SPECTRUM OF GROUP COHOMOLOGY AND SUPPORT VARIETIES

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0. Introduction

It is a great privilege to reflect on the results and influence of Daniel Quillen's two papers in the Annals (1971) entitled "The spectrum of an equivariant cohomology ring, I, II" [26], [27]. As with other papers by Dan, these are very clearly written and reach their conclusions with elegance and efficiency. The object of study is the equivariant cohomology algebra of a compact Lie group G acting on a reasonable topological space. A case of particular interest is the action of a finite group on a point, in which case the ring in question is the cohomology algebra of the finite group. Dan writes: "It is the purpose of this series of papers to relate the invariants attached to such a ring by commutative algebra to the family of elementary abelian p-subgroups of G."

What follows is a brief introduction to Dan's results and methods, followed by an idiosyncratic discussion of some subsequent developments.

The specific goal of Dan's first paper [26] is to give a proof of a conjecture by M. Atiyah (unpublished) and R. Swan [31] concerning the Krull dimension of the mod-p cohomology of a finite group. Those familiar with Dan's style will not be surprised that in reaching his goal he lays out clearly and concisely the foundations for equivariant cohomology as introduced by A. Borel [7]. Although we do not address the many topological applications of equivariant cohomology or recent developments using equivariant theories in algebraic geometry, we would remiss if we did not point out that this paper establishes the definitions and techniques which a generation of mathematicians have actively pursued.

The second paper [26] opens the way to many subsequent developments in the cohomology and representation theory of groups and related structures. Namely, Dan identifies up to (Zariski) homeomorphism the spectrum of the cohomology of a finite group in terms of its elementary abelian p-subgroups. More recent developments which are an outgrowth of Dan's methods are the subject of the latter part of this survey. The lasting impact of this second paper is in large part due to its vision of investigating group theory using the commutative algebra of the cohomology of groups, leading to new roles for algebraic geometry and triangulated categories in the study of representation theory.

1. Equivariant cohomolgy and results of Quillen

Quillen worked in the generality of a compact Lie group G acting on a paracompact space X. He was inspired by the work of Atiyah and G. Segal on equivariant K-theory [3]. He studied the equivariant cohomology (as introduced by A. Borel

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in [7]) of X with coefficients in a field k of characteristic $p > 0$, $H^*(EG \times_G X, k)$. In the introduction to the first paper, Quillen remarks that consideration of equivariant cohomology is necessary even if one's interest is the special case $H^*(BG, k)$ (i.e., with X a point). Subsequently, equivariant cohomology can be replaced by more algebraic arguments given by Quillen [28] and Quillen and Venkov [29] using the Evens' norm map [15], as explained in detail in [16]. Thus, for applications to the study of cohomology and representations of finite groups, Quillen's equivariant cohomology theory is no longer required. That said, the interested reader will find important applications of this theory in algebraic topology, for example in [22].

Quillen uses the Leray spectral sequence to study equivariant cohomology. For example, he extends the approach to finite generation of cohomology used by Venkov which entails the embedding of the given compact group G into some unitary group $U = U_n$. He considers the Leray spectral sequence

$$E_2^{s,t} = H^s(U/G; H^t(X,k)) \;\Rightarrow\; H^{s+t}(U \times_G X, k)$$

for the fiber sequence $X \to U \times_G X \to U/G$ to conclude that if $H^*(X)$ is finitely generated over k, then $H^*(EG \times_G X, k)$ is a finite module over $H^*(X, k)$.

Quillen introduces the notion of an F-isomorphism of graded commutative algebras over k; namely, $f : R \to S$ is an F-isomorphism if every homogeneous element of R in the kernel of f is nilpotent and every homogeneous element in S has some p-th power in the image of f.

Recall that a finite group E is said to be an elementary abelian p-groups if it is isomorphic to a product of copies of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. The cohomology algebra of an elementary abelian p-group is easy to determine using the Künneth Theorem: if $p = 2$, the cohomology algebra is a polynomial algebra on $Hom_k(E, k)$ placed in dimension 1; if $p > 2$, the cohomology algebraic is the tensor algebra of an exterior algebra on $Hom_k(E, k)$ placed in dimension 1 and a symmetric algebra on $Hom_k(E, k)$ placed in dimension 2. To prove the Atiyah-Swan Conjecture, Quillen considers the sheaf $\mathcal{H}^t$ on X/G associated to the presheaf $U \mapsto H^t(EG \times_G q^{-1}(U), k)$, where $q : X \to X/G$ is the quotient map and $U \subset X$ ranges over the open subsets of X. Quillen compares the sheaf $\mathcal{H}^t$ to the "compatible family" $\{X^E \to H^*(BE, k)\}$, where $X^E \subset X$ is the fixed point space of E acting on X, and where E ranges over elementary abelian p-groups of G.

Quillen applies the Leray spectral sequence

$$E_2^{s,t} = H^s(X/G; \mathcal{H}^t) \;\Rightarrow\; H^{s+t}(EG \times_G X; k)$$

and a localization argument to conclude that (specialized to the case in which X is a point and G is a finite group) the natural homomorphism

$$H^*(BG, k) \longrightarrow \varprojlim_{E \subset G} H^*(BE, k) \tag{1}$$

is an F-isomorphism, where the limit is indexed by the category whose objects are elementary abelian p-groups of G and whose maps are compositions of conjugations and inclusions. The injectivity aspect of this F-isomorphism is the important statement that a cohomology class in $H^*(G, k)$ is nilpotent if it vanishes upon restriction to all elementary abelian subgroups $E \subset G$. Another proof of this "detection up to nilpotents" was subsequently found by Quillen and Venkov [29], a proof not using equivariant cohomology.

In other words, "up to F-isomorphism" the cohomology algebra of an arbitrary finite group G is determined by the "inverse system" of the cohomology algebras of the elementary subgroups of G. This is especially remarkable, for computations of $H^i(G,k)$ for relatively easy finite groups G and relatively small degree i are often far beyond our means of computation.

As a corollary of the F-isomorphism (1), Quillen concludes:

Theorem 1.1. *(Atiyah-Swan Conjecture) Let G be a finite group and k a field of characteristic $p > 0$. Then the Krull dimension of $H^*(BG,k)$ equals the maximal rank of an elementary abelian p-group of G.*

In the second of these two important papers [26] , Quillen continues his investigation of properties of the cohomology ring $H^*(BG,k)$ (and, more generally, $H^*(EG \times^G X,k)$), considering the maximal ideal spectrum of these cohomology algebras. For $p > 2$, Quillen considers only the cohomology in even degree, thereby avoiding problems with graded-commutativity. For notational simplicity, we restrict below to the case $p > 2$ and denote by $H^\bullet(BG,k)$ the graded subalgebra generated by cohomology classes of even degree.

If E is an elementary abelian p-group, then each element of $g \in Hom(E,\mathbb{Z}/p)$ can be identified with an element of $\gamma \in H^1(BE,k)$ whose Bockstein $\beta(\gamma)$ is an element of $H^2(BE,k)$. We set $\epsilon_E \in H^{2p^r-2}(E,k)$ equal to the product of these elements $\beta(\gamma)$ ranging over all $0 \neq g \in Hom(E,\mathbb{Z}/p)$. Observe that the computation of $H^*(E,k)$ for an elementary abelian p-group immediately implies that $V(E)$ is an affine space of dimension equal to the rank of E.

Proposition 1.2. *Let $V(E)$ be the maximal ideal spectrum of $H^\bullet(E,k)$, where E is an elementary abelian p-group E. With ϵ_E as above, the zero locus $Z(\epsilon_E) \subset V(E)$ is the union of the images $V(F) \to V(E)$ as $F \subset E$ ranges over all proper subgroups.*

The complement of the zero loci $Z(\epsilon_E) \subset V(E)$ serve as strata in the following *Quillen's Stratification Theorem*, stated here in the special case in which G is a finite group and X is a point.

Theorem 1.3. *(Stratification Theorem) Let G be a finite group. Then the maximal ideal spectrum of $H^\bullet(G,k)$ is the disjoint union of locally closed affine subvarieties,*

$$V(G) \simeq \coprod_E i_E(V(E)^+), \quad V(E)^+ = V(E) - Z(\epsilon_E),$$

where the union is indexed by conjugacy classes of elementary abelian p-subgroups of G. Here, $i_E : V(E) \to V(G)$ is the natural map induced by the inclusion $E \subset G$.

Moreover, the Weyl group $W_E = N_G(E)/C_G(E)$ of E in G acts freely on $V(E)^+$ over its image in $V(G)$.

In geometric terms, Theorem 1.3 asserts that the maximal ideal spectrum of an arbitrary finite group is the union of quotients by finite groups of affine spaces indexed by the conjugacy classes of maximal elementary abelian subgroups, patched together along inclusions of smaller elementary abelian subgroups.

Quillen further investigates the cohomology algebra $H^\bullet(BG,k)$ using the action of the Steenrd algebra. This work anticipates much further work on the role of the Steenrod algebra in the study of classifying spaces of compact groups (e.g., [24]).

Theorem 1.4. *(Steenrod Stability) A prime ideal of $H^\bullet(BG,k)$ is the kernel of the restriction map to some elementary abelian subgroup of G if and only if it is homogeneous and stable under the Steenrod operations $\mathcal{P}^i, i \geq 0$.*

As Quillen points out, this implies that the only stable homogeneous prime of a stratum $i_E(V(E)^+)$ is the generic point. He continues: "In some intuitive sense at least, this means that the stratra are the orbits of the spectrum for the action of automorphisms of the cohomology functor."

To conclude this section, I mention an interesting result from the thesis of J. Duflot, who worked under Quillen's direction to further explore the perspective of Quillen's fundamental papers.

Theorem 1.5. [14] *Let G be a finite group with a unique elementary abelian p-subgroup of maximal rank and assume that this subgroup is central. Then $H^\bullet(BG,k)$ is Cohen-Macaulay.*

2. Cohomological support and rank varieties

Quillen's papers discussed in the previous section concern the cohomology algebra $H^*(BG,k)$ where k is a field of characteristic $p > 0$ on which G is assumed to act trivially. In this section, we describe how Quillen's approach has led to an investigation of representations of G on finite dimensional k-vector spaces (or, equivalently, of finite dimensional kG-modules). Throughout this section, G will denote an arbitrary finite group. The work described here was done in the 1980's, more than ten years after Quillen's papers; as mentioned below, this relies on basic results of Chouinard and Dade proved in the mid-1970's.

Perhaps the first extension of Quillen's work to non-trivial kG-modules M is a theorem of J. Alperin and L. Evens [1] which establishes that the complexity $c_G(M)$ of M (i.e., the rate of growth of a minimal projective resolution of M) equals the maximum of $c_E(M)$ as E varies over the elementary abelian subgroups of M. For M the trivial representation k, this theorem reduces to Theorem 1.1. A precursor to the theorem of Alperin and Evens is a theorem of L. Chouinard [12] which asserts that a kG-module M is projective if and only if the restrictions of M along each elementary abelian p-subgroup $E \subset G$ is projective.

In [2] and [4], Alperin-Evens and G. Avrunin extended the geometric approach of Quillen's second paper [27] to non-trivial kG-modules. Let $|G|$ denote the maximal ideal spectrum of $H^\bullet(G,k)$. Alperin and Evens introduced the cohomological support variety $|G|_M$ of a kG-module M, defined to be the variety of the annihilator ideal inside the commutative algebra $H^\bullet(G,k)$ of the $H^\bullet(G,k)$-module $Ext^*_{kG}(M,M)$. It is useful to identify this annihilator ideal with the kernel of the natural map $H^\bullet(G,k) \to Ext^*_{kG}(M,M)$. Alperin and Evens prove that $|G|_M$ is the union indexed by (conjugacy classes of) elementary abelian subgroups $E \subset G$ of the images of cohomological support varieties $|E|_M$; for $M = k$, this reduces to Theorem 1.3.

The functor sending a kG-module M to the geometric invariant $|G|_M$ has good properties. Some of these are established by reducing the study of M to its restrictions to elementary abelian subgroups and then using Carlson's rank variety $V(E)_M$ for a kE-module [8]. The construction of J. Carlson is motivated by the theorem of E. Dade [13] which asserts that a kE-module M (for an elementary abelian p-group E) is projective if and only if M is projective when restricted to all cyclic shifted subgroups of kE (i.e., subalgebras isomorphic to the group algebra of the cyclic group $\mathbb{Z}/p\mathbb{Z}$). The points of the rank variety $V(E)_M$ correspond to those shifted subgroups (with respect to a chosen set of generators for the augmentation ideal of kE) restricted to which M is free. As conjectured by Carlson and proved by G. Avrunin and L. Scott [5], $|E|_M$ can be identified with $V(E)_M$.

The representation theory of an elementary abelian p-group E of rank ≥ 2 is "wild" except in the special case that $p = 2$ and the rank of E is also 2. Thus, even for elementary abelian p-groups of rank 2, there are many interesting questions about the representation theory of E (e.g., [11]).

An important construction arising from Quillen's geometric point of view is the construction by Carlson of L_ζ-modules [9]. Let $\zeta \in H^{2m}(G,k)$ be a homogeneous cohomology class of even degree and view ζ as a function on $|G|$, the maximal ideal spectrum of the cohomology algebra $H^\bullet(G,k)$. As in the special case of elementary abelian p-groups, for any finite dimensional kG-module M set $|G|_M \subset |G|$ to be the closed subvariety associated to the annihilator ideal of $H^\bullet(G,k)$ of the graded module $Ext^*_{kG}(M,M)$. Then L_ζ has the property that $|G|_{L_\zeta} \subset |G|$ is the zero locus of ζ. One consequence of this is that $Z \subset |G|$ is of form $|G|_M$ for some (finite dimensional) M if and only if Z is a Zariski closed, conical subset of $|G|$.

3. Extensions and refinements

In this final section, we briefly discuss how considerations of $H^*(G,k)$ by Quillen in his foundational papers and the support/rank varieties of Alperin, Evens, and Carlson have been extended and refined in subsequent years. There has been a great deal of activity in the consideration of these extensions and refinements; our discussion is an unbalanced account omitting mention of the contributions of many mathematicians.

In a series of papers in the 1980's, B. Parshall and the author led an investigation into the cohomology and support varieties of p-restricted Lie algebras. Such Lie algebras $\mathfrak{g}$ have a restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ which is a finite dimensional, co-commutative Hopf algebra (as is the group algebra of a finite group). A restricted representation of $\mathfrak{g}$ is alternatively viewed as a $\mathfrak{u}(\mathfrak{g})$-module. Work of Friedlander-Parshall, H. Andersen, J. Jantzen, and others led to a natural extension to p-restricted Lie algebras $\mathfrak{g}$ of most of the results previously proved for representations for finite groups. Of particular interest is that there is a comparison of cohomological support varieties (defined exactly as in the context of finite groups) and rank varieties (defined in terms of restrictions of $\mathfrak{u}(\mathfrak{g})$-modules to 1-dimensional Lie algebras generated by a p-nilpotent element of $\mathfrak{g}$) [17].

This "Lie algebra variant" has several aspects which distinguish it from the context of finite groups. For example, there is no evident class of subalgebras of $\mathfrak{g}$ (such as nilpotent subalgebras) which plays the role of the class of elementary abelian subgroups of G. Another interesting aspect is that the cohomological support variety associated to $\mathfrak{g}$ is the p-nilpotent cone of $\mathfrak{g}$, an algebraic variety with more interesting geometry than the cohomological support variety of a finite group. Many computations of cohomological support varieties have been achieved using results from the theory of algebraic groups (e.g., [25]).

The previous results for p-restricted Lie algebras were extended to infinitesimal group schemes in two papers by A. Suslin, C. Bendel, and the author [30] in 1997. Infinitesimal group schemes are group schemes whose coordinate algebras are finite dimensional, local k-algebras; infinitesimal group schemes of height 1 correspond naturally to p-restricted Lie algebras (see [23]). One notable aspect of this study of infinitesimal group schemes is the comparison once again between cohomological support varieties and a form of rank varieties, this time formulated in terms of 1-parameter subgroups. As for finite groups and the special case of p-restricted Lie algebras, a key step in this analysis is the proof that cohomology modulo nilpotents is detected by restriction to 1-parameter subgroups. Underlying this work is the theorem of Suslin and the author [21] asserting the the cohomology algebra of any finite group scheme G is finite dimensional and the Ext-algebra $Ext^*_G(M,M)$ is noetherian over $H^*(G,k)$.

In the past 5 years, a suitable generalization of rank varieties for all finite group schemes has been developed by J. Pevtsova and the author (e.g., [18]), addressing the heretofore absence of rank varieties for finite groups other than elementary abelian groups. The introduction of π-points (encompassing cyclic shifted subgroups and 1-parameter subgroups) has led to refinements of support varieties for kG-modules even in the case of finite groups. For example, work of Pevtsova, Suslin, and the author [20] verified that the maximal Jordan types of a kG-module M are a set of intrinsic invariants for M, much finer than the support variety of M. This has led to an interesting class of kG-modules, those of constant Jordan type, whose study was initiated in [10].

We conclude this discussion with one further development, another geometric invariant which has evolved from Quillen's papers. Namely, support varieties and Jordan types of a kG-module M are "local invariants", determined by the restrictions of M to π-points (small subalgebras of kG). In [19], Pevtsova and the author associate algebraic vector bundles to certain kG-modules with G an infinitesimal group scheme. Not only do associated vector bundles distinguish certain modules whose local invariants are the same, but their construction gives an explicit way to produce vector bundles. The reader is referred to [6] for a discussion of realization of bundles on projective spaces.

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