AN APPROACH TO INTERSECTION THEORY ON SINGULAR VARIETIES USING MOTIVIC COMPLEXES

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Abstract. We introduce techniques of Suslin, Voevodsky, and others into the study of singular varieties. Our approach is modeled after Goresky-MacPherson intersection homology. We provide a formulation of perversity cycle spaces leading to perversity homology theory and a companion perversity cohomology theory based upon generalized cocycle spaces. These theories lead to conditions on pairs of cycles which can be intersected and a suitable equivalence relation on cocycles/cycles enabling pairings on equivalence classes. We establish suspension and splitting theorems, as well as a localization property. Some examples of intersections on singular varieties are computed.

Contents

0. Introduction

In this paper, we initiate an investigation of pairings on cycle groups on singular algebraic varieties over a field. We utilize the approach to motivic cohomology developed by A. Suslin and V. Voevodsky [?], blended with the philosophy of intersection homology theory as introduced by M. Goresky and R. MacPherson [?]. An important source of insight for the approach we take comes from "semi-topological cohomology and homology," especially from the foundations of Lawson homology due to H. B. Lawson [?].

Our goal is to provide contexts in which there is a good formulation of the intersection product of cycles on singular varieties. This is an age-old problem, one that motivated the original introduction of cohomology and in some sense culminated with intersection homology theory for stratified topological spaces. In the context of algebraic varieties, the moving techniques for stratified spaces (for

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example, those of [?]) do not apply. Indeed, we know of no means of improving intersections occurring within the singular locus of a given variety.

If X is a smooth projective variety, then Poincaré duality provides a ring structure on the singular homology of X. This product admits a purely algebro-geometric description on the fundamental classes of algebraic cycles α and β: by the method of Chow, one can move α within its rational equivalence class (to α', say) so that α' and β intersect properly (i.e., in the expected dimension). For proper intersections on a smooth variety, multiplicities may be defined purely algebraically, for example by the Tor-formula of Serre. The homology class of the cycle class $\alpha \bullet \beta$ represents the product of the homology classes of α and β.

If X is singular, then its homology groups typically cannot be endowed with a reasonable ring structure. The intersection homology of Goresky-MacPherson rectifies this by defining groups $IH_*^{\overline{p}}(X)$ which, roughly speaking, are the homology groups of a complex of chains with controlled incidence with the singular locus of X. There are intersection pairings $IH_r^{\overline{p}}(X) \otimes IH_s^{\overline{q}}(X) \to IH_{r+s-\dim(X)}^{\overline{p}+\overline{q}}(X)$ (provided some conditions are satisfied) which, in case $r+s = \dim(X)$, become perfect after tensoring with the rationals. The challenge which originally motivated us was to extend the picture of the previous paragraph, namely the description of the intersection product of algebraic cycle classes, to intersection homology of singular varieties.

Previous approaches to this problem have not led to an intersection pairing lifting the Goresky-MacPherson pairing. P. Gajer defined a semi-topological version of intersection homology and established some of its structural properties [?]. A. Corti and M. Hanamura gave a definition of intersection Chow groups by incorporating information obtained from a resolution of singularities [?]; they provided also a motivic lifting of the decomposition theorem of [?] assuming various conjectures on algebraic cycles [?]. J. Wildeshaus used weight structures to define a motivic intersection complex, and proved its existence in some cases [?]. In the topological setting, intersection homology may be defined geometrically, using a subcomplex of the complex of singular chains [?], or sheaf-theoretically, using the constructible derived category [?]. In the algebraic setting, it would be interesting to relate our geometrically oriented approach to the categorical constructions.

Introducing cycle (and cocycle) spaces and defining homotopy pairings on these spaces guides the formulation of equivalence relations on cycles and gives pairings on homotopy groups. The equivalence relations which arise are necessarily finer than rational equivalence: even if one restricts attention to cycles which meet "properly" and whose intersection meets the singular locus properly, one must take care in defining equivalence relations so that cap and cup product pairings are well defined on equivalence classes. Our primary interest is the intersection of fundamental classes of algebraic cycles, corresponding to a pairing on connected components of our cycle and cocycle spaces.

We work with an algebraic variety X equipped with a stratification; such a stratification might arise from a resolution of singularities of X or a "platification" of a family of coherent sheaves on X. Fixing a perversity function $\overline{p}$, we introduce perversity cycles on X and generalized cocycles on X with values in Y. These are cycles which meet the strata of X (or $X \times Y$) in a manner controlled by $\overline{p}$. The discrete abelian groups of perversity cycles (and generalized cocycles) for a given variety X determine presheaves which lead to singular complexes (i.e., simplicial abelian groups) as first conceived by Suslin (see [**?**]). Our homology/cohomology theories are the homotopy groups of these singular complexes, doubly graded in a manner compatible with the grading in motivic homology and cohomology [**?**].

We show that our theories satisfy good properties including suspension isomorphisms ($\mathbb{A}^1$-invariance), a splitting theorem, and a suitable form of localization. These theorems enable our definition of a cup product in perversity cohomology, extending the cup product in motivic cohomology. We then proceed to establish a cap product relating perversity cohomology and perversity homology, extending the usual cap product relating cohomology and homology. To do this, we introduce the condition $(*, \overline{c})$ on a pair of cycles and a perversity $\overline{c}$ which permits a sensible intersection of cycles meeting especially nicely; this intersection product is compatible with that of Goresky-MacPherson intersection homology.

For the reader's convenience, we briefly outline the contents of each section of this paper.

In Section 1, we revisit various sheaves and presheaves of relative cycles as investigated by Suslin and Voevodsky. We discuss to what extent and how these sheaves are represented by Chow varieties. These (pre)sheaves are defined on (Sch/k) so that we may apply results of Voevodsky on sheaves for the cdh-topology; the cycle sheaves are evaluated on the standard cosimplicial scheme whose constituents are affine spaces Δ^n. In fact, one is naturally led to another of Voevodsky's Grothendieck topologies, the h-topology, when the characteristic of the ground field is positive.

We begin our study of cycles on a stratified (possibly singular) variety X in Section 2. Following Goresky-MacPherson, we fix a "perversity" $\overline{p}$ and consider U-relative cycles on $U \times X$ whose specializations at points $u \in U$ meet the strata of X_u in codimension controlled by the perversity $\overline{p}$. Applying our sheaves to $\Delta^\bullet$, we obtain our perversity motivic homology groups $H_n^{\overline{p}}(X, \mathbb{Z}(r))$ as the homotopy groups of the associated simplicial abelian group (or, equivalently, as the homology of the associated chain complex). There is a natural map to motivic Borel-Moore homology $H_n^{\overline{p}}(X, \mathbb{Z}(r)) \to H_n^{BM}(X, \mathbb{Z}(r))$ induced by an inclusion of simplicial abelian groups. Furthermore, when our ground field k is the complex field $\mathbb{C}$, we verify in Proposition 2.6 that there is a natural map from the bidegree perversity homology group corresponding to π_0 to the Goresky-MacPherson intersection homology group. In Theorem 2.11, we use techniques of Voevodsky (supplemented by recent results of S. Kelly and A. Suslin) to prove a form of localization for our perversity motivic homology groups.

A central theme of our work is the interplay between the sheaf-theoretic foundations of Suslin-Voevodsky and constructions using Chow varieties as first considered by Lawson in [**?**]. In particular, in Section 3, we employ the constructions introduced by Lawson to prove suspension theorems for our homology groups. These theorems are first proved in Theorem 3.1 for projective varieties (for Chow varieties are defined for projective varieties) and then extended to quasi-projective varieties using the localization theorem of the previous section. The proofs require verification that "Lawson moving constructions" preserve perversity of cycles.

In Section 4, we relate our groups to the problem of intersecting cycles on a stratified singular variety. We introduce the condition $(*, \overline{c})$ on a pair of cycles which allows (static) intersection with good properties, especially suitable behavior with respect to specialization. For example, Corollary 4.4 verifies that this intersection commutes with specialization, the formal analogue of being continuous. We analyze in detail the resulting intersection pairing for the standard example due to Zobel of the cone on $\mathbb{P}^1 \times \mathbb{P}^1$.

The "generalized cocycles" introduced in Definition 5.1 pair well with perversity cycles. Perversity cycles satisfy an incidence condition with the strata of a given stratified variety X, whereas generalized cocycles on X with values in some Y are cycles on $X \times Y$ more general than cocycles (i.e., not necessarily equidimensional over X) whose fiber dimensions over points of X are controlled by the perversity $\overline{p}$. Algebraic cocycles first appeared in work of the first author and Lawson [**?**] as an algebraic model for cocycles in algebraic topology; our groups are a stratified variant of groups briefly considered by the first author and Gabber in [**?**] (a more sophisticated form of which is presented in the paper of the first author and Voevodsky [**?**].) These bivariant perversity motivic cohomology groups satisfy a suspension theorem (Theorem 5.7) which leads to perversity motivic cohomology by setting the covariant variable equal to a projective space. As we describe, generalized cocycles arise from resolutions and from coherent sheaves (with stratification determined by the resolution or the sheaf).

In the final section of this paper, we lay the foundations for applications by establishing a cup product on perversity motivic cohomology and a cap product pairing relating perversity motivic cohomology and perversity motivic homology. For example, in Theorem 6.4 we establish the (motivic) perversity version of the splitting theorems established for semi-topological cohomology by the first author and Lawson. These constructions suffice to formulate intersection products in simple situations (for example, for a variety X with isolated singularites). We conclude by verifying in Proposition 6.19 that our cap product pairing is compatible with the intersection product in intersection homology.

Throughout, we work over an infinite field k of characteristic $p \geq 0$. We shall assume the k is perfect, though as remarked in Remark 1.1 the hypothesis that k be perfect should be unnecessary in view of recent (not yet unavailable in print) work of A.Suslin For us, a k-scheme is a separated scheme of finite type over k, and a variety is an integral k-scheme. If $char(k) = p$, then at times we must invert p in the coefficients of our theory (which has the effect of taking the h-sheafiification of the cdh-sheaves of cycles).

1. Roadmap for various presheaves

sec:one

We employ a plethora of presheaves and sheaves of algebraic cycles. Our invariants are homotopy groups of simplicial abelian groups (equivalently, homology groups of associated normalized chain complexes) obtained by evaluating an abelian (pre)sheaf on a cosimplicial scheme. Our geometric constructions are correspondences among Chow varieties of r-dimensional cycles on a projective variety X. The presheaves represented by Chow varieties are closely related to the Suslin-Voevodsky presheaves $z_{equi}(X, r)$ and $z(X, r)$. To extend our results to quasi-projective X, we employ the technology developed by Suslin and Voevodsky for sheaves for the cdh-topology.

For a scheme X, the Suslin-Voevodsky cdh-sheaf $z(X, r)$ on $(Sch/k)_{cdh}$ sends a k-scheme U to the abelian group of U-relative cycles on $U \times X$ (of relative dimension r) with well-defined specializations and universally integral coefficients [**?**, Lemma 3.3.9]. If k admits resolution of singularities, this sheaf has the important *localization property* (see [**?**, Thm. 4.3.1], [**?**, Remark 5.10]): if $Y \hookrightarrow X$ is closed with Zariski open complement U, then the triple of simplicial abelian groups

$$z(Y,r)(\bullet) \ \to \ z(X,r)(\bullet) \ \to \ z(U,r)(\bullet)$$

determines a distinguished triangle of Suslin complexes

loc (1.0.1) $$C_*(z(Y,r)) \ \to \ C_*(z(X,r)) \ \to \ C_*(z(U,r)) \to C_*(z(Y,r))[1].$$

Assume now that X is projective and consider the subsheaf $z^{eff}(X,r) \ \subset \ z(X,r)$ whose value on U is the monoid of those U-relative cycles which are effective. When $\mathrm{char}(k) = 0$, cycles in $z^{eff}(X,r)(U)$ can be identified with the graphs of homomorphisms from the semi-normalization of U into the Chow monoid $\mathcal{C}_r(X)$,

eq:eff (1.0.2)
$$z^{eff}(X,r)(U)[1/p] \ \simeq \mathrm{Hom}(U^{sn}, \mathcal{C}_r(X)) \ \simeq \mathrm{Hom}(U^{sn}, \mathcal{C}_r(X)^{sn}), \quad \mathrm{char}(k) = 0;$$

this is the h-representability of the sheaf $z^{eff}(X,r)$ [**?**, Cor. 4.4.13] and the fact that $z^{eff}(X,r) \to z^{eff}(X,r)_h$ and $z(X,r) \to z(X,r)_h$ are isomorphisms in characteristic zero [**?**, Thm. 4.2.2].

In arbitrary characteristic, the h-sheafifications may be computed using continuous algebraic maps ([**?**, 4.1], [**?**, Cor. 4.4.13]), so that

$$z^{eff}(X,r)_h(U) \ \simeq \mathrm{Hom}(U, \mathcal{C}_r(X))_h \ \simeq \mathrm{Hom}_{\mathrm{c.alg}}(U, \mathcal{C}_r(X)).$$

(This h-sheafification admits a description as a limit of morphisms of schemes even though in positive characteristic the object h-representing a sheaf is not unique; see [**?**, Prop. 3.2.11].) Notice also that p is invertible in $\mathrm{Hom}_{\mathrm{c.alg}}(U, \mathcal{C}_r(X))$ if U is equidimensional, since then the relative Frobenius $F_{U/k} : U \to U^{(1)}$ is generically flat of degree $p^{\dim_k(U)}$, and the continuous algebraic map $U^{(1)} \xleftarrow{F_{U/k}} U \xrightarrow{f} \mathcal{C}_r(X)$ corresponds to $1/p^{\dim_k(U)} \cdot f$. Since $z^{eff}(X,r)[1/p]$ and $z(X,r)[1/p]$ are h-sheaves [**?**, Thm. 4.2.2], this implies $z^{eff}(X,r)_h(U)[1/p] \ \simeq \mathrm{Hom}_{\mathrm{c.alg}}(U, \mathcal{C}_r(X))$ for $U \in Sm/k$.

The presheaf $z^{eff}(X,r)$ admits a reasonable description in terms of Chow varieties before inverting p. If X is projective and $\mathrm{char}(k) = p > 0$, then for U smooth and quasi-projective, the subgroup $z^{eff}(X,r)(U) \subset \mathrm{Hom}(U, \mathcal{C}_r(X))$ consists of those morphisms $f : U \to \mathcal{C}_r(X)$ such that, for every generic point $\eta \in U$, the cycle classified by $f(\eta)$ is defined over $k(f(\eta))$; in general, the field of definition of the cycle classified by $f(u)$ is a finite radicial extension of $k(f(u))$ [**?**, Prop. 2.3]. For example, if $a, b \in k$ are such that $k(a^{1/p}, b^{1/p})$ has degree p^2 over k, then the zero cycle $p \cdot \langle a^{1/p}, b^{1/p}, -1\rangle \in \mathbb{P}^2$ determines a map $\mathrm{Spec}(k) \to \mathcal{C}_0(\mathbb{P}^2)$ even though $p \cdot (a^{1/p}, b^{1/p}, -1)$ is not k-rational.

Now we consider possibly ineffective cycles, retaining the hypothesis that X be projective. A subtlety arises in comparing the presheaf $z^{eff}(X,r)^+$ to the sheaf $z(X,r)$, one that arises because not every element of $z(X,r)(U)$ is a difference of elements of $z^{eff}(X,r)(U)$. In general, there is an intermediate presheaf

$$z^{eff}(X,r)^+ \ \subset \ z_{equi}(X,r) \ \subset z(X,r)$$

consisting of U-relative cycles on X each component of which has relative dimension r. Examples show that $z_{equi}(X,r)(U)$ can strictly contain $z^{eff}(X,r)^+(U)$ and be strictly contained in $z(X,r)(U)$. Nevertheless, by [**?**, Cor. 3.4.4] we have

$$z^{eff}(X,r)^+(U) \ = z_{equi}(X,r)(U), \quad U \text{ geometrically unibranch.}$$

Consequently,

$$z_{equi}(X,r)_{|(Sm/k)} \;\subseteq\; (\mathrm{Hom}(-,\mathcal{C}_r(X))^+)_{|(Sm/k)}$$

with equality if $\mathrm{char}(k)=0$ and with image consisting of morphisms satisfying the field of definition condition described above if $\mathrm{char}(k)=p>0$. Moreover, by [**?**, Prop. 4.2.10],

$$z_{equi}(X,r)_{cdh} \;\xrightarrow{\sim}\; z(x,r).$$

The cycle sheaves $z(X,r)$ and $z(X,r)_h$ for X projective can be described in terms of continuous algebraic maps to the group completion $\mathcal{Z}_r(X) := \mathcal{C}_r(X)^2/R$ of the Chow monoid $\mathcal{C}_r(X)$; here, R is the usual relation $(V,W)\sim(V',W')$ if and only if $V+W'=W+V'$ ([**?**, 4.1], [**?**, Prop. 4.4.15]). We remind the reader that a continuous algebraic map to the group completion is (up to a bicontinuous algebraic map) a pair of *rational* maps to the Chow monoids which induces a well-defined set-theoretic map (on $\overline{k}$-points) to $\mathcal{Z}_r(X_{\overline{k}})$. This permits fibers of dimension $> r$ (which may occur outside the domains of definition of the rational maps) to cancel. Then as sheaves on (Sch/k), we have ([**?**, 4.1], [**?**, Prop. 4.4.15]):

$$z(X,r) \;=\; z(X,r)_h \;=\; \mathrm{Hom}_{\mathrm{c.alg}}(-,\mathcal{Z}_r(X)), \quad \mathrm{char}(k)=0$$

$$z(X,r)[1/p] \;=\; z(X,r)_h \;=\; \mathrm{Hom}_{\mathrm{c.alg}}(-,\mathcal{Z}_r(X))[1/p], \quad \mathrm{char}(k)=p.$$

By the above, $\mathrm{Hom}_{\mathrm{c.alg}}(U,\mathcal{Z}_r(X))[1/p] = \mathrm{Hom}_{\mathrm{c.alg}}(U,\mathcal{Z}_r(X))$ if U is equidimensional. Furthermore, for $\mathrm{char}(k)=p$ and $U\in Sm/k$, the image $z(X,r)(U) \subset \mathrm{Hom}_{\mathrm{c.alg}}(U,\mathcal{Z}_r(X))$ consists of those continuous algebraic maps $U\to\mathcal{Z}_r(X)$ which are induced by a pair of morphisms from an open dense subset of U (i.e., the bicontinuous algebraic map is an isomorphism), both of which satisfy the field of definition condition.

tie **Remark 1.1.** What ties all this together is a fundamental result of Voevodsky [**?**, Thm. 5.5(2)] which asserts that the map of presheaves $z_{equi}(X,r)\to z(X,r)$ induces a quasi-isomorphism on associated Suslin complexes provided k admits resolution of singularities. This is supplemented by a theorem of S.Kelly [**?**, Thm 5.3.1] based on O. Gabber's theorem on the existence of smooth alterations of degree prime to ℓ, [**?**, 1.3], [**?**, 3.2.1]; Kelly's theorem extends Voevodsky's Theorem to perfect fields of characteristic $p>0$, establishing that $z_{equi}(X,r)[1/p]\to z(X,r)[1/p]$ induces induces a quasi-isomorphism on associated Suslin complexes. Furthermore, the techniques of A. Suslin [**?**] should extend the validity of this quasi-isomorphism $z_{equi}(X,r)[1/p]\overset{\cong}{\to} z(X,r)[1/p]$ to any (infinite) field k of characteristic $p>0$.

Since Chow varieties are defined for projective varieties, one needs a localization property of the form (1.0.1) in order to extend arguments using Chow varieties to apply to quasi-projective varieties. The importance of this quasi-isomorphism is that it enables the localization property (1.0.1) for $z(X,r)$ to be "transported" to the presheaves $z_{equi}(X,r)\otimes\mathbb{Z}[1/p$

In Section 5, we consider a "bivariant" version of these constructions. Namely, we consider quasi-projective varieties X, Y of pure dimension d,n. We have subpresheaves and subsheaves

$$z^{t,eff}(X,Y) \;\subset\; z^{eff}(X\times Y, d+n-t), \quad z^t(X,Y) \;\subset z(X\times Y, d+n-t)$$

which guide us to various "cohomological" theories on X (taking Y to be projective space).

This paper is concerned with versions of these presheaves and sheaves for stratified varieties and a given perversity. Thus, the presheaves and sheaves we consider will be elaborations of the ones mentioned above, taking into account the stratification and perversity.

sec:two

2. Perversity cycles

We assume X and Y are equidimensional k-schemes of dimension d and n respectively.

A *stratified variety* is a variety X equipped with a filtration by closed subsets $X^d \hookrightarrow X^{d-1} \hookrightarrow \cdots \hookrightarrow X^2 \hookrightarrow X^1 \hookrightarrow X$ such that X^i has codimension at least i in X. If X and Y are stratified, we say $f : Y \to X$ is a *stratified morphism* if f is a morphism of schemes such that $f(Y^i) \subseteq X^i$ for all i. A *perversity* is a non-decreasing sequence of integers $p_1, p_2, \ldots, p_d$ such that $p_1 = 0$ and, for all i, p_{i+1} equals either p_i or $p_i + 1$. Perversities are denoted $\overline{p}, \overline{q}$, etc. The perversities we consider range from the zero perversity $\overline{0}$ with $p_i = 0$ for all i, to the top perversity $\overline{t}$, with $p_i = i-1$ for all i. Our convention differs from that of Goresky-MacPherson [**?**, 1.3] since over the complex numbers our strata always have even real dimension; our p_i corresponds to their p_{2i}.

Let $Z_r(X)$ denote the group of r-dimensional algebraic cycles on X. Suppose X is stratified. We say an r-cycle α is *of perversity* $\overline{p}$ (or satisfies the perversity condition $\overline{p}$) if for all i, the dimension of the intersection $|\alpha| \cap X^i$ is no larger than $r - i + p_i$. When the codimension of X^i in X is exactly i, the perversity of a cycle measures its failure to meet properly the closed sets occurring in the stratification of X. Let $Z_{r,\overline{p}}(X) \subset Z_r(X)$ denote the group of r-dimensional cycles of perversity $\overline{p}$ on the stratified variety X. Often, X^1 is taken to be the singular locus of X, and then the condition $p_1 = 0$ means that no component of the cycle is contained in the singular locus.

Since elements of $z(X,r)(U)$ are required to have well-defined specializations for $u \in U$, we may define subpresheaves by imposing incidence conditions on the fibers over all $u \in U$. Let T be a locally closed subset in X and p an integer. For $U \in Sch/k$, we define $z(X,r)_{T,p}(U) \subseteq z(X,r)(U)$ to be the subgroup of U-relative cycles $\alpha \hookrightarrow U \times X$ satisfying the additional condition that, for all $u \in U$, the intersection of the support of α_u with T_u in X_u has excess at most p. This condition is topological, hence insensitive to the field of definition of the various α_u's.

If $f : U' \to U$ is a morphism in Sch/k and $\alpha \in z(X,r)(U)$ is a cycle, then for all $u' \in U'$, by functoriality the cycle $(f^*\alpha)_{u'}$ coincides with the cycle $(\alpha_u)_{u'}$ where $f(u') = u$. Since the morphism $f_{u'} : \mathrm{Spec}(k(u')) \to \mathrm{Spec}(k(u))$ is universally open, by [**?**, Lemma 3.3.8(1)] the support of $(\alpha_u)_{u'}$ is the base change via $f_{u'}$ of the support of α_u. Therefore the assignment $U \mapsto z(X,r)_{T,p}(U)$ defines a presheaf $z(X,r)_{T,p}(-) \subseteq z(X,r)(-)$. The behavior of supports under base change also implies that if $f : U' \to U$ is an h-cover and $\alpha \in z(X,r)(U)$ satisfies $f^*\alpha \in z(X,r)_{T,p}(U')$, then $\alpha \in z(X,r)_{T,p}(U)$. Therefore $z(X,r)_{T,p} \subseteq z(X,r)$ is a cdh-subsheaf.

Similarly we may define a sheaf

tp

$$z(X,r)_{\mathcal{T},p}(-) \;\subseteq\; z(X,r)(-) \tag{2.0.1}$$

where $\mathcal{T}$ is a collection of locally closed subsets of X and p is a $\mathbb{Z}_{\geq 0}$-valued function on $\mathcal{T}$: we require the excess with $T \in \mathcal{T}$ to be bounded by $p(T)$. We refer to such a pair $(\mathcal{T}, p)$ as an *incidence datum* on X. The equidimensional version is denoted $z_{equi}(X,r)_{\mathcal{T},p}$. If $(\mathcal{T}, p)$ and $(\mathcal{T}', p')$ are incidence data with $\mathcal{T} \subseteq \mathcal{T}'$ and $p'|_{\mathcal{T}} \leq p$, there is a canonical presheaf inclusion $z(X,r)_{\mathcal{T}',p'} \subseteq z(X,r)_{\mathcal{T},p}$.

If X is stratified and $\overline{p}$ is a perversity, we denote by $z_{equi}(X,r)_{\overline{p}}$ and $z(X,r)_{\overline{p}}$ the subpresheaves of $z(X,r)$ consisting of cycles whose excess intersection with X^i is bounded by p_i (for all i). (If $\mathcal{T}$ is the set of strata of the stratified variety X, then we have used $z(X,r)_{\overline{p}}$ to denote $z(X,r)_{\mathcal{T},\overline{p}}$.) Put differently, $\alpha \in z(X,r)(U)$ belongs to $z(X,r)_{\overline{p}}(U)$ if for all $u \in U$, the specialization α_u belongs to $Z_{r,\overline{p}}(X_u)$.

Lemma 2.1. *The cdh-sheafification of $z_{equi}(X,r)_{\mathcal{T},p}$ is $z(X,r)_{\mathcal{T},p}$. Therefore,*

$$z(X,r)_{\overline{p}} \;\cong\; (z_{equi}(X,r)_{\overline{p}})_{cdh}.$$

Proof. The cdh-sheafification of $z_{equi}(X,r)$ is $z(X,r)$ [**?**, Thm. 4.2.9], so any cycle $\alpha \in z(X,r)_{\mathcal{T},p}(U)$ belongs to $z_{equi}(X,r)(U')$ for some cdh cover $p : U' \to U$. Since the support of $\alpha_{u'}$ coincides with that of $\alpha_{p(u')}$, in fact the base change of α lies in $z_{equi}(X,r)_{\mathcal{T},p}(U')$. □

We prove two elementary functoriality properties for $X \mapsto z(X,r)_{\overline{p}}$. We remark that proper push-forward is defined only under restrictive conditions; since disjoint closed sets in the source of a morphism may have images which intersect, the push-forward of a perversity cycle via a stratified morphism need not satisfy the same perversity condition.

Proposition 2.2. *Let $f : W \to X$ be a flat, stratified morphism of relative dimension e. Then for any perversity $\overline{p}$ and any $r \geq 0$, f induces maps of (pre)sheaves*

$$f^* : z(X,r)_{\overline{p}} \to z(W,r+e)_{\overline{p}}, \quad f^* : z_{equi}(X,r)_{\overline{p}} \to z_{equi}(W,r+e)_{\overline{p}}.$$

If $f : W \to X$ is a proper morphism with the property that $W^{i-c_i} = f^{-1}(X^i)$ for some perversity $\overline{c}$, then for any perversity $\overline{p}$ and any $r \geq 0$, f induces maps of (pre)sheaves

$$f_* : z(W,r)_{\overline{p}} \to z(X,r)_{\overline{p}*\overline{c}}, \quad f_* : z_{equi}(W,r)_{\overline{p}} \to z_{equi}(X,r)_{\overline{p}*\overline{c}},$$

where $\overline{p} * \overline{c}$ is the perversity with $(\overline{p} * \overline{c})_i = p_{i-c_i} + c_i$.

In particular, if $i : W \to X$ is a closed immersion and $i(W)$ meets each X^i properly, then such proper push-forward maps exist for i if we take each c_i equal to 0.

The pull-back and push-forward operations are compatible.

Proof. The existence statements for the presheaves with no perversity condition are [**?**, Lemma 3.6.4] (flat pull-back) and [**?**, Cor. 3.6.3] (proper push-forward). Therefore the first assertion follows from the observation that for any locally closed subset $T \subset X$, any flat map $f : W \to X$, and any r-cycle β on X, we have that $|f^*(\beta)| \cap f^{-1}(T) = f^{-1}(|\beta| \cap T)$. The second assertion follows from the observation that, for any r-cycle α on W, we have $\dim |f(\alpha) \cap X^i| \leq r - i + c_i + p_{i-c_i}$ since $|f(\alpha \cap W^{i-c_i})| = |f(\alpha) \cap X^i)|$.

The flat pull-back and proper push-forward transformations are compatible [**?**, Prop. 3.6.5]. □

The algebraic n-simplex is the affine variety $\operatorname{Spec}(k[x_0, \ldots, x_n]/\sum_i x_i - 1)$ and is denoted by Δ^n. The schemes Δ^n fit together into a cosimplicial scheme $\Delta^\bullet$. If

$\mathcal{F}$ is an abelian presheaf on Sm/k, we denote by $\mathcal{F}(\bullet)$ the simplicial abelian group obtained by evaluation at $\Delta^\bullet$. For example, $z(X,r)_{\overline{p}}(\bullet)$ denotes the simplicial abelian group whose abelian group of n-simplices is $z(X,r)_{\overline{p}}(\Delta^n)$. We denote by $C_*(\mathcal{F})$ (the "Suslin complex" of $\mathcal{F}$) the normalized chain complex of $\mathcal{F}(\bullet)$; thus, $\pi_i(\mathcal{F}(\bullet)) = H_i(C_*(\mathcal{F}))$.

simpl **Remark 2.3.** A map $\phi : \mathcal{F} \to \mathcal{G}$ of abelian presheaves on Sm/k induces maps

$$\phi_\bullet : \mathcal{F}(\bullet) \to \mathcal{G}(\bullet), \quad \phi_* : C_*(\mathcal{F}) \to C_*(\mathcal{G}).$$

The map $\phi_\bullet$ is a homotopy equivalence if and only if the map ϕ_* is a quasi-isomorphism.

For $n \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$, the Borel-Moore motivic homology $H_n^{BM}(X,\mathbb{Z}(r))$ of $X \in Sch/k$ is the homology in degree $n-2r$ of the complex $C_*(z(X,r))$; for $r < 0$, $H_n^{BM}(X,\mathbb{Z}(r))$ is the homology of $C_*(z(X \times \mathbb{A}^{-r}, 0))$ in degree $n-2r$ [**?**, 4.3, 9.1]. This motivates the following definition.

Definition 2.4. The perversity $\overline{p}$ (Borel-Moore) motivic homology of a stratified variety X, written $H_n^{\overline{p}}(X,\mathbb{Z}(r))$, is the homology in degree $n-2r$ of the complex $C_*(z(X,r))_{\overline{p}}$. Equivalently, $H_n^{\overline{p}}(X,\mathbb{Z}(r)) \equiv \pi_{n-2r}(z(X,r)_{\overline{p}}(\bullet))$.

The group $H_{2r}^{BM}(X,\mathbb{Z}(r))$ is the Chow group $A_r(X)$ of r-dimensional cycles on X. The group $H_{2r}^{\overline{p}}(X,\mathbb{Z}(r))$ admits a similar description.

Arp **Proposition 2.5.** *Consider $W_0,\ W_1 \in Z_{r,\overline{p}}(X) = z(X,r)_{\overline{p}}(k)$. The following are equivalent:*

1. *$W_0,\ W_1$ determine the same element in $\pi_0(z(X,r)_{\overline{p}}(\bullet)) = H_{2r}^{\overline{p}}(X,\mathbb{Z}(r))$.*
2. *$W_0,\ W_1$ determine the same element in $\pi_0(z_{equi}(X,r)_{\overline{p}}(\bullet))$.*
3. *There exists an $(r+1)$-dimensional cycle $\mathcal{W} \hookrightarrow X \times \mathbb{A}^1$ satisfying the following properties:*
   (i.) $\mathcal{W}$ is flat over $\mathbb{A}^1$;
   (ii.) for all $t \in \mathbb{A}^1$, $\mathcal{W}_t \in Z_{r,\overline{p}}(X_t)$; and
   (iii.) $W_0 = \mathcal{W} \bullet (X \times 0)$ and $W_1 = \mathcal{W} \bullet (X \times 1)$.
4. *There exists an effective $(r+1)$-dimensional cycle $\mathcal{W} \hookrightarrow X \times \mathbb{A}^1$ satisfying (i.) and (ii.), and a cycle $E \in Z_{r,\overline{p}}(X)$ such that $\mathcal{W} \bullet (X \times 0) = W_0 + E$ and $\mathcal{W} \bullet (X \times 1) = W_1 + E$.*

If $W_0,\ W_1$ satisfy these conditions, then we say that they are rationally equivalent as r-cycles of perversity $\overline{p}$, written $W_0 \sim_{\overline{p}} W_1$. We denote by $A_{r,\overline{p}}(X)$ the quotient of $Z_{r,\overline{p}}(X)$ by the relation $\sim_{\overline{p}}$:

pervChow (2.5.1)
$$A_{r,\overline{p}}(X) \equiv Z_{r,\overline{p}}(X)/\sim_{\overline{p}} = H_{2r}^{\overline{p}}(X,\mathbb{Z}(r)).$$

Proof. The equivalence of (1) and (2) follows from the observation that relative cycles are automatically flat (hence equidimensional) over a smooth base of dimension ≤ 1.

To show the equivalence of the second and third conditions, observe that elements of $z_{equi}(X,r)_{\overline{p}}(\Delta^1)$ are in bijective correspondence with $(r+1)$-dimensional cycles $\mathcal{W} \hookrightarrow X \times \mathbb{A}^1$ satisfying the conditions (i.) and (ii.) of the third condition.

The equivalence of the third and fourth conditions is essentially verified in [**?**, Ex. 1.6.2]. $\square$

Forgetting the stratification of X determines a group homomorphism from $A_{r,\overline{p}}(X)$ to rational equivalence classes of r-cycles on X, $A_{r,\overline{p}}(X) \to A_r(X)$, which need not be injective or surjective.

The following proposition establishes a perverse cycle class map from our perversity $\overline{p}$ Chow group to the Goresky-MacPherson group. We ignore a slight notational conflict; our p_i corresponds to p_{2i} in the Goresky-MacPherson convention. We use the geometric model for intersection homology as developed in [**?**, 1.3]: instead of considering the usual complex of (locally finite) chains, one considers the subcomplex of chains whose excess intersection with the strata is controlled by $\overline{p}$, and with boundary satisfying a similar condition. The homology groups of this complex are the intersection homology groups of perversity $\overline{p}$; these turn out to be independent of the stratification, as established via the sheaf-theoretic approach in [**?**, §4, Cor. 1].

Our original hope was to define purely algebro-geometrically a pairing $A_{r,\overline{p}}(X) \times A_{s,\overline{q}}(X) \to A_{r+s-d,\overline{p}+\overline{q}}(X)$ which agrees with the Goresky-MacPherson pairing via the perverse cycle class map. The construction of such a pairing, and the study of the dependence of our groups on the stratification, seem to require additional geometric input.

prop:GM **Proposition 2.6.** *Let X be a stratified variety of dimension d over $\mathbb{C}$, and suppose the stratification is sufficiently fine to compute the intersection homology groups $IH^{\overline{p}}_*(X)$. Let $H^{\overline{p}}_{2r}(X,\mathbb{Z}(r))$ (2.5.1) denote the perversity $\overline{p}$ Chow group with respect to the same stratification. Then there is a canonical perverse cycle class map*

$$c : H^{\overline{p}}_{2r}(X,\mathbb{Z}(r)) \to IH^{\overline{p}}_{2r}(X,\mathbb{Z}).$$

Proof. If α is an algebraic cycle in $Z_{r,\overline{p}}(X)$, then a triangulation of α determines a cycle in the intersection chain complex. It suffices to show that if $\alpha \sim_{\overline{p}} \alpha'$, then the difference $\alpha - \alpha'$ goes to zero in $IH^{\overline{p}}_{2r}(X,\mathbb{Z})$. If $\alpha \sim_{\overline{p}} \alpha'$, then there exists a cycle $\mathcal{W}$ on $X \times \mathbb{P}^1$ such that $\mathcal{W}_0 = \alpha + E$ and $\mathcal{W}_1 = \alpha' + E$, with $\alpha, \alpha', E \in Z_{r,\overline{p}}(X)$.

We equip $X \times \mathbb{P}^1$ with the stratification given by pulling back the stratification of X. We claim $\mathcal{W}$ determines a class in $IH^{\overline{p}}_{2r+2}(X \times \mathbb{P}^1,\mathbb{Z})$. This follows from the observation that if $Y \hookrightarrow X$ is a Cartier divisor, β is an $(r+1)$-dimensional cycle on X not contained in Y, $T \hookrightarrow X$ is closed, and the r-cycle $\beta \cap Y$ has excess $\leq e$ with $T \cap Y \hookrightarrow Y$ in Y, then β itself has excess $\leq e$ with $T \hookrightarrow X$.

We utilize the intersection pairing

$$H^2(X \times \mathbb{P}^1,\mathbb{Z}) \times IH^{\overline{p}}_{2r+2}(X \times \mathbb{P}^1,\mathbb{Z}) \to IH^{\overline{p}}_{2r}(X \times \mathbb{P}^1,\mathbb{Z}).$$

The pair $(X \times 0, \mathcal{W})$ intersects properly in each stratum $X^i \times \mathbb{P}^1$ since $X^i \times 0$ does not contain $\mathcal{W} \cap (X^i \times \mathbb{P}^1)$, and the same holds for the pair $(X \times \infty, \mathcal{W})$. Therefore the product $[X \times 0] \cdot [\mathcal{W}]$ is represented by the class of $\mathcal{W}_0$, and similarly $[X \times \infty] \cdot [\mathcal{W}] = [\mathcal{W}_\infty]$. The divisors $X \times 0, X \times \infty \hookrightarrow X \times \mathbb{P}^1$ determine the same class in $H^2(X \times \mathbb{P}^1,\mathbb{Z})$, so $[\mathcal{W}_0] = [\alpha + E] = [\alpha' + E] = [\mathcal{W}_\infty] \in IH^{\overline{p}}_{2r}(X \times \mathbb{P}^1,\mathbb{Z})$, hence $[\alpha] - [\alpha'] = 0 \in IH^{\overline{p}}_{2r}(X \times \mathbb{P}^1,\mathbb{Z})$.

There are push-forward morphisms $0_*, \infty_* : IH^{\overline{p}}_{2r}(X,\mathbb{Z}) \to IH^{\overline{p}}_{2r}(X \times \mathbb{P}^1,\mathbb{Z})$ and a projection morphism $p_* : IH^{\overline{p}}_{2r}(X \times \mathbb{P}^1,\mathbb{Z}) \to IH^{\overline{p}}_{2r}(X,\mathbb{Z})$ [**?**, Proof of Prop. 2.1]. Both 0 and ∞ are sections to p, so $p_* \circ 0_*$ and $p_* \circ \infty_*$ are both the identity, and this completes the proof. □

Remark 2.7. For X projective, C. Flannery constructed a morphism from the homotopy groups of the space of algebraic cycles of some perversity (i.e., semi-topological intersection homology groups), to the Goresky-MacPherson groups [**?**].

We now use Voevodsky's results on the cohomology of pretheories to relate $z(X,r)_{\overline{p}}(\bullet)$ and $z_{equi}(X,r)_{\overline{p}}(\bullet)$. A pretheory is a presheaf equipped with push-forward maps along relative divisors in relative smooth curves (over smooth bases). Here we show the subpresheaves defined by incidence data are in fact subpretheories.

pretheory

Lemma 2.8. *Let X be an equidimensional k-scheme, and let $(\mathcal{T},p)$ be an incidence datum on X. The subpresheaves $z_{equi}(X,r)_{\mathcal{T},p}$ and $z(X,r)_{\mathcal{T},p}$ are subpretheories inside $z_{equi}(X,r)$ and $z(X,r)$.*

Proof. We recall that both $z_{equi}(X,r)$ and $z_{equi}(X,r)_{cdh} = z(X,r)$ admit canonical structures of pretheories (in the sense of Voevodsky) in such a way that the canonical morphism $z_{equi}(X,r) \to z_{equi}(X,r)_{cdh}$ is a morphism of pretheories [**?**, Remark 5.10]. The pretheory structure is defined using intersection followed by push-forward along a finite morphism [**?**, Prop. 5.7]. For notational simplicity we treat here only the case $z(X,r)$. So suppose U is a smooth k-scheme, $C \to U$ is a smooth curve, and $Z \in c_{equi}(C/U,0)$ with morphisms $f : Z \to C$ and $p : Z \to U$. (We use c instead of z to indicate the support of Z is proper over U; see [**?**, Lemma 3.3.9].) For $W \in z(X,r)(C)$, we first form the intersection W_Z of $W \hookrightarrow C \times X$ with the Cartier divisor $Z \times X \hookrightarrow C \times X$. The cycle $\phi_{C/U}(Z)(W) \in z(X,r)(U)$ is then the push-forward $(p \times \mathrm{id})_*(W_Z)$ of W_Z along (the proper morphism) $p \times \mathrm{id} : Z \times X \to U \times X$. In particular, the support of $\phi_{C/U}(Z)(W)$ at $u \in U$ is contained in the union of the supports of W_c for $c \in f(p^{-1}(u))$. Therefore $W \in z(X,r)_{\mathcal{T},p}(C)$ implies $\phi_{C/U}(Z)(W) \in z(X,r)_{\mathcal{T},p}(U)$, and the subpresheaves $z_{equi}(X,r)_{\mathcal{T},p}$ and $z(X,r)_{\mathcal{T},p}$ are subpretheories inside $z_{equi}(X,r)$ and $z(X,r)$. □

Remark 2.9. The pretheory structure may be phrased as a coherent system of morphisms $c_{equi}(C/U,0) \to \mathrm{Hom}(z(X,r)(C), z(X,r)(U))$ for all relative curves $C \to U$. A presheaf with transfers $\mathcal{F}$ has push-forwards along all $Z \in c(U \times Y/U,0)$ for $U,Y \in Sm/k$, i.e., is equipped with a coherent system of morphisms $c(U \times Y/U,0) \to \mathrm{Hom}(\mathcal{F}(Y),\mathcal{F}(U))$, hence has more structure than a pretheory [**?**, Prop. 3.1.11]. Since the construction in the proof of Lemma 2.8 works with $Z \in c(U \times Y/U,0)$, the presheaves $z_{equi}(X,r)_{\overline{p}}$ and $z(X,r)_{\overline{p}}$ are in fact presheaves with transfers.

vanishing

Proposition 2.10. *Let X be a quasi-projective variety, and let $(\mathcal{T},p)$ be an incidence datum on X. The canonical morphism $z_{equi}(X,r)_{\mathcal{T},p} \to z(X,r)_{\mathcal{T},p}$ induces a quasi-isomorphism of Suslin complexes*

$$C_*(z_{equi}(X,r)_{\mathcal{T},p}) \to C_*(z(X,r)_{\mathcal{T},p}), \quad \mathrm{char}(k) = 0;$$

$$C_*(z_{equi}(X,r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \to C_*(z(X,r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p], \quad \mathrm{char}(k) = p.$$

Proof. Since the canonical morphism of pretheories $z_{equi}(X,r)_{\mathcal{T},p} \to z(X,r)_{\mathcal{T},p}$ becomes an isomorphism after cdh-sheafification, Voevodsky's results on the cohomology of pretheories imply $C_*(z_{equi}(X,r)_{\mathcal{T},p}) \to C_*(z(X,r)_{\mathcal{T},p})$ is a quasi-isomorphism [**?**, Thm. 5.5(2)] provided that k admits resolution of singularities. To prove the second assertion for (infinite) fields of positive characteristic, we appeal to Remark 1.1. □

The key additional property satisfied by $z(X,r)$ and not $z_{equi}(X,r)$ is the following localization property. By [?, Thm. 4.3.1], if $i : X_\infty \hookrightarrow X$ is a closed immersion with open complement $j : U \subset X$, there is an exact sequence of cdh-sheaves:

exact-cdh
$$0 \to z(X_\infty, r) \xrightarrow{i_*} z(X,r) \xrightarrow{j^*} z(U,r) \to 0. \tag{2.10.1}$$

There is *not* such a short exact sequence with $z(-)$ replaced by $z_{equi}(-)$.

cdh ses
Theorem 2.11. *Let X be a quasi-projective variety, and let $(\mathcal{T}, p)$ be an incidence datum on X. Suppose $j : X \subset \overline{X}$ is an open immersion with $\overline{X}$ projective, and let $i : X_\infty \hookrightarrow \overline{X}$ denote the closed complement. The exact sequence of cdh-sheaves:*

short
$$0 \to z(X_\infty, r) \xrightarrow{i_*} z(\overline{X}, r)_{\mathcal{T},p} \xrightarrow{j^*} z(X,r)_{\mathcal{T},p} \tag{2.11.1}$$

determines the following distinguished triangle of Suslin complexes

$$C_*(z_{equi}(X_\infty, r)) \xrightarrow{i_*} C_*(z_{equi}(\overline{X}, r)_{\mathcal{T},p}) \xrightarrow{j^*} C_*(z_{equi}(X,r)_{\mathcal{T},p})$$
$$\to C_*(z_{equi}(X_\infty, r))[1]$$

if $\mathrm{char}(k) = 0$*;*

$$C_*(z_{equi}(X_\infty, r)) \otimes \mathbb{Z}[1/p] \xrightarrow{i_*} C_*(z_{equi}(\overline{X}, r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \xrightarrow{j^*} C_*(z_{equi}(X,r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p]$$
$$\to C_*(z_{equi}(X_\infty, r))[1] \otimes \mathbb{Z}[1/p]$$

if $\mathrm{char}(k) = p > 0$.

Proof. The exactness of the asserted exact sequence of sheaves is clear except at the final term. Given $\alpha \in z(X,r)_{\mathcal{T},p}(U) \subset z(X,r)(U)$, by (2.10.1) there exists a cdh-cover $p : U' \to U$ and an element $\overline{\alpha'} \in z(\overline{X}, r)(U')$ such that $j^*(\overline{\alpha'}) = p^*(\alpha)$. But $p^*(\alpha) \in z(X,r)_{\mathcal{T},p}(U')$ by definition $\overline{\alpha'} \in z(\overline{X}, r)_{\mathcal{T},p}(U')$. The distinguished triangle with z_{equi} replaced by z follows from [?, Thm. 5.5(2)]. The asserted statement now follows from Proposition 2.10. □

Similarly, we have Mayer-Vietoris distinguished triangles; in contrast with Theorem **??**, these distinguished triangles apply fully to perversity cycles. The proof requires merely notational changes of the proof of that theorem.

MV
Corollary 2.12. *Let X be a quasi-projective variety, and let $(\mathcal{T}, p)$ be an incidence datum on X. Consider Zariski open subsets U_1, U_2 of X with $U_1 \cup U_2 = X$ and $U_1 \cap U_2 = U_{1,2}$. The exact sequence of cdh-sheaves:*

short
$$0 \to z(X,r)_{\mathcal{T},p} \xrightarrow{i_*} z(U_1, r)_{\mathcal{T},p} + z(U_2, r)_{\mathcal{T},p} \xrightarrow{j^*} z(U_{1,2}, r)_{\mathcal{T},p} \tag{2.12.1}$$

determines Mayer-Vietoris distinguished triangles of Suslin complexes

$$C_*(z_{equi}(X,r)_{\mathcal{T},p}) \xrightarrow{i_*} C_*(z_{equi}(U_1, r)_{\mathcal{T},p}) \oplus C_*(z_{equi}(U_2, r)_{\mathcal{T},p}) \xrightarrow{j^*}$$
$$C_*(z_{equi}(U_{1,2}, r)_{\mathcal{T},p}) \to C_*(z_{equi}(X,r)_{\mathcal{T},p}))[1]$$

if $\mathrm{char}(k) = 0$*;*

$$C_*(z_{equi}(X,r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \xrightarrow{i_*} C_*(z_{equi}(U_1, r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \oplus C_*(z_{equi}(U_2, r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \xrightarrow{j^*}$$
$$C_*(z_{equi}(U_{1,2}, r)_{\mathcal{T},p}) \otimes \mathbb{Z}[1/p] \to C_*(z_{equi}(X,r)_{\mathcal{T},p}))[1] \otimes \mathbb{Z}[1/p].$$

if $\mathrm{char}(k) = p > 0$.

sec:three

3. Suspension theorems

In this section, we adapt the proof of Lawson [?] (formulated in more algebraic terms in [?] and adopted further in [?]) to establish "Lawson suspension theorems" ($\mathbb{A}^1$-invariance) for perversity cycles.

Let X be a projective variety of dimension d equipped with an embedding $X \hookrightarrow \mathbb{P}^N$. There is an induced embedding $\Sigma(X) \hookrightarrow \Sigma(\mathbb{P}^N) = \mathbb{P}^{N+1}$, where $\Sigma(-)$ denotes the algebraic suspension. If $\mathbb{P}^N \hookrightarrow \Sigma(\mathbb{P}^N)$ is defined by the vanishing of the suspension coordinate, then the identification $X = \Sigma(X) \cap \mathbb{P}^N$ allows us to view subvarieties of X as subvarieties of $\mathbb{P}^{N+1}$. If $X' \subset X$ is an open subscheme of a projective variety X with complement X_∞, then we define $\Sigma(X') \equiv \Sigma(X) - \Sigma(X_\infty)$; this is an open subscheme of $\Sigma(X)$.

If $(\mathcal{T}, p)$ is an incidence datum on X, then we define $\Sigma(\mathcal{T}) := \{\Sigma(X^i)\}_{X^i \in \mathcal{T}}$ and we consider both $(\mathcal{T}, p)$ and $(\Sigma(\mathcal{T}), p)$ as incidence data on $\Sigma(X)$.

Our arguments in this section use geometric constructions on the Chow monoids and therefore our results concern presheaves of equidimensional cycles and their cdh-sheafifications. To obtain the results for $z_{equi}(X, r)_{\mathcal{T},p}$, we apply the functor $\mathrm{Hom}\,(-, \mathcal{C}_{r+1}(\Sigma(X)))^+$ (or the subfunctor of morphisms satisfying the field of definition condition) to our constructions. For $z(X, r)_{\mathcal{T},p}$, we apply the functor $\mathrm{Hom}_{\mathrm{c.alg}}(-, \mathcal{Z}_{r+1}(\Sigma(X)))$ or its field of definition subfunctor. For both cases we observe our constructions respect the field of definition condition.

thm:proj-susp **Theorem 3.1.** *Let X be a projective variety, and let $(\mathcal{T}, p)$ be an incidence datum on X. The fiberwise suspension morphism of presheaves*

$$\Sigma_X : z_{equi}(X, r)_{\mathcal{T},p} \to z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}$$

sending an effective cycle $W \subset U \times X$ to the effective cycle $\Sigma_X(W) \subset U \times \Sigma(X)$ induces a homotopy equivalence

basic
$$z_{equi}(X, r)_{\mathcal{T},p}(\bullet) \stackrel{\sim}{\to} z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}(\bullet). \tag{3.1.1}$$

The fiberwise suspension also induces a homotopy equivalence

$$z(X, r)_{\mathcal{T},p}(\bullet) \stackrel{\sim}{\to} z(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}(\bullet).$$

We establish this homotopy equivalence (3.1.1) by factoring Σ_X as a composition

factor
$$z_{equi}(X, r)_{\mathcal{T},p} \to z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p} \to z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}, \tag{3.1.2}$$

showing in Proposition 3.3 (respectively, in Proposition 3.4) that the first (resp., second) morphism induces a homotopy equivalence upon evaluation at $\Delta^\bullet$. The presheaf $z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}$ consists of cycles meeting X properly, and having excess intersection with X^i no larger than p_i.

Before giving the proof of Theorem 3.1, we state explicitly the special case of primary interest, the suspension isomorphism for perversity cycles on a stratified projective variety.

cor:susp **Corollary 3.2.** *Let X be a stratified projective variety, and let $\overline{p}$ be a perversity. Equip $\Sigma(X)$ with the stratification $\{\Sigma(X^i)\}$, where $\{X^i\}$ is the given stratification of X. Fiberwise suspension induces homotopy equivalences*

$$\Sigma_X : z_{equi}(X, r)_{\overline{p}}(\bullet) \stackrel{\sim}{\to} z_{equi}(\Sigma(X), r+1)_{\overline{p}}(\bullet) \text{ and}$$

$$\Sigma_X : z(X, r)_{\overline{p}}(\bullet) \stackrel{\sim}{\to} z(\Sigma(X), r+1)_{\overline{p}}(\bullet).$$

The proof of our first homotopy equivalence uses the technique of deformation to the normal cone (see [?, Ch. 5]), called "holomorphic taffy" by Lawson in [?].

prop:iso1 **Proposition 3.3.** *Retain the notation and hypotheses of Theorem 3.1. The morphism* $\Sigma_X : z_{equi}(X, r)_{\mathcal{T},p} \to z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}$ *induces a homotopy equivalence* $z_{equi}(X, r)_{\mathcal{T},p}(\bullet) \xrightarrow{\sim} z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(\bullet)$. *The same result holds for the cdh-sheafification.*

Proof. Let $\mathcal{C}_{r+1,d}(\Sigma(X))_X$ denote the open subset of the Chow variety consisting of cycles α such that $\alpha \cap X$ has dimension r, i.e., α is not contained in X. The suspension morphism $\Sigma_X : \mathcal{C}_{r,d}(X) \to \mathcal{C}_{r+1,d}(\Sigma(X))$ factors through $\mathcal{C}_{r+1,d}(\Sigma(X))_X$. As shown in [?, Prop. 3.2], there is a continuous algebraic map (i.e., a morphism on semi-normalizations)

$$\varphi : \mathcal{C}_{r+1,d}(\Sigma(X))_X \times \mathbb{A}^1 \to \mathcal{C}_{r+1,d}(\Sigma(X))_X$$

with the following properties [?, Prop. 3.2]. (Here φ_t denotes the restriction of φ to $\mathcal{C}_{r+1,d}(\Sigma(X))_X \times \{t\}$.)

(1) φ_0 is the identity on $\mathcal{C}_{r+1,d}(\Sigma(X))_X$;
(2) φ_1 has image contained in $\Sigma_X(\mathcal{C}_{r,d}(X))$, in fact $\varphi_1(\alpha) = \Sigma_X(\alpha \cap X)$; and
(3) φ_t acts as the identity on $\Sigma_X(\mathcal{C}_{r,d}(X))$ for all $t \in \mathbb{A}^1$, in fact φ_t (for $t \neq 1$) is induced by an automorphism of $\mathbb{P}^{N+1}$ fixing the suspension hyperplane $\mathbb{P}^N$.
(4) φ does not depend on the degree d.

From properties (2) and (3) it follows that φ preserves the field of definition of a cycle. For $t \neq 1$, we use that the automorphism is defined over the ground field k. For $t = 1$, the operation $\alpha \mapsto \Sigma_X(\alpha \cap X)$ may be described as eliminating all instances of the suspension coordinate in the equations defining α.

We adapt this construction as follows. For any $U \in Sm/k$, let $z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(U) \subset z_{equi}(\Sigma(X), r+1)(U)$ consist of those U-relative cycles with the property that each specialization meets X properly and meets $X^i \in \mathcal{T}$ with excess at most p_i. We proceed to show that φ induces a morphism of presheaves (on Sm/k):

eq:phi (3.3.1) $$\varphi_{\mathcal{T},p} : z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(-) \to z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(- \times \mathbb{A}^1)$$

with the following properties:

(1) $(\varphi_{\mathcal{T},p})_0$ is the identity on $z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}$;
(2) $(\varphi_{\mathcal{T},p})_1$ has image contained in $\Sigma_X(z_{equi}(X, r)_{\mathcal{T},p})$, in fact $(\varphi_{\mathcal{T},p})_1(\alpha) = \Sigma_X(\alpha \cap X)$ for any $\alpha \in z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(U)$; and
(3) $(\varphi_{\mathcal{T},p})_t$ acts as the identity on $\Sigma_X(z_{equi}(X, r)_{\mathcal{T},p})$ for all $t \in \mathbb{A}^1$, in fact $(\varphi_{\mathcal{T},p})_t$ (for $t \neq 1$) is induced by an automorphism of $\mathbb{P}^{N+1}$ fixing the suspension hyperplane $\mathbb{P}^N$.

(Here $(\varphi_{\mathcal{T},p})_t$ denotes $\varphi_{\mathcal{T},p}$ followed by restriction to $(- \times \{t\})$.) There is a canonical inclusion of presheaves of abelian monoids on Sm/k:

$$z^{eff}_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(-) \to \mathrm{Hom}(-, \mathcal{C}_{r+1}(\Sigma(X))_X)$$

which induces

$$z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}(-) \to \mathrm{Hom}(-, \mathcal{C}_{r+1}(\Sigma(X))_X)^+.$$

Now φ induces a natural transformation

eq:Phi
$$\mathrm{Hom}(-,\mathcal{C}_{r+1}(\Sigma(X))_X)^+ \to \mathrm{Hom}(-\times\mathbb{A}^1,\mathcal{C}_{r+1}(\Sigma(X))_X)^+ \tag{3.3.2}$$

sending a morphism $f : U \to \mathcal{C}_{r+1}(\Sigma(X))_X$ to the composition $\varphi\circ(f\times \mathrm{id}_{\mathbb{A}^1}) : U\times\mathbb{A}^1 \to \mathcal{C}_{r+1}(\Sigma(X))_X$. We claim this restricts to our desired morphism $\varphi_{\mathcal{T},p}$. Properties (2) and (3) of φ imply that for all $\alpha\in\mathcal{C}_{r+1}(\Sigma(X))_X$ and all $t\in\mathbb{A}^1$, we have $\varphi_t(\alpha)\cap X^i = \alpha\cap X^i$, so the incidence conditions with the sets appearing in $\mathcal{T}$ are preserved. We have already observed that φ preserves the field of definition of a cycle.

We are now in a position to apply [**?**, Lemma 6.6], and this completes the proof for the *equi*-theory. If we work with continuous algebraic maps into $\mathcal{Z}_{r+1}(\Sigma(X))$ instead of $\mathrm{Hom}(-,\mathcal{C}_{r+1}(\Sigma(X))_X)^+$, we obtain the result for the (non-equidimensional) cdh theory $z(-,-)$. □

The proof of our second homotopy equivalence uses the technique first introduced by Lawson in [**?**], which he calls "magic fans."

prop:iso2
Proposition 3.4. *Retain the notation and hypotheses of Theorem 3.1. The canonical inclusion $z_{equi}(\Sigma(X),r{+}1)_{X,\mathcal{T},p}\to z_{equi}(\Sigma(X),r{+}1)_{\Sigma(\mathcal{T}),p}$ induces a homotopy equivalence*

$$z_{equi}(\Sigma(X),r+1)_{X,\mathcal{T},p}(\bullet) \xrightarrow{\sim} z_{equi}(\Sigma(X),r+1)_{\Sigma(\mathcal{T}),p}(\bullet).$$

The same result holds for the cdh-sheafification.

Proof. Let $\alpha\in Z_{r,\overline{p}}(X)$, and suppose $\alpha=\alpha^+-\alpha^-$, where α^+ and α^- are effective cycles with no components in common. Then $Z_{r,\le d,\overline{p}}(X)\subset Z_{r,\overline{p}}(X)$ consists of those cycles α such that $\deg(\alpha^+)\le d$ and $\deg(\alpha^-)\le d$ (with respect to the given closed embedding $X\subset\mathbb{P}^N$). Since the degree is invariant under field extensions, this pointwise condition defines a subpresheaf in our cycle presheaves.

As shown in [**?**, Prop. 3.5], for every $d\ge 0$, there exists an integer e_d such that for every $e\ge e_d$ there exists a morphism of semi-normal schemes

eq:psi
$$\psi_e : \mathcal{C}_{r+1,\le d}(\Sigma(X))\times\mathbb{A}^1 \to \mathcal{C}_{r+1,\le de}(\Sigma(X)) \tag{3.4.1}$$

with the following properties:

(1) $\psi_e(\alpha,0)=e\cdot\alpha$ for all $\alpha\in\mathcal{C}_{r+1,\le d}(\Sigma(X))$; and
(2) $\psi_e(\alpha,t)\in\mathcal{C}_{r+1,\le de}(\Sigma(X))_X$, for all $\alpha\in\mathcal{C}_{r+1,\le d}(\Sigma(X))$ and all $t\neq 0\in\mathbb{A}^1$.

Since the $\mathbb{A}^1$ corresponds a family of divisors defined over the ground field, and the suspension and projection operations preserve the field of definition of a cycle, the morphism ψ_e preserves the field of definition condition.

For ease of exposition we introduce some notation. Let $\mathcal{F}'_{\le d}$ denote the presheaf $z^{eff}_{equi}(\Sigma(X),r+1,\le d)_{X,\mathcal{T},p}$, and let $\mathcal{F}_{\le d}$ denote the presheaf $z^{eff}_{equi}(\Sigma(X),r+1,\le d)_{\Sigma(\mathcal{T}),p}$. We have the following commutative diagram of canonical inclusions of presheaves on Sm/k:

$$\begin{tikzcd}
\mathcal{F}'_{\le d} \arrow[r] \arrow[d] & \mathrm{Hom}(-,C_{r+1,\le d}(\Sigma(X))_X) \arrow[d] \\
\mathcal{F}_{\le d} \arrow[r] & \mathrm{Hom}(-,C_{r+1,\le d}(\Sigma(X)))
\end{tikzcd}$$

We let $z_{equi}(\Sigma(X),r{+}1,\le d)_{\Sigma(\mathcal{T}),p}$ denote the quotient of $\mathcal{F}_{\le d}\times\mathcal{F}_{\le d}$ by the evident relation: $(a,b)\sim(a',b')$ if $a+b'=a'+b$ as cycles. Note that $z_{equi}(\Sigma(X),r+$

$1)_{\Sigma(\mathcal{T}),p} = \cup_d z_{equi}(\Sigma(X), r+1, \leq d)_{\Sigma(\mathcal{T}),p}$. We employ the analogous notation with the subscript $X, \mathcal{T}, p$.

We claim that ψ_e of (3.4.1) restricts to a morphism of presheaves $(\psi_e)_{\mathcal{T},p}$: $\mathcal{F}_{\leq d}(-) \to \mathcal{F}_{\leq de}(- \times \mathbb{A}^1)$ with the following properties:

(1) $((\psi_e)_{\mathcal{T},p})_0(\alpha) = e \cdot \alpha$ for all $\alpha \in \mathcal{F}_{\leq d}(U)$; and
(2) $((\psi_e)_{\mathcal{T},p})_t(\alpha) \in \mathcal{F}'_{de}(U)$ for all $\alpha \in \mathcal{F}_{\leq d}(U)$ and all $t \neq 0 \in \mathbb{A}^1$.

Since the operation ψ_e affects only the suspension coordinate, it follows that

$$(\psi_e)_t(\alpha) \cap \Sigma(X^i) = (\psi_e)_t(\alpha \cap \Sigma(X^i))$$

for all $\alpha \in \mathcal{C}_{r+1}(\Sigma(X))$, $t \in \mathbb{A}^1$. The right hand side is controlled by hypothesis, and a bound on the dimension of the left hand side defines $\mathcal{F}_{\leq de}$. Therefore ψ_e restricts to a morphism on the subpresheaf $\mathcal{F}_{\leq d}$.

The first property is immediate from the corresponding condition of ψ_e. The second property means that ψ_e improves the incidence with $X^i \hookrightarrow \Sigma(X^i)$ and with $X \hookrightarrow \Sigma(X)$. The improvement with $X \hookrightarrow \Sigma(X)$ is due to [**?**, Prop. 3.5], and the incidence with X^i is handled similarly. Namely, given a bounded family of cycles $\{\alpha\}$ on $\Sigma(X)$ satisfying the $(\Sigma(\mathcal{T}), p)$ condition, we consider the bounded families of $(r+1-i+p_i)$-dimensional cycles $\{|\alpha \cap \Sigma(X^i)|\}$ for $i = 1, \ldots, d$. Following [**?**, Prop. 3.5] we find a $\mathbb{P}^1$-family of hypersurfaces (of large degree e depending on these bounded families) through $e \cdot \mathbb{P}^{N+1}$ such that no member (besides $e \cdot \mathbb{P}^{N+1}$) contains any of the cycles in the bounded families. This guarantees the moved cycle satisfies the (stronger) $(X, \mathcal{T}, p)$ condition.

The rest is formal. The morphism of presheaves

$$(\mathcal{F}_{\leq d} \times \mathcal{F}_{\leq d})(-) \to z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}(- \times \mathbb{A}^1)$$

defined by

$$(a, b) \mapsto ((\psi_{e+1})_{\mathcal{T},p}(a) - (\psi_e)_{\mathcal{T},p}(a)) \ - \ ((\psi_{e+1})_{\mathcal{T},p}(b) - (\psi_e)_{\mathcal{T},p}(b))$$

determines a natural transformation

psi_e

$$z_{equi}(\Sigma(X), r+1, \leq d)_{\Sigma(\mathcal{T}),p}(-) \to z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}(- \times \mathbb{A}^1) \tag{3.4.2}$$

which relates the identity (at $t = 0$) to a morphism factoring (for all $t \neq 0$) through $z_{equi}(\Sigma(X), r+1)_{X,\mathcal{T},p}$. Now [**?**, Lemma 6.6] completes the proof, as in the conclusion of the proof of Proposition 3.3. □

We next extend Theorem 3.1 and Corollary 3.2 to quasi-projective varieties. The proof employs the localization theorem for $z(X, r)_{\mathcal{T},p}$ and the comparison of $z_{equi}(X, r)_{\mathcal{T},p}$ with $z(X, r)_{\mathcal{T},p}$, and thus requires that k admits resolution of singularities. Localization provides us with the distinguished triangles of Proposition 2.11 which we use to reduce the case of X quasi-projective to the consideration of the projective closure $\overline{X}$ of X and the projective complement $X_\infty = \overline{X} - X$.

thm:quasi-susp

Theorem 3.5. *Let X be a quasi-projective variety, and let $(\mathcal{T}, p)$ be an incidence datum on X. The morphism of presheaves*

$$\Sigma_X : z_{equi}(X, r)_{\mathcal{T},p} \to z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}$$

induces a homotopy equivalence

$$z_{equi}(X, r)_{\mathcal{T},p}(\bullet) \cong z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}(\bullet), \quad char(k) = 0,$$

$$z_{equi}(X, r)_{\mathcal{T},p}[1/p](\bullet) \cong z_{equi}(\Sigma(X), r+1)_{\Sigma(\mathcal{T}),p}[1/p](\bullet), \quad char(k) = p.$$

Proof. By Proposition 2.10, we may replace z_{equi} by z.

Choose a projective compactification $\overline{X}$ of X, and regard $(\mathcal{T}, p)$ as an incidence datum on $\overline{X}$. The morphism $\Sigma_{X_\infty} : z(X_\infty, r) \to z(\Sigma(X_\infty), r+1)$ induces a quasi-isomorphism of Suslin complexes by the usual $\mathbb{A}^1$-homotopy invariance [**?**, Thm. 8.3(1)] and the isomorphism of sheaves $z(\Sigma(X_\infty), r+1) \cong z(X_\infty \times \mathbb{A}^1, r+1)$. The morphism $\Sigma_{\overline{X}} : z(\overline{X}, r)_{\mathcal{T},p} \to z(\Sigma(\overline{X}), r+1)_{\Sigma(\mathcal{T}),p}$ induces a homotopy equivalence after evaluation at $\Delta^\bullet$ by Theorem 3.1.

The suspension map determines a map of distinguished triangles of Suslin complexes as in Proposition 2.11 which determines a map of long exact sequences of homology groups. Thus, the 5-Lemma enables us to conclude quasi-isomorphisms of Suslin complexes. The proof is now completed by referring to Remark 2.3. □

Without perversities, there are isomorphisms of cdh-sheaves

$$z(X \times \mathbb{P}^t, r+t)/z(X \times \mathbb{P}^{t-1}, r+t) \cong z(X \times \mathbb{A}^t, r+t) \cong z(\Sigma^t(X), r+t).$$

With the added complexity of perversity, there is a canonical injective morphism of presheaves

good closure

$$(3.5.1) \qquad z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}}/z_{equi}(X \times \mathbb{P}^{t-1}, r+t)_{\overline{p}} \to z_{equi}(X \times \mathbb{A}^t, r+t)_{\overline{p}}$$

We denote the image of 3.5.1 by

$$z_{equi}(X \times \mathbb{A}^t, r+t)_{\overline{p},cl} \ \subset \ z_{equi}(X \times \mathbb{A}^t, r+t)_{\overline{p}},$$

consisting of those perversity $\overline{p}$ cycles α on $X \times \mathbb{A}^t$ such that the closure $\overline{\alpha} \hookrightarrow X \times \mathbb{P}^t$ is of perversity $\overline{p}$; we say that a cycle in the image of 3.5.1 has good closure. If $\mathcal{O}_X(1) \simeq \mathcal{O}_X$, then $z_{equi}(X \times \mathbb{A}^1, r+1)_{\overline{p}} \cong z_{equi}(\Sigma(X), r+1)_{\overline{p}}$ because the vertex of $\Sigma^t(X)$ does not influence the incidence datum.

The basic argument of the proof of the following theorem is that the constructions ϕ of Proposition 3.3 and ψ of Proposition 3.4 for $\pi^* : z_{equi}(X, r) \to z_{equi}(\Sigma^t, r)$ preserve this property of good closure.

slice

Theorem 3.6. *Let X be a stratified quasi-projective variety, and let $\overline{p}$ be a perversity. The canonical morphism of presheaves*

$$\pi^* : z_{equi}(X, r)_{\overline{p}} \ \to \ z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}}/z_{equi}(X \times \mathbb{P}^{t-1}, r+t)_{\overline{p}}$$

induces a homotopy equivalence if $char(k) = 0$:

$$\pi^*(\bullet) : (z_{equi}(X, r)_{\overline{p}}(\bullet) \ \stackrel{\sim}{\to} \ (z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}}(\bullet)/(z_{equi}(X \times \mathbb{P}^{t-1}, r+t)_{\overline{p}}(\bullet);$$

if $char(k) = p$, then

$$(z_{equi}(X, r)_{\overline{p}} \otimes \mathbb{Z}[1/p])(\bullet) \ \stackrel{\sim}{\to}$$

$$(z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}} \otimes \mathbb{Z}[1/p])(\bullet)/(z_{equi}(X \times \mathbb{P}^{t-1}, r+t)_{\overline{p}} \otimes \mathbb{Z}[1/p])(\bullet).$$

In particular, the homotopy inverse of $\pi^*(\bullet)$ determines a slice map

slice1

$$(3.6.1) \qquad sl : z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}}(\bullet) \ \to \ z_{equi}(X, r)_{\overline{p}}(\bullet), \quad char(k) = 0,$$

slice2

(3.6.2)

$$sl : z_{equi}(X \times \mathbb{P}^t, r+t)_{\overline{p}} \otimes \mathbb{Z}[1/p](\bullet) \ \to \ z_{equi}(X, r)_{\overline{p}}\mathbb{Z}[1/p](\bullet), \quad char(k) = p.$$

PROBLEM !!!

Proof. Applying Corollary 2.12 and the 5-Lemma, we easily reduce to the case in which $\mathcal{O}_X(1) \simeq \mathcal{O}_X$. Therefore, we may assume that $X \times \mathbb{A}^t \subset \Sigma^t(X)$ with complement $\mathbb{P}^{t-1}$. As discussed above, it then suffices to show that $\pi^* : z_{equi}(X, r)_{\overline{p}}(\bullet) \to z_{equi}(\Sigma^t(X), r+t)_{\overline{p},cl}(\bullet)$ is a homotopy equivalence (with p inverted if $char(k) = p$).

We proceed to consider the special case $t = 1$; the general case follows similarly, using parametrized versions of the ϕ of Proposition 3.3 and ψ of Proposition 3.4 to treat the iterated suspension Σ^t. For $\alpha \hookrightarrow \Sigma(X)$ we denote by $\overline{\alpha} \hookrightarrow X \times \mathbb{P}^1$ the birational transform of α via $X \times \mathbb{P}^1 \dashrightarrow \Sigma(X)$. We denote by $v \in \Sigma(X)$ the vertex of the suspension.

Suppose $\alpha \hookrightarrow \Sigma(X)$ is a perversity $\overline{p}$ cycle with good closure, and such that $\alpha \cap X$ is a perversity $\overline{p}$ cycle. For $t \neq 1$, $(\varphi_{\mathcal{T},p})_t$ acts by scaling the suspension coordinate, so preserves the closure condition; for $t = 1$ we obtain the suspension of $\alpha \cap X$, which has good closure by assumption. This handles the deformation to the normal cone.

We claim if $\alpha \hookrightarrow \Sigma(X)$ is a cycle of perversity $\overline{p}$ with good closure, then $\psi_e(\alpha, t)$ also has good closure. The transform $\overline{\alpha} \hookrightarrow X \times \mathbb{P}^1$ contains the point (x, ∞) if and only if the line $\Sigma(x) \hookrightarrow \Sigma(X)$ is tangent to α at the vertex $v \in \Sigma(X)$. Therefore it suffices to show that

$$\Sigma(x) \subset T_v(\psi_e(\alpha, t)) \Rightarrow \Sigma(x) \subset T_v(\alpha)$$

for all $t \in \mathbb{A}^1$, for then the support of $\overline{\psi_e(\alpha, t)}$ along $X \times \infty$ is contained in the support of $\overline{\alpha}$ along $X \times \infty$.

The points v, x_1, x_2 lie on a line ℓ in $\mathbb{P}^{N+2}$. For any $p \neq x_1 \in \ell$ we have a canonical identification

$$T_p(\Sigma(\alpha)) = T_v(\alpha) \oplus S,$$

where the summand S corresponds to the direction of ℓ. We lose the summand S after slicing with D, and then we find $T_v(\psi_e(\alpha, t)) \subseteq T_v(\alpha)$. □

sec:four

4. Intersection product under condition $(*, \overline{c})$

Let X be a possibly singular variety of pure dimension d with smooth locus X^{sm} open in X and singular locus $X_{sing} = X - X^{sm}$. Let V, W be closed irreducible subvarieties of X of dimension r, s respectively and assume that the dimension of the intersection of the supports $|V| \cap |W|$ is $\leq r + s - d$ (i.e., V, W intersect properly). Assume that no component of $|V| \cap |W| \cap X_{sing}$ has dimension $\geq r+s-d$. Then we justify in Theorem 4.2 our view that a good candidate for $V \bullet W$ on X is the closure in X of the usual intersection product of $V \cap X^{sm}$, $W \cap X^{sm}$ on the smooth variety X^{sm}.

With this in mind, we first formalize a stratified version of "proper" intersection of cycles on a possibly singular variety X.

def:star

Definition 4.1. Let X be a stratified variety of pure dimension d, let α, β be algebraic cycles on X of dimension r, s, and let $\bar{c}$ be a perversity. Then (α, β) is said to satisfy condition $(*, \overline{c})$ provided that

$$\dim(|\alpha| \cap |\beta| \cap X^i) \leq r + s - d - (i - c_i), \quad \text{for all } i.$$

As we shall see in Section 6, such pairs are provided by cycles of perversity $\overline{p}$ and generalized cocycles of perversity $\overline{q}$, if $\overline{p} + \overline{q} \leq \bar{t}$.

thm:star **Theorem 4.2.** *Let X be a stratified variety of pure dimension d with the property that $X_{sing} \subset X^1$. Let $z_{r*s,\overline{c}}(X) \subset z(X,r)\times z(X,s)$ denote the subsheaf on (Sch/k) consisting of pairs satisfying condition $(*,\overline{c})$. Then the closure of the intersection pairing on the smooth locus of X defines a morphism of functors on (Sch/k):*

$$\bullet : z_{r*s,\overline{c}}(X) \ \to \ z(X, r+s-d)_{\overline{c}}.$$

Proof. A pair $(\alpha,\beta) \in z(X,r)(U) \times z(X,s)(U)$ belongs to $z_{r*s,\overline{c}}(X)(U)$ provided every specialization (α_u,β_u) satisfies $(*,\overline{c})$ on X_u. If $U' \to U$ is a morphism in Sch/k, then the specialization of $(\alpha,\beta)_{U'}$ at $u' \in U'$ has support equal to the base change via $u' \to u$ of the support of $|\alpha_u| \cap |\beta_u|$, hence satisfies $(*,\overline{c})$. Therefore the condition $(*,\overline{c})$ defines a presheaf.

The morphism of functors is determined by the intersection product on the smooth locus of X. For the moment assume U is integral with generic point η. We send (α,β) to $\alpha\bullet\beta$, defined to be the closure in $X\times U$ of the $r+s-d$-dimensional cycle $(\alpha_\eta)^{sm} \bullet_{X_\eta^{sm}} (\beta_\eta)^{sm}$ in X_η^{sm}. This is a cycle on $X \times U$ whose generic points lie over η, so we need to show it has well-defined specializations.

Every pair (α_u,β_u) satisfies $(*,\overline{c})$, therefore $|\alpha_u| \cap |\beta_u|$ has its generic points in X_u^{sm} for every $u \in U$. The intersection product on smooth varieties is compatible with specialization, so the specialization of $\alpha\bullet\beta$ along a fat point (x_0,x_1) over $u \in U$ is the closure in X_u of the intersection product of $((x_0,x_1)^*(\alpha))^{sm}$ and $((x_0,x_1)^*(\beta))^{sm}$ in X_u^{sm}. By hypothesis the specializations of α and β are independent of the choice of fat point, so the same is true of $\alpha\bullet\beta$. Since the intersection product preserves integral coefficients, if α and β have universally integral coefficients then so must $\alpha\bullet\beta$.

If U has several irreducible components, we define $\alpha\bullet\beta$ by the procedure above on each component. Where the components of U meet, the specializations agree since they may be described in terms of specializations of α and β, which agree by hypothesis. □

Remark 4.3. Typically $z(X,r)_{\overline{p}} \times z(X,s)_{\overline{q}} \not\subseteq z_{r*s,\overline{c}}(X)$ for any perversity $\overline{c}$, so the intersection product of Theorem 4.2 does not provide a pairing $A_{r,\overline{p}}(X)\times A_{s,\overline{q}}(X) \to A_{r+s-d,\overline{p}+\overline{q}}(X)$ in general.

We make explicit the following special case of the functoriality of Theorem 4.2. In fact, much of the above proof of Theorem 4.2 can be interpreted as confirming the commutativity of the diagram in the following corollary.

specialization **Corollary 4.4.** *Retain the notation and hypotheses of Theorem 4.2. Let C be a smooth and connected curve, let $\eta \in C$ be the generic point of C, and let $\gamma \in C$ be a closed point of C. Then the following diagram commutes*

$$\begin{tikzcd}
z_{r*s,\overline{c}}(X)(\eta) \arrow[r, "\bullet"] & z(X,r+s-d)_{\overline{c}}(\eta) \\
z_{r*s,\overline{c}}(X)(C) \arrow[r, "\bullet"] \arrow[u] \arrow[d] & z(X,r+s-d)_{\overline{c}}(C) \arrow[u] \arrow[d] \\
z_{r*s,\overline{c}}(X)(\gamma) \arrow[r, "\bullet"] & z(X,r+s-d)_{\overline{c}}(\gamma)
\end{tikzcd}$$

Example 4.5. We consider a simple example due to Zobel [?] of a singular variety X on which there is no decent intersection product on usual rational equivalence classes of cycles. Namely, X is the cone on a quadric surface $Q \hookrightarrow \mathbb{P}^3$, i.e., on $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$. We refer to the unique singular point of X as its vertex v.

We use the "obvious" stratification, namely, $v = X^3 = X^2 = X^1 \hookrightarrow X$. Since $p_3 \leq p_2 + 1$ and $p_3 \leq p_1 + 2$, the condition on the incidence with X^3 determines the perversity. Therefore we abuse notation and write $\overline{p}$ for any perversity with $p_3 = p$, where $p \in \{0, 1, 2\}$.

By the $\mathbb{A}^1$-invariance of Chow groups, we have:

- $A_2(X) = A_{2,\overline{1}}(X) = A_{2,\overline{2}}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, with generators corresponding to cones on the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$; and
- $A_1(X) = A_{1,\overline{2}}(X) \cong \mathbb{Z}$, with generator corresponding to the cone on a point in $\mathbb{P}^1 \times \mathbb{P}^1$.

The classes of the lines $L = \mathbb{P}^1 \times q$, $M = p \times \mathbb{P}^1 \hookrightarrow Q \hookrightarrow X$ are equal in $A_1(X)$, and each generates. Note that each is rationally equivalent to $N = C(p \times q)$. The lines L and M are contained in X^{sm} but N is not. Consider the divisor $D = C(\mathbb{P}^1 \times q')$ in $A_2(X)$ for some $q' \neq q$. We have $|D| \cap |L| = \emptyset$ while $|D| \cap |M| = p \times q' \in Q \hookrightarrow X$, and surely the coefficient of $p \times q'$ should be 1. Therefore, there is no reasonable pairing $A_2(X) \times A_1(X) \to A_0(X) \xrightarrow{\deg} \mathbb{Z}$, even if we consider only intersections which occur in the smooth locus of X. Note that Theorem 4.2 implies that any rational equivalence between L and M passes through the vertex, and that the classes of L and M must be distinct in $A_{1,\overline{0}}(X)$,

We proceed to compute the intersection pairing (guaranteed by Theorem 4.2) on the intersection Chow groups.

To calculate the zero perversity groups, we use that X is birational to $\mathbb{P}^1 \times Q$, and that geometry away from the vertex corresponds to geometry away from $\infty \times Q$. Taking the birational transform of divisors and rational equivalences (all missing the vertex) identifies $A_{2,\overline{0}}(X)$ with the relative Picard group $\mathrm{Pic}(\mathbb{P}^1 \times Q, \infty \times Q)$. We have $\mathrm{Pic}(\mathbb{P}^1 \times Q, \infty \times Q) \cong \mathbb{Z}$ (generated by $\mathcal{O}(1)$ of the fiber of $\mathbb{P}^1 \times Q \to Q$) since line bundles pulled back from Q have nontrivial intersections with the divisor $\infty \times Q$. In essence we use the exact sequence

$$\Gamma(\mathbb{P}^1 \times Q, \mathcal{O}^*) \to \Gamma(\infty \times Q, \mathcal{O}^*) \to \mathrm{Pic}(\mathbb{P}^1 \times Q, \infty \times Q) \to \mathrm{Pic}(\mathbb{P}^1 \times Q) \to \mathrm{Pic}(\infty \times Q)$$

in which the first arrow is an isomorphism and the last may be identified with a projection $\mathbb{Z}^3 \to \mathbb{Z}^2$. The map $\mathbb{Z} \cong A_{2,\overline{0}}(X) \to A_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ sends 1 to $(1,1)$. Theorem 4.2 yields a pairing (in the notation of (2.5.1))

$$A_{2,\overline{0}}(X) \times A_1(X) \to A_0(X) \cong \mathbb{Z}, \quad (D, \alpha) \mapsto \deg(\mathcal{O}(D)|_\alpha).$$

The same assignment determines a pairing $A_{2,\overline{0}}(X) \times A_{1,\overline{0}}(X) \to A_0(X) \cong \mathbb{Z}$; we proceed to calculate the group $A_{1,\overline{0}}(X)$ by a similar procedure. The birational transform identifies $A_{1,\overline{0}}(X) = A_{1,\overline{1}}(X)$ with 1-cycles on $\mathbb{P}^1 \times Q$ disjoint from $\infty \times Q$, modulo rational equivalences avoiding $\infty \times Q$. To calculate this group, note that an integral 1-cycle C disjoint from $\infty \times Q$ must be contained in $p \times Q$ for some $p \neq \infty \in \mathbb{P}^1$. Such 1-cycles C, C' (contained in $p \times Q, p' \times Q$ respectively) are rationally equivalent on $\mathbb{P}^1 \times Q$ if and only if they are rationally equivalent avoiding $\infty \times Q$. Since $C \hookrightarrow p \times Q$ can be moved (avoiding $\infty \times Q$) to $0 \times Q$, say, we find $A_{1,\overline{0}}(X) \cong A_1(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The map $\mathbb{Z} \oplus \mathbb{Z} \cong A_{1,\overline{0}}(X) = A_{1,\overline{1}}(X) \to A_{1,\overline{2}}(X) = A_1(X) \cong \mathbb{Z}$ sends both $(1,0)$ and $(0,1)$ to 1. The pairing $A_2(X) \times A_{1,\overline{0}}(X) \to A_0(X) \cong \mathbb{Z}$ may be thought of as sending (D,C) to the degree of $\mathcal{O}(D \cap X^{sm})|_C$ since the Weil divisor D is Cartier along $C \hookrightarrow X^{sm}$.

There are also pairings between divisors. Intersection with a Cartier divisor determines pairings $A_{2,\overline{0}}(X) \times A_{2,\overline{0}}(X) \to A_{1,\overline{0}}(X)$ and $A_{2,\overline{0}}(X) \times A_{2,\overline{1}}(X) \to A_{1,\overline{1}}(X)$. Finally, there is a pairing $A_{2,\overline{1}}(X) \times A_{2,\overline{1}}(X) \to A_{1,\overline{2}}(X)$ which is the closure of the intersection product formed in the smooth locus, given in coordinates by $(a,b),(c,d) \mapsto ad + bc$.

Example 4.6. More generally, if Y is the cone on a smooth projective variety X of dimension $d-1$, given the stratification $v = Y^d = \cdots = Y^1$, we have the following computation of the intersection Chow groups and product of Theorem 4.2. We write $\overline{p}$ for any perversity with $p_d = p$. There are two types of groups:

- $A_{r,\overline{p}}(Y) = A_r(Y) \cong A_{r-1}(X)$ (for $r > 0$ and $r-d+p \geq 0$, so that incidence with the vertex is allowed), and
- $A_{r,\overline{p}}(Y) = A_{r,\overline{0}}(Y) \cong A_r(X)$ (for $r \geq 0$ and $r-d+p < 0$, so that incidence with the vertex is disallowed).

There are three kinds of pairings:

- $A_{r,\overline{p}}(Y) \times A_{s,\overline{q}}(Y) \to A_{r+s-d,\overline{p}+\overline{q}}(Y)$, with $p \geq d-r$ and $q \geq d-s$, provided $r+s-d \geq 1$; via the identification above this product is given by the intersection product on X:
$$A_{r-1}(X) \times A_{s-1}(X) \xrightarrow{\bullet_X} A_{r-1+s-1-(d-1)}(X).$$
- $A_{r,\overline{p}}(Y) \times A_{s,\overline{q}}(Y) \to A_{r+s-d,\overline{p}+\overline{q}}(Y)$, with $p < d-r$ and $q \geq d-s$; this is given by
$$A_r(X) \times A_{s-1}(X) \xrightarrow{\bullet_X} A_{r+s-1-(d-1)}(X) \cong A_{r+s-d,\overline{0}}(Y)$$
followed by the canonical morphism $A_{r+s-d,\overline{0}}(Y) \to A_{r+s-d,\overline{p}+\overline{q}}(Y)$.
- $A_{r,\overline{p}}(Y) \times A_{s,\overline{q}}(Y) \to A_{r+s-d,\overline{p}+\overline{q}}(Y)$, with $p < d-r$ and $q < d-s$; this is given by
$$A_r(X) \times A_s(X) \xrightarrow{\bullet_X} A_{r+s-(d-1)}(X) \cong A_{r+s-d+1,\overline{0}}(Y)$$
followed by intersecting with the Cartier divisor $X \hookrightarrow Y$, which maps $A_{r+s-d+1,\overline{0}}(Y)$ to $A_{r+s-d,\overline{0}}(Y)$.

  This is an instance of the following well-known general principle. If $i : X \hookrightarrow Y$ is a Cartier divisor, and $a, b \in A_*(X)$, then $i_*(a) \cdot i_*(b) = i_*(a \cdot b) \cdot X$ in $A_*(Y)$ provided both sides are defined. This identity follows from the projection formula, the associativity of the intersection product, and the self-intersection formula [**?**, Cor. 6.3].

sec:perverse

5. Generalized cocycles

In this section, X will denote a quasi-projective variety of pure dimension d and Y will denote a quasi-projective variety of pure dimension n. We assume that X is equipped with a stratification such that the singular locus X_{sing} of X is contained in X^1. In Definition 5.1, we define the cdh-sheaf on Sm/k of codimension t cocycles of perversity $\overline{p}$ on X with values in Y, $z^{t,\overline{p}}(X,Y)$. Following this definition for a general quasi-projective variety Y, we shall often assume that Y is projective so that we can interpret $z^{s,\overline{p}}(X,Y)$ in terms of maps to Chow varieties.

We recall that an effective algebraic t-cocycle on X with values in Y is the cycle $Z_f \hookrightarrow X \times Y$ associated with some morphism $f : X \to \mathcal{C}_{n-t}(Y)$. Part of the motivation for considering such cocycles is that the i-th homotopy group of some formulation of the "space" of t-cocycles on X with values in $\mathbb{P}^t$ modulo $(t-1)$-cocycles on X with values in $\mathbb{P}^{t-1}$ represents $H^{2t-i}(X,\mathbb{Z}(i))$ as in [**?**] (or, in the semi-topological context, $L^tH^{2t-i}(X)$ as in [**?**]). An important feature of cocycle groups is that there are natural cup product pairings on cocycle groups and cap product pairings relating cocycle groups and cycle groups.

We proceed to develop a theory of "generalized cocycles" on a stratified variety X with values in Y. As the name suggests, an effective generalized cocycle is given by weakening the condition that it is the graph of some morphism; instead, in the case Y is projective, we require that it be the graph of some rational map $f : X \dashrightarrow \mathcal{C}_{n-t}(Y)$.

One should view generalized cocycles on X as cycles (on $X \times Y$ for some Y) which are generically equidimensional over X (i.e., generically satisfy the cocycle condition) and whose failure to be equidimensional over strata of X is governed by a perversity $\overline{p}$. Thus, there is an additional constraint on a generalized cocycle of a given perversity $\overline{p}$ to be a generalized cocycle of some perversity $\overline{q} < \overline{p}$, with usual cocycles satisfying the full equidimensionality condition. The cap product pairing of Section 6 will show that a generalized cocycle of perversity $\overline{p}$ taken together with a cycle of perversity $\overline{q}$ will essentially satisfy the condition $(*,\overline{c})$ with $\overline{c} = \overline{p}+\overline{q}$. As the perversity condition $\overline{p}$ of the generalized cocycle is weakened (i.e., as $\overline{p}$ increases), such a weakened generalized cocycle pairs with the perversity $\overline{q}$ cycles satisfying a stronger perversity condition (i.e., $\overline{q}$ decreases).

One formal difference between cycle theories and cocycle theories is that one should not expect localization in the contravariant variable X. Thus, the proof of the suspension theorem for generalized cocycle spaces does not proceed by first considering X projective and then using localization. Instead, one assumes that the covariant variable Y is projective and observes that the constructions of algebraic homotopies as in Section 3 can be employed on Chow varieties of Y.

If X is stratified, then $X \times Y$ inherits a stratification from that of X, with $(X \times Y)^i \equiv X^i \times Y$. We define the group of perversity $\overline{p}$ cocycles on X with values in Y,

$$Z^{t,\overline{p}}(X,Y) \ \subseteq \ Z_{d+n-t,\overline{p}}(X \times Y),$$

to be the group of $(d+n-t)$-dimensional cycles α on $X \times Y$ with the property that for $x \in X^i - X^{i+1}$, the dimension of $|\alpha| \cap |x \times Y|$ is no larger than $n-t+p_i$ (for $i = 1,\dots,d$), and for $x \in X - X^1$ the dimension of $|\alpha| \cap |x \times Y|$ is $n-t$. Because this condition is a constraint on the support $|\alpha|$ of α, this does not permit "large" fibers to cancel. Roughly speaking, a cycle lies in $Z_{d+n-t,\overline{p}}(X \times Y)$ if its excess with each stratum $X^i \times Y$ is not too large; it lies in the smaller group $Z^{t,\overline{p}}(X,Y)$ if in addition this excess is distributed evenly over each stratum $X^i - X^{i+1}$.

def:cocycle **Definition 5.1.** Let $\overline{p}$ be a perversity, and let t be an integer $0 \le t \le n$. We define

$$z^{t,\overline{p}}(X,Y) \ \subseteq \ z(X \times Y, d+n-t)_{\overline{p}} \ \subseteq \ z(X \times Y, d+n-t)$$

to be the subpresheaf (on Sch/k) whose value on U consists of U-relative cycles with $\mathbb{Z}$-coefficients $W \hookrightarrow U \times X \times Y$ such that for all $u \in U$, the specialization $W_u \in Z_{d+n-t}(X_u \times Y)$ belongs to $Z^{t,\overline{p}}(X_u, Y)$. We define the subpresheaves $z^{t,\overline{p}}_{equi}(X,Y) \subseteq z_{equi}(X \times Y, d+n-t)_{\overline{p}}$ similarly.

We denote by $z^{t,\overline{p}}(X,Y)[1/p]$ the presheaf $z^{t,\overline{p}}(X,Y) \otimes \mathbb{Z}[1/p]$, and extend this abbreviated notation to other presheaves.

We define the bivariant perversity $\overline{p}$ motivic cohomology group of bidegree (i,t) to be the group

$$H^{i,t,\overline{p}}(X,Y) \equiv \pi_{2t-i}(z^{t,\overline{p}}(X,Y)(\bullet)).$$

These groups are contravariantly functorial with respect to flat, stratified morphisms $f : X' \to X$, and covariantly functorial with respect to proper morphisms $g : Y \to Y'$: we have $f^* : H^{i,t,\overline{p}}(X,Y) \to H^{i,t,\overline{p}}(X',Y)$ and $g_* : H^{i,t,\overline{p}}(X,Y) \to H^{i+2r,r+t,\overline{p}}(X,Y')$, where $r = \dim(Y') - \dim(Y)$.

hyperplane **Lemma 5.2.** *Let X be a stratified quasi-projective variety and let $\overline{p}$ be a perversity. The homotopy class of the map*

$$i_\ell : z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet) \ \to \ z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)$$

induced by the embedding $\ell : \mathbb{P}^{t-1} \hookrightarrow \mathbb{P}^t$ of a hyperplane is independent of the choice of hyperplane ℓ (i.e., independent of the choice of linear embedding).

Similarly, the homotopy class of the quotient map

$$p_\ell : z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet) \ \to z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet)$$

is independent of the choice of hyperplane ℓ.

Proof. Let $\ell, \ell' : \mathbb{P}^{t-1} \to \mathbb{P}^t$ be two linear embeddings and let $\theta \in PGL_{t+1}$ satisfy the condition that $\theta \circ \ell = \ell'$. Choose a map $f : \mathbb{A}^1 \to PGL_{t+1}$ with $f(0) = \mathrm{id}$, $f(1) = \theta$. The action of PGL_{t+1} on $\mathbb{P}^t$ and the morphism f determine a morphism $\mathbb{A}^1 \times \mathbb{P}^t \to \mathbb{P}^t$. Pulling back along this morphism determines a morphism of sheaves

$$\Theta : z^{t,\overline{p}}(X,\mathbb{P}^t)(-) \ \to \ z^{t,\overline{p}}(X,\mathbb{P}^t)(- \times \mathbb{A}^1)$$

such that the composition

$$\Theta \circ i_\ell : z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(-) \ \to \ z^{t,\overline{p}}(X,\mathbb{P}^t)(- \times \mathbb{A}^1)$$

is a homotopy relating i_ℓ (restriction to $(- \times \{0\})$) and $i_{\ell'}$ (restriction to $(- \times \{1\})$). By [**?**, Lem 4.1], this implies the first assertion.

To prove the second observation, observe that we have a commutative square

theta-commute (5.2.1)

$$\begin{tikzcd}
z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet) \arrow[r, "p_\ell"] \arrow[d, "\theta"'] & z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet) \arrow[d, "\overline{\theta}"'] \\
z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet) \arrow[r, "p_{\ell'}"'] & z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet).
\end{tikzcd}$$

Here, $\overline{\theta}$ is the map on quotients induced by θ; both θ, $\overline{\theta}$ are isomorphisms. Since θ is homotopic to the identity, we conclude that p_ℓ, $p_{\ell'}$ are homotopic. □

defn:pervcoh **Definition 5.3.** We define the simplicial abelian group

$$z^{t,\overline{p}}(X)(\bullet) := z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet);$$

this is canonical by Lemma 5.2. The perversity $\overline{p}$ motivic cohomology groups are then defined to be its homotopy groups:

$$H^{i,\overline{p}}(X,\mathbb{Z}(t)) \equiv \pi_{2t-i}(z^{t,\overline{p}}(X)(\bullet)).$$

Remark 5.4. Voevodsky localization as in Remark 1.1 implies that there are canonical homotopy equivalences

$$(z^{t,\overline{p}}(X,\mathbb{P}^t)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1}))_{cdh}(\bullet) \to z^{t,\overline{p}}(X)(\bullet), \quad \mathrm{char}(k)=0,$$

$$(z^{t,\overline{p}}(X,\mathbb{P}^t)[1/p]/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})[1/p])_{cdh}(\bullet) \to z^{t,\overline{p}}(X)[1/p](\bullet), \quad \mathrm{char}(k)=p$$

giving an interpretation of $H^{i,t,\overline{p}}(X)$ as the Suslin homology of the quotient cdh-sheaf.

FV compare

Remark 5.5. If X is smooth, k admits resolutions of singularities, and $\overline{p}$ is the zero perversity, then we recover the motivic cohomology groups of Friedlander-Voevodsky: $H^{i,t,\overline{0}}(X) = H^i(X,\mathbb{Z}(t))$. This follows from [**?**, Prop. 6.4, Thm. 8.1, Thm. 8.2]. One reason this comparison is likely to fail for singular X is that the zero perversity condition on a cycle does not imply it has well-defined specializations (let alone with universally integral coefficients), whereas the groups $H^i(X,\mathbb{Z}(t))$ are defined using cycles which have well-defined specializations for all $x \in X$. If X is smooth, then the zero perversity condition on a cycle (i.e., the condition used to define $H^{i,\overline{0}}(X,\mathbb{Z}(t))$) implies it has well-defined specializations for all $x \in X$ by [**?**, Cor. 3.4.5].

The following proposition relates generalized cocycles to Chow varieties when the covariant variable is projective.

cocycle basic

Proposition 5.6. *Let X be a stratified quasi-projective variety of dimension d and Y, T be projective varieties of dimension n, m respectively. Let $W \hookrightarrow U \times X \times Y$ be an element of $z^{t,\overline{p}}(X,Y)(U)$.*

(1) *For every $u \in U$, every component of the specialization W_u is the closure of the cycle associated to a rational map $f_u : X_u \dashrightarrow \mathcal{C}_{n-t}(Y)$ defined on $(X-X^1)_u$.*

(2) *For any fat point (x_0,x_1,R) over $u \in U$ there is a rational map $\widetilde{f} : X_R \dashrightarrow \mathcal{C}_{n-t}(Y)$ defined on $(X-X^1)_R$ such that the compositions (set $K := \mathrm{Frac}\, R$):*

$$X_k \xrightarrow{\mathrm{id}_X \times x_0} X_R \overset{\widetilde{f}}{\dashrightarrow} \mathcal{C}_{n-t}(Y) \quad \text{and} \quad X_K \to X_R \overset{\widetilde{f}}{\dashrightarrow} \mathcal{C}_{n-t}(Y)$$

coincide with

$$X_k \to X_u \overset{f_u}{\dashrightarrow} \mathcal{C}_{n-t}(Y) \quad \text{and} \quad X_K \to X_{\eta_U} \overset{f_{\eta_U}}{\dashrightarrow} \mathcal{C}_{n-t}(Y).$$

(3) *For any continuous algebraic map $g : \mathcal{C}_{n-t}(Y) \to \mathcal{C}_{m-s}(T)$, the closure of the cycle associated to $g \circ f_{\eta_U} : X_{\eta_U} \dashrightarrow \mathcal{C}_{m-s}(T)$, denoted W_g, is an element of $z^{s,\overline{p}}(X,T)(U)$.*

(4) *For any continuous algebraic map $h : \mathcal{C}_{n-t}(Y) \times \mathbb{A}^1 \to \mathcal{C}_{m-s}(T)$, the closure of the cycle associated to $h \circ (f_{\eta_U} \times \mathrm{id}_{\mathbb{A}^1}) : X_{\eta_U} \times \mathbb{A}^1 \dashrightarrow \mathcal{C}_{m-s}(T)$, denoted $(W_{\mathbb{A}^1})_h$, is an element of $z^{t,\overline{p}}(X,T)(U \times \mathbb{A}^1)$. The formation of $(W_{\mathbb{A}^1})_h$ is compatible with restriction to $t \in \mathbb{A}^1$ in the sense that the image of $(W_{\mathbb{A}^1})_h$ in $z^{t,\overline{p}}(X,T)(U \times \{t\})$ coincides with W_{h_t}.*

Proof. Let $W'_u \hookrightarrow X_u \times Y$ denote a component of the specialization of W at some $u \in U$. Since $X - X^1$ is smooth, the restriction $W'_u|_{(X-X^1)_u}$ is an element of $z_{equi}(Y, n-t)((X-X^1)_u)$ [**?**, Cor. 3.4.5], and there is a canonical inclusion $z_{equi}(Y, n-t)((X-X^1)_u) \subseteq \mathrm{Hom}((X-X^1)_u, \mathcal{C}_{n-t}(Y))^+$. This establishes (1).

The perversity condition implies that for any $u \in U$, all of the generic points of W_u lie in $(X-X^1)_u \times Y$, so to verify (2) we may restrict to $X - X^1$, where all of the rational maps are defined. Since Y is projective the pullbacks on $z_{equi}(Y, n-t)$ correspond to composition of morphisms to Chow varieties.

Now we show W_g has well-defined specializations. The specializations are determined by the generic points of the cycle $(W_g)_\eta \hookrightarrow X_\eta \times Y$, where η denotes the union $\cup\eta_U$ of the generic points of U. But both W_η and $(W_g)_\eta$ have their generic points in $(X-X^1)_\eta \times Y$, so we may restrict to $X - X^1$. Since specialization corresponds to restriction of morphisms to Chow varieties, the specializations of $(W_g)_\eta$ are determined by those of W_η. Since the latter do not depend on the fat point, the former are independent as well.

To verify (3), it remains to show the perversity condition is preserved. We may assume U is the spectrum of a field. Let $X' \hookrightarrow X \times \mathcal{C}_{n-t}(Y)$ be the graph of the rational map, and let $\pi : X' \to X$, $c : X' \to \mathcal{C}_{n-t}(Y)$ denote the induced morphisms. For any $x \in X$ we have the following formulas for the dimensions of the fibers W_x, $(W_g)_x$:

$$\dim(W_x) = (n-t) + \dim(\mathrm{im}(c : \pi^{-1}(x) \to \mathcal{C}_{n-t}(Y)))$$
$$\dim((W_g)_x) = (m-s) + \dim(\mathrm{im}(g \circ c : \pi^{-1}(x) \to \mathcal{C}_{m-s}(T)))$$

Clearly $\dim(\mathrm{im}(g \circ c : \pi^{-1}(x) \to \mathcal{C}_{m-s}(T))) \leq \dim(\mathrm{im}(c : \pi^{-1}(x) \to \mathcal{C}_{n-t}(Y)))$, so the perversity of W_g is no worse than that of W. The verification of (4) is similar and we omit the details. □

We denote by $z^{t,\overline{p}}(X, \Sigma(Y))_Y \subset z^{t,\overline{p}}(X, \Sigma(Y))$ the subpresheaf consisting of U-relative cycles W none of whose specializations $W_u \hookrightarrow X_u \times \Sigma(Y)$ have components contained in the Cartier divisor $X_u \times Y \hookrightarrow X_u \times \Sigma(Y)$, and satisfy the property that $W_u \cap (X_u \times Y)$ belongs to $Z^{t,\overline{p}}(X_u, Y)$.

In the proof of the following theorem, we employ the same moving constructions which we used in the proof of Theorem 3.1.

thm:suspgen

Theorem 5.7. *Let X be a stratified quasi-projective variety, let Y be a projective variety, and let $\overline{p}$ be a perversity. Equip $\Sigma(X)$ with the stratification $\{\Sigma(X^i)\}$, where $\{X^i\}$ is the given stratification of X. Fiberwise suspension induces homotopy equivalences*

$$\Sigma_Y : z^{t,\overline{p}}(X, Y)(\bullet) \ \overset{\sim}{\to}\ z^{t,\overline{p}}(X, \Sigma(Y))(\bullet).$$

Therefore we have an induced isomorphism $H^{i,t,\overline{p}}(X, Y) \cong H^{i,t,\overline{p}}(X, \Sigma(Y))$.

Proof. The overall strategy is similar to that employed in the proof of Theorem 3.1: deformation to the normal cone and the projecting cones construction provide $\mathbb{A}^1$-homotopies and allow us to conclude that each of the morphisms:

gen cocycle factor

$$z^{t,\overline{p}}(X, Y)(\bullet) \xrightarrow{\Sigma_Y} z^{t,\overline{p}}(X, \Sigma(Y))_Y(\bullet) \to z^{t,\overline{p}}(X, \Sigma(Y))(\bullet) \tag{5.7.1}$$

is a homotopy equivalence. We explain why the constructions given in the proofs of Propositions 3.3 and 3.4 suffice, and we do not repeat the arguments which require only modification of notation.

The deformation to the normal cone of Proposition 3.3 defines a continuous algebraic map $\varphi : \mathcal{C}_{n-t}(\Sigma(Y))_Y \times \mathbb{A}^1 \to \mathcal{C}_{n-t}(\Sigma(Y))_Y$. By Proposition 5.6(4), this provides a morphism:

$$\varphi : z^{t,\overline{p}}(X, \Sigma(Y))_Y(-) \to z^{t,\overline{p}}(X, \Sigma(Y))_Y(- \times \mathbb{A}^1).$$

Let φ_t denote the composition of φ with restriction to $(- \times \{t\})$. We must show:

- φ_0 is the identity,
- φ_1 has image contained in $\Sigma_Y(z^{t,\overline{p}}(X,Y))$, and
- φ_t acts as the identity on $\Sigma_Y(z^{t,\overline{p}}(X,Y))$ for all $t \in \mathbb{A}^1$.

The morphism φ_0 is induced by the identity on the Chow variety, and $W = W_{\mathrm{id}}$, so the first property is clear. The third property follows for a similar reason.

To see that the second property holds, note that any specialization $(W_{\varphi_1})_u$ is associated to the rational map $X_u \dashrightarrow \mathcal{C}_{n-t+1}(\Sigma(Y)) \xrightarrow{\varphi_1} \Sigma_Y(\mathcal{C}_{n-t}(Y)) \hookrightarrow \mathcal{C}_{n-t+1}(\Sigma(Y))$. Therefore $(W_{\varphi_1})_u|_{X-X^1}$ is a suspension, and the closure of a suspension is a suspension (namely, it is the suspension of the closure!). Alternatively, the fiber of $(W_{\varphi_1})_u$ over $x \in X$ is the image of $(W_{\varphi_1 \circ c})_u \cap (\pi^{-1}(x) \times Y) \to x \times Y$, and $\varphi_1 \circ c : X' \to \mathcal{C}_{n-t+1}(\Sigma(Y))$ factors through $\Sigma_Y(\mathcal{C}_{n-t}(Y))$, so all of the fiber cycles of $W_{\varphi_1 \circ c} \to X'$ are suspensions. The image is therefore a suspension as well. This proves the generalized cocycles analogue of Proposition 3.3 and establishes that the first arrow in 5.7.1 is a homotopy equivalence.

We proceed to analyze the second arrow in 5.7.1. The projecting cones are slightly more delicate for the simple reason that $\mathcal{C}_{n-t+1}(\Sigma(Y))_Y \subset \mathcal{C}_{n-t+1}(\Sigma(Y))$ is open rather than closed, so that we cannot conclude that X lands in $\mathcal{C}_{n-t+1}(\Sigma(Y))_Y$ simply because $X - X^1$ does. The construction of Proposition 3.4 provides a morphism:

$$\psi := \psi_e : z^{t,\overline{p}}(X, \Sigma(Y), \leq d)(-) \to z^{t,\overline{p}}(X, \Sigma(Y), \leq de)(- \times \mathbb{A}^1)$$

where d bounds the degree of the cycles on Y and e depends on d. We must show:

- ψ_0 is e times the identity,
- ψ_t carries $z^{t,\overline{p}}(X, \Sigma(Y), \leq d)$ into $z^{t,\overline{p}}(X, \Sigma(Y), \leq de)_Y$ for general $t \in \mathbb{A}^1$.

We have a morphism $\psi : \mathcal{C}_{n-t+1,\leq d}(\Sigma(Y)) \times \mathbb{A}^1 \to \mathcal{C}_{n-t+1,\leq de}(\Sigma(Y))$ which restricts to a closed immersion (namely, e times the identity) at $t = 0$. Therefore there is an open subscheme $S \subset \mathbb{A}^1$ such that ψ_t is a closed immersion for $t \in S$ by [**?**, Lemma I.1.10.1]. We may assume $1 \in S$, and then given $W \in z^{t,\overline{p}}(X, \Sigma(Y))(U)$, our task is to show $W_{\psi_1} \in z^{t,\overline{p}}(X, \Sigma(Y))_Y(U)$.

Since ψ_1 is a closed immersion, the graph of $X \dashrightarrow \mathcal{C}_{n-t+1,\leq d}(\Sigma(Y)) \xrightarrow{\psi_1} \mathcal{C}_{n-t+1,\leq de}(\Sigma(Y))_Y \subset \mathcal{C}_{n-t+1,\leq de}(\Sigma(Y))$ is isomorphic to the graph $X' \hookrightarrow X \times \mathcal{C}_{n-t+1,\leq d}(\Sigma(Y))$. This implies all of the specializations of the cycle $W_{\psi_1} \hookrightarrow U \times X \times \Sigma(Y)$ are covered by (birational, proper) surjections $(W_{\psi_1 \circ c})_u \to (W_{\psi_1})_u$. The support of $(W_{\psi_1 \circ c})_u$ over some $x' \in X'$ is the cycle $\psi_1(c(x'))$, and none of these $(n-t+1)$-dimensional cycles are contained in $Y \hookrightarrow \Sigma(Y)$. Therefore, the cycle $(W_{\psi_1 \circ c})_u \cap (\pi^{-1}(x)_u \times \Sigma(Y))$ is not contained in $X_u \times Y \hookrightarrow X_u \times \Sigma(Y)$. □

We will need the following particular case of the proper push-forward morphism. If X is a stratified variety and $i : Y \hookrightarrow Y'$ is a closed immersion of pure codimension c, then the push-forward along i determines a morphism of presheaves $z^{t,\overline{p}}(X,Y) \to z^{t+c,\overline{p}}(X,Y')$. In particular, the inclusion of a hyperplane $i : \mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^s$ induces a morphism $i_* : z^{s-1,\overline{p}}(X, \mathbb{P}^{s-1}) \to z^{s,\overline{p}}(X, \mathbb{P}^s)$ of presheaves on Sch/k. The existence of i_* follows from the existence of proper push-forward functors on the presheaves $z(X,r)$ and $z_{equi}(X,r)$ [**?**, Cor. 3.6.3]. Alternatively, i_* is the morphism provided by Proposition 5.6(3) for the continuous algebraic map $\mathcal{C}_0(\mathbb{P}^{s-1}) \to \mathcal{C}_0(\mathbb{P}^s)$.

lem:com

Lemma 5.8. *Let X be a stratified quasi-projective variety, and let $\overline{p}$ be a perversity. The following square is homotopy commutative:*

sigma-commute

$$\begin{tikzcd} z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet) \arrow[r] \arrow[d, "\Sigma^i"'] & z^{t,\overline{p}}(X,\mathbb{P}^{t})(\bullet) \arrow[d, "\Sigma^i"'] \\ z^{t-1,\overline{p}}(X,\mathbb{P}^{t+i-1})(\bullet) \arrow[r] & z^{t,\overline{p}}(X,\mathbb{P}^{t+i})(\bullet). \end{tikzcd} \tag{5.8.1}$$

Proof. The two compositions of the square (5.8.1) are given by first embedding $\mathbb{P}^{t-1}$ in $\mathbb{P}^t$, then suspending i-times; and by first suspending i-times, then embedding $\mathbb{P}^{t+i-1}$ in $\mathbb{P}^{t+i}$. These are readily seen to be related by an $\mathbb{A}^1$-family of automorphisms of $\mathbb{P}^{t+i}$, and the required homotopy is obtained by composing with these automorphisms. $\square$

iterated suspension

Theorem 5.9. *Let X be a stratified quasi-projective variety, and let $\overline{p}$ be a perversity. The fiberwise suspension map (with respect to $\mathbb{P}^t$) induces a homotopy equivalence*

$$z^{t,\overline{p}}(X)(\bullet) \ \xrightarrow{\sim}\ z^{t,\overline{p}}(X,\mathbb{P}^{t+i})(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t+i-1})(\bullet).$$

Proof. This follows from Theorem 5.7, by applying the 5-Lemma to the map of short exact sequences (arising from Definition 5.3) of the form

$$0 \to z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet) \to z^{t,\overline{p}}(X,\mathbb{P}^{t})(\bullet) \to z^{t,\overline{p}}(X)(\bullet) \to 0$$

determined by Lemma 5.8. $\square$

Two natural sources of cocycles are flat morphisms and vector bundles. Here we explain how arbitrary morphisms and coherent sheaves give rise to generalized cocycles (for a stratification and perversity determined by the morphism and sheaf respectively).

Morphisms. Let X and Y be quasi-projective k-varieties. If $f : Y \to X$ is a dominant flat morphism, then taking the cycle associated to the scheme-theoretic fiber $f^{-1}(x)$ determines an effective d-cocycle on X with values in Y. As we see in the following example, general morphisms provide examples of generalized cocycles.

Example 5.10. With the notation as above, we define

$$\mathbb{Z}\operatorname{Hom}^{\overline{p}}(-\times Y, X) \ \subset\ z^{d,\overline{p}}(X,Y)(-)$$

to be the subsheaf whose value on U is the free abelian group on the morphisms $f : U \times Y \to X$ with the property that the induced map $f_u : Y_u \to X_u$ is dominant and the transpose of the graph $\Gamma^t_{f_u} \subset X_u \times Y_u$ lies in $Z^{d,\overline{p}}(X_u, Y_u)$ (for all $u \in U$).

Our next proposition shows how $\mathbb{Z}\operatorname{Hom}^{\overline{p}}(Y,X)$ acts on generalized cocycles.

morphisms act

Proposition 5.11. *Let X, Y be projective varieties, let W be a quasi-projective variety, suppose X is stratified, and let $\overline{p}$ be a perversity. Then there is a natural pairing given by proper push-forward*

$$\mathbb{Z}\operatorname{Hom}^{\overline{p}}(Y,X) \times z^{t,\overline{0}}(Y,W) \to z^{d+t-n,\overline{p}}(X,W).$$

Proof. It suffices to define the pairing for a pair $(f, \beta) \in \mathrm{Hom}^{\overline{p}}(Y, X)(U) \times z^{t,\overline{0}}(Y, W)(U)$ consisting of a morphism $f : U \times Y \to X$ and a cycle $\beta \hookrightarrow U \times Y \times W$ with specializations β_u equidimensional over Y_u. Now f induces a proper morphism $f : U \times Y \times W \to U \times X \times W$, and we claim $f_*(\beta)$ belongs to $z^{t,\overline{p}}(X, W)(U)$. Set $w = \dim(W)$. By hypothesis, for any $(u, y) \in U \times Y$, we have $\dim(|\beta_u|_y) = w - t$. Therefore, for any $(u, x) \in U \times X$, we have $\dim(|f_*(\beta)_u|_x) \leq \dim(f^{-1}(x)) + w - t$. By assumption, $x \in X^i - X^{i+1}$ implies $\dim(f^{-1}(x)) \leq (n - d) + p_i$, and the claim follows. The formation of $f_*(\beta)$ is functorial in U, so the pairing defines a natural transformation. □

As mentioned in the introduction, cycle classes on a resolution determine generalized cocycles on the variety being resolved. We say a morphism $f : Y \to X$ determines a stratification S and perversity $\overline{p}$ if f does not belong to $\mathrm{Hom}^{\overline{q}}(Y, X)$ for any stricter incidence datum $(T, \overline{q})$, with T a stratification.

Proposition 5.12. *If $f : Y \to X$ is a resolution of singularities, push-forward along f defines a morphism $H^{BM}_{2n-i}(Y, \mathbb{Z}(n-t)) \to H^{i,\overline{p}}(X, \mathbb{Z}(t))$ for the stratification and perversity determined by the resolution (and hence for any less strict incidence datum).*

Proof. We have a push-forward $f_* : H^{i,\overline{0}}(Y, \mathbb{Z}(t)) \to H^{i,\overline{p}}(X, \mathbb{Z}(t))$ by Proposition 5.11, an identification $H^{i,\overline{0}}(Y, \mathbb{Z}(t)) \cong H^i(Y, \mathbb{Z}(t))$ by Remark 5.5, and Friedlander-Voevodsky duality $H^i(Y, \mathbb{Z}(t)) \cong H^{BM}_{2n-i}(Y, \mathbb{Z}(n - t))$ [**?**, Thms. 8.2, 8.3(1)]. □

Coherent sheaves. Suppose $\mathcal{F}$ is a globally generated coherent sheaf on X with generic rank r. There is an exact sequence of sheaves on X:

$$0 \to \mathcal{K} \to H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F} \to 0.$$

If $U \subset X$ is the locus over which $\mathcal{F}$ is locally free, then the projectivization of the locally free sheaf $\mathcal{K}|_U$ may be viewed as an element of $Z^r(U, \mathbb{P}^n)$ with $n = h^0(X, \mathcal{F}) - 1$.

We shall show in Proposition 5.13 below that the closure in $X \times \mathbb{P}^n$ of this $\mathbb{P}^{r-1}$-bundle over U, denoted $\mathbb{P}(\mathcal{K})$, is an element of $Z^{r,\overline{p}}(X, \mathbb{P}^n)$ for a stratification and perversity which may be expressed in terms of $\mathcal{F}$ itself. Namely, stratify X according to the rank-jumping behavior of $\mathcal{F}$. Then there exists a sequence of integers $p_1, \ldots, p_d$ such that $x \in X^i$ if and only if $\mathrm{rk}(\mathcal{F}|_x) \geq r+p_i$ and $x \in X^i - X^{i+1}$ if and only if $\mathrm{rk}(\mathcal{F}|_x) \leq r + p_i$. We say this stratification and perversity are determined by $\mathcal{F}$.

coh1 **Proposition 5.13.** *Let $\mathcal{F}$ be a globally generated coherent sheaf on X with generic rank r, and set $n = h^0(X, \mathcal{F}) - 1$. Then $\mathbb{P}(\mathcal{K}) \in Z^{r,\overline{p}}(X, \mathbb{P}^n)$ for the stratification and perversity determined by $\mathcal{F}$.*

Proof. Let $\mathbb{P}(\mathcal{F}) \hookrightarrow X \times \mathbb{P}^n$ denote the closure in $X \times \mathbb{P}^n$ of the $\mathbb{P}^{r-1}$-bundle over U classified by the surjection $H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_U \to \mathcal{F}|_U$; the $\mathbb{P}^n$ which appears here is dual to the one which houses $\mathbb{P}(\mathcal{K})$. Then the fiber of $\mathbb{P}(\mathcal{F})$ over $x \in X$ is contained in the projectivization of the vector space $\mathcal{F}|_x$, in fact $\mathbb{P}(\mathcal{F})$ is the main component of the (possibly reducible) projectivization of $\mathcal{F}$, hence the perversity of $\mathbb{P}(\mathcal{F})$ is controlled by the rank-jumping behavior of $\mathcal{F}$.

To prove the lemma, then, it suffices to show the perversity of $\mathbb{P}(\mathcal{K})$ is identical to that of $\mathbb{P}(\mathcal{F})$. Let $X' \hookrightarrow X \times \mathbb{G}(n - r, n)$ denote the graph of the rational map

$X \dashrightarrow \mathbb{G}(n-r,n)$ determined by $\mathcal{K}|_U$, and let $X'' \hookrightarrow X \times \mathbb{G}(r-1,n)$ denote the graph of the map determined by $\mathcal{F}|_U$.

Note that $\mathbb{P}(\mathcal{K})$ is the push-forward via $X' \to X$ of the codimension r cocycle on X' with values in $\mathbb{P}^n$ classified by the morphism $X' \to \mathbb{G}(n-r,n)$, and similarly $\mathbb{P}(\mathcal{F})$ is the push-forward via $X'' \to X$ of the cocycle determined by $X'' \to \mathbb{G}(r-1,n)$. Furthermore $X' \cong X''$ via the isomorphism $\mathbb{G}(n-r,n) \cong \mathbb{G}(r-1,n)$.

Let $F'_x \hookrightarrow X', F''_x \hookrightarrow X''$ denote the fibers over $x \in X$. The dimension of the fiber of $\mathbb{P}(\mathcal{K})$ over $x \in X$ is equal to the dimension of the image of the morphism $F'_x \to X' \to \mathbb{G}(n-r,n)$ plus $n-r$. Similarly the dimension of the fiber of $\mathbb{P}(\mathcal{F})$ over $x \in X$ is equal to the dimension of the image of $F''_x \to X'' \to \mathbb{G}(r-1,n)$ plus $r-1$. By the previous paragraph, $F'_x \cong F''_x$ compatibly with the isomorphisms of Grassmannians, hence the perversities agree. □

We denote by $z^{r,\overline{p}}(X,\mathbb{P}^\infty)(\bullet)$ the simplicial abelian group $\operatorname{colim}_n z^{r,\overline{p}}(X,\mathbb{P}^n)(\bullet)$. Note that the transition maps in the colimit are the suspension weak equivalences $\Sigma_{\mathbb{P}^n} : z^{r,\overline{p}}(X,\mathbb{P}^n)(\bullet) \to z^{r,\overline{p}}(X,\mathbb{P}^{n+1})(\bullet)$.

Proposition 5.14. *The class of $\mathbb{P}(\mathcal{K})$ in $\pi_0(z^{r,\overline{p}}(X,\mathbb{P}^\infty)(\bullet))$ is independent of the choice of generating sections of $\mathcal{F}$.*

Proof. Suppose given exact sequences

$$0 \to \mathcal{K}_f \to H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_X \xrightarrow{f} \mathcal{F} \to 0,$$

$$0 \to \mathcal{K}_g \to H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_X \xrightarrow{g} \mathcal{F} \to 0.$$

The sections $t \cdot f + (1-t) \cdot g$ determine an exact sequence of coherent sheaves on $X \times \mathbb{A}^1$ (let $p : X \times \mathbb{A}^1 \to X$ denote the projection):

$$0 \to \mathcal{K}_{\mathbb{A}^1} \to H^0(X,\mathcal{F}) \otimes_k \mathcal{O}^2_{X \times \mathbb{A}^1} \to p^*\mathcal{F} \to 0.$$

The perversities of $\mathbb{P}(p^*\mathcal{F})$ and $\mathbb{P}(\mathcal{K}_{\mathbb{A}^1})$ agree by the argument in the previous proposition, and the perversity of $\mathbb{P}(p^*\mathcal{F})$ (for the product stratification) is the same as that of $\mathbb{P}(\mathcal{F})$ itself. Therefore $\mathbb{P}(\mathcal{K}_{\mathbb{A}^1})$ belongs to $z^{r,\overline{p}}(X,\mathbb{P}^{2n+1})(\Delta^1)$. Furthermore $\mathbb{P}(\mathcal{K}_{\mathbb{A}^1})_0 = \Sigma^{n+1}\mathbb{P}(\mathcal{K}_g)$ and $\mathbb{P}(\mathcal{K}_{\mathbb{A}^1})_1 = \Sigma^{n+1}\mathbb{P}(\mathcal{K}_f)$ since any additional components in the fibers at $t = 0,1$ would violate the perversity condition, hence the elements agree in $\pi_0(z^{r,\overline{p}}(X,\mathbb{P}^{2n+1})(\bullet))$. □

sec:cup-cap

6. Cup product and cap product

The geometric operation underlying our cup product is the join. The semi-topological precursor (in the absence of perversities) of our product is the cup product pairing on semi-topological cohomology defined using the fiberwise join [**?**, Thm. 6.1]; building on this, an algebraic version for smooth varieties is developed in [**?**, Prop. 8.6].

Definition 6.1. Let V be a k-scheme. Given $\alpha \hookrightarrow V \times \mathbb{P}^t$ and $\beta \hookrightarrow V \times \mathbb{P}^s$, let $J_V(\alpha,\beta) \hookrightarrow V \times \mathbb{P}^{t+s+1}$ denote their fiberwise join. If α (resp. β) is an integral subscheme whose ideal sheaf is locally generated by $\{f(x,t)\}$ (resp. $\{g(x,s)\}$), then $J_V(\alpha,\beta)$ is the (integral) subscheme with ideal sheaf locally generated by $\{f(x,t); g(x,s)\}$. (Here the x's are coordinates on V, the t's are coordinates on $\mathbb{P}^t$, and the s's are coordinates on $\mathbb{P}^s$.) We define the join of a general pair of cycles α, β by linear extension.

join **Proposition 6.2.** *Let X be a stratified quasi-projective variety. The join defines a morphism of functors on Sch/k*

$$z^{s,\overline{p}}(X,\mathbb{P}^s) \times z^{t,\overline{q}}(X,\mathbb{P}^t) \to z^{s+t,\overline{p}+\overline{q}}(X,\mathbb{P}^{s+t+1})$$

and similarly for the equi-theory.

Proof. We send the pair $(\alpha,\beta) \in z^{s,\overline{p}}(X,\mathbb{P}^s)(U) \times z^{t,\overline{q}}(X,\mathbb{P}^t)(U)$ to the fiberwise join $J := J_{U\times X}(\alpha,\beta) \hookrightarrow U \times X \times \mathbb{P}^{s+t+1}$ described above.

The join defines a continuous algebraic map $\mathcal{C}_0(\mathbb{P}^s) \times \mathcal{C}_0(\mathbb{P}^t) \to \mathcal{C}_1(\mathbb{P}^{s+t+1})$ determined by sending (p,q) to the line connecting $i_s(p)$ and $i_t(q)$, where i_s (resp. i_t) identifies $\mathbb{P}^s$ (resp. $\mathbb{P}^t$) with the "first" $s+1$ (resp. "last" $t+1$) coordinates of $\mathbb{P}^{s+t+1}$ [**?**, (6.1.1)].

The generic points of the join are in one-to-one correspondence with pairs of generic points of the cycles being joined. Since the generic points of α_{η_U} and β_{η_U} lie in $(X - X^1)_{\eta_U}$, the same is true of J_{η_U}. Therefore it suffices to show the restriction of J to $X - X^1$ has well-defined specializations for all $u \in U$. But on $X - X^1$, all of the specializations α_u , β_u are given by morphisms $f_u : (X - X^1)_u \to \mathcal{C}_0(\mathbb{P}^s)$, $g_u : (X - X^1)_u \to \mathcal{C}_0(\mathbb{P}^t)$.

Therefore, on $X - X^1$, every specialization J_u is the cycle determined by the morphism $f_u \# g_u : (X - X^1)_u \to \mathcal{C}_0(\mathbb{P}^s) \times \mathcal{C}_0(\mathbb{P}^t) \to \mathcal{C}_1(\mathbb{P}^{s+t+1})$. The basic compatibility of morphisms to Chow varieties and pullbacks of cycles (as discussed in the proof of Proposition 5.6) implies J has well-defined specializations. From the definition it is clear that the join preserves integrality of the cycle coefficients.

Now we verify J has the required incidence properties, which is a pointwise condition on U. The relative join is compatible with base change [**?**, Remark 1.3.3(2)]. Therefore, if $x \in X$, the support of $J(\alpha_u,\beta_u)_x \hookrightarrow u \times x \times \mathbb{P}^{t+s+1}$ coincides with the support of $J(|\alpha_u|_x, |\beta_u|_x) \hookrightarrow u \times x \times \mathbb{P}^{t+s+1}$. In particular if $\alpha_u \in Z^{s,\overline{p}}(X_u,\mathbb{P}^s)$ and $\beta_u \in Z^{t,\overline{q}}(X_u,\mathbb{P}^t)$, then the dimension of the fiber of J_u over $x \in X^i - X^{i+1}$ is less than or equal to $p_i + q_i + 1$, as desired. □

We remind the reader that a $\mathbb{Z}$-bilinear pairing $A_\bullet \times B_\bullet \to C_\bullet$ of simplicial abelian groups factors as a map of simplicial sets through the smash product of $A_\bullet$ and $B_\bullet$,

$$A_\bullet \times B_\bullet \;\to\; A_\bullet \wedge B_\bullet \;\to\; C_\bullet$$

and thus determines a pairing on homotopy groups

smash (6.2.1) $$\pi_i(A_\bullet) \otimes \pi_j(B_\bullet) \;\to \pi_{i+j}(C_\bullet).$$

Next we relate the "total" groups $z^{s,\overline{p}}(X,\mathbb{P}^s)(\bullet)$ to the "pure" groups $z^{i,\overline{p}}(X)(\bullet)$ (Definition 5.3) which isolate the cycles on $X \times \mathbb{P}^s$ with no component supported on a hyperplane. The proof here follows closely the proof of [**?**, Thm. 2.10].

For positive integers s,t with $s \geq t$, and K algebraically closed, there is a morphism

$$\pi : SP^s(\mathbb{P}^1_K) \;\to\; SP^{\binom{s}{t}}(SP^t(\mathbb{P}^1_K))$$

sending the cycle $\sum_{i\in I} z_i$ to the cycle $\sum_{J\subset I,|J|=t}(\sum_{j\in J} z_j)$. By Galois descent, the same formula defines a morphism assuming that K is perfect, or if one works with cycles with $\mathbb{Z}[1/p]$-coefficients instead of $\mathbb{Z}$-coefficients. (In characteristic zero, one should ignore all instances of $1/p$ which appear in the statements below.) Since the symmetric product $SP^m(X)$ of a normal variety X is normal, the symmetric products which appear coincide with the weak normalizations of the Chow varieties $\mathcal{C}_{0,m}(X)$. Therefore π induces a continuous algebraic map $\pi : \mathcal{C}_0(\mathbb{P}^s) \to \mathcal{C}_0(\mathbb{P}^t)$.

prop:symm **Proposition 6.3.** *Let X be a stratified quasi-projective variety. For every $t \leq s$, there are natural maps of presheaves*

$$z^{s,\overline{p}}(X, \mathbb{P}^s)(-) \;\to\; z^{t,\overline{p}}(X, \mathbb{P}^t)(-)$$

with the property that for any choice of linear embeddings $\mathbb{P}^{t-1} \subset \mathbb{P}^t \subset \mathbb{P}^s$ the composition

$$z^{t,\overline{p}}(X, \mathbb{P}^t)(\bullet) \;\to\; z^{s,\overline{p}}(X, \mathbb{P}^s)(\bullet) \;\to\; z^{t,\overline{p}}(X, \mathbb{P}^t)(\bullet) \;\to\; z^{t,\overline{p}}(X)(\bullet)$$

is homotopy equivalent to the defining projection of Definition 5.3.

Proof. Proposition 5.6(3) implies that π induces, for $s \geq t$, a natural transformation $p : z^{s,\overline{p}}(X, \mathbb{P}^s) \to z^{t,\overline{p}}(X, \mathbb{P}^t)$. The flag $\mathbb{P}^0 \hookrightarrow \mathbb{P}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{P}^s$ induces a nested sequence of presheaves:

$$z^{0,\overline{p}}(X, \mathbb{P}^0) \subset z^{1,\overline{p}}(X, \mathbb{P}^1) \subset \ldots \subset z^{s,\overline{p}}(X, \mathbb{P}^s).$$

It suffices to show the composition $p \circ i : z^{t,\overline{p}}(X, \mathbb{P}^t) \subset z^{s,\overline{p}}(X, \mathbb{P}^s) \to z^{t,\overline{p}}(X, \mathbb{P}^t)$ is equal to $\mathrm{id} + \psi$, where $\psi : z^{t,\overline{p}}(X, \mathbb{P}^t) \to z^{t,\overline{p}}(X, \mathbb{P}^t)$ is a morphism factoring through $z^{t-1,\overline{p}}(X, \mathbb{P}^{t-1})$. For any $\alpha \in z^{t,\overline{p}}(X, \mathbb{P}^t)(U)$ and any $u \in U$, the specialization α_u restricts to a cocycle on $(X - X^1)_u$ with values in $\mathbb{P}^t$. It follows from [**?**, Lemma 2.11] that the restriction $j^*((p \circ i)(\alpha_u) - \alpha_u)$ of $(p \circ i)(\alpha_u) - \alpha_u$ to $(X - X^1)_u$ lies in $(X - X^1)_u \times \mathbb{P}^{t-1}$. The morphism $p \circ i$ is compatible with the open immersion $j : X - X^1 \subset X$, and $X - X^1$ contains all of the generic points of $(p \circ i)(\alpha_u) - \alpha_u$. Therefore the closure of $j^*((p \circ i)(\alpha_u) - \alpha_u)$, namely $(p \circ i)(\alpha_u) - \alpha_u$, is contained in $X_u \times \mathbb{P}^{t-1}$. ☐

thm:splitting **Theorem 6.4.** *Let X be a stratified quasi-projective variety. The maps of Proposition 6.3 induce a homotopy equivalence*

eq:split (6.4.1)
$$z^{s,\overline{p}}(X, \mathbb{P}^s)(\bullet) \;\xrightarrow{\sim}\; \prod_{i=0}^{s} z^{i,\overline{p}}(X)(\bullet)$$

which is functorial with respect to flat, stratified morphisms.

Proof. The evaluation of the nested sequence of presheaves at $\Delta^\bullet$ induces a nested sequence of simplicial abelian groups:

$$z^{0,\overline{p}}(X, \mathbb{P}^0)(\bullet) \subset z^{1,\overline{p}}(X, \mathbb{P}^1)(\bullet) \subset \cdots \subset z^{s,\overline{p}}(X, \mathbb{P}^s)(\bullet).$$

Proposition 6.3 in conjunction with Theorem **??** implies the formal hypotheses of [**?**, Prop. 2.13] are satisfied. The construction involves only the "targets" $\mathbb{P}^0, \ldots, \mathbb{P}^s$, hence are compatible with flat pull-back via stratified morphisms. ☐

Remark 6.5. One can replace the $\mathbb{P}^s$ on the left hand side of the weak equivalence of Theorem 6.4 with $\mathbb{P}^r$ (for any $r \geq s$) by appealing to a suspension theorem and observing our constructions are compatible with suspension.

prop:short **Proposition 6.6.** *Choose a hyperplane $\mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^s$ and a non-negative integer m. Then there is a split short exact sequence of homotopy groups*

$$0 \to \; \pi_m(z^{s-1,\overline{p}}(X, \mathbb{P}^{s-1})(\bullet)) \to \pi_m(z^{s,\overline{p}}(X, \mathbb{P}^s)(\bullet)) \to \pi_m(z^s(X)(\bullet)) \to 0.$$

Proof. The short exact sequence of simplicial abelian groups

$$0 \to \; z^{s-1,\overline{p}}(X,\mathbb{P}^{s-1})(\bullet) \to z^{s,\overline{p}}(X,\mathbb{P}^{s})(\bullet) \to \frac{z^{s,\overline{p}}(X,\mathbb{P}^{s})(\bullet)}{z^{s-1,\overline{p}}(X,\mathbb{P}^{s-1})(\bullet)} \to 0$$

induces a long exact sequence in homotopy groups (because a surjective homomorphism of simplicial abelian groups is a Kan fibration). This long exact sequence splits into split short exact sequences as asserted thanks to Theorem 6.4. $\square$

cupproduct

Theorem 6.7. *The fiberwise join pairings of Proposition 6.2 determine natural (with respect to X) "cup product pairings"*

$$\cup : H^{i,\overline{p}}(X,\mathbb{Z}(s)) \otimes H^{j,\overline{q}}(X,\mathbb{Z}(t)) \;\to\; H^{i+j,\overline{p}+\overline{q}}(X,\mathbb{Z}(s+t)).$$

Proof. Consider the composition

$$\begin{aligned}
&\pi_{2s-i}(z^{s,\overline{p}}(X,\mathbb{P}^{s})(\bullet)) \otimes \pi_{2t-j}(z^{t,\overline{q}}(X,\mathbb{P}^{t})(\bullet)) \to \\
&\to \pi_{2(s+t)-i-j}(z^{s+t,\overline{p}+\overline{q}}(X,\mathbb{P}^{s+t+1})(\bullet)) \to \\
&\to \; \pi_{2(s+t)-i-j}(z^{s+t,\overline{p}+\overline{q}}(X,\mathbb{P}^{s+t+1})(\bullet)/z^{s+t-1,\overline{p}+\overline{q}}(X,\mathbb{P}^{s+t})(\bullet))
\end{aligned}$$

given by the map induced by fiberwise join followed by the projection. To prove the theorem, we consider $\mathbb{P}^s\#\mathbb{P}^{t-1}$, $\mathbb{P}^{s-1}\#\mathbb{P}^t$ inside $\mathbb{P}^s\#\mathbb{P}^t = \mathbb{P}^{s+t+1}$ and observe that this composition sends both

$$\pi_{2s-i}(z^{s,\overline{p}}(X,\mathbb{P}^{s})(\bullet)) \otimes \pi_{2t-j}(z^{t-1,\overline{q}}(X,\mathbb{P}^{t-1})(\bullet)) \qquad \text{and}$$

$$\pi_{2s-i}(z^{s-1,\overline{p}}(X,\mathbb{P}^{s-1})(\bullet)) \otimes \pi_{2t-j}(z^{t,\overline{q}}(X,\mathbb{P}^{t})(\bullet))$$

to 0. This follows immediately from the short exact sequence of Proposition 6.6 and the independence statement of Lemma 5.2. $\square$

The following proposition can be seen as having its origins in a semi-topological version given in [**?**, Thm 2.6].

smooth pairing

Proposition 6.8. *Let X be a stratified quasi-projective variety, and let Y be a smooth quasi-projective variety of dimension n. Restriction of correspondences determines a morphism of presheaves:*

$$z^{t,\overline{p}}(X,Y) \times z(X,r)_{\overline{q}} \to z(X\times Y, r+n-t)_{\overline{p}+\overline{q}}$$

and therefore a pairing:

$$H^{i,\overline{p}}(X,Y;Z(t)) \otimes H^{\overline{q}}_m(X,\mathbb{Z}(r)) \to H^{\overline{p}+\overline{q}}_{2n+m-i}(X\times Y,\mathbb{Z}(r+n-t)).$$

Proof. Given $\alpha,\beta \in z^{t,\overline{p}}(X,Y)(U) \times z(X,r)_{\overline{q}}(U)$, the dimension of α_u over any $x \in X^i - X^{i+1}$ is less than or equal to $n-t+p_i$. The dimension of $\beta_u \cap X^i_u$ is less than or equal to $r-i+q_i$. Therefore the support of $|\alpha|\cap|\beta\times Y|\cap(X^i - X^{i+1}\times Y)$ has dimension no larger than $(r-i+q_i)+(n-t+p_i) = (r+n-t)-i+(p_i+q_i)$. Then Theorem 4.2 implies the closure of the intersection product formed in $X^{sm}\times Y$ belongs to $z(X\times Y, r+n-t)_{\overline{p}+\overline{q}}(U)$, as desired. $\square$

sing pairing

Corollary 6.9. *Let X be a stratified quasi-projective variety, and let Y be a quasi-projective variety of dimension n. Restriction of correspondences determines a morphism of presheaves:*

$$z^{t,\overline{p}}(X,Y) \times z(X,r)_{\overline{q}} \to z(X\times Y, r+n-t)_{\overline{p}+\overline{q}}.$$

Proof. Embed Y as a closed subvariety of codimension c of some open subvariety $\mathbb{P}$ of a projective space. The restriction of the pairing

$$z^{t+c,\overline{p}}(X,\mathbb{P}) \times z(X,r)_{\overline{q}} \to z(X \times \mathbb{P}, r + \dim(\mathbb{P}) - t - c)_{\overline{p}+\overline{q}}$$

provided by Proposition 6.8 to the subpresheaf $z^{t,\overline{p}}(X,Y) \times z(X,r)_{\overline{q}}$ factors through $z(X \times Y, r+n-t)_{\overline{p}+\overline{q}}$. □

Remark 6.10. The restriction of the pairing of Corollary 6.9 to the subsheaf $\mathbb{Z}\operatorname{Hom}^{\overline{p}}(Y,X) \subset z^{d,\overline{p}}(X,Y)$ may be thought of as sending a pair $(f,\beta) \in \operatorname{Hom}^{\overline{p}}(Y,X)(U) \times z(X,r)_{\overline{q}}(U)$ to the pull-back of $\beta \hookrightarrow U \times X$ along $f : U \times Y \to U \times X$. (Strictly speaking we intersect the graph of f with the pull-back of β to $U \times X \times Y$.)

Combining Corollary 6.9 with the slice construction of Theorem 3.6, we can now formulate a *cap product pairing* relating generalized cycles and perversity cycles.

prop:cap

Proposition 6.11. *Let X be a stratified quasi-projective variety. The pairing of Proposition 6.8 determines the pairing of simplicial abelian groups*

cap pairing

$$z^{t,\overline{p}}(X)(\bullet) \times z(X,r)_{\overline{q}}(\bullet) \to z(X, r-t)_{\overline{p}+\overline{q}}(\bullet); \tag{6.11.1}$$

taking homotopy groups, we obtain the cap product pairing

cap pairing2

$$H^{i,\overline{p}}(X,\mathbb{Z}(t)) \otimes H_m^{\overline{q}}(X,\mathbb{Z}(r)) \ \to\ H_{m-i}^{\overline{p}+\overline{q}}(X,\mathbb{Z}(r-t)). \tag{6.11.2}$$

Proof. The pairing of Proposition 6.8 specializes to

$$z^{t,\overline{p}}(X,\mathbb{P}^t) \times z(X,r)_{\overline{q}} \to z(X \times \mathbb{P}^t, r+n-t)_{\overline{p}+\overline{q}}$$

and therefore determines a paring of simplicial abelian groups

$$(z^{t,\overline{p}}(X)(\bullet) = z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet)) \times z(X,r)_{\overline{q}}(\bullet) \to$$

$$z(X \times \mathbb{P}^t, r)_{\overline{p}+\overline{q}}(\bullet)/z(X \times \mathbb{P}^{t-1}, r)_{\overline{p}+\overline{q}}(\bullet).$$

Composing this pairing with the slice map of Theorem 3.6, we obtain (6.11.1).

The pairing (6.11.2) is obtained by applying $\pi_{2t-i}(-)\otimes\pi_{m-2r}(-) \to \pi_{m-i-(2r-2t)}(-)$ to (6.11.1). □

q

duality map

Corollary 6.12. *Assume X is of pure dimension d. Specializing the pairing of Proposition* 6.11 *to the fundamental class ($r=d$), we obtain a natural map*

$$\mathcal{I} : z^{t,\overline{p}}(X)(\bullet) \ \to\ z(X, d-t)_{\overline{p}}(\bullet)$$

and therefore a canonical morphism:

$$\mathcal{I} : H^{i,t,\overline{p}}(X) \to H_{2d-i}^{\overline{p}}(X,\mathbb{Z}(d-t)).$$

This map is given by the composition of the natural inclusion $z_{equi}^{t,\overline{p}}(X,\mathbb{P}^) \subset z_{equi}(X \times \mathbb{P}^t, d)_{\overline{p}}$ of Definition 5.1 followed by the slice map (3.6.1).

Should the duality map of Corollary 6.12 be an isomorphism as in the following Question, the cap product pairing of Proposition 6.11 would become a pairing on perversity homology groups. We pose the evident question for which even partial answers would be of considerable interest guess **Question 6.13.** Assume that X is a normal, connected quasi-projective variety of dimension d. Does there exists a stratification $\mathcal{T}$ of X such that the map

$$\mathcal{I} : H^{i,t,\overline{p}}(X) \to H^{\overline{p}}_{2d-i}(X, \mathbb{Z}(d-t)),$$

is an isomorphism for all $i, \overline{p}$, thereby enabling intersection products

$$H^{\overline{p}}_{2d-i}(X, \mathbb{Z}(d-t)) \otimes H^{\overline{q}}_{m}(X, \mathbb{Z}(r)) \ \to \ H^{\overline{p}+\overline{q}}_{m-i}(X, \mathbb{Z}(r-t))?$$

To answer affirmatively this Question, one needs a perversity version of the duality theorem for smooth varieties given in [**?**]. One possible approach is to choose a resolution of singularities $\pi : \tilde{X} \to X$ and take the stratification of X associated to this resolution. Given an $r-t$-cycle ζ on X of perversity $\overline{p}$, one should move the proper transform $\widetilde{\zeta \times \mathbb{P}^t}$ on $\tilde{X} \times \mathbb{P}^t$. The moved cycle η on $\tilde{X} \times \mathbb{P}^t$ should intersect properly the proper transforms of the strata on $X \times \mathbb{P}^t$. By definition of the perversity $\overline{p}$ associated to the resolution $\pi : \tilde{X} \to X$, $\pi_*(\eta)$ should lie in $Z^t(X, \mathbb{P}^t)_{\overline{p}}$. We need to verify that $\pi_*(\eta)$ represents $\zeta \times \mathbb{P}^t \in Z(X \times \mathbb{P}^t, r)_{\overline{p}}$.

As one would expect, our cup and cap products are related by the map $\mathcal{I}$ of Corollary 6.12.

END OF REVISION

Given t hyperplanes $\underline{H} := H_1, \ldots, H_t$ in $\mathbb{P}^t$, and $\alpha \in Z^{t,\overline{p}}(X, \mathbb{P}^t)$, we denote by $\alpha \cdot \underline{H} \in Z_{d-t}(X)$ the cycle obtained by intersecting α with the $X \times H_i$'s, provided this intersection is proper. The following proposition will be used repeatedly in the remainder of this paper to justify that "slicing" a perversity $\overline{p}$ generalized cocycle on $X \times \mathbb{P}^t$ with a sufficiently generic linear space $\underline{H}$ determines a cycle on X of perversity $\overline{p}$.

stratumwise proper **Proposition 6.14.** *Let $\alpha \in Z^{t,\overline{p}}(X, \mathbb{P}^t)$, and suppose given a closed set $C_i \hookrightarrow X^i - X^{i+1}$ for each i. For a generic collection of hyperplanes $\underline{H}$, we have the following (for all i):*

- *the support $|\alpha \cdot \underline{H}| \cap (X^i - X^{i+1})$ has dimension no larger than $d-t-i+p_i$, so that $\alpha \cdot \underline{H} \in Z_{d-t,\overline{p}}(X)$; and*
- *the supports of $|\alpha \cdot \underline{H}| \cap (X^i - X^{i+1})$ and C_i meet properly in $X^i - X^{i+1}$, i.e., the intersection has codimension at least $t - p_i$ in C_i.*

Proof. For simplicity of exposition we assume α has fiber dimension exactly p_i over $X^i - X^{i+1}$. We choose H_1 so that it does not contain (i.e., intersects properly) the generic points of $\alpha^i - \alpha^{i+1}$ (viewed as cycles of dimension p_i on $\mathbb{P}^t_K$, with $K = k(\eta_{X^i - X^{i+1}})$) for any i. In general, choose H_{j+1} so that it does not contain any of the generic points of any of the $(\alpha \cap_{a=1}^{j} H_a)^i - (\alpha \cap_{a=1}^{j} H_a)^{i+1}$. Then

$$\dim(|\alpha \cdot \underline{H}| \cap (X^i - X^{i+1})) \leq \dim(\alpha \cap (X^i - X^{i+1})) - t = (d - i + p_i) - t$$

as claimed.

To ensure the hyperplanes $\underline{H}$ satisfy the second property, we must choose H_1 so that in addition it does not contain any generic points of $(\alpha^i - \alpha^{i+1}) \cap (C_i \times \mathbb{P}^t)$ for any i, and similarly at the subsequent stages. By the definition of a generalized cocycle we have $\dim((\alpha^i - \alpha^{i+1}) \cap (C_i \times \mathbb{P}^t)) = \dim(C_i) + p_i$. Our choice of hyperplanes implies $\dim(\alpha\cdot\underline{H}\cap(C_i\times\mathbb{P}^t)) = \dim(C_i)+p_i-t$, so the intersection has codimension $t-p_i$ in C_i. This concludes the proof. $\square$

Remark 6.15. The second assertion of Proposition 6.14 means that, up to homotopy, a generalized cocycle of perversity $\overline{p}$ behaves like a perversity $\overline{p}$ cycle which can be moved in each stratum. In the application, the C_i's will be obtained by slicing some $\beta \in Z^{s,\overline{q}}(X)$ with generic hyperplanes $H_1,\dots,H_s$ in $\mathbb{P}^s$, or as the incidences $\beta\cap(X^i - X^{i+1})$ of some $\beta \in Z_{d-s,\overline{q}}(X)$.

cup module cap

Proposition 6.16. *Via the map $\mathcal{I}$ of Corollary* 6.12*, the cup product of Theorem* 6.7 *is compatible with the cap product of Proposition* 6.11 *in the sense that the following diagram is commutative.*

$$\begin{array}{ccc} H^{i,s,\overline{p}}(X)[1/p]\otimes H^{j,t,\overline{q}}(X)[1/p] & \xrightarrow{\cup} & H^{i+j,s+t,\overline{p}+\overline{q}}(X)[1/p] \\ \downarrow{\scriptstyle 1\otimes\mathcal{I}} & & \downarrow{\scriptstyle \mathcal{I}} \\ H^{i,s,\overline{p}}(X)[1/p]\otimes H^{\overline{q}}_{2d-j}(X,\mathbb{Z}[1/p](d-t)) & \xrightarrow{\cap} & H^{\overline{p}+\overline{q}}_{2d-i-j}(X,\mathbb{Z}[1/p](d-s-t)) \end{array}$$

Proof. The proof combines some simple geometry with the fact that the suspension isomorphism $z(X\times\mathbb{P}^t,r)_{\overline{p}}(\bullet)/z(X\times\mathbb{P}^{t-1},r-1)_{\overline{p}}(\bullet)\cong z(X,r-t)_{\overline{p}}(\bullet)$ is independent (up to homotopy) of the choice of t hyperplanes in $\mathbb{P}^t$, whose proof follows the same lines as that of Lemma 5.2.

For $\alpha\in z^{s,\overline{p}}(X,\mathbb{P}^s)(U)$ and $\beta\in z^{t,\overline{q}}(X,\mathbb{P}^t)(U)$, the join $J_{U\times X}(\alpha,\beta)\in z^{s+t,\overline{p}+\overline{q}}(X\times\mathbb{P}^{s+t+1})(U)$ may be described as the intersection in $U\times X\times\mathbb{P}^{s+t+1}$ of $\Sigma^{t+1}_{\mathbb{P}^s}(\alpha)$ and $\Sigma^{s+1}_{\mathbb{P}^t}(\beta)$. Thus, if α and β happen to represent perversity cohomology classes, the composition that factors through $H^{i+j,s+t,\overline{p}+\overline{q}}(X)$ is represented geometrically by $\Sigma^{t+1}_{\mathbb{P}^s}(\alpha)\bullet_{s+t+1}\Sigma^{s+1}_{\mathbb{P}^t}(\beta)\bullet_{s+t+1}h^{s+t+1}$, where h is the hyperplane class on $\mathbb{P}^{s+t+1}$ and the subscript on the product indicates that intersections are formed in $\Delta^\bullet\times X\times\mathbb{P}^{s+t+1}$.

The image of β in $z(X,d-t)_{\overline{q}}(U)$ is represented by $\beta\bullet_t x^t$, where x is the hyperplane class on $\mathbb{P}^t$. Therefore, again assuming we have representatives of perversity cohomology classes, the composition that factors through $H^{i,s,\overline{p}}(X)\otimes H^{\overline{q}}_{2d-j}(X,\mathbb{Z}(d-t))$ is represented by $\alpha\bullet_s p^*(\beta\bullet_t x^t)\bullet_s y^s$, where y is the hyperplane class on $\mathbb{P}^s$ and $p: X\times\mathbb{P}^s\to X$ is the projection. Here we use the flexibility provided by Proposition 6.14 to choose hyperplane classes on $\mathbb{P}^s$ which are adapted to $p^*(\beta\bullet_t x^t)$.

Let $q: X\times\mathbb{P}^t\to X$ denote the projection. Then $\Sigma^{t+1}_{\mathbb{P}^s}\circ p^*$ and $\Sigma^{s+1}_{\mathbb{P}^t}\circ q^*$ coincide as operations from cycles on X to cycles on $X\times\mathbb{P}^{s+t+1}$. By the suspension theorem, the class of β in $H^{\overline{q}}_{2d-j}(X,\mathbb{Z}(d-t))$ coincides with the class of $q^*(\beta\bullet_t x^t)$. We suppress the source of the suspension morphism from the notation. For classes α and β we have: $\alpha\bullet_s p^*(\beta\bullet_t x^t)\bullet_s y^s$
$= \Sigma^{t+1}(\alpha\bullet_s p^*(\beta\bullet_t x^t))\bullet_{s+t+1}h^{s+t+1}$
$= \Sigma^{t+1}(\alpha)\bullet_{s+t+1}\Sigma^{t+1}(p^*(\beta\bullet_t x^t))\bullet_{s+t+1}h^{s+t+1}$
$= \Sigma^{t+1}(\alpha)\bullet_{s+t+1}\Sigma^{s+1}(q^*(\beta\bullet_t x^t))\bullet_{s+t+1}h^{s+t+1}$
$= \Sigma^{t+1}(\alpha)\bullet_{s+t+1}\Sigma^{s+1}(\beta)\bullet_{s+t+1}h^{s+t+1}$, as desired. $\square$

Example 6.17. Consider X with only isolated singularities and stratified by $X_{sing}=X^d=\cdots=X^1$. If the singularities of X are isolated, we have an equality of presheaves $z^{t,\overline{p}}_{equi}(X,\mathbb{P}^t)=z_{equi}(X\times\mathbb{P}^t,d)_{\overline{p}}$. Therefore the map $\mathcal{I}$ is a homotopy equivalence:

$$
\begin{aligned}
z^{t,\overline{p}}(X)(\bullet) &\equiv z^{t,\overline{p}}(X,\mathbb{P}^t)(\bullet)/z^{t-1,\overline{p}}(X,\mathbb{P}^{t-1})(\bullet)\\
&= z(X\times\mathbb{P}^t,d)_{\overline{p}}(\bullet)/z(X\times\mathbb{P}^{t-1},d)_{\overline{p}}(\bullet)\\
&\cong z(X\times\mathbb{A}^t,d)_{\overline{p},\mathrm{cl}}(\bullet)\\
&\cong z(X\times\mathbb{A}^t,d)_{\overline{p}}(\bullet)\\
&\cong z(X,d-t)_{\overline{p}}(\bullet).
\end{aligned}
\tag{6.17.1}
$$

Therefore, if X has only isolated singularities, then the pairing of Theorem 6.7 provides a pairing on our intersection variants of motivic homology $H_m^{\overline{p}}(X,\mathbb{Z}(r))$.

We now assume that $k=\mathbb{C}$ and establish the compatibility of our pairings with those defined by Goresky-MacPherson. We first establish the cohomological analogue of the perverse cycle map $c: H_{2r}^{\overline{p}}(X,\mathbb{Z}(r))\to IH_{2r}^{\overline{p}}(X,\mathbb{Z})$ of Proposition **??**.

Definition 6.18. Assume the stratification of the complex, stratified variety X is sufficiently fine as in Proposition 2.6. We define

$$\tilde{c}=c\circ\pi : H^{2r,r,\overline{p}}(X)\to H_{2r}^{\overline{p}}(X,\mathbb{Z}(r))\to IH_{2r}^{\overline{p}}(X,\mathbb{Z})$$

where π is the map given by applying π_0 to the composition

$$z^{t,\overline{p}}(X)(\bullet)\to z(X\times\mathbb{P}^t,d)_{\overline{p}}(\bullet)/z(X\times\mathbb{P}^{t-1},d)_{\overline{p}}(\bullet)\to z(\Sigma^t(X),d)_{\overline{p}}.$$

The morphism of simplicial abelian groups

$$z^{t,\overline{p}}(X)(\bullet)\to z(X\times\mathbb{P}^t,d)_{\overline{p}}(\bullet)/z(X\times\mathbb{P}^{t-1},d)_{\overline{p}}(\bullet)\to z(\Sigma^t(X),d)_{\overline{p}}\xleftarrow{\cong} z(X,d-t)_{\overline{p}}(\bullet)$$

induces a morphism (provided the stratification of X is sufficiently fine)

$$c:\pi_0(z^{t,\overline{p}}(X)(\bullet))\to\pi_0(z(X,d-t)_{\overline{p}}(\bullet))\equiv A_{d-t,\overline{p}}(X)\to IH_{2(d-t)}^{\overline{p}}(X).$$

prop:compat **Proposition 6.19.** *Via the perverse cycle class map c, the pairing in Proposition* **??** *is compatible with the pairing in intersection homology. In other words, the following diagram is commutative:*

$$
\begin{tikzcd}
\pi_0(z^{t,\overline{p}}(X)(\bullet))\otimes\pi_0(z(X,r)_{\overline{q}}(\bullet)) \arrow[r] \arrow[d, "c\otimes c"] & \pi_0\left(\frac{z(X\times\mathbb{P}^t,r)_{\overline{p}+\overline{q}}(\bullet)}{z(X\times\mathbb{P}^{t-1},r)_{\overline{p}+\overline{q}}(\bullet)}\right) \arrow[r, "\cong"] & \pi_0(z(X,r-t)_{\overline{p}+\overline{q}}(\bullet)) \arrow[d, "c"] \\
IH_{2(\dim_{\mathbb{C}}(X)-t)}^{\overline{p}}(X)\otimes IH_{2r}^{\overline{q}}(X) \arrow[rr, "\cap"] & & IH_{2(r-t)}^{\overline{p}+\overline{q}}(X)
\end{tikzcd}
$$

Proof. First, the suspension homotopy equivalence $\frac{z(X\times\mathbb{P}^t,r)_{\overline{p}+\overline{q}}(\bullet)}{z(X\times\mathbb{P}^{t-1},r)_{\overline{p}+\overline{q}}(\bullet)}\cong z(X,r-t)_{\overline{p}+\overline{q}}(\bullet)$ is induced by the geometric operation of slicing with t hyperplanes. The resulting isomorphism is independent of the choice of hyperplanes since any two choices can be connected by an affine line, as in the proof of Lemma 5.2.

Given $a\in\pi_0(z^{t,\overline{p}}(X)(\bullet))$ and $b\in\pi_0(z(X,r)_{\overline{q}}(\bullet))$ (with representatives $\alpha\hookrightarrow X\times\mathbb{P}^t$ and $\beta\hookrightarrow X$), the class of $a\otimes b$ in $\pi_0(\frac{z(X\times\mathbb{P}^t,r)_{\overline{p}+\overline{q}}(\bullet)}{z(X\times\mathbb{P}^{t-1},r)_{\overline{p}+\overline{q}}(\bullet)})$ is represented by $\alpha\bullet(\beta\times\mathbb{P}^t)$. After choosing a sequence of hyperplanes $\underline{H}$ adapted to α (in the sense of Proposition 6.14, with $C_i=\beta^i-\beta^{i+1}$), we see that the class of $a\otimes b$ in $\pi_0(z(X,r-t)_{\overline{p}+\overline{q}}(\bullet))$ is represented by $(\alpha\cdot\underline{H})\bullet\beta$.

Since $\alpha\cdot\underline{H}$ and β intersect properly in each stratum, $c((\alpha\cdot\underline{H})\bullet\beta)$ is represented by the closure of their intersection in the smooth locus. Therefore it suffices to show

$c((\alpha \cdot \underline{H}) \cap c(\beta)$ has the same representative. The intersection homology pairing between chains intersecting properly in each stratum is determined by the cup product of the corresponding cohomology classes in the smooth locus [**?**, 2.1]. In the smooth locus, the intersection product maps to the cup product of cohomology classes. $\square$

References

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