FILTRATIONS, 1-PARAMETER SUBGROUPS, AND RATIONAL INJECTIVITY

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Abstract. We investigate rational G-modules M for a linear algebraic group G over an algebraically closed field k of characteristic $p > 0$ using filtrations by sub-coalgebras of the coordinate algebra $k[G]$ of G. Even in the special case of the additive group $\mathbb{G}_a$, interesting structures and examples are revealed. The "degree" filtration we consider for unipotent algebraic groups leads to a "filtration by exponential degree" applicable to rational G modules for any linear algebraic group G of exponential type; this filtration is defined in terms of 1-parameter subgroups and is related to support varieties introduced recently by the author for such rational G-modules. We formulate in terms of this filtration a necessary and sufficient condition for rational injectivity for rational G-modules. Our investigation leads to the consideration of two new classes of rational G-modules: those that are "mock injective" and those that are "mock trivial".

0. Introduction

Beginning with the very special case of the additive group $\mathbb{G}_a$, we consider the filtration by degree on rational $\mathbb{G}_a$-modules which enables us to better understand the intriguing category $(\mathbb{G}_a\text{-}Mod)$ of rational $\mathbb{G}_a$-modules. This filtration leads to a similarly defined filtration by degree on rational U_N-modules, where $U_N \subset GL_N$ is the closed subgroup of strictly upper triangular matrices, and determines a filtration on rational U-modules for a closed linear subgroup $U \subset U_N$. We then initiate the study of a less evident filtration on rational G-modules for G a linear algebraic group of exponential type. For rational U_N-modules for the unipotent algebraic group U_N, this filtration of exponential degree is a comparable to the more elementary filtration by degree we first consider. Throughout, we fix an algebraic closed field k of characteristic $p > 0$ and consider (smooth) linear algebraic groups over k together with their rational actions on k-vector spaces.

In some sense, this paper is a sequel to the author's recent paper [4] in which a theory of support varieties $M \mapsto V(G)_M$ was constructed for rational G-modules. The construction of the filtration by exponential degree $\{M_{[0]} \subset M_{[1]} \subset \cdots \subset M\}$ uses restrictions of M to 1-parameter subgroups $\mathbb{G}_a \to G$, and thus is based upon actions of G on M at p-unipotent elements of G. The role of 1-parameter subgroups to study rational G-modules was introduced in [3]; the property of p-unipotent degree introduced in [3, 2.5] is the precursor to our filtration by exponential degree. The origins of this approach to filtrations lie in considerations of support varieties

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for modules for infinitesimal group schemes, varieties which are defined in terms of p-nilpotent actions on such modules.

The basic theme of this paper is to investigate rational G-modules through their restrictions via 1-parameter subgroups $\mathbb{G}_a \to G$. Whereas the support variety construction $M \mapsto V(G)_M$ is defined in terms of restrictions of M to a p-nilpotent operator associated to each 1-parameter subgroup of G, our present approach involves the full information of the restriction of M to all 1-parameter subgroups by using filtrations on the category of rational G-modules. We give a necessary and sufficient condition for rational injectivity of a rational G-module, something we have not succeeded in doing using support varieties. Moreover, these filtrations enable us to formulate and study the classes of mock injective modules (those whose restrictions to every Frobenius kernel are injective) and of mock trivial modules (those for which actions at 1-parameter subgroups are trivial). These modules are somewhat elusive to construct, but can be shown to exist in great numbers and have interesting properties.

Perhaps the groups of most interest are reductive groups, especially simple groups of classical type. For such groups G, it is instructive to compare the approach in this paper and in [4] with traditional considerations of weights for the action of a maximal torus $T \subset G$ on a rational G-module. Whereas consideration of weights for T are highly suitable in classifying irreducible rational G-modules, the action at p-unipotent elements has the potential of recognizing extensions of such modules. Although our filtration by exponential degree involves actions at p-unipotent elements of G, Example 3.14 shows that a bound on the T-weights for a rational G-module M determines a bound on the exponential degree of M for G reductive (where T is a maximal torus for G). We emphasize that the filtration by exponential degree applies to rational modules for unipotent groups whose rational modules are not equipped with a torus action.

We sketch the contents of this paper. We begin with the most elementary example $G = \mathbb{G}_a$. Indeed, this effort was in part motivated by the prospect of establishing a "local criterion" for a rational $\mathbb{G}_a$-module M to be rationally injective using the support variety $V(\mathbb{G}_a)_M$, or equivalently using the restrictions of M to all Frobenius kernels $\mathbb{G}_{a(r)}$ of $\mathbb{G}_a$. Proposition 2.11 provides counter-examples to our (unwritten) conjecture that rational injectivity of rational $\mathbb{G}_a$-modules is detected in this "local fashion." The filtration we consider for rational $\mathbb{G}_a$-modules arises from a filtration of the coordinate algebra $k[\mathbb{G}_a]$ by sub-coalgebras $k[\mathbb{G}_a]_{<d}$. As observed in Proposition 1.12, the category of comodules for the sub-coalgebra $k[\mathbb{G}_a]_{<p^r}$ is naturally isomorphic to the category of rational modules for the infinitesimal kernels $\mathbb{G}_{a(r)}$ of the linear algebraic group $\mathbb{G}_a$. This correspondence is very special, following from the simple observation that the restriction maps $k[\mathbb{G}_a] \to k[\mathbb{G}_{a(r)}]$ are split as maps of coalgebras.

In Section 2, we extend our consideration of filtrations to rational U-modules for a unipotent algebraic group U equipped with an embedding in some U_N. To investigate some properties of a rational U-module M, we find it more useful to consider the submodules $M_{<p^r} \subset M$ occurring in the filtration of M than to consider restrictions of M to Frobenius kernels $U_{(r)}$. For example, Proposition 2.13 gives a necessary and sufficient condition for a rational U-module M to be injective which is formulated in terms of the filtration $\{M_{<d}, d > 0\}$ of M. For non-abelian U, the sub-coalgebras $k[U]_{<p^r} \subset k[U]$ which we use to define this degree filtration are not

well related to the coordinate algebras of infinitesimal kernels $U_{(r)}$; nevertheless, Proposition 2.10 provides some comparison of $k[U]_{<p^r}$ and $k[U_{(r)}]$.

The key construction of this paper is that of the sub-coalgebras $(k[G])_{[d]} \subset k[G]$ in Definition 3.4 for G a linear algebraic group of exponential type. For such G, we introduce in Definition 3.10 the filtration $\{M_{[d]} \subset M, d \geq 0\}$ for any rational G-module M. As shown in Proposition 3.8, this filtration is equivalent to that provided in Section 2 in the special case $G = U_N,\ N \leq p$; in particular, equivalent to the elementary filtration considered for rational $\mathbb{G}_a$-modules. In a few examples of finite dimensional rational G-modules M, we find an explicit value for d such that $M = M_{[d]}$. Theorem 3.15 provides a list of properties for the filtration $\{M_{[d]},\ d \geq 0\}$ of a rational G module M with a structure of exponential type. In particular, this is a filtration by rational G-submodules of M, satisfies various aspects of functoriality, and is independent of the structure of exponential type on G. The filtration is finite for finite dimensional rational modules and has expected functoriality properties. Proposition 3.17 gives a relation of this filtration to the theory of support varieties for G as formulated in [4].

In Proposition 4.1, we establish basic properties of the functors $(-)_{[d]}$ determining our filtration of rational G-modules. Using some of these properties, we give in Proposition 4.2 a necessary and sufficient condition for M to be a rationally injective G-module in terms of its filtration $\{M_{[d]},\ d \geq 0\}$. Much of Section 4 is devoted to formulating and then investigating the classes of "mock injective" modules and "mock trivial" modules, rational G-modules with interesting properties. Mock injectives are rational G-modules whose restrictions to all Frobenius kernels $G_{(r)}$ are injective $G_{(r)}$-modules. Every injective rational G-module is mock injective, but somewhat surprisingly there are mock injectives which are not injective. Mock trivial modules are rational G-modules whose restriction along any 1-parameter subgroup of G is trivial. Both these classes satisfy familiar closure properties. [Note: In [14], W. Hardesty, D. Nakano, and P. Sobaje provide further analysis of mock injectives, establishing necessary and sufficient conditions on G for the existence of mock injective modules which are not injective G-modules.]

Finally, we conclude in Proposition 4.12 with a Grothendieck spectral sequence relating the right derived functors $R^t(-)_{[d]}$ of the filtration functor $(-)_{[d]}$ for a given degree d with the rational cohomology of G for any linear algebraic group G of exponential type.

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1. Rational modules for the additive group $\mathbb{G}_a$

We recall that $\mathbb{G}_a$ (the additive group) has coordinate algebra $k[T]$ equipped with the coproduct

$$\Delta : k[T] \ \to k[T] \otimes k[T], \quad T \mapsto (T \otimes 1 + 1 \otimes T).$$

In particular, this coproduct on $k[T]$ gives $k[T]$ the structure of a rational $\mathbb{G}_a$-module (which is rationally injective). One can view that action as follows: for every commutative k-algebra R and for every $a \in \mathbb{G}_a(R) = R$, the action of a on $f(T) \in R[T]$ is given by $a \circ f(T) \ = \ f(a+T)$.

The r-th Frobenius kernel $\mathbb{G}_{a(r)}$ of $\mathbb{G}_a$,

$$i_r : \mathbb{G}_{a(r)} \ \subset \ \mathbb{G}_a,$$

is the closed subgroup scheme with coordinate algebra given by $i_r^* : k[\mathbb{G}_a] = k[T] \to k[T]/T^{p^r} = k[\mathbb{G}_{a(r)}]$ and group algebra (i.e., k-linear dual of $k[\mathbb{G}_{a(r)}]$) denoted by $k\mathbb{G}_{a(r)}$. Using the notation of [12], we let $v_0, \ldots, v_{p^r-1}$ be the k-basis of $k\mathbb{G}_{a(r)}$ dual to the standard basis $\{T^j, 0 \leq j < p^r\}$ of $k[T]/T^{p^r}$. Denote v_{p^s} by u_s. If $j = \sum_{\ell=0}^{r-1} j_\ell p^\ell,\ 0 \leq j_\ell < p$, then

$$v_j \;=\; \frac{u_0^{j_0} \cdots u_{r-1}^{j_{r-1}}}{j_0! \cdots j_{r-1}!}.$$

Notation 1.1. (see [12]) With notation as above,

$$k\mathbb{G}_{a(r)} \;\simeq\; k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

For any $r, s > 0$, the quotient map

$$q_{r,s} : k[\mathbb{G}_{a(r+s)}] \cong k[T]/T^{p^{r+s}} \to k[\mathbb{G}_{a(r)}] \cong k[T]/T^{p^r}$$

sending T to T is a Hopf algebra map, whose dual we denote by

$$i_{r,s} : k\mathbb{G}_{a(r)} \to k\mathbb{G}_{a(r+s)}, \quad u_i \mapsto u_i,\ i < r.$$

The colimit

$$k\mathbb{G}_a \;\equiv\; \varinjlim_r k\mathbb{G}_{a(r)} \;\simeq k[u_0, \ldots, u_n, \ldots]/(u_0^p, \ldots, u_n^p, \ldots),$$

is the group algebra (or hyperalgebra or algebra of distributions at the identity) of $\mathbb{G}_a$.

The following evident lemma makes explicit the action of $k\mathbb{G}_a$ on a rational G-module M.

Lemma 1.2. *Let M be a rational $\mathbb{G}_a$-module given by the comodule structure $\Delta_M : M \to M \otimes k[\mathbb{G}_a]$. For $\phi \in k\mathbb{G}_a$,*

$$\phi(m) \;=\; ((1 \otimes \phi) \circ \Delta_M)(m). \tag{1.2.1}$$

Consequently, the action of $v_j \in k\mathbb{G}_a$ on the rational $\mathbb{G}_a$-module M is determined by the formula

$$\Delta_M(m) \;=\; \sum_j v_j(m) \otimes T^j. \tag{1.2.2}$$

In particular, the action of v_j on $f(T) = \sum_{n \geq 0} a_n T^n \in k[\mathbb{G}_a] \simeq k[T]$ is given by

$$v_j(f(T)) \;=\; \sum_{n \geq j} a_n \binom{n}{j} T^{n-j}, \tag{1.2.3}$$

since

$$\Delta_{k[\mathbb{G}_a]}(T^n) \;=\; (T \otimes 1 + 1 \otimes T)^n = \sum_{j \geq 0} \binom{n}{j} T^{n-j} \otimes T^j.$$

Using (1.2.2), we immediately identify those $k\mathbb{G}_a$-modules which arise as rational $\mathbb{G}_a$-modules.

Proposition 1.3. *Let M be a $k\mathbb{G}_a$-module satisfying the following condition:*

$$\forall\ m \in M,\ \exists \text{ only finitely many } v_j \text{ acting non-trivially on } m. \tag{1.3.1}$$

Then the $k\mathbb{G}_a$-module structure on M (i.e., the action of each $v_j \in k\mathbb{G}_a$ on M) arises from the rational $\mathbb{G}_a$-module structure

$$M \ \to M \otimes k[\mathbb{G}_a], \quad m \in M \mapsto \sum_j v_j(m) \otimes T^j.$$

Conversely, any rational $\mathbb{G}_a$-module satisfies condition (1.3.1).

We make explicit the following useful consequence of Proposition 1.3.

Corollary 1.4. *Let M be a rational $\mathbb{G}_a$-module and $S \subset M$ be a subset. Then the rational $\mathbb{G}_a$-submodule generated by S, $\langle \mathbb{G}_a \cdot S\rangle$, is spanned by $\{v_j(s); s \in S, j \geq 0\}$.*

In particular, the rational $\mathbb{G}_a$-submodule $\langle \mathbb{G}_a \cdot f(T)\rangle \ \subset \ k[\mathbb{G}_a] \simeq k[T]$ generated by $f(T)$ is the subspace of $k[T]$ spanned by $\{v_j(f(T))\}$ as given in (1.2.3).

Proof. The span of $\{v_j(s); s \in S, j \geq 0\}$ is clearly a $k\mathbb{G}_a$-submodule of M. Thus, the corollary follows from Proposition 1.3. □

Using a theorem of E. Kummer [9] (see also [7]), we obtain the following explicit description of the rational $\mathbb{G}_a$-submodule $\langle \mathbb{G}_a \cdot T^n\rangle$.

Proposition 1.5. *The rational $\mathbb{G}_a$-submodule $\langle \mathbb{G}_a \cdot T^n\rangle \ \subset k[T]$ has a k-basis consisting of those T^m such that adding $n-m$ to m involves no carries in base-p arithmetic. In other words, if we write n in base-p as $\sum_{i\geq 0} n_i p^i$ with $0 \leq n_i < p$ for all i, then $\langle \mathbb{G}_a \cdot T^n\rangle$ is spanned by those T^m for which $m = \sum_{i \geq 0} m_i p^i$ with $m_i \leq n_i$.*

Proof. By (1.2.3), $v_j(T^n)$ is a non-zero multiple of T^{n-j} if and only if p does not divide $\binom{n}{j}$. Kummer's theorem asserts that the maximal p-th power dividing $\binom{n}{j}$ equals the number of carries in base-p arithmetic arising in adding j to $n-j$. □

Example 1.6. We can easily construct many non-isomorphic rational $\mathbb{G}_a$-module structures on the underlying vector space of $k[T]$. Namely, for each $i \geq 0$, choose $n_i \geq 0$ with $\varinjlim_i n_i = \infty$ and choose $g_i(\underline{u}) \in k\mathbb{G}_a \simeq k[u_0, u_1, \ldots]/(u_0^p, u_1^p, \ldots)$ such that $g_i(\underline{u})$ is a polynomial in the u_j's with $j \geq n_i$. The $k\mathbb{G}_a$-module structure on $k[T]$ given by setting the action of u_i on $k[T]$ to be that of $g_i(\underline{u})$ on $k[\mathbb{G}_a] \simeq k[T]$ with its structure arising from the coproduct of $\mathbb{G}_a$ is a rational $\mathbb{G}_a$-module by Proposition 1.3 (since only finite many $g_i(\underline{u})$'s act non-trivially on a given T^n by (1.2.3)).

Remark 1.7. Different choices of $g_i(\underline{u})$ in the preceding example can lead to isomorphic rational $\mathbb{G}_a$-modules. For example, let $\theta : \mathbb{N} \to \mathbb{N}$ be a bijection and set $g_i(\underline{u}) = u_{\theta(i)}$. The resulting module structure on $k[T]$ is isomorphic to that of $k[\mathbb{G}_a]$ via the k-linear isomorphism $\Theta_\Sigma : k[T] \to k[T]$ sending the monomial $\prod_{i>0}(T^{p^i})^{d_i}$ to $\prod_{i>0}(T^{p^{\theta(i)}})^{d_i}$ where $0 \leq d_i < p-1$

The following elementary proposition justifies the functor $(-)_{<d}$ of Definition 1.9.

Proposition 1.8. *For any rational $\mathbb{G}_a$-module M and any $\phi \in k\mathbb{G}_a$,*

$$M_\phi \ \equiv \ \{m \in M : \phi(m) = 0\} \ \subset \ M$$

is a rational $\mathbb{G}_a$-submodule of M. Moreover, if $f : M \to N$ is a map of rational $\mathbb{G}_a$-modules, then f restricts to $M_\phi \to N_\phi$.

Proof. To show that $M_\phi \subset M$ is a rational $\mathbb{G}_a$-submodule it suffices by Proposition 1.3 to show that $\psi \cdot M_\phi \subset M_\phi$ for any $\psi \in k\mathbb{G}_a$. This follows immediately from the commutativity of $\mathbb{G}_a$ (implying the commutativity of $k\mathbb{G}_a$).

The second assertion concerning a map $f : M \to N$ of rational $\mathbb{G}_a$-modules follows from the fact that f necessarily commutes with the action of ϕ. □

Proposition 1.8 enables the formulation of many natural filtrations on $(\mathbb{G}_a - Mod)$. The motivation for considering the following is given by Proposition 1.11.

Definition 1.9. For any $d \geq 1$, we define the idempotent endo-functor

$$(-)_{<d} : (G\text{-}Mod) \ \to \ (G\text{-}Mod), \quad M \ \mapsto \ M_{<d} \equiv \{m \in M : v_j(m) = 0, j \geq d\}.$$

In other words, $m \in M_{<d}$ if and only if $\Delta(m) \in M \otimes k[T]_{<d}$.

For any rational $\mathbb{G}_a$-module M, we consider the *degree filtration*

$$M_{<1} \ \subset M_{<2} \ \subset M_{<3} \ \subset \ \cdots \ \subset M$$

of M by rational $\mathbb{G}_a$-submodules.

The following proposition, established in [4], follows easily from the observation that the coproduct $\Delta_M : M \to M \otimes k[\mathbb{G}_a]$ defining the rational $\mathbb{G}_a$-module structure on M sends a finite dimensional subspace of M to a finite dimensional subspace of $M \to M \otimes k[\mathbb{G}_a]$.

Proposition 1.10. [4, 2.6] *Each finite dimensional rational $\mathbb{G}_a$-module lies in the image of $(-)_{<d}$ for d sufficiently large. Consequently,*

1. *For any rational $\mathbb{G}_a$-module M, $M = \cup_d M_{<d}$.*
2. *If M is finite dimensional, then $M \ = \ M_{<d}$ for $d >> 0$.*

Unlike for other linear groups considered in later sections, the filtration on the coordinate algebra $[\mathbb{G}_a] = k[T]$ of $\mathbb{G}_a$ can be viewed as a coalgebra splitting of restriction maps $k[\mathbb{G}_{(a)}] \ \to \ k[\mathbb{G}_{a(r)}]$ as observed in the next proposition.

Proposition 1.11. *For each $d > 0$, the rational submodule $j_d : k[T]_{<d} \subset k[T]$ is a sub-coalgebra.*

Moreover,

$$pr_{p^r} \circ j_d : k[T]_{<p^r} \subset k[T] \to k[T]/T^{p^r}, \ r > 0$$

is an isomorphism of coalgebras.

Proof. The fact that j_d is an embedding of coalgebras follows from the form of the coproduct $\Delta : k[T] \to K[T] \otimes k[T]$ which sends T^n to $(T \otimes 1 + 1 \otimes T)^n$; thus Δ applied to a polynomial $f(T)$ of degree $< d$ is mapped to $\sum_i g_i \otimes h_i$ with the degree of each g_i and each $h_i < d$.

The fact that $pr_d \circ j_d$ is injective (and thus an isomorphism by dimension considerations) is evident by inspection. For $d = p^r$, one easily checks that the coalgebra structure on $k[T]$ induces a coalgebra structure on $k[T]/T^{p^r}$. □

We summarize some of the relationships between various functors on rational $\mathbb{G}_a$-modules. We denote the abelian category of such rational modules either by ($\mathbb{G}_a$-Mod) or by ($k[\mathbb{G}_a]$-$coMod$); we denote the category of rational modules for the infinitesimal group scheme $\mathbb{G}_{a(r)}$ either by ($\mathbb{G}_{a(r)}$-Mod) or by ($k[\mathbb{G}_{a(r)}]$-$coMod$).

We denote by

$$\rho_r : (\mathbb{G}_a\text{-}Mod) \ \to \ (\mathbb{G}_{a(r)}\text{-}Mod)$$

the restriction functor sending a rational $\mathbb{G}_a$-module M with coporoduct $M \to M \otimes k[\mathbb{G}_a]$ to the comodule for $k[\mathbb{G}_{a(r)}]$ with coproduct defined by composition with the projection $pr_{p^r} : k[\mathbb{G}_a] \to k[\mathbb{G}_{a(r)}]$.

Proposition 1.12. *Consider the full subcategory $\iota_d : (k[\mathbb{G}_a]_{<d}\text{-}coMod) \subset (k[\mathbb{G}_a]\text{-}coMod)$ of rational $\mathbb{G}_a$-modules M whose coproduct is of the form $M \to M \otimes k[\mathbb{G}_a]_{<d}$.*

1. *The image of ι_d consists of rational $\mathbb{G}_a$-modules M such that $M = M_{<d}$.*
2. *For any $d > 0$, ι_d is left adjoint to functor $(k[\mathbb{G}_a]\text{-}coMod) \to (k[\mathbb{G}_a]_{<d}\text{-}coMod)$ given by $(-)_{<d}$.*
3. *For any $r > 0$, the composition*

$$\rho_r \circ \iota_{p^r} : (\mathbb{G}_a\text{-}Mod)_{<p^r} \stackrel{\sim}{\to} Mod(\mathbb{G}_{a(r)})$$

is an equivalence of categories.

Proof. The first statement is essentially a tautology. The fact that ι_d is left adjoint to $(-)_{<d}$ follows from the observation that if $f : M \to N$ is a map of $k\mathbb{G}_a$-modules and if $\phi \in k\mathbb{G}_a$ vanishes on M, then f factors (uniquely) through $N_\phi \subset N$.

The last statement is a consequence of the isomorphism $pr_{p^r} \circ j_{p^r} : k[T]_{<p^r} \stackrel{\sim}{\to} k[T]/T^{p^r}$ of Proposition 1.11. Namely, viewing ρ_r and ι_{p^r} as functors on categories of comodules, ι_{p^r} is determined on comodules by composing with j_{p^r} and ρ_{p^r} is determined by composing with pr_{p^r}. □

2. Rational modules for unipotent groups

Let U_N denote the unipotent radical of the standard (upper triangular) Borel subgroup of the general linear group GL_N. Then $k[U_N]$ is a polynomial algebra on the strictly upper triangular coordinate functions $\{x_{i,j};\ 1 \leq i < j \leq N\}$. We equip $k[U_N]$ with the grading determined by setting the degree of each $x_{i,j}$ equal to 1. A closed embedding $U \subset U_N$ of linear algebraic groups is said to be *linear* if the ideal $I_U \subset k[U_N]$ defining $U \subset U_N$ is generated by functions $f(x_{i,j})$ of degree 1. This implies that the maximal ideal at the identity of U, $\mathfrak{m}_U \subset k[U]$, is generated by m elements where $m = dim(U)$, so that $k[U]$ can be identified with the symmetric algebra $S^\bullet(\mathfrak{m}_U/\mathfrak{m}_U^2)$.

For any $d > 0$, we set $k[U_N]_{<d} \subset k[U_N]$ equal to the subspace of polynomials (functions on U_N) of degree $< d$.

Proposition 2.1. *Let $i : U \to U_N$ be a closed embedding of linear algebraic groups. Set $k[U]_{<d} \subset k[U]$ equal to the image under i^* of $k[U_N]_{<d} \subset k[U_N]$. The map of Hopf algebras $i^* : k[U_N] \to k[U]$ induces for each $d > 0$ a map of coalgebras*

$$k[U_N]_{<d} \to k[U]_{<d}.$$

Proof. The coproduct $\Delta_{U_N} : k[U_N] \to k[U_N] \otimes k[U_N]$ of the Hopf algebra $k[U_N]$ is a map of algebras, determined by

$$\Delta_{U_N}(x_{i,j}) \ = \ (x_{i,j} \otimes 1) + (\sum_{t;i<t<j} x_{i,t} \otimes x_{t,j}) + (1 \otimes x_{i,j}).$$

In particular, if $f \in k[U_N]$ has degree $< d$ and if $\Delta_{U_N}(f) = \sum_i f'_i \otimes f''_i$, then each f'_i and each f''_i also has degree $< d$.

Because i^* is a map of Hopf algebras, i^* determines a commutative square of algebras

$$\begin{tikzcd} k[U_N] \arrow[r, "\Delta_{U_N}"] \arrow[d, "i^*"'] & k[U_N] \otimes k[U_N] \arrow[d, "i^* \otimes i^*"] \\ k[U] \arrow[r, "\Delta_U"'] & k[U] \otimes k[U]. \end{tikzcd} \tag{2.1.1}$$

A simple diagram chase implies that (2.1.1) restricts to a commutative square

$$\begin{tikzcd} k[U_N]_{<d} \arrow[r, "\Delta_{U_N}"] \arrow[d, "i^*"'] & k[U_N]_{<d} \otimes k[U_N]_{<d} \arrow[d, "i^* \otimes i^*"] \\ k[U]_{<d} \arrow[r, "\Delta_U"'] & k[U]_{<d} \otimes k[U]_{<d}. \end{tikzcd} \tag{2.1.2}$$

□

In particular, each subspace $k[U]_{<d} \subset k[U]$ is a rational U-module with coaction $k[U]_{<d} \to k[U]_{<d} \otimes k[U]$ given by the coalgebra structure on $k[U]_{<d}$.

Definition 2.2. Let U be a linear algebraic group provided with a closed embedding $i : U \to U_N$ for some N. For any rational U-module M and any $d > 0$, we define

$$M_{<d} \equiv \{m \in M : \Delta_M(m) \in M \otimes k[U]_{<d}\}. \tag{2.2.1}$$

The *degree filtration* on M is the filtration

$$M_{<1} \subset M_{<2} \subset M_{<3} \subset \cdots M.$$

If $M = M_{<d}$, then we say that M has filtration degree $< d$.

The following proposition will prove useful at many points; in particular, it implies that the degree filtration of (2.2.1) is a filtration by rational $\mathbb{G}_a$-modules.

Proposition 2.3. *Let C be a coalgebra over k and $i : B \subset C$ a right coideal (i.e., $\Delta_C : C \to C \otimes C$ restricts to $\Delta_B : B \to B \otimes C$). For any right C-comodule M (i.e., $\Delta_M : M \to M \otimes C$), the subspace*

$$M' \equiv \Delta_M^{-1}(M \otimes B) \subset M$$

is a right C-subcomodule of M. Moreover, if $i : B \subset C$ is a sub-coalgebra, then M' is a right B-comodule.

In particular, let G be a linear algebraic group and let $B \subset k[G]$ be a right coideal (i.e., a rational G-submodule of $k[G]$). Then for any rational G-module M, the subspace $M' \equiv \Delta_M^{-1}(M \otimes B) \subset M$ is a rational G-submodule.

Proof. The comodule structure map $\Delta_M : M \to M \otimes C$ for M is a map of right C-comodules provided that the right C-comodule structure on $M \otimes C$ is given by sending $m \otimes c$ to $m \otimes \Delta_C(c)$. Since $i : B \subset C$ is a right coideal, $1 \otimes i : M \otimes B \subset M \otimes C$ is a right C-comodule. We claim that the pre-image $M' \equiv \Delta_M^{-1}(M \otimes B)$ (in the abelian category of right C-modules) of the right C-subcomodule $M \otimes B \subset M \otimes C$ with respect to the map $\Delta_M : M \to M \otimes C$ of right C-comodules is a right C-comodule as asserted. Namely, the kernel $K \subset M \oplus (M \otimes B)$ of the map of right C-modules $(\Delta_M, (1 \otimes i)) : M \oplus (M \otimes B) \to M \otimes C$ maps isomorphically via projection onto the first summand of $M \oplus (M \otimes B)$ to $M' \subset M$ since $1 \otimes i : M \otimes B \to M \otimes C$ is injective. Furthermore, the right C-coproduct $\Delta_{M'} : M' \to M' \otimes C$ (the restriction of Δ_M) has image in $M \otimes B$ by definition of M'.

If $B \subset C$ is a sub-coalgebra, then the right C-comodule structure on $M \otimes B$, $\Delta_{M \otimes B} : M \otimes B \to M \otimes B \otimes C$, is a right B-comodule structure and thus restricts to a right B-comodule structure on M'.

Specializing the previous argument to $C = k[G]$, we get the second assertion concerning rational G-modules. □

Remark 2.4. To understand the statement of Proposition 2.3, it may be useful to consider the special case in which G is a discrete group acting on a k-vector space M, and $B \subset C$ is taken to be the inclusion of group algebras $k[G/H] \subset k[G]$ for some normal subgroup $H \subset G$. In this case, $M' \subset M$ is the subspace of elements $m' \in M$ with the following property: if $gm = m'$ for some $g \in G, m \in M$, then $ghm = m'$ for every $h \in H$.

Specializing C to $k[U]$ and B to $k[U]_{<d}$ in Propostion 2.3, we conclude the following.

Proposition 2.5. *Let U be a linear algebraic group provided with a closed embedding into some U_N and let M be a rational U-module.*

(1) *For any $d > 0$, the subspace $M_{<d} \subset M$ of a rational U-module M is a rational U-submodule of M whose structure arises from a comodule structure for the sub-coalgebra $k[U]_{<d}$ of $k[U]$.*
(2) *Conversely, if $N \subset M$ is a rational U-module whose structure arises from a comodule structure for $k[U]_{<d}$, then $N = N_{<d}$.*
(3) *In particular, the degree filtration $\{M_{<d}, d > 0\}$ is a filtration of M by rational U-submodules.*

Example 2.6. Let M be a rational G-module, with $G = U_2 \subset GL_2$ (isomorphic to $\mathbb{G}_a$). Then the degree filtration on M as formulated in Definition 2.2 equals that in Definition 1.9 for M viewed as a $\mathbb{G}_a$-module).

Example 2.7. Let M be a polynomial GL_N-module, homogeneous of degree $d-1$ (i.e., a comodule for $k[\mathbb{M}_N]_{d-1} \subset k[GL_N]$), and consider M via restriction as a rational U_N-module. Then M has filtration degree $< d$. This follows immediately by observing that restriction of M to U_N has coproduct $M \to M \otimes k[U_N]$ equal to the composition $(1_M \otimes \pi) \circ \Delta_M : M \to M \otimes k[\mathbb{M}_N] \to M \otimes k[U_N]$, where $\pi : k[\mathbb{M}_N] \to k[U_N]$ sends $x_{i,j}$ with $i < j$ to $x_{i,j}$; $x_{i,i}$ to 1; and $x_{i,j}$ with $i > j$ to 0.

The following proposition asserts that $(-)_{<d}$ is an idempotent functor determining a right adjoint to the embedding $(k[U]_{<d}\text{-}coMod) \hookrightarrow (k[U]\text{-}coMod)$. This is a generalization of Proposition 1.12.1 and itself is generalized in Proposition 4.1.

Proposition 2.8. *Let U be a linear algebraic group provided with a closed embedding $U \to U_N$ for some N. Then for any $d > 0$ and any rational U-module M*

(1) $M_{<d} = (M_{<d})_{<d}$ *(where $M_{<d}$ is given in (2.2.1));*
(2) *the natural embedding given by the inclusion of coalgebras $k[U]_{<d} \to k[U]$,*

$$\iota_d : (k[U]_{<d}\text{-}coMod) \hookrightarrow (k[U]\text{-}coMod),$$

is left adjoint to the functor

$$(-)_{<d} : (k[U]\text{-}coMod) \to (k[U]_{<d}\text{-}coMod), \quad M \mapsto M_{<d}.$$

Proof. By Proposition 2.3, $\Delta_{M_{<d}} : M_{<d} \to M_{<d} \otimes k[U]$ has image in $M_{<d} \otimes k[U]_{<d}$. Thus,

$$(M_{<d})_{<d} \ = \ (\Delta_{M_{<d}})^{-1}(M_{<d} \otimes k[U]_{<d}) \ = \ M_{<d}.$$

If $M = M_{<d}$ and N are rational U-modules, then any map $f : M \to N$ of rational U-modules fits in a commutative square

(2.8.1)
$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "f"'] & M \otimes k[U_N]_{<d} \arrow[d, "f \otimes i_d"] \\ N \arrow[r, "\Delta_N"'] & N \otimes k[U]. \end{tikzcd}$$

Consequently, f factors uniquely through $N_{<d}$; this means that $(-)_{<d}$ is right adjoint to ι_d. □

The fact that $(-)_{<d}$ admits an exact left adjoint immediately implies that it sends injectives to injectives as we make explicit in the following corollary of Proposition 2.8.

Corollary 2.9. *For any rationally injective U-module L and any $d > 0$, $L_{<d}$ is an injective object of $(k[U]_{<d}$-$coMod)$.*

For any linear algebraic group G with coordinate algebra $k[G]$, we denote by $G^{(r)}$ the linear algebraic group whose coordinate algebra $k[G^{(r)}]$ is the base change $k \otimes_k k[G]$ along the p^r-th power map $k \to k$. The r-th Frobenius map is the natural map $F^r : k[G^{(r)}] \to k[G]$ of k-algebras sending $1 \otimes f$ to f^{p^r}; for G defined over $\mathbb{F}_p$, we may view this as an endomorphism of $k[G]$. We define the r-th Frobenius kernel

$$G_{(r)} \ \equiv ker\{F^r : G \to G^{(r)}\} \subset \ G,$$

so that $k[G_{(r)}] \ = \ k[G] \otimes_{k[G^{(r)}]} k$, where $k[G^{(r)}] \to k$ is the counit of $k[G^{(r)}]$. Thus, we may identify $k[G_{(r)}]$ with $k[G]/(f^{p^r}, f \in \mathfrak{m}_G)$, the quotient of $k[G]$ by the ideal generated by p^r-th powers of elements in the maximal ideal at the identity (i.e., generated by f^{p^r} for all $f \in k[G]$ with $f(1) = 0$). The quotient map $k[G] \twoheadrightarrow k[G_{(r)}]$ is a map of Hopf algebras.

The following proposition should be compared and contrasted with Proposition 1.11.

Proposition 2.10. *Let U be a connected linear algebraic group provided with a closed embedding $i : U \to U_N$ for some N. Let m denote the dimension of U. Then for any $r > 0$*

1. *The composition $k[U]_{<p^r} \subset \ k[U] \ \twoheadrightarrow \ k[U_{(r)}]$ is injective.*
2. *The dimension of $k[U_{(r)}]$ equals p^{rm}.*
3. *The composition*
$$k[U]_{<p^r \cdot \frac{N(N-1)}{2}} \ \subset \ k[U] \ \twoheadrightarrow \ k[U_{(r)}]$$
is surjective.
4. *If the closed embedding $i : U \subset U_N$ is linear, then the dimension of $k[U]_{<p^r}$ equals $\binom{m+p^r-1}{p^r}$; if $m > 1$, then this is not divisible by p^{rm}.*

Proof. Every non-zero element of $\mathfrak{m}_U$ has positive degree in $k[U]$ (i.e., is not constant), so that $k[U]_{<p^r} \cap \mathfrak{m}_U^{p^r} \ = \ 0$; this proves (1).

The computation in (2) of the dimension of $k[U_{(r)}]$ can be verified as follows. For $r = 1$, $k[U_{(1)}]$ is dual to the restricted enveloping algebra of $\mathfrak{u} = Lie(U)$ and therefore has dimension equal to p^m. Furthermore, the quotient $U_{(r)}/U_{(r-1)}$ is isomorphic to $U_{(1)}$ for $r > 1$, so the computation is concluded using induction on r.

The commutativity of the following diagram with surjective vertical maps

$$
\begin{tikzcd}
k[U_N]_{<p^{r+s}} \arrow[r] \arrow[d] & k[U_N] \arrow[r] \arrow[d] & k[U_{N(r)}] \arrow[d] \\
k[U]_{<p^{r+s}} \arrow[r] & k[U] \arrow[r] & k[U_{(r)}]
\end{tikzcd}
\tag{2.10.1}
$$

reduces the proof of (3) to the case that $U = U_N$. In this case, $m = \frac{N(N-1)}{2}$. We view $k[U_N]_{<p^r \cdot m} \to k[U_{N(r)}]$ as the surjective map from the space of polynomials in m variables spanned by monomials of total degree $< p^r \cdot m$ to the space of polynomials in the same variables spanned by monomials not divisible by the p^r-th power of any variable.

To prove (4), observe that the dimension of those polynomial functions of total degree $< p^r$ in m variables equals the dimension of those homogeneous polynomial functions of degree p^r in $m+1$ variables. One checks recursively that the latter dimension equals $\binom{m+p^r-1}{p^r-1}$; this is not a p-th power provided that $m \geq 1$. $\square$

For some time, we tried to prove the following injectivity criterion for rational $\mathbb{G}_a$-modules: if M is a rational $\mathbb{G}_a$-module whose restriction to each $\mathbb{G}_{a(r)}$ is injective, then M is injective. As the following proposition makes clear, this "support variety criterion for injectivity" fails miserably not just for $\mathbb{G}_a$ but for any connected unipotent algebraic group.

Proposition 2.11. *Let U be a connected unipotent algebraic group of positive dimension and let $U \subset G$ be a closed embedding of U in a reductive group G. Then $k[G]$ is not injective as a rational U-module, whereas the restriction of $k[G]$ to each Frobenius kernel $U_{(r)}$ of U is injective.*

Proof. As shown in [2, 2.1,4.5] (see also Proposition 4.4 below), for a closed subgroup H of a reductive algebraic group G to satisfy the condition that $k[G]$ is injective as a rational $k[H]$-module, it is necessary and sufficient that H be reductive. In particular, $k[G]$ is not injective as a rational U-module.

On the other hand, $k[G]$ is rationally injective as a G-module and thus projective as an $kG_{(r)}$-module for any $r > 0$. Since $kG_{(r)}$ is free as a $kU_{(r)}$-module (see, for example, [8]), we conclude that $k[G]$ is projective (in fact, free) when restricted to each Frobenius kernel $U_{(r)}$. $\square$

In order to provide a necessary and sufficient criterion of for rational injectivity for rational U-modules, we mention the following structure property for rationally injective modules for a unipotent algebraic group U. We remind the reader $V \otimes k[G]$ is rationally injective for any affine algebraic group G and any k-vector space V (see [8]).

Proposition 2.12. *Let U be a connected, unipotent algebraic group. Let L be a rationally injective U-module and set $L_0 = H^0(U, L)$. There exists a map $f : L \to L_0 \otimes k[U]$ of rational U-modules whose restriction to L_0 identifies L_0 with*

$L_0 \otimes 1 \subset L_0 \otimes k[U]$; in particular, such a map f is injective when restricted to $L_0 \subset L$.

Moreover, a map $f : L \to L_0 \otimes k[U]$ of rational U-modules is an isomorphism if and only if the induced map $H^0(f) : H^0(U, L) = L_0 \to H^0(U, L_0 \otimes k[U])$ is an isomorphism of vector spaces.

Proof. The existence of a map $f : L \to L_0 \otimes k[U]$ of rational U-modules which "is the identity on L_0" is an immediate consequence of the extension mapping property of the rationally injective $k[U]$-module $L_0 \otimes k[U]$ applied to the monomorphism $L_0 \subset L$. This map is necessarily injective because it is injective on the socle of L.

Clearly, the condition that $H^0(f)$ be an isomorphism is necessary for f to be an isomorphism. Since U is unipotent, $H^0(U, L)$ is the socle of both L and $L_0 \otimes k[U]$. Thus, the condition that $H^0(f)$ be injective implies that f is itself injective because a non-trivial kernel of f would have to meet the socle of L non-trivially. If f is injective, the rational injectivity of L implies the existence of some $g : L_0 \otimes k[U] \to L$ with $g \circ f = id$. On the other hand, if $L_0 \otimes k[U] \simeq L \oplus L'$ with $L' \neq 0$, then $H^0(f)$ is not surjective. □

If U is a unipotent algebraic group, rational injectivity of a rational U-module L can be detected on submodules $L_{<p^r}$ by arguing by induction on the dimension of the socle of L. This is done in the following proposition, which motivates the criterion of Proposition 4.2 for an arbitrary linear algebraic group of exponential type.

Proposition 2.13. *Let U be a linear algebraic group provided with a closed embedding $i : U \to U_N$ for some N. Then a rational U-module L is rationally injective if and only if $L_{<p^r} \subset L$ is injective as a $k[U]_{<p^r}$-comodule for each $r > 0$.*

In particular, if $U \simeq \mathbb{G}_a$, then a rational $\mathbb{G}_a$-module L is rationally injective if and only if for all $r > 0$ the restriction of $L_{<p^r} \subset L$ to $k\mathbb{G}_{a(r)}$ (via $\rho_r \circ \iota_{<p^r}$ of Proposition 1.12) is free.

Proof. By Corollary 2.9, the restriction of a rationally injective U-module is injective as a $k[U]_{<p^r}$-comodule for each $r > 0$. For $U \simeq \mathbb{G}_a$, Proposition 1.12(3) tells us that injective $k[\mathbb{G}_a]_{<p^r}$-comodules are injective (equivalently, free) $k\mathbb{G}_{a(r)}$-modules.

To prove the converse, consider some rational U-module L such that $L_{<p^r} \subset L$ is injective as a $k[U]_{<p^r}$-comodule for each $r > 0$. Let L_0 denote the socle of L. As in Proposition 2.12, let $f : L \to L_0 \otimes k[U]$ be some injective map of rational U-modules extending $L_0 \subset L_0 \otimes k[U]$. We proceed to show that f is an isomorphism. Namely, for each $r > 0$, we use the injectivity of $L_{<p^r}$ as a $k[U]_{<p^r}$-comodule to extend the identity map $L_{<p^r} \to L_{<p^r}$ along the monomorphism $f_r : L_{<p^r} \to L_0 \otimes k[U]_{<p^r}$ to some $k[U]_{<p^r}$-comodule homomorphism $g_r : L_0 \otimes k[U]_{<p^r} \to L_{<p^r}$. The fact that g_r extends the identity map implies that g_r is surjective. On the other hand, g_r induces an isomorphism on socles, so must be injective. Thus, each f_r is an isomorphism (with inverse g_r). Since $f = \varinjlim_r f_r$, we conclude that f is an isomorphism. □

Remark 2.14. Proposition 2.13 stands in contrast with Proposition 2.10: $k[U]_{<p^r}$ is not free as a $U_{(r)}$-module if the dimension of U is greater than 1.

3. Filtrations on rational G-modules for G of exponential type

Throughout this section, G denotes a connected linear algebraic group with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ the variety of r-tuples $(B_0, \ldots, B_{r-1})$ of pairwise commuting, p-nilpotent elements of $\mathfrak{g}$; in other words, each B_i satisfies $B_i^{[p]} = 0$ and each pair B_i, B_j satisfies $[B_i, B_j] = 0$. We denote by $V_r(G)$ the variety of height r infinitesimal 1-parameter subgroups $\mathbb{G}_{a(r)} \to G$ of G.

We begin by recalling from [4, 1.6] the definition of a structure of exponential type on a linear algebraic group, a definition which extends the formulation in [12] of an embedding $G \subset GL_n$ of exponential type. Up to isomorphism (as made explicit in [4, 1.7]), if such a structure exists then it is unique.

Definition 3.1. [4, 1.6] Let G be a linear algebraic group with Lie algebra $\mathfrak{g}$. A structure of exponential type on G is a morphism of k-schemes

(3.1.1) $$\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \to G, \quad (B, s) \mapsto \mathcal{E}_B(s)$$

such that

(1) For each $B \in \mathcal{N}_p(\mathfrak{g})(k)$, $\mathcal{E}_B : \mathbb{G}_a \to G$ is a 1-parameter subgroup.
(2) For any pair of commuting p-nilpotent elements $B, B' \in \mathfrak{g}$, the maps $\mathcal{E}_B, \mathcal{E}_{B'} : \mathbb{G}_a \to G$ commute.
(3) For any commutative k-algebra A, any $\alpha \in A$, and any $s \in \mathbb{G}_a(A)$, $\mathcal{E}_{\alpha \cdot B}(s) = \mathcal{E}_B(\alpha \cdot s)$.
(4) Every 1-parameter subgroup $\psi : \mathbb{G}_a \to G$ is of the form
$$\mathcal{E}_{\underline{B}} \equiv \prod_{s=0}^{r-1} (\mathcal{E}_{B_s} \circ F^s)$$
for some $r > 0$, some $\underline{B} \in \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$; furthermore, $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \to V_r(G)$, $\underline{B} \mapsto \mathcal{E}_{\underline{B}} \circ i_r$ is an isomorphism for each $r > 0$.

A connected linear algebraic group which admits a structure of exponential type is said to be a **linear algebraic group of exponential type**.

Moreover, $H \subset G$ is said to be an embedding of *exponential type* if H is equipped with the structure of exponential type given by restricting that provided to G; in particular, we require $\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \to G$ to restrict to $\mathcal{E} : \mathcal{N}_p(\mathfrak{h}) \times \mathbb{G}_a \to H$.

The following explicit example [12, 1.2] helps to illuminate Definition 3.1.

Example 3.2. Let $\mathfrak{gl}_N = Lie(GL_N)$. Then the pairing

(3.2.1)
$$\mathcal{E} : \mathcal{N}_p(\mathfrak{gl}_N) \times \mathbb{G}_a \to GL_N, \quad (B, t) \mapsto exp_B(t) \equiv 1 + tB + \frac{(tB)^2}{2} + \cdots + \frac{(tB)^{p-1}}{(p-1)!}$$

defines a structure of exponential type on GL_N.

Many familiar linear algebraic groups are linear algebraic groups of exponential type as recalled in the following example.

Example 3.3. The following linear algebraic groups are of exponential type.

- Any simple algebraic group of classical type, any parabolic subgroup of such a group, any unipotent radical of such a parabolic subgroup is of exponential type [12, 1.8].
- Any reductive algebraic group G with Coxeter number $h(G) \leq p$ and PSL_p not a factor of $[G, G]$ is of exponential type [11].

- Any unipotent radical of a parabolic subgroup of SL_N or a product of commuting root groups in SL_n.
- Any unipotent algebraic group U of nilpotent class $< p$ is of exponential type, with the Campbell-Hausdorff-Baker formula determining the exponential structure.

We introduce a filtration $\{M_{[d]}, d > 0\}$ on a rational G module M for G a linear algebraic group of exponential type. For $G = U_N$ with $N \leq p$, this filtration is comparable to the filtration of Definition 2.2 as seen in Remark 3.12; this filtration is a natural extension of the condition that a rational G module have exponential degree $< p^r$ as introduced in [4, 4.5].

We first define $\{k[G]_{[d]}, d > 0\}$ on the coordinate algebra $k[G]$ of a linear algebraic group G of exponential type; this is shown in Proposition 3.5 to be a filtration by sub-coalgebras.

Definition 3.4. Let G be a linear algebraic group of exponential type, $\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \to G$. For any $d \geq 0$, we define $(k[G])_{[d]} \subset k[G]$ as follows:

$$(k[G])_{[d]} \equiv \{f \in k[G] : (\mathcal{E}_{B*}(v_j))(f) = 0, \ \forall B \in \mathcal{N}_p(\mathfrak{g}), j > d\}.$$

In other words, $(k[G])_{[d]}$ is the pre-image under $\mathcal{E}^* : k[G] \to k[\mathbb{G}_a] \otimes k[\mathcal{N}_p]$ of $(k[\mathcal{N}_p])[T]_{<d+1}$.

In what follows, we shall employ the notation

$$\mathcal{E}_B^* : k[G] \to k[\mathbb{G}_a], \quad \mathcal{E}_{B*} : k\mathbb{G}_a \to kG$$

for the maps on coordinate algebras and group algebras induced by $\mathcal{E}_B \mathbb{G}_a \to G$.

Proposition 3.5. *Let G be a linear algebraic group of exponential type. For any $d \geq 0$, $(k[G])_{[d]} \subset k[G]$ is a sub-coalgebra. In particular, $(k[G])_{[d]}$ is a rational G-submodule of $k[G]$.*

Moreover, $(k[G])_{[0]}$ is a sub-Hopf algebra of $k[G]$.

Proof. Let $f \in k[T]$, $B \in \mathcal{N}_p(\mathfrak{g})$. Because $\mathcal{E}_B : \mathbb{G}_a \to G$ is a morphism of algebraic groups,

$$\mathcal{E}_B^*(\Delta_G(f)) = \Delta_{\mathbb{G}_a}(\mathcal{E}_B^*(f)) \in k[\mathbb{G}_a] \otimes k[\mathbb{G}_a].$$

On the other hand, if $p(T) = \mathcal{E}_B^*(f) \in k[\mathbb{G}_a]$ has degree $< d+1$, then $\Delta_{\mathbb{G}_a}(p(T))$ is of the form $\sum_i p_i(T) \otimes p_i'(T)$ with each $p_i(T)$, $p_i'(T)$ having degree $< d+1$. Thus, the coproduct of $k[G]$ restricts to a coproduct on $(k[G])_{[d]}$.

The multiplicative structure of the commutative k-algebra $k[G]$ restricts to a multiplicative structure $(k[G])_{[0]} \otimes (k[G])_{[0]} \to (k[G])_{[0]}$ (since the product of two polynomials in $k[T]$ of degree < 1 is again of degree < 1), thereby verifying that $(k[G])_{[0]}$ is a sub-Hopf algebra of $k[G]$. □

We investigate the relationship between the filtration of Definition 3.4 to that of Proposition 2.1. We point out the following elementary computation: for $B \in U_N$, $exp_B(x_{i,j}) \in k[T]$ equals the (i,j)-th entry of the matrix $1 + BT + B^2 \cdot T^2/2 + \cdots B^{p-1}T^{p-1}/(p-1)!$.

Proposition 3.6. *Let U be a linear algebraic group provided with a closed embedding $i : U \subset U_N$ of exponential type. For any $d > 0$,*

$$k[U]_{<d} \subset (k[U])_{[(p-1)(d-1])}.$$

On the other hand, if $k \neq k[U]_{[0]} \subset k[U]$ (e.g., $U = U_N$ with $N > p$; see Example 3.7), then $k[U]_{[0]}$ is not contained in $k[U]_{<d}$ for any d.

Proof. In the special case $U = U_N$, $\mathcal{E}_B^*(x_{i,j}), 1 \leq i < j \leq N$, is the polynomial in T whose coefficient of T^n is $1/(n!)$ times the (i,j)-th entry of B^n for any n, $1 \leq n < p$. Thus, the ring homomorphism $\mathcal{E}_B^*$ sends a polynomial in the $x_{i,j}$ of degree $\leq d-1$ to a polynomial in T of degree $\leq (p-1)(d-1)$. Hence $\mathcal{E}_{B*}(v_j)$ applied to $f \in k[U_N]_{<d}$ is 0 for $j > (p-1)(d-1)$. For $U \subset U_N$ of exponential type, this argument restricts to U. This establishes the inclusion $k[U]_{<d} \subset (k[U])_{[(p-1)(d-1)]}$.

If there exists a non-constant function $f \in k[U]_{[0]}$, then $\tilde{f} \in k[U_N]$ mapping to f must have positive degree. Thus, powers of f (also in the Hopf algebra $k[U]_{[0]}$) have arbitrarily large degree. □

The following examples point out that even for U_N the comparison of the filtrations of Definition 3.4 and 2.1 is not entirely straight-forward.

Example 3.7. Let $U = U_3$ and consider $f = 2x_{1,3} - x_{1,2}x_{2,3} \in k[U_3]$. Then for all $B \in U_3$, the degree of $exp_B^*(f) \in k[T]$ is ≤ 1 whereas $f \notin k[U_3]_{<2}$.

Let $U = U_N$ with $N > p$. Then $f = \prod_{i=1}^{N-1} x_{i,i+1}$ satisfies the property that $exp_B^*(f) = \prod_{i=1}^{N-1} exp_B^*(x_{i,i+1}) = 0$ for any B such that $B^p = 0$. Therefore $f \in (k[U_N])_{0]}$.

In the special case $U = U_N$, $N \leq p$, we verify that our two filtrations are equivalent.

Proposition 3.8. *Assume that $N \leq p$. Then*

$$(k[U_N])_{[e-1]} \subset k[U_N]_{<d} \subset (k[U_N])_{[(p-1)(d-1]}$$

provided that $e(N-1) < d$.

Proof. To prove the inclusion $(k[U_N])_{[e-1]} \subset k[U_N]_{<d}$ for $N \leq p$, for each $f \in k[U_N]$ of degree $D \geq d$ in the matrix functions $x_{i,j}$ (with $i < j$) we must exhibit a strictly upper triangular matrix B such that $exp_B^*(f) \in k[T]$ has degree $\geq e$. We write

$$f = \sum_{\underline{d}} a_{\underline{d}} x_{\underline{d}}, \quad x_{\underline{d}} = \prod_{i<j} x_{i,j}^{d_{i,j}}, \quad deg(\underline{d}) = \sum d_{i,j}.$$

We say that a monomial $x_{\underline{d'}}$ is a contraction of another monomial $x_{\underline{d}}$ if $x_{\underline{d'}}$ can be obtained from $x_{\underline{d}}$ by iterated replacement of a string of factors of the form $x_{i,s_1}, x_{s_1,s_2}, \ldots, x_{s_\ell,j}$ by the single factor $x_{i,j}$. We say that a monomial $x_{\underline{e}}$ appearing in f (i.e., such that $a_{\underline{e}} \neq 0$) is reduced for f provided that each monomial $x_{\underline{d}}$ of the same degree as $x_{\underline{e}}$ satisfying $d_{i,j} \neq 0$ only if $e_{i,j} \neq 0$ (but not necessarily appearing in f) has no contraction of smaller degree appearing in f.

Starting with a monomial $x_{\underline{D}}$ appearing of f of top degree D, we identify by following repeated contractions some reduced monomial $x_{\underline{e}}$ appearing in f of degree e with $e(N-1) \geq D \geq d$. We consider matrices B with the property that $b_{i,j} \neq 0$ only if $e_{i,j} \neq 0$ for this reduced monomial $x_{\underline{e}}$ of degree e. We claim that the coefficient of T^e in $exp_B^*(f)$ equals the sum

$$\sum_{\underline{d}} a_{\underline{d}} (\prod_{i<j} b_{i,j}^{d_{i,j}}), \tag{3.8.1}$$

where the sum is taken over all monomials $x_{\underline{d}}$ of f of degree e with $d_{i,j} \neq 0$ only if $e_{i,j} \neq 0$. First, observe that for any monomial $x_{\underline{d'}}$, we have $exp_B^*(x_{\underline{d'}}) = \prod(\exp_B^*(x_{i,j}))^{d'_{i,j}}$. This implies that if $x_{\underline{d'}}$ has degree $> e$, then the coefficient of T^e in $exp_B^*(x_{\underline{d'}})$ is 0. Moreover, if $x_{\underline{d}}$ has degree $= e$, then the coefficient of T^e in

$exp^*_B(x_{\underline{d}})$ equals $a_{\underline{d}}(\prod_{i<j} b_{i,j}^{d_{i,j}})$. Finally, if $x_{\underline{d'}}$ has degree $< e$, then the coefficient of T^e in $exp^*_B(x_{\underline{d'}})$ is non-zero only if some factor $x_{i,j}$ is non-zero on a power B^s of B with $s > 0$; this implies that $x_{\underline{d'}}$ is a contraction of some monomial $x_{\underline{d}}$ of degree e with $d_{i,j} \neq 0$ only if $e_{i,j} \neq 0$; since $x_{\underline{e}}$ is assumed to be reduced, this implies that $x_{\underline{d'}}$ does not appear in f.

We view (3.8.1) as a polynomial in the variables $b_{i,j}$ with (i,j) running through pairs $1 \leq i < j \leq N$ such that $e_{i,j} \neq 0$ for our chosen $x_{\underline{e}}$ for the given f of degree D. Since this polynomial is not constant, we may find values for the $b_{i,j}$'s constituting a matrix B such that $exp^*_B(f)$ has non-zero coefficient of T^e; in other words, $f \notin (k[U_N])_{[e-1]}$.

The second inclusion is a special case of Proposition 3.6. □

One consequence of the following proposition is that $(k[G])_{[d]}$ is never finite dimensional if G is a non-trivial reductive algebraic group, since for reductive G the kernel ideal of $k[G] \to k[\mathcal{U}_p]$ is an infinite dimensional vector space. Indeed, for G reductive, the Krull dimension of $k[G]$ equals the Krull dimension of $k[\mathcal{U}]$ plus the rank of G, and the Krull dimension of $k[\mathcal{U}]$ is greater or equal to the Krull dimension of $k[\mathcal{U}_p]$. Here, and in the proposition below, $\mathcal{U} \subset G$ is the closed subvariety of unipotent elements and $\mathcal{U}_p \subset \mathcal{U}$ is the closed subvariety of elements whose p-th power is the identity.

Proposition 3.9. *Let G be a linear algebraic group of exponential type and let $\mathcal{U}_p(G) \subset G$ denote the closed variety of p-unipotent elements of G. Then $\mathcal{E}^* : k[G] \to k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a]$ factors through an embedding*

$$\overline{\mathcal{E}^*} : k[\mathcal{U}_p(G)] \hookrightarrow k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a]. \tag{3.9.1}$$

Consequently, the augmentation ideal of $(k[G])_{[0]}$ (i.e., the functions $f \in (k[G])_{[0]} \subset k[G]$ such that $f(id_G) = 0$) equals the ideal in $k[G]$ of those functions on G which vanish on $\mathcal{U}_p(G) \subset G$.

Proof. If $f \in k[G]$ vanishes on $\mathcal{U}_p(G)$ (i.e. lies in the ideal defining the closed subvariety $\mathcal{U}_p(G) \subset G$), then $\mathcal{E}^*_B(f) = 0$ for all $B \in \mathcal{N}_p(\mathfrak{g})$ since $\mathcal{E}_B : \mathbb{G}_a \to G$ factors through $\mathcal{U}_p$. This implies that $\mathcal{E}^*$ factors through $k[\mathcal{U}_p(G)]$, for if not then there exists some $f \in k[G]$ vanishing on $\mathcal{U}_p(G)$ such that $\mathcal{E}^*(f) = \sum_i f_i \otimes T^i \in k[\mathcal{N}_p(\mathfrak{g})] \otimes k[T]$ is non-zero; for such an f and some i with $f_i \neq 0$, find $B \in \mathcal{N}_p(\mathfrak{g})$ such that $f_i(B) \neq 0$; then $\mathcal{E}^*_B(f) \neq 0$, contradicting the assumption that $\mathcal{E}^*(f) = 0$. On the other hand, this induced map $k[\mathcal{U}_p(G)] \to k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a]$ is injective, for its composition with $id \otimes eval_1 : k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a] \to k[\mathcal{N}_p(\mathfrak{g})]$ is an isomorphism by condition (4) of Definition 3.1.

In particular, we have shown that the kernel of $\mathcal{E}^* : k[G] \to k[\mathbb{G}_a] \otimes k[\mathcal{N}_p(\mathfrak{g})]$ equals the kernel of the restriction map $k[G] \to k[\mathcal{U}_p]$ which equals the ideal of those functions on G which vanish on $\mathcal{U}_p(G) \subset G$. On the other hand, $(k[G])_{[0]}$ consists of those $f \in k[G]$ such that $\mathcal{E}^*_B(f)$ is constant (i.e., lies in k for all B). Therefore, the augmentation ideal of $(k[G])_{[0]}$ equals the ideal of $\mathcal{U}_p(G)$. □

We introduce the filtration by exponential degree on a rational G-module, an "extension" of the degree filtration on a rational U-module given in Definition 2.2.

Definition 3.10. Let G be a linear algebraic group of exponential type and let M be a rational G-module. For any $d \geq 0$, we define

$$M_{[d]} \equiv \{m \in M : \Delta_M(m) \in M \otimes (k[G])_{[d]}\}. \tag{3.10.1}$$

The filtration *by exponential degree* on M is the filtration

$$M_{[0]} \subset M_{[1]} \subset \cdots \subset M.$$

We say that M has *exponential degree* $\leq d$ if $M = M_{[d]}$.

Proposition 3.11. *With notation as in Definition 3.10, $M_{[d]} \subset M$ consists of those elements $m \in M$ such that $(\mathcal{E}_B)_*(v_j) \in kG$ vanishes on m for all $B \in \mathcal{N}_p(\mathfrak{g})$ and all $j > d$.*

Proof. If $\Delta_M(m) \in M \otimes k[G]$ lies in $M \otimes (k[G])_{[d]}$, then the composition

$$(1 \otimes ev_B \otimes 1) \circ \mathcal{E}^* \circ \Delta_M : M \to M \otimes k[G] \to M \otimes k[\mathcal{N}_p(\mathfrak{g})] \otimes k[T] \to M \otimes k[T]$$

has image in $M \otimes k[T]_{\leq d}$ and equals $\mathcal{E}_B^* \circ \Delta_M(m)$; thus, $(\mathcal{E}_B)_*(v_j)$ applied to m vanishes for all $B \in \mathcal{N}_p(\mathfrak{g})$ and all $j > d$.

Conversely, if $\Delta_M(m) = \sum_\alpha m_\alpha \otimes f_\alpha \in M \otimes k[G]$ with some f_α of degree $> d$, then

$$\mathcal{E}^*(\Delta_M(m)) = \sum_\alpha m_\alpha \otimes \sum_j g_{\alpha,j} \otimes T^j \in M \otimes k[\mathcal{N}_p(\mathfrak{g})] \otimes k[T]$$

with some $g_{\alpha,j} \neq 0$ for $j > d$. Then for any $B \in \mathcal{N}_p(\mathfrak{g})$ such that $g_{\alpha,j}(B) \neq 0$, $\mathcal{E}_B^*(v_j)(m) = \sum_{\alpha,j} g_{\alpha,j}(B) m_\alpha \otimes T^j \neq 0$. □

Remark 3.12. Let G be a linear algebraic group of exponential type and let M be a rational G-module. The condition that M has exponential degree $\leq p^r - 1$ is equivalent to the condition that M has exponential degree $< p^r$ in the sense of [4, 4.5].

Take G to equal U_N for some $N \leq p$ and M be a rational U-module. By Propositions 2.5 and 3.8, the condition $M = M_{<d}$ in the sense of Definition 2.2 implies that $M = M_{[(p-1)(d-1)]}$ in the sense of Definition 3.10. Similarly, the condition that $M = M_{[e-1]}$ in the sense of Definition 3.10 implies that $M = M_{<d}$ in the sense of Definition 2.2 provided that $e(N-1) < d$.

The following are natural examples of rational G-modules of (explicitly) bounded exponential degree.

Example 3.13. Let $S(N,d)$ denote the Schur algebra, so that the linear dual of $S(N,d)$ is the coalgebra $k[\mathbb{M}_N]_d$, the vector space of polynomials homogeneous of degree d in the variables $\{x_{i,j}, 1 \leq i,j \leq N\}$. We verify that

$$k[\mathbb{M}_N]_d \hookrightarrow k[GL_N]_{[(p-1)d]}. \tag{3.13.1}$$

Namely, if B is a p-nilpotent, $N \times N$ matrix, then $exp_B^*(x_{i,j}) \in k[T] = k[\mathbb{G}_a]$ is the (i,j)-th entry of the matrix $1 + BT + B^2T^2/2 + \cdots + B^{p-1}T^{p-1}/(p-1)!$ which has degree $\leq p-1$ (as a polynomial in T). Thus, if $f \in k[GL_N]$ is homogenous of degree d in the $x_{i,j}$ (i.e., in the image of $k[\mathbb{M}_N]_d$), then $exp_B^*(f)$ has degree $\leq (p-1)d$.

Consequently, if M is a polynomial representation of GL_N homogeneous of degree d (i.e., a comodule for $k[\mathbb{M}_N]_d$), then $M = M_{[(p-1)d]}$.

Example 3.14. Let G be a reductive group with a structure of exponential type and let M be a rational G-module all of whose high weights μ satisfy the condition that $2\sum_{j=1}^{l} \langle \mu, \omega_j^\vee \rangle < p^r$. Here, $\{\omega_1, \ldots, \omega_\ell\}$ is the set of fundamental dominant weights of G (with respect to some $T \subset B \subset G$) and

$$\omega_j^\vee = 2\omega_j / \langle \alpha_j, \alpha_j \rangle.$$

Then $M = M_{[p^r-1]}$ as seen in [3, 2.7] following [1, 4.6.2].

We provide various properties of our filtration by exponential degree of rational G-modules.

Theorem 3.15. *Let G be a linear algebraic group of exponential type and let M be a rational G-module.*

(1) *The abelian category of comodules for the coalgebra $(k[G])_{[d]}$ equals the full subcategory of $(G\text{-}Mod)$ consisting of those rational G-modules of exponential degree $\leq d$ (i.e., M such that $M = M_{[d]}$).*
(2) *The filtration $\{M_{[d]},\ d \geq 0\}$ of M is independent of the choice of structure of exponential type for G.*
(3) *If M is a finite dimensional rational G-module, then $M = M_{[d]}$ for $d >> 0$.*
(4) *$M \ = \ \cup_d M_{[d]}$ for any rational G-module M.*
(5) *The filtration of $k[G]$ by exponential degree, $\{(k[G])_{[d]},\ d \geq 0\}$, is finite if and only if $\mathcal{N}_p(\mathfrak{g}) = \{0\}$.*
(6) *If M has exponential degree $\leq d$, then its Frobenius twist $M^{(1)}$ has exponential degree $\leq pd$.*
(7) *If M' has exponential degree $\leq d$ and if $f : M' \to M$ is a map of rational G-modules, then $f(M') \ \subset \ M_{[d]}$.*
(8) *If $f : M' \hookrightarrow M$ is an inclusion of rational G-modules and if $m' \in M' \backslash M'_{[d]}$, then $f(m') \in M \backslash M_{[d]}$.*
(9) *If $j : H \subset G$ is an embedding of exponential type and if M has exponential degree $\leq d$ as a rational G-module, then the restriction to H of M has exponential degree $\leq d$ as a rational H-module.*

Proof. Property (1) is merely a rephrasing of the condition that a rational G-module M satisfies the condition $M = M_{[d]}$.

Property (2) follows from [4, 1.7]; property (3) is established in [4, 2.6]. Since any rational G-module is a union of its finite dimensional submodules, property (4) follows from property (3).

If $\mathcal{N}_p(\mathfrak{g}) = 0$, then by condition (4) of Definition 3.1 there are no non-trivial 1-parameter subgroups $\mathbb{G}_a \to G$ so that $k[G] = k[G]_{[0]}$. Conversely, If $\psi : \mathbb{G}_a \to G$ is a non-trivial 1-parameter subgroup, then $\psi^* : k[G] \to k[\mathbb{G}_a]$ has infinite dimensional image so that for each $d > 0$ there exist $f \in k[G]$ which do not lie in $k[G]_{[d]}$.

The Frobenius twist of a rational G-module M, $M^{(1)}$, has as its coproduct structure $\Delta_{M^{(1)}} : M^{(1)} \to M^{(1)} \otimes k[G]$ the composition

$$(1_M \otimes F) \circ (\Delta_M)^{(1)} : M^{(1)} \to M^{(1)} \otimes k[G]^{(1)} \to M^{(1)} \otimes k[G]$$

where $F : k[G]^{(1)} \to k[G]$ is the k-linear map sending $\alpha \otimes f \in k \otimes_\phi k[G]$ to $\alpha f^p \in k[G]$. For any 1-parameter subgroup $\psi : \mathbb{G}_a \to G$, $\psi^*(f^p) = (\psi^*(f))^p \in k[\mathbb{G}_a]$, so that the image under $F : k[G]^{(1)} \to k[G]$ of $(k[G])_{[d]})^{(1)}$ lies in $(k[G])_{[pd]}$, thereby establishing property (6).

Properties (7) and (8) are easy consequences Definition 3.10 and the fact that a map $f : M' \to M$ is a map of $k[G]$-comodules.

Finally, the condition that $j : H \subset G$ be an embedding of exponential type (see Definition 3.1) implies the commutativity of the square

(3.15.1)
$$\begin{array}{ccc} k[G] & \xrightarrow{\mathcal{E}^*} & k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a] \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*\otimes 1} \\ k[H] & \xrightarrow[\mathcal{E}^*]{} & k[\mathcal{N}_p(\mathfrak{h})] \otimes k[\mathbb{G}_a]. \end{array}$$

The surjectivity of j^* together with the commutativity of (3.15.1) implies that j^* restricts to $j^*_d : (k[G])_{[d]} \to (k[H])_{[d]}$. Thus, if the coproduct $\Delta_M : M \to M \otimes k[G]$ for the rational G-module M factors through $M \otimes (k[G])_{[d]}$, then the coproduct for M restricted to H factors through $M \otimes (k[H])_{[d]}$. □

A key definition of [4] is that of the (p-nilpotent) action at a 1-parameter subgroup $\mathcal{E}_{\underline{B}} : \mathbb{G}_a \to G$ of a linear algebraic group G of exponential type acting on a rational G-module M. In [4, 2.6.1], this is defined to be the action of $\sum_{s\geq 0}(\mathcal{E}_{B_s})_*(u_s) \in kG$ acting on M.

Definition 3.16. [4, 4.4] Let G of a linear algebraic group G of exponential type and M a rational G-module. Then the **support variety** of M, $V(G)_M \subset V(G)$, is the subvariety of those 1-parameter subgroups $\mathcal{E}_{\underline{B}} : \mathbb{G}_a \to G$ at which the action of G on M is not free (in the sense that M is not free $k[t]/t^p$-module with t acting as $\sum_{s\geq 0}(\mathcal{E}_{B_s})_*(u_s)$).

For any $\underline{B} \in \mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g}))$ and any $r > 0$, we set $\Lambda_r(\underline{B})$ equal to $(B_{r-1}, B_{r-2}, \ldots, B_0)$. Thus, $\mathcal{E}_{\Lambda_r(\underline{B})} : \mathbb{G}_a \to G$ can be viewed as an infinitesimal 1-parameter subgroup $\mathbb{G}_{a(r)} \to G_{(r)}$. As shown in [4, 4.3], the π-points $k[u]/u^p \to kG_{(r)}$ given by sending u to $\sum_{s=0}^{r-1}(\mathcal{E}_{B_s})_*(u_s)$ and to $(\mathcal{E}_{\Lambda_r(\underline{B})})_*(u_{r-1})$ are equivalent. One consequence of this equivalence of π-points is the close relationship between support varieties as defined in Definition 3.16 for a linear algebraic group G and the support varieties for the Frobenius kernels $G_{(r)}$ as defined for any finite group scheme. This enables the proof (given in [4, 4.6.1]) of the following proposition giving a consequence involving support varieties of the hypothesis that a rational G-modules has exponential degree $< p^r$.

Proposition 3.17. *([4, 4.6.1]) Let G be a linear algebraic group of exponential type and let M be a rational G-module such that $M = M_{[d]}$. If $p^r > d$ (so that M has exponential degree $< p^r$), then the support variety $V(G)_M$ of M satisfies*
$$V(G)_M \ = \ \Lambda_r^{-1}(V_r(G)_M).$$
Here, $\Lambda_r : \mathcal{C}_\infty(\mathcal{N}_p(\mathfrak{g})) \to \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ sends $\underline{B}$ to $(B_{r-1}, B_{r-2}, \ldots, B_0)$.

Proof. This follows immediately from Remark 3.12 and Proposition 4.6.1 of [4]. □

The following simple example makes clear that the condition that $M = M_{[d]}$ is not equivalent to some condition on the support variety of M. Conceptually, the support variety of M is the locus of 1-parameter subgroups ψ at which the p-nilpotent action at ψ is *not free*, whereas the condition $M = M_{[d]}$ is the condition on the *triviality of the action* of $\mathcal{E}_{B*}(v_j)$ on M for all $B \in \mathcal{N}_p(\mathfrak{g})$ and all $j > d$.

Example 3.18. Consider the 2-dimensional rational $\mathbb{G}_a$-module Y_R with basis $\{v, w\}$ whose $k\mathbb{G}_a$-module structure is given by $u_s(w) = 0, \forall s \geq 0$; $u_s(v) = w, s \leq$

$R,\ u_s(v) = 0, \forall s > R$. If $p > 2$, then $V(\mathbb{G}_a)_{Y_R} = V(\mathbb{G}_a) = \mathbb{A}^\infty$ for each $R > 0$. On the other hand, Y_R has exponential degree $\leq R$ but does not have exponential degree $\leq R-1$ for each $R > 0$.

4. Mock Injectives, mock trivials, and the functor $(-)_{[d]}$

We begin this final section with a list of properties for the "filtration by exponential degree" functor $(-)_{[d]} : (G\text{-}Mod) \to ((k[G])_{[d]}\text{-}coMod)$. These enable us to extend the rational injectivity criterion for rational G-modules with G a unipotent algebraic group (given in Proposition 2.13) to rational G-modules for G an arbitrary linear algebraic group of exponential type.

We then briefly consider two new classes of rational G-modules: those that are "mock injective" (have trivial support varieties) and those that are "mock trivial" (those with trivial p-unipotent action). It is relatively easy to show the existence of mock injectives and mock trivials, more difficult to construct specific examples and study general properties of these classes. We conclude with a brief consideration of the right derived functors of the filtration functors $(-)_{[d]}$, observing that they occur on the E_2-page of a Grothendieck spectral sequence converging to rational cohomology.

Proposition 4.1. *Let G be a linear algebraic group of exponential type and $d \geq 0$ a non-negative integer.*

1. *The natural embedding $\iota_d : ((k[G])_{[d]}\text{-}coMod) \subset (G\text{-}Mod)$ is exact and fully faithful.*
2. *The functor $(-)_{[d]} : (G\text{-}Mod) \to ((k[G])_{[d]}\text{-}coMod),\ M \mapsto M_{[d]}$ is left exact and idempotent (in the sense that $(-)_{[d]} = (-)_{[d]} \circ \iota_d \circ (-)_{[d]}$).*
3. *The natural embedding $\iota_d : ((k[G])_{[d]}\text{-}coMod) \subset (G\text{-}Mod)$ is left adjoint to $(-)_{[d]}$.*
4. *The category $((k[G])_{[d]}\text{-}coMod)$ has enough injectives; in other words, for every rational G-module M of exponential degree $\leq d$, there exists an inclusion of rational G-modules $M \hookrightarrow L$ of exponential degree $\leq d$ with L an injective object of $((k[G])_{[d]}\text{-}coMod)$.*

Proof. The fact that ι_d is fully faithful follows from Theorem 3.15(7); the exactness statement of Property (1) is clear. The idempotence of Property (2) follows directly from the definition of the category $((k[G])_{[d]}\text{-}coMod)$ (see Theorem 3.15(1)). The left exactness of Property (2) is an immediate consequence of the definition of $(-)_{[d]}$. Property (3) is proved exactly as the adjoint property is proved in Proposition 2.8.

To prove Property (4), recall that $(G\text{-}Mod)$ has enough injectives. If M is a rational G-module of exponential degree $\leq d$ and if $j : M \to I$ is an embedding of M into a rationally injective G-module I, then j factors through $L \equiv I_{[d]}$ (by Theorem 3.15(7)). Since ι_d is an exact left adjoint to $(-)_{[d]}$, L is an an injective object of $((k[G])_{[d]}\text{-}coMod)$. $\square$

The following necessary and sufficient criterion for rational injectivity is an extension of Proposition 2.13.

Proposition 4.2. *Let G be a linear algebraic group of exponential type. Then the following are equivalent for a rational G-module L.*

1. *L is rationally injective.*
2. *For each $d \geq 0$, $L_{[d]}$ is injective in $((k[G])_{[d]}\text{-}coMod)$.*

(3) *For some strictly increasing sequence of non-negative integers* $\{d_i,\ i \geq 0\}$, $L_{[d_i]}$ *is injective in* $((k[G])_{[d_i]}$*-coMod*$)$ *for all* $i \geq 0$.

Proof. If L is injective, then Proposition 4.1(4) implies that $L_{[d]} \subset L$ is an injective object of $((k[G])_{[d]}$-*coMod*$)$. Namely, this is a formal consequence of the fact that $(-)_{[d]}$ has an exact left adjoint. Thus, condition (1) implies condition (2) which clearly implies condition (3).

Assume now that the rational G-module has the property that each $L_{[d_i]} \subset L$ is an injective object of $((k[G])_{[d_i]}$-*coMod*$)$ for all $i \geq 0$. Let $M' \to M$ be an inclusion of rational G-modules and observe that $(M')_{[d_i]} = M' \cap M_{[d_i]}$; let $f' : M' \to L$ be a map of rational G-modules. Set $N_i = (M')_{[d_i]} + M_{[d_{i-1}]} \subset M_{[d_i]}$ (with $N_{-1} = 0$). We inductively define $f_i : N_i \to L_{[d_i]}$ extending $f'_{[d_i]} + f_{i-1}$ using the injectivity of $L_{[d_i]}$ as an object of $((k[G])_{[d_i]}$-*coMod*$)$. Using Theorem 3.15(4), we define $f : M \to L$ extending f' to be $\varinjlim_i f_i : M = \varinjlim_i N_i \to L$. □

Definition 4.3. Let G be a connected, linear algebraic group and M a rational G-module. Then M is said to be **mock injective** if the restriction of M to each Frobenius kernel $G_{(r)}$ is injective.

In particular, if G is a linear algebraic group of exponential type, a rational G-module M is mock injective if and only if $V(G)_M = 0$ (by [4, 6.1]).

The following proposition contrasts the behavior of injectives and mock injectives. The first statement is merely a restatement of Theorem 4.3 and Corollary 4.5 of [2]. We follow the terminology of [2] by saying that the algebraic group G is reductive if and only if its connected component G^0 is a central product of a torus and a connected, semi-simple algebraic group.

Proposition 4.4. *Let G be a linear algebraic group and H a closed subgroup.*

(1) *The regular representation $k[G]$ of G when restricted to H is rationally injective if and only if G/H is an affine variety. In particular, if G is reductive, then $k[G]$ is rationally injective as a rational H module if and only if H is reductive.*
(2) *$k[G]$ is always a mock injective H-module.*
(3) *If H is unipotent and M is a rationally injective H-module, then $ind_H^G(M)$ is mock injective.*

Proof. As mentioned above, (1) is a restatement of Theorem 4.3 and Corollary 4.5 of [2]. The proof of (2) is given in the proof of Proposition 2.11 (with the notational change of replacing U by H in that proof. Finally, (3) follows from (2) and Proposition 2.12 which asserts for H unipotent that any rationally injective H-module L is isomorphic to $H^0(H, L) \otimes k[H]$. □

Corollary 4.5. *Let G be linear algebraic group which is not reductive. Then there exist mock injective G-modules which are not rationally injective.*

Proof. If G is not reductive and $G \subset GL_N$ is a closed embedding of G into some GL_N, then we may apply Proposition 4.4.1 to conclude that $k[GL_N]$ is mock injective but not injective as a rational G-module. □

The reader should consult the recent paper [14] for examples of mock injectives for reductive groups.

We next list a few closure properties of the category of mock injectives.

Proposition 4.6. *Let G be a connected, linear algebraic group of exponential type.*

(1) *If M_1, M_2 are rational G-modules which are mock injective, then $M_1 \otimes M_2$ is mock injective.*
(2) *If M_1, M_2 are rational G-modules which are mock injective and if*
$$0 \to \ M_1 \ \to \ M \to \ M_3 \to 0$$
is a short exact sequence of rational G-modules, then M is also mock injective.
(3) *An arbitrary direct sum of mock injective rational G-modules $\{M_i, \ i \in I\}$, $\bigoplus_{i \in I} M_i$, is mock injective.*

Proof. We use the criterion for mock injectivity mentioned in Definition 4.3. With this criterion, (1) and (2) follow immediately from Theorem 4.6 of [4], as does property (3) if I is finite. To prove (3) for any arbitrary indexing set I, we use the observation that for $\underline{B} \notin V(G)_M$ it suffices to show that the action of $\sum_s \mathcal{E}_{B_s *}(u_s)$ determines a free action of $k[t]/t^p$ on M; this condition is clearly inherited by arbitrary direct sums. □

We now introduce the class of mock trivial G-modules.

Definition 4.7. Let G be a linear algebraic group with a structure of exponential type. Then a rational G-module M is said to be **mock trivial** if $M = M_{[0]}$; equivalently, if the coproduct structure $\Delta : M \to M \otimes k[G]$ factors through $M \to M \otimes k[G]_{[0]}$.

Proposition 4.8. *Let G be a linear algebraic group with a structure of exponential type. Then a rational G-module M is* **mock trivial** *if and only if the pull-back of M along any 1-parameter subgroup $\psi : \mathbb{G}_a \to G$ is trivial.*

This implies in particular that the p-nilpotent action of G at every 1-parameter subgroup $\psi \in V(G)$ is trivial which in turn implies that the support variety of M, $V(G)_M$, equals all of $V(G)$.

Proof. By Proposition 3.11, M is mock trivial if and only if $\mathcal{E}_B^*(M)$ is trivial as $\mathbb{G}_a$-module for all 1-parameter subgroups of the form $\mathcal{E}_B : \mathbb{G}_a \to G$, and this is the case if and only if $(\mathcal{E}_B \circ F^i)^*(M)$ is trivial as a $\mathbb{G}_a$-module for all $\mathcal{E}_B : \mathbb{G}_a \to B$ and all $i \geq 0$. Since any 1-parameter subgroup of G is of the form $\mathcal{E}_{\underline{B}} = \prod_s \mathcal{E}_{B_s} \circ F^s$ and since the action of $(\prod_s \mathcal{E}_{B_s} \circ F^s)_*(v_j)$ on M equals the product of the actions of $(\mathcal{E}_{B_s})_*(v_{j-p^s})$ on M, we conclude that $\psi^*(M)$ is a trivial $\mathbb{G}_a$-module whenever M is mock trivial. The converse is clear from the first equivalence mentioned at the beginning of this proof.

As recalled prior to Definition 3.16, the p-nilpotent action of G at $\mathcal{E}_{\underline{B}} = \prod_s \mathcal{E}_{B_s} \circ F^s$ is defined in [4, 2.9.1] to be the action of $(\sum_{s \geq 0} \mathcal{E}_{B_s})_*(u_s)$. By the preceding paragraph, if M is mock trivial, this action is trivial. This immediately implies that $V(G)_M = V(G)$ for any mock trivial rational G-module. □

We state a few properties of the class of mock trivial G-modules. Of course, even the class of trivial G-modules need not be closed under extensions.

Proposition 4.9. *Let G be a linear algebraic group with a structure of exponential type.*

(1) *If $G \neq U_p(G)$, then $k[G]_{[0]}$ is a non-trivial sub Hopf algebra of $k[G]$ and thus is an indecomposable mock trivial module, but not trivial.*

(2) *If $G \neq U_p(G)$, then there exist finite dimensional rational G-modules which are mock trivial but not trivial.*
(3) *If $H \subset G$ is an embedding of exponential type such that $x^p = 1,\ x \in H$ (i.e., $H = U_p(H)$ and if M is a mock trivial rational G-module, then M restricted to H is trivial.*
(4) *If M is a rational G-module such that $M = M_{[0]}$ and if $M' \subset M$ is rational G-submodule, then $M' = (M')_{[0]}$; similarly, any quotient $\overline{M}$ of a rational G-module such that $M = M_{[0]}$ also satisfies $\overline{M} = (\overline{M})_{[0]}$.*
(5) *A colimit $\varinjlim_\alpha L_\alpha$ of mock trivial G-modules L_α is again mock trivial.*

Proof. By Proposition 3.9, $k[G]_{[0]}$ is $G \neq U_p(G)$ and by Proposition 3.5 $k[G]_{[0]}$ is a sub Hopf algebra of $k[G]$; in particular, $k[G]_{[0]}$ is a non-trivial rational G-module M such that $M = M_{[0]}$. Since the socle of $k[G]_{[0]}$ is 1 dimensional, $k[G]_{[0]}$ is indecomposable. To prove (2), we apply (1) and recall that any rational G-module is a union of its finite dimensional submodules.

If $H \subset G$ is an embedding of exponential type, then $\mathcal{E} : \mathcal{N}_p(G) \times \mathbb{G}_a \to G$ restricts to $\mathcal{E}_H : \mathcal{N}_p(H) \times \mathbb{G}_a \to H$;. this implies that if the comodule structure for M has the property that it arises from a coproduct $M \to M \otimes k[G]_{[0]}$, then the restriction to H has the property that the coproduct arises from a coproduct $M \to M \otimes k[H]_{[0]}$. Thus, Property (3) follows from Proposition 3.11 (in the special case $d = 1$).

Properties (4) and (5) are evident properties of the abelian category of $k[G]_{[0]}$ comodules. □

Remark 4.10. Explicit examples of non-trivial mock trivial modules can be constructed using Proposition 3.9. Namely, for a given G, one considers the ideal $I_{\mathcal{U}} \subset k[G]$ of functions vanishing on the closed subvariety $\mathcal{U}_p \subset G$ of p-unipotent elements. By Proposition 3.9, $I_{\mathcal{U}} \subset (k[G])_{[0]}$. For any non-empty subspace $V \subset I_{\mathcal{U}}$, the G-submodule $G \cdot V \subset (k[G])_{[0]}$ generated by V is a non-trivial mock trivial G-module; if V is finite dimensional, $G \cdot V$ is also finite dimensional.

In the following proposition, we view (closed) points of the support variety of a finite group scheme as equivalence classes of p-points as in [5] rather than use 1-parameter subgroups which provide distinguished representatives of equivalence classes of $\mathfrak{p}$-points as in [12]. Since we do not consider the scheme structure of support varieties, we do not use the language and technology of π-points found in [6]. For a finite group scheme H, $P(H)$ consists of the closed points of the scheme $\Pi(H)$ of π-points; the points of $P(H)$ are equivalence classes of p-points of H.

Proposition 4.11. *Let G be a linear algebraic group of exponential type. If a rational G-module M is mock trivial then the restriction of M to each Frobenius kernel $G_{(r)}$ satisfies the condition that for every $[\alpha] \in P(G_{(r)})$ there exists a representative $\alpha : k[t]/t^p \to kG_{(r)}$ such that $\alpha^*(M)$ is trivial as a $k[t]/t^p$-module.*

Proof. There is a natural homeomorphism relating $P(G_{(r)})$ to $V_r(G_{(r)}) \simeq \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ given by sending $\underline{B} \in \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ to the p-point $\alpha_{\underline{B}} : k[t]/t^p \to kG_{(r)},\ t \mapsto \mathcal{E}_{(B_0,\dots,B_{r-1})}(u_{r-1})$. As shown in [4, 4.3], the equivalence class of $[\alpha_{\underline{B}}] \in P(G_{(r)})$ is also represented by the π-point $k[t]/t^p \to kG_{(r)},\ t \mapsto \sum_{i=0}^{r-1} (\mathcal{E}_{B_i})_*(u_{r-1-i})$ (see the discussion after Definition 3.16). If M is mock trivial, then each $\sum_{i=0}^{r-1} (\mathcal{E}_{B_i})_*(u_{r-1-i})$ acts trivially on M. □

We conclude with Grothendieck spectral sequences relating rational cohomology to the structures we have considered. We view the t-th right derived functor of the left exact functor $(-)_{[d]}$,

$$R^t(-)_{[d]} : (G\text{-}Mod) \ \to \ ((k[G])_{[d]}\text{-}coMod),$$

as "derived filtrations functors".

Proposition 4.12. *Let G be a linear algebraic group of exponential type. Denote by $(k\text{-}Mod)$ the abelian category of k-vector spaces. For any $d \geq 0$, there is a natural identification of functors*

$$H^0(G,-) \ \simeq \ Hom_{(k[G])_{[d]}\text{-}coMod)}(k,-) \circ (-)_{[d]} : (G\text{-}Mod) \to (k\text{-}Mod)$$

leading to a spectral sequence

$$R^s Hom_{(k[G])_{[d]}\text{-}coMod)}(k,-) \circ R^t(-)_{[d]}(M) \ \Rightarrow H^{s+t}(G,M).$$

Proof. The asserted identification of the composition $Hom_{(k[G])_{[d]}\text{-}coMod)}(k,-) \circ (-)_{[d]}$ with $H^0(G,-)$ is made by observing that both send a rational G-module M to the subspace M^G of invariant elements (which consists of those $m \in M$ such that $\Delta_M(m) = m \otimes 1 \in M \otimes k[G]$).

Since the functor $(-)_{[d]}$ has an exact left adjoint by Proposition 4.1(3) and therefore sends injectives to injectives, the Grothendieck spectral sequence for a composition of left exact functors applies and has the asserted form (see [13]). □

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