# ELEMENTARY SUBALGEBRAS OF LIE ALGEBRAS

JON F. CARLSON*, ERIC M. FRIEDLANDER**, AND JULIA PEVTSOVA***

Abstract. We initiate the investigation of the projective varieties $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of dimension $r$ of a ($p$-restricted) Lie algebra $\mathfrak{g}$ for various $r \geq 1$. These varieties $\mathbb{E}(r, \mathfrak{g})$ are the natural ambient varieties for generalized support varieties for restricted representations of $\mathfrak{g}$. We identify these varieties in special cases, revealing their interesting and varied geometric structures. We also introduce invariants for a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module $M$, the local $(r, j)$-radical rank and local $(r, j)$-socle rank, functions which are lower/upper semicontinuous on $\mathbb{E}(r, \mathfrak{g})$. Examples are given of $\mathfrak{u}(\mathfrak{g})$-modules for which some of these rank functions are constant.

## 0. Introduction

We say that a Lie subalgebra $\epsilon \subset \mathfrak{g}$ of a $p$-restricted Lie algebra $\mathfrak{g}$ over a field $k$ of characteristic $p$ is *elementary* if it is abelian with trivial $p$-restriction. Thus, if $\epsilon$ has dimension $r$, then $\epsilon \simeq \mathfrak{g}_a^{\oplus r}$ where $\mathfrak{g}_a$ is the one dimensional Lie algebra of the additive group $\mathbb{G}_a$. This paper is dedicated to the study of the projective variety $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of $\mathfrak{g}$ for some positive integer $r$ and its relationship to the representation theory of $\mathfrak{g}$.

We have been led to the investigation of $\mathbb{E}(r, \mathfrak{g})$ through considerations of cohomology and modular representations of finite group schemes. Recall that the structure of a restricted representation of $\mathfrak{g}$ on a $k$-vector space is equivalent to the structure of a module for the restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$ (a cocommutative Hopf algebra over $k$ of dimension $p^{\dim(\mathfrak{g})}$). A key precursor of this present work is the identification of the spectrum of the cohomology algebra $\mathrm{H}^*(\mathfrak{u}(\mathfrak{g}), k)$ with the $p$-nilpotent cone $\mathcal{N}_p(\mathfrak{g})$ achieved in [19], [28], [1], [42]. The projectivization of $\mathcal{N}_p(\mathfrak{g})$ equals $\mathbb{E}(1, \mathfrak{g})$. More generally, $\mathbb{E}(r, \mathfrak{g})$ is the orbit space under the evident $\mathrm{GL}_r$-action on the variety of $r$-tuples of commuting, linearly independent, p-nilpotent elements of $\mathfrak{g}$. Our interest in $\mathbb{E}(r, \mathfrak{g})$ and its close connections with the representation theory of $\mathfrak{g}$ can be traced back through the work of many authors to the fundamental papers of Daniel Quillen who established the important geometric role that elementary abelian $p$-subgroups play in the cohomology theory of finite groups [37].

It is interesting to observe that the theory of cohomological support varieties for restricted $\mathfrak{g}$-representations (i.e., $\mathfrak{u}(\mathfrak{g})$-modules) as considered first in [20] has evolved into the more geometric study of $\pi$-points as introduced by the second and

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*Date*: May 2, 2018.

2000 *Mathematics Subject Classification.* 17B50, 16G10.

*Key words and phrases.* restricted Lie algebras, algebraic vector bundles.

* partially supported by the NSF grant DMS-1001102.

** partially supported by the NSF grant DMS-0909314 and DMS-0966589.

*** partially supported by the NSF grant DMS-0800930 and DMS-0953011.

third authors in [21]. This latter work closed a historical loop, relating cohomological considerations to earlier work on cyclic shifted subgroups as investigated by Everett Dade [15] and the first author [9].

For $r > 1$ and $\mathfrak{g}$ the Lie algebra of a connected reductive group $G$, $\mathbb{E}(r, \mathfrak{g})$ is closely related to the spectrum of cohomology of the $r$-th Frobenius kernel $G_{(r)}$ of $G$ (see [41] for classical simple groups $G$; [32], [39] for more general types). Work of Alexander Premet concerning the variety of commuting, nilpotent pairs in $\mathfrak{g}$ [36] gives considerable information about $\mathbb{E}(2, \mathfrak{g})$. Much less is known for larger $r$'s, although work in progress indicates the usefulness of considering the representation theory of $\mathfrak{g}$ when investigating the topology of $\mathbb{E}(r, \mathfrak{g})$.

We consider numerous examples of restricted Lie algebras $\mathfrak{g}$ in Section 1, and give some explicit computations of $\mathbb{E}(r, \mathfrak{g})$. Influenced by the role of maximal elementary abelian $p$-subgroups in the study of the cohomology of finite groups, we are especially interested in examples of $\mathbb{E}(r, \mathfrak{g})$ considered in Section 2 for which $r$ is maximal among the dimensions of elementary subalgebras of $\mathfrak{g}$. For simple Lie algebras over a field of characteristic 0, Anatoly Malcev determined this maximal dimension [31] which is itself an interesting invariant of $\mathfrak{g}$. Our computations verify that the Grassmann variety of $n$ planes in a $2n$-dimensional $k$-vector space maps bijectively (via a finite, radicial morphism) to $\mathbb{E}(n^2, \mathfrak{gl}_{2n})$; similar results apply to the computation of $\mathbb{E}(n(n+1), \mathfrak{gl}_{2n+1})$ and $\mathbb{E}\left(\frac{(n+1)n}{2}, \mathfrak{sp}_{2n}\right)$. As we point out in Section 2, these maps turn out to be isomorphisms of varieties. The reader interested in the description of $\mathbb{E}(r, \mathfrak{g})$ for other types of simple Lie algebras $\mathfrak{g}$ can find them in a forthcoming paper [35]. We also provide some computations for restricted Lie algebras not arising from reductive groups.

We offer several explicit motivations for considering $\mathbb{E}(r, \mathfrak{g})$ in addition to the fact that these projective varieties are of intrinsic interest. Some of these motivations are pursued in Sections 3 and 4 where (restricted) representations of $\mathfrak{g}$ come to the fore. We point to the forthcoming paper [13], which utilizes the discussion of this current work in an investigation of coherent sheaves and algebraic vector bundles on $\mathbb{E}(r, \mathfrak{g})$.

- The varieties $\mathbb{E}(r, \mathfrak{g})$ are the natural ambient varieties in which to define generalized support varieties for restricted representations of $\mathfrak{g}$ (as in [22]).

- Coherent sheaves on $\mathbb{E}(r, \mathfrak{g})$ are naturally associated to arbitrary (restricted) representations of $\mathfrak{g}$ (see [13]).

- For certain representations of $\mathfrak{g}$ including those of constant Jordan type, the associated coherent sheaves are algebraic vector bundles on $\mathbb{E}(r, \mathfrak{g})$ (see [13]).

- Determination of the (Zariski) topology of $\mathbb{E}(r, \mathfrak{g})$ is an interesting challenge which can be informed by the representation theory of $\mathfrak{g}$.

The isomorphism type of the restriction $\epsilon^* M$ of a $\mathfrak{u}(\mathfrak{g})$-module $M$ to an elementary subalgebra $\epsilon$ of dimension 1 is given by its Jordan type, which is a partition of the dimension of $M$. On the other hand, the classification of indecomposable modules of an elementary subalgebra of dimension $r > 1$ is a wild problem (except in the special case in which $r = 2 = p$), so that the isomorphism types of $\epsilon^* M$ for $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ do not form convenient invariants of a $\mathfrak{u}(\mathfrak{g})$-module $M$. Following the approach undertaken in [12], we consider the dimensions of the radicals and socles of such restrictions, $\dim \operatorname{Rad}^j(\epsilon^* M)$ and $\dim \operatorname{Soc}^j(\epsilon^* M)$, for $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ and any $j$ with $1 \leq j \leq (p-1)r$. As we establish in Section 3, these dimensions give upper/lower semi-continuous functions on $\mathbb{E}(r,\mathfrak{g})$. In particular, they lead to "generalized rank varieties" refining those introduced in [23]. We achieve some computations of these generalized rank varieties $\mathbb{E}(r,\mathfrak{g})_M$ for two families of $\mathfrak{u}(\mathfrak{g})$-modules $M$: the $L_\zeta$ modules which play an important role in the theory of support varieties (see, for example, [4, 5.9]) and induced modules.

One outgrowth of the authors' interpretation of cohomological support varieties in terms of $\pi$-points (as in [21]) is the identification of the interesting classes of modules of constant Jordan type and constant $j$-rank for $1 \leq j < p$ (see [11]). As already seen in [12], this has a natural analogue in the context of elementary subalgebras of dimension $r > 1$. In Section 4, we give examples of $\mathfrak{u}(\mathfrak{g})$-modules of constant $(r,j)$-radical rank and of constant $(r,j)$-socle rank. This represents a continuation of investigations initiated by the authors in [11], [23] (see also [2], [6], [5], [7], [10], [14], [18], and others).

Although we postpone consideration of Lie algebras over fields of characteristic 0, we remark that much of the formalism of Sections 1 and 3, and many of the examples in Sections 2 are valid (and often easier) in characteristic 0. On the other hand, some of our results and examples, particularly in Section 4, require that $k$ have positive characteristic.

In a sequel to this work (see [13]) we show that $\mathfrak{u}(\mathfrak{g})$-modules of constant $(r,j)$-radical rank and of constant $(r,j)$-socle rank determine vector bundles on $\mathbb{E}(r,\mathfrak{g})$. Of particular interest are those $\mathfrak{u}(\mathfrak{g})$-modules not equipped with large groups of symmetries. We anticipate that the investigation of such modules may provide algebraic vector bundles with interesting properties.

Throughout, $k$ is an algebraically closed field of characteristic $p > 0$. All Lie algebras $\mathfrak{g}$ considered in this paper are assumed to be finite dimensional over $k$ and $p$-restricted; a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is assumed to be closed under $p$-restriction. Without explicit mention to the contrary, all $\mathfrak{u}(\mathfrak{g})$-modules are finite dimensional.

We thank Steve Mitchell and Monty McGovern for useful discussions pertaining to the material in Section 2. We also thank the referee for many useful comments and suggestions.

## 1. The subvariety $\mathbb{E}(r,\mathfrak{g})$ of $\operatorname{Grass}(r,\mathfrak{g})$

We begin by formulating the definition of $\mathbb{E}(r,\mathfrak{g})$ of the variety of elementary subalgebras of $\mathfrak{g}$ and establishing the existence of a natural closed embedding of $\mathbb{E}(r,\mathfrak{g})$ into the projective variety $\operatorname{Grass}(r,\mathfrak{g})$ of $r$-planes of the underlying vector space of $\mathfrak{g}$. Once these preliminaries are complete, we introduce various examples which reappear frequently, here and in [13].

Let $V$ be an $n$-dimensional vector space and $r < n$ a positive integer. We consider the projective variety $\operatorname{Grass}(r,V)$ of $r$-planes of $V$. We choose a basis for $V$, $\{v_1,\ldots,v_n\}$; a change of basis has the effect of changing the Plücker embedding (1.1.2) by a linear automorphism of $\mathbb{P}(\Lambda^r(V))$. We represent a choice of basis $\{u_1,\ldots,u_r\}$ for an $r$-plane $U \subset V$ by an $n \times r$-matrix $(a_{i,j})$, where $u_j = \sum_{i=1}^n a_{i,j} v_i$. Let $\mathbb{M}^\circ_{n,r} \subset \mathbb{M}_{n,r}$ denote the open subvariety of the affine space $\mathbb{M}_{n,r} \simeq \mathbb{A}^{nr}$ consisting of $n \times r$ matrices of (maximal) rank $r$ and set $p : \mathbb{M}^\circ_{n,r} \longrightarrow \operatorname{Grass}(r,V)$ equal to the map sending a rank $r$ matrix $(a_{i,j})$ to the $r$-plane spanned by $\{\sum_{i=1}^n a_{i,1} v_i, \ldots, \sum_{i=1}^n a_{i,r} v_i\}$.

We summarize a few useful, well known facts about $\mathrm{Grass}(r,V)$. Note that there is a natural (left) action of $\mathrm{GL}_r$ on $\mathbb{M}_{n,r}$ via multiplication by the inverse on the right.

**Proposition 1.1.** *For any subset $\Sigma \subset \{1,\ldots,n\}$ of cardinality $r$, set $U_\Sigma \subset \mathrm{Grass}(r,V)$ to be the subset of those $r$-planes $U \subset V$ with a representing $n\times r$ matrix $A_U$ whose $r\times r$ minor indexed by $\Sigma$ (denoted by $\mathfrak{p}_\Sigma(A_U)$) is non-zero. Then we have the following:*

(1) $p : \mathbb{M}^\circ_{n,r} \to \mathrm{Grass}(r,V)$ *is a principal $\mathrm{GL}_r$-torsor, locally trivial in the Zariski topology;*

(2) *Sending an $r$-plane $U \in U_\Sigma$ to the unique $n\times r$-matrix $A^\Sigma_U$ whose $\Sigma$-submatrix (i.e., the $r\times r$-submatrix whose rows are those of $A^\Sigma_U$ indexed by elements of $\Sigma$) is the identity determines a section of $p$ over $U_\Sigma$:*

$$s_\Sigma : U_\Sigma \to \mathbb{M}^\circ_{n.r}; \tag{1.1.1}$$

(3) *The Plücker embedding*

$$\mathfrak{p} : \mathrm{Grass}(r,V) \hookrightarrow \mathbb{P}(\Lambda^r(\mathbb{V})), \quad U \mapsto [\mathfrak{p}_\Sigma(A_U)] \tag{1.1.2}$$

*sending $U\in U_\Sigma$ to the $\binom{n}{r}$-tuple of $r\times r$-minors of $A^\Sigma_U$ is a closed immersion of algebraic varieties;*

(4) *$U_\Sigma \subset \mathrm{Grass}(r,V)$ is a Zariski open subset, the complement of the zero locus of $\mathfrak{p}_\Sigma$, and is isomorphic to $\mathbb{A}^{r(n-r)}$.*

Elementary subalgebras as defined below play the central role in what follows.

**Definition 1.2.** An elementary subalgebra $\epsilon \subset \mathfrak{g}$ of dimension $r$ is a Lie subalgebra of dimension $r$ which is commutative and has $p$-restriction equal to 0. We define

$$\mathbb{E}(r,\mathfrak{g}) \ = \ \{\epsilon \subset \mathfrak{g} : \ \epsilon \text{ elementary subalgebra of dimension } r\}$$

We denote by $\mathcal{N}_p(\mathfrak{g}) \subset \mathfrak{g}$ the closed subvariety of $p$-nilpotent elements of $\mathfrak{g}$ (that is, $\mathcal{N}_p(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, x^{[p]} = 0\}$), by $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \subset (\mathcal{N}_p(\mathfrak{g}))^{\times r}$ the variety of $r$-tuples of $p$-nilpotent, pairwise commuting elements of $\mathfrak{g}$, and by $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \subset \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))$ the open subvariety of linearly independent $r$-tuples of $p$-nilpotent, pairwise commuting elements of $\mathfrak{g}$.

**Notation 1.3.** For an algebraic group $G$ with Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, we consistently use the adjoint action of $G$ on $\mathbb{E}(r,\mathfrak{g})$. Explicitly, for an $r$-dimensional elementary subalgebra $\epsilon \subset \mathfrak{g}$, and for $g \in G$, we denote by $g\cdot\epsilon \in \mathbb{E}(r,\mathfrak{g})$ the $r$-dimensional elementary subalgebra defined as follows:

$$G\cdot\epsilon := \{\mathrm{Ad}(g)x \,|\, x \in \epsilon\}$$

Consequently, we use $G\cdot\epsilon$ to denote the orbit of $\epsilon\in\mathbb{E}(r,\mathfrak{g})$ under this action.

**Proposition 1.4.** *Let $\mathfrak{g}$ be a Lie algebra of dimension $n$, let $r$ be a positive integer, $1 \le r < n$, and let $\mathrm{Grass}(r,\mathfrak{g})$ be the projective variety of $r$-planes of $\mathfrak{g}$, where we view $\mathfrak{g}$ as a vector space. There exists a natural cartesian square*

$$\begin{tikzcd} \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \arrow[r, hook] \arrow[d] & \mathbb{M}^\circ_{n,r} \arrow[d, "p"] \\ \mathbb{E}(r,\mathfrak{g}) \arrow[r, hook] & \mathrm{Grass}(r,\mathfrak{g}) \end{tikzcd} \tag{1.4.1}$$

*whose vertical maps are* $\mathrm{GL}_r$*-torsors locally trivial for the Zariski topology and whose horizontal maps are closed immersions. In particular,* $\mathbb{E}(r,\mathfrak{g})$ *has a natural structure of a projective algebraic variety, as a reduced closed subscheme of* $Grass(r,\mathfrak{g})$*.*

*If* $G$ *is a linear algebraic group with* $\mathfrak{g} = \mathrm{Lie}(G)$*, then* $\mathbb{E}(r,\mathfrak{g}) \hookrightarrow \mathrm{Grass}(r,\mathfrak{g})$ *is a* $G$*-stable embedding with respect to the adjoint action of* $G$*.*

*Proof.* The horizontal maps of (1.4.1) are the evident inclusions, the left vertical map is the restriction of $p$. Clearly, (1.4.1) is cartesian; in particular, $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \subset \mathbb{M}^\circ_{n,r}$ is stable under the action of $\mathrm{GL}_r$.

To prove that $\mathbb{E}(r,\mathfrak{g}) \subset \mathrm{Grass}(r,\mathfrak{g})$ is closed, it suffices to verify for each $\Sigma$ that $(\mathbb{E}(r,\mathfrak{g}) \cap U_\Sigma) \subset U_\Sigma$ is a closed embedding. The restriction of (1.4.1) above $U_\Sigma$ takes the form

(1.4.2)
$$\begin{tikzcd}
\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \cap p^{-1}(U_\Sigma) \arrow[r] \arrow[d] & p^{-1}(U_\Sigma) \arrow[r, "\sim"] \arrow[d, "p"] & U_\Sigma \times \mathrm{GL}_r \arrow[d, "pr"] \\
\mathbb{E}(r,\mathfrak{g}) \cap U_\Sigma \arrow[r] & U_\Sigma \arrow[r, equal] & U_\Sigma
\end{tikzcd}$$

Consequently, to prove that $\mathbb{E}(r,\mathfrak{g}) \subset \mathrm{Grass}(r,\mathfrak{g})$ is closed and that $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \to \mathbb{E}(r,\mathfrak{g})$ is a $\mathrm{GL}_r$-torsor which is locally trivial for the Zariski topology it suffices to prove that $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \subset \mathbb{M}^\circ_{n,r}$ is closed.

It is clear that $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \subset \mathbb{M}_{n,r}$ is a closed subvariety since it is defined by the vanishing of the Lie bracket and the $p$-operator $(-)^{[p]}$ both of which can be expressed as polynomial equations on the matrix coefficients. Hence, $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ = \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \cap \mathbb{M}^\circ_{n,r}$ is closed in $\mathbb{M}^\circ_{n,r}$.

If $\mathfrak{g} = \mathrm{Lie}(G)$, then the (diagonal) adjoint action of $G$ on $n\times r$-matrices $\mathfrak{g}^{\oplus r}$ sends a matrix whose columns pair-wise commute and which satisfies the condition that $(-)^{[p]}$ vanishes on these columns to another matrix satisfying the same conditions (since $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ preserves both the Lie bracket and the $p^{th}$-power). Thus, $\mathbb{E}(r,\mathfrak{g})$ is $G$-stable. □

**Remark 1.5.** Let $V$ be a $k$-vector space of dimension $n$, and let $V^\# = \mathrm{Hom}_k(V,k)$ denote its linear dual. Consider $\mathbb{V} \equiv \mathrm{Spec}\, S^*(V^\#) \simeq \mathbb{G}_a^{\times n}$, the vector group on the (based) vector space $V$. Then $\mathrm{Lie}(\mathbb{V}) \simeq \mathfrak{g}_a^{\oplus n}$ and we have an isomorphism of algebras
$$\mathfrak{u}(\mathrm{Lie}\,\mathbb{V}) \;\simeq\; \mathfrak{u}(\mathfrak{g}_a^{\oplus n}) \;\simeq\; k[t_1,\ldots,t_n]/(t_1^p,\ldots,t_n^p).$$
Let $E = (\mathbb{Z}/p)^{\times n}$ be an elementary abelian $p$-group of rank $n$ and choose an embedding of $V$ into the radical $\mathrm{Rad}(kE)$ of the group algebra of $E$ such that the composition with the projection to $\mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$ is an isomorphism. This choice determines an isomorphism
$$\mathfrak{u}(\mathrm{Lie}(\mathbb{V})) \xrightarrow{\sim} kE\ .$$
With this identification, the investigations of [12] are special cases of considerations of this paper.

**Example 1.6.** For any (finite dimensional, $p$-restricted) Lie algebra,
$$\mathbb{E}(1,\mathfrak{g}) \;\simeq\; \mathrm{Proj}\, k[\mathcal{N}_p(\mathfrak{g})]$$
as shown in [42], where $k[\mathcal{N}_p(\mathfrak{g})]$ is the (graded) coordinate algebra of the $p$-null cone of $\mathfrak{g}$. If $G$ is reductive with $\mathfrak{g} = \mathrm{Lie}(G)$ and if $p$ is good for $G$, then $\mathcal{N}_p(\mathfrak{g})$ is

irreducible and equals the $G$-orbit $G\cdot\mathfrak{u}$ of the nilpotent radical of a specific parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ (see [34, 6.3.1]).

**Example 1.7.** Let $G$ be a connected reductive algebraic group, let $\mathfrak{g} = \operatorname{Lie} G$, and assume that $p \geq h$, the Coxeter number of $G$, and that the derived subgroup of $G$ is simply connected. The assumption on $p$ implies that $\mathcal{N}_p(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$, the null cone of $\mathfrak{g}$. Finally, we exclude the case when $G$ is of type $A_1$ to ensure that $\mathbb{E}(2, \mathfrak{g})$ is non-empty.

As shown by A. Premet in [36], $\mathcal{C}_2(\mathcal{N}(\mathfrak{g}))$ is equidimensional with irreducible components enumerated by the distinguished nilpotent orbits of $\mathfrak{g}$; in particular, $\mathcal{C}_2(\mathcal{N}(\mathfrak{gl}_n))$ is irreducible. This easily implies that $\mathbb{E}(2, \mathfrak{g})$ is an equidimensional variety, irreducible in the special case $\mathfrak{g} = \mathfrak{gl}_n$. Since $\dim \mathbb{E}(2, \mathfrak{g}) = \dim \mathcal{C}_2(\mathcal{N}_p(\mathfrak{g})) - \dim \mathrm{GL}_2$, $\dim \mathbb{E}(2, \mathfrak{g}) = \dim[G, G] - 4$. In particular, $\mathbb{E}(2, \mathfrak{gl}_n)$ has dimension $n^2 - 5$ for $p \geq n$.

**Example 1.8.** Assume that $p > 2$. Let $\mathfrak{u}_3 \subset \mathfrak{gl}_3$ denote the Lie subalgebra of strictly upper triangular matrices and take $r = 2$. Then a 2-dimensional elementary Lie subalgebra $\epsilon \subset \mathfrak{u}_3$ is spanned by $E_{1,3}$ and another element $X \in \mathfrak{u}_3$ not a scalar multiple of $E_{1,3}$. We can further normalize the basis of $\epsilon$ by subtracting a multiple of $E_{1,3}$ from $X$, so that $X = a_{1,2}E_{1,2} + a_{2,3}E_{2,3}$. Thus, 2-dimensional elementary Lie subalgebras $\epsilon \subset \mathfrak{u}$ are parametrized by points $\langle a_{1,2}, a_{2,3}\rangle \in \mathbb{P}^1$, so that $\mathbb{E}(2, \mathfrak{u}_3) \simeq \mathbb{P}^1$.

In this case, $\mathfrak{u}_3$ is the Lie algebra of the unipotent radical of the Borel subgroup $B_3 \subset \mathrm{GL}_3$ of upper triangular matrices. The adjoint action of $\mathrm{GL}_3$ on $\mathfrak{gl}_3$ induces the action of $B_3$ on $\mathbb{E}(2, \mathfrak{u}_3)$ since $B_3$ stabilizes $\mathfrak{u}_3$. With respect to this action of $B_3$, $\mathbb{E}(2, \mathfrak{u}_3)$ is the union of an open dense orbit consisting of regular nilpotent elements of the form $a_{1,2}E_{1,2} + a_{2,3}E_{2,3}$, with $a_{1,2} \neq 0 \neq a_{2,3}$; and two closed orbits. The open orbit is isomorphic to the 1-dimensional torus $\mathbb{G}_m \subset \mathbb{P}^1$ and the two closed orbits are single points $\{0\}, \{\infty\}$.

We thank the referee for the following observation.

**Proposition 1.9.** *Let $G$ be a semisimple algebraic group, and $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of $G$. Let $r$ be the Lie rank of $\mathfrak{g}$. Assume $p \geq h$, where $h$ the Coxeter number of $G$. Let $\epsilon_{reg} \in \mathbb{E}(r, \mathfrak{g})$ be an elementary subalgebra containing a regular element of $\mathfrak{g}$. Then $G \cdot \epsilon_{reg} \subset \mathbb{E}(r, \mathfrak{g})$ is an open orbit.*

*Proof.* Let $X$ be a regular nilpotent element. Recall that the nilpotent part of the centralizer of $X$ in $\mathfrak{g}$ is generated by $\langle X, X^2, \ldots, X^r\rangle$. Hence, there exists an elementary algebra $\epsilon_{reg}$ of dimension $r$ contaning $X$. Let $Z$ be the complement of the regular nilpotent orbit in $\mathcal{N}_p(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ (that is, $Z$ is the closure of the subregular orbit). Observe that any $r$-tuple of nilpotent commuting matrices of $\mathfrak{g}$ containing a *regular* nilpotent element has to be conjugate to $(X, X^2, \ldots, X^r)$ under the action of $G \times \mathrm{GL}_r$. This implies that the diagram (1.4.1) extends as follows:

$$\begin{tikzcd}
\mathcal{C}_r(Z)^\circ \arrow[r, hook] \arrow[d] & \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ \arrow[r, hook] \arrow[d] & \mathbb{M}^\circ_{n,r} \arrow[d, "p"] \\
\mathbb{E}(r, \mathfrak{g}) - G \cdot \epsilon_{reg} \arrow[r, hook] & \mathbb{E}(r, \mathfrak{g}) \arrow[r, hook] & \mathrm{Grass}(r, \mathfrak{g}).
\end{tikzcd}$$

Since $\mathcal{C}_r(Z)^\circ$ is a closed $\mathrm{GL}_r$-stable subset of $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))^\circ$, we conclude that $G \cdot \epsilon_{reg}$ is open in $\mathbb{E}(r, \mathfrak{g})$. $\square$

**Example 1.10.** We consider the algebraic group $G = \mathrm{GL}_n$ and some $r,\ 1 \le r < n$. Let $\mathfrak{u}_{r,n-r} \subset \mathfrak{gl}_n$ denote the Lie subalgebra of $n \times n$ matrices $(a_{i,j})$ with $a_{i,j} = 0$ unless $1 \le i \le r,\ r+1 \le j \le n$. Then $\mathfrak{u}_{r,n-r} \subset \mathfrak{gl}_n$ is an elementary subalgebra of dimension $r(n-r)$. The argument given in [33, §5] applies in our situation to show that $\mathfrak{u}_{r,n-r}$ is a maximal elementary subalgebra (that is, not contained in any other elementary subalgebra).

Let $X \subset \mathbb{E}(r(n-r), \mathfrak{gl}_n)$ denote the $\mathrm{GL}_n$-orbit of $\mathfrak{u}_{r,n-r}$ (as defined in Notation 1.3). Let $P_r$ be the standard parabolic subgroup of $\mathrm{GL}_n$ defined by the equations $a_{i,j} = 0$ for $i > r, j \le n-r$. Since $P_r$ is the stabilizer of $\mathfrak{u}_{r,n-r}$ under the adjoint action of $\mathrm{GL}_n$, $X = G \cdot \mathfrak{u}_{r,n-r} \simeq \mathrm{GL}_n / P_r \simeq \mathrm{Grass}(r,n)$. Since $X$ is projective, it is a closed $\mathrm{GL}_n$-stable subvariety of $\mathbb{E}(r(n-r), \mathfrak{gl}_n)$.

We next give examples of $p$-restricted Lie algebras which are not the Lie algebras of algebraic groups.

**Example 1.11.** Let $\phi : \mathfrak{gl}_{2n} \to k$ be a semi-linear map (so that $\phi(av) = a^p\phi(v)$), and consider the extension of $p$-restricted Lie algebras, split as an extension of Lie algebras (see [19, 3.11]):

$$(1.11.1) \qquad 0 \to k \to \widetilde{\mathfrak{gl}}_{2n} \to \mathfrak{gl}_{2n} \to 0, \quad (b,x)^{[p]} = (\phi(x), x^{[p]}).$$

Then $\mathbb{E}(n^2+1, \widetilde{\mathfrak{gl}}_{2n})$ can be identified with the subvariety of $\mathrm{Grass}(n, 2n)$ consisting of those elementary subalgebras $\epsilon \subset \mathfrak{gl}_{2n}$ of dimension $n^2$ such that the restriction of $\phi$ to $\epsilon$ is 0 (or, equivalently, such that $\epsilon$ is contained in the kernel of $\phi$).

**Example 1.12.** (1). Consider the general linear group $\mathrm{GL}_n$ and let $V$ be the defining representation. Let $\mathbb{V}$ be the vector group associated to $V$ as in Remark 1.5. We set

$$(1.12.1) \qquad G_{1,n} \overset{\text{def}}{=\!=} \mathbb{V} \rtimes \mathrm{GL}_n, \qquad \mathfrak{g}_{1,n} \overset{\text{def}}{=\!=} \mathrm{Lie}\, G_{1,n}$$

Any subspace $\epsilon \subset V$ of dimension $r < n$ can be considered as an elementary subalgebra of $\mathfrak{g}_{1,n}$. Moreover, the $G_{1,n}$-orbit of $\epsilon \in \mathbb{E}(r, \mathfrak{g}_{1,n})$ can be identified with $\mathrm{Grass}(r, V)$.

(2). More generally, let $H$ be an algebraic group, $W$ be a rational representation of $H$, and $\mathbb{W}$ be the vector group associated to $W$. Let $G \equiv \mathbb{W} \rtimes H$, and let $\mathfrak{h} = \mathrm{Lie}\, H$. A subspace $\epsilon \subset W$ of dimension $r < \dim W$ can be viewed as an elementary subalgebra of $\mathfrak{g}$. Moreover, the $G$-orbit of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ can be identified with the $H$-orbit of $\epsilon$ in $\mathrm{Grass}(r, W)$.

We conclude this section by giving a straightforward way to obtain additional computations from known computations of $\mathbb{E}(r, \mathfrak{g})$. The proof is immediate.

**Proposition 1.13.** *Let $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_s$ be finite dimensional $p$-restricted Lie algebras and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. Then there is a natural morphism of projective varieties*

$$(1.13.1) \qquad \mathbb{E}(r_1, \mathfrak{g}_1) \times \cdots \times \mathbb{E}(r_s, \mathfrak{g}_s) \longrightarrow \mathbb{E}(r, \mathfrak{g}), \quad r = \sum r_i,$$

*sending $(\epsilon_1 \subset \mathfrak{g}_1, \ldots, \epsilon_s \subset \mathfrak{g}_s)$ to $\epsilon_1 \oplus \cdots \oplus \epsilon_s \subset \mathfrak{g}$. Moreover, if $r_i$ is the maximum of the dimensions of the elementary subalgebras of $\mathfrak{g}_i$ for each $i, 1 \le i \le s$, then this morphism is bijective.*

**Corollary 1.14.** *In the special case of Proposition 1.13 in which each $\mathfrak{g}_i \simeq \mathfrak{sl}_2$, $r_1 = \cdots = r_s = 1$, (1.13.1) specializes to*

$$(\mathbb{P}^1)^{\times r} \ \simeq\ \mathbb{E}(r, \mathfrak{sl}_2^{\oplus r}).$$

*Proof.* This follows from the fact that $\mathbb{E}(1,\mathfrak{sl}_2) = \operatorname{Proj} k[\mathcal{N}(\mathfrak{sl}_2)] \simeq \mathbb{P}^1$ (see, for example, [22]). □

## 2. Elementary subalgebras of maximal dimension

The study of maximal abelian subalgebras in complex semi-simple Lie algebras has a long history, dating back at least to the work of Schur in the general linear case at the turn of last century [38]. The dimensions of maximal abelian subalgebras of a complex simple Lie algebra are known thanks to the classical work of Malcev [31]. Malcev's arguments apply to the positive characteristic case with little modification showing that the maximal dimensions he determined also give maximal dimensions of elementary subalgebras of simple Lie algebras of types A, B, C, D, E, F, G at least for $p$ good. In this paper, we reproduce this calculation for types A and C.

As pointed out to us by S. Mitchell, our investigation of Lie algebras over fields of positive characteristic is closely related to the study Barry [3] who considered the analogous problem of identifying maximal elementary abelian subgroups of Chevalley groups. Subsequent work by Milgram and Priddy [33] in the case of the general linear groups guided some of our calculations.

The reader finds below consideration of $\mathbb{E}(r,\mathfrak{g})$ for several families of $p$-restricted Lie algebras $\mathfrak{g}$ and $r$ the maximal dimension of an elementary subalgebra of $\mathfrak{g}$.

- Heisenberg Lie algebras (Proposition 2.2)
- The general linear Lie algebra $\mathfrak{gl}_n$ (Theorems 2.7 and 2.8).
- The symplectic Lie algebra $\mathfrak{sp}_{2n}$. (Theorem 2.12).
- The Lie algebra of a maximal parabolic of $\mathfrak{gl}_n$ (Theorem 2.13).
- The Lie algebras of Example 1.12(1) (Corollary 2.14).

In what follows, we consider a connected reductive algebraic group $G$ over $k$. We choose a Borel subgroup $B = U \cdot T \subset G$, thereby fixing a basis of simple roots $\Delta \subset \Phi$ and the subset of positive roots $\Phi^+$. For a simple root $\alpha \in \Delta$, we denote by $P_\alpha$, $\mathfrak{p}_\alpha$, the corresponding standard maximal parabolic subgroup and its Lie algebra. We write

$$\mathfrak{p}_\alpha \; = \; \mathfrak{h} \oplus \sum_{\beta \in \Phi_I^- \cup \Phi^+} k x_\beta,$$

where $x_\beta$ is the root vector corresponding to the root $\beta$ and $\Phi_I$ is the root subsystem generated by the subset $\Delta \backslash \{\alpha\}$. We follow the convention in [8, ch.6] in the numbering of simple roots. For $\mathfrak{g} = \operatorname{Lie}(G)$ we denote by $\mathfrak{h} \subset \mathfrak{g}$ the Cartan algebra given by $\mathfrak{h} = \operatorname{Lie}(T)$ and write $\mathfrak{g} \; = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, the standard triangular decomposition.

We begin by recalling the explicit nature of the *Heisenberg Lie algebras* which not only constitutes our first example but also reappear in the inductive analysis of other examples.

**Definition 2.1.** A ($p$-)restricted Lie algebra $\mathfrak{g}$ is a Heisenberg restricted Lie algebra if the center $\mathfrak{z}$ of $\mathfrak{g}$ is one dimensional, $\mathfrak{g}/\mathfrak{z}$ is an elementary Lie algebra and if the $p$-power operation vanishes on $\mathfrak{g}$.

The requirement that the $p$-restriction map vanish on a Heisenberg algebra means that only example in the case that $p = 2$ is the trivial example: $\mathfrak{g} = \mathfrak{z}$. More generally, if $p = 2$ then any restricted Lie algebra with vanishing restriction map is an elementary algebra.

Let $\mathfrak{g}$ be a Heisenberg restricted Lie algebra. Then $\mathfrak{g}$ admits a basis

$$\{x_1, \dots x_{n-1}, y_1, \dots y_{n-1}, y_n\} \tag{2.1.1}$$

such that $y_n$ generates the one dimensional center $\mathfrak{z}$ of $\mathfrak{g}$ and

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{i,j} y_n \quad 1 \le i, j \le n-1.$$

Let $W = \mathfrak{g}/\mathfrak{z}$, let $\phi : \mathfrak{g} \to W$ be the projection map, and let $\sigma : W \to \mathfrak{g}$ be a $k$-linear right splitting of $\phi$. For $x, y \in W$, let $\langle x, y\rangle$ be the coefficient of $y_n$ in $[\sigma(x), \sigma(y)] \in \mathfrak{z} = k y_n$. So defined, $\langle -, -\rangle$ gives $W$ a symplectic vector space structure.

We recall that a subspace $L$ of a symplectic vector space $W$ is said to be Lagrangian if $L$ is an isotropic subspace (i.e., if the pairing of any two elements of $L$ is 0) of maximal dimension. We denote by $\mathrm{LG}(n, W)$ the *Lagrangian Grassmannian* of $W$, the homogeneous space parameterizing the Lagrangian subspaces of $W$. Note that, if $L$ is a Lagrangian subspace of $W = \mathfrak{g}/\mathfrak{z}$, for $\mathfrak{g}$ and $\mathfrak{z}$ as in the previous paragraph, then the inverse image $\phi^{-1}(L) \subseteq \mathfrak{g}$ is an elementary Lie algebra.

**Proposition 2.2.** *Let $\mathfrak{g}$ be a Heisenberg restricted Lie algebra of dimension $2n-1$. Equip $W = \mathfrak{g}/\mathfrak{z}$ with the symplectic form as above.*

(1) *The maximal dimension of an elementary subalgebra of $\mathfrak{g}$ is $n$.*
(2) $\mathbb{E}(n, \mathfrak{g}) \simeq \mathrm{LG}(n-1, W)$.

*Proof.* Let $\phi : \mathfrak{g} \to W = \mathfrak{g}/\mathfrak{z}$ be the projection map. Observe that if a subalgebra $\epsilon$ of $\mathfrak{g}$ is elementary then $\phi(\epsilon)$ is an isotropic linear subspace of $W$. Since $\dim \phi(\epsilon) + \dim \phi(\epsilon)^{\perp} = \dim W$ (where $\phi(\epsilon)^{\perp}$ denotes the orthogonal complement with respect to the symplectic form) and $\phi(\epsilon) \subset \phi(\epsilon)^{\perp}$ since $\phi(\epsilon)$ is isotropic, we get that $\dim \phi(\epsilon) \le (\dim W)/2 = n-1$, and, consequently, $\dim \epsilon \le n$. Moreover, the equality holds if and only if $\epsilon/\mathfrak{z}$ is a Lagrangian subspace of $W$. Hence, $\mathbb{E}(n, \mathfrak{g}) \simeq \mathrm{LG}(n-1, W)$. □

**Example 2.3.** We give various Lie-theoretic contexts in which the Heisenberg Lie algebras arise. In every case, assume that $p > 2$.

(1) Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, and let $\mathfrak{p}_J \subset \mathfrak{g}$ be the standard parabolic subalgebra defined by the subset $J = \{\alpha_2, \dots, \alpha_{n-1}\}$ of simple roots, that is, $\mathfrak{p}_J = \mathfrak{h} \oplus \bigoplus\limits_{\alpha \in \Phi_J^- \cup \Phi^+} k x_\alpha$, where $\Phi_J$ is the root subsystem of $\Phi$ generated by the subset of simple roots $J$. Then the unipotent radical $\mathfrak{u}_J = \bigoplus\limits_{\alpha \in \Phi^+ \backslash \Phi_J^+} k x_\alpha$ of $\mathfrak{p}_J$ is a Heisenberg restricted Lie algebra of dimension $2n-1$. In matrix terms, this is the subalgebra of strictly upper triangular matrices with non-zero entries in the top row or the rightmost column.
(2) Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\mathfrak{p} = \mathfrak{p}_{\alpha_1}$ be the maximal parabolic subalgebra corresponding to the simple root $\alpha_1$. Let $\gamma_n = 2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n$ be the highest long root, and let further

$$\beta_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i, \quad \gamma_{n-i} = \gamma_n - \beta_i. \tag{2.3.1}$$

Then $\mathfrak{u}_{\alpha_1}$, the nilpotent radical of $\mathfrak{p}_{\alpha_1}$ is a Heisenberg Lie algebra, and the basis $\{x_{\beta_1}, \dots, x_{\beta_{n-1}}, x_{\gamma_{n-1}}, \dots, x_{\gamma_1}, x_{\gamma_n}\}$ satisfies the conditions required in (2.1.1).
(3) *Type $E_7$.* Let $\mathfrak{p} = \mathfrak{p}_{\alpha_1}$. Then the nilpotent radical of $\mathfrak{p}$ is a Heisenberg Lie algebra. .

**Remark 2.4.** The referee has pointed out that all of the above examples fit into a general pattern. Let $G$ be a simple algebraic group. Suppose that $\alpha$ is the positive root of maximum height. If $\beta$ is any other positive root, then $(\beta, \alpha^\vee)$ is one of 0, 1 or 2, and has value 2 if and only if $\beta = \alpha$. Then the direct sum of the root subspaces of $\mathfrak{g} = \mathrm{Lie}(G)$ spanned by $x_\beta$ with $(\beta, \alpha^\vee) > 0$ is a Heisenberg restricted Lie algebra provided $p > 2$.

The following well known property of parabolic subgroups is used frequently.

**Lemma 2.5.** *Let $G$ be a simple algebraic group and $P$ be a standard parabolic subgroup of $G$. Let $\mathfrak{p} = \mathrm{Lie}(P)$ and $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Unless $G$ is of type $A_1$ and $p = 2$, we have $[\mathfrak{u}, \mathfrak{p}] = \mathfrak{u}$.*

*Proof.* Since $\mathfrak{u}$ is a Lie ideal in $\mathfrak{p}$, we have $[\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{u}$. For the opposite inclusion, it suffices to show that for any simple root $\alpha$ such that $x_\alpha \in \mathfrak{u}$, we have $x_\alpha \in [\mathfrak{h}, \mathfrak{u}]$. Except for the situation excluded in the statement of the lemma, we can always find a simple root $\beta$ such that the entry $\langle \alpha, \beta \rangle$ of the Cartan matrix of $\mathfrak{g}$ is non-zero. Hence, $[h_\beta, x_\alpha]$ is a non-zero multiple of $x_\alpha$, and we conclude that $\mathfrak{u} \in [\mathfrak{h}, \mathfrak{u}]$. □

In the examples that follow, the closed subvariety $\mathbb{E}(r, \mathfrak{g}) \subset \mathrm{Grass}(r, \mathfrak{g})$ is a single orbit or a disjoint union of two orbits for $G$. Such an orbit $G \cdot \epsilon$ can be described set-theoretically via the orbit map $\pi : G \to \mathbb{E}(r, \mathfrak{g})$, $g \mapsto g \cdot \epsilon$. In order to use this observation to identify $\mathbb{E}(r, \mathfrak{g})$ as a homogenous space $G/\mathrm{Stab}_G(\epsilon)$ (or a disjoint union of two homogeneous spaces), we need to know that the orbit map is separable. The following remark addresses this issue.

**Remark 2.6.** Let $G$ be an algebraic group and $X$ be a $G$-variety, both defined over an algebraically closed field $k$. For $x \in X$, the orbit map $\pi_x : G \to G \cdot x \subset X$ determines a homeomorphism $\overline{\pi}_x : G/G_x \to G \cdot x$ where $G_x$ is the (reduced) stabilizer of $x$. This is an isomorphism of varieties if the map $\pi_x$ is separable (equivalently, if the tangent map $d\pi_x$ at the identity is surjective). In [13, 3.7] we show that when $p > 2h - 2$ where $h$ is the Coxeter number of a semi-simple algebraic group $G$, the orbit map $G \to G \cdot \epsilon \subset \mathrm{Grass}(r, \mathfrak{g})$ under the adjoint action of $G$ on $\mathrm{Grass}(r, \mathfrak{g})$ is separable. This implies that the homeomorphisms of (2.7)(3), (2.8)(3) and (2.9) are isomorphisms of varieties at least when $p > 2n - 2$; and that the homeomorphism of Theorem 2.12 is an isomorphism at least for $p > 4n - 2$.

We point out that in a forthcoming paper [35], the authors show that the orbit map $G \to G \cdot \epsilon \subset \mathrm{Grass}(r, \mathfrak{g})$ is always separable in types A, B, C, D removing the restriction on $p$. Hence, the maps in (2.7)(3), (2.8)(3) and (2.9) are, in fact, isomorphisms for any $p$.

We consider the special linear Lie algebra $\mathfrak{sl}_n = \mathrm{Lie}(\mathrm{SL}_n)$ in two parallel theorems, one for $n$ even and the other for $n$ odd. We denote by $\mathfrak{u}_n = \mathrm{Lie}(U)$ the nilpotent radical of the Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B)$. We also use the notation $P_{r,n-r}$, $\mathfrak{p}_{r,n-r}$, and $\mathfrak{u}_{r,n-r}$ for the maximal parabolic corresponding to the simple root $\alpha_r$, its Lie algebra, and its nilpotent radical.

The first parts of both Theorem 2.7 and Theorem 2.8 are well-known in the context of maximal elementary abelian subgroups in $\mathrm{GL}_n(\mathbb{F}_p)$ (see, for example, [24] or [33]). We use the approach of [33] to compute conjugacy classes.

**Theorem 2.7.** *Assume $p > 2$, and $m \geq 1$.*

(1) *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{sl}_{2m}$ is $m^2$.*

(2) *Any elementary abelian subalgebra of dimension $m^2$ is conjugate to $\mathfrak{u}_{m,m}$, the nilpotent radical of the standard maximal parabolic $P_{m,m}$.*
(3) *There is a finite, radicial morphism* $\operatorname{Grass}(m,2m) \to \mathbb{E}(m^2, \mathfrak{sl}_{2m})$ *, inducing a homeomorphism on Zariski spaces; this morphism is an isomorphism if $p > 4m-2$.*

*Proof.* We prove the following statement by induction: any elementary subalgebra of $\mathfrak{sl}_{2m}$ has dimension at most $m^2$ and any subalgebra of such dimension inside the nilpotent radical $\mathfrak{u}_{2m}$ (the subalgebra of strictly upper triangular $2m \times 2m$-matrices) must coincide with $\mathfrak{u}_{m,m}$. This implies claims (1) and (2) of the theorem.

The statement is clear for $m = 1$. Assume it is proved for $m-1$. Let $\epsilon$ be an elementary subalgebra of $\mathfrak{sl}_{2m}$. Since $\epsilon$ consists of nilpotent matrices, it can be conjugated into upper-triangular form by Engel's theorem. Let $J = \{\alpha_2, \ldots, \alpha_{2m-2}\}$ and let $\mathfrak{u}_J$ be the nilpotent radical of the standard parabolic $P_J$ determined by $J$. Since $[\mathfrak{u}_{2m}, \mathfrak{u}_J] \subset \mathfrak{u}_J$, this is a Lie ideal in $\mathfrak{u}_{2m}$.

We consider extension

$$0 \longrightarrow \mathfrak{u}_J \longrightarrow \mathfrak{u}_{2m} \longrightarrow \mathfrak{u}_{2m}/\mathfrak{u}_J \simeq \mathfrak{u}_{2m-2} \longrightarrow 0.$$

Pictorially, the Lie algebras can be represented as follows, where $\mathfrak{u}_J$ is in the positions marked by $*$ in the first array and $\mathfrak{u}_{2m-2}$ is isomorphic to the Lie algebra with the positions marked by $*$ in the second.

$$\mathfrak{u}_J : \begin{array}{cccccc|c} 0 & * & * & \ldots & * & * \\ \hline 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & * \\ . & . & . & \cdots & & * \\ & & & & & * \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array} ; \qquad \mathfrak{u}_{2m-2} : \begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \hline 0 & 0 & * & * & \cdots & * & 0 \\ 0 & 0 & 0 & * & \cdots & * & 0 \\ . & . & . & . & \cdots & * & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & \cdots & * & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}$$

By induction, the dimension of the projection of $\epsilon$ onto $\mathfrak{u}_{2m-2}$ is at most $(m-1)^2$, and this dimension is attained if and only if the image of $\epsilon$ under the projection is the subalgebra of $\mathfrak{u}_{2m-2}$ of all block matrices of the form $\begin{pmatrix} 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix}$, where $\mathbf{A}$ is an $(m-1) \times (m-1)$ matrix. Since $\mathfrak{u}_J$ is a Heisenberg Lie algebra of dimension $4m-3$ (see Example 2.3(1)), Proposition 2.2 implies that the maximal elementary subalgebra of $\mathfrak{u}_J$ has dimension $2m-1$. Hence, $\dim \epsilon \le (m-1)^2 + 2m - 1 = m^2$.

Now let's assume that $\epsilon$ has the maximal dimension $m^2$ and is upper-triangular. Our goal is to show that $\epsilon = \mathfrak{u}_{m,m}$. The argument in the previous paragraph implies that every element in $\epsilon \subset \mathfrak{sl}_{2m}$ has the form

$$\begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & * \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.7.1}$$

for some $\mathbf{v_i}, (\mathbf{w_i})^T \in k^{m-1}$.

Let $\begin{pmatrix} 0 & \mathbf{v_2'} & \mathbf{v_1'} & * \\ 0 & 0 & 0 & \mathbf{w_1'} \\ 0 & 0 & 0 & \mathbf{w_2'} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ be an element in $\epsilon \cap \mathfrak{u}_J$. Taking a bracket of this element with a general element in $\epsilon$ of the form as in (2.7.1), we get

$$\begin{pmatrix} 0 & 0 & \mathbf{v_2'A} & * \\ 0 & 0 & 0 & \mathbf{Aw_2'} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The assumption that $\epsilon$ has maximal dimension $m^2$ implies that for any $m-1 \times m-1$ matrix $A$, there is an element in $\epsilon$ of the form (2.7.1). Since $\epsilon$ is abelian, we conclude that $\mathbf{v_2'A} = 0$, $\mathbf{Aw_2'} = 0$ for any $\mathbf{A} \in M_{m-1}$. Hence, $\mathbf{v_2'} = 0$, $\mathbf{w_2'} = 0$ which implies that $\epsilon \cap \mathfrak{u}_J \subset \mathfrak{u}_{m,m}$. Moreover, for the dimension to be maximal, we need $\dim \epsilon \cap \mathfrak{u}_J = 2m - 1$. Hence, for any $\mathbf{v_1}, (\mathbf{w_1})^T \in k^{m-1}$, the matrix

$$\begin{pmatrix} 0 & 0 & \mathbf{v_1} & 0 \\ 0 & 0 & 0 & \mathbf{w_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in $\epsilon$.

It remains to show that for an arbitrary element of $\epsilon$, necessarily of the form (2.7.1), we must have $\mathbf{v_2} = 0, \mathbf{w_2} = 0$. We prove this by contradiction. Suppose $\begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & * \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon$ with $\mathbf{v_2} \neq 0$. Subtracting a multiple of $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which is necessarily in $\epsilon$, we get that $M = \begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & 0 \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ belongs to $\epsilon$. Since $\mathbf{v_2} \neq 0$, we can find a vector $(\mathbf{w_1})^T \in k^{m-1}$ such that $\mathbf{v_2} \cdot (\mathbf{w_1})^T \neq 0$. As observed above, we have $M' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mathbf{w_1})^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in $\epsilon$. Therefore, $[M, M']$ has a non-trivial entry $\mathbf{v_2} \cdot (\mathbf{w_1})^T$ in the $(1, 2m)$ spot which contradicts commutativity of $\epsilon$. Hence, $\mathbf{v_2} = 0$. Similarly, $\mathbf{w_2} = 0$. This finishes the proof of the claim.

To show (3), let $\widetilde{P}$ denote the stabilizer of $\mathfrak{u}_{m,m}$ under the adjoint action of $\mathrm{SL}_{2m}$, so that $\mathrm{SL}_{2m}/\widetilde{P} \simeq \mathrm{SL}_{2m} \cdot \mathfrak{u}_{m,m}$. By (2) and the fact that $P_{m,m}$ normalizes its unipotent radical $U_{m,m}$, and, hence, stabilizes $\mathfrak{u}_{m,m}$, the orbit map $\mathrm{SL}_{2m} \to \mathrm{SL}_{2m} \cdot \mathfrak{u}_{m,m} = \mathbb{E}(m^2, \mathfrak{sl}_{2m})$ factors as $\mathrm{SL}_{2m} \to \mathrm{SL}_{2m}/P_{m,m} \to \mathrm{SL}_{2m}/\widetilde{P}$. Since $P_{m,m}$ is maximal among (reduced) algebraic subgroups of $\mathrm{SL}_{2m}$, we conclude that $\widetilde{P}_{\mathrm{red}} = P_{m,m}$. Consequently, we conclude that

$$\mathrm{Grass}(m, 2m) = \mathrm{SL}_{2m}/P_{m,m} \to \mathrm{SL}_{2m}/\widetilde{P} = \mathbb{E}(m^2, \mathfrak{sl}_{2m})$$

is a torsor for the infinitesimal group scheme $\widetilde{P}/P_{m,m}$ and thus is finite and radicial.

The second assertion of (3) (that the map $\mathrm{Grass}(m, 2m) \to \mathbb{E}(m^2, \mathfrak{sl}_{2m})$ is an isomorphism for $p > 4m-2$), is verified in [13, 3.7] as explained in Remark 2.6. $\square$

**Theorem 2.8.** *Assume $m > 1$, $p > 2$.*

(1) *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{sl}_{2m+1}$ is $m(m+1)$.*
(2) *There are two distinct conjugacy classes of such elementary subalgebras, represented by $\mathfrak{u}_{m,m+1}$ and $\mathfrak{u}_{m+1,m}$.*
(3) *There is a finite radicial morphism*

$$\mathrm{Grass}(m, 2m+1) \sqcup \mathrm{Grass}(m, 2m+1) \longrightarrow \mathbb{E}(m(m+1), \mathfrak{sl}_{2m+1})$$

*inducing a homeomorphism on Zariski spaces; this morphism is an isomorphism for $p > 4m$.*

*Proof.* Let $\mathfrak{u}_3$ be the Heisenberg Lie algebra of strictly upper-triangular $3 \times 3$ matrices. By Proposition 2.2, $\mathbb{E}(2, \mathfrak{u}_3) \simeq \mathrm{LG}(1,2) \simeq \mathbb{P}^1$, and the maximal dimension is 2. In the following complete list of maximal elementary subalgebras of $\mathfrak{u}_3$, we separate the algebras $\mathfrak{u}_{1,2}$ and $\mathfrak{u}_{2,1}$ for easy referencing later in the proof.

- $\mathfrak{u}_{1,2} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \right\}$,
- $\mathfrak{u}_{2,1} = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \right\}$,
- a one-parameter family $\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & xa \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \right\}$ for a fixed $x \in k^*$.

We prove the following statements by induction: For any $m > 1$, an elementary subalgebra of $\mathfrak{sl}_{2m+1}$ has dimension at most $m(m+1)$. Any subalgebra of such dimension inside $\mathfrak{u}_{2m+1}$ must coincide either with $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$. This implies (1) and (2).

Base case: $m = 2$. Any elementary subalgebra can be conjugated to the upper-triangular form. So it suffices to prove the statement for an elementary subalgebra $\epsilon$ of $\mathfrak{u}_5$, the Lie algebra of strictly upper triangular $5 \times 5$ matrices. As in the proof of Theorem 2.7, we consider a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{u}_J \longrightarrow \mathfrak{u}_5 \xrightarrow{\mathrm{pr}} \mathfrak{u}_3 \longrightarrow 0$$

where $J = \{\alpha_2, \alpha_3\}$ (and, hence, $\mathfrak{u}_J \subset \mathfrak{u}_5$ is the subalgebra of upper triangular matrices with zeros everywhere except for the top row and the rightmost column). Since $\dim(\mathrm{pr}(\epsilon)) \leq 2$ by the remark above, and $\dim(\epsilon \cap \mathfrak{u}_J) \leq 4$ by Proposition 2.2(1), we get that $\dim \epsilon \leq 6$. For the equality to be attained, we need $\mathrm{pr}(\epsilon)$ to be one of the two dimensional elementary subalgebras listed above. If $\mathrm{pr}(\epsilon) = \mathfrak{u}_{2,1}$ then arguing exactly as in the proof for the even dimensional case, we conclude that $\epsilon = \mathfrak{u}_{3,2} \subset \mathfrak{u}_5$. Similarly, if $\mathrm{pr}(\epsilon) = \mathfrak{u}_{1,2}$, then $\epsilon = \mathfrak{u}_{2,3}$. We now assume that

$$\mathrm{pr}(\epsilon) = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & xa \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \}.$$

Let $A' = \begin{pmatrix} 0 & a_{12} & a_{13} & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon \cap \mathfrak{u}_J$, and let $A = \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & a & b & * \\ 0 & 0 & 0 & xa & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon$.
Then

$$[A', A] = \begin{pmatrix} 0 & 0 & aa_{12} & xaa_{13} + ba_{12} & * \\ 0 & 0 & 0 & 0 & -aa_{35} - ba_{45} \\ 0 & 0 & 0 & 0 & -xaa_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\epsilon$ is abelian, and since the values of $a, b$ run through all elements of $k$, we conclude that $a_{12} = a_{13} = a_{35} = a_{45} = 0$. Therefore, $\dim \epsilon \cap \mathfrak{u}_J \leq 3$ and $\dim \epsilon \leq 5$. Hence, the maximum is not attained in this case. This finishes the proof in the base case $m = 2$.

We omit the induction step since it is very similar to the even dimensional case proved in Theorem 2.7.

To prove (2), we observe that $\mathfrak{u}_{m,m+1}$ and $\mathfrak{u}_{m+1,m}$ are not conjugate under the adjoint action of $\mathrm{SL}_{2m+1}$ since their nullspaces in the standard representation of $\mathfrak{sl}_{2m+1}$ have different dimensions.

Finally, statement (3) follows from (1) and (2) as in the end of the proof of Theorem 2.7. □

We make the immediate observation that the results of Theorems 2.7 and 2.8 apply equally well to $\mathfrak{gl}_n$.

**Corollary 2.9.** *Assume $p > 2$.*

(1) *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{gl}_n$ is $\lfloor \frac{n^2}{4} \rfloor$.*
(2) *For any $m \geq 1$, there is a finite radicial morphism*

$$\mathrm{Grass}(m, 2m) \longrightarrow \mathbb{E}(m^2, \mathfrak{gl}_{2m})$$

*inducing a homeomorphism on Zariski spaces; this morphism is an isomorphism for $p > 4m - 2$.*
(3) *For any $m \geq 2$, there is a finite radicial morphism*

$$\mathrm{Grass}(m, 2m+1) \sqcup \mathrm{Grass}(m, 2m+1) \longrightarrow \mathbb{E}(m(m+1), \mathfrak{gl}_{2m+1})$$

*inducing a homeomorphism on Zariski spaces; this morphism is an isomorphism for $p > 4m$.*

**Remark 2.10.** In the case $n = 3$, excluded above, the variety $\mathbb{E}(2, \mathfrak{gl}_3)$ is irreducible (see Example 3.20).

To make analogous calculations in the symplectic case, we need the following technical observation.

**Lemma 2.11.** *Let $\epsilon$ be an elementary subalgebra of the symplectic Lie algebra $\mathfrak{sp}_{2m}$. There exists an element $g \in \mathrm{Sp}_{2m}$ such that $g\epsilon g^{-1}$ belongs to the nilpotent radical of the standard Borel subalgebra of $\mathfrak{sp}_{2m}$.*

*Proof.* Let $V$ be a $2m$-dimensional symplectic space with a basis $\{x_1, \dots, x_m, y_m, \dots y_1\}$ such that the symplectic form with respect to this basis has the standard matrix

$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. A complete isotropic flag is a nested sequence of subspaces of the form:

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = V_m^\perp \subset V_{m-1}^\perp \subset \ldots \subset V_1^\perp \subset V$$

such that $\dim V_i = i$. The condition that $V_i \subseteq V_i^\perp$ implies that each $V_i$ is isotropic. The standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sp}_{2m}$ (such as in [17, 12.5]) is characterized as the stabilizer of the standard complete isotropic flag in $V$, meaning the flag with $V_i$ spanned by $\{x_1, \ldots, x_i\}$ (so that $V_i^\perp$ is spanned by $\{x_1, \ldots, x_n, y_n, \ldots, y_{n-i-1}\}$). Thus, each $V_i$, as given, has the property that $\mathfrak{b}V_i \subseteq V_i$. Any two complete isotropic flags are conjugate by an element of $\mathrm{Sp}_{2n}$. Therefore if we show that the subalgebra $\epsilon$ stabilizes a complete isotropic flag, then some conjugate of $\epsilon$ is contained in a standard Borel subalgebra of $\mathfrak{sp}_{2m}$, as asserted.

Constructing a complete isotropic flag that is invariant under $\epsilon$ is a straightforward inductive exercise. We begin with $i = 0$. Assume for some $i$ an isotropic $\epsilon$-invariant subspace $V_i \subseteq V_i^\perp$ has been constructed. Choose $V_{i+1}$ to be any subspace such that $V_i \subset V_{i+1}$ and $V_{i+1}/V_i$ is an $\epsilon$-invariant subspace of dimension one in $V_i^\perp/V_i$. Since $\epsilon$ is an elementary Lie algebra, its restricted enveloping algebra $\mathfrak{u}(\epsilon)$ is a local ring and, hence, $V_{i+1}/V_i$ always has such a 1-dimensional invariant subspace. Note that $V_{i+1}$ is isotropic because it is contained in $V_i^\perp$ and $V_i$ is isotropic. Continuing this process to step $n$ constructs an $\epsilon$-invariant complete isotropic flag. □

**Theorem 2.12.** *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and assume that $p \neq 2$. Then*

(1) *For any elementary subalgebra $\epsilon$ of $\mathfrak{g}$, $\dim \epsilon \leq \frac{n(n+1)}{2}$.*
(2) *Any elementary subalgebra $\epsilon$ of maximal dimension $\frac{n(n+1)}{2}$ is conjugate to $\mathfrak{u}_{\alpha_n}$ under the adjoint action of $\mathrm{Sp}_{2n}$.*
(3) *The orbit map $\mathrm{Sp}_{2n} \to \mathrm{Sp}_{2n} \cdot \mathfrak{u}_{\alpha_n}$ determines a finite radicial morphism $\mathrm{Sp}_{2n}/P_{\alpha_n} \longrightarrow \mathbb{E}(\frac{n(n+1)}{2}, \mathfrak{sp}_{2n})$. For $p > 4n-2$, this morphism is an isomorphism.*

*Proof.* We prove by induction that the statement of the theorem holds for a Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ of any reductive group of type $C_n$. The statement is trivial for $n = 1$.

Assume the statement is proven for $n-1$. Let $G$ be a reductive group of type $C_n$ and let $\mathfrak{g} = \mathrm{Lie}\, G$. Recall that we follow the convention of [8] for numbering of simple roots, so that the Dynkin diagram for $\mathfrak{g}$ looks as follows:

(2.12.1)
$$\underset{1}{\circ} \text{---} \underset{2}{\circ} \text{---} \underset{3}{\circ} \quad \cdots \quad \underset{n-2}{\circ} \text{---} \underset{n-1}{\circ} \Longleftarrow \underset{n}{\circ}$$

Let $\mathfrak{p}_{\alpha_1} = \mathfrak{l}_{\alpha_1} \oplus \mathfrak{u}_{\alpha_1}$ be the maximal parabolic subalgebra corresponding to the simple root $\alpha_1$ with the Levi factor $\mathfrak{l}_{\alpha_1}$ and the nilpotent radical $\mathfrak{u}_{\alpha_1}$. To obtain the Dynkin diagram for $\mathfrak{l}_{\alpha_1}$ we simply remove the first node from (2.12.1). Hence, $\mathfrak{l}_{\alpha_1}$ is a reductive Lie algebra of type $C_{n-1}$, and we can apply inductive hypothesis to it.

Let $\mathfrak{u}_{\mathfrak{l}_{\alpha_1}}$ be the nilpotent radical of the standard Borel subalgebra of $\mathfrak{l}_{\alpha_1}$, and $\mathfrak{u}_{\mathfrak{g}}$ be the nilpotent radical of the Borel subalgebra of $\mathfrak{g}$. We have a short exact sequence

$$0 \longrightarrow \mathfrak{u}_{\alpha_1} \longrightarrow \mathfrak{u}_{\mathfrak{g}} \xrightarrow{\mathrm{pr}} \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \longrightarrow 0.$$

Let $\epsilon$ be an elementary subalgebra of $\mathfrak{g}$. Since $\epsilon$ consists of nilpotent matrices, it can be conjugated into the standard Borel subalgebra of $\mathfrak{g}$ by Lemma 2.11. Furthermore, since every element of $\epsilon$ is $p$-nilpotent, such a conjugate necessarily belongs to the nilpotent radical $\mathfrak{u}_\mathfrak{g}$. Hence, we may assume that $\epsilon \subset \mathfrak{u}_\mathfrak{g}$. By the induction hypothesis, $\dim \mathrm{pr}(\epsilon) \leq \frac{n(n-1)}{2}$. Since $\mathfrak{u}_{\alpha_1}$ is a Heisenberg Lie algebra of dimension $2n-1$ (see Example 2.3(2)), Proposition 2.2 implies that $\dim \mathfrak{u}_{\alpha_1} \cap \epsilon \leq n$. Hence, $\dim \epsilon \leq n + \frac{n(n-1)}{2}$. This proves (1).

To prove (2), we observe that the induction hypothesis implies that for an elementary subalgebra $\epsilon$ to attain the maximal dimension, we must have that

$$\mathrm{pr}\downarrow_\epsilon : \epsilon \longrightarrow \mathfrak{u}_{\mathfrak{l}_{\alpha_1}}$$

is surjective onto $\mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$, the nilpotent radical of the parabolic of $\mathfrak{l}_{\alpha_1}$ corresponding to $\alpha_n$.

Let $\{x_{\beta_i}, x_{\gamma_i}\}$ be a basis of $\mathfrak{u}_{\alpha_1}$ as defined in (2.3.1). Let $x = \sum\limits_1^{n-1} b_i x_{\beta_i} + \sum\limits_1^{n} c_i x_{\gamma_i} \in \mathfrak{u}_{\alpha_1} \cap \epsilon$. We want to show that $x \in \mathfrak{u}_{\alpha_n}$ or, equivalently, that $b_i = 0$ for $1 \leq i \leq n-1$.. Assume, to the contrary, that $b_i \neq 0$ for some $i$, $1 \leq i \leq n-1$. Let $\mu = \gamma_{n-1} - \beta_i = \alpha_2 + \ldots + \alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n$. Then $x_\mu \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n} \subset \mathrm{pr}(\epsilon)$. Therefore, there exists $y = x' + x_\mu \in \epsilon$ for some $x' \in \mathfrak{u}_{\alpha_1}$. Note that $[x, x'] \subset [\mathfrak{u}_{\alpha_1}, \mathfrak{u}_{\alpha_1}] = k x_{\gamma_n}$, and that $\mu + \gamma_i$ is never a root, and $\mu + \beta_j$ is not a root unless $j = i$. Hence,

$$[x, y] = [x, x'] + [x, x_\mu] = c x_{\gamma_n} + b_i [x_{\beta_i}, x_\mu] = c x_{\gamma_n} + b_i c_{\beta_i \mu} x_{\gamma_{n-1}} \neq 0.$$

Here, $c_{\beta_i\mu}$ is the structure constant from the equation $[x_{\beta_i}, x_\mu] = c_{\beta_i\mu} x_{\beta_i + \mu} = c_{\beta_i\mu} x_{\gamma_{n-1}}$. This structure constant is not zero because the only elements in the $\beta_i$ string through $\mu$ are $\mu$ and $\mu + \beta_i$ (See [40], Theorem 1(d)). Thus, we have a contradiction with the commutativity of $\epsilon$. Hence, $b_i = 0$ for all $i$, $1 \leq i \leq n-1$, and, therefore, $\mathfrak{u}_{\alpha_1} \cap \epsilon \subset \mathfrak{u}_{\alpha_n}$. Moreover, since we assume that $\dim \epsilon$ is maximal, we must have $\dim \mathfrak{u}_{\alpha_1} \cap \epsilon = n$, and, therefore, $\mathfrak{u}_{\alpha_1} \cap \epsilon = \bigoplus\limits_{i=1}^n k x_{\gamma_i}$.

Now let $x + a$ be any element in $\epsilon$ where $x \in \mathfrak{u}_{\alpha_1}$ and $a \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$. We need to show that $x \in \mathfrak{u}_{\alpha_n}$, that is, $x \in \bigoplus\limits_{i=1}^n k x_{\gamma_i}$. Let $x = \sum b_i x_{\beta_i} + \sum c_i x_{\gamma_i}$ and assume to the contrary that $b_i \neq 0$ for some $i$. Note that $[x_{\gamma_j}, \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}] = 0$ for any $j$, $1 \leq j \leq n$, since both $x_{\gamma_j}$ and any $a \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$ are linear combinations of root vectors for roots that have coefficient by $\alpha_n$ equal to 1. Hence, $[x + a, \gamma_{n-i}] = b_i [x_{\beta_i}, \gamma_{n-i}] \neq 0$. Again, we have contradiction. Therefore, $\epsilon \subset \mathfrak{u}_{\alpha_n}$. This proves (2).

To establish (3), we first note that $P_{\alpha_n}$ is the (reduced) stabilizer of $\mathfrak{u}_{\alpha_n}$ under the adjoint action of $\mathrm{Sp}_{2n}$. Arguing as in the end of the proof of Theorem 2.7, we conclude that the orbit map $\mathrm{Sp}_{2n} \to \mathrm{Sp}_{2n} \cdot \mathfrak{u}_{\alpha_n}$ induces a finite radicial morphism

$$\mathrm{Sp}_{2n} / P_{\alpha_n} \simeq \mathrm{Sp}_{2n} \cdot \mathfrak{u}_{\alpha_n}.$$

Since the Coxeter number of $\mathrm{Sp}_{2n}$ is $2n$, the final statement that the above map is an isomorphism for $p > 4n-2$ follows from [13, 3.7] as discussed in Remark 2.6. □

In the last calculation of this section we show that any Grassmannian $\mathrm{Grass}(a, b)$ can be realized as $\mathbb{E}(r, \mathfrak{g})$ or one of the to connected components of $\mathbb{E}(r, \mathfrak{g})$ if we let $\mathfrak{g}$ be a maximal parabolic subgroup of type $A$.

**Theorem 2.13.** *Assume that $p > 2$, and that $n \geq 4$. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$, and suppose that $r \leq m$. Let $\mathfrak{p}_{r,n-r}$ be the standard maximal parabolic subalgebra of $\mathfrak{sl}_n$ corresponding to the simple root $\alpha_r$. Then the maximal dimension of an elementary subalgebra of $\mathfrak{p}_{r,n-r}$ is $\left\lfloor \frac{n^2}{4} \right\rfloor$. The corresponding variety of elementary subalgebras is homeomorphic to* $\mathrm{Grass}(m, n-r)$ *if $n$ is even and* $\mathrm{Grass}(m, n-r) \sqcup \mathrm{Grass}(m, n-r)$ *if $n$ is odd.*

*Proof.* We consider the case of $n = 2m+1$ odd. The even case is similar.

Theorem 2.8 implies immediately that $\dim \epsilon \leq m(m+1) = \left\lfloor \frac{n^2}{4} \right\rfloor$ for any elementary subalgebra $\epsilon \subset \mathfrak{p}_{r,n-r}$. Since $\mathfrak{u}_{m,m+1}$ is a subalgebra of $\mathfrak{p}_{r,n-r}$, we have equality in the maximal case.

To compute the variety, we first show that any elementary subalgebra of maximal dimension is conjugate to either $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$ under the adjoint action of $P_{r,n-r}$. Let $\epsilon \subset \mathfrak{p}_{r,n-r}$ be an elementary subalgebra of maximal dimension..

By Theorem 2.8, $\epsilon$ is conjugate to $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$ under the adjoint action of $\mathrm{SL}_{2m+1}$. Assume that $\epsilon = g\mathfrak{u}_{m+1,m}g^{-1}$ for some $g \in \mathrm{SL}_{2m+1}$ (the case of $\mathfrak{u}_{m,m+1}$ is strictly analogous). We proceed to show that there exists $\widehat{g} \in P_{r,n-r}$ such that $\epsilon = \widehat{g}\mathfrak{u}_{m+1,m}\widehat{g}^{-1}$.

Let $W(\mathrm{SL}_{2m+1}) \simeq N_{\mathrm{SL}_{2m+1}}(T)/C_{\mathrm{SL}_{2m+1}}(T)$ be the Weyl group, $B_{2m+1}$ be the Borel subgroup of $\mathrm{SL}_{2m+1}$, and $U_{2m+1}$ be the unipotent radical of $B_{2m+1}$ . For an element $w \in W(\mathrm{SL}_{2m+1})$, we denote by $\widetilde{w}$ a fixed coset representative of $w$ in $N_{\mathrm{SL}_{2m+1}}(T)$.

Using the Bruhat decomposition, we can write $g = g_1\widetilde{w}g_2$ where $g_1 \in U_{2m+1}$, $g_2 \in B_{2m+1}$, and $w \in W(\mathrm{SL}_{2m+1})$. Since both $\mathfrak{u}_{m+1,m}$ and $P_{r,n-r}$ are stable under the conjugation by $U_{2m+1}$ and $B_{2m+1}$, it suffices to prove the statement for $g = \widetilde{w}$, where $w$ is a Weyl group element. We make the standard identifications $W(\mathrm{SL}_{2m+1}) \simeq S_{2m+1}$, $W(L_{r,n-r}) \simeq S_r \times S_{n-r}$ and $W(L_{m+1,m}) \simeq S_{m+1} \times S_m$ where $L_{i,j}$ is the Levi factor of the standard parabolic $P_{i,j}$.

We further decompose

$$S_{2m+1} = W(\mathrm{SL}_{2m+1}) = \bigsqcup_{s \in S_r \times S_{n-r} \backslash S_{2m+1}/(S_{m+1} \times S_m)} (S_r \times S_{n-r})s(S_{m+1} \times S_m)$$

into double cosets. Now let $w$ be an element of $W$ such that $\epsilon = \widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1} \in \mathfrak{p}_{r,n-r}$. To prove that $\epsilon$ is conjugate to $\mathfrak{u}_{m+1,m}$ by an element of $P_{r,n-r}$, it suffices to show that $w$ belong to the identity double coset. Indeed, if that is the case, then we can write $w = w_1w_2$ with $w_1 \in S_r \times S_{n-r}$, $w_2 \in S_{m+1} \times S_m$. Since $w_2$ acts trivially on $\mathfrak{u}_{m+1,m}$, we get that $\epsilon = \widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1} = \widetilde{w}_1\mathfrak{u}_{m+1,m}\widetilde{w}_1^{-1}$ which is satisfactory since $\widetilde{w}_1 \in P_{r,n-r}$.

To establish that $w$ is in the identity double coset, we first prove the following claim:

*Claim.* Suppose $\widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1} \subset \mathfrak{p}_{r,n-r}$. Then for any $i$, $1 \leq i \leq r$,

$$w^{-1}(i) \leq m+1.$$

*Proof of the Claim.* We prove the claim by contradiction. Let $E_{\ell j}$ denote the matrix with 1 at the entry $(\ell, j)$ and 0 everywhere else. Suppose there exists $i$, $1 \leq i \leq r$, such that $w^{-1}(i) = j > m+1$. Since the permutation $w$ is bijective, and

$r < m+1$, we can find an index $\ell$ such that

$$\begin{cases} 1 \leq \ell \leq m+1 \\ w(\ell) \geq r+1. \end{cases}$$

The conditions on $\ell$ now imply that

$$\begin{cases} E_{\ell j} \in \mathfrak{u}_{m+1,m}, \\ \widetilde{w} E_{\ell j} \widetilde{w}^{-1} = E_{w(\ell)w(j)} = E_{w(\ell)i} \notin \mathfrak{p}_{r,n-r}, \end{cases}$$

a contradiction. This finishes the proof of the claim.

The Claim implies that for any $i$, $1 \leq i \leq r$, the transposition $(i, w^{-1}(i))$ is in $S_{m+1} \times \mathrm{id} \subset S_{2m+1}$. Therefore, multiplying $w$ by such transposition on the right preserves both the double coset representative of $w$ and the property that $\epsilon = \widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1}$. Hence, $w$ is in the same double coset as a permutation which acts trivially on the first $r$ entries. But such permutations are in the identity double coset and, hence, so is $w$. This finishes the proof of the claim that $\mathfrak{u}_{m+1,m}$ can be conjugated to $\epsilon$ with an element in $P_{r,n-r}$.

The above discussion implies that

$$\mathbb{E}(m(m+1), \mathfrak{p}_{r,n-r}) = P_{r,n-r} \cdot \mathfrak{u}_{m+1,m} \sqcup P_{r,n-r} \cdot \mathfrak{u}_{m,m+1}.$$

The (reduced) stabilizer of $\mathfrak{u}_{m+1,m}$ in $P_{r,n-r}$ is $P_{r,m+1-r,m} = P_{m+1,m} \cap P_{r,n-r}$. Hence, the orbit map $P_{r,n-r} \twoheadrightarrow P_{r,n-r} \cdot \mathfrak{u}_{m+1,m}$ induces a homeomorphism

$$\mathrm{Grass}(m, n-r) \cong P_{r,n-r}/P_{r,m+1-r,m} \xrightarrow{\sim} P_{r,n-r} \cdot \mathfrak{u}_{m+1,m},$$

and similarly for the other component. □

Theorem 2.13 has the following immediate corollary.

**Corollary 2.14.** *Let $\mathfrak{g}_{1,2m} \subset \mathfrak{gl}_{2m+1}$ be as defined in Example 1.12(1). The maximal dimension of an elementary subalgebra of $\mathfrak{g}_{1,2m}$ is $m(m+1)$. For $m \geq 2$, $\mathbb{E}(m(m+1), \mathfrak{g}_{1,2m})$ is homeomorphic to* $\mathrm{Grass}(m, 2m) \sqcup \mathrm{Grass}(m-1, 2m)$.

## 3. Radicals, socles, and geometric invariants for $\mathfrak{u}(\mathfrak{g})$-modules

As throughout this paper, $\mathfrak{g}$ denotes a finite dimensional $p$-restricted Lie algebra over $k$. We recall that $\mathfrak{g}$ is the Lie algebra Lie($\underline{\mathfrak{g}}$) of a uniquely defined infinitesimal group scheme $\underline{\mathfrak{g}}$ of height 1 (see, for example, [16]). In [42], *a rank variety* $V(G)_M$ was constructed for any finite dimensional representation $M$ of the infinitesimal group scheme $G$. The variety $V(G)_M$ is a closed subset of $V(G)$, the variety of (infinitesimal) 1-parameter subgroups of $G$. As shown in [42], these rank varieties can be identified with cohomological support varieties defined in terms of the action of $\mathrm{H}^*(G,k)$ on $\mathrm{Ext}^*_G(M,M)$.

For infinitesimal group schemes $G$ of height 1 (i.e., of the form $\underline{\mathfrak{g}}$ for some finite dimensional $p$-restricted Lie algebra), we consider more complete invariants of representations of $G$ which one can think of as more sophisticated variants of "higher rank varieties." Our investigations follow that of our earlier paper [12] in which we considered representations of elementary abelian $p$-groups. Because the group algebra $k(\mathbb{Z}/p^{\times r})$ is isomorphic to the restricted enveloping algebra $\mathfrak{u}(\mathfrak{g}_a^{\oplus r})$ of the Lie algebra $\mathfrak{g}_a^{\oplus r}$ (commutative, with trivial $p$-restriction), that investigation is in fact a very special case of what follows.

We use our earlier work for elementary abelian $p$-groups as a guide for the study of $\mathfrak{u}(\mathfrak{g})$-modules for an arbitrary $\mathfrak{g}$. In particular, rather than considering isomorphism types of a given module upon restriction to elementary subalgebras of a given rank $r$, we consider dimensions of the radicals (respectively, socles) of such restrictions. A key result is Theorem 3.13 which verifies that these dimensions are lower (resp., upper) semi-continuous. As seen in Theorem 3.17, this implies that the non-maximal radical and socle varieties associated to a $\mathfrak{u}(\mathfrak{g})$-module $M$ are closed.

The following is a natural extension of the usual support variety in the case $r = 1$ (see [20]) and of the variety $\mathrm{Grass}(r, V)_M$ of [12, 1.4] for $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$. If $\epsilon \subset \mathfrak{g}$ is an elementary subalgebra and $M$ a $\mathfrak{u}(\mathfrak{g})$-module, then we shall denote by $\epsilon^* M$ the restriction of $M$ to $\mathfrak{u}(\epsilon) \subset \mathfrak{u}(\mathfrak{g})$.

**Definition 3.1.** For any $\mathfrak{u}(\mathfrak{g})$-module $M$ and any positive integer $r$, we define

$$\mathbb{E}(r, \mathfrak{g})_M \;=\; \{\epsilon \in \mathbb{E}(r, \mathfrak{g}); \epsilon^* M \text{ is not projective}\}.$$

In particular,

$$\mathbb{E}(1, \mathfrak{g})_M \;=\; \mathrm{Proj}\, k[V(\underline{\mathfrak{g}})_M] \;\subset\; \mathrm{Proj}\, k[V(\underline{\mathfrak{g}})] \;=\; \mathbb{E}(1, \mathfrak{g})$$

is the projectivization of the closed subvariety of $V(\underline{\mathfrak{g}}) = \mathcal{N}_p(\mathfrak{g})$ consisting of those one dimensional Lie subalgebras (with trivial $p$-restriction) restricted to which $M$ is not projective.

The following proposition tells us that the geometric invariant $M \mapsto \mathbb{E}(r, \mathfrak{g})_M$ can be computed in terms of the more familiar (projectivized) support variety $\mathbb{E}(1, \mathfrak{g})_M \;=\; \mathrm{Proj}(V(\underline{\mathfrak{g}})_M)$.

**Proposition 3.2.** *For any $\mathfrak{u}(\mathfrak{g})$-module $M$ and positive integer $r$,*

$$\mathbb{E}(r, \mathfrak{g})_M \;=\; \{\epsilon \in \mathbb{E}(r, \mathfrak{g});\; \epsilon \cap V(\underline{\mathfrak{g}})_M \neq 0\} \tag{3.2.1}$$

*where the intersection $\epsilon \cap V(\underline{\mathfrak{g}})_M$ is as subvarieties of $\mathfrak{g}$.*

*Proof.* By definition, $\epsilon \in \mathbb{E}(r, \mathfrak{g})_M$ if and only if $\epsilon^* M$ is not free which is the case if and only if $V(\underline{\epsilon})_{\epsilon^* M} \neq 0$. Since $\epsilon \subset \mathfrak{g}$ induces an isomorphism

$$V(\underline{\epsilon})_{\epsilon^*(M)} \xrightarrow{\sim} V(\underline{\epsilon}) \cap V(\underline{\mathfrak{g}})_M$$

(see [20]), this is equivalent to $\epsilon \cap V(\underline{g})_M \neq 0$. □

**Proposition 3.3.** *For any $\mathfrak{u}(\mathfrak{g})$-module $M$ and for any $r \geq 1$,*

$$\mathbb{E}(r, \mathfrak{g})_M \;\subset\; \mathbb{E}(r, \mathfrak{g})$$

*is a closed subvariety.*

*Moreover, if $G$ is an algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$ and if $M$ is a rational $G$-module, then $\mathbb{E}(r, \mathfrak{g})_M \;\subset\; \mathbb{E}(r, \mathfrak{g})$ is $G$-stable.*

*Proof.* Let $\mathrm{Proj}\, \epsilon \subset \mathbb{E}(1, \mathfrak{g})$ be the projectivization of the linear subvariety $\epsilon \subset \mathfrak{g}$. Let $X_M = \{\epsilon \in \mathrm{Grass}(r, \mathfrak{g}) \mid \mathrm{Proj}\, \epsilon \cap \mathbb{E}(1, \mathfrak{g})_M \neq \emptyset\}$. Then $X_M \subset \mathrm{Grass}(r, \mathfrak{g})$ is a closed subvariety (see [25, ex. 6.14]). Since $\mathbb{E}(r, \mathfrak{g})_M = \mathbb{E}(r, \mathfrak{g}) \cap X_M$ by Prop. 3.2, we conclude that $\mathbb{E}(r, \mathfrak{g})_M$ is a closed subvariety of $\mathbb{E}(r, \mathfrak{g})$.

For $\mathfrak{g} = \mathrm{Lie}(G)$, $M$ a rational $G$-module, and $x \in G$, denote by $M^x$ the module $M$ twisted by $x$. For $\epsilon \in \mathbb{E}(r, \mathfrak{g})$, denote by $\epsilon^x$ the image of $\epsilon$ under the adjoint action of $x$ on $\mathbb{E}(r, \mathfrak{g})$. The adjoint action by $x^{-1}$ induces an isomorphism

$\alpha_{x^{-1}} : \mathfrak{u}(\epsilon^x) \xrightarrow{\sim} \mathfrak{u}(\epsilon)$, and the pull-back of $M$ along this isomorphism equals $(\epsilon^x)^*(M^x)$. Since $M \simeq M^x$ as $\mathfrak{u}(\mathfrak{g})$-modules, we conclude that $\mathbb{E}(r,\mathfrak{g})_M$ is $G$-stable. □

Proposition 3.2 implies the following result concerning the realization of subsets of $\mathbb{E}(r,\mathfrak{g})$ as subsets of the form $X = \mathbb{E}(r,\mathfrak{g})_M$. We remind the reader of the definition of the module $L_\zeta$ associated to a cohomology class $\zeta \in \mathrm{H}^n(\mathfrak{u}(\mathfrak{g}),k)$: $L_\zeta$ is the kernel of the map $\zeta : \Omega^n(k) \to k$ determined by $\zeta$, where $\Omega^n(k)$ is the $n^{th}$ Heller shift of the trivial module $k$ (see [4] or Example 4.7).

**Corollary 3.4.** *A subset $X \subset \mathbb{E}(r,\mathfrak{g})$ has the form $X = \mathbb{E}(r,\mathfrak{g})_M$ for some $\mathfrak{u}(\mathfrak{g})$-module $M$ if and only if there exists a closed subset $Z \subset \mathbb{E}(1,\mathfrak{g})$ such that*

$$X \;=\; \{\epsilon \in \mathbb{E}(r,\mathfrak{g});\ \operatorname{Proj}\epsilon \cap Z \neq \emptyset\}. \tag{3.4.1}$$

*Moreover, such an $M$ can be chosen to be a tensor product of modules $L_\zeta$ with each $\zeta$ of even cohomological degree.*

*Proof.* We recall that any closed, conical subvariety of $V(\underline{\mathfrak{g}})$ (i.e., any closed subvariety of $\mathbb{E}(1,\mathfrak{g})$) can be realized as the (affine) support of a tensor product of modules $L_\zeta$ (see [20]) and that the support of any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module is a closed, conical subvariety of $V(\underline{\mathfrak{g}})$. Thus, the proposition follows immediately from Proposition 3.2. □

**Example 3.5.** As one specific example of Corollary 3.4, we take some even degree cohomology class $0 \neq \zeta \in \mathrm{H}^{2m}(\mathfrak{u}(\mathfrak{g}),k)$ and $M = L_\zeta$. We identify $V(\underline{\mathfrak{g}})$ with the spectrum of $\mathrm{H}^{\mathrm{ev}}(\mathfrak{u}(\mathfrak{g}),k)$ (for $p>2$), so that $\zeta$ is a (homogeneous) algebraic function on $V(\underline{\mathfrak{g}})$. Thus $V(\underline{\mathfrak{g}})_{L_\zeta} = Z(\zeta) \subset V(\underline{\mathfrak{g}})$ (see [42, Theorem 7.5]), the zero locus of the function $\zeta$. Then,

$$\mathbb{E}(r,\mathfrak{g})_{L_\zeta} \;=\; \{\epsilon \in \mathbb{E}(r,\mathfrak{g});\ \epsilon \cap Z(\zeta) \neq \{0\}\}.$$

On the other hand, if $\zeta \in \mathrm{H}^{2m+1}(\mathfrak{u}(\mathfrak{g}),k)$ has odd degree and $p>2$, then $V(\underline{\mathfrak{g}})_{L_\zeta} = V(\underline{\mathfrak{g}})$, so that $\mathbb{E}(r,\mathfrak{g})_{L_\zeta} \;=\; \mathbb{E}(r,\mathfrak{g})$.

**Remark 3.6.** As pointed out in [12, 1.10] in the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus 3}$ and $r=2$, not every closed subset $X \subset \mathbb{E}(r,\mathfrak{g})$ has the form (3.4.1).

**Example 3.7.** We consider another computation of $\mathbb{E}(r,\mathfrak{g})_M$. Let $G$ be a reductive group and assume that $p$ is good for $G$. Let $\lambda$ be a dominant weight and consider the induced module $M = \mathrm{H}^0(\lambda) = \operatorname{Ind}_B^G \lambda$. By a result of Nakano, Parshall, and Vella [34, 6.2.1], $V(\underline{\mathfrak{g}})_{\mathrm{H}^0(\lambda)} = G\cdot \mathfrak{u}_J$, where $\mathfrak{u}_J$ is the nilpotent radical of a suitably chosen parabolic subgroup $P_J \subset G$. Then,

$$\mathbb{E}(r,\mathfrak{g})_{\mathrm{H}^0(\lambda)} \;=\; G\cdot\{\epsilon \in \mathbb{E}(r,\mathfrak{g});\ \epsilon \cap \mathfrak{u}_J \neq \{0\}\}.$$

We now proceed to consider invariants of $\mathfrak{u}(\mathfrak{g})$-modules associated to $\mathbb{E}(r,\mathfrak{g})$ which for $r>1$ are not determined by the case $r=1$. As before, for a given $M$ and a given $r \geq 1$, we consider the restrictions $\epsilon^*(M)$ for $\epsilon \in \mathbb{E}(r,\mathfrak{g})$.

**Definition 3.8.** Let $\mathfrak{g}$ be a $p$-restricted Lie algebra and $M$ a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module. For any $r \geq 1$, any $\epsilon \in \mathbb{E}(r,\mathfrak{g})$, and any $j, 1 \leq j \leq (p-1)r$, we consider

$$\mathrm{Rad}^j(\epsilon^*(M)) \;=\; \sum_{j_1+\cdots+j_r=j} \mathrm{Im}\{u_1^{j_1}\cdots u_r^{j_r} : M \to M\}$$

and

$$\mathrm{Soc}^j(\epsilon^*(M)) \;=\; \bigcap_{j_1+\cdots+j_r=j} \mathrm{Ker}\{u_1^{j_1}\cdots u_r^{j_r} : M \to M\},$$

where $\{u_1,\ldots,u_r\}$ is a basis for $\epsilon$.

For each $r \geq 1$ and each $j, 1 \leq j \leq (p-1)r$, we define the local $(r,j)$-radical rank of $M$ and the local $(r,j)$-socle rank of $M$ to be the (non-negative) integer valued functions

$$\epsilon \in \mathbb{E}(r,\mathfrak{g}) \;\mapsto\; \dim \mathrm{Rad}^j(\epsilon^*(M))$$

and

$$\epsilon \in \mathbb{E}(r,\mathfrak{g}) \;\mapsto\; \dim \mathrm{Soc}^j(\epsilon^*(M))$$

respectively.

**Remark 3.9.** If $M$ is a $\mathfrak{u}(\mathfrak{g})$-module, we denote by $M^\# = \mathrm{Hom}_k(M,k)$ the dual of $M$ whose $\mathfrak{u}(\mathfrak{g})$-module structure arises from that on $M$ using the antipode of $\mathfrak{u}(\mathfrak{g})$. Thus, if $X \in \mathfrak{g}$ and $f \in M^\#$, then $(X \circ f)(m) = -f(X \circ m)$. If $i : L \subset M$ is a $\mathfrak{u}(\mathfrak{g})$-submodule, then we denote by $L^\perp \subset M^\#$ the submodule defined as the kernel of $i^\# : M^\# \to L^\#$. We remind the reader that

$$\mathrm{Soc}^j(\epsilon^*(M^\#)) \;\simeq\; (\mathrm{Rad}^j(\epsilon^* M))^\perp \tag{3.9.1}$$

(as shown in [12, 2.2]).

The following elementary observation enables us to conclude in [13] that the constructions of §4 determine vector bundles on $G$-orbits of $\mathbb{E}(r, \mathrm{Lie}\, G)$.

**Proposition 3.10.** *If $\mathfrak{g} = \mathrm{Lie}(G)$ and $M$ is a rational $G$-module, then the local $(r,j)$-radical rank of $M$ and the local $(r,j)$-socle rank of $M$ are constant on $G$-orbits of $\mathbb{E}(r,\mathfrak{g})$.*

*Proof.* Let $g \in G$, and let $\epsilon \in \mathbb{E}(r,\mathfrak{g})$. We denote by $\epsilon^g \in \mathbb{E}(r,\mathfrak{g})$ the image of $\epsilon$ under the adjoint action of $G$ on $\mathbb{E}(r,\mathfrak{g})$, and let $g \cdot (-) : M \to M$ be the action of $G$ on $M$. Observe that

$$g : M \xrightarrow{m \mapsto gm} M^g$$

defines an isomorphism of rational $G$-modules, where the action of $x \in G$ on $m \in M^g$ is given by the action of $gxg^{-1}$ on $m$ (with respect to the $G$-module structure on $M$). Thus, the proposition follows from the observation that the pull-back of $\epsilon^{g*}(M^g)$ equals $\epsilon^*(M)$ under the isomorphism given by conjugation by $g$: $\mathfrak{u}(\epsilon) \xrightarrow{\sim} \mathfrak{u}(\epsilon^g)$ . □

The following discussion leads to Theorem 3.13 which establishes the lower and upper semi-continuity of local $(r,j)$-radical rank and local $(r,j)$-socle rank respectively.

**Notation 3.11.** We fix a basis $\{x_1,\ldots,x_n\}$ of $\mathfrak{g}$ and use it to identify $M_{n,r} \simeq \mathfrak{g}^{\oplus r}$ (as vector spaces). Let $\Sigma \subset \{1,\ldots,n\}$ be an $r$-subset. Recall the section $s_\Sigma : U_\Sigma \to \mathbb{M}^\circ_{n,r}$ of (1.1.1) that sends an $r$-plane $\epsilon \in U_\Sigma$ to the $n \times r$ matrix $A^\Sigma_\epsilon$ with the $r \times r$ submatrix corresponding to $\Sigma$ being the identity and the columns generating the plane $\epsilon$. Extend the map $s_\Sigma$ to $s_\Sigma : U_\Sigma \to \mathbb{M}_{n,r}$ and consider the induced map on coordinate algebras:

$$k[\mathbb{M}_{n,r}] = k[T_{i,s}] \xrightarrow{s_\Sigma^*} k[U_\Sigma\ ] \tag{3.11.1}$$

We define

$$T_{i,s}^\Sigma \equiv s_\Sigma^*(T_{i,s})$$

It follows from the definition that $T_{i,s}^\Sigma = \delta_{\alpha^{-1}(i),s}$ for $i \in \Sigma$, where $\alpha : \{1, \ldots, r\} \to \Sigma$ is the function with $\alpha(1) < \cdots < \alpha(r)$, and that $T_{i,s}^\Sigma$ for $i \notin \Sigma$ are algebraically independent generators of $k[U_\Sigma]$.

Let $V_\Sigma \equiv \mathbb{E}(r, \mathfrak{g}) \cap U_\Sigma$. We define the set $\{Y_{i,s}^\Sigma\}$ of algebraic generators of $k[V_\Sigma]$ as images of $\{T_{i,s}^\Sigma\}$ under the map of coordinate algebras induced by the closed immersion $V_\Sigma \subset U_\Sigma$:

$$k[U_\Sigma] \twoheadrightarrow k[V_\Sigma]\ , \quad T_{i,s}^\Sigma \mapsto Y_{i,s}^\Sigma$$

It again follows that $Y_{i,s}^\Sigma = \delta_{\alpha^{-1}(i),s}$, for $i \in \Sigma$ and $\alpha$ as above. For each $\epsilon \in V_\Sigma \subset U_\Sigma$ (implicitly assumed to be a $k$-rational point), we have

$$Y_{i,s}^\Sigma(\epsilon) = T_{i,s}^\Sigma(\epsilon) = s_\Sigma^*(T_{i,s}^\Sigma)(\epsilon) = T_{i,s}(s_\Sigma(\epsilon)).$$

Hence,

$$A_\epsilon^\Sigma = [Y_{i,s}^\Sigma(\epsilon)]. \tag{3.11.2}$$

**Definition 3.12.** For a $\mathfrak{u}(\mathfrak{g})$-module $M$, and for a given $s, 1 \leq s \leq r$, we define the endomorphism of $k[V_\Sigma]$–modules

$$\Theta_s^\Sigma \equiv \sum_{i=1}^n x_i \otimes Y_{i,s}^\Sigma : M \otimes k[V_\Sigma] \to M \otimes k[V_\Sigma], \tag{3.12.1}$$

via

$$m \otimes 1 \mapsto \sum_i x_i m \otimes Y_{i,s}^\Sigma.$$

We refer the reader to [26, III.12] for the definition of an upper/lower semi-continuous function on a topological space.

**Theorem 3.13.** *Let $M$ be a $\mathfrak{u}(\mathfrak{g})$-module, $r$ a positive integer, and $j$ an integer satisfying $1 \leq j \leq (p-1)r$. Then the local $(r,j)$-radical rank of $M$ is a lower semi-continuous function and the local $(r,j)$-socle rank of $M$ is an upper semicontinuous function on $\mathbb{E}(r, \mathfrak{g})$.*

*Proof.* It suffices to show that the local $(r,j)$-radical rank of $M$ is lower semi-continuous when restricted along each of the open immersions $V_\Sigma \subset \mathbb{E}(r, \mathfrak{g})$. For $\epsilon \in V_\Sigma$ with residue field $K$, the specialization of $\Theta_s^\Sigma$ at $\epsilon$ defines a linear operator $\Theta_s^\Sigma(\epsilon) = \sum_{i=1}^n Y_{i,s}^\Sigma(\epsilon) x_i$ on $M_K$:

$$m \mapsto \Theta_s^\Sigma(\epsilon) \cdot m = \sum_{i=1}^n Y_{i,s}^\Sigma(\epsilon) x_i m.$$

Since the columns of $[Y_{i,s}^\Sigma(\epsilon)]$ generate $\epsilon$ by (3.11.2), we get that

$$\mathrm{Rad}(\epsilon^* M) \ = \ \sum_{s=1}^r \mathrm{Im}\{\Theta_s^\Sigma(\epsilon) : M_K \to M_K\} \tag{3.13.1}$$

and

$$\text{(3.13.2)} \qquad \operatorname{Rad}^j(\epsilon^* M) \;=\; \sum_{j_1+\cdots+j_r=j} \operatorname{Im}\{\Theta_1^\Sigma(\epsilon)^{j_1}\dots\Theta_r^\Sigma(\epsilon)^{j_r} : M_K \to M_K\} =$$

$$\operatorname{Im}\{\bigoplus_{j_1+\cdots+j_r=j} \Theta_1^\Sigma(\epsilon)^{j_1}\dots\Theta_r^\Sigma(\epsilon)^{j_r} : M_K^{\oplus r(j)} \to M_K\}$$

where $r(j)$ is the number of ways to write $j$ as the sum of non-negative integers $j_1+\cdots+j_r$. Hence, the usual argument for lower semicontinuity of the dimension of images of a homomorphism of finitely generated free modules applied to the $k[V_\Sigma]$-linear map

$$\bigoplus_{j_1+\cdots+j_r=j} (\Theta_1^\Sigma)^{j_1}\dots(\Theta_r^\Sigma)^{j_r} : (M\otimes k[V_\Sigma])^{\oplus r(j)} \to M\otimes k[V_\Sigma].$$

enables us to conclude that the function

$$\text{(3.13.3)} \qquad \epsilon\in\mathbb{E}(r,\mathfrak{g}) \;\mapsto\; \dim\operatorname{Rad}^j(\epsilon^* M) \quad \text{is lower semi-continuous.}$$

The upper semi-continuity of socle ranks now follows by Remark 3.9. □

**Remark 3.14.** To get some understanding of the operators $\Theta_s^\Sigma(\epsilon)$ occurring in the proof of Theorem 3.13, we work out the very special case in which $\mathfrak{g}=\mathfrak{g}_a\oplus\mathfrak{g}_a$, $r=1$ (so that $\mathbb{E}(r,\mathfrak{g})=\mathbb{P}^1$), and $j=1$. We fix a basis $\{x_1,x_2\}$ for $\mathfrak{g}$ which induces the identification $\mathfrak{g}\simeq\mathbb{A}^2$. The two possibilities for $\Sigma\subset\{1,2\}$ are $\{1\},\{2\}$. Let $k[T_1,T_2]$ be the coordinate ring for $\mathbb{A}^2$ (corresponding to the fixed basis $\{x_1,x_2\}$. Let $\Sigma=\{1\}$. We have $V_{\{1\}}=U_{\{1\}}=\{[a:b]\,|\,a\neq 0\}\simeq\mathbb{A}^1$ and the section $s_{\{1\}}:V_{\{1\}}\to\mathbb{A}^2$ given explicitly as $[a:b]\mapsto(1,b/a)$. The corresponding map of coordinate algebras as in (3.11.1) is given by

$$k[\mathbb{A}^2]=k[T_1,T_2]\to k[V_{\{1\}}]\simeq k[\mathbb{A}^1]$$
$$T_1\mapsto 1, T_2\mapsto s_{\{1\}}^*(T_2)$$

Then for a $\mathfrak{u}(\mathfrak{g})$-module $M$, $\epsilon=\langle a,b\rangle\in\mathbb{P}^1$ with $a\neq 0$, and $m\in M$, we have

$$\text{(3.14.1)} \qquad \Theta^{\{1\}} \;=\; x_1\otimes 1 + x_2\otimes s_{\{1\}}^*(T_2) : M\otimes k[V_{\{1\}}]\to M\otimes k[V_{\{1\}}];$$

$$\Theta^{\{1\}}(\epsilon) \;=\; x_1+\frac{b}{a}x_2, \quad m\mapsto x_1(m)+\frac{b}{a}x_2(m).$$

We extend the formulation of "generalized support varieties" introduced in [23] for $r=1$ and in [12] for elementary abelian $p$-groups (or, equivalently, for $\mathfrak{g}=\mathfrak{g}_a^{\oplus r}$) to any $r$ and an arbitrary $p$-restricted Lie algebra $\mathfrak{g}$.

**Definition 3.15.** For any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module $M$, any positive integer $r$, and any $j$, $1\leq j\leq(p-1)r$, we define

$$\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}} \;\equiv\; \{\epsilon\in\mathbb{E}(\mathrm{r},\mathfrak{g}) : \dim(\mathrm{Rad}^{\mathrm{j}}(\epsilon^*\mathrm{M})) < \max_{\epsilon'\in\mathbb{E}(\mathrm{r},\mathfrak{g})} \dim\mathrm{Rad}^{\mathrm{j}}(\epsilon'^*\mathrm{M})\}$$

$$\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}} \;\equiv\; \{\epsilon\in\mathbb{E}(\mathrm{r},\mathfrak{g}) : \dim(\mathrm{Soc}^{\mathrm{j}}(\epsilon^*\mathrm{M})) > \min_{\epsilon'\in\mathbb{E}(\mathrm{r},\mathfrak{g})} \dim\mathrm{Soc}^{\mathrm{j}}(\epsilon'^*\mathrm{M})\}$$

These notions are somewhat similar to the support varieties. For example, we have the following.

**Lemma 3.16.** *Suppose that* $\mathbb{E}(r,\mathfrak{g})_M\neq\mathbb{E}(r,\mathfrak{g})$. *Then* $\mathbb{R}\mathrm{ad}^1(\mathrm{r},\mathfrak{g})_{\mathrm{M}} \;\simeq\; \mathbb{E}(\mathrm{r},\mathfrak{g})_{\mathrm{M}} \;\simeq\; \mathbb{S}\mathrm{oc}^1(\mathrm{r},\mathfrak{g})_{\mathrm{M}}$.

*Proof.* The hypothesis implies that there exists an elementary subalgebra $\epsilon$ such that $\epsilon^*M$ is a free $\mathfrak{u}(\epsilon)$-module. Let $n = \dim(M)/p^r$. Then we have an isomophism of $\mathfrak{u}(\epsilon)$-modules, $\mathfrak{u}(\epsilon)^n \simeq \epsilon^*M$. If on the other hand, $\mathfrak{c}$ is an elementary subalgebra that does not act freely on $M$, then any homomoprhism $\mathfrak{u}(\mathfrak{c})^n \longrightarrow \mathfrak{c}^*M$ must fail to be surjective, as otherwise it would be an isomorphism. It follows that the dimension of $\mathfrak{c}^*M/\operatorname{Rad}(\mathfrak{c}^*M)$ is larger than $n$, by Nakayama's Lemma. So $\dim(\operatorname{Rad}(\mathfrak{c}^*M))$ is less than $\dim(\operatorname{Rad}(\epsilon^*M))$. The proof that $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}} = \mathbb{E}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ is a dual argument. □

**Theorem 3.17.** *Let $M$ be a finite dimensional $\mathfrak{g}$-module, and let $r, j$ be positive integers such that $1 \leq j \leq (p-1)r$. Then $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ are proper closed subvarieties in $\mathbb{E}(r, \mathfrak{g})$.*

*Proof.* Follows immediately from Theorem 3.13. □

One approach to our first application, requires the following elementary fact.

**Lemma 3.18.** *Let $k[x_1, \ldots, x_n]$ be a polynomial ring, let $x_1^{i_1} \ldots x_n^{i_n}$ be a monomial of degree $i$ and assume that $p = char\, k > i$. There exist linear polynomials without constant term $\lambda_0, \ldots, \lambda_m$ on the variables $x_1, \ldots, x_n$, and scalars $a_0, \ldots, a_m \in k$ such that*

$$x_1^{i_1} \ldots x_n^{i_n} = a_0\lambda_0^i + \ldots + a_m\lambda_m^i.$$

*Proof.* It suffices to prove the statement for $n = 2$, thanks to an easy induction argument (with respect to $n$). Hence, we assume that we have only two variables, $x$ and $y$.

Let $\lambda_j = jx + y$ for $j = 0, \ldots, i$, so that we have $i+1$ equalities of the form $(jx+y)^i = \lambda_j^i$ for $j = 0, \ldots, i$. Treating monomials on $x, y$ as variables, we interpret this as a system of $i+1$ equations on $i+1$ variables with the matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 1 & i & \ldots & \binom{i}{j} & \ldots & i & 1 \\ 2^i & 2^{i-1}i & \ldots & 2^{i-j}\binom{i}{j} & \ldots & 2i & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ i^i & i^{i-1}i & \ldots & i^{i-j}\binom{i}{j} & \ldots & i^2 & 1 \end{pmatrix}$$

By canceling the coefficient $\binom{i}{j}$ in the $(j+1)$-st column (which is non-trivial since $p > i$) we reduce the determinant of this matrix to a non-trivial Vandermonde determinan t. Hence, the matrix is invertible. We conclude the monomials $x^jy^{i-j}$ can be expressed as linear combinations of the free terms $\lambda_0^i, \ldots, \lambda_i^i$. □

Determination of the closed subvarieties $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ of $\mathbb{E}(r, \mathfrak{g})$ appears to be highly non-trivial. The reader will find a few computer-aided calculations in [12] for $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$. The following proposition presents some information for $\mathbb{E}(n-1, \mathfrak{gl}_n)$.

**Proposition 3.19.** *Assume that $p \geq n$. Let $X \in \mathfrak{gl}_n$ be a regular nilpotent element, and let $\epsilon \in \mathbb{E}(n-1, \mathfrak{gl}_n)$ be an $n-1$-plane with basis $\{X, X^2, \ldots, X^{n-1}\}$. Then $\mathrm{GL}_n \cdot \epsilon$ is an open $\mathrm{GL}_n$-orbit for $\mathbb{E}(n-1, \mathfrak{gl}_n)$.*

*Proof.* Let $V$ be the defining $n$-dimensional representation of $\mathfrak{gl}_n$. Let $\epsilon'$ be any elementary Lie subalgebra of $\mathfrak{gl}_n$ of dimension $n-1$. If $\epsilon'$ contains a regular nilpotent element $Y$, then $\epsilon'$ has basis $\{Y, Y^2, \dots, Y^{n-1}\}$, since the centralizer of a regular nilpotent element in $\mathfrak{gl}_n$ is generated as a linear space by the powers of that nilpotent element. Hence, in this case $\epsilon'$ is conjugate to the fixed plane $\epsilon$. Moreover, $\mathrm{Rad}^{n-1}(\epsilon'^*V) = \mathrm{Im}\{Y^{n-1} : V \to V\}$, and, hence, $\dim \mathrm{Rad}^{n-1}(\epsilon'^*V) = 1$.

Suppose $\epsilon'$ does not contain a regular nilpotent element. Then for any matrix $Y \in \epsilon'$, we have $Y^{n-1} = 0$. Lemma 3.18 implies that any monomial of degree $n-1$ on elements of $\epsilon'$ is trivial. Therefore, $\mathrm{Rad}^{n-1}(\epsilon'^*V) = 0$. We conclude that $\mathrm{GL}_n \cdot \epsilon$ is the complement to $\mathbb{R}\mathrm{ad}^{\mathrm{n}-1}(\mathrm{n}-1, \mathfrak{gl}_\mathrm{n})_\mathrm{V}$ in $\mathbb{E}(n-1, \mathfrak{gl}_n)$. Theorem 3.13 now implies that $\mathrm{GL}_n \cdot \epsilon$ is open. □

**Example 3.20.** In this example we describe the geometry of $\mathbb{E}(2, \mathfrak{gl}_3)$ making an extensive use of the $\mathrm{GL}_3$-action. Further calculations involving more geometry are currently being investigated.

Assume that $p > 3$. Fix a regular nilpotent element $X \in \mathfrak{gl}_3$. Let $\epsilon_1 = \langle X, X^2 \rangle$ be the 2-plane in $\mathfrak{gl}_3$ with the basis $X, X^2$, and let

$$C_1 = \mathrm{GL}_3 \cdot \epsilon_1 \subset \mathbb{E}(2, \mathfrak{gl}_3)$$

be the orbit of $\epsilon_1$ in $\mathbb{E}(2, \mathfrak{gl}_3)$. By Proposition 1.9 or by Proposition 3.19 this is an open subset of $\mathbb{E}(2, \mathfrak{gl}_3)$. Since $\mathbb{E}(2, \mathfrak{gl}_3)$ is irreducible (see Example 1.7), $C_1$ is dense. We have $\dim C_1 = \dim \overline{C_1} = \dim \mathbb{E}(2, \mathfrak{gl}_3) = 4$.

The closure of $C_1$ contains two more (closed) $\mathrm{GL}_3$ stable subvarieties, each one of dimension 2. They are the $\mathrm{GL}_3$ saturations in $\mathbb{E}(2, \mathfrak{gl}_3)$ of the elementary subalgebras $\mathfrak{u}_{1,2}$ (spanned by $E_{1,2}$ and $E_{1,3}$), and $\mathfrak{u}_{2,1}$ (spanned by $E_{1,3}$ and $E_{2,3}$). Since the stabilizer of $\mathfrak{u}_{1,2}$ (resp. $\mathfrak{u}_{2,1}$) is the standard parabolic $P_{1,2}$ (resp. $P_{2,1}$), the corresponding orbit is readily identified with $\mathrm{GL}_3 / P_{1,2} \simeq \mathrm{Grass}(2,3) = \mathbb{P}^2$ (resp., $\mathrm{GL}_3 / P_{2,1} \simeq \mathbb{P}^2$) (see Remark 2.6).

**Proposition 3.21.** *Let $\mathfrak{u}$ be a $p$-restricted Lie algebra with trivial $p$-restriction map. Then the locus of elementary subalgebras $\epsilon \in \mathbb{E}(r, \mathfrak{u})$ such that $\epsilon$ is maximal (that is, not properly contained in any other elementary subalgebra of $\mathfrak{u}$) is an open subset of $\mathbb{E}(r, \mathfrak{u})$.*

*Proof.* If no maximal elementary subalgebras are contained in $\mathbb{E}(r, \mathfrak{u})$, then the statement is clear. Hence, we may assume that there is at least one maximal elementary subalgebra $\epsilon \in \mathbb{E}(r, \mathfrak{u})$.

Regard $\mathfrak{u}$ as acting on itself by the adjoint representation. Note that we necessarily have $\epsilon \subset \mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}}))$. Moreover, our hypothesis that $x^{[p]} = 0$ for any $x \in \mathfrak{u}$ implies that this inclusion is an equality if and only if $\epsilon$ is a maximal elementary subalgebra. Hence,

$$\dim \mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}})) \geq \dim \epsilon = r$$

with equality if and only if $\epsilon$ is maximal. We conclude that the locus of elementary subalgebras $\epsilon \in \mathbb{E}(r, \mathfrak{u})$ such that $\epsilon$ is nonmaximal equals the nonminimal socle variety $\mathbb{S}\mathrm{oc}^1(\mathrm{r}, \mathfrak{u})_{\mathfrak{u}_{\mathrm{ad}}}$. The statement now follows from Theorem 3.17. □

## 4. Modules of constant $(r, j)$-radical rank and/or constant $(r, j)$-socle rank

In previous work with coauthors, we have considered the interesting class of modules of constant Jordan type (see, for example, [11]). In the terminology of this paper, these are $\mathfrak{u}(\mathfrak{g})$-modules $M$ with the property that the isomorphism type of $\epsilon^*M$ is independent of $\epsilon \in \mathbb{E}(1, \mathfrak{g})$. In the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$, further classes of special modules were considered by replacing this condition on the isomorphism type of $\epsilon^*M$ for $\epsilon \in \mathbb{E}(1, \mathfrak{g}_a^{\oplus n})$ by the "radical" or "socle" type of $\epsilon^*M$ for $\epsilon \in \mathbb{E}(r, \mathfrak{g}_a^{\oplus n})$.

In this section, we consider $\mathfrak{u}(\mathfrak{g})$-modules of constant $(r, j)$-radical rank and constant $r$-radical type (and similarly for socles). As already seen in [12] in the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$, the variation of radical and socle behavior for $r > 1$ can be quite different. Moreover, having constant $r$-radical type does not imply the constant behavior for a different $r$.

As we investigate in [13], a $\mathfrak{u}(\mathfrak{g})$-module of constant $(r, j)$-radical rank or constant $(r, j)$-socle rank determines a vector bundle on $\mathbb{E}(r, \mathfrak{g})$, thereby providing good motivation for studying such modules. While a great many examples of such $\mathfrak{u}(\mathfrak{g})$-modules, some well known, can be constructed from rational $G$-modules, there are numerous others which do not arise in this way. Some examples are given in 4.9, 4.10 and 4.11. Although identifying the associated vector bundles is hard, some such vector bundles might prove to be of geometric importance.

**Definition 4.1.** Fix integers $r > 0$ and $j, 1 \le j < (p-1)r$. A $\mathfrak{u}(\mathfrak{g})$-module $M$ is said to have constant $(r, j)$-radical rank (respectively, $(r, j)$-socle rank) if the dimension of $\mathrm{Rad}^j(\epsilon^*M)$ (respectively, $\mathrm{Soc}^j(\epsilon^*M)$) is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

We say that $M$ has constant $r$-radical type (respectively, $r$-socle type) if $M$ has constant $(r, j)$-radical rank (respectively, $(r, j)$-socle rank) for all $j$.

**Remark 4.2.** For $r > 1$, the condition that the $r$-radical type of $M$ is constant does not imply that the isomorphism type of $\epsilon^*M$ is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. The condition that $\dim \mathrm{Rad}^j(\epsilon^*(M)) = \dim \mathrm{Rad}^j(\epsilon'^*M)$ for all $j$ is much weaker than the condition that $\epsilon^*M \simeq \epsilon'^*M$. Indeed, examples are given in [12] (for $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$) of modules $M$ whose $r$-radical type is constant but whose $r$-socle type is not constant. In particular, the isomorphism type of $\epsilon^*M$ for such $M$ varies with $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

**Proposition 4.3.** *A $\mathfrak{u}(\mathfrak{g})$-module $M$ has constant $(r, j)$-radical rank (respectively, $(r, j)$-socle rank) if and only if $\mathbb{R}\mathrm{ad}^j(r, \mathfrak{g})_M = \emptyset$ (resp., $\mathbb{S}\mathrm{oc}^j(r, \mathfrak{g})_M = \emptyset$.)*

*Proof.* This follows from the fact that there is a non-maximal radical rank if and only if the radical rank is not constant, a non-minimal socle rank if and only if the socle rank is not constant. □

**Proposition 4.4.** *Let $G$ be an affine algebraic group, and let $\mathfrak{g} = \mathrm{Lie}(G)$. If $\mathbb{E}(r, \mathfrak{g})$ consists of a single $G$-orbit, then any finite dimensional rational $G$-module has constant $r$-radical type and constant $r$-socle type.*

*Proof.* This follows immediately from Proposition 3.10. □

**Remark 4.5.** We point out that examples arising from Proposition 4.4 have much stronger properties than constant radical or socle rank: they have the same isomorphism type restricted to any elementary subalgebra of dimension $r$. On the other hand, using $L_\zeta$-modules, we give examples in Propositions 4.10, 4.11 of modules which have constant radical types but do not arise from a single $G$-orbit and don't even have $G$-structure.

**Example 4.6.** If $P$ is a finite dimensional projective $\mathfrak{u}(\mathfrak{g})$-module, then $\epsilon^*P$ is a projective (and thus free) $\mathfrak{u}(\epsilon)$-module for any elementary subalgebra $\epsilon \subset \mathfrak{g}$. Thus, the $r$-radical type and $r$-socle type of $P$ are constant.

**Example 4.7.** Let $\mathfrak{g}$ be a $p$-restricted Lie algebra. Recall that $\Omega^s(k)$ for $s > 0$ is the kernel of $P_{s-1} \xrightarrow{d} P_{s-2}$, where $d$ is the differential in the minimal projective resolution $P_* \to k$ of $k$ as a $\mathfrak{u}(\mathfrak{g})$-module; if $s < 0$, then $\Omega^s(k)$ is the cokernel of $I^{-s-2} \xrightarrow{d} I^{-s-1}$, where $d$ is the differential in the minimal injective resolution $k = I^{-1} \to I^*$ of $k$ as a $\mathfrak{u}(\mathfrak{g})$-module. Then for any $s \in \mathbb{Z}$, the $s$-th Heller shift $\Omega^s(k)$ has constant $r$-radical type and constant $r$-socle type for each $r > 0$.

Namely, for any $\epsilon \in \mathbb{E}(r, \mathfrak{g})$, $\epsilon^*(\Omega^s(k))$ is the direct sum of the $s$-th Heller shift of the trivial module $k$ and a free $\mathfrak{u}(\epsilon)$-module (whose rank is independent of the choice of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$).

The following example is one of many we can realize using Proposition 4.4.

**Example 4.8.** Let $\mathfrak{g} = \mathfrak{gl}_{2n}$ and $r = n^2$. If $M$ is any finite dimensional rational $\mathrm{GL}_{2n}$-module, then it has constant $r$-radical type and constant $r$-socle type by Corollary 2.9.

In Example 4.8, the dimension $r$ of elementary subalgebras $\epsilon \subset \mathfrak{g}$ is maximal. We next consider an example of non-maximal elementary subalgebras.

**Example 4.9.** Choose $r > 0$ such that no elementary subalgebra of dimension $r$ in $\mathfrak{g}$ is maximal. Let $\zeta \in \widehat{\mathrm{H}}^n(\mathfrak{u}(\mathfrak{g}), k)$ for $n < 0$ be an element in negative Tate cohomology. Consider the associated short exact sequence

$$0 \longrightarrow k \longrightarrow E \longrightarrow \Omega^{n-1}(k) \longrightarrow 0. \tag{4.9.1}$$

Then $E$ has constant $r$-radical rank and constant $r$-socle rank for every $j$, $1 \leq j \leq (p-1)r$.

Namely, we observe that the restriction of the exact sequence (4.9.1) to $\epsilon$ splits for every $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. This splitting is a consequence of [12, 3.8] (stated for an elementary abelian $p$-group and equally applicable to any elementary subalgebra $\mathfrak{f} \subset \mathfrak{g}$ which strictly contains $\epsilon$). The assertion is now proved with an appeal to Example 4.7.

We next proceed to consider modules $L_\zeta$, adapting to the context of $p$-restricted Lie algebras the results of [12, §5].

**Proposition 4.10.** *(see* [12, 5.5]*) Suppose that we have a non-zero cohomology class $\zeta \in \mathrm{H}^m(\mathfrak{u}(\mathfrak{g}), k)$ satisfying the condition that*

$$Z(\zeta) \ \subset \ \mathcal{N}_p(\mathfrak{g}) \ \subset \ \mathfrak{g}$$

*does not contain a linear subspace of dimension $r$ for some $r \geq 1$. Then the $\mathfrak{u}(\mathfrak{g})$-module $L_\zeta$ has constant $r$-radical type.*

*Proof.* Consider $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. The hypothesis implies that $\epsilon$ is not contained in $Z(\zeta)$. Hence, $\zeta\downarrow_\epsilon \in \mathrm{H}^m(\mathfrak{u}(\epsilon), k)$ is not nilpotent. Recall that $\mathrm{H}^*(\mathfrak{u}(\epsilon), k) \simeq \mathrm{H}^*(\mathbb{Z}/p^{\times r}, k)$ is a tensor product of a symmetric and an exterior algebras on $r$ generators. Therefore, a non-nilpotent element is not a zero divisor. Proposition 5.3 of [12] applied to $\epsilon$ implies that

$$\mathrm{Rad}(L_{\zeta\downarrow_\epsilon}) = \mathrm{Rad}(\Omega^n(\epsilon^* k)), \tag{4.10.1}$$

where $\Omega^n(\epsilon^* k)$ is the $n$-th Heller shift of the trivial $\mathfrak{u}(\epsilon)$-module. We note that the statement and proof of [12, Lemma 5.4] generalizes immediately to the map

$\mathfrak{u}(\epsilon) \twoheadrightarrow \mathfrak{u}(\mathfrak{g})$ yielding the statement that $\dim \operatorname{Rad}(\epsilon^*(L_\zeta)) - \dim \operatorname{Rad}(L_{\zeta\downarrow_\epsilon}) = \dim \operatorname{Rad}(\epsilon^*(\Omega^n(k))) - \dim \operatorname{Rad}(\Omega^n(\epsilon^* k))$ is independent of $\epsilon$ whenever $\zeta \downarrow_\epsilon \neq 0$. Combined with (4.10.1), this allows us to conclude that

$$\dim \operatorname{Rad}(\epsilon^*(L_\zeta)) = \dim \operatorname{Rad}(\epsilon^*(\Omega^n(k))).$$

Since $\epsilon^*(L_\zeta)$ is a submodule of $\epsilon^*(\Omega^n(k))$ this further implies that equality of radicals

$$\operatorname{Rad}^j(\epsilon^*(L_\zeta)) = \operatorname{Rad}^j(\epsilon^*(\Omega^n(k)))$$

for all $j > 0$. Since $\Omega^n(k)$ has constant $r$-radical type by Example 4.7, we conclude that the same holds for $L_\zeta$.

□

Utilizing another result of [12], we obtain a large class of $\mathfrak{u}(\mathfrak{g})$-modules of constant radical type.

**Proposition 4.11.** *Let $d$ be a positive integer, sufficiently large compared to $r$ and $\dim \mathfrak{g}$. There exists some $0 \neq \zeta \in \mathrm{H}^{2d}(\mathfrak{u}(\mathfrak{g}), k)$ such that $L_\zeta$ has constant $r$-radical type.*

*Proof.* The embedding $V(\underline{\mathfrak{g}}) \simeq \operatorname{Spec} \mathrm{H}^{\mathrm{ev}}(\mathfrak{u}(\mathfrak{g}), k) \hookrightarrow \mathfrak{g}$ (for $p > 2$) is given by the natural map $S^*(\mathfrak{g}^{\#}[2]) \twoheadrightarrow \mathrm{H}^*(\mathfrak{u}(\mathfrak{g}), k)$ determined by the Hochschild construction $\mathfrak{g}^{\#} \twoheadrightarrow \mathrm{H}^2(\mathfrak{u}(\mathfrak{g}), k)$ (see, for example, [19]). (Here, $\mathfrak{g}^{\#}[2]$ is the vector space dual to the underlying vector space of $\mathfrak{g}$, placed in cohomological degree 2.) As computed in [12, 5.7], the set of all homogeneous polynomials $F$ of degree $d$ in $S^*(\mathfrak{g}^{\#}[2])$ such that the zero locus $Z(F) \subset \operatorname{Proj}(\mathfrak{g})$ does not contain a linear hyperplane isomorphic to $\mathbb{P}^{r-1}$ is dense in the space of all polynomials of degree $d$ for $d$ sufficiently large. Let $\zeta$ be the restriction to $\operatorname{Proj} k[V(\underline{\mathfrak{g}})]$ of such an $F$ in $S^*(\mathfrak{g}^{\#}[2])$; since such an $F$ can be chosen from a dense subset of homogeneous polynomials of degree $d$, we may find such an $F$ whose associated restriction $\zeta$ is non-zero. Now, we may apply Proposition 4.10 to conclude that $L_\zeta$ has constant $r$-radical type. □

**Remark 4.12.** A bound on $d$ can be deduced from the proof of [12, 5.7]: $d$ must satisfy the inequality: $\binom{r+d-1}{d-1} > (r+1)(n-r-1)$ where $n = \dim \mathfrak{g}$.

The following closure property for modules of constant radical and socle types is an extension of a similar property for modules of constant Jordan type.

**Proposition 4.13.** *Suppose $\mathbb{E}(r, \mathfrak{g})$ is connected. Let $M$ be a $\mathfrak{u}(\mathfrak{g})$-module of constant $(r, j)$-radical rank (respectively, constant $(r, j)$-socle rank) for some $r, j$. Then any $\mathfrak{u}(\mathfrak{g})$-summand $M'$ of $M$ also has constant $(r, j)$-radical rank (resp., constant $(r, j)$-socle rank).*

*Proof.* Write $M = M' \oplus M''$, and set $m$ equal to the $(r, j)$-radical rank of $M$. Since the local $(r, j)$-radical types of $M'$, $M''$ are both lower semicontinuous by Theorem 3.13 and since the sum of these local radical types is a constant function, we conclude that both $M'$, $M''$ have constant $(r, j)$-radical rank.

The argument for $(r, j)$-socle rank is essentially the same. □

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Department of Mathematics, University of Georgia, Athens, GA
*E-mail address*: `jfc@math.uga.edu`

Department of Mathematics, University of Southern California, Los Angeles, CA
*E-mail address*: `ericmf@usc.edu, eric@math.northwestern.edu`

Department of Mathematics, University of Washington, Seattle, WA
*E-mail address*: `julia@math.washington.edu`