if M is a projective G -module if and only if M is an injective G -module.
- (2) $|G|_{M \oplus N} = |G|_M \cup |G|_N$.
- (3) $|G|_{M \otimes N} = |G|_M \cap |G|_N$.
- (4) For any short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ and any permutation σ of $\{1, 2, 3\}$, $|G|_{\sigma(1)} \subset |G|_{\sigma(2)} \cup |G|_{\sigma(3)}$.

The theory of support varieties has not only given information about the representation theory of G but also has led to new classes of modules. We mention the “modules of constant Jordan type” introduced by J. Carlson, J. Pevtsova, and the author [13] based on the “well-definedness of maximal Jordan type” established by J. Pevtsova, A. Suslin, and the author [32]. We point out the paper of J. Carlson, Z. Lin, and D. Nakano [15] which gives an interesting relationship between the cohomological support variety for $G(\mathbb{F}_p)$ and for $G_{(1)}$ for finite dimensional G -modules with G equal to some simple algebraic group.

Remark 3.9. If G is a linear algebraic group, then we face the following daunting problems in adopting the techniques of cohomological support varieties to the representation theory of linear algebraic groups.

- If G is a simple algebraic group, then $H^i(G, k)$ vanishes in positive dimensions by the vanishing theorem of G. Kempf [40].
- On the other hand, if U is a non-trivial linear algebraic group which is unipotent, then $H^\bullet(G, k)$ is not finitely generated.
- We are unaware of a result which can play the role of Quillen’s detection theorem stating that cohomology of a finite group is detected modulo nilpotents on elementary subgroups (see Theorem 3.5).

In recent work, the author has explored unipotent algebraic groups with the view that, unlike simple algebraic groups, these should have “enough cohomology”. Unfortunately, this appear not to be the case even for the Heisenberg group $U_3 \subset GL_3$ of upper triangular elements. A tentative framework has been developed by the author in which a cohomological support theory $M \mapsto V^{coh}(G)_M$ is formulated

using a continuous approximation of the rational cohomology $H^\bullet(G, k)$ of a linear algebraic group G . This naturally maps to the support theory $M \mapsto V(G)_M$ discussed in Lecture IV provided that G is of “exponential type”. For $G = \mathbb{G}_a$, this map is an isomorphism for all finite dimensional \mathbb{G}_a -modules. However, even for the Heisenberg group U_3 , the two theories are quite different. For example, the image of $V^{coh}(G)_M \rightarrow V(G)_M$ is contained in the G -invariants of $V(G)$.

4. LECTURE IV: SUPPORT VARIETIES FOR LINEAR ALGEBRAIC GROUPS

In his final lecture, the author presented his construction $M \mapsto V(G)_M$ of support varieties for G -modules M , where G is a linear algebraic group “of exponential type”. The beginnings of this theory can be found in [21] and some applications in [23]. The theory succeeds in that the support varieties defined here extend those for infinitesimal kernels, have many of the expected properties (see Theorem 4.13), and are formulated intrinsically for those linear algebraic groups for which the theory applies. One interesting aspect of this theory is that it leads to new and apparently interesting classes of (infinite-dimensional) G -modules.

One failure of the theory we present is that there are G -modules M which are not injective but for which $V(G)_M = 0$. We hope that this theory will be refined, perhaps using “formal 1-parameter subgroups” mentioned at the end of this lecture.

As for the first three lectures, we begin by providing the outline of this fourth lecture given to participants.

IV.A 1-parameter subgroups.

- i.) Group homomorphisms $\mathbb{G}_a \rightarrow G$;
- ii.) Examples of \mathbb{G}_a and GL_N ;
- iii.) Springer isomorphisms and groups of exponential type;
- iv.) SFB for $G_{(r)}$.

IV.B Linear algebraic groups of exponential type.

- i.) Defns; Sobaje’s theorem;
- ii.) p -nilpotent operator α_B ;
- iii.) Jordan types.

IV.C Support varieties.

- i.) $V(G), V(G)_M$;
- ii.) Example of \mathbb{G}_a ;
- iii.) Properties.

IV.D Special modules.

- i.) Mock injective modules;
- ii.) Mock trivial modules;
- iii.) Modules for realization of subspaces of $V(G)$.

IV.E Some open problems.

- i.) Formal 1-parameter subgroups and injectivity;

- ii.) Finite generation of cohomology of sub-coalgebras;
- iii.) Detecting rational cohomology modulo nilpotents.

Topics for discussion/projects:

IV.A Work through the exponential map for GL_N ; work through some details of proofs found in [57], [58].

IV.B Investigate 1-parameter groups for Sp_{2n} .

IV.C Work out examples for \mathbb{G}_a , for induced modules, for homogeneous varieties.

IV.D Investigate the question of what $C \subset V(G)$ can be realized as $V(G)_M$ for some (possibly infinite dimensional) G -module M .

4.1. 1-parameter subgroups. In this subsection, we discuss 1-parameter subgroups of linear algebraic groups. These 1-parameter subgroups might more formally be called *unipotent* 1-parameter subgroups. After giving the definition and some examples, we give the definition of a linear algebraic group of exponential type. For such a group G , the set $V(G)$ of 1-parameter subgroups is the set of k -points of an ind-scheme $\mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g}))$ defined in terms of the restricted Lie algebra \mathfrak{g} of G . As we mention, most of the familiar linear algebraic groups are groups of exponential type.

We begin by recalling from [57] the affine scheme $V_r(G)$ of height r infinitesimal 1-parameter subgroups of an affine group scheme G over k .

Definition 4.1. Let G be an affine group scheme over k and r a positive integer. Then the functor sending a commutative k -algebras A to the set of morphisms (over $\text{Spec } A$) of group schemes of the form $\mathbb{G}_{a(r),A} \rightarrow G_A$ is representable by an affine group scheme $V_r(G)$. Here, G_A is the base change $G \times_{\text{Spec } k} \text{Spec } A$ of G .

In particular, $V_r(G)(k)$ is the set of height r infinitesimal 1-parameter subgroups $\mu : \mathbb{G}_{a(r)} \rightarrow G$.

For any affine group scheme G over k , $V_r(G) = V_r(G_{(r)})$.

Definition 4.2. Let G be a linear algebraic group over k . Then a 1-parameter subgroup is a morphism of group schemes over k of the form $\psi : \mathbb{G}_a \rightarrow G$. We denote by $V(G)$ the set of 1-parameter subgroups of G .

Restriction to $G_{(r)}$ determines a natural map $V(G) \rightarrow (V_r(G))(k) = (V_r(G_{(r)}))(k)$ from $V(G)$ to the set of infinitesimal 1-parameter subgroups $\mathbb{G}_{a(r)} \rightarrow G$, the set of k -points of the affine scheme $V(G_{(r)})$ of Definition 4.1.

Example 4.3. (1) Take $G = \mathbb{G}_a$. A 1-parameter subgroup $\mathbb{G}_a \rightarrow \mathbb{G}_a$ is determined by a map of coordinate algebras $k[T] \leftarrow k[T]$ given by sending T to an additive polynomial; namely a polynomial of the form $\sum_{i \geq 0} a_i T^{p^i}$. (The condition that the map $k[T] \leftarrow k[T]$ sending T to $p(T)$ is a map of Hopf algebras is equivalent to the condition that $p(T)$ be of this form.) Thus, $V(\mathbb{G}_a)$ is the set of k -points of the affine ind-scheme \mathbb{A}^∞ , the set of all sequences $\underline{a} = (a_0, a_1, \dots, a_n, \dots)$ with the property that $a_N = 0$ for N sufficiently large (i.e., “finite sequences”).

(2) Take $G = GL_N$. Then a 1-parameter subgroup $\psi : \mathbb{G}_a \rightarrow GL_N$ has associated map on coordinate algebras $k[\{X_{i,j}\}, \det^{-1}] \rightarrow k[T]$ which must be

compatible with the coproducts Δ_{GL_N} and $\Delta_{\mathbb{G}_a}$. As shown in [57], such a 1-parameter subgroup corresponds to a finite sequence $\underline{A} = (A_0, A_1, \dots, A_N, \dots)$ of p -nilpotent $N \times n$ matrices (i.e., p -nilpotent elements of \mathfrak{gl}_N) which pairwise commute. To such a finite sequence \underline{A} , the associated 1-parameter subgroup is the morphism of algebraic groups

$$\prod_{i \geq 0} \exp_{A_i} \circ F^i : \mathbb{G}_a \rightarrow \mathbb{G}_a \rightarrow GL_N, \quad r \in R \mapsto \prod_{i \geq 0} \exp_{A_i}(r^{p^i}) \in GL_N(R),$$

where

$$\exp_A(s) = 1 + s \cdot A + (s^2/2) \cdot A^2 + \dots + (s^{p-1}/(p-1)!) \cdot A^{p-1}.$$

Thus, $V(GL_N)$ is the set of affine k points of the ind-scheme $\mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{gl}_N)) = \varinjlim_r \mathcal{C}_r(\mathcal{N}_p(\mathfrak{gl}_N))$, where $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{gl}_N)) \simeq V_r(GL_N)$ represents the functor of r -tuples of p -nilpotent, pair-wise commuting $N \times N$ matrices.

Definition 4.4. Let \mathfrak{g} be a finite dimensional restricted Lie algebra over k . Denote by $\mathcal{N}_p(\mathfrak{g})$ the subvariety of \mathfrak{g} (viewed as an affine space) consisting of $X \in \mathfrak{g}$ with $X^{[p]} = 0$. We define the affine k -scheme $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ to be the subvariety of $(\mathcal{N}_p(\mathfrak{g}))^{\times r}$ consisting of r -tuples (B_0, \dots, B_r) satisfying

$$[B_i, B_j] = B_i^{[p]} = B_j^{[p]} = 0, \quad 0 \leq i, j \leq r.$$

We define $\mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g}))$ to be the ind-scheme $\varinjlim_r \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$.

Our construction of support varieties only applies to a linear algebraic group G which is of **exponential type**. This condition is the condition that $V(G)$ can be naturally identified with the set of k points of $\mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g}))$ as is the case for $G = GL_N$. The following definition of [22] is an extension of the concept in [57] of an embedding $G \subset GL_N$ of exponential type.

Definition 4.5. Let G be a linear algebraic group over k with Lie algebra \mathfrak{g} . A structure of exponential type on G is a morphism of k -schemes

$$(4.5.1) \quad \mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \rightarrow G, \quad (B, s) \mapsto \mathcal{E}_B(s)$$

such that

- (1) For each $B \in \mathcal{N}_p(\mathfrak{g})(k)$, $\mathcal{E}_B : \mathbb{G}_a \rightarrow G$ is a 1-parameter subgroup.
- (2) For any pair of commuting p -nilpotent elements $B, B' \in \mathfrak{g}$, the maps $\mathcal{E}_B, \mathcal{E}_{B'} : \mathbb{G}_a \rightarrow G$ commute.
- (3) For any commutative k -algebra A , any $\alpha \in A$, and any $s \in \mathbb{G}_a(A)$, $\mathcal{E}_{\alpha \cdot B}(s) = \mathcal{E}_B(\alpha \cdot s)$.
- (4) Every 1-parameter subgroup $\psi : \mathbb{G}_a \rightarrow G$ is of the form

$$\mathcal{E}_{\underline{B}} \equiv \prod_{s=0}^{r-1} (\mathcal{E}_{B_s} \circ F^s)$$

for some $r > 0$, some $\underline{B} \in \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$; furthermore, $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \rightarrow V_r(G)$, $\underline{B} \mapsto \mathcal{E}_{\underline{B}} \circ i_r$ is an isomorphism for each $r > 0$.

A linear algebraic group over k which admits a structure of exponential type is said to be a **linear algebraic group of exponential type**.

Moreover, a closed subgroup $H \subset G$ is said to be an embedding of *exponential type* if H is equipped with the structure of exponential type given by restricting

that provided to G ; in particular, we require $\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \rightarrow G$ to restrict to $\mathcal{E} : \mathcal{N}_p(\mathfrak{h}) \times \mathbb{G}_a \rightarrow H$.

Up to isomorphism, if such a structure exists then it is unique.

Example 4.6. There are many examples of linear algebraic groups of exponential type.

- (1) Any classical simple linear algebraic group G over k (i.e., of type A, B, C or D) and the unipotent radical of any parabolic subgroup defined of such a group G over \mathbb{F}_p as remarked in [57].
- (2) Any simple linear algebraic group G provided that p is separably good for G (see [54]).
- (3) Any term of the lower central series of the unipotent radical of a parabolic subgroup defined over \mathbb{F}_p of a simple algebraic group G , provided p is separably good for G (see [52] plus [54]).

4.2. p -nilpotent operators. We begin this subsection by briefly recalling the theory of π -points for finite group schemes developed by J. Pevtsova and the author.

If G is a linear algebraic group over k and M a rational $G_{(r)}$ -module, then the geometric formulation $V_r(G_{(r)})_M$ of $|G_{(r)}|_M$ is obtained by associating to every point of $V_r(M)$ a p -nilpotent operator on M . In the following definition, we use field extensions K/k to capture the scheme structure of $V_r(G_{(r)})_M$.

Definition 4.7. Let G be a finite group scheme with group algebra kG (the k -linear dual to the coordinate algebra $k[G]$). Then a π -point is a left flat K -linear map of algebras $\alpha_K : K[t]/T^p \rightarrow KG$ for some field extension K/k with the property that α_K factors through $KC_K \rightarrow KG$ for some unipotent subgroup scheme $C_K \subset G_K$.

For a suitable equivalence relation on π -points, the set of equivalence classes of π -points of G is naturally identified with the set of non-tautological homogeneous prime ideals of $H^\bullet(G, k)$. Indeed, one can put a scheme structure $\Pi(G)$ on equivalence classes of π -points which is formulated in terms of the category of G -modules (and not using homological algebra) so that $\Pi(G)$ is isomorphic to $\text{Proj } H^\bullet(G, k)$ as a k -scheme [29].

For any G -module M , the “local action” on M at the π -point $\alpha : K[t]/T^p \rightarrow KG$ is the action of $\alpha_{K^*}(T)$ on $M_K \equiv M \otimes K$ (equivalently, the action of $T \in k[T]/T^p$ on $\alpha_K^*(M_K)$).

For any G -module M , the “ Π -support variety” $\Pi(G)_M$ of M consists of those equivalence classes of π -points $\alpha : K[t]/T^p \rightarrow KG$ for which $\alpha^*(M_K)$ is *not* free as a $K[T]/T^p$ -module.

The fact that $\Pi(G)_M$ is well defined (that the condition that an equivalence of class of π -points can be tested on any representative of that equivalence class) was justified by the work of J. Pevtsova, and the author in [28].

The following definition of “local action” of $G_{(r)}$ on M at an infinitesimal 1-parameter subgroup is implicit in [58].

Definition 4.8. Let G be a linear algebraic group of exponential type, let $\underline{B} = (B_0, \dots, B_{r-1})$ be a k -point of $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$, and let M a $G_{(r)}$ -module. Then the local action of $G_{(r)}$ on M at $\mathcal{E}_{\underline{B}}$ is defined to be the local action at the π -point $\mu_{\underline{B}} \equiv \mathcal{E}_{\underline{B}^*} \circ \epsilon_r : k[T] \rightarrow k\mathbb{G}_{a(r)} \rightarrow kG_{(r)}$ sending T to $\mathcal{E}_{\underline{B}^*}(u_{r-1})$. (The map $\epsilon_r : k[T]/T^p \rightarrow k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(\{u_i^p\})$ is the map of k -algebras sending T to u_{r-1} ; this is a Hopf algebra map if and only if $r = 1$.)

Consequently,

$$V_r(G)_M = V(G_{(r)}) \simeq \{\mathcal{E}_{\underline{B}} \in V_r(G) : \mu_{\underline{B}}^*(M) \text{ is not free}\}.$$

After much experimentation, the author introduced in [21] the following definition of the local action at a 1-parameter subgroup $\mathcal{E}_{\underline{B}}$ of a linear algebraic group of exponential type G acting on a G -module M . This definition is not formulated in terms of $\mathcal{E}_{\underline{B}}^*(M)$. The justification of the somewhat confusing “twist” (i.e., a reordering of $\underline{B} = (B_0, \dots, B_r \dots)$) is implicit in Proposition 4.10, which shows that the restriction to Frobenius kernels of this definition gives a “functionally equivalent” formulation of “local action” as that given in Definition 4.8.

Definition 4.9. Let G be a linear algebraic group of exponential type, equipped with an exponentiation $\mathcal{E} : \mathcal{N}_p \times \mathbb{G}_a \rightarrow G$. Let M be a rational G -module and $\underline{B} = (B_0, B_1, \dots, B_n, \dots) \in \mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g}))$ be a finite sequence. Then the action of G on M at $\mathcal{E}_{\underline{B}} : \mathbb{G}_a \rightarrow G \in V(G)$ is defined to be the action of

$$(4.9.1) \quad \sum_{s \geq 0} (\mathcal{E}_{B_s})_*(u_s) = \sum_{s \geq 0} (\mathcal{E}_{B_s} \circ F_s)_*(u_0).$$

One checks that this action is in fact p -nilpotent, thereby defining

$$(4.9.2) \quad \alpha_{\underline{B}} : k[u]/u^p \rightarrow kG, \quad \underline{B} \in \mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g})); \quad u \mapsto \sum_{s \geq 0} (\mathcal{E}_{B_s})_*(u_s).$$

The close connection of Definition 4.9 and the theory of π -points briefly summarized in Definition 4.7 is given by the following result of [22] based upon an argument of P. Sobaje [53].

Proposition 4.10. [22, 4.3] *Let G be a linear algebraic group of exponential type, equipped with an exponentiation $\mathcal{E} : \mathcal{N}_p \times \mathbb{G}_a \rightarrow G$. For any $r > 0$ and any $\underline{B} \in \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$, the π -points of $G_{(r)}$*

$$\mu_{\underline{B}} = \mathcal{E}_{\underline{B}} \circ \epsilon_r : k[T]/T^p \rightarrow k\mathbb{G}_{a(r)} \rightarrow kG_{(r)}, \quad \alpha_{\Lambda_r(\underline{B})} : k[u]/u^p \rightarrow kG_{(r)}$$

are equivalent, where $\Lambda_r(B_0, \dots, B_{r-1}) = (B_{r-1}, \dots, B_0)$.

This equivalence of π -points enables a comparison of support varieties for finite group schemes and the definition we now give of support varieties for linear algebraic groups of exponential type. Indeed, it enables a comparison of the “generalized support varieties” introduced by J. Pevtsova and the author in [30] using the local data of the full Jordan type of a $k[u]/u^p$ -module rather than merely whether or not such a module is free.

4.3. The support variety $V(G)_M$. Much of this subsection is copied from the author’s paper [22]. After giving the definition of the support variety $V(G)_M$ of a rational G -module M of a linear algebraic group of exponential type, we review many of the properties of this construction. The first property of Theorem 4.13 tells us that $V(G)_M$ can be recovered from $V(G_{(r)})_M$ for $r \gg 0$ provided that M is finite dimensional. On the other hand, for M infinite dimensional the support variety $V(G)_M$ provides information about M not detected by any Frobenius kernel.

Definition 4.11. Let G be a linear algebraic group equipped with a structure of exponential type and let M be a G -module. We define the **support variety** of M

to be the subset $V(G)_M \subset V(G)$ consisting of those $\mathcal{E}_{\underline{B}}$ such that $\alpha_{\underline{B}}^*(M)$ is not free as a $k[u]/u^p$ -module, where $\alpha_{\underline{B}} : k[u]/u^p \rightarrow kG$ is defined in (4.9.2).

For a finite dimensional G -module M , we define the **Jordan type** of M at the 1-parameter subgroup $\mathcal{E}_{\underline{B}}$ to be

$$JT_{G,M}(\mathcal{E}_{\underline{B}}) \equiv JT\left(\sum_{s \geq 0} (\mathcal{E}_{B_s})_*(u_s), M\right),$$

the Jordan type of the local action of G on M at $\mathcal{E}_{\underline{B}}$ (see Definition 4.9). For such a finite dimensional G -module M , $V(G)_M \subset V(G)$ consists of those 1-parameter subgroups $\mathcal{E}_{\underline{B}}$ such that some block of the Jordan type of M at $\mathcal{E}_{\underline{B}}$ has size $< p$.

The following definition is closely related to the formulation of p -nilpotent degree given in [21, 2.6].

Definition 4.12. (cf. [21, 2.6]) Let G be a linear algebraic group equipped with a structure of exponential type and let M be a G -module. Then M is said to have exponential degree $< p^r$ if $(\mathcal{E}_{\underline{B}})_*(u_s)$ acts trivially on M for all $s \geq r$, all $B \in \mathcal{N}_p(\mathfrak{g})$.

As observed in [22], every finite dimensional G -module M has exponential degree $< p^r$ for r sufficiently large.

Theorem 4.13. [22, 4.6] *Let G be a linear algebraic group equipped with a structure of exponential type and M a rational G -module*

- (1) *If M has exponential degree $< p^r$, then $V(G)_M = \Lambda_r^{-1}(V_r(G)_M(k))$.*
- (2) *If M is finite dimensional, then $V(G)_M \subset V(G)$ is closed.*
- (3) $V(G)_{M \oplus N} = V(G)_M \cup V(G)_N$.
- (4) $V(G)_{M \otimes N} = V(G)_M \cap V(G)_N$.
- (5) *If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence of rational G -modules, then the support variety $V(G)_{M_i}$ of one of the M_i 's is contained in the union of the support varieties of the other two.*
- (6) *If G admits an embedding $i : G \hookrightarrow GL_N$ of exponential type, then*

$$V(G)_{M^{(1)}} = \{\mathcal{E}_{(B_0, B_1, B_2, \dots)} \in V(G) : \mathcal{E}_{(B_1^{(1)}, B_2^{(1)}, \dots)} \in V(G)_M\}.$$

(Here, $M^{(1)}$ is the Frobenius twist of M , as formulated in Definition 2.14.)

- (7) *For any $r > 0$, the restriction of M to $kG_{(r)}$ is injective (equivalently, projective) if and only if the intersection of $V(G)_M$ with the subset $\{\psi_{\underline{B}} : B_s = 0, s > r\}$ inside $V(G)$ equals $\{\mathcal{E}_0\}$.*
- (8) $V(G)_M \subset V(G)$ is a $G(k)$ -stable subset.

Remark 4.14. In [22, 4.6], property (6) was proved under the assumption that $i : G \hookrightarrow GL_N$ be defined over \mathbb{F}_p . This is unnecessary, for [22, 1.11] also does not require i to be defined over \mathbb{F}_p . Namely, one uses the diagram

$$(4.14.1) \quad \begin{array}{ccccc} \mathbb{G}_a & \xrightarrow{\psi_{\underline{B}}} & G & \xrightarrow{F} & G^{(1)} \\ & & \downarrow i & & \downarrow i^{(1)} \\ & & GL_N & \xrightarrow{F} & GL_N^{(1)} = GL_N \end{array}$$

to reduce to verifying both [22, 1.11] and [22, 4.6] in the special case $G = GL_N$.

We repeat a remark which suggests how to formulate support varieties in the category of strict polynomial functors.

Remark 4.15. [22, 4.7] A special case of Theorem 4.13 is the case $G = GL_n$ and M a polynomial GL_n -module homogenous of some degree as in Example 2.6.5. In particular, Theorem 4.13 provides a theory of support varieties for modules over the Schur algebra $S(n, d)$ for $n \geq d$.

A few examples of such support varieties are given in [22]. We give another interesting example here, an infinite dimensional version of J. Carlson's L_ζ -modules.

We consider a class of examples Q_ζ associated to rational cohomology classes $\zeta \in H^\bullet(G, k)$. Given a linear algebraic group G and some choice of injective resolution $k \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow \dots$ of rational G -modules, we set $\Omega^{-2d}(k)$ equal to the quotient of I^{2d-1} modulo the image of I^{2d-2} . The restriction of $\Omega^{-2d}(k)$ to some Frobenius kernel $G_{(r)}$ of G is equivalent in the stable category of $G_{(r)}$ -modules to $\Omega_{(r)}^{-2d}(k)$ defined as the quotient of $I_{(r)}^{2d-1}$ modulo the image of $I_{(r)}^{2d-2}$ for a minimal injective resolution $k \rightarrow I_{(r)}^0 \rightarrow \dots \rightarrow I_{(r)}^n \rightarrow \dots$ of rational $G_{(r)}$ -modules.

Proposition 4.16. *Let G be a linear algebraic group equipped with a structure of exponential type. Consider a rational cohomology class $\zeta \in H^{2d}(G, k)$ represented by a map $\tilde{\zeta} : k \rightarrow \Omega^{-2d}(k)$ of rational G -modules. We define Q_ζ to be the cokernel of $\tilde{\zeta}$, thus fitting in the short exact sequence of rational G -modules*

$$(4.16.1) \quad 0 \rightarrow k \xrightarrow{\tilde{\zeta}} \Omega^{-2d}(k) \rightarrow Q_\zeta \rightarrow 0.$$

Then

$$V(G)_{Q_\zeta} = \bigcup_r \{ \mathcal{E}_{\underline{B}} \in V_r(G) : (\alpha_{\Lambda_r(\underline{B})})^*(\zeta) = 0 \in H^{2d}(k[t]/t^p, k) \}.$$

Proof. Since $\Omega_{(r)}^{-2d}(k)$ is stably equivalent as a $G_{(r)}$ -module to the restriction to $G_{(r)}$ of the rational G -module $\Omega^{-2d}(k)$, we conclude that the restriction of Q_ζ to $G_{(r)}$ is stably equivalent to the finite dimensional $G_{(r)}$ -module Q_{ζ_r} (associated to $\zeta_r \in H^{2d}(G_{(r)}, k)$, the restriction of ζ) fitting in the short exact sequence

$$0 \rightarrow k \rightarrow \Omega_{(r)}^{-2d}(k) \rightarrow Q_{\zeta_r} \rightarrow 0.$$

By definition, the Carlson L_{ζ_r} -module introduced in [11] fits in the distinguished triangle

$$L_{\zeta_r} \rightarrow \Omega_{(r)}^{2d}(k) \rightarrow k \rightarrow \Omega_{(r)}^{-1}(L_\zeta)$$

whose $[-2d]$ -shift is the distinguished triangle

$$\Omega_{(r)}^{-2d}(L_{\zeta_r}) \rightarrow k \rightarrow \Omega_{(r)}^{-2d}(k) \rightarrow \Omega_{(r)}^{-2d-1}(L_{\zeta_r}).$$

Thus, we conclude that Q_{ζ_r} is stably equivalent to $\Omega_{(r)}^{-2d-1}(L_{\zeta_r})$, and thus has the same support as a $G_{(r)}$ -module as the support of L_{ζ_r} .

Observe that $V(G)_{Q_\zeta} \cap V_r G$ equals $V(G_{(r)})_{Q_{\zeta_r}}$. The identification of $V(G)_{L_\zeta}$ now follows from [29, 3.7] which asserts that

$$V(G_{(r)})_{L_{\zeta_r}} = \{ \mu : \mathbb{G}_{a(r)} \rightarrow G_{(r)}, (\mu \circ \epsilon_r)^*(\zeta_r) = 0 \}.$$

□

4.4. Classes of rational G -modules. The π -point approach to support varieties for a finite group scheme G naturally led to the formulation of the class of G -modules of constant Jordan type (and more general modules of constant j -type for $1 \leq j < r$). For an infinitesimal group scheme G , J. Pevtsova and the author in [31] showed how to construct various vector bundles on $V(G)$ associated to a G -module M of constant Jordan type. We refer the reader to the book by D. Benson [6] and the paper by D. Benson and J. Pevtsova [7] for an exploration of vector bundles constructed in this manner for elementary abelian p -groups $E \simeq \mathbb{Z}/p^{\times s}$. This is a “special case” of an infinitesimal group scheme because the representation theory of $\mathbb{Z}/p^{\times s}$ is that of the height 1 infinitesimal group scheme $\mathbb{G}_{a(1)}^{\times s}$. We also mention that J. Carlson, J. Pevtsova, and the author introduced in [14] a construction which produced vector bundles on Grassmann varieties associated to modules of constant Jordan type as well as to more general modules, those of constant j -type.

In this subsection, we briefly mention three interesting classes of (infinite dimensional) G -modules for G a linear algebraic group of exponential type. Consideration of special classes of G -modules is one means of obtaining partial understanding of the wild category $Mod_k(G)$.

Throughout this subsection G will denote a linear algebraic group of exponential type.

Definition 4.17. We say that M is mock injective if M is not injective but $V(G)_M = 0$.

As the author showed in [23], using results of E. Cline, B. Parshall, and L. Scott [16] on the relationship of induced modules (see Definition 2.9) to injectivity, such mock injectives exist for any unipotent algebraic group which is of exponential type. Necessary and sufficient conditions on G for the existence of mock injectives can be found in [34], once again using induction.

Definition 4.18. We say that M is mock trivial if the local action of G on M is trivial for all 1-parameter subgroups $\mathcal{E}_B : \mathbb{G}_a \rightarrow G$.

In [23], the author shows how to construct mock trivial G -modules for any G which is not unipotent.

Definition 4.19. We say that M is of mock exponential degree $< p^r$ if there exists some $r > 0$ such that $V(G)_M = \Lambda_r^{-1}(V(G_{(r)})_M)$.

This class of G -modules includes all finite dimensional G -modules.

4.5. Some questions of possible interest. In this final subsection, we mention some questions which might interest the reader, some of which concern the special classes of G -modules defined in the previous subsection.

Question 4.20. *For certain linear algebraic groups G (e.g., \mathbb{G}_a), can one describe the monoid (under tensor product) of mock injective G -modules with 1-dimensional socle?*

Question 4.21. *Can one characterize G -modules of bounded mock exponential degree using G -modules which are extensions of mock injective modules by finite dimensional modules?*

Question 4.22. *What conditions on a subset $X \subset V(G)$ imply good properties of the subcategory of $\text{Mod}_k(G)$ consisting of those G -modules M with $V(G)_M \subset X$ for some G -module M ?*

Question 4.23. *What are (necessary and/or sufficient) conditions on a subset $X \subset V(G)$ to be of the form $V(G)_M$?*

Question 4.24. *Do there exist rational cohomology classes $\zeta \in H^d(G, k)$ for some G and some $d > 0$ which are not nilpotent but which satisfy the condition that $\mathcal{E}_B^*(\zeta) = 0 \in H^*(k[T]/T^p, k)$ for all $\mathcal{E}_B \in V(G)$?*

Question 4.25. *As in [23], we have natural filtrations of $k[G]$ by subalgebras $C \subset k[G]$. Especially for G unipotent, can we prove finiteness theorems for $\text{Ext}_{C\text{-coMod}}^*(M, M)$ for M a finite dimensional rational G -module which inform questions about $\text{Ext}_G^*(M, M)$?*

We conclude with a possible ‘‘improvement’’ of our support theory $M \mapsto V(G)_M$ for linear algebraic groups of exponential type. The formulation of $V(G)$ presented in this text (and in [22]) is that of a colimit $\varinjlim_r V_r(G)$, where $V_r(G) \simeq \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$.

What follows is an alternative support theory, $M \mapsto \widehat{V(G)}_M$. We remind the reader of the affine scheme $V_r(G)$ given in Definition 4.1 for any affine group scheme G : the set of A -points of $V_r(G)$ is the set of the morphisms of group schemes $\mathbb{G}_{a(r),A} \rightarrow G_A$ over $\text{Spec} A$.

Definition 4.26. Let G be a linear algebraic group. For each $r > 0$, we define the restriction morphism $V_{r+1}(G) \rightarrow V_r(G)$ by restricting the domain of a height $r + 1$ 1-parameter subgroup $\mathbb{G}_{a(r+1),A} \rightarrow G_A$ to $\mathbb{G}_{a(r),A} \subset \mathbb{G}_{a(r+1),A}$.

Thus, $\{V_r(G), r > 0\}$ is a pro-object of affine schemes.

If G is a linear algebraic group of exponential type, the restriction map $V_{r+1}(G) \rightarrow V_r(G)$ is given by the projection $\mathcal{C}_{r+1}(\mathcal{N}_p(\mathfrak{g})) \rightarrow \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ onto the first r factors.

Definition 4.27. For G a linear algebraic group of exponential type, we define

$$\widehat{V(G)} = \varprojlim_r \{V(G_{(r)})(k)\}$$

equipped with the topology of the inverse limit of the Zariski topologies on the sets of k -points $V(G_{(r)})(k)$.

We view an element of $\widehat{V(G)}$ as a ‘‘formal 1-parameter subgroup’’ given as an infinite product $\widehat{\mathcal{E}}_B = \prod_{s=0}^{\infty} \mathcal{E}_{B_s} \circ F^s$.

One proves the following proposition by using the following observation: for any coaction $\Delta_M : M \rightarrow M \otimes k[G]$ and any $m \in M$, there exists a positive integer $s(m)$ such that the composition

$$u_s \circ \mathcal{E}_B^* \circ \Delta_M : M \rightarrow M \otimes k[G] \rightarrow M \otimes k[T] \rightarrow k$$

vanishes on m for all $B \in \mathcal{N}_p(\mathfrak{g})$ and all $s \geq s(m)$.

Proposition 4.28. *Let G a linear algebraic group of exponential type and M a G -module. Then for any $\widehat{\mathcal{E}}_B \in \widehat{V(G)}$ and any $m \in M$, the infinite sum $\sum_{s=0}^{\infty} (\mathcal{E}_{B_s})_*(u_s)$ applied to m is finite (i.e. $(\mathcal{E}_{B_s})_*(u_s)$ applied to m vanishes for $s \gg 0$).*

Consequently, $\sum_{s=0}^{\infty} (\mathcal{E}_{B_s})_*(u_s)$ defines a p -nilpotent operator $\psi_{\underline{B},M} : M \rightarrow M$. We define

$$\widehat{V(G)}_M \equiv \{\widehat{\mathcal{E}}_{\underline{B}} : \text{not all blocks of } \psi_{\underline{B},M} \text{ have size } p\}.$$

We conclude with the following questions concerning $M \mapsto \widehat{V(G)}_M$.

Question 4.29. *Does use of formal 1-parameter subgroups provide necessary and sufficient conditions for injectivity of a G -module?*

Does use of formal 1-parameter subgroups provide necessary and sufficient conditions for a G -module to be of bounded exponential degree?

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