VECTOR BUNDLES ASSOCIATED TO LIE ALGEBRAS

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Abstract. We introduce and investigate a functorial construction which associates coherent sheaves to finite dimensional (restricted) representations of a restricted Lie algebra $\mathfrak{g}$. These are sheaves on locally closed subvarieties of the projective variety $\mathbb{E}(r,\mathfrak{g})$ of elementary subalgebras of $\mathfrak{g}$ of dimension r. We show that representations of constant radical or socle rank studied in [5] which generalize modules of constant Jordan type lead to algebraic vector bundles on $\mathbb{E}(r,\mathfrak{g})$. For $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of an algebraic group G, rational representations of G enable us to realize familiar algebraic vector bundles on G-orbits of $\mathbb{E}(r,\mathfrak{g})$.

0. Introduction

In [5], the authors introduced the projective algebraic variety $\mathbb{E}(r,\mathfrak{g})$ of *elementary subalgebras* $\epsilon \subset \mathfrak{g}$ of dimension r of a given finite dimensional restricted Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic $p > 0$. An elementary subalgebra $\epsilon \subset \mathfrak{g}$ is a commutative Lie subalgebra restricted to which the p-th power operator $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ is trivial. In this paper, we explore the connections between geometric structures on these varieties $\mathbb{E}(r,\mathfrak{g})$ and restricted representations of $\mathfrak{g}$.

We recall that the category of restricted representations of $\mathfrak{g}$ is equivalent to the category of modules for the finite dimensional associative k-algebra $\mathfrak{u}(\mathfrak{g})$, the restricted enveloping algebra of $\mathfrak{g}$. We construct coherent sheaves and algebraic vector bundles on $\mathbb{E}(r,\mathfrak{g})$ associated to finite dimensional $\mathfrak{u}(\mathfrak{g})$-modules, extending considerations in [9] (the case $r = 1$) and [4] (the special case in which $\mathfrak{g}$ is itself an elementary Lie algebra).

For a given finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M and a given $r > 0$, we consider radicals and socles of M restricted to an elementary subalgebra ϵ of $\mathfrak{g}$ as $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ varies. In Section 1 we show that for any locally closed subvariety $X \subset \mathbb{E}(r,\mathfrak{g})$, any $j > 0$ and any $\mathfrak{u}(\mathfrak{g})$-module M, there are sheaves $\mathcal{K}er^{j,X}(M), \mathcal{I}m^{j,X}(M)$ on X which depend functorially on M and whose generic fiber is identified naturally with the j-th socle or the j-th radical of M. Consequently, these "image" and "kernel" sheaves encode considerable information about the action of $\mathfrak{g}$ on M, with local input the action of elementary subalgebras ϵ on M. Much is known about the modules for an elementary Lie algebra (the category of which is equivalent to the more familiar category of $k(\mathbb{Z}/p\mathbb{Z}^{\oplus r})$-modules), even though this category is wild if $r > 1$, unless $r = 2, p = 2$.

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We present two different but equivalent constructions of the sheaves $\mathcal{K}er^{j,X}(M), \mathcal{I}m^{j,X}(M)$. Our first construction uses equivariant descent and natural operators on coherent sheaves on the Stiefel variety which is a GL-torsor over a Grassmannian. The second construction involves a straight-forward patching technique, making use of the standard affine charts on the Grassmannian and allowing for an easy identification of the generic fiber. It is the patching construction that allows us to show in Section 2 how modules of constant Jordan type (and, more generally, modules of constant r-radical rank and constant r-socle rank) lead to vector bundles on $\mathbb{E}(r, \mathfrak{g})$.

As in [9] and [5], we envision the consideration of detailed "new" invariants for $\mathfrak{u}(\mathfrak{g})$-modules. Namely, for a given finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M, a given $r > 0$, and a given locally closed subvariety $X \subset \mathbb{E}(r, \mathfrak{g})$, we consider the classes in $K_0'(X)$ of our image and kernel sheaves on X. For example, if $\mathfrak{g} = \mathrm{Lie}(G)$ for an affine algebraic group G and if M a rational G-module, $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ are G-equivariant algebraic vector bundles provided that $X \subset \mathbb{E}(r, \mathfrak{g})$ is a G-orbit. One can ask which classes of coherent sheaves and vector bundles can be realized as image and kernel sheaves of $\mathfrak{u}(\mathfrak{g})$-modules. Computations are difficult, which is to be expected granted the subtleties which already arise in the case in which $\mathfrak{g}$ is itself an elementary Lie algebra as seen in [4].

A second type of application should arise from the explicit nature of our construction of coherent sheaves from the data of a finitely generated $\mathfrak{u}(\mathfrak{g})$-module. For various types of $\mathfrak{u}(\mathfrak{g})$-modules M and for certain subvarieties $X \subset \mathbb{E}(r, \mathfrak{g})$, we obtain vector bundles; since X is typically singular, such explicit constructions should provide insight into the difficult challenge of understanding algebraic vector bundles on singular varieties.

Yet another application is the explicit construction of familiar vector bundles such as tangent and cotangent bundles on certain projective varieties in terms of $\mathcal{K}er^{j,X}(M)$, $\mathcal{I}m^{j,X}(M)$, and other similarly constructed sheaves. In Section 4, we investigate examples arising from rational modules for an affine algebraic group, whereas in Section 5 we provide further examples which do not arise from actions of an algebraic group.

The paper is organized as follows. In Section 1 we present our two constructions of the image and kernel sheaves and show that they are equivalent. Following the construction, we suggest in Section 2 methods to extract geometric invariants for a $\mathfrak{u}(\mathfrak{g})$-module M arising from our image and kernel sheaves on $\mathbb{E}(r, \mathfrak{g})$ for various r. The challenge, which appears to lend itself to only incremental progress, is to search among all the geometric data one obtains for computable invariants which distinguish many classes of modules and suggests families of modules worthy of further study.

If M is the restriction of a rational G-module, the action of G on $\mathbb{E}(r, \mathfrak{g})$ equips the coherent sheaves $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$ on a G-stable subvariety $X \subset \mathbb{E}(r, \mathfrak{g})$ with the structure of G-equivariant coherent sheaves on X. In Section 3, we focus on the context in which $X = G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$ is a G-orbit and M a rational G-module; in this case, the image and kernel sheaves $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$ are G-equivariant algebraic vector bundles on X. If the orbit map $\phi_\epsilon : G \to \mathbb{E}(r, \mathfrak{g})$ is separable so that $G \cdot \epsilon$ is isomorphic to $H = G/G_\epsilon$, then we identify in Theorem 3.6 the vector bundles $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ on $G \cdot \epsilon$ as the H-equivariant vector bundles obtained by induction starting with the representations of H on $\mathrm{Rad}^j(\epsilon^*M)$, $\mathrm{Soc}^j(\epsilon^*M)$, where ϵ^*M denotes the restriction of M to ϵ. Using this identification, we realize many familiar vector bundles as image and kernel bundles associated to rational G-modules. Hence, for $\mathfrak{g} = \mathrm{Lie}(G)$ and M a rational G-module, we get explicitly determined algebraic vector bundles associated to M which we can view as invariants of M for each choice of $r > 0$, $X \subset \mathbb{E}(r, \mathfrak{g})$ and j, such that $1 \leq j \leq (p-1)r$.

The final section of this paper is devoted to vector bundles which arise from the semi-direct product of an algebraic group H with a vector group associated to a rational H-module W. We consider image and kernel bundles for (non-rational) representations of $\mathfrak{g}_{W,H} = \mathrm{Lie}(W \rtimes H)$. Many of the examples of our recent paper [4] are reinterpreted and extended using this construction. As we show in Theorem 5.9 and its corollary, most homogeneous bundles on H-orbits inside $\mathrm{Grass}(r, W) \subset \mathbb{E}(r, \mathfrak{g}_{W,H})$ are realized as image bundles in this manner.

Throughout, k is an algebraically closed field of characteristic $p > 0$. All Lie algebras $\mathfrak{g}$ considered in this paper are assumed to be finite dimensional over k and p-restricted; all Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ will be assumed to be closed under p-restriction. We use the terminology "affine algebraic group" to refer to a reduced group scheme represented by a finitely generated, integral k-algebra $k[G]$. We use the terminology "rational representation" of an affine group scheme G to mean a comodule for the coalgebra $k[G]$; we shall sometimes refer to such rational representations informally as G-modules. Without explicit mention to the contrary, all G-modules are finite dimensional.

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1. The coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ on $\mathbb{E}(r, \mathfrak{g})$

sec:cohsheaves

Let $\mathfrak{g}$ be a restricted Lie algebra of dimension n. We recall from [5, 1.3] that an *elementary* subalgebra of $\mathfrak{g}$ is an abelian restricted Lie subalgebra with trivial p-restriction. For some r, $0 < r < n$, we consider the Grassmann variety $\mathrm{Grass}(r, \mathfrak{g})$ of r-planes in $\mathfrak{g}$ which is viewed as an n-dimensional vector space over k. The subset of $\mathrm{Grass}(r, \mathfrak{g})$ consisting of those r-planes $\epsilon \subset \mathfrak{g}$ which are elementary subalgebras constitute a closed subvariety: $\mathbb{E}(r, \mathfrak{g}) \subset \mathrm{Grass}(r, \mathfrak{g})$. Let $\mathbb{M}_{n,r}$ denote the affine space of $n \times r$ matrices over k and let $\mathbb{M}^\circ_{n,r}$ be the open subset consisting of those matrices that have maximal rank.

For each finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M and each j, $1 \leq j \leq (p-1)r$, we construct the coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ on $\mathbb{E}(r, \mathfrak{g})$. The reader should keep in mind that these image and kernel sheaves are not images and kernels of the action of $\mathfrak{u}(\mathfrak{g})$, but rather globalizations of images and kernels of local (with respect to the Zariski topology on $\mathbb{E}(r, \mathfrak{g})$) actions on M.

Indeed, we provide two independent constructions. The first is by equivariant descent for the GL_r torsor $\mathbb{M}^\circ_{n,r} \longrightarrow \mathrm{Grass}(r, \mathfrak{g})$ (the Stiefel variety over $\mathrm{Grass}(r, \mathfrak{g})$). In the special case $r = 1$ (so that $\mathbb{G}_m$ replaces GL_r), this is implicit in the original construction of vector bundles for infinitesimal group schemes given in [9]. Each construction has its advantages: that of equivariant descent is quickly seen to be well defined independent of choices, that of patching leads to an identification of fibers.

We employ the natural action of GL_r on $\mathfrak{g}^{\oplus r}$ given by $(a_{i,j}) \in \mathrm{GL}_r(k)$ acting as $(a_{i,j}) \otimes \mathrm{Id}$ on $k^{\oplus r} \otimes \mathfrak{g}$. This action induces an action of GL_r on $\mathfrak{g}^{\times r}$, the affine variety associated to $\mathfrak{g}^{\oplus r}$ (isomorphic to the affine space $\mathbb{A}^{nr}$). We set $(\mathfrak{g}^{\times r})^o \subset \mathfrak{g}^{\times r}$ to be the open subvariety of those r-tuples of elements of $\mathfrak{g}$ that are linearly independent. We denote by $\mathcal{N}_p(\mathfrak{g}) \subset \mathfrak{g}$ the closed subvariety of $\mathfrak{g}$ (viewed as affine n-space) consisting of those $X \in \mathfrak{g}$ with $X^{[p]} = 0$. We further denote by $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \subset (\mathcal{N}_p(\mathfrak{g}))^{\times r}$ the closed subvariety of r-tuples $(X_1, \ldots, X_r)$ which are pairwise commuting (as well as p-nilpotent), and by $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o$ the subset of those r-tuples that are also linearly independent.

We consider the following diagram of quasi-projective varieties over k with Cartesian (i.e., pull-back) squares. We have

diag (1.0.1)
$$\begin{tikzcd} \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \arrow[r] & \mathfrak{g}^{\times r} \\ \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o \arrow[r] \arrow[u] \arrow[d] & (\mathfrak{g}^{\times r})^o \arrow[u] \arrow[d] \\ \mathbb{E}(r, \mathfrak{g}) \arrow[r] & \mathrm{Grass}(r, \mathfrak{g}) \end{tikzcd}$$

where upper vertical maps are open immersions, lower vertical maps are quotient maps by the GL_r actions, and horizontal maps are closed immersions.

We choose a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ which determines an identification $k[\mathbb{M}_{n,r}] \simeq k[\mathfrak{g}^{\times r}]$, where $\mathbb{M}_{n,r}$ is the vector group (isomorphic to $\mathbb{A}^{nr}$) of $n \times r$ matrices. Under this identification, the matrix function $T_{i,s} \in k[\mathbb{M}_{n,r}]$ is sent to $x_i^{\#} \circ \mathrm{pr}_s : \mathfrak{g}^{\times r} \to k$ defined as first projecting to the s^{th} factor and then applying the linear dual of x_i. We set $Y_{i,s} \in k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]$ to be the image of the matrix function $T_{i,s}$ under the surjective map

$$k[\mathbb{M}_{n,r}] \simeq k[\mathfrak{g}^{\times r}] \twoheadrightarrow k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))], \quad T_{i,s} \mapsto Y_{i,s}.$$

For any s, $1 \le s \le r$, we define

thetass (1.0.2)
$$\Theta_s \equiv \sum_{i=1}^{n} x_i \otimes Y_{i,s} \in \mathfrak{g} \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]$$

and use the same notation to denote the operator

$$\Theta_s : M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))] \to M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))], \quad \Theta_s(m \otimes f) = \sum_{i=1}^{n} x_i m \otimes Y_{i,s} f$$

for any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M.

prop:indept **Proposition 1.1.** *The operator Θ_s of (1.0.2) does not depend upon the choice of basis of $\mathfrak{g}$.*

Proof. Let $\{y_1, \ldots, y_n\}$ be another choice of basis of $\mathfrak{g}$, and set $Z_{i,s}$ equal to the image of $T_{i,s}$ under the surjective map $k[\mathbb{M}_{n,r}] \to k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]$ determined by this choice. Let $(a_{i,j}) \in \mathrm{GL}_n(k)$ be the change of basis matrix, so that $y_j = \sum_i a_{i,j} x_i$. Since $Y_{i,s}$'s are the images of the linear duals to x_i's under the projection $k[\mathbb{M}_{n,r}] \to$

$k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]$ (and similarly for $Z_{i,s}$), we conclude that $Z_{j,s} = \sum_i b_{j,i} Y_{i,s}$ where $(b_{i,j}) = (a_{i,j})^{-1}$. To prove the proposition, it suffices to observe that

$$\sum_j y_j \otimes Z_{j,s} \ \equiv \ \sum_j (\sum_i a_{i,j} x_i) \otimes (\sum_i b_{j,i} Y_{i,s}) \ = \ \sum_i x_i \otimes Y_{i,s}.$$

This follows directly from the fact that $(a_{i,j}) \cdot (b_{i,j})$ is equal to the identity matrix. $\square$

Let $j : X \subset \mathbb{E}(r,\mathfrak{g})$ be a locally closed embedding, and denote by $\widetilde{X} \to X$ the restriction of the GL_r-torsor $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o \to \mathbb{E}(r,\mathfrak{g})$ to X so that there is a Cartesian square

eq:wtX (1.1.1)
$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{j}"] \arrow[d] & \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o \arrow[d] \\ X \arrow[r, hook, "j"] & \mathbb{E}(r,\mathfrak{g}). \end{tikzcd}$$

We specialize (1.0.2) by defining

thetaX (1.1.2)
$$\Theta_s^{\widetilde{X}} : M \otimes \mathcal{O}_{\widetilde{X}} \ \to \ M \otimes \mathcal{O}_{\widetilde{X}}, \quad \Theta_s^{\widetilde{X}}(m \otimes f) = \sum_{i=1}^n x_i m \otimes \widetilde{j}^*(Y_{i,s}) f.$$

defn:descent **Definition 1.2.** For any finite-dimensional $\mathfrak{u}(\mathfrak{g})$-module M, and any j, with $1 \leq j \leq (p-1)r$, we define the following submodules of $M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]$:

$$\mathrm{Im}\{\Theta^j, M\} = \mathrm{Im}\{\sum_{\sum j_\ell = j} \Theta_1^{j_1} \cdots \Theta_r^{j_r} : (M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))])^{\oplus r(j)} \to M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))]\},$$

$$\mathrm{Ker}\{\Theta^j, M\} = \mathrm{Ker}\{[\Theta_1^{j_1} \cdots \Theta_r^{j_r}]_{\sum j_\ell = j} : M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))] \to (M \otimes k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))])^{\oplus r(j)}\},$$

where $r(j)$ is the number of ways to write j as a sum of r nonnegative integers.

For any locally closed subset $X \subset \mathbb{E}(r,\mathfrak{g})$, we define the following coherent sheaves on $\widetilde{X}$:

$$\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Im}\{\sum_{\Sigma j_\ell = j} (\Theta_1^{\widetilde{X}})^{j_1} \cdots (\Theta_r^{\widetilde{X}})^{j_r} : (M \otimes \mathcal{O}_{\widetilde{X}})^{\oplus r(j)} \to M \otimes \mathcal{O}_{\widetilde{X}}\},$$

$$\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Ker}\{[(\Theta_1^{\widetilde{X}})^{j_1} \cdots (\Theta_r^{\widetilde{X}})^{j_r}]_{\Sigma j_\ell = j} : M \otimes \mathcal{O}_{\widetilde{X}} \to (M \otimes \mathcal{O}_{\widetilde{X}}^{\oplus r(j)}\}.$$

rem:indept **Remark 1.3.** By Proposition 1.1, $\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\}$, $\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\}$ do not depend upon our choice of basis for $\mathfrak{g}$.

Let G be an affine algebraic group (such as GL_r) and X an algebraic variety on which G acts. A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be G-equivariant if there is an algebraic (i.e., functorial with respect to base change from k to any finitely generated commutative k-algebra R) action of G on $\mathcal{F}$ compatible with the action of G on X: for all open subsets $U \subset X$ and every $h, g \in G(R)$, an $\mathcal{O}_X(U_R)$-isomorphism ${}^g(-) : \mathcal{F}(U_R) \to \mathcal{F}(U_R^{g^{-1}})$ such that ${}^h(-) \circ {}^g(-) = {}^{gh}(-)$. This is equivalent to the following data: an isomorphism $\theta : \mu^*\mathcal{F} \stackrel{\sim}{\to} p^*\mathcal{F}$ (where $\mu,\ p : G \times X \to X$ are the action and projection maps) together with a cocycle condition on the pull-backs of θ to $G \times G \times X$ insuring that ${}^h(-) \circ {}^g(-) = {}^{gh}(-)$.

The argument of [4, Lemma 6.7] applies without change to show the following:

le:equiv **Lemma 1.4.** *Let M be a $\mathfrak{u}(\mathfrak{g})$-module. For any locally closed subset $X \subset \mathbb{E}(r, \mathfrak{g})$, $\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\}$, $\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\}$ are GL_r-invariant $\mathcal{O}_{\widetilde{X}}$-submodules of $M \otimes \mathcal{O}_{\widetilde{X}}$.*

The relevance of Lemma 1.4 to our consideration of coherent sheaves on $\mathbb{E}(r, \mathfrak{g})$ becomes evident in view of the following categorical equivalence.

prop:categ **Proposition 1.5.** *There is a natural equivalence of categories*

eq:equiv1
$$\eta : \mathrm{Coh}^{\mathrm{GL}_r}(\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o) \xrightarrow{\sim} \mathrm{Coh}(\mathbb{E}(r, \mathfrak{g})) \tag{1.5.1}$$

between the category of GL_r-equivariant coherent sheaves on $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o$ and the category of coherent sheaves on $\mathbb{E}(r, \mathfrak{g})$.

Moreover, (1.5.1) restricts to an equivalence of categories

eq:equiv2
$$\eta_X : \mathrm{Coh}^{\mathrm{GL}_r}(\widetilde{X}) \xrightarrow{\sim} \mathrm{Coh}(X) \tag{1.5.2}$$

for any locally closed subset $X \in \mathbb{E}(r, \mathfrak{g})$ and $\widetilde{X} \to X$ as in (1.1.1).

Proof. This follows from the observation that $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o \to \mathbb{E}(r, \mathfrak{g})$ is a GL_r-torsor. See, for example, [4, 6.5]. □

Proposition 1.5 immediately gives our construction of image and kernel sheaves.

thm:equiv3 **Theorem 1.6.** *Let M be a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module, let $X \subset \mathbb{E}(r, \mathfrak{g})$ be a locally closed subvariety, and let j be a positive integer with $j \leq (p-1)r$. Then the GL_r-invariant $\mathcal{O}_{\widetilde{X}}$-submodules of $M \otimes \mathcal{O}_{\widetilde{X}}$ of Lemma 1.4,*

$$Im\{\Theta^{j,\widetilde{X}}, M\}, \quad \mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\}$$

determine coherent subsheaves of $M \otimes \mathcal{O}_X$ on X:

$$\mathcal{I}m^{j,X}(M), \quad \mathcal{K}er^{j,X}(M)$$

via the categorical equivalence η_X of (1.5.2).

Let G be an affine algebraic group over k. Then the structure of a rational G-module on a k-vector space M is the data of a functorial action of $G(R)$ on $M \otimes R$ for all finitely generated commutative k-algebras R. This readily implies that if G acts on an algebraic variety X and if M is a finite dimensional rational G-module, then $M \otimes \mathcal{O}_X$ is a G-equivariant coherent O_X-module with G acting diagonally on the tensor product.

If $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of an affine algebraic group, then G acts on $\mathfrak{g}^{\times r}$ by the diagonal adjoint action and this action commutes with that of GL_r. This observation leads to the following refinement of Theorem 1.6.

cor:equiv **Corollary 1.7.** *Let G be an affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, $X \subset \mathbb{E}(r, \mathfrak{g})$ a G-stable locally closed subvariety, and M a rational G-module. Then $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$ are G-equivariant sheaves on X for any j with $1 \leq j \leq (p-1)r$.*

Proof. The diagonal action of G on $\mathfrak{g}^{\times r}$ determines an action of G on $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o$ and thus on $\widetilde{X} \subset \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^o$ over the G-stable subvariety $X \subset \mathbb{E}(r, \mathfrak{g})$. We thus may consider the category of G-equivariant coherent sheaves on $\widetilde{X}$. If M is a rational G-module, then the maps (1.1.2) are maps of G-equivariant coherent sheaves on $\widetilde{X}$; consequently, $\mathrm{Im}\{\Theta^{j,\tilde{X}}, M\}$, $\mathrm{Ker}\{\Theta^{j,\tilde{X}}, M\}$ of Definition 1.2 are G-equivariant coherent sheaves on $\widetilde{X}$. Since the action of GL_r on these G-equivariant coherent sheaves on $\widetilde{X}$ commutes with this action of G, the equivalence η_X in (1.5.2) (given by descent) sends these G-equivariant coherent sheaves on $\widetilde{X}$ to G-equivariant coherent sheaves on X. $\square$

The special case of Corollary 1.7 in which X is a G-orbit is of particular interest since any G-equivariant sheaf on such an X is a G-equivariant vector bundle.

We now proceed to identify these image and kernel subsheaves of $\mathcal{O}_X \otimes M$ when restricted to open subsets $X \cap \mathcal{U}_\Sigma$, where $\{\mathcal{U}_\Sigma\}$ is a standard affine open covering of $\mathrm{Grass}(r,n)$ such that the GL_r-torsor $\rho : \mathbb{M}^\circ_{n,r} \to \mathrm{Grass}(r,n)$ splits over each $\mathcal{U}_\Sigma$.

Once again, we choose a basis $x_1, \dots, x_n$ for $\mathfrak{g}$ as a k-vector space, thereby identifying $\mathrm{Grass}(r,\mathfrak{g})$ with $\mathrm{Grass}(r,n)$. Let $\Sigma \subset \{1,\dots,n\}$ range over the subsets of cardinality r. For a given Σ, let $\rho^{-1}(\mathcal{U}_\Sigma) \subset \mathbb{M}^\circ_{n,r}$ be a subset of those $n \times r$ matrices whose $r \times r$-minor with rows indexed by elements of Σ has non-vanishing determinant. Thus, $\mathcal{U}_\Sigma \subset \mathrm{Grass}(r,n)$ consists of those r-planes in n-space which project onto r-space via the map which forgets the coordinates not indexed by elements of Σ. We define the section

$$s_\Sigma : \mathcal{U}_\Sigma \ \to \ \rho^{-1}(\mathcal{U}_\Sigma)$$

by sending an r-plane $L \in \mathcal{U}_\Sigma$ to the unique $n \times r$-matrix $\widetilde{L}$ satisfying the conditions that $\rho(\widetilde{L}) = L$ and that the $r \times r$-minor of $\widetilde{L}$ with rows indexed by elements of Σ is the identity matrix I_r. In particular, $\rho^{-1}(\mathcal{U}_\Sigma) \to \mathcal{U}_\Sigma$ is a trivial GL_r-torsor for any Σ.

note3 **Notation 1.8.** For $\Sigma = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$, s_Σ provides an identification of $k[\mathcal{U}_\Sigma]$ with the quotient

eq:YSigma
$$k[\mathbb{M}_{n,r}] = k[T_{i,j}]_{1\le i\le n, 1\le j\le r} \ \longrightarrow \ k[Y^\Sigma_{i,j}]_{i\notin\Sigma, 1\le j\le r} = k[\mathcal{U}_\Sigma] \tag{1.8.1}$$

sending $T_{i,j}$ to 1, if $i = i_j \in \Sigma$; to 0 if $i = i_{j'} \in \Sigma$ and $j \neq j'$; and to $Y^\Sigma_{i,j}$ otherwise. For notational convenience, we set $Y^\Sigma_{i,j}$ equal to 1, if $i \in \Sigma$ and $i = i_j$, and we set $Y^\Sigma_{i,j} = 0$ if $i = i_{j'} \in \Sigma$ and $j \neq j'$.

As in (1.0.2), we define

eq:Theta
$$\Theta^\Sigma_s \equiv \sum_{i=1}^n x_i \otimes Y^\Sigma_{i,s} : M \otimes k[\mathcal{U}_\Sigma] \to M \otimes k[\mathcal{U}_\Sigma], \tag{1.8.2}$$

by

$$m \otimes 1 \mapsto \sum_i x_i(m) \otimes Y^\Sigma_{i,s}.$$

For any closed subset $i : W \subset \mathrm{Grass}(r,\mathfrak{g}) \simeq \mathrm{Grass}(r,n)$, set $W_\Sigma = W \cap \mathcal{U}_\Sigma$ and

$$Y^{W,\Sigma}_{i,j} \ = \ i^*(Y^\Sigma_{i,j}).$$

We define

eq:ThetaW
$$\Theta^{W,\Sigma}_s \equiv \sum_{i=1}^n x_i \otimes Y^{W,\Sigma}_{i,s} : M \otimes k[W_\Sigma] \to M \otimes k[W_\Sigma], \tag{1.8.3}$$

by

$$m \otimes 1 \mapsto \sum_i x_i(m) \otimes Y^{W,\Sigma}_{i,s}.$$

defn:localj

Definition 1.9. Let M be a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module, let $W \subset \mathbb{E}(r, \mathfrak{g})$ be a closed subset, let $\Sigma \subset \{1, \ldots, n\}$ be a subset of cardinality r, and choose j such that $1 < j \leq (p-1)r$. We define the following $k[W_\Sigma]$-submodules of the free module $M \otimes k[W_\Sigma]$:

$$\mathcal{K}er^{j,W_\Sigma}(M) \equiv \operatorname{Ker}\{\bigoplus_{j_1+\cdots+j_r=j} (\Theta_1^{W,\Sigma})^{j_1} \ldots (\Theta_r^{W,\Sigma})^{j_r} : M \otimes k[W_\Sigma] \to (M \otimes k[W_\Sigma])^{\oplus r(j)}\}$$

$$\mathcal{I}m^{j,W_\Sigma}(M) \equiv \operatorname{Im}\{\sum_{j_1+\cdots+j_r=j} (\Theta_1^{W,\Sigma})^{j_1} \ldots (\Theta_r^{W,\Sigma})^{j_r} : (M \otimes k[W_\Sigma])^{\oplus r(j)} \to M \otimes k[W_\Sigma]\}$$

where $r(j)$ is the number of ways j can be written as a sum of r nonnegative integers, $j = j_1 + \cdots + j_r$.

We identify these $k[W_\Sigma]$-submodules of $M \otimes k[W_\Sigma]$ with coherent subsheaves of the free $\mathcal{O}_W$-module $M \otimes \mathcal{O}_W$ restricted to the affine open subvariety $W_\Sigma \subset W$.

thm:compare

Theorem 1.10. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, $X \subset \mathbb{E}(r, \mathfrak{g})$ be a locally closed subset, $W = \overline{X}$ be the closure of X, and r, j be positive integers with $j \leq (p-1)r$. Then*

$$\mathcal{I}m^{j,X}(M) \subset M \otimes \mathcal{O}_X \quad \textit{restricted to } X \cap \mathcal{U}_\Sigma \subset X$$

equals

$$\mathcal{I}m^{j,W_\Sigma}(M) \subset M \otimes \mathcal{O}_{W_\Sigma} \quad \textit{restricted to } X \cap \mathcal{U}_\Sigma \subset W_\Sigma.$$

The analogous identification of restrictions of $\mathcal{K}er^{j,X}(M) \subset M \otimes \mathcal{O}_X$ are also valid.

Proof. It suffices to show that the asserted equalities of subsheaves of $M \otimes \mathcal{O}_X$ on X are valid when restricted to each open chart $\mathcal{U}_\Sigma \cap X$ of X as Σ runs through subsets of cardinality r in $\{1, 2, \ldots, n\}$. Moreover, it suffices to verify the equality of subsheaves of $M \otimes \mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$ obtained by pulling back these restrictions along the map $\mathbb{M}^\circ_{n,r} \to \operatorname{Grass}(r, n)$. These equalities are verified by comparing the formulation of Θ_s in (1.0.2) with that of Θ_s^Σ in (1.8.2); namely, Θ_s^Σ is the restriction along the section $s_\Sigma : \mathcal{U}_\Sigma \to \rho^{-1}(\mathcal{U}_\Sigma)$ of Θ_s. $\square$

The following proposition identifies the "generic" fibers of the image and kernel sheaves. This is particularly useful when the locally closed subset $X \subset \mathbb{E}(r, \mathfrak{g})$ is an orbit closure. Here and throughout the paper, we denote by $\epsilon^* M$ the restriction of a $\mathfrak{g}$-module M to the subalgebra $\epsilon \subset \mathfrak{g}$.

prop:fibers

Proposition 1.11. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, $X \subset \mathbb{E}(r, \mathfrak{g})$ be a locally closed subset, $W = \overline{X}$ be the closure of X, and r, j be positive integers with $j \leq (p-1)r$. For any $\Sigma \subset \{1, \ldots, n\}$ of cardinality r there exists an open dense subset $U \subset X \cap W_\Sigma$ such that for any point $\epsilon \in U$ with residue field K there are natural identifications*

$$\mathcal{I}m^{j,X}(M)_\epsilon = \mathcal{I}m^j(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} K \;=\; \operatorname{Rad}^j(\epsilon^*(M_K)),$$

$$\mathcal{K}er^{j,X}(M)_\epsilon = \mathcal{K}er^j(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} K \;=\; \operatorname{Soc}^j(\epsilon^*(M_K)).$$

Proof. Since X is open dense in W, we may assume that $W = X$. For $\epsilon \in W_\Sigma$ a generic point, the given identifications are immediate consequences of the exactness of localization and Definition 1.9. The fact that these identifications apply to an open subset now follows from the generic flatness of the $k[W_\Sigma]$-modules $\mathcal{I}m^j(M)_{W_\Sigma}$, $\mathcal{K}er^j(M)_{W_\Sigma}$. $\square$

Remark 1.12. For an elementary example of the failure of the isomorphism $\mathcal{K}er^j(M)_\epsilon \simeq \mathrm{Soc}^j(\epsilon^*M)$ outside of an open subset of W_Σ, we consider $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_a$, $r = 1$, and $j = 1$. Let $\{x_1, x_2\}$ be a fixed basis of $\mathfrak{g}$, and let M be the four dimensional module with basis $\{m_1, \ldots, m_4\}$, such that $x_1m_1 = m_4,\ x_1m_2 = x_1m_3 = x_1m_4 = 0$ and $x_2m_1 = m_3,\ x_2m_2 = m_4,\ x_2m_3 = x_2m_4 = 0$. We can picture M as follows:

$$\begin{tikzcd}
m_2 \arrow[dr, "x_2"] & & m_1 \arrow[dl, "x_1"] \arrow[dr, "x_2"] & \\
& m_4 & & m_3.
\end{tikzcd}$$

The kernel of

$$x_1 \otimes 1 + x_2 \otimes T_2^{\{1\}} : M \otimes k[T_2^{\{1\}}] \to M \otimes k[T_2^{\{1\}}]$$

(as in Definition 1.9) with $j = 1$ is a free $k[T_2^{\{1\}}]$-module of rank 2, generated by $m_3 \otimes 1$ and $m_4 \otimes 1$. The specialization of this module at the point $\epsilon = kx_1$ (letting $T_2 \to 0$) is a vector space of dimension 2. This is a proper subspace of $\mathrm{Soc}(\epsilon^*(M))$ which is spanned by m_2, m_3, m_4.

Remark 1.13. In this paper, we concentrate on the variety of elementary subalgebras $\mathbb{E}(r, \mathfrak{g})$ and its G-orbits. Nevertheless, the formalism of the equivariant descent construction of the sheaves $\mathcal{I}m^X(M)$, $\mathcal{K}er^X(M)$ works equally well for any locally closed subvariety $X \subset \mathrm{Grass}(r, \mathfrak{g})$, since the commutativity condition that defines $\mathbb{E}(r, \mathfrak{g})$ does not enter into the construction of the image and kernel sheaves for $j = 1$. Moreover, one can easily check that the identification of the image and kernel sheaves on the affine charts as in Theorem 1.10 and identification of the generic fibers as in Proposition 1.11 remain valid.

2. Geometric invariants of $\mathfrak{u}(\mathfrak{g})$-modules

sec:geominv

In this section, we discuss various invariants of finite dimensional $\mathfrak{u}(\mathfrak{g})$-modules M which involve consideration of the projective varieties $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of $\mathfrak{g}$. We begin by recalling various closed subvarieties of $\mathbb{E}(r, \mathfrak{g})$ introduced in [9] associated to M, before considering the image and kernel sheaves of Section 1. As always in this paper, $\mathfrak{g}$ denotes a finite dimensional restricted Lie algebra over k. For a Lie subalgebra $\epsilon \subset \mathfrak{g}$, we denote by ϵ^*M the restriction of a $\mathfrak{g}$-module M to ϵ.

defn:subvar **Definition 2.1.** ([5, 3.2],[5, 3.15]) Let M be a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module, r a positive integer, and j an integer satisfying $1 \le j \le (p-1)r$. We define the following closed subvarieties of $\mathbb{E}(r, \mathfrak{g})$ associated to M. Let

eq:supp (2.1.1) $$\mathbb{E}(r, \mathfrak{g})_M \equiv \{\epsilon \in \mathbb{E}(r, \mathfrak{g}) : \epsilon^*M \text{ is not projective}\},$$

eq:rad (2.1.2) $$\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}} \equiv \{\epsilon \in \mathbb{E}(\mathrm{r}, \mathfrak{g}) : \dim(\mathrm{Rad}^{\mathrm{j}}(\epsilon^*\mathrm{M})) < \mathrm{Max}\,\mathrm{R_j}\},$$

and

eq:soc (2.1.3) $$\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}} \equiv \{\epsilon \in \mathbb{E}(\mathrm{r}, \mathfrak{g}) : \dim(\mathrm{Soc}^{\mathrm{j}}(\epsilon^*\mathrm{M})) > \mathrm{Min}\,\mathrm{S_j}\},$$

where $\mathrm{Max}\,R_j$ is the maximum value of $\dim(\mathrm{Rad}^j(\epsilon'^*M))$ and $\mathrm{Min}\,S_j$ is the minimum value of $\dim(\mathrm{Soc}^j(\epsilon'^*M))$, as ϵ' ranges over all elements of $\mathbb{E}(r, \mathfrak{g})$.

The closed subvariety $\mathbb{E}(1,\mathfrak{g})_M$ equals the (projectivized) support (or, equivalently, rank) variety of M considered by various authors (e.g., [8]). If $\mathbb{E}(1,\mathfrak{g})_M \neq \mathbb{E}(1,\mathfrak{g})$, then

$$\mathbb{E}(1,\mathfrak{g})_M \;=\; \mathbb{R}\mathrm{ad}(1,\mathfrak{g})_{\mathrm{M}} \;=\; \mathbb{S}\mathrm{oc}(1,\mathfrak{g})_{\mathrm{M}}.$$

For $r = 1$, these subvarieties were introduced in [10]; for general r, j they were defined in [5, 3.1].

The reader is directed to [4, 4.6] for an interesting example of a module M for $\mathfrak{u}(\mathfrak{g})$ with $\mathfrak{g}$ elementary for which $\mathbb{R}\mathrm{ad}^1(2,\mathfrak{g})_{\mathrm{M}} \neq \emptyset,\ \mathbb{S}\mathrm{oc}^1(2,\mathfrak{g})_{\mathrm{M}} = \emptyset$.

The following theorem emphasizes the additional information given by the image and kernel sheaves of Section 1.

thm:bundle

Theorem 2.2. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, r and j be positive integers, such that $j \leq (p-1)r$. Set $Z = \mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}}$ (resp, $Z = \mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}}$) and let $X = \mathbb{E}(r,\mathfrak{g}) \setminus Z$ denote the Zariski open subset of $\mathbb{E}(r,\mathfrak{g})$ given as the complement of Z.*

Then $\mathcal{I}m^{j,X}(M) = \mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M)_{|X}$ and $\mathcal{K}er^{j,X}(M) = \mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)_{|X}$ are algebraic vector bundles on X.

Moreover, the fiber of $\mathcal{I}m^{j,X}(M)$ (reps., $\mathcal{K}er^{j,X}(M)$) at any $\epsilon \in X$ is naturally identified with $\mathrm{Rad}^j(\epsilon^* M)$ (resp. $\mathrm{Soc}^j(\epsilon^* M)$).

Proof. It suffices to restrict to an arbitrary $\Sigma \subset \{1,\ldots,n\}$ of cardinality r and prove that the $\mathcal{O}_{\mathcal{U}_\Sigma \cap X}$-modules $\mathcal{I}m^j(M)_{|\mathcal{U}_\Sigma\cap X}$ (resp., $(\mathcal{K}er^j(M)_{|\mathcal{U}_\Sigma\cap X})$ are locally free. Here, $\mathcal{U}_\Sigma \subset \mathrm{Grass}(r,\mathfrak{g})$ is as in Notation 1.8. We set W equal to the closure of X in $\mathbb{E}(r,\mathfrak{g})$.

Set $\Theta_s^\Sigma(\epsilon)$ for $\epsilon \in \mathcal{U}_\Sigma$ equal to the specialization of Θ_s^Σ (as given in (1.8.2)) at the point ϵ. Thus, $\Theta_s^\Sigma(\epsilon)$ is the map $\Theta_s^\Sigma \otimes_{k[\mathcal{U}_\Sigma]} k$ defined by tensoring along evaluation at ϵ, $k[\mathcal{U}_\Sigma] \to k$. Since specialization is right exact,

$$\mathrm{Coker}\{\sum_{\sum j_i = j} (\Theta_1^\Sigma)^{j_1}\cdots(\Theta_r^\Sigma)^{j_r}\} \otimes_{k[\mathcal{U}_\Sigma]} k \;=\; \mathrm{Coker}\{\sum_{\sum j_i = j} \Theta_1^\Sigma(\epsilon)^{j_1}\cdots\Theta_r^\Sigma(\epsilon)^{j_r}\}.$$

This equals

$$\mathrm{Coker}\{\sum_{\sum j_i = j} (\Theta_1^{W,\Sigma})^{j_1}\cdots(\Theta_r^{W,\Sigma})^{j_r}\} \otimes_{k[W_\Sigma]} k$$

for $\epsilon \in W_\Sigma = \mathcal{U}_\Sigma \cap W$. Exactly as in the proof of [4, 6.2], the hypothesis that $\dim \mathrm{Rad}^j(\epsilon^* M)$ is the same for any $\epsilon \in X$ implies that $\mathrm{Coker}\{\sum_{\sum j_i = j} (\Theta_1^\Sigma)^{j_1}\cdots(\Theta_r^\Sigma)^{j_r}\}_{|\mathcal{U}_\Sigma\cap X}$ is a locally free $\mathcal{O}_{\mathcal{U}_\Sigma\cap X}$-module. The short exact sequence

$$0 \longrightarrow \mathcal{I}m^j(M)_{W_\Sigma} \longrightarrow (M \otimes k[W_\Sigma])^{\oplus r(j)} \longrightarrow \mathrm{Coker}\{\sum_{\sum j_i = j} (\Theta_1^{W,\Sigma})^{j_1}\cdots(\Theta_r^{W,\Sigma})^{j_r}\} \longrightarrow 0$$

localized at $\mathcal{U}_\Sigma \cap X$ implies that $\mathcal{I}m^j(M)_{|\mathcal{U}_\Sigma\cap X}$ is locally free, and also enables the identification of the fiber above $\epsilon \in \mathcal{U}_\Sigma \cap X$.

The proof for $\mathcal{K}er^j(M)$ is a minor adaptation of above; see also the proof of Theorem 6.2 of [4]. $\square$

We recall from [5, 4.1], that a $\mathfrak{u}(\mathfrak{g})$-module is said to have constant (r,j)-radical rank (respectively, (r,j)-socle rank) if the dimension of $\mathrm{Rad}^j(\epsilon^* M)$ (respectively, $\mathrm{Soc}^j(\epsilon^* M)$) is independent of $\epsilon \in \mathbb{E}(r,\mathfrak{g})$. As an immediate corollary of Theorem 2.2, we verify that $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ (respectively, $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$) is an algebraic vector bundle on $\mathbb{E}(r,\mathfrak{g})$ provided that M has constant (r,j)-radical rank (respectively, constant (r,j)-socle rank).

cor:bundle **Corollary 2.3.** *Let M be an $\mathfrak{u}(\mathfrak{g})$-module of constant (r,j)-radical rank (respectively, (r,j)-socle rank). Then the coherent sheaf $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ (resp., $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$) is an algebraic vector bundle on $\mathbb{E}(r,\mathfrak{g})$.*

Moreover, the fiber of $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ (respectively, $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$) at ϵ is naturally identified with $\mathrm{Rad}^j(\epsilon^*M)$ (respectively, $\mathrm{Soc}^j(\epsilon^*M)$).

Proof. The condition of constant (r,j)-radical rank (respectively, (r,j)-socle rank) implies that $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}} = \emptyset$ (respectively, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}} = \emptyset$). Hence, the corollary is a special case of Theorem 2.2 with $X = \mathbb{E}(r,\mathfrak{g})$. □

ex:unip **Example 2.4.** Let $\mathfrak{u}$ be a nilpotent restricted Lie algebra such that $x^{[p]} = 0$ for any $x \in \mathfrak{u}$, and let $\mathfrak{u}$ also denote the adjoint module of $\mathfrak{u}$ on itself. Assume that $\mathfrak{u}$ has a maximal elementary subalgebra of dimension r. We see that $X = \mathbb{E}(r,\mathfrak{u})\backslash\mathbb{S}\mathrm{oc}(\mathrm{r},\mathfrak{u})_{\mathfrak{u}}$ is the open subvariety of $\mathbb{E}(r,\mathfrak{u})$ consisting of all maximal elementary subalgebras of dimension r. That is, if $\epsilon \in X$ is maximal, then $\mathrm{Soc}(\epsilon^*(\mathfrak{u})) = \epsilon$, and otherwise the dimension of $\mathrm{Soc}(\epsilon^*(\mathfrak{u}))$ is larger than r. Applying Theorem 2.2 we conclude that $\mathcal{K}er^{1,X}(\mathfrak{u}) \subset \mathfrak{u}\otimes\mathcal{O}_X$ is isomorphic to the restriction along $X \subset \mathbb{E}(r,\mathfrak{u}) \subset \mathrm{Grass}(r,\mathfrak{u})$ of the canonical rank r subbundle $\gamma_r \subset \mathfrak{u}\otimes\mathcal{O}_{\mathrm{Grass}(r,\mathfrak{u})}$.

If we take $\mathfrak{u}$ to be the Heisenberg algebra $\mathfrak{u}_3$ (the Lie subalgebra of strictly upper triangular matrices in $\mathfrak{gl}_3$), then $\mathbb{E}(2,\mathfrak{u}_3) \simeq \mathbb{P}^1$ whenever $p \geq 3$ and every $\epsilon \in \mathbb{E}(2,\mathfrak{u}_3)$ is maximal. In this case,

$$\mathcal{K}er^{1,\mathbb{E}(2,\mathfrak{u}_3)}(\mathfrak{u}_3) \;\simeq\; \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1} \;\subset\; \mathfrak{u}_3\otimes\mathcal{O}_{\mathbb{P}^1}.$$

The following proposition refines the analysis given in [9] of projective modules on $\mathfrak{sl}_2^{\oplus r}$. We implicitly use the isomorphism $\mathbb{E}(r,\mathfrak{sl}_2^{\oplus r}) \simeq (\mathbb{P}^1)^{\times r}$ of [5, 1.12].

Proposition 2.5. *Let $\mathfrak{g} = \mathfrak{sl}_2^{\oplus r}$ and let $\pi_s : \mathfrak{g} \to \mathfrak{sl}_2$ be the s-th projection, $1 \leq s \leq r$. Assume $p \geq 3$. For each λ, $0 \leq \lambda \leq p-1$, let P_λ be the indecomposable projective $\mathfrak{u}(\mathfrak{sl}_2)$-module of highest weight λ. Then for each $(\lambda,s) \neq (\lambda',s')$, there exists some j such that the vector bundle $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(\pi_s^*(P_\lambda))$ on $\mathbb{E}(r,\mathfrak{g})$ is not isomorphic to $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(\pi_{s'}^*(P_{\lambda'}))$.*

Proof. Observe that $\mathrm{Soc}^j(\epsilon^*(\pi_s^*M)) = \mathrm{Soc}^j(\epsilon_s^*M)$ for any $\mathfrak{u}(\mathfrak{sl}_2)$-module M and any $j, 1 \leq j \leq (p-1)r$, where $\epsilon = (\epsilon_1,\ldots,\epsilon_r) \in \mathbb{E}(r,\mathfrak{sl}_2^{\oplus r})$; in particular, the action of ϵ on $\epsilon^*(\pi_s^*M)$ factors through ϵ_s. This implies that $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(\pi_s^*(P_\lambda)) \simeq \pi_s^*(\mathcal{K}er^{j,\mathbb{E}(1,\mathfrak{sl}_2)}(P_\lambda))$. The proposition now follows from the computation given in [9, 6.3]. □

A challenge in defining invariants for a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M is to select some natural values from the vast family of coherent sheaves having the form $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$ parametrized by $r > 0$, $1 \leq j \leq (p-1)r$, and $X \subset \mathbb{E}(r,\mathfrak{g})$ locally closed. For this purpose, it is natural to consider Grothendieck groups as recalled below.

Definition 2.6. Let X be a quasi-projective variety over k. We denote by $K_0(X)$ the Grothendieck group of algebraic vector bundles on X (i.e., locally free, coherent sheaves of $\mathcal{O}_X$-modules). We denote by $K_0'(X)$ the Grothendieck group of coherent sheaves of $\mathcal{O}_X$-modules. We recall that tensor product (of $\mathcal{O}_X$-modules) provides

$K_0(X)$ with the structure of a commutative ring and $K'_0(X)$ with the structure of a $K_0(X)$-module.

Remark 2.7. The group $K_0(X)$ is very difficult to compute if X is a singular variety. On the other hand, $K'_0(-)$ satisfies various general properties which make it more accessible to computation, especially the property of localization proved by D. Quillen [19]. If X is smooth, the natural map $K_0(X) \to K'_0(X)$ is an isomorphism.

defn:kprime

Definition 2.8. Let M be a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module. For each $r > 0$, and each j such that $1 \leq j \leq (p-1)r$, we define

$$im^{j,r}(M),\ ker^{j,r}(M) \ \in \ K'_0(\mathbb{E}(r,\mathfrak{g}))$$

to be the classes of the coherent sheaves $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M),\ \mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ of $\mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}$-modules.

Example 2.9. We consider $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_{2m}) = \mathfrak{gl}_{2m}$ and $r = m^2$. Recall that $\mathbb{E}(m^2, \mathfrak{gl}_{2m})$ is a single GL_{2m}-orbit, isomorphic to $\mathrm{Grass}(m, 2m)$ when $p \nmid 2m$ (see [5, §2]). Thus,

$$K'_0(\mathbb{E}(m^2, \mathfrak{gl}_{2m})) \simeq K_0(\mathrm{Grass}(m, 2m)) \simeq \mathbb{Z}^{\binom{2m}{m}}$$

(see, for example, [18, 2.2] for the last isomorphism).

Remark 2.10. If the finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M is either of constant (r,j)-radical rank or of constant (r,j)-socle rank as in Corollary 2.3, then the classes $im^{j,r}(M),\ ker^{j,r}(M) \ \in \ K'_0(\mathbb{E}(r,\mathfrak{g}))$ of Definition 2.8 lift to classes in $K_0(\mathbb{E}(r,\mathfrak{g}))$ and these lifting are more natural choices of invariants.

If $\mathbb{E}(r,\mathfrak{g})$ is singular, the vector bundles $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M)$, $\mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ are especially interesting, for construction of vector bundles on singular varieties is particularly challenging. It is worthy of note that in most examples given in [4] these image and kernel bundles are not G-equivariant for some algebraic group G.

On the other hand, the lack of calculations for $K_0(X)$ for such singular X means that we have few if any means of identifying the classes in $K_0(\mathbb{E}(r,\mathfrak{g}))$ of $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M),\ \mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$.

In the context in which G is an affine algebraic group and M a rational G-module, we can refine the invariants of Definition 2.8 by using the G-equivariant K'-theory, $X \mapsto K'_0(G;X)$, of R. Thomason [24]. In this context, the *representation ring* $R(G)$ in incorporated into the invariants. (Recall that $R(G)$ is the free abelian group on the irreducible representations of G.)

rem:kprimeG

Remark 2.11. Let G be an affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, and M a finite dimensional rational G-module. For each $r > 0$, each j with $1 \leq j \leq (p-1)r$, we can define

$$im^{j,r;G}(M),\ ker^{j,r;G}(M) \ \in \ K'_0(G; \mathbb{E}(r,\mathfrak{g}))$$

to be the classes of the G-equivariant coherent sheaves $\mathcal{I}m^{j,\mathbb{E}(r,\mathfrak{g})}(M),\ \mathcal{K}er^{j,\mathbb{E}(r,\mathfrak{g})}(M)$ of $\mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}$-modules.

The interested reader is referred to the excellent survey paper of A. Merkurjev [16] for much useful background about G-equivariant K'-theory. In particular, we point out the following result:

$$\mathbb{Z} \otimes_{R(G)} K'_0(G;X) \ \simeq K'_0(X).$$

sec:Gorbit

3. Vector bundles on G-orbits of $\mathbb{E}(r,\mathfrak{g})$: constructions

Our explicit examples of algebraic vector bundles involve considerations of image, cokernel, and kernel sheaves associated to a rational G-module on G-orbits of $\mathbb{E}(r,\mathfrak{g})$, where G is an algebraic group and $\mathfrak{g}$ is the Lie algebra of G. In this section we develop and recall the techniques which allow us to calculate some examples in the next section. In particular, Theorem 3.1 verifies that the image and kernel sheaves determine algebraic vector bundles on G-orbits inside $\mathbb{E}(r,\mathfrak{g})$. These vector bundles are interpreted in Theorem 3.6 in terms of the well-known induction functor from rational H-modules to vector bundles on G/H for an appropriate subgroup $H \subset G$. This latter theorem is the main computational tool that we apply in Section 4.

Let $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ be an elementary subalgebra, and let $X = G\cdot\epsilon$ be the G-orbit of ϵ in $\mathbb{E}(r,\mathfrak{g})$. Then X is open in its closure and, hence, to any finite-dimensional rational G-representation M and any j, $1 \leq j \leq (p-1)r$, we can associate coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ on X as in Theorem 1.6. Let $\mathcal{C}oker^{j,X}(M)$ denote the quotient sheaf $(M\otimes\mathcal{O}_X)/\mathcal{I}m^{j,X}(M)$.

thm:orbit2

Theorem 3.1. *Let G be an affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, and M a rational G-module. Let $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ be an elementary subalgebra of rank r, and let $X = G\cdot\epsilon \subset \mathbb{E}(r,\mathfrak{g})$ be the orbit of ϵ under the adjoint action of G.*

Then

$$\mathcal{I}m^{j,X}(M),\quad \mathcal{K}er^{j,X}(M),\quad \mathcal{C}oker^{j,X}(M)$$

are algebraic vector bundles on X.

Moreover, we have natural identifications as H-modules

$$\mathcal{I}m^{j,X}(M)_\epsilon \;\simeq\; \mathrm{Rad}^j(\epsilon^*M),\quad \mathcal{I}m^{j,X}(M)_\epsilon \;\simeq\; \mathrm{Soc}^j(\epsilon^*M),$$

where H is the (reduced) stabilizer of $\epsilon\in X$.

Proof. Since X is a G-stable locally closed subset of $\mathbb{E}(r,\mathfrak{g})$, the coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{C}oker^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ are G-equivariant by Corollary 1.7. If $x = g\cdot\epsilon$ for some $g\in G$, then the action of g on one of these sheaves sends the fiber at ϵ isomorphically to the fiber at x. Since X is Noetherian, we conclude that the sheaves are locally free (see, for example, [9, 4.11] or [12, 5. ex. 5.8]).

The action of G on $\mathbb{E}(r,\mathfrak{g})$ determines for each $g\in G,\ x\in X$ an isomorphism $g: \mathcal{O}_{X,x}\to\mathcal{O}_{X,g^{-1}x}$. Together with the action of G on M, this determines the (diagonal) action $g: M\otimes\mathcal{O}_{X,x}\to M\otimes\mathcal{O}_{X,g^{-1}x}$. In particular, this determines an action of H on $M\otimes\mathcal{O}_{X,\epsilon}$. Since the action of G is $\mathcal{O}_X$-linear, this determines actions of H on the fibers at ϵ, $\mathcal{I}m^{j,X}(M)_\epsilon$, $\mathcal{I}m^{j,X}(M)_\epsilon$, of the coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{I}m^{j,X}(M)$. As is readily checked using the explicit description of the action just given, this action on the fibers is that determined by the action of $H\subset G$ on M. The second assertion now follows using the isomorphisms of Proposition 1.11. $\square$

The quotient of an affine algebraic group G by a closed subgroup H is representable by variety G/H (see [14, I.5.6(8)]). The following "sheaf-theoretic induction functor" enables a reasonably explicit description of G-equivariant coherent sheaves on G/H.

prop:cL **Proposition 3.2.** *Let G be an affine algebraic group, $H \subset G$ a closed subgroup. For each (finite dimensional) rational H-module W, consider the sheaf of $\mathcal{O}_{G/H}$-modules $\mathcal{L}_{G/H}(W)$ which sends an open subset $U \subset G/H$ to*

cL (3.2.1) $$\mathcal{L}_{G/H}(W)(U) \;=\; \{\textit{sections of } G \times^H W \to G/H \textit{ above } U\}.$$

(1) *So defined, $W \mapsto \mathcal{L}_{G/H}(W)$ induces an equivalence of categories*

l:eq (3.2.2) $$\left\{ \begin{array}{c} \textit{finite dimensional} \\ \textit{rational } H\textit{-modules} \end{array} \right\} \quad \sim \quad \left\{ \begin{array}{c} G\textit{-equivariant algebraic} \\ \textit{vector bundles on } G/H \end{array} \right\}$$

(2) *If W is the restriction of a rational G-module, then $\mathcal{L}_{G/H}(W)$ is isomorphic to $W \otimes \mathcal{O}_{G/H}$, a free coherent sheaf of $\mathcal{O}_{G/H}$-modules.*

(3) *$\mathcal{L}_{G/H}(-)$ is exact and commutes with tensor powers $(-)^{\otimes^i}$, duals $(-)^{\#}$, symmetric powers $S^i(-)$, divided powers $\Gamma^i(-)$, exterior powers $\Lambda^i(-)$, and Frobenius twists $(-)^{(i)}$.*

Proof. A discussion of the functor $\mathcal{L}_{G/H}(-)$ of (3.2.1) can be found in in [14, I.5.8, I.5.9] as well as in [18, §1].

The key property of this functor in our context of $H \subset G$ is that $\mathcal{L}_{G/H}(W)$ as in (1) is a locally free sheaf on G/H. This follows from the fact that $p : G \to G/H$ is an H-torsor in the fppf topology (see [6, III.4.1.8]) and that the sheaf of sections of such an H-torsor is locally trivial in the Zariski topology by [6, III.4. 2.4].

The fact that $\mathcal{L}_{G/H}(-)$ is G-equivariant and induces the equivalence of categories (3.2.2) is observed in [18, §1] where the functors inducing this equivalence are made explicit. Statement (2) easily follows from the equivalence (3.2.2) and is also shown in [14, 5.12.(3)] in greater generality than we need here.

To prove (3), we refer the reader to the proof of [14, II.4.1] in the special case in which $H \subset G$ is a parabolic subgroup of a reductive algebraic group; this restriction on $H \subset G$ is used only to insure that $\mathcal{L}_{G/H}(W)$ is locally free which is verified above for more general $H \subset G$. $\square$

The following proposition is an essentially immediate corollary of Proposition 3.2, especially (3.2.2).

prop:cL2 **Proposition 3.3.** *Let G be an affine algebraic group, and $H \subset G$ be a closed subgroup. Let $\mathcal{E}$ be a G-equivariant vector bundle on G/H; set W equal to the fiber of $\mathcal{E}$ over the coset $eH \in G/H$ and equip this fiber with the H-module structure obtained by restricting the action of G to H (which stabilizes the fiber over $eH \in G/H$). Then there is a unique G-equivariant isomorphism*

$$\mathcal{L}_{G/H}(W) \quad \simeq \quad \mathcal{E}$$

which is the identity map on fibers over the coset $eH \in G/H$.

We point out the following immediate consequence of Proposition 3.3.

cor:cansub **Corollary 3.4.** *Let W be a rational representation of H of dimension r, V a rational representation of G, and $W \subset V$ be a monomorphism of H-modules. Then $\mathcal{L}_{G/H}(W) \;\subset\; \mathcal{L}_{G/H}(V) = V \otimes \mathcal{O}_{G/H}$ naturally corresponds to a map $f : G/H \to \mathrm{Grass}(r, V)$ sending the orbit gH to the subspace $g \cdot W \subset V$. Under this correspondence, we have*

$$\mathcal{L}_{G/H}(W) = f^*(\gamma_r),$$

where γ_r *is the canonical rank* r *subbundle on* $\mathrm{Grass}(r, V)$*. Moreover, the embedding* $\mathcal{L}_{G/H}(W) \subset \mathcal{L}_{G/H}(V) = V \otimes \mathcal{O}_{G/H}$ *is the pull-back via* f *of the canonical embedding* $\gamma_r \subset V \otimes \mathcal{O}_{\mathrm{Grass}(r,V)}$.

ex:standard **Example 3.5.** We identify some standard bundles using the functor $\mathcal{L}$ in the special case $G = \mathrm{GL}(V) = \mathrm{GL}_n$ and $P = P_{r,n-r}$, a maximal parabolic with the Levi factor $L \simeq \mathrm{GL}_r \times \mathrm{GL}_{n-r}$. Set $X = \mathrm{Grass}(r, V) = G/P$ and denote by $W \subset V$ the subspace of dimension r stabilized by P; the action of P on W is given by composition of the projection $P \to L \to \mathrm{GL}(W)$. Then $\mathcal{L}_{G/P}(W) \subset V \otimes \mathcal{O}_{G/P}$ corresponds to the G-equivariant isomorphism $f : G/P \xrightarrow{\sim} \mathrm{Grass}(r, V)$ sending the identity coset to $W \subset V$. Thus, by Corollary 3.4 we have isomorphisms

deltagamma
$$\gamma_r \simeq \mathcal{L}_{G/P}(W) \subset \mathcal{L}_{G/P}(V) \simeq V \otimes \mathcal{O}_X, \quad \delta_{n-r} \simeq \mathcal{L}_{G/P}((V/W)^{\#}), \tag{3.5.1}$$

where γ_r (resp., δ_{n-r}) is the canonical rank r (resp., rank $n - r$) subbundle on X.

Observe that we have a short exact sequence of algebraic vector bundles on X:

$$0 \longrightarrow \gamma_r \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \delta_{n-r}^{\vee} \longrightarrow 0\ , \tag{3.5.2}$$

where we denote by $\mathcal{E}^{\vee}$ the dual sheaf to $\mathcal{E}$. If $F(-)$ is one of the functors of Proposition 3.2.(3), then Proposition 3.2.(3) implies that

$$F(\gamma_r) \simeq \mathcal{L}_{G/P}(F(W)).$$

Combining Theorem 3.1 and Proposition 3.3, we conclude the following "identifications" of the vector bundles on a G-orbit in $\mathbb{E}(r, \mathfrak{g})$ associated to a rational G-module. The proof follows immediately from these propositions.

thm:functor **Theorem 3.6.** *Let* G *be an algebraic group and* M *be a rational* G*-module. Set* $\mathfrak{g} = \mathrm{Lie}(G)$*, and let* r *be a positive integer. Let* $X \equiv G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$ *be a* G*-orbit and set* $H \subset G$ *to be the (reduced) stabilizer of* $\epsilon \in \mathbb{E}(r, \mathfrak{g})$*.*

We assume that $X \simeq G/H$*, and consider* $\mathcal{L}_{G/H} : H$*-mod* $\to G/H$*-bundles as in* (3.2.1)*. For any* j*,* $1 \leq j \leq (p-1)r$*, we have the following isomorphisms of* G*-equivariant vector bundles*

$$\mathcal{I}m^{j,X}(M) \simeq \mathcal{L}_{G/H}(\mathrm{Rad}^j(\epsilon^* M)), \quad \mathcal{K}er^{j,X}(M) \simeq \mathcal{L}_{G/H}(\mathrm{Soc}^j(\epsilon^* M))$$

as subbundles of the trivial bundle $\mathcal{L}_{G/H}(M) = M \otimes \mathcal{O}_X$*, where* $\mathrm{Rad}^j(\epsilon^* M)$*,* $\mathrm{Soc}^j(\epsilon^* M)$ *are endowed with the action of* H *induced by the action of* G *on* M*.*

If the orbit map $\phi_\epsilon : G \to G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$ is separable, then ϕ_ϵ induces an isomorphism $\overline{\phi}_\epsilon : G/H \simeq G \cdot \epsilon$ (see, for example, [15]). To complement Theorem 3.6, we give the following criterion for the separability of the orbit map.

thm:sep **Theorem 3.7.** *Let* G *be a simple algebraic group whose Coxeter number* h *satisfies* $p > 2h - 2$*. Then for any* r*-dimensional subspace* $\epsilon \subset \mathfrak{g}$ *whose (adjoint) orbit* $G \cdot \epsilon$ *is closed in* $\mathrm{Grass}(r, \mathfrak{g})$*, the orbit map*

$$\phi_\epsilon : G \to G \cdot \epsilon \subset \mathrm{Grass}(r, \mathfrak{g})$$

is separable.

Proof. Since $p > 2h - 2$, the prime p does not divide the order of the finite covering $G \to \mathrm{Ad}(G)$ of G over its adjoint form (see [2, Planche I - X, VI]) and thus p does not divide the degree of the field extension $k(G)/k(\mathrm{Ad}(G))$; consequently,, this covering map is separable. Moreover, $G \cdot \epsilon = \mathrm{Ad}(G) \cdot \epsilon$. Consequently, we may assume that $G = \mathrm{Ad}(G)$. In other words, that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is injective.

The orbit map $\psi_\epsilon : \mathrm{GL}(\mathfrak{g}) \to \mathrm{Grass}(r, \mathfrak{g})$ is locally trivial (see, for example, [14, II.1.10]); indeed, this orbit map is a P-torsor, where $P \subset \mathrm{GL}(\mathfrak{g})$ is a standard parabolic subgroup identified here as the stabilizer of ϵ with respect to the action of $\mathrm{GL}(\mathfrak{g})$ on $\mathrm{Grass}(r, \mathfrak{g})$ given by left multiplication. We consider the following commutative diagram

sq (3.7.1)
$$\begin{tikzcd} P_\epsilon \arrow[r] \arrow[d, hook] & G \arrow[r, "\phi_\epsilon"] \arrow[d, hook, "Ad"] & G \cdot \epsilon \arrow[d, hook] \\ P \arrow[r] & \mathrm{GL}(\mathfrak{g}) \arrow[r, "\psi_\epsilon"] & \mathrm{Grass}(r, \mathfrak{g}) \end{tikzcd}$$

where $P_\epsilon \subset G$ is the (reduced) stabilizer of ϵ.

We recall that G/P_ϵ is quasi-projective since P_ϵ is a closed (reduced) subgroup of G (see [1, II.6.8]). On the other hand, $G/P_\epsilon \to G \cdot \epsilon$ is proper (in fact, finite), and $G \cdot \epsilon \subset \mathrm{Grass}(r, \mathfrak{g})$ is assumed to be closed and thus proper over k, so that G/P_ϵ is proper over k as well as quasi-projective. Thus, G/P_ϵ is projective which means that $P_\epsilon \subset G$ is a parabolic subgroup. By construction, $P_\epsilon = G \cap P$.

To prove that the orbit map $\phi_\epsilon : G \to G \cdot \epsilon$ is separable it suffices to show that the tangent map $d\phi_\epsilon$ at the identity is surjective (see [22, 4.3.7]). Let $\mathfrak{p}_\epsilon = \mathrm{Lie}\, P_\epsilon$, and let $\mathfrak{g} = \mathfrak{u}_\epsilon^- \oplus \mathfrak{p}_\epsilon$. We proceed to prove that the k-linear map of vector spaces

eq:comp (3.7.2)
$$\mathfrak{u}_\epsilon^- \hookrightarrow \mathfrak{g} \xrightarrow{(d\phi_\epsilon)_1} \mathbb{T}_\epsilon(G \cdot \epsilon)$$

is injective and thus by dimension reasons an isomorphism. This will imply the subjectivity of $d\phi_\epsilon$ at the identity.

Since ψ_ϵ is separable, $\ker(d\psi_\epsilon)_1 = \mathrm{Lie}(P)$. Suppose that $X \in \mathfrak{u}_\epsilon^- \cap \ker(d\phi_\epsilon)_1$. The commutativity of (3.7.1) and the separability of ψ_ϵ implies that such an X must be in $\mathrm{Lie}(P)$. We recall that a subgroup $H \subset \mathrm{GL}_N$ of exponential type in the sense of [23] has the property that for any p-nilpotent $Y \in \mathrm{Lie}(H)$ the 1-parameter subgroup $\exp_Y : \mathbb{G}_a \to \mathrm{GL}_N$ factors through H. As essentially observed in [23], $P \to \mathrm{GL}(\mathfrak{g})$ is an embedding of exponential type. Moreover, as verified in [17, 7.4.1] for G simple of adjoint type and $p > 2h-2$, the embedding $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is also of exponential type. Thus, $\exp_X : \mathbb{G}_a \to \mathrm{GL}(\mathfrak{g})$ factors (uniquely) via some 1-parameter subgroup $e_X : \mathbb{G}_a \to G \cap P = P_\epsilon$. This implies that $X = d(\exp_X)(1) = d(e_X)(1) \in \mathfrak{p}_\epsilon$. Since $X \in u_\epsilon^-$, we conclude $X = 0$. In other words, (3.7.2) is injective. □

cor:gln **Corollary 3.8.** *Assume $p > 2n - 2$. Then for any r-dimensional subspace $\epsilon \subset \mathfrak{gl}_n$ whose (adjoint) orbit $\mathrm{GL}_n \cdot \epsilon$ is closed in $\mathrm{Grass}(r, \mathfrak{gl}_n)$, the orbit map $\mathrm{GL}_n \to \mathrm{GL}_n \cdot \epsilon$ is separable.*

Proof. Since the standard map $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a $\mathbb{G}_m$-torsor, the separability of the orbit map for GL_n follows immediately from Theorem 3.7. □

The following proposition, a generalization of [4, 7.9], enables us to identify kernel bundles provided we know corresponding image bundles and vice versa.

dual **Proposition 3.9.** *Retain the notation and hypotheses of Theorem 3.6. Then there is a natural short exact sequence of vector bundles on $G/H \simeq X \subset \mathbb{E}(r, \mathfrak{g})$*

duality (3.9.1)
$$0 \longrightarrow \mathcal{K}er^{j,X}(M^\#) \longrightarrow (M^\#) \otimes \mathcal{O}_X \longrightarrow (\mathcal{I}m^{j,X}(M))^\vee \longrightarrow 0.$$

Proof. The proof is a repetition of that of [4, 7.9]. By [4, 2.2], the sequence

duality1 (3.9.2) $$0 \longrightarrow \mathrm{Soc}^j(\epsilon^*(M^\#)) \longrightarrow M^\# \longrightarrow (\mathrm{Rad}^j(\epsilon^* M))^\# \longrightarrow 0.$$

is an exact sequence of H-modules. Applying the functor $\mathcal{L}$ to (3.9.2) (which preserves exactness by Proposition 3.2) and appealing to Theorem 3.6, we conclude the exactness of (3.9.1). □

In the next proposition we remind the reader of some standard constructions of bundles using the operator $\mathcal{L}$ in addition to γ and δ mentioned in Example 3.5.

tan-cot **Proposition 3.10.** *Let G be a reductive algebraic group and let P be a standard parabolic subgroup. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{p} = \mathrm{Lie}(P)$, and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$.*

(1) *The tangent bundle of G/P is isomorphic to $\mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p})$,*
$$\mathbb{T}_{G/P} \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}).$$
(2) *Assume that $\mathfrak{g}$ has a nondegenerate G-invariant symmetric bilinear form (such as the Killing form). Then the cotangent bundle of G/P is isomorphic to $\mathcal{L}_{G/P}(\mathfrak{u})$:*
$$\Omega_{G/P} = \mathbb{T}^\vee_{G/P} \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{u}),$$
where $\mathfrak{u}$ is viewed as P-module via the restriction of the adjoint action of P on $\mathfrak{p}$.

(3) *For $G = \mathrm{SL}_n$ with $p \nmid n$, $P = P_{r,n-r}$, and $X = G/P = \mathrm{Grass}(r,n)$, we have*
$$\mathbb{T}_X \simeq \gamma_r^\vee \otimes \delta_{n-r}^\vee, \quad \Omega_X \simeq \gamma_r \otimes \delta_{n-r}.$$
(4) *For $G = \mathrm{Sp}_{2n}, P = P_{\alpha_n}$, and $Y = G/P = \mathrm{LG}(n,V)$, we have*
$$\mathbb{T}_Y \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \;\simeq\; S^2(\gamma_n^\vee).$$
Moreover, if $p > 3$, then
$$\Omega_Y \;\simeq\; S^2(\gamma_n).$$

Proof. (1) See [14, II.6.1].

(2) This follows from (1) together with the isomorphism of P-modules $(\mathfrak{g}/\mathfrak{p})^\# \simeq \mathfrak{u}$, guaranteed by the existence of a nondegenerate form.

(3) We have $\mathfrak{g} = \mathrm{End}(V)$. Let $e_1, \dots, e_n$ be a basis of V, and choose a linear splitting of the sequence $0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$ sending V/W to the subspace generated by $e_{r+1}, \dots, e_n$ (see notation introduced in Example 3.5). We have
$$\mathrm{End}(V) = \mathrm{Hom}(W, V/W) \oplus \mathrm{Hom}(V/W, W) \oplus \mathrm{Hom}(W,W) \oplus \mathrm{Hom}(V/W, V/W),$$
where the sum of the last three summands is a P-stable subspace isomorphic to $\mathfrak{p}$. Hence, we have an isomorphism of P-modules: $\mathfrak{g}/\mathfrak{p} \simeq \mathrm{Hom}(W, V/W) \simeq W^\# \otimes V/W$. Therefore,
$$\mathbb{T}_X \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq \mathcal{L}(W^\#) \otimes \mathcal{L}(V/W) = \gamma_r^\vee \otimes \delta_{n-r}^\vee.$$
Consequently,
$$\Omega_X \simeq \gamma_r \otimes \delta_{n-r},$$
since the Killing form is nondegenerate on $\mathfrak{sl}_n$ for $p \not| n$.

(4). In this case, W is an isotropic subspace of V, and $W \simeq (V/W)^{\#}$. Then $\mathfrak{g}/\mathfrak{p} \simeq \mathrm{Hom}_{Sym}(W, V/W) \simeq S^2(W^{\#})$. Hence,

$$\mathbb{T}_Y \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq S^2(\gamma_n^{\vee}).$$

If $p > 3$, there exists a non-degenerate Sp_{2n}-invariant form on $\mathfrak{g}$ (see [21]). Hence, we can dualize to obtain the last asserted isomorphism. □

sec:examples

4. Vector bundles on G-orbits of $\mathbb{E}(r, \mathfrak{g})$: examples

We now work out some specific examples of vector bundles on G-orbits of $\mathbb{E}(r, \mathfrak{g})$ associated to $\mathfrak{g}$-modules. Our first example is for GL_n-orbits of $\mathbb{E}(m, \mathfrak{gl}_n)$.

In the following two propositions we make an assumption that $\mathrm{GL}_n \cdot \mathfrak{u}_{r,n-r} \equiv \mathrm{Grass}(r, V)$. Since $P_{r,n-r}$ is the reduced stabilizer of $\mathfrak{u}_{r,n-r}$, this assumption is equivalent to the separability of the orbit map $\mathrm{GL}_n \to \mathrm{GL}_n \cdot \mathfrak{u}_{r,n-r}$. Hence, it is satisfied for $p > 2n - 2$ by Theorem 3.7.

prop:can-gen

Proposition 4.1. *Let $G = \mathrm{GL}(V) \simeq \mathrm{GL}_n$ for some $n \geq 2$, and set $\epsilon = \mathfrak{u}_{r,n-r} \in \mathbb{E}(r(n-r), \mathfrak{gl}_n)$, the subalgebra of all matrices with nonzero entries only in the top r rows and right-most $n - r$ columns, for some $r < n$. Assume that $p > 2n - 2$, so that by Corollary 3.8 the orbit map $\phi_\epsilon : \mathrm{GL}_n \to \mathrm{GL}_n \cdot \mathfrak{u}_{r,n-r} \equiv X$ is separable. Thus, ϕ_ϵ is isomorphic to the P-torsor $\mathrm{GL}_n \to \mathrm{Grass}(r, n)$, where $P = P_{r,n-r}$ is the standard parabolic of type $(r, n - r)$.*

We have the following isomorphisms of algebraic vector bundles on X:

(1) $\mathcal{I}m^{1,X}(V) \simeq \mathcal{K}er^{1,X}(V) \simeq \gamma_r$,
$\mathcal{I}m^{j,X}(V) = 0$ *for* $j > 1$.
(2) $\mathcal{C}oker^{1,X}(V) \simeq \delta^{\vee}_{n-r}$,
$\mathcal{C}oker^{j,X}(V) = 0$ *for* $j > 1$.
(3) $\mathcal{K}er^{1,X}(\Lambda^{n-1}(V)) \simeq \mathcal{I}m^X(\Lambda^{n-1}(V)) \simeq \delta^{\vee}_{n-r}$,
$\mathcal{I}m^{j,X}(\Lambda^{n-1}(V)) = 0$ *for* $j > 1$.

Proof. Choose a basis $e_1, \ldots, e_n$ for V so that both $\mathrm{Rad}(\epsilon^* V)$ and $\mathrm{Soc}(\epsilon^* M)$ are the subspace $W \subset V$ spanned by $e_1, \ldots, e_r$. That is, $\mathrm{Rad}(\epsilon^* V) = \mathrm{Soc}(\epsilon^* M) = W$ as $P_{r,n-r}$-modules in the notation of Example 3.5. Hence, Theorem 3.6 implies that

$$\mathcal{I}m^{1,X}(V) \simeq \mathcal{L}_{G/P}(W) = \gamma_r \quad \text{and} \quad \mathcal{K}er^{1,X}(V) \simeq \mathcal{L}_{G/P}(W) = \gamma_r.$$

This proves the first part of (1). The vanishing $\mathcal{I}m^{j,X}(V) = 0$ follows immediately from the fact that $\mathrm{Rad}^j(\epsilon^*(V)) = 0$ for $j \geq 2$.

Part (2) follows from the exactness of (3.5.1). To prove part (3), observe that $det : \mathrm{GL}_n \to \mathrm{GL}_1$ splits because $p > n$, so that we may view $\mathbb{E}(r, \mathfrak{gl}_n) = \mathbb{E}(r, \mathfrak{sl}_n)$ as the orbit of ϵ under SL_n. Since $\Lambda^{n-1} V \simeq V^{\#}$ as SL_n-modules, part (3) follows from Proposition 3.9. □

prop:can-gen2

Proposition 4.2. *We retain the hypotheses and notation of Proposition 4.1. For any positive integer $m \leq n - r$,*

(1) $\mathcal{I}m^{m,X}(V^{\otimes m}) = \gamma_r^{\otimes m}$,
(2) $\mathcal{I}m^{m,X}(S^m(V)) = S^m(\gamma_r)$,
(3) $\mathcal{I}m^{m,X}(\Lambda^m(V)) = \Lambda^m(\gamma_r)$.

Proof. Write $\mathfrak{u}(\epsilon) = k[t_{i,j}]/(t_{i,j}^p)$, $1 \leq i \leq r$, $r + 1 \leq j \leq n$. The action of $t_{i,j}$ on V is given by the rule $t_{i,j} e_j = e_i$ and $t_{i,j} e_\ell = 0$ for $\ell \neq j$. Let $W = \mathrm{Rad}(\epsilon^* V)$ as in the proof of Prop. 4.1. On a tensor product $M \otimes N$ of modules the action is given by $t_{i,j}(v \otimes w) = t_{i,j} v \otimes w + v \otimes t_{i,j} w$; thus $\mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$ is contained in the subspace of $V^{\otimes m}$ spanned by all elements $e_{i_1} \otimes \cdots \otimes e_{i_m}$, where $1 \le i_1, \ldots, i_m \le r$, which is $W^{\otimes m}$. On the other hand, for any sequence $i_1, \ldots, i_m$, with $1 \le i_1, \ldots, i_m \le r$, we have that

eq:tensor
$$e_{i_1} \otimes \cdots \otimes e_{i_m} \;=\; (t_{i_1, r+1} \ldots t_{i_m, r+m})(e_{r+1} \otimes \cdots \otimes e_{r+m}) \tag{4.2.1}$$

since $r + m \le n$. Hence, $\mathrm{Rad}^m(\epsilon^*(V^{\otimes m})) = W^{\otimes m}$. Therefore, the equality $\mathcal{I}m^{m,X}(V^{\otimes m}) = \mathcal{L}_{G/P}(W^{\otimes m}) = \gamma_r^{\otimes m}$ follows from Proposition 3.2.3, Theorem 3.6, and Example 3.5.

To show (2), note that the action of $\mathfrak{u}(\epsilon)$ on $S^m(V)$ is induced by the action on $V^{\otimes m}$ via the projection $V^{\otimes m} \twoheadrightarrow S^m(V)$. Hence, the formula (4.2.1) is still valid in $S^m(V)$, and implies the inclusion $S^m(W) \subset \mathrm{Rad}^m(\epsilon^*(S^m(V))$. The reverse inclusion is immediate just as in the tensor powers case. Therefore, $\mathrm{Rad}^m(\epsilon^*(S^m(V))) = S^m(W)$, and we conclude the equality $\mathcal{I}m^{m,X}(S^m(V)) = S^m(\gamma_r)$ appealing to Theorem 3.6.

The proof for exterior powers is completely analogous. □

Remark 4.3. The restriction on m in Proposition 4.2 is not sharp. For example, if $n = 4$ and $r = 2$, then it is straight forward to see that $\mathcal{I}m^3(V^{\otimes 3}) \simeq \gamma_2^{\otimes 3}$ provided $p > 2$. On the other hand, if $n = 3$ and $r = 2$, then $\mathcal{I}m(V^{\otimes 2})$ is a proper subbundle of $\gamma_2^{\otimes 2}$, regardless of the prime.

We use some standard Lie-theoretic notation for the remainder of this section. Let G be a simple algebraic group and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a fixed set of simple roots corresponding to a fixed torus T inside a Borel subgroup B. We follow the convention in [2, ch.6] in the numbering of simple roots. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $\mathfrak{h} = \mathrm{Lie}\, T$ be the Cartan subalgebra. For a simple root $\alpha \in \Delta$, we denote by P_α, $\mathfrak{p}_\alpha$ the corresponding standard maximal parabolic subgroup and its Lie algebra.

We provide a calculation analogous to Proposition 4.1 and 4.2 for the symplectic group Sp_{2n}. The only maximal parabolic subgroup P_α in standard form whose unipotent radical is abelian corresponds to the longest simple root: namely, $P = P_{\alpha_n}$ for α_n the unique long simple root. Equivalently, P_{α_n} is the unique cominuscule parabolic subgroup of Sp_{2n} in standard form, as in Definition 4.5.

If we view $G = \mathrm{Sp}_{2n}$ as the group of automorphisms of a symplectic vector space V of dimension $2n$ with chosen symplectic basis $\{x_1, \ldots, x_n, y_n, \ldots, y_1\}$, then P_{α_n} is the stabilizer of the totally isotropic subspace spanned by $\{x_1, \ldots, x_n\}$.

We recall from [5, 2.12] that $m = \binom{n+1}{2}$ is the dimension of each maximal elementary subalgebra of $\mathfrak{sp}_{2n}$. For $p > 4n - 2$, we have an isomorphism

lagrange
$$\mathbb{E}(m, \mathfrak{sp}_{2n}) \;\simeq\; \mathrm{Sp}_{2n}/P_{\alpha_n} \;=\; \mathrm{LG}(n, V) \tag{4.3.1}$$

(as follows from Theorem 3.7 and [5, 2.5, 2.9]) where $\mathrm{LG}(n, V)$ is the Lagrangian Grassmannian of maximal isotropic subspaces of the defining representation V.

symplec **Proposition 4.4.** *Consider $G = \mathrm{Sp}_{2n}$ and its defining representation V (of dimension $2n$); assume $p > 4n - 2$. Let $P_{\alpha_n} \subset \mathrm{Sp}_{2n}$ be the maximal parabolic subgroup in standard form corresponding to the longest root as described above, and let $\mathfrak{p} = \mathrm{Lie}(P_{\alpha_n})$. Let ϵ be the nilpotent radical of $\mathfrak{p}$, an elementary subalgebra of $\mathfrak{sp}_{2n}$ of dimension $m = \binom{n+1}{2}$. As in (4.3.1), let*

$$Y = \mathbb{E}(m, \mathfrak{sp}_{2n}) \;\simeq \mathrm{LG}(n, V),$$

and let $\gamma_n \subset \mathcal{O}_Y^{\oplus 2n}$ be the canonical subbundle of rank n. We have the following natural identifications of algebraic vector bundles on Y:

(1) $\mathcal{I}m^{1,Y}(V) \simeq \gamma_n, \quad \mathcal{I}m^{j,Y}(V) = 0$ *for* $j > 1$.

(2) $\mathcal{I}m^{1,Y}(\Lambda^{2n-1}(V)) \simeq \gamma_n^\vee, \quad \mathcal{I}m^{j,Y}(\Lambda^{2n-1}(V)) = 0$ *for* $j > 1$.

(3) *For* $m \leq n$,
 (a) $\mathcal{I}m^{m,Y}(V^{\otimes m}) = (\gamma_n)^{\otimes m}$,
 (b) $\mathcal{I}m^{m,Y}(S^m(V)) = S^m(\gamma_n)$,
 (c) $\mathcal{I}m^{m,Y}(\Lambda^m(V)) = \Lambda^m(\gamma_n)$.

Proof. We view Sp_{2n} as the stabilizer of the symplectic form defined by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

so that $\mathfrak{sp}_{2n}$ is the set of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $D = -A^T$ and B and C are $n \times n$ symmetric matrices. Then $\mathfrak{p} \subset \mathfrak{sp}_{2n}$ is defined by $C = 0$ (this can be easily verified from the explicit description of roots and roots spaces as in, for example, [7, 12.5]). We view V as the space of column vectors on which these matrices act from the left, and give V the standard basis $e_1, \ldots, e_{2n}$.

The restricted enveloping algebra of ϵ has the form $k[t_{i,j}]/(t_{i,j}^p)$ where $1 \leq i \leq n$ and $n+i \leq j \leq 2n$. The generator $t_{i,j}$ acts on V by the matrix $E_{i,j}$ if $j = n+i$ and by $E_{i,j} + E_{j-n,i+n}$ otherwise. Here, $E_{i,j}$ is the matrix with 1 in the (i,j) position and 0 elsewhere. Thus we have that

eq:relations
$$t_{i,j}e_j = e_i, \quad t_{i,j}e_{i+n} = e_{j-n}, \quad \text{and} \quad t_{i,j}e_\ell = 0 \tag{4.4.1}$$

whenever $\ell \neq j, i+n$. These relations immediately imply that $\mathrm{Rad}(\epsilon^* V) = \mathrm{Soc}(\epsilon^*(V)) = W$ where $W \subset V$ is the P-stable subspace generated by $e_1, \ldots, e_n$. Moreover, we also have that $\mathrm{Rad}^j(\epsilon^* V) = \mathrm{Soc}^j(\epsilon^* V) = 0$ for any $j > 1$. Applying Theorem 3.6, we get

$$\mathcal{I}m^{1,Y}(V) = \mathcal{K}er^{1,Y}(V) \simeq \mathcal{L}_{\mathrm{Sp}_{2n}/P_{\alpha_n}}(W) = \gamma_n, \quad \mathcal{I}m^{j,Y}(V) = \mathcal{K}er^{j,Y}(V) = 0 \text{ for } j > 1.$$

Part (2) follows from (1) and the fact that $\Lambda^{2n-1}(V)$ is the dual of $\mathfrak{g}$-module V (since G has no non-trivial 1-dimensional rational representation).

We proceed to show that $\mathcal{I}m^{m,Y}(V^{\otimes m}) \simeq (\gamma_n)^{\otimes m}$ for $m \leq n$. We note that as in the proof of Proposition 4.1, it is only necessary to show that $(\mathrm{Rad}(\epsilon^* V))^{\otimes m} \subseteq \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$, since the reverse inclusion is obvious.

Since $\mathrm{Rad}^2(\epsilon^* V) = 0$, the action of $\mathrm{Rad}^m(\mathfrak{u}(\epsilon))$ on $V^{\otimes m}$ is given by the formula

formula
$$(t_{i_1,n+j_1} \cdots t_{i_m,n+j_m})(e_{s_1} \otimes \cdots \otimes e_{s_m}) = \sum_{\pi \in \Sigma_m} t_{i_{\pi(1)},n+j_{\pi(1)}} e_{s_1} \otimes \cdots \otimes t_{i_{\pi(1)},n+j_{\pi(m)}} e_{s_m}. \tag{4.4.2}$$

To prove the inclusion $(\mathrm{Rad}(\epsilon^* V))^{\otimes m} \subseteq \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$, we need to show that for any m-tuple of indices $(i_1, \ldots, i_m)$, $1 \leq i_j \leq n$, we have $e_{i_1} \otimes \cdots \otimes e_{i_m} \in \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$. We first show the following

<u>*Claim*</u>. For any simple tensor $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in $(\mathrm{Rad}(\epsilon^* V))^{\otimes m}$, there exists a permutation $w \in \Sigma_m$ such that $e_{w(i_1)} \otimes \cdots \otimes e_{w(i_m)} \in \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$.

We proceed to prove the claim. Let $e_{i_1} \otimes \cdots \otimes e_{i_m}$ be any simple tensor in $(\mathrm{Rad}(\epsilon^* V))^{\otimes m}$. Applying a suitable permutation $\pi \in \Sigma_m$ to $(1, \dots, m)$, we may assume that $(i_1, \dots, i_m)$ has the form $(i_1^{a_1}, i_2^{a_2}, \dots, i_\ell^{a_\ell})$ where $i_1 > i_2 > \cdots > i_\ell$ and $a_1 + \ldots + a_\ell = m$. Applying yet another permutation, we may assume that the string of indices $(i_1, \dots, i_m)$ has the form

$$(i_1, i_2, \dots, i_\ell, i_1^{a_1 - 1}, \dots, i_\ell^{a_\ell - 1}),$$

with $i_1 > i_2 > \cdots > i_\ell$. To this string of indices we associate the string of indices $j_1, \dots, j_m$ by the following rule:

$$j_1 = i_1, j_2 = i_2, \dots, j_\ell = i_\ell$$

and $(j_{\ell+1}, \dots, j_m)$ is a subset of $m - \ell$ distinct numbers from $\{1, \dots, n\} \backslash \{i_1, i_2, \dots, i_\ell\}$. We claim that

eq:perm
$$(t_{i_1, n+j_1} \cdots t_{i_m, n+j_m})(e_{n+j_1} \otimes \cdots \otimes e_{n+j_m}) = e_{i_1} \otimes \cdots \otimes e_{i_m}. \tag{4.4.3}$$

Indeed, relations (4.4.1) imply that $t_{i_1,n+j_1} e_{n+j_1} \otimes \cdots \otimes t_{i_m,n+j_m} e_{n+j_m} = e_{i_1} \otimes \cdots \otimes e_{i_m}$. We need to show that all the other terms in (4.4.2) are zero. To have $t_{i_s,n+j_s} e_{n+j_r} \neq 0$, we must have either $j_s = j_r$ or $i_s = j_r$. By the choice of $(j_1, \dots, j_m)$, the second condition $i_s = j_r$ implies that $s = r$ and, hence, $j_s = j_r$. Therefore, $t_{i_s,n+j_s} e_{n+j_r} \neq 0$ if and only if $j_s = j_r$. Since by construction all $(j_1, \dots, j_m)$ are distinct, we conclude that $t_{i_{\pi(1)},n+j_{\pi(1)}} e_{n+j_1} \otimes \cdots \otimes t_{i_{\pi(1)},n+j_{\pi(m)}} e_{n+j_m} \neq 0$ if and only if π is the identity permutation which proves (4.4.3). This finishes the proof of the claim.

Now let $e_{i_1} \otimes \cdots \otimes e_{i_m}$ be an arbitrary tensor with $1 \leq i_j \leq n$. As we just proved, there exist $w \in \Sigma_m$ and indices $j_1, \dots, j_m$ such that

eq:perm2
$$(t_{w(i_1),n+j_1} \cdots t_{w(i_m),n+j_m})(e_{n+j_1} \otimes \cdots \otimes e_{n+j_m}) = e_{w(i_1)} \otimes \cdots \otimes e_{w(i_m)}. \tag{4.4.4}$$

The formula (4.4.2) implies that if we apply w^{-1} to (4.4.4) we get the desired result, that is

$$(t_{i_1,n+w^{-1}(j_1)} \cdots t_{i_m,n+w^{-1}(j_m)})(e_{n+w^{-1}(j_1)} \otimes \cdots \otimes e_{n+w^{-1}(j_m)}) = e_{i_1} \otimes \cdots \otimes e_{i_m}.$$

Therefore, $e_{i_1} \otimes \cdots \otimes e_{i_m} \in \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$. The statement for symmetric and exterior powers follows just as in Proposition 4.2. $\square$

comin **Definition 4.5.** For α a simple root, the (maximal) parabolic P_α of G is called *cominuscule* if α enters with coefficient at most 1 in any positive root.

The cominuscule parabolics appear naturally in our study of elementary subalgebras because of the following equivalent description.

le:comm **Lemma 4.6.** [20, Lemma 2.2] *Let G be a simple algebraic group and P be a proper standard parabolic subgroup. Assume $p \neq 2$ whenever $\Phi(G)$ has two different root lengths. Then the nilpotent radical of $\mathfrak{p} = \mathrm{Lie}(P)$ is abelian if and only if P is a cominuscule parabolic.*

The following is a complete list of cominuscule parabolics for simple groups (see, for example, [3] or [20]):

(1) Type A_n. P_α for any $\alpha \in \{\alpha_1, \dots, \alpha_n\}$.
(2) Type B_n. P_{α_1}.
(3) Type C_n. P_{α_n} (α_n is the unique long simple root).

(4) Type D_n. P_α for $\alpha \in \{\alpha_1, \alpha_{n-1}, \alpha_n\}$.
(5) Type E_6. P_α for $\alpha \in \{\alpha_1, \alpha_6\}$.
(6) Type E_7. P_{α_7}.

For types E_8, F_4, G_2 there are no cominuscule parabolics.

prop:comin

Proposition 4.7. *Let G be a simple algebraic group and P_α be a maximal parabolic subgroup of G. Denote by $\mathfrak{p}$ the Lie algebra $\mathrm{Lie}(P_\alpha)$ and by $\mathfrak{u}$ the nilpotent radical of $\mathfrak{p}$.*

(1) *If G has type B or C, assume that $p \neq 2,3$. Then $\mathfrak{u}$ is an elementary subalgebra if and only if P_α is cominuscule.*
(2) *Assume $p \neq 2$. Then $[\mathfrak{u}, \mathfrak{p}] = \mathfrak{u}$.*
(3) *If P_α is cominuscule, then $\mathfrak{p} = [\mathfrak{u}, \mathfrak{g}]$.*

Proof. If $\mathfrak{u}$ is elementary then, in particular, it is abelian and, hence, P_α is cominuscule by [20, 2.2]. Conversely, assume P_α is cominuscule. Then $\mathfrak{u}$ is abelian by [20, 2.2]. Since G is a simple algebraic group, each root space g_α is one dimensional generated by a root vector x_α. We have $x_\alpha^{[p]} = 0$ since $p\alpha$ is not a root. Since root vectors generate $\mathfrak{u}$ and $\mathfrak{u}$ is abelian, we conclude that the $[p]$-th power operation is trivial on $\mathfrak{u}$. Hence, $\mathfrak{u}$ is elementary.

To prove (2), observe that because $\mathfrak{u}$ is a Lie ideal in $\mathfrak{p}$, we have $[\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{u}$. By the structure theory for classical Lie algebras, for any $\alpha \in \Phi^+$ there exists $h_\alpha \in \mathfrak{h}$ such that $[h_\alpha, x_\alpha] = 2x_\alpha$. Hence, $\mathfrak{u} = [\mathfrak{h}, \mathfrak{u}] \subset [\mathfrak{p}, \mathfrak{u}]$.

Finally, we proceed to prove (3). Let $P = P_{\alpha_i}$, let $I = \Delta \backslash \{\alpha_i\}$ and let $\Phi_I \subset \Phi$ be the root system corresponding to the subset of simple roots I. We have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-$ where $\mathfrak{u}^- = \sum\limits_{\beta \in \Phi^+ \backslash \Phi_I^+} k x_{-\beta}$. Note that $\Phi^+ \backslash \Phi_I^+$ consists of all positive roots into which α_i enters with coefficient 1. Let $\beta \in \Phi^+ \backslash \Phi_I^+$ and let γ be any root. If $\beta + \gamma$ is a root, then α_i enters into $\beta + \gamma$ with coefficient 0 or 1. Therefore, $x_{\beta+\gamma} \notin \mathfrak{u}^-$. Hence, $[x_\beta, x_\gamma] \in \mathfrak{p}$. Since x_β for $\beta \in \Phi^+ \backslash \Phi_I^+$ generate $\mathfrak{u}$, we conclude that $[\mathfrak{u}, \mathfrak{g}] \subset \mathfrak{p}$.

For the opposite inclusion, we first show that $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$. Let $S \subset \Phi^+ \backslash \Phi_I^+$ be the set of all positive roots of the form $a_1\alpha_1 + \ldots + a_n\alpha_n$ such that $a_i = 1$ and $a_j \in \{0,1\}$ for all $j \neq i$. For any subset $J \subset \Delta$ of simple roots such that the subgraph of the Dynkin diagram corresponding to J is connected, we have that $\sum\limits_{\alpha_j \in J} \alpha_j$ is a root ([2, VI.1.6, Cor. 3 of Prop. 19]). This easily implies that for any simple root α_j, $j \neq i$, we can find $\beta_1, \beta_2 \in S$ such that $\beta_2 - \beta_1 = \alpha_j$. Hence, $\{\beta\}_{\beta \in S}$ generate the integral root lattice $\mathbb{Z}\Phi$. Consider the simply laced case first (A, D, E). Since the bijection $\alpha \to \alpha^\vee$ is linear in this case, we conclude that $\{\beta^\vee\}_{\beta \in S}$ generate the integer coroot lattice $\mathbb{Z}\Phi^\vee$. This, in turn, implies that $\{h_\beta\}_{\beta \in S}$ generate the integer form $\mathrm{Lie}(T_\mathbb{Z})$ of the Lie algebra $\mathrm{Lie}(T) = \mathfrak{h}$ over $\mathbb{Z}$, and, therefore, generate $\mathfrak{h} = \mathrm{Lie}(T_\mathbb{Z}) \otimes_\mathbb{Z} k$ over k (see [14, II.I.11]).

In the non-simply laced case (B or C), the relation $\beta_1 - \beta_2 = \alpha_j$ leads to $c_1\beta_1^\vee - c_2\beta_2^\vee = c_3\alpha_j^\vee$ where $c_1, c_2, c_3 \in \{1,2\}$. Hence, in this case $\{\beta\}_{\beta \in S}$ generate the lattice $\mathbb{Z}[\frac{1}{2}]\Phi^\vee$. Since $p \neq 2$, this still implies that $\{h_\beta\}_{\beta \in S}$ generate $\mathfrak{h} = \mathrm{Lie}(T_\mathbb{Z}) \otimes_\mathbb{Z} k$ over k.

In either case, since $h_\beta = [x_\beta, x_{-\beta}] \in [\mathfrak{u}, \mathfrak{g}]$ for $\beta \in S$, we conclude that $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$.

The inclusion $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$ implies $[\mathfrak{p}, \mathfrak{h}] \subset [\mathfrak{p}, [\mathfrak{u}, \mathfrak{g}]]$. Hence, by the Jacobi identity, we have

$$[\mathfrak{p}, \mathfrak{h}] \subset [\mathfrak{p}, [\mathfrak{u}, \mathfrak{g}]] = [[\mathfrak{p}, \mathfrak{u}], \mathfrak{g}]] + [\mathfrak{u}, [\mathfrak{p}, \mathfrak{g}]] = [\mathfrak{u}, \mathfrak{g}] + [\mathfrak{u}, \mathfrak{p}] \subset [\mathfrak{u}, \mathfrak{g}].$$

Consequently, $\mathfrak{p} = [\mathfrak{p}, \mathfrak{h}] + \mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$. □

We next show how to realize the tangent bundle of G/P for a cominuscule parabolic P of a simple algebraic group G as a cokernel bundle.

prop:tan

Proposition 4.8. *Let G be a simple algebraic group, and let P be a cominuscule parabolic subgroup of G. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{p} = \mathrm{Lie}(P)$, and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Assume that $G \cdot \mathfrak{u} \subset \mathbb{E}(\dim(\mathfrak{u}), \mathfrak{g})$ is isomorphic to G/P (for example, assume p satisfies the conditions of Theorem 3.7). We have isomorphisms of vector bundles on G/P:*

$$\mathcal{I}m^{1,G/P}(\mathfrak{g}) \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{p})$$

and

$$\mathcal{C}oker^{1,G/P}(\mathfrak{g}) \;\simeq\; \mathbb{T}_{G/P}.$$

Proof. Let $X = G/P$, and let $\epsilon = \mathfrak{u}$. Then $\mathrm{Rad}(\epsilon^*\mathfrak{g}) = [u, \mathfrak{g}] = \mathfrak{p}$ by Prop. 4.7. Theorem 3.1 and Proposition 3.2 give an isomorphism

$$\mathcal{I}m^{1,X}(\mathfrak{g}) \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{p})$$

as bundles on X. Applying Proposition 3.2 again, we conclude that the short exact sequence of rational P-modules $0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p} \to 0$ determines a short exact sequence of bundles on X:

$$0 \to \mathcal{L}_{G/P}(\mathfrak{p}) \;\to\; \mathfrak{g} \otimes \mathcal{O}_X \;\to\; \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \to 0.$$

Applying Proposition 3.10, we conclude that

$$\mathcal{C}oker^{1,X}(\mathfrak{g}) \;\simeq\; (\mathfrak{g} \otimes \mathcal{O}_X)/\mathcal{I}m^{1,X}(\mathfrak{g}) \simeq (\mathfrak{g} \otimes \mathcal{O}_X)/\mathcal{L}_{G/P}(\mathfrak{p}) \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq \mathbb{T}_{G/P}.$$

□

We offer some other interesting bundles coming from the adjoint representation of $\mathfrak{g}$.

prop:cominbund

Proposition 4.9. *Assume $p \neq 2$. Under the assumptions of Proposition 4.8, we have*

$$\mathcal{I}m^{2,G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(\mathfrak{u}),$$

where $\mathfrak{u}$ is viewed as a submodule of $\mathfrak{p}$ under the adjoint action of P.

Proof. Let $\epsilon = \mathfrak{u}$. By Proposition 4.7, $\mathrm{Rad}^2(\epsilon^*\mathfrak{g}) = [\mathfrak{u}, [\mathfrak{u}, \mathfrak{g}]] = \mathfrak{u}$. Hence, $\mathcal{I}m^{2,G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(\mathfrak{u})$ by Theorem 3.6. □

In the next three examples we specialize Proposition 4.9 to the simple groups of types A, B, and C.

cot-sln

Example 4.10. Let $G = \mathrm{SL}_n$, let $P = P_{r,n-r}$ be the standard maximal parabolic corresponding to the simple root α_r, and let $X = G \cdot \mathfrak{u} \subset \mathbb{E}(r(n-r), \mathfrak{sl}_n)$. Assume $X \simeq G/P = \mathrm{Grass}(r, n)$ (e.g., $p > 2n - 2$). We have an isomorphism of vector bundles on X

$$\mathcal{I}m^{2,X}(\mathfrak{g}) \simeq \Omega_X \simeq \gamma_r \otimes \delta_{n-r}.$$

Indeed, this follows immediately from Propositions 4.9 and 3.10(3).

Example 4.11. Let $G = \mathrm{SO}_{2n+1}$ be a simple algebraic group of type B_n so that $\mathfrak{g} = \mathfrak{so}_{2n+1}$, and let $P = P_{\alpha_1}$ be the standard cominuscule parabolic subgroup of G. We choose the symmetric form, the Cartan matrix, and the simple roots as in [7, 12.3]. Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p} = \mathrm{Lie}(P)$, and set $X = G\cdot\mathfrak{u} \subset \mathbb{E}(2n-1, \mathfrak{g})$. Assume $p > 4n-2$. Then X is isomorphic to G/P by Theorem 3.7.

We claim that

$$\mathcal{I}m^{2,X}(\mathfrak{g}) \ = \ \mathcal{L}_{G/P}(\mathfrak{u}) \ \simeq \ \mathcal{L}_{G/P}(V_{2n-1}) \ \simeq \ \Omega_{\mathbb{P}^{2n-1}}.$$

Here, V_{2n-1} is the natural module for the block of the Levy factor of P which has type B_{n-1}. More precisely, we have $P = LU$ where L is the Levi factor and U is the unipotent radical. The Levi factor L is a block matrix group with blocks of size 2 and $2n-1$. Factoring out the subgroup concentrated in the block of size 2, we get a simple algebraic group isomorphic to SO_{2n-1}. We take V_{2n-1} to be the standard module for this group inflated to the parabolic P.

To justify these claims, we note that the isomorphism $\mathcal{I}m^{2,X}(\mathfrak{g}) = \mathcal{L}_{G/P}(\mathfrak{u})$ is the content of Proposition 4.9, whereas the isomorphism $\mathcal{I}m^{2,X}(\mathfrak{g}) = \mathcal{L}_{G/P}(V_{2n-1})$ follows from an isomorphism of P-modules $\mathfrak{u} \simeq V_{2n-1}$ which can be checked by direct inspection. The asserted isomorphism $\mathcal{I}m^{2,X}(\mathfrak{g}) \ \simeq \ \Omega_{\mathbb{P}^{2n-1}}$ follows from Proposition 3.10.2, since the condition on p guarantees the existence of a nondegenrate invariant form on $\mathfrak{g} = \mathrm{Lie}(G)$ (see [21]).

cot-spn **Example 4.12.** Let $G = \mathrm{Sp}_{2n}$, $P = P_{\alpha_n}$, and assume that $p > 4n-2$. We have an isomorphism of vector bundles on $\mathbb{E}(\binom{n+1}{2}, \mathfrak{g}) \simeq \mathrm{LG}(n, V)$:

$$\mathcal{I}m^2(\mathfrak{g}) \simeq S^2(\gamma_n).$$

Just as in the previous examples, this follows immediately from [5, 2.12], [5, 2.9], and Theorem 3.7 which allow us to identify $\mathbb{E}(\binom{n+1}{2}, \mathfrak{g})$ with $\mathrm{LG}(n, V)$, and Propositions 4.9 and 3.10(4). Proposition 3.10 is applicable here since for $p > 3$ there exists a nondegenerate Sp_{2n}-invariant symmetric bilinear form on $\mathfrak{sp}_{2n}$ (see [21, p.48]).

The following example complements Example 4.10, evaluating kernel bundles rather than image bundles.

prop:kernel **Proposition 4.13.** *Let $G = \mathrm{SL}_n$, and let $P = P_{r,n-r} \subset G$ be a cominuscule parabolic. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{p} = \mathrm{Lie}(P)$, and let $\mathfrak{l}$, $\mathfrak{u}$ be the Levi subalgebra and the nilpotent radical of $\mathfrak{p}$. Let $X = G\cdot\mathfrak{u} \subset \mathbb{E}(r, \mathfrak{g})$ where $r = \dim\mathfrak{u}$. Assume $p > 2n-2$ so that X is isomorphic to G/P. Then we have an isomorphism of bundles on $X \simeq G/P$:*

$$\mathcal{K}er^{1,X}(\mathfrak{g}) \ \simeq \ \mathcal{L}_{G/P}(\mathfrak{u}) \simeq \Omega_X.$$

Proof. Let $\epsilon = \mathfrak{u}$ which is elementary by Proposition 4.7. We have $\mathrm{Soc}(\epsilon^*(\mathfrak{g})) = C_{\mathfrak{g}}(\mathfrak{u})$, the centralizer of $\mathfrak{u}$ in $\mathfrak{g}$. Since $\mathfrak{p}$ is the normalizer of $\mathfrak{u}$, we have $C_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{p}$. Moreover, since $\mathfrak{u} \subset \mathfrak{p}$ is a Lie ideal, so is $C_{\mathfrak{g}}(\mathfrak{u})$. Since $\mathfrak{p}/\mathfrak{u} \simeq \mathfrak{l}$ is reductive, we conclude that $C_{\mathfrak{g}}(\mathfrak{u})/\mathfrak{u} = C_{\mathfrak{l}}(\mathfrak{u})$ belongs to the center of $\mathfrak{l}$. We claim that this center is trivial.

Note that in the usual matrix representation, $\mathfrak{p}$ is the set of all matrices $(a_{i,j})$ with $a_{i,j} = 0$ whenever both $i > r$ and $j \le n-r$. Thus $\mathfrak{u} = \mathfrak{u}_{r,n-r}$ consists of all matrices which are nonzero only in the upper r rows and rightmost $n-r$ columns, and $\mathfrak{l}$ is the algebra of all $n \times n$ matrices that are nonzero only in the upper left

$r \times r$ block and the lower right $(n-r) \times (n-r)$ block. So the center of $\mathfrak{l}$ consists of all matrices of the form

$$c \;=\; \begin{pmatrix} aI_r & 0 \\ 0 & bI_{n-r} \end{pmatrix}$$

were $a, b \in k$ have the property that $ra + (n-r)b = 0$ and I_r is the $r \times r$ identity matrix. If $x = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}$, where X is an $r \times (n-r)$ block, then an easy calculation yields that $[c, x] = cx - xc = (a-b)x$. Since p does not divide n, we conclude that ℓ does has a trivial center, and, hence, $\dim C_{\mathfrak{l}}(\mathfrak{u}) = 0$.

Since $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, and no elements in $\mathfrak{l}$ centralize $\mathfrak{u}$, we conclude that $C_{\mathfrak{g}}(\mathfrak{u}) \simeq u \oplus C_{\mathfrak{l}}(\mathfrak{u})$. Combining this with Theorem 3.6 we get the following isomorphisms:

$$\mathrm{Ker}^{1,G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(C_{\mathfrak{g}}(\mathfrak{u})) \simeq \mathcal{L}_{G/P}(\mathfrak{u}).$$

Since our assumption on p implies that the Trace form on $\mathfrak{sl}_n$ is non-degenerate, we conclude that $\mathcal{L}_{G/P}(\mathfrak{u})$ is isomorphic to the cotangent bundle $\Omega_{G/P}$ by Proposition 3.10. □

sec:Gsemi 5. Vector bundles associated to semi-direct products

In this section, we provide a reinterpretation of "GL-equivariant kE-modules" considered in [4] as modules for the subgroup scheme $G_{(1),n} = \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n$ of the algebraic group $\mathbb{V} \rtimes \mathrm{GL}_n$ of Example 1.10 of [5]. This leads to consideration of rational representations for semi-direct product subgroup schemes $\mathbb{W}_{(1)} \rtimes H$ of the affine algebraic group $\mathbb{W} \rtimes H$, where H is any affine algebraic group and W is any faithful rational H-representation.

The representations of $G_{(1),n}$ and $\mathbb{W}_{(1)} \rtimes H$ we consider do not typically extend to the algebraic groups $\mathbb{V} \rtimes \mathrm{GL}_n$ and $\mathbb{W} \rtimes H$.

not:V **Notation 5.1.** Throughout this section, V is an n-dimensional vector space with chosen basis, so that we may identify $\mathrm{GL}(V)$ with GL_n and V with the defining representation of GL_n. Let $\mathbb{V} = \mathrm{Spec}(S^*(V^\#)) \simeq \mathbb{G}_a^{\oplus n}$ be the vector group associated to V, and let $\mathbb{V}_{(1)} \simeq (\mathbb{G}_{a(1)})^{\oplus n}$ be the first Frobenius kernel of $\mathbb{V}$. The standard action of GL_n on V induces an action on the vector group $\mathbb{V}$. Moreover, it is straightforward that this action stabilizes the subgroup scheme $\mathbb{V}_{(1)} \subset \mathbb{V}$. Hence, we can form the following semi-direct products:

eq:V (5.1.1)
$$G_{1,n} \overset{\text{def}}{=\!=} \mathbb{V} \rtimes \mathrm{GL}_n \qquad G_{(1),n} \overset{\text{def}}{=\!=} \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n\,.$$

Let

eq:g (5.1.2)
$$\mathfrak{g}_{1,n} \overset{\text{def}}{=\!=} \mathrm{Lie}(G_{(1),n}) = \mathrm{Lie}(G_{1,n})\,.$$

We view $V \simeq \mathrm{Lie}(\mathbb{V}_{(1)}) \subset \mathfrak{g}_{1,n}$ as an elementary subalgebra of $\mathfrak{g}_{1,n}$ which is also a Lie ideal stable under the adjoint action of $G_{1,n}$.

For any r-dimensional subspace $\epsilon \subset V \subset \mathfrak{g}_{1,n}$, we consider the adjoint action of $G_{1,n}$ on ϵ. Here, V is stable under the adjoint action, and the action of $\mathbb{V}$ on V is trivial. Moreover, the restriction of this adjoint action on ϵ to $\mathrm{GL}_n \subset G_{1,n}$ can be identified with the action of GL_n on $\epsilon \in \mathrm{Grass}(r, n)$ determined by left multiplication by $n \times n$ matrices on a column vector. This left multiplication map

$\mathrm{GL}_n \to \mathrm{Grass}(r,n)$ is locally a product projection and thus separable. Thus, the orbit map $\phi_\epsilon : G_{1,n} \to \mathbb{E}(r,\mathfrak{g}_{1,n})$ can be identified with the composition

$$G_{1,n} \to \mathrm{GL}_n \to \mathrm{Grass}(r,n) \subset \mathbb{E}(r,\mathfrak{g}_{1,n})$$

and thus induces an isomorphism

$$G_{1,n}\cdot\epsilon \simeq \mathrm{Grass}(r,n).$$

In particular, the orbit map restricted to GL_n, $\phi_\epsilon : \mathrm{GL}_n \to G_{1,n}\cdot\epsilon$, is separable.

We recall the notion of a GL-equivariant kE-module considered in [4].

Definition 5.2. Let E be an elementary abelian p-group of rank n and choose some linear map $V \to \mathrm{Rad}(kE)$ such that the composition $V \to \mathrm{Rad}(kE) \to \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$ is an isomorphism. This determines an identification $kE \simeq S^*(V)/\langle v^p, v \in V\rangle$. Then M is said to be a GL-*equivariant* kE*-module* (in the terminology of [4, 3.5]) if M is provided with two pairings

pair (5.2.1) $$S^*(V)/\langle v^p, v\in V\rangle \otimes M \to M, \quad \mathrm{GL}(V)\times M \to M$$

such that the second pairing makes M into a rational $\mathrm{GL}(V)$-module and the first pairing is $\mathrm{GL}(V)$-equivariant with respect to the diagonal action of $\mathrm{GL}(V)$ on $S^*(V)/\langle v^p, v\in V\rangle \otimes M$.

As the next proposition explains, the consideration of GL-equivariant kE-modules has a natural interpretation as $G_{(1),n}$-representations for $G_{(1),n} = \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n$.

prop:semi **Proposition 5.3.** *There is a natural equivalence of categories between the category of rational modules for the group scheme $G_{(1),n}$ and the category of "*GL*-equivariant kE-modules".*

Proof. Assume that we are given a functorial action of the semi-direct product

$$(G_{(1),n})(A) = \mathbb{V}_{(1)}(A) \rtimes \mathrm{GL}_n(A) \quad \text{on } M\otimes A$$

as A runs over commutative k-algebras. We view this as a group action of pairs $(v,g) = (v,1)\cdot(0,g)$ on M. Since $(0,g)\cdot(v,1) = ({}^gv,g) = ({}^gv,1)\cdot(0,g)$ in the semi-direct product, we conclude for any $m\in M$ that the action of $(0,g)$ on $(v,1)\circ m$ equals the action of $({}^gv,1)$ on $(0,g)\circ m$. This is precisely the condition that the action of $\mathbb{V}_{(1)}\times M \to M$ is GL_n-equivariant for the diagonal action of GL_n on $\mathbb{V}_{(1)}\times M$. Consequently, once the identification $kE \simeq k\mathbb{V}_{(1)} = \mathfrak{u}(\mathrm{Lie}(\mathbb{V}))$ is chosen, giving a GL_n-equivariant action $kE\times M \to M$ is the same as giving actions of $\mathbb{V}_{(1)}$ and GL_n on M which satisfy the condition that this pair of actions determines an action of the semi-direct product.

Conversely, given a GL_n-equivariant kE-module N, it is straightforward to check that the actions of GL_n and $kE \simeq k\mathbb{V}_{(1)}$ determine an action of $G_{(1),n}$ on the underlying vector space of N. □

Observe that we have GL_n acting on $\mathfrak{g}_{1,n}$ by restricting the adjoint action of $G_{1,n}$ on its Lie algebra to $\mathrm{GL}_n \subset G_{1,n}$. This, in turn, makes $\mathbb{E}(r,\mathfrak{g}_{1,n})$ into a GL_n-variety. We next observe that rational $G_{(1),n}$-representations (even those which are not restrictions of $G_{1,n}$-representations) lead to GL_n-equivariant sheaves on $\mathrm{Grass}(r,V)$.

For the rest of this section, we will require slight generalizations of several statements occurring earlier in the paper. the proofs of these generalizations are identical to the ones used to show the original conclusions.

rem:gen

Remark 5.4. Let $\widetilde{G}$ be an affine group scheme, let $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$, and let $G \hookrightarrow \widetilde{G}$ be a closed, reduced algebraic subgroup of $\widetilde{G}$. Let $\epsilon \in \mathbb{E}(r, \mathfrak{g})$, and let $X = G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$ be the orbit of ϵ under the action of G on $\mathbb{E}(r, \mathfrak{g})$ induced by the adjoint action of $\widetilde{G}$ on $\mathfrak{g}$. Let M a rational $\widetilde{G}$-module. Then proofs of Corollary 1.7, Theorem 3.1, and Theorem 3.6 apply to prove the corresponding statements for $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$, and $\mathcal{C}oker^{j,X}(M)$ in this slightly modified context.

In particular, the aforementioned results are applicable to the situation $\widetilde{G} = G_{1,n}$ and $G = \mathrm{GL}_n$.

Using the GL_n equivariance of image and kernel sheaves, we obtain the following comparison.

thm:identify1

Theorem 5.5. *Let $M_{|E}$ denote a kE-module associated to a rational $G_{(1),n}$-module M. Choose some $r, 1 \le r < n$, and some j with $1 \le j \le (p-1)r$. Let $\epsilon \subset V \subset \mathfrak{g}_{1,n}$ be an r-dimensional subspace. Then there are natural identifications of GL_n-equivariant vector bundles on $X = \mathrm{GL}_n \cdot \epsilon \simeq \mathrm{Grass}(r, V) \subset \mathbb{E}(r, \mathfrak{g}_{1,n})$,*

$$\mathcal{I}m^{j,X}(M) \simeq \mathcal{I}m^j(M_{|E}), \quad \mathcal{K}er^{j,X}(M) \simeq \mathcal{K}er^j(M_{|E}),$$

where the vector bundles $\mathcal{I}m^j(M_{|E})$, $\mathcal{K}er^j(M_{|E})$ on $\mathrm{Grass}(r, V)$ are those constructed in [4].

Proof. The vector bundle $\mathcal{I}m^{j,X}(M)$ on $X \subset \mathbb{E}(r, \mathfrak{g}_{1,n})$ is GL_n-equivariant by Corollary 1.7 with the fiber at the point $\epsilon \in X$ isomorphic to $\mathrm{Rad}^j(\epsilon^* M)$ by Theorem 3.1 (see also Remark 5.4). As proved in [4, 7.5], the vector bundle $\mathcal{I}m^j(M_{|E})$ on $\mathrm{Grass}(r, V)$ is also GL_n-equivariant with fiber over $\epsilon \in \mathrm{Grass}(r, V)$ also isomorphic to $\mathrm{Rad}^j(\epsilon^* M)$. Hence, $\mathcal{I}m^{j,X}(M) \simeq \mathcal{I}m^j(M_{|E})$ by Proposition 3.3.

The argument for the kernels is strictly analogous. □

As an immediate corollary of Theorem 5.5, we conclude the following interpretation of the computations of [4]. The modules N, M, R of the following proposition are rational $G_{(1),n}$-modules which do not extend to rational $G_{1,n}$-modules. For example, the $G_{(1),n}$ action on $N = S^*(V)/S^{*\ge j+1}(V)$ when restricted to $\mathbb{V}_{(1)}$ increases the degree of elements of N, whereas the GL_n structure is a direct sum of actions on each symmetric power $S^i(V)$. See [4, 3.6] for details of the $G_{(1),n}$ – structures on N, M, R.

perspect

Proposition 5.6. [4, 7.12,7.11,7.14] *Let $\epsilon \subset V$ be an r-plane for some integer r, $1 \le r \le n$, and let $X = \mathrm{GL}_n \cdot \epsilon \simeq \mathrm{Grass}(r, n)$ be the orbit of $\epsilon \in \mathbb{E}(r, \mathfrak{g}_{1,n})$. We have the following isomorphisms of GL_n-equivariant vector bundles on $\mathrm{Grass}(r, n)$:*

(1) For the rational $G_{(1),n}$-module $N = S^*(V)/S^{*\ge j+1}(V)$ and for any j with $1 \le j \le p-1$,

$$\mathcal{I}m^{j,X}(N) \ \simeq \ S^j(\gamma_r),$$

where γ_r is the canonical rank r subbundle of the trivial rank n bundle on $\mathrm{Grass}(r, n)$.

(2) For the rational $G_{(1),n}$-module $M = \mathrm{Rad}^r(\Lambda^*(V))/\mathrm{Rad}^{r+2}(\Lambda^*(V))$,

$$\mathcal{K}er^{1,X}(M) \ \simeq \ \mathcal{O}_X(-1) \oplus \mathcal{O}_X^{\binom{n}{r+1}}.$$

(3) For the rational $G_{(1),n}$-module $R = S^{r(p-1)}(V)/\langle S^{r(p-1)+2}(V); v^p, v \in \mathbb{V}\rangle$,

$$\mathcal{K}er^{1,X}(R) \;\simeq\; \mathcal{O}_X(1-p) \oplus (\mathrm{Rad}(R) \otimes \mathcal{O}_X).$$

We point out that specializing Proposition 5.6.1 to the case $j = 1$ gives a realization of the canonical subbundle γ_r on the Grassmannian as an image bundle of the $G_{(1),n}$-module $S^*(V)/S^{\geq 2}(V)$ different from the realization of γ_r given in Proposition 4.1.1.

Our new examples of vector bundles arise by considering subgroup schemes of $G_{(1),n}$ which we now introduce.

defn:W **Definition 5.7.** Let H be an algebraic group and W a faithful, finite dimensional rational H-module of dimension n; let $\mathbb{W}$ be the associated vector group ($\simeq \mathbb{G}_a^{\oplus n}$) equipped with the action of H. Let

$$G_{W,H} \;\equiv\; \mathbb{W}_{(1)} \rtimes H \subset \mathbb{W} \rtimes H,$$

and let

$$\mathfrak{g}_{W,H} = \mathrm{Lie}(G_{W,H}).$$

For any subspace $\epsilon \subset W$ of dimension r, we identify the $\mathbb{W} \rtimes H$-orbit (i.e., adjoint orbit) of $\epsilon \in \mathbb{E}(r, \mathfrak{g}_{W,H})$ with $Y = H \cdot \epsilon \subset \mathrm{Grass}(r, W) \subset \mathbb{E}(r, \mathfrak{g}_{W,H})$, where H acts on $\mathrm{Grass}(r, W)$ as the restriction of the standard quotient $\mathrm{GL}(W) \to \mathrm{Grass}(r, W)$.

If $\rho : H \to \mathrm{GL}_n$ defines the representation of H on W, then ρ induces closed embeddings

$$\mathbb{W} \rtimes H \;\subset\; G_{1,n}, \quad G_{W,H} \;\subset\; G_{(1),n}.$$

We next apply Corollary 1.7 together with Remark 5.4 to $\widetilde{G} = G_{W,H}$ and $G = H$ to obtain the following equivariance statement.

prop:W **Corollary 5.8.** *Using the notation and terminology of Definition 5.7, let $Y = H \cdot \epsilon \subset \mathrm{Grass}(r, W) \subset \mathbb{E}(r, g_{W,H})$ be the $W \rtimes H$-orbit of some $\epsilon \subset W$, a dimension r subspace of the rational $G_{W,H}$-module W. Let M be a finite dimensional rational $G_{W,H}$-module. For any j, $1 \leq j \leq (p-1)r$, the image and kernel sheaves*

$$\mathcal{I}m^{j,Y}(M), \;\mathcal{K}er^{j,Y}(M)$$

on Y are H-equivariant algebraic vector bundles.

As in Definition 5.7, we consider the adjoint action of H on $\epsilon \in \mathrm{Grass}(r, W)$. Let $H_\epsilon \subset H$ denote the (reduced) stabilizer of ϵ and let

eq:phi (5.8.1) $$\phi : H/H_\epsilon \longrightarrow Y = H \cdot \epsilon$$

denote the morphism of varieties induced by the orbit map $H \to H \cdot \epsilon \subset \mathrm{Grass}(r, W)$. We recall that ϕ is always a homeomorphism, and it is an isomorphism of varieties if the orbit map is separable.

We easily extend the computations of Proposition 5.6 by considering the rational $G_{(1),n}$-modules N, M, R upon restriction to $G_{W,H} \subset G_{(1),n}$. If $i : Y \subset X$ is an embedding of a locally closed subvariety Y in a quasi-projective variety X and if $\mathcal{E}$ is an algebraic vector bundle on X, then we denote by $\mathcal{E}_{|Y}$ the restriction $i^*\mathcal{E}$ of $\mathcal{E}$ to X. Simillarly, if $i : H \to G$ is a closed embedding of affine group schemes and M is a rational G-module, then we denote by $M_{|H}$ the restriction of M to H.

thm:H **Theorem 5.9.** *Retain the context and notation of Definition 5.7, and assume that the map $\phi : H/H_\epsilon \to Y = H\cdot\epsilon$ of* (5.8.1) *is an isomorphism. We have the following isomorphisms of H-equivariant vector bundles on $Y \subset$* Grass(r, W)*:*

(1) For the rational $G_{(1),n}$-modules $N = S^*(W)/S^{*\geq j+1}(W)$ and any j such that $1 \leq j \leq p-1$,

$$\mathcal{I}m^{j,Y}(N_{|G_{W,H}}) \;\simeq\; S^j(\gamma_r)_{|Y},$$

where γ_r denotes the canonical rank r subbundle on Grass(r, W)*.*

(2) For the rational $G_{(1),n}$-modules $M = \mathrm{Rad}^r(\Lambda^*(W)/\mathrm{Rad}^{r+2}(\Lambda^*(W))$,

$$\mathcal{K}er^{1,Y}(M_{|G_{W,H}}) \;\simeq\; \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y^{\binom{n}{r+1}}.$$

(3) For the rational $G_{(1),n}$-modules $R = S^{r(p-1)}(W)/\langle S^{r(p-1)+2}(W); v^p, v \in V\rangle$,

$$\mathcal{K}er^{1,Y}(R_{|G_{W,H}}) \;\simeq\; \mathcal{O}_Y(1-p) \oplus (\mathrm{Rad}(R) \otimes \mathcal{O}_Y).$$

Proof. Let L be a rational representation of $G_{(1),n}$. Theorem 3.6 implies that the fibers above $\epsilon \in Y$ of $\mathcal{I}m^{j,Y}(L_{|G_{W,H}})$, $\mathcal{I}m^{j,X}(L)_{|Y}$ are both isomorphic to $\mathrm{Rad}^j(\epsilon^* L)$ as modules for $H_\epsilon \subset H$. Since both $\mathcal{I}m^{j,Y}(L_{|G_{W,H}})$, $\mathcal{I}m^{j,X}(L)_{|Y}$ are H-equivariant coherent sheaves on $Y \simeq H/H_\epsilon$, we conclude that they are isomorphic by Theorem 3.6.

The first statement now follows immediately from Proposition 5.6 and the above observation applied to N. The proofs for (2) and (3) are completely analogous. □

We restate as a corollary the following special case of Theorem 5.9(1) We can interpret this corollary as saying for any affine algebraic group H and any subgroup $S \subset H$ which is the stabilizer of some r-dimensional subspace ϵ of an H-module W that the H-equivariant vector bundle $\mathcal{L}_{H/S}(\epsilon)$ on H/S can be realized as $\mathcal{I}m^{1,Y}(M)$ for some $G_{W,H}$-representation M.

cor:im1 **Corollary 5.10.** *Let H be an affine algebraic group, and let W be a finite dimensional rational H-module. Choose an r-dimensional subspace $\epsilon \subset W$, let $S \subset H$ be the (reduced) stabilizer of ϵ, and assume that $\phi : H/S \to Y$ induced by the orbit map is an isomorphism. Then there exists a rational $G_{W,H}$-module M such that*

$$\mathcal{I}m^{1,Y}(M) \;\simeq\; (\gamma_r)_{|Y} \;\simeq\; \mathcal{L}_{H/S}(\epsilon)$$

as H-equivariant algebraic vector bundles on $Y \subset$ Grass$(r, W) \subset \mathbb{E}(r, g_{W,H})$*. Here, γ_r is the canonical rank r subbundle of $W \otimes \mathcal{O}_{\mathrm{Grass}(r,W)}$.*

Proof. The isomorphism $\mathcal{I}m^{1,Y}(M) \;\simeq\; (\gamma_r)_{|Y}$ is a special case of Theorem 5.9(1) for $j = 1$.

Note that the given action of H on W induces an action on Grass(r, W) and also makes the canonical subbundle γ_r on Grass(r, W) H-equivariant. The action of S on the fiber of γ_r (an r-dimensional subspace of W) above the point $\epsilon \in$ Grass(r, W) is the restriction of the action of H on W. Similarly, the action of S on the fiber of $\mathcal{L}_{H/S}(\epsilon)$ above the point $eH \in H/S$ is the restriction to S acting on this fiber of the action of H on W. Hence, $\mathcal{L}_{H/S}(\epsilon) \simeq (\gamma_r)_{|Y}$ by Prop. 3.3. □

In the following proposition, we consider the evident semi-direct product $G_{\mathfrak{h},H} \equiv \mathfrak{h} \rtimes H$ determined by the adjoint action of H on $\mathrm{Lie}(H) = \mathfrak{h}$.

prop:cot **Proposition 5.11.** *Let H be a simple algebraic group, and assume that $p > 2h-2$ where h is the Coxeter number for H. Let P be a standard cominuscule parabolic of H, let $\mathfrak{u}$ be the nilradical of $\mathrm{Lie}(P)$, and let $Y = H \cdot \mathfrak{u} \subset \mathrm{Grass}(\dim(\mathfrak{u}), \mathfrak{h})$ be the orbit of $\mathfrak{u}$ under the adjoint action of H.*

Let $\Omega_{H/P}$ be the cotangent bundle on $H/P \simeq Y$. Then for any j, $1 \leq j \leq p-1$, there exists a rational $G_{\mathfrak{h},H}$-module N such that

$$\mathcal{I}m^{j,Y}(N) \;\simeq\; S^j(\Omega_{H/P}).$$

Proof. Recall that P is the (reduced) stabilizer of $\mathfrak{u} \subset \mathfrak{g}$ for the adjoint action of G on $\mathfrak{g}$. Theorem 3.7 implies that the orbit map induces an isomorphism $H/P \simeq Y$. Let $r = \dim \mathfrak{u}$. By Theorem 5.9(1), we can find a rational $G_{\mathfrak{h},H}$-module N such that $\mathcal{I}m^{j,Y}(N) \simeq S^j(\gamma_r)_{|Y} = S^j((\gamma_r)_{|Y})$ where γ_r is the canonical rank r subbundle on $\mathrm{Grass}(r, \mathfrak{h})$. As shown in Corollary 5.10, $(\gamma_r)_Y \;\simeq\; \mathcal{L}_{H/S}(\mathfrak{u})$ which is isomorphic to Ω_Y by Proposition 3.10(2). The latter is applicable since the assumption on p implies that $\mathfrak{h}$ admits a nondegenerate, H-invariant, symmetric bilinear form (see [21, p.48]).

□

We apply Proposition 5.11 to simple groups of type C_n to obtain the following realization results for bundles on the Lagrangian Grassmannian (cf. Propositions 4.4 and 3.10).

Sp2m **Example 5.12.** Assume $p > 4n-2$. Take $H = \mathrm{Sp}_{2n}$, and let $\epsilon \subset \mathfrak{h} = \mathfrak{sp}_{2n}$ be the nilpotent radical of the Lie algebra of the standard cominuscule parabolic subgroup $P_{\alpha_n} \subset H$; let $r = \dim \epsilon = \binom{n+1}{2}$. Consider $Y = H \cdot \epsilon \simeq \mathrm{LG}(n,n) \simeq \mathbb{E}(r, \mathfrak{h})$ as in (4.3.1). Then there is a rational $G_{\mathfrak{h},H}$-module N such that

$$\mathcal{I}m^{j,Y}(N_{|G_{\mathfrak{h},H}}) \;\simeq\; S^j(\Omega_Y) \;\simeq\; S^{2j}(\gamma_n)$$

for any j, $1 \leq j \leq p-1$. Here, γ_n is the canonical rank n subbundle on $\mathrm{LG}(n,n)$.

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