RESTRICTIONS TO $G(\mathbb{F}_p)$ AND $G_{(r)}$ OF RATIONAL G-MODULES

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to Andrei Suslin, with great admiration

Abstract. We fix a prime p and consider a connected reductive algebraic group G over a perfect field k which is defined over $\mathbb{F}_p$. Let M be a finite dimensional rational G-module M, a comodule for $k[G]$. We seek to somewhat unravel the relationship between the restriction of M to the finite Chevalley subgroup $G(\mathbb{F}_p) \subset G$ and the family of restrictions of M to Frobenius kernels $G_{(r)} \subset G$. In particular, we confront the conundrum that if M is the Frobenius twist of a rational G-module N, $M = N^{(1)}$, then the restrictions of M and N to $G(\mathbb{F}_p)$ are equal whereas the restriction of M to $G_{(1)}$ is trivial. Our analysis enables us to compare support varieties (and the finer non-maximal support varieties) for $G(\mathbb{F}_p)$ and $G_{(r)}$ of a rational G-module M where the choice of r depends explicitly on M.

0. Introduction

Our aim is to provide some understanding of the relationship of the restrictions of a finite dimensional rational G-module M to $G(\mathbb{F}_p)$ and $G_{(r)}$. We consider reductive groups G defined over a field k of characteristic $p > 0$ and equipped with the data of an $\mathbb{F}_p$-structure. We require that p be at least as large as the Coxeter number of G, an assumption which much simplifies arguments and might in fact be necessary for such a comparison. As we discuss in the final section, our results apply to provide information about $G(\mathbb{F}_q)$ for q an arbitrary p^{th}-power.

Representations of a Lie group are faithfully reflected by their linearizations as representations of the Lie algebra, but this is far from correct for modular representation theory. Instead, rational representations of a smooth, affine group G correspond to locally finite representations of the hyperalgebra $\varinjlim_r kG_{(r)}$ of distributions supported at the identity of G. This motivates our search for a direct relationship between the action of elements of the finite discrete group $G(\mathbb{F}_p) \subset G$ and the action of the distributions of bounded height supported at the identity of G. Propositions 4.2 establishes an explicit relationship, one which relies on the association to an element $x \in G(\mathbb{F}_p)$ of order p the 1-parameter subgroup $\phi_x : \mathbb{G}_a \to G$ whose construction is due to G. Seitz [18] and D. Testerman [23].

The role of 1-parameter subgroups in the theory of support varieties for infinitesimal groups schemes was explored in earlier work of C. Bendel, A. Suslin, and the author in [21], [22], and this is the underlying foundation of our approach. We employ the point of view developed by the author and J. Pevtsova of π-points and

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π-point spaces $\Pi(G)$ for finite group schemes G such as $G(\mathbb{F}_p)$ and $G_{(r)}$; this perspective enables us to establish a natural relationship between invariants of $G(\mathbb{F}_p)$ and $G_{(r)}$. Using the association $x \mapsto \phi_x$, we construct in Theorem 3.5 a natural embedding

$$\Psi : \Pi(G(\mathbb{F}_p)) \ \to \ (\Pi(G_{(r)}))/G(\mathbb{F}_p), \quad r > 0.$$

In Proposition 3.6, we give an interpretation of the global p-nilpotent operator defined and studied by J. Pevtsova and the author in [9] which relates this operator to Ψ.

As seen in Theorem 4.5, the support variety of M as a $G(\mathbb{F}_p)$-module can be identified with its image under this map Ψ provided that r is sufficiently large. We associate a new invariant, $s(M)$, which gives an upper bound on how large r must be. In Proposition 2.7, we bound $s(M)$ in terms of the weights of M. On the other hand, we should emphasize that weights of a rational G-module M are determined by the action of semi-simple elements in the endomorphism algebra of M, whereas our analysis addresses the action of p-nilpotent elements. Indeed, one can view the consideration of cohomology and support varieties as a study of nilpotent actions, in sharp contrast to the classification of irreducible modules in terms of weights.

Our methods apply to finer invariants than support varieties. Namely, we compare the maximal Jordan types (as introduced by J. Pevtsova, A. Suslin, and the author in [11]) of a rational G-module restricted to $G(\mathbb{F}_p)$ and $G_{(r)}$. This comparison, given in Theorem 4.11, enables us to compare non-maximal support varieties, subvarieties of π-point spaces which are refinements of support varieties.

This work has been motivated by the challenge of interpreting and extending work of J. Carlson, Z. Lin, and D. Nakano in [3] which in turn was based on earlier work of Z. Lin, and D. Nakano in [15]. With the benefit of the technology of π-points and the use of 1-parameter subgroups, we require neither geometric properties of nilpotent cones nor knowledge of centralizers of unipotent elements to reprove and strengthen their results.

Unless stated to the contrary, p will denote an arbitrary prime number, $q = p^d$ an arbitrary p^{th} power, $\mathbb{F}_q$ the finite field of order q, k an arbitrary field of characteristic p, and $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ with chosen embeddings $\mathbb{F}_q \subset \overline{\mathbb{F}}_p$.

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1. Recollections

An affine group scheme G over k is said to be an affine algebraic group over k if it is smooth over k; G is said to be a finite group scheme if its coordinate algebra $k[G]$ is finite dimensional over k; a finite group scheme G is said to be an infinitesimal group scheme if its coordinate algebra is local.

We denote by $\phi : k \to k$ the p^{th}-power map, the "arithmetic Frobenius", and let ϕ^d denote its d^{th}-power. For any scheme X over k, we denote by $X^{(d)}$ the base change of X along ϕ^d, $X^{(1)} = X \times_{\phi^d} \operatorname{Spec} k$. Thus, the coordinate algebra of the affine group scheme $G^{(d)}$ equals $k \otimes_{\phi^d} k[G]$.

The Frobenius map $F^d : G \to G^{(d)}$ is the map of k-group schemes given by $F^{d*} : k \otimes_{\phi^d} k[G] \to k[G],\ a \otimes f \ \mapsto \ a \cdot f^q$ (where $q = p^d$ as usual) (see [12]).

Definition 1.1. Assume that $\mathbb{F}_q \subset k$. An $\mathbb{F}_q$-structure on an affine group scheme G over k is the data of a sub Hopf algebra $\mathbb{F}_q[G] \ \subset \ k[G]$ such that

$$k[G] \ \simeq \ k \otimes_{\mathbb{F}_q} \mathbb{F}_q[G].$$

If the affine group scheme G over k is provided with an $\mathbb{F}_q$-structure, then we say that G is defined over $\mathbb{F}_q$.

We begin with the following (presumably well known) observation, which gives an "intrinsic" condition on G for it to be defined over $\mathbb{F}_q$ and clarifies what we mean by the "Frobenius endomorphism" on G.

Proposition 1.2. *Let G be an affine group scheme over k with $\mathbb{F}_q \subset k$. The data of an $\mathbb{F}_q$-structure on G is equivalent to the data of an isomomorphism $\Phi : k[G] \ \to \ k \otimes_{\phi^d} k[G]$ of Hopf algebras over k with the property that $k[G] \ \equiv \ k \otimes_{\mathbb{F}_q} (k[G])^{\Phi}$, where $(k[G])^{\Phi}$ is defined as the $\mathbb{F}_q$-subalgebra of $k[G]$ consisting of those f such that $\Phi(f) = 1 \otimes_{\phi^d} f$.*

Such an $\mathbb{F}_q$-structure on G determines an isomorphism $G^{(d)} \ \to \ G$ of group schemes over k. Thus, if G is defined over $\mathbb{F}_q$, then we may (and will) view the Frobenius map F^d as an endomorphism of G.

Proof. Identify $k[G]$ with $k \otimes_{\mathbb{F}_q} \mathbb{F}_q[G]$. Then we define $\Phi : k[G] \ \to \ k \otimes_{\phi^d} k[G]$ by sending $a \otimes f \in k \otimes_{\mathbb{F}_q} \mathbb{F}_p[G]$ to $a \otimes 1 \otimes f \in k \otimes_{\phi^d} k \otimes_{\mathbb{F}_q} \mathbb{F}_p[G] \equiv k \otimes_{\phi^d} k[G]$. The fact that Φ is an isomorphism is readily check using the fact that k-linear map $k \to k \otimes_{\phi^d} k,\ a \mapsto a \otimes_{\phi^d} 1$ is an isomorphism; for example, $1 \otimes_{\phi^d} b$ is the image of b^q. Observe that $\mathbb{F}_q[G] \subset k \otimes_{\mathbb{F}_q} \mathbb{F}_q[G]$ is the $\mathbb{F}_q$-subalgebra consisting of those f such that $\Phi(f) = 1 \otimes_{\phi^d} f$.

Given an $\mathbb{F}_q$-structure on G, then the isomorphism $G^{(d)} \to G$ is given by $\Phi : k[G] \to k[G^{(d)}]$. □

Our objective is to establish relationships between representations of the finite group schemes $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of G and the infinitesimal group schemes $G_{(r)}$ which we now recall. The reader is referred to [5, 3.6] for a more detailed discussion of the finite group $G(\mathbb{F}_q)$ of fixed points of F^d on G.

Definition 1.3. Let G be an affine group scheme over k. For any $r \geq 0$, we denote by $G_{(r)}$ the infinitesimal group scheme over k of height r given by

$$G_{(r)} \ \equiv \ \mathrm{Ker}\{F^r : G \to G^{(r)}\}.$$

If G is defined over F_q with d dividing r, then $G_{(r)}$ equals the kernel of $F^r : G \to G$.

For G defined over $\mathbb{F}_q$, we denote by $G(\mathbb{F}_q)$ the finite group

$$G(\mathbb{F}_q) \ \equiv \ \{x \in G(k) |\ F^d(x) \cdot x^{-1} = 1\}.$$

We next recall the definition of the distribution algebra of an affine group scheme.

Definition 1.4. Let G be an affine group scheme over k. Then a distribution of G (with support at the identity $1 \in G$) is a k-linear map $\phi : k[G] \to k$ which vanishes on some power of the the maximal ideal $m_1 \subset k[G]$. The algebra of distributions of G is denoted $Dist(G)$. The reader is referred to [14, I.7] for a detailed discussion.

Let M be a (rational) G-module; in other words, M is a comodule for $k[G]$ whose structure is given by the k-linear map $\nabla_M : M \to M \otimes G$. Then $Dist(G)$ acts on M as follows:

$$Dist(G) \times M \to M, \quad (\phi, m) \mapsto \sum_i \phi(f_i) m_i \quad where \ \ \nabla(m) = \sum_i m_i \otimes f_i.$$

If $H \subset G$ is a closed subgroup scheme, then $Dist(H) \subset Dist(G)$. We readily identify $Dist(G_{(r)}) \subset Dist(G)$ as the *group algebra* of $G_{(r)}$, the Hopf dual of the coordinate algebra $k[G_{(r)}]$; we typically use the notation $kG_{(r)}$ to denote $Dist(G_{(r)})$. On the other hand, if G is discrete then $Dist(G) \simeq k$ since we are considering distributions supported at the identity.

Example 1.5. Let $G = \mathbb{G}_a$, the additive group with coordinate algebra $k[t]$. Let $(d/dt)^{(i)} : k[t] \to k$ denote the k-linear map sending t^j to 0 for $j \neq i$ and t^i to 1. Then $Dist(\mathbb{G}_a)$ is the divided power algebra spanned by $(d/dt)^{(i)}, i \geq 0$, with algebra generators

$$u_j \ \equiv \ (d/dt)^{(p^j)}, \quad 0 \leq j \tag{1}$$

Moreover, $Dist(\mathbb{G}_{a(r)}) \subset Dist(\mathbb{G}_a)$ is the subalgebra generated by $u_j, 0 \leq j < r$.

We require the following elementary observation in order to initiate our comparision of actions of $G(\mathbb{F}_p)$ and $G_{(r)}$.

Proposition 1.6. *Let M be a finite dimensional rational $\mathbb{G}_a$-module. Then the action of $Dist(\mathbb{G}_a)$ on M is the trivial extension of an action of $Dist(\mathbb{G}_{a(r)}) = k\mathbb{G}_{a(r)}$ for $r >> 0$.*

Proof. Let M a rational $\mathbb{G}_a$-module, given by the coaction $\nabla_M : M \to M \otimes k[t]$. Then the k-linear action $Dist(\mathbb{G}_a) \times M \to M$ is given by $((d/dt)^{(i)}, m) \mapsto m_i$ where $\nabla_M = \sum m_i \otimes t^i$. If M is finite dimensional, then $\nabla_M(M) \subset M \otimes k[t]$ is finite dimensional, so that the action of $(d/dt)^{(i)}$ on M is trivial for $i >> 0$. □

We remind the reader that the prime 2 is *bad* for a simple algebraic group G over k if G is not of type A_ℓ, that both $p = 2, 3$ are bad if G is of E_6, E_6, F_4, G_2 and that 2, 3, 5 are bad for G of type E_8. If G is semi-simple, the prime p is bad for G if it is bad for some factor of its simply connected cover. Otherwise, p is said to be *good* for G.

As considered in [18], a subgroup A of a semi-simple algebraic group G over k is a group of type A_1 if A is a closed subgroup isomorphic to SL_2 or PSL_2. Such a group A of type A_1 is said to be *good* provided the weights of its maximal torus for the action of A on the Lie algebra of G are all at most $2p-2$. As shown by G. Seitz in [18, 1.1], if G is simple, p is good for G, and G is not of type A_n, then the restriction of the adjoint representation to a good A_1 in G is a tilting module; for $G = SL_n$, the restriction of the action of a good A_1 on the Lie algebra of GL_{n+1} is also a tilting module.

We shall depend heavily on the following theorem of G. Seitz, which in turn depends upon work of D. Testerman [23].

Theorem 1.7. *(G. Seitz,* [18, 1.3]*) Let G be a simple algebraic group over a perfect field k, with p good for G, and let $x \in G(k)$ have order p. Then there is a unique 1-dimensional unipotent k-subgroup $U \subset G$ containing x such that U is contained in a good $A_1 \subset G_{\overline{k}}$, where $G_{\overline{k}/k}$ is the base change of G to an algebraic closure $\overline{k}$ of*

k. Consequently, there is a unique monomorphism $\phi_x : \mathbb{G}_a \to G$ over k with image in a good $A_1 \subset G_{\overline{k}}$ and satisfying $\phi_x(1) = x$.

If $x_1, \ldots, x_s \in G(k)$ are elements of order p which pairwise commute, then there is an abelian unipotent k-subgroup $\underline{E} \subset G$ through which each ϕ_{x_i} factors. Moreover, if G is defined over $\mathbb{F}_p$ and if $x_1, \ldots, x_s \in G(\mathbb{F}_p)$ generate an elementary abelian p-group $E \subset G(\mathbb{F}_p)$ of rank s, then the associated 1-dimensional unipotent k-subgroups $U_1, \ldots, U_s \subset G$ generate an abelian unipotent subgroup $\underline{E} \subset G$ of rank s.

Remark 1.8. In [18, 1.3], Seitz proves the existence of a unique 1-dimensional unipotent subgroup $U \subset G$ containing x contained in a good A_1 under the hypothesis that k is algebraically closed. If k is perfect, but not algebraically closed, the Galois group $Gal(\overline{k}, k)$ acts on $G(\overline{k})$ with fixed group k. Since the Galois conjugate of a 1-dimensional unipotent subgroup $\overline{U}$ of $G_{\overline{k}}$ contained in a good A_1 of $G_{\overline{k}}$ is again a 1-dimensional unipotent subgroup of $G_{\overline{k}}$ contained in a good A_1, the uniqueness of $\overline{U} \subset G_{\overline{k}}$ implies that $\overline{U}$ is $Gal(\overline{k}, k)$-invariant and thus the base change of a 1-dimensional unipotent k-subgroup $U \subset G$ contained in a good $A_1 \subset G_{\overline{k}}$.

If $x_1, \ldots, x_s$ are pairwise commuting, unipotent elements of order p, then each U_i is contained in the centralizer of each x_j; thus, each U_i commutes with U_j, so that $U_1, \ldots, U_s$ generate an abelian, unipotent subgroup $\underline{E} \subset G$.

If G is defined over $\mathbb{F}_p$ and if each $x_i \in G(\mathbb{F}_p)$, then we may take $k = \mathbb{F}_p$ to conclude that each U_i is defined over $\mathbb{F}_p$. Because $E \subset \underline{E}(\mathbb{F}_p)$, $\underline{E}$ must have dimension $\geq s$; since $\underline{E}$ is generated by s 1-dimensional unipotent groups, we conclude that $\underline{E}$ has dimension equal to s.

We recall that a prime p is good for a reductive algebraic group if it is good for every factor of its commutator $G' = [G, G]$. We shall frequently impose the following condition on G.

Definition 1.9. Let G be an affine algebraic group over a field k. Then G is said to be *suitable* if k is perfect, if G is a connected, reductive algebraic group over k, if p is good for G, and if the degree of the simply connected covering group $G_{sc} \to G$ is prime to p.

Corollary 1.10. *Let G be a suitable affine algebraic group. Then the assertions of Theorem 1.7 (as formulated for simple groups) also apply to G.*

Proof. As above, we may assume k is algebraically closed. First, assume that G is semi-simple, so that $\pi : G_{sc} \to G$ has degree prime to p and G_{sc} is a product of simple groups. Clearly, the assertions of Theorem 1.7 also apply to $(G')_{sc}$. Moreover, any $x \in G(k)$ with $x^p = 1$ lifts to $x' \in G_{sc}$ with $(x')^p = 1$ and any two liftings are related by an automorphism of $(G')_{sc}$ over G. Thus, $\phi_{x'} : \mathbb{G}_a \to G_{sc}$ determines $\pi \circ \phi_x : \mathbb{G}_a \to G$ independent of the choice of lifting x'. The properties of $\phi_{x'}$ given in Theorem 1.7 imply the same properties for ϕ_x.

Now, consider a general connected, reductive group G covered by $\tau : R \times G' \to G$ where R is a central torus of G, $G' = [G, G]$ is semi-simple, and the degree of this finite covering is again prime to p. Then the previous lifting argument applies equally to $\phi_x : \mathbb{G}_a \to G$ obtained as $\tau \circ \phi_{\tilde{x}}$ for any $\tilde{x} \in R \times G'$ with $\tau(\tilde{x}) = x$. □

The following definition appears related to the concepts of *saturation* as considered in [19] and of *exponential type* as considered in [21].

Definition 1.11. Let G be as a suitable affine algebraic group, let $H \subset G$ be a closed algebraic k-subgroup which is connected and smooth over k, and let $\overline{k}$ be an algebraic closure of k. Then we say that H *satisfies condition (S)* if for every $x \in H(\overline{k})$ of order p the map $\phi_x : \mathbb{G}_a \to G$ of Corollary 1.10 factors through H.

This condition (S) is not satisfied by all $H \subset G$. A simple example (suggested by G. Seitz) in which condition (S) is not satisfied is given by taking $G = SL_2 \times SL_2$, $H = SL_2$, and the embedding $H \to G$ given by $1 \times F$. Then $x \times x \in SL(2, \mathbb{F}_p) \times SL(2, \mathbb{F}_p)$ is an element with p^{th}-power equal to 1, but $\phi_{x\times x} : \mathbb{G}_a \to SL_2 \times SL_2$ does not factor through H, where x is the upper triangular unipotent matrix with a 1 in the $(1, 2)$-position.

Example 1.12. As observed in [18], if G is of classical type, $x \in G(k)$ of order p, and $e = x - 1$ as an endomorphism of the natural representation, then

$$(2)\quad \phi_x(t) = 1 + te + \frac{t(t-1)}{2}e^2 + \cdots + \frac{t(t-1)\cdots(t-p+1)}{(p-1)!}e^{p-1} \in G(k), \quad t \in k.$$

Thus, for such G, if H is either a parabolic subgroup of G or the unipotent radical of a parabolic, then H satisfies condition (S).

As observed in [19, §4], if we write

$$log(x) \;=\; \sum_{0<i<p} (-1)^{i+1} e^i/i, \quad e = x - 1, \quad x^p = 1,$$

then

$$(3)\qquad \phi_x(t) \;=\; exp(t \cdot logx) \;=\; \sum_{0\le i<p} \frac{(t \cdot logx)^i}{i!}.$$

We recall that the Coxeter number $h(G)$ of a simple algebraic group G over k is the height of the longest root plus 1 of the root system of $G_{\overline{k}}$. (Equivalently, we may consider the root system of some form of G split over k.) If $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is the set of simple roots of $G_{\overline{k}}$ and R^+ the set of positive roots for this root system, then $h(G) - 1 \;=\; max_{\alpha \in R^+}\{\sum n_i : \alpha = \sum n_i\alpha_i \in \Pi\}$. For a reductive group, we define $h(G)$ to be the maximum of the Coxeter numbers of some split form of each simple factor of $[G, G]$. Observe that the condition $p \ge h(G)$ implies both that p is good for G and that the degree of $G_{sc} \to G$ is prime to p.

The multiplicative property established below plays a crucial role in our construction of a map between support varieties for $G(\mathbb{F}_p)$ and for $G_{(r)}$. The hypothesis that $p \ge h(G)$ which we require to prove our comparison results appears in the proof of this proposition.

Proposition 1.13. *Let G be a suitable affine algebraic group. For any element $x \in G(k)$ with $x^p = 1$, let $\phi_x : \mathbb{G}_a \to G$ be as in Corollary 1.10. If $x, y \in G(k)$ with $x^p = y^p = 1$ and $[x, y] \;=\; 1$, then*

$$(4)\qquad \phi_x \cdot \phi_y \;=\; \phi_{xy} : \mathbb{G}_a \to G.$$

and

$$(5)\qquad \phi_x \cdot \phi_y \;=\; \phi_y \cdot \phi_x : \mathbb{G}_a \to G.$$

Proof. We may assume k is algebraically closed. Let $G \subset GL_N$ be an embedding defined over k, Let $B = U \cdot T \subset G$ be a split Borel subgroup, and let $B_N = U_N \cdot T_N \subset GL_N$ with $B \subset B_N$. Define filtrations on $U(k)$ (respectively, $U_N(k)$) by setting $F^iU(k)$ (resp., $F^iU_N(k)$) to be the subgroup generated by root subgroups U_α with α a positive root of G (resp., GL_N) of height $\geq i$. Then $U \subset U_N$ restricts to $F^iU \subset F^iU_N$.

Our hypothesis $p \geq h(G)$ implies that $F^pU(k) = 0$. Assume that $[x, y] = 1$. We readily check that if $x = 1 + e \in F^iU_N$ and if $y = 1 + f \in F^jU_N$, then $1 + ef \in F^{i+j}U_N$. This tells us that if $x = 1 + e$, $y = 1 + f \in G(k)$ satisfy $x^p = y^p = 1$, then $e^if^j = 0$ for $i + j \geq p$. The equality $\phi_{xy} = \phi_x \cdot \phi_y$ now follows from [19, Prop.9]. Thus,

$$\phi_x \cdot \phi_y = \phi_{x \cdot y} = \phi_{y \cdot x} = \phi_y \cdot \phi_x.$$

□

Let $\mathcal{U}$ and $\mathcal{U}_p$ (respectively, $\mathcal{N}$ and $\mathcal{N}_p$) denote the subvarieties of unipotent elements and p-unipotent elements of G (resp., subvarieties of nilpotent elements and p-nilpotent elements of $Lie(G)$). We conclude this section by recalling from [20, 3.1] under the hypotheses of Corollary 1.10 that there are isomorphisms, "Springer isomorphisms" , $\exp : \mathcal{N} \xrightarrow{\sim} \mathcal{U}$. The condition that $p \geq h(G)$ for the reductive group G implies that $\mathcal{N} = \mathcal{N}_p$ and $\mathcal{U}_p = \mathcal{U}$.

2. 1-parameter subgroups

Theorem 1.7 exhibits 1-parameter subgroups of G associated to elements of order p in $G(k)$. As established C. Bendel, A. Suslin, and the author in [21], [22], (infinitesimal) 1-parameter subgroups provide an alternate interpretation of cohomological invariants of $kG_{(r)}$-modules. In this section, we investigate further the role of 1-parameter subgroups in the representation theory of Frobenius kernels.

We use the familiar notation of $H^\bullet(G, k)$ to denote the commutative k-algebra $H^*(G, k) = Ext^*_G(k, k)$ if $p = 2$ and to denote the even dimensional subalgebra $H^{ev}(G, k) \subset H^*(G, k)$ if $p > 2$.

In the theorem below, the map of k-algebras (but not of Hopf algebras, for $r > 1$)

$$\epsilon : k[u]/u^p \rightarrow k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p), \quad u \mapsto u_{r-1}, \tag{6}$$

(where u_j is the distribution $(d/dt)^{(p^j)}$) makes its first appearance. As seen in the next section, ϵ provides the link between 1-parameter subgroups and π-points of $G_{(r)}$.

Theorem 2.1. [22] *Let G be an infinitesimal group scheme over k of height $\leq r$. Then the functor sending a commutative k-algebra A to the set of maps $\mu_A : \mathbb{G}_{a(r),A} \to G_A$ of group schemes over* $\operatorname{Spec} A$ *is represented by an affine k-scheme $V(G)$. Thus, a scheme-theoretic point of $V(G)$ with residue field K corresponds to a 1-parameter subgroup of the form $\mu_K : \mathbb{G}_{a(r),K} \to G_K$.*

The closed subspaces of $V(G)$ are the subsets of the form

$$\begin{aligned} V(G)_M &= \{\mu_K \in V(G) \,|\, \mu_K^*(M_K) \textit{ is not free as } \mathbb{G}_{a(r),K} - module\} \\ &= \{\mu_K \in V(G) \,|\, (\mu_K \circ \epsilon)^*(M_K) \textit{ is not free as } K[u]/u^p - module\} \end{aligned}$$

for some finite dimensional kG-module M.

There is a natural p-isogeny

$$\Phi : V(G) \;\to\; \operatorname{Spec} H^\bullet(G,k) \equiv |G| \tag{7}$$

with the property that $\Phi_G(V(G)_M) \;=\; Z(ann_{H^\bullet(G,k)} Ext^*_G(M,M)) \equiv |G|_M$ *for any finite dimensional* kG*-module* M*.*

In Theorem 2.1, the infinitesimal group scheme is assumed to have height $\leq r$, yet the notation $V(G)$ does not refer to r. This is justified by the observation that if G has height $\leq r$ then any 1-parameter subgroup $\mathbb{G}_{a(r+1),A} \to G_A$ factors uniquely through the projection $\mathbb{G}_{a(r+1),A} \to \mathbb{G}_{a(r+1),A}/\mathbb{G}_{a(1),A} \simeq \mathbb{G}_{a(r),A}$.

We recall that $V(G)$ admits a natural grading associated to the action of $\mathbb{G}_a$ on the domain $\mathbb{G}_{a(r)}$ of a 1-parameter subgroup $\mu : \mathbb{G}_{a(r)} \to G$:

$$\mathbb{G}_a \times V(G) \;\to\; V(G), \quad (s,\mu) \mapsto \mu(s \cdot -) \tag{8}$$

In the special case $G = \mathbb{G}_{a(r)}$, the result of s acting on $\mu : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ given by $t \mapsto a_0 t + a_1 t^p + \cdots a_{r-1} t^{p^{r-1}}$ is the 1-parameter subgroup given by $t \mapsto a_0 t + a_1 s^p t^p + \cdots + a_{r-1} s^{p^{r-1}} t^{p^{r-1}}$.

This non-linearity of the action of $\mathbb{G}_a$ on the 1-parameter subgroups of $\mathbb{G}_{a(r)}$ (parameterized by r-tuples $(a_0, \ldots, a_{r-1})$) can be a source of some confusion. As we see in the following example, k-linearity is retained for the 1-parameter subgroups of the form ϕ_x.

Example 2.2. Let G be a suitable affine algebraic group. Then the uniqueness property of $x \mapsto \phi_x$ implies the following linearity. For any $x \in G(k)$ of order p, the 1-parameter subgroup $\phi_x : \mathbb{G}_a \to G$ satisfies

$$\phi_{\phi_x(s)} \;=\; s \cdot \phi_x \;\equiv\; \phi_x(s \cdot -) \tag{9}$$

for any $s \in k$.

Example 2.3. ([21, 1.7,1.8]) Let $G = GL_N$ or a semi-simple algebraic group over k which is a product classical types, and let $\mathfrak{g} = Lie(G)$. Then a 1-parameter subgroup $\phi : \mathbb{G}_a \to G$ is (uniquely) of the form

$$t \mapsto exp(t\alpha_0) \cdot exp(t^p \alpha_1) \cdots exp(t^{p^{r-1}} \alpha_{(r-1)}), \tag{10}$$

where r is some positive integer, $\alpha_0, \ldots \alpha_{r-1}$ are pair-wise commuting p-nilpotent elements of $\mathfrak{g}$ with entries in k, and where $exp(\alpha) = 1 + \alpha + \frac{\alpha^2}{2} + \cdots + \frac{\alpha^{p-1}}{(p-1)!}$ (cf. (3)). Let $V_r(G)$ be the k-scheme representing 1-parameter subgroups of G of the form (10). For a given r, the restriction of such 1-parameter subgroups to $\mathbb{G}_{a(r)}$ determines an isomorphism

$$V_r(G) \;\stackrel{\sim}{\to}\; V(G_{(r)}),$$

identified in [21] as the scheme of r-tuples of p-nilpotent, pairwise commuting elements of $\mathfrak{g}$.

Proposition 2.4. *Let* G *be an algebraic group of classical type which is suitable (in the sense of Definition 1.9). Sending* $x \in \mathcal{U}_p(G)$ *to the 1-parameter subgroup* $\phi_x : \mathbb{G}_a \to G$ *determines a rational map*

$$\Phi : \mathcal{U}_p(G) \;\to\; V_1(G)$$

which is defined and injective on geometric points. In particular, if $\mathcal{U}_p$ is normal (for example, if $p \geq h(G)$), then Φ is an injective morphism and thus defines the injective morphism $\Phi : \mathcal{U}_p(G) \to V(G_{(r)})$ for any $r > 0$.

More generally, if G is a suitable affine algebraic group equipped with an embedding $G \subset GL_N$ with p not dividing N, then sending $x \in \mathcal{U}_p(G)$ to the 1-parameter subgroup $\phi_x : \mathbb{G}_a \to G$ determines a rational map $\Phi : \mathcal{U}_p(G) \to V_1(GL_N)$. This rational map is injective on geometric points, sending a geometric point of $\mathcal{U}_p(G)$ to a geometric point of $V_1(GL_N)$ whose restriction to any $V(GL_{N(r)})$ lies in the image of $V(G_{(r)})$.

Furthermore, for any closed algebraic subgroup $H \subset G$ which satisfies condition (S) of Definition 1.11, $\Phi \circ i$ sends geometric points of $\mathcal{U}_p(H)$ to K-points of $V_1(H)$.

Proof. We first assume that G is a suitable affine algebraic group of classical type. To define Φ as a rational map on the irreducible variety $\mathcal{U}_p(G)$, we must give the image of the generic point of $\mathcal{U}_p(G)$. If $\eta : \operatorname{Spec} K \to \mathcal{U}_p(G)$ is the generic point corresponding to some $x_\eta \in G(K)$ with $x^p = 1$, we define $\Phi(\eta) : \operatorname{Spec} K \to V_1(G)$ to be 1-parameter subgroup $\phi_{x_\eta} : \mathbb{G}_{a,K} \to G_K$. We show that Φ is a morphism, by showing that it is induced by a functor on finitely generated commutative k-algebras R. Namely, $\mathcal{U}_p(G)$ applied to R is the subset of $G(R)$ consisting of elements with p^{th}-power 1. For any such $x \in G(R)$, write $e = x - 1$ as an endomorphism of the natural module (over R) for the classical group G, and define $\Phi(u) \in Hom(\operatorname{Spec} R, V_1(G))$ to be the 1-parameter subgroup given by formula (2). Since $\phi_x(t) = exp(t \cdot logx)$ as in (3), Φ is injective when restricted to $G_{(r)}$ for any $r > 0$.

More generally, let G be an arbitrary suitable affine algebraic group. Choose some embedding $G \subset GL_N$ with p not dividing N. Let $\operatorname{Spec} K \to \mathcal{U}_p(G)$ be a geometric point, corresponding to an element $x \in G(K)$ with K algebraically closed (indeed, K perfect would suffice). Then Corollary 1.10 associates to x a uniquely determined $\phi_x : \mathbb{G}_{a,K} \to G_K$. Since the composition of ϕ_x with $i : G_K \subset GL_{N,K}$ must be the 1-parameter subgroup of $GL_{N,K}$ associated to $x \in G(K) \subset GL(N,K)$, the restriction of $i \circ \phi_x$ to $\mathbb{G}_{a,K}$ must be the image of x under the composition $\phi_x \circ i : \mathcal{U}_p(G) \to \mathcal{U}_p(GL_N) \to V_1(GL_N)$. In particular, Φ is injective on geometric points of $\mathcal{U}_p(G)$.

If a rational map is defined at every geometric point and has normal domain, then it is a morphism. As established in [1, §9], $\mathcal{U}$ is normal provided that G satisfies the hypotheses of Corollary 1.10, so that $\mathcal{U}_p$ is normal provided that $p \geq h(G)$.

This argument applies without change to any closed algebraic subgroup $H \subset G$ satisfying condition (S) of Definition 1.11. □

The following definition is justfied by Proposition 1.6 which tells us that for a given finite dimensional $\mathbb{G}_a$-module M the distributions $(d/dt)^{(i)}$ vanish on M for $i >> 0$.

Definition 2.5. Let G be a suitable affine algebraic group defined over an algebraically closed field k. Let $\eta : \operatorname{Spec} K \to \mathcal{U}_p(G)$ denote the generic point of the p-unipotent variety of G, and let $\overline{\eta} : Spec\overline{K} \to \mathcal{U}_p(G)$ be any geometric point lying over η. For any finite dimensional rational G-module M, we define $s(M)$ to be the least integer s such that $(d/dt)^{(i)}$ vanishes on $\phi^*_{x_{\overline{\eta}}}(M_{\overline{K}})$ for all $i \geq s$. We call $s(M)$ the *p-nilpotent degree* of M.

If G is defined over $\mathbb{F}_q$, then we define $s_{\mathbb{F}_q}(M)$ to be the the least integer s such that $(d/dt)^{(i)}$ vanishes on $\phi^*_x(M)$ for all $i \geq s$ and all $x \in G(\mathbb{F}_q)$ with $x^p = 1$.

We initiate an investigation of this p-nilpotent degree of M.

Proposition 2.6. *Let G be as in Definition 2.5; in particular, we assume k is algebraically closed. For any finite dimension rational G-module M, the integer $s(M)$ of Definition 2.5 is the least integer s such that for all k-points $x \in \mathcal{U}_p(G)$ and all $i \geq s$ the distribution $(d/dt)^{(i)}$ vanishes on $\phi_x^*(M)$.*

In particular, if G is defined over $\mathbb{F}_q$, then $s_{\mathbb{F}_q}(M) \leq s(M)$.

Proof. We first assume G is of classical type, and set L equal to an algebraic closure of the field of fractions of $\mathcal{U}_p(G)$. Let $\theta_x : \operatorname{Spec} R_x \to \mathcal{U}_p(G)$ be the strict Hensel local ring at the point $x : \operatorname{Spec} k \to \mathcal{U}_p(G)$, so that L is (isomorphic to) the field of fractions of R_x. By Proposition 2.4, θ_x determines $\phi_{\theta_x} : \mathbb{G}_{a,R_x} \to G_{R_x}$. The $k[G]$-comodule structure on M determines the coproduct $\nabla_{R_x} : M \otimes R_x \to M \otimes R_x[t]$. Clearly, if the image of $\nabla_{R_x} \otimes L$ inside $M \otimes L[t]$ lies in $M \otimes L[t]_{<s}$, then the image of $\nabla_{R_x} \otimes k$ lies in $M \otimes k[t]_{<s}$. Applying the identification of the action of $(d/dt)^{(i)}$ given in Proposition 1.6, we conclude that $s(M)$ is greater or equal to the least integer s such that for all k-points $x \in \mathcal{U}_p(G)$ and all $i \geq s$ the distribution $(d/dt)^{(i)}$ vanishes on $\phi_x^*(M)$.

On the other hand, let $\nabla_L : M_L \to M_L \otimes_L L[t]$ be the $L \otimes k[G]$-comodule structure on $M_L = M \otimes L$ induced by the $k[G]$-comodule structure on M and set

$$\nabla_L(m \otimes 1) \;=\; \sum_{j=0}^{s(M)-1} a_i m_j \otimes t^i \in M \otimes L[t].$$

Choose some k-rational point $x \in \mathcal{U}_p(G)$ with $a_{s(M)-1}$ a unit of R_x. Applying once again the identification of the action of $(d/dt)^{(i)}$ given in Proposition 1.6, we conclude that the action of $(d/dt)^{(s(M)-1)}$ on $\phi_x^*(M)$ is given by sending $m \in M$ to $\overline{a}_{s(M)-1} m_{s(M)-1}$, where $\overline{a}_{s(M)-1}$ is the (necessarily, non-zero) image of $a_{s(M)-1)}$ under $R_x \to k$. Combining this non-vanishing of $(d/dt)^{(s(M)-1)}$ on $\phi_x^*(M)$ with the previous paragraph, we conclude that $s(M)$ equals the least integer s such that for all k-points $x \in \mathcal{U}_p(G)$ and all $i \geq s$ the distribution $(d/dt)^{(i)}$ vanishes on $\phi_x^*(M)$.

We now consider any reductive G satisfying the condition of Definition 2.5 and choose some closed embedding $i : G \subset GL_N$. As above, we consider $\theta_x : \operatorname{Spec} R_x \to \mathcal{U}_p(G)$ and observe that the composition $i \circ \theta_x : \operatorname{Spec} R_x \to \mathcal{U}_p(GL_N)$ determines $\phi_{\theta_x} : \mathbb{G}_{a,R_x} \to GL_{N,R_x}$. By Proposition 2.4, $\phi_{\theta_x} \otimes_{R_x} L$ factors through G_L and ϕ_{θ_x} factors through G, so that ϕ_{θ_x} factors through G_{R_x}. The proof now proceeds as above for G classical. $\square$

The following bound on $s(M)$ is suggested by the discussion of [3, 4.6].

Proposition 2.7. *Let G be a suitable affine algebraic group over an algebraically closed field k. Let M be a rational G-module and d be a positive integer chosen sufficiently large that every high weight λ of M as a G-module satisfies*

$$\sum_{j=1}^{\ell} \langle \lambda, \omega_j^\vee \rangle \;<\; d, \tag{11}$$

where $\omega_1, \ldots, \omega_\ell$ are the fundamental dominant weights of G with respect to some split Borel subgroup. Then

$$s(M) \quad\leq\quad 2d. \tag{12}$$

Proof. Let $x \in G(k)$ satisfy $x^p = 1$, let $\phi_x : \mathbb{G}_a \to G$ be Seitz's 1-parameter subgroup associated to x, and let $A_x \subset G$ be a good A_1 containing the image of ϕ_x. By Dynkin's theorem [4, 5.6.6] as observed in [3, 4.6.2], the T_x-weight $wt_x(v_\mu)$ of a T-eigenvector v_μ of a rational G-module M of weight μ satisfies

$$-2d < -2\sum_{j=1}^{\ell}\langle\lambda,\omega_j^\vee\rangle \ \leq \ wt_x(v_\mu) \ \leq \ 2\sum_{j=1}^{\ell}\langle\lambda,\omega_j^\vee\rangle < 2d.$$

Let T_x denote the torus of A_x, and choose a split Borel subgroup $B = U\cdot T$ with $T_x \subset T$. Thus, T_x-eigenspaces of M are sums of T-eigenspaces of M, so that the above inequalities apply as well to T_x-weight of any T_x-eigenvector v_μ.

The coproduct $M \to M \otimes k[G]$ is T_x-equivariant, (i.e., is a map of $Dist(T_x)$-modules) provided $k[G]$ is equipped with the adjoint action, so that

$$M \ \to \ M \otimes k[G] \ \to \ M \otimes k[A_x] \ \to \ M \otimes k[U_x] \equiv M \otimes k[X] \tag{13}$$

preserves weights with respect to T_x. Thus if $m \in M$ is a T_x-eigenvector of weight ℓ and if the composition of (13) sends m to $\sum_i m_i \otimes X^i$, then m_i has T_x-weight $\ell + 2i$ (since X as an element of the A_x-module $k[X]$ has weight -2).

We conclude that if v_μ is a T_x-eigenvector of M of weight μ, then $(d/dt)^{(i)}v_\mu$ is a T_x-eigenvector with T_x-weight satisfying

$$-2d < wt_x((d/dt)^{(i)}v_\mu) < 2d, \quad -2d + 2i < wt_x((d/dt)^{(i)}v_\mu).$$

We thus conclude that $(d/dt)^{(i)}v_\mu = 0$ for any $i \geq 2d$, so that $s(M) \leq 2d$. □

As expected, $s(M)$ depends only upon conjugacy classes of p-nilpotent elements as verified in the next proposition.

Proposition 2.8. *Let G be a suitable affine algebraic group over an algebraically closed field k, and let M be a rational G-module. Choose a representative x_C of each conjugacy class C of p-unipotent elements of $G(k)$. Then $s(M)$ is the least integer s such that the distribution $(d/dt)^{(i)}$ vanishes on $\phi_{x_C}^*(M)$ for all $i \geq s$ and all x_C.*

Assume, in addition, that G is defined over $\mathbb{F}_q$ and choose a representative $x_{C'}$ of each conjugacy class C' of p-unipotent elements of $G(\mathbb{F}_q)$. Then $s_{\mathbb{F}_q}(M)$ is the least integer s such that the distribution $(d/dt)^{(i)}$ vanishes on $\phi_{x_{C'}}^*(M)$ for all $i \geq s$ and all $x_{C'}$.

Proof. By Proposition 2.6, it suffices to verify that if $x, y : \operatorname{Spec} k \to \mathcal{U}_p(G) \subset G$ are conjugate with $y = gxg^{-1}$, then $(d/dt)^{(i)}$ vanishes on $\phi_x^*(M)$ for all $i \geq s$ if and only if $(d/dt)^{(i)}$ vanishes on $\phi_y^*(M) = \phi_x(M^g)$ for all $i \geq s$. This follows immediately from the observation that for any $g \in G(k)$ the rational G-module M is isomorphic to its g-conjugate M^g.

If G is defined over $\mathbb{F}_q$ and $x \in G(\mathbb{F}_q)$, then Seitz shows in [18, 9.1] that ϕ_x is defined over $\mathbb{F}_q$. If $g \in G(\mathbb{F}_q)$, then the g-conjugate M^g of M is isomorphic to M as a $G(\mathbb{F}_q)$-module, so that $\phi_{x*}((d/dt)^{(i)})$ vanishes on M if and only if $\phi_{gxg^{-1}*}((d/dt)^{(i)})$ vanishes on M. □

We next consider the behavior of the p-nilpotent degree $s(M)$ with respect to certain operations on rational G-modules. This enables us to provide further upper bounds for $s(M)$.

Proposition 2.9. *Let G be a suitable affine algebraic group defined over an algebraically closed field k, and let M, N be finite dimensional rational G-modules.*

1. *If N is a submodule of M, then $s(N) \leq s(M)$.*
2. *$s(M) = s(M^{\#})$, where $M^{\#} = Hom_k(M,k)$ is the k-linear dual of M.*
3. *$s(M \oplus N) = max\{s(M), s(N)\}$.*
4. *$s(M \otimes N) \leq s(M) + s(N)$.*
5. *For any $n > 0$, each of $s(M^{\otimes n})$, $s(S^n(M))$, $s(\Lambda^n(M))$ is less than or equal to $n \cdot s(M)$.*
6. *If G is defined over $\mathbb{F}_p$, then $s_{\mathbb{F}_p}(M^{(1)}) = p \cdot s_{\mathbb{F}_p}(M)$, where $M^{(1)}$ is the Frobenius twist of M.*

Proof. Statement (1) is immediate from the observation that the action of $(d/dt)^{(i)}$ on $\phi_x^*(N_{k(x)})$ is the restriction of the action on $\phi_x^*(M_{k(x)})$. Statements (2) and (3) follow from the fact that $\phi_x^*(-)$ from rational G-modules to rational $\mathbb{G}_{a,k(x)}$-modules commutes with taking duals and direct sums.

If M, N are rational $\mathbb{G}_a$-modules, then the action of $(d/dt)^{(\ell)}$ on $M \otimes N$ is given by $\sum_{i+j=\ell}(d/dt)^{(i)} \otimes (d/dt)^{(j)}$. Thus, statements (4) and (5) follow from the observation that $\phi_x^*(-)$ also commutes with tensor products, symmetric powers, and exterior powers.

To prove statement (6), observe that if G is defined over $\mathbb{F}_p$ and $x \in G(\mathbb{F}_p)$, then $\phi_x^*(-)$ commutes with the Frobenius twist. □

In the following example, we see that the bound of Proposition 2.7 is far from sharp.

Example 2.10. Let $M = S_\lambda$ be the irreducible SL_2-module of high weight λ, $0 \leq \lambda < p$. Then S_λ is the natural representation of SL_2 on $k[x,y]_\lambda$, homogeneous polynomials in two variables of degree λ. Thus, $s(S_\lambda) \leq \lambda$ by Proposition 2.9(5), whereas Proposition 2.7 gives the bound $s(S_\lambda) \leq 2(\lambda+1)$. If M is an arbitrary irreducible rational SL_2-module, then $M \simeq S(\lambda_0) \otimes S(\lambda_1)^{(1)} \otimes \cdots \otimes S(\lambda_r)^{(r)}$ so that Proposition 2.9(4) tells us that $s(M)$ satisfies $s(M) \leq \sum_{i=0}^r p^i \lambda_i$.

3. π-points and 1-parameter subgroups

Our perspective on support varieties is that developed by the author and J. Pevtsova in [7], [8]. The advantage of this perspective is that it gives a uniform treatment of support varieties for finite groups and Frobenius kernels. In Theorem 3.5, we define the natural map

$$\Psi_G : \Pi(G(\mathbb{F}_p)) \ \to \ (\Pi(G_{(r)}))/G(\mathbb{F}_p)$$

whose restriction to abelian unipotent groups $\underline{E}$ associated to elementary abelian p-groups $E \subset G(\mathbb{F}_p)$ is very explicit. Namely, a π-point of $E(\mathbb{F}_p)$ of the form $\alpha_x : k[u]/u^p \to kE$, $u \mapsto [x]-1$ is sent to the π-point $\beta_x : k[u]/u^p \to k\underline{E}_{(r)}$, $u \mapsto \phi_x(u_0 + \cdots + u_{r-1})$ associated to the 1-parameter subgroup $\phi_x \circ \sigma$.

For the reader's convenience, we recall the definition of π-points of G and the π-point scheme $\Pi(G)$ for a finite group scheme G over k.

Definition 3.1. ([8]) Let G be a finite group scheme over k.

1. A π-point of G is a (left) flat map of K-algebras $\alpha_K : K[u]/u^p \to KG$ for some field extension K/k with the property that there exists a unipotent

abelian closed subgroup scheme $i : C_K \subset G_K$ defined over K such that α_K factors through $i_* : KC_K \to KG_K = KG$.

(2) Two π-points $\alpha_K : K[u]/u^p \to KG$, $\beta_L : L[u]/u^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if they satisfy the following condition for all finite dimensional kG-modules M: $\alpha_K^*(M_K)$ is free as $K[u]/u^p$-module if and only if $\beta_L^*(M_L)$ is free as an $L[u]/u^p$-module.

The Π-point scheme $\Pi(G)$ is a scheme of finite type over k whose points are equivalence classes of π-points of G. A subset $Y \subset \Pi(G)$ is closed if and only if there exists a finite dimensional kG-module M such that Y equals

$$\Pi(G)_M = \{[\alpha_K] \mid \alpha_K^*(M_K) \text{ is not free as a } K[u]/u^p\text{-module}\}.$$

The scheme structure on $\Pi(G)$ is given in [8] in terms of the stable module category of kG-modules.

To relate the scheme of 1-parameter subgroups $V(G)$ and the scheme of 1-parameter π-points $\Pi(G)$, we recall the following theorem

Theorem 3.2. [8, 7.5] *Let G be a finite group scheme over k. Then there is a natural isomorphism of schemes over k*

$$\operatorname{Proj} H^\bullet(G,k) \;\simeq\; \Pi(G).$$

Moreover, for any finite dimensional kG-module M, this isomorphism restricts to

$$\operatorname{Proj} H^\bullet(G,k)/Ann_{H^\bullet(G,k)}(Ext^*(M,M)) \;\simeq\; \Pi(G)_M.$$

Combining Theorem 3.2 and the p-isogeny (7) of Theorem 2.1, we conclude the existence of the natural p-isogeny for G an infinitesimal group scheme of height $\leq r$

$$\Phi : \operatorname{Proj} k[V(G)] \;\to\; \operatorname{Proj} H^\bullet(G,k) \;\xrightarrow{\sim}\; \Pi(G) \tag{14}$$

which sends $\phi : \mathbb{G}_{a(r)} \to G$ to the π-point $\phi_* \circ \epsilon : k[u]/u^p \to k\mathbb{G}_{a(r)} \to kG$, where

$$\epsilon : k[u]/u^p \;\to\; k\mathbb{G}_{a(r)}, \quad u \mapsto u_{r-1}. \tag{15}$$

In the following proposition, we pre-compose ϕ_x with the distinguished 1-parameter subgroup of $\mathbb{G}_{a(r)}$:

$$\sigma : \mathbb{G}_{a(r)} \;\to\; \mathbb{G}_{a(r)}, \quad t \mapsto t + t^p + \cdots + t^{p^{r-1}}. \tag{16}$$

The role of σ is that $\sigma_*(u_{r-1}) = \sum_{i=0}^{r-1} u_i \in k\mathbb{G}_{a(r)}$, so that

$$((\phi \circ \sigma)_* \circ \epsilon)(u) \;=\; \phi(u_0 + \ldots + u_{r-1}).$$

Proposition 3.3. *Let G be a suitable affine algebraic group. Consider an elementary abelian p-subgroup $E \subset G(\mathbb{F}_p)$ of rank s, let $\{1 = y_0, y_1, \ldots, y_{p^s-1}\}$ be a listing of the elements of E, and let $\mathbb{G}_a^{\times s} \cong \underline{E} \subset G$ be the unipotent abelian subgroup generated by the $\phi_{y_j} : \mathbb{G}_a \to G$.*

Define the k-linear map $\mathcal{L} : \operatorname{Rad}(kE) \to \operatorname{Rad}(k\underline{E}_{(r)})$ by sending $[y_i] - 1$ to $(\phi_{y_i} \circ \sigma)_*(u_{r-1})$:

$$\mathcal{L}(\sum_{i=1}^{p^s-1} a_i([y_i]-1)) \;\equiv\; (\sum_{i=1}^{p^s-1} a_i(\phi_{y_i} \circ \sigma))_*(u_{r-1}) \;=\; \sum_{i=1}^{p^s-1} a_i(\phi_{y_i})_*(u_0 + \cdots + u_{r-1}) \tag{17}$$

Then (17) determines a k-linear map

$$\tilde{\mathcal{L}} : \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE) \;\to\; \operatorname{Rad}(k\underline{E}_{(r)}). \tag{18}$$

Moreover, the induced map $\overline{\mathcal{L}} : \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE) \ \to\ \mathrm{Rad}(k\underline{E}_{(r)})/\mathrm{Rad}^2(k\underline{E}_{(r)})$ *is injective for all* $r \geq 1$ *and is an isomorphism if* $r = 1$.

Proof. To prove that $\mathcal{L}$ of (17) determines $\tilde{\mathcal{L}}$ as in (18), we must show that $\mathcal{L}$ vanishes on $\mathrm{Rad}^2(kE)$. Since elements of the form $([y_i]-1)([y_j]-1)$ span $\mathrm{Rad}^2(kE)$, it suffices to prove that $\mathcal{L}$ vanishes on elements of this form. Observe that

$$([y_i] - 1)([y_j] - 1) = ([y_i + y_j] - 1) - ([y_i] - 1) - ([y_j] - 1),$$

so that $\tilde{\mathcal{L}}(([y_i] - 1)([y_j] - 1))$ equals

$$(\phi_{y_i+y_j} - \phi_{y_i} - \phi_{y_j})_*(u_0 + \cdots + u_{r-1}).$$

Thus, the required vanishing follows from (1.13).

For $r = 1$, $\overline{\mathcal{L}}$ is a k-linear map between k-vector spaces of the same dimension carrying a basis on the left to a basis on the right. For $r > 1$, the the map $\overline{\mathcal{L}}$ has a section provided by projecting down to $V(\underline{E}_{(1)})$. □

Corollary 3.4. *Retain the notation and hypotheses of Proposition 3.3. Then*

$$\overline{\mathcal{L}} : \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE) \ \to\ \mathrm{Rad}(k\underline{E}_{(r)})/\mathrm{Rad}^2(k\underline{E}_{(r)}) \tag{19}$$

naturally determines an embedding of (Zariski) spaces of equivalence classes of π*-points*

$$\Psi_{\underline{E}} : \Pi(E(\mathbb{F}_p)) \ \to\ \Pi(\underline{E}_{(r)}). \tag{20}$$

So defined, $\Psi_{\underline{E}}$ *sends the equivalence class of the* π*-point*

$$\alpha_x : K[u]/u^p \to K\underline{E}(\mathbb{F}_p),\ u \mapsto x - 1$$

to the equivalence class of the π*-point*

$$\beta_x = (\phi_x \circ \sigma)_* \circ \epsilon : K[u]/u^p \to K\underline{E}_{(r)}$$

for any $1 \neq x \in E(\mathbb{F}_p)$ *of order* p*, where* ϵ *and* σ *are given in (6) and (16).*

For $r = 1$*, this embedding is an isomorphism.*

Proof. For an elementary abelian p-group E, equivalence classes of π-points in $\Pi(E)$ are represented by maps of K-algebras $K[u]/u^p \to KE$ sending u to some element of $\mathrm{Rad}(KE)\backslash \mathrm{Rad}^2(KE)$; the equivalence relation on such maps is generated by pairs of maps differing by a non-zero scalar multiple, pairs of maps sending u to elements of $\mathrm{Rad}(kE)$ differing by an element of $\mathrm{Rad}^2(KE)$, and pairs of maps which become equal after a common base extension (see [8, 7.5]). Since $k\underline{E}_{(r)}$ is isomorphic to the group algebra of an elementary abelian group (of rank equal to r times the rank of E), we have the same description of $\Pi(\underline{E}_{(r)})$ in terms of elements of $\mathrm{Rad}(K\underline{E}_{(r)}))/\mathrm{Rad}^2(K\underline{E}_{(r)}))$.

Thus, $\overline{\mathcal{L}}$ (and its base extensions to fields K/k) naturally induces $\Psi_{\underline{E}} : \Pi(E) \to \Pi(\underline{E}_{(r)})$. By (17), $\Psi_{\underline{E}}$ sends the equivalence class of the π-point α_x corresponding to $[x] - 1 \in \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$ to the equivalence class of the π-point $k[u]/u^p \to k\underline{E}_{(r)}$ sending u to $\phi_{x*}(u_0 + \cdots u_{r-1}) \ = \ ((\phi_x \circ \sigma)_* \circ \epsilon)(u)$; thus, $\Psi_{\underline{E}}([\alpha_x) = [\beta_x]$ as asserted.

By Propositions 18 and 3.6, $\Psi_{\underline{E}}$ is an isomorphism for $r = 1$ and an embedding for $r \geq 1$. □

The following theorem is an extension/refinement of [3, Thm.4] and [7, 5.8]. Observe that if G is an affine algebraic group defined over $\mathbb{F}_q$, then $G(\mathbb{F}_q)$ naturally acts (by conjugation) on G and thus on $H^\bullet(G_{(r)}, k)$ for any $r \geq 1$ and thus on $\Pi(G_{(r)})$.

Theorem 3.5. *Let G be a suitable affine algebraic group, and let $H \subset G$ be a connected, smooth, closed algebraic subgroup satisfying condition (S) (cf. Definition 1.11; for example take $H = G$). Assume that $H \subset G$ is defined over $\mathbb{F}_p$. For any $r > 0$, there is a well-defined embedding of (Zariski) spaces of equivalence classes of π-points*

$$\Psi_H : \Pi(H(\mathbb{F}_p)) \ \to \ \Pi(H_{(r)})/H(\mathbb{F}_p), \tag{21}$$

whose restriction $\Psi_{\underline{E}}$ to any elementary abelian p-group $E \subset H(\mathbb{F}_p)$ is given by Corollary 3.4.

Proof. As defined in Corollary 3.4, the restriction to $\Pi(E')$ of $\Psi_{|\underline{E}}$ clearly equals $\Psi_{|\underline{E}'}$ whenever $E' < E$.

Quillen's stratification theorem (see [17], [7, 3.6]) implies that

$$\Pi(H(\mathbb{F}_p)) \ \simeq \ \varinjlim_{E < H(\mathbb{F}_p)} \ \Pi(E),$$

where the colimit is indexed by the category whose objects are elementary abelian p-subgroups of $H(\mathbb{F}_p)$ and whose maps are compositions of inclusions and conjugations by elements of $H(\mathbb{F}_p)$. (Thus the π-point $\alpha_x : k[u]/u^p \to kE(\mathbb{F}_p) \subset kH(\mathbb{F}_p), u \mapsto x - 1$ is equivalent to the π-point $\alpha_{x^h} : k[u]/u^p \to kE^h(\mathbb{F}_p) \subset kH(\mathbb{F}_p), u \mapsto x^h - 1$ for any $h \in H(\mathbb{F}_p)$.)

Applying the uniqueness of $x \mapsto \phi_x$, we conclude that the action of $h \in H(\mathbb{F}_p)$ sends $\phi_x : \mathbb{G}_{a(r)} \to H$ to $\phi_{x^h} : \mathbb{G}_{a(r)} \to H$. Thus, the conjugate α_{x^h} of α_x is mapped to the conjugate by h of β_x. We conclude that the colimit of the maps $\Psi_{\underline{E}}$ induces the continuous, injective map $\Psi_H : \Pi(H(\mathbb{F}_p)) \to \Pi(H_{(r)})/H(\mathbb{F}_p)$. □

It is natural to ask whether the map Ψ of (21) is a morphism of schemes. To check this, one would have to investigate more carefully the scheme structure of $\Pi(G(\mathbb{F}_p)) \simeq \operatorname{Proj} H^\bullet(G(\mathbb{F}_p), k)$. With this question in mind, we investigate further the operation of sending a 1-parameter subgroup $\phi : \mathbb{G}_{a(r)} \to G_{(r)}$ to the π-point $\phi_* \circ \epsilon : k[u]/u^p \to kG_{(r)}, \ u \mapsto \phi(u_{r-1})$.

We recall from [9, 2.2] the global p-nilpotent operator

$$\Theta_G : k[G] \to k[V(G)] \tag{22}$$

associated to an infinitesimal group scheme G of height $\leq r$. This is a k-linear functional, but not a homomorphism of algebras. We can identify Θ_G (as in [9, 2.2.2]) with the image of u under the composition

$$k[u]/u^p \ \stackrel{\epsilon\otimes 1}{\to} \ k\mathbb{G}_{a(r)} \otimes k[V(G)] \ \stackrel{\mathcal{U}_{G,*}}{\to} \ kG \otimes k[V(G)] \tag{23}$$

where $\mathcal{U}_{G,*}$ has the property that its base change from $k[V(G)]$ to some commutative k-algebra A is the 1-parameter subgroup $\phi_A : \mathbb{G}_{(a(r),A} \to G_A$ represented by that $k[V(G)] \to A$.

Proposition 3.6. *Let G be an infinitesimal group scheme of height $\leq r$. Consider the map of commutative k-algebras $S^*(k[G]) \ \to \ k[V(G)]$ induced by Θ_G of (22), with the corresponding map of affine schemes $\mathcal{S} : V(G) \to kG$. This map sends an A-valued point of $V(G)$ given by $\phi_A : \mathbb{G}_{a(r),A} \to G_A$ to $\phi_A(u_{r-1}) \in A \otimes kG$.*

Furthermore, $S^*(k[G]) \to k[V(G)]$ is a map of graded algebras of degree p^{r-1}, where $k[V(G)]$ is graded as in [21, 1.23]*; thus the associated morphism*

$$\mathcal{S} : V(G) \to kG$$

has homogeneous degree p^{r-1}. In the special case $G = \mathbb{G}_{a(r)}$, if $\phi_{a_0,\dots,a_{r-1}} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$, then

(24)

$$\mathcal{S}(\phi) \;=\; (\phi_{a_0,\dots,a_{r-1}})(u_{r-1}) \;=\; a_{r-1}u_0 + a_{r-2}^p u_1 + \cdots + a_0^{p^{r-1}} u_{r-1} + g(u_0,\dots,u_{r-1})$$

where g is a polynomial in $\{u_0,\dots,u_{r-1}\}$ with vanishing constant and linear terms, and where u_i is given homogeneous degree p^i.

More generally, if $G = \underline{E}_{(r)}$ for some abelian unipotent group $\underline{E} \simeq \mathbb{G}_a^s$, then $\mathcal{S}$ induces

$$\overline{\mathcal{S}} : V(\underline{E}_{(r)}) \to \mathrm{Rad}(k\underline{E}_{(r)})/\,\mathrm{Rad}^2(k\underline{E}_{(r)}) \tag{25}$$

which can be identified with the p-isogeny $\Phi : V(\underline{E}_{(r)}) \;\to\; \mathrm{Spec}(H^\bullet(\underline{E}_{(r)},k)_{red})$ of Theorem 2.1 using the natural isomorphism

$$\mathrm{Rad}(k\underline{E}_{(r)})/\,\mathrm{Rad}^2(k\underline{E}_{(r)}) \;\simeq\; \mathrm{Spec}(H^\bullet(\underline{E}_{(r)},k)_{red}).$$

Proof. To check that the image under $\mathcal{S}$ of the A-valued point $\phi_A : \mathbb{G}_{a(r),A} \to G_A$ of $V(G)$ equals image of u under the composition $\phi_A \circ \epsilon : k[u]/u^p \to AG_{a(r),A} \to AG_A$, we simply specialize (23) along this point.

As proved in [9, 2.10], Θ_G as a k-linear functional is homogeneous of degree p^{r-1}. This is equivalent to the statement that the induced map $S^*(k[G]) \to k[V(G)]$ is a map of graded algebras of degree p^{r-1}. The formula (24) is given in the proof of [22, 6.5].

Since $(\phi_A \circ \epsilon)(u) \in \mathrm{Rad}(AG)$, $\mathcal{S}$ factors through $\mathrm{Rad}(kG) \subset kG$. The identification of $\overline{\mathcal{S}} : V(\underline{E}_{(r)}) \to \mathrm{Rad}(k\underline{E}_{(r)})/\,\mathrm{Rad}^2(k\underline{E}_{(r)}) \simeq H^\bullet(\underline{E}_{(r)},k)_{red}$ for $\underline{E} \simeq \mathbb{G}_a^{\times s}$ with Φ of Theorem 2.1 is given in [21, 1.14, 1.15]. (In making this comparison and comparing degrees of the corresponding maps of graded algebras, it is useful to recall that the Bockstein $\beta : H^1(\mathbb{Z}/p,k) \to H^2(\mathbb{Z}/p,k)$ as a map of schemes is of degree p). □

4. Comparing actions of $G(\mathbb{F}_p)$ and $G_{(r)}$ at π-points

In Theorem 4.5, we compare support varieties of a rational G-module when restricted to $G(\mathbb{F}_p)$ and $G_{(r)}$. After recalling maximal Jordan types and the non-maximal support varieties, we provide in Theorem 4.11 a stronger result which involves the comparison of maximal Jordan types.

The following elementary proposition is the key to our comparison of actions of $G(\mathbb{F}_p)$ and $G_{(r)}$ on a rational G-module.

Proposition 4.1. *Let G be a connected affine algebraic group over k and M a rational G-module given by $\rho : G \to GL_n$, where $n = dim(M)$. Let $\phi : \mathbb{G}_a \to G$ be a 1-parameter subgroup and set $x = \phi(1) \in G(k)$. Then the action of x on M equals that of the action of $(\rho\circ\phi)_*(\sum_{i\geq 0}(d/dt)^{(i)}) \in End_k(M)$, where $(d/dt)^{(i)}$, $i \geq 0$ are the "standard" distributions on $\mathbb{G}_a$ supported at 0.*

Proof. The action of x on M is equal to that of $1 \in \mathbb{G}_a(k)$ on $\phi^*(M)$. Let $\nabla_{M,\phi} : M \to M \otimes k[t]$ denote the composition

$$\nabla_{M,\phi} = (1\otimes\phi^*)\circ(1\otimes\rho^*)\circ\nabla_n :\; M \to M\otimes k[GL_n] \to M\otimes k[G] \to M\otimes k[\mathbb{G}_a]$$

which defines the rational action of $\mathbb{G}_a$ on $\phi^*(M)$, where ∇_n is the standard comodule action on the natural defining representation for GL_n. The action of $1 \in \mathbb{G}_a(k)$ on $m \in \phi^*(M)$ is given by evaluating $\nabla_{M,\phi}(m) = \sum_i m_i \otimes t^i$ at $t = 1$. In other words, this action is given by applying the distribution $\sum_{i\geq 0}(d/dt)^{(i)} \in Dist(\mathbb{G}_a)$ to $\nabla_{M,\phi}(m)$.

By functoriality, this action is given by evaluating $(\rho \circ \phi)_*(\sum_{i\geq 0}(d/dt)^{(i)}) \in Dist(GL_n)$ on $\nabla_n(m) \in M \otimes k[GL_n]$. By definition, this is the action of $(\rho \circ \phi)_*(\sum_{i\geq 0}(d/dt)^{(i)})$ viewed as an element of $End_k(M)$. □

We now determine a first relationship between the actions of p-nilpotent elements of $G(k)$ and actions of infinitesimal 1-parameter subgroups of G on rational G-modules.

Proposition 4.2. *Let G be an algebraic group over k and M be a finite dimensional rational G-module, given by $\rho : G \to GL_n$. Consider a 1-parameter subgroup $\phi : \mathbb{G}_a \to G$, and set $x = \phi(1) \in G(k)$ assume $x^p = 1$. Choose some $r > 0$ such that the action of $(\rho \circ \phi)_*((d/dt)^{(i)})$ on M is trivial for all $i \geq p^r$.*

Consider the π-points

$$\alpha_x : k[u]/u^p \to kG(k) \quad u \mapsto x - 1,$$
$$\beta_x = (\phi \circ \sigma)_* \circ \epsilon : k[u]/u^p \to kG_{a(r)} \to kG_{(r)}.$$

Then

$$\rho_*(\alpha_x(u)),\ \rho_*(\beta_x(u)) \ \in \ im\{(\rho \circ \phi)_* : Rad(kG_{a(r)}) \to End_k(M)\},$$

and

$$\rho_*(\alpha_x(u)) - \rho_*(\beta_x(u)) \ \in \ im\{(\rho \circ \phi)_* : Rad^2(kG_{a(r)}) \to End_k(M)\}.$$

Proof. Proposition 4.1 asserts that the image of $u + 1 \in kG(k)$ as a distribution on GL_n and thus in $End_k(M)$ equals $\sum_{i\geq 0}^{p^r-1}(\rho \circ \phi)_*((d/dt)^{(i)})$. Hence,

$$\rho_*(\alpha_x(u)) = \sum_{i=1}^{p^r-1} (\rho \circ \phi)_*((d/dt)^{(i)}) \ \in \ End_k(M). \tag{26}$$

Since the image of u under $\sigma_* \circ \epsilon$ is $\sum_{j=0}^{r-1}(d/dt)^{(p^j)}$,

$$\rho_*(\beta_x((u)) = \sum_{j=0}^{r-1}(\rho \circ \phi)_*((d/dt)^{(p^j)}). \tag{27}$$

The assertions now follow from the fact recalled in Example 1.6 that $k\mathbb{G}_{a(r)}$ is a divided power algebra on $\{(d/dt)^{(p^j)},\ j \geq 0\}$. □

With the aid of [3, Prop.8], we can weaken the hypothesis on r in Proposition 4.2 if we are concerned only with the question of whether pull-backs via α_x, β_x of M are free.

Proposition 4.3. *Let G be an affine algebraic group over k, $x \in G(k)$ a non-trivial element of order p, and $\phi : \mathbb{G}_a \to G$ a 1-parameter subgroup with $\phi_x(1) = x$. Let M be a finite dimensional rational G-module, given by $\rho : G \to GL_n$. Choose $r > 0$ such that the action of $(\rho \circ \phi)_*((d/dt)^{(i)})$ on M is trivial for all $i \geq (p-1)p^r$.*

Consider the π-points

$$\alpha_x : k[u]/u^p \to kG(k) \quad u \mapsto x - 1,$$

$$\beta_x = (\phi \circ \sigma)_* \circ \epsilon : k[u]/u^p \to kG_{a(r)} \to kG_{(r)}$$

as in Proposition 4.2. Then $\alpha_x^*(M)$ is free (as a $k[u]/u^p$-module) if and only if $\beta_x^*(M)$ is free.

Proof. The assertion is the comparison of two $k[u]/u^p$-modules, the first given by the action of u as in (26) and the second by the action of u as in (27). Proposition 8 of [3] asserts that if $x, y \in End(M)$ with $x \neq 0$, $x^p = 0 = y^{p-1}$ and if M is free over the group algebra $k\langle 1+x\rangle$, then M is free over the group algebra $k\langle 1+x+y\rangle$ (and hence vice versa). Since $(d/dt)^{s(p^j)}$ for any $s > j$ and $(d/dt)^{p^r}$ have $(p-1)^{st}$-power equal to 0, the assertion that these two $k[u[/u^p$-modules are either both free or both not free follows by repeated applictions of [3, Prop.8]. $\square$

Corollary 4.4. *Let $\underline{E} \simeq \mathbb{G}_a^{\times s}$ be an abelian unipotent algebraic group over $\mathbb{F}_p$ and consider $x_1, \ldots, x_s \in \underline{E}(\mathbb{F}_p)$ which generate $\underline{E}(\mathbb{F}_p)$. Define $\phi_{x_j} : \mathbb{G}_a \to G$ to be the embedding with $\phi_{x_j}(1) = x_j$ for each j, $1 \leq j \leq s$. Let M be a finite dimensional rational $\underline{E}$-module, given by $\rho : \underline{E} \to GL_n$. Choose $r > 0$ such that the action of $(\rho \circ \phi_{x_j})_*((d/dt)^{(i)})$ on M is trivial for all j and all $i \geq (p-1)p^r$.*

Then for any $0 \neq (a_1, \ldots, a_s) \in k^{\times s}$, $(\sum_{j=1}^s a_j \alpha_{x_j})^*(M)$ is free (as a $k[u]/u^p$-module) if and only if $(\sum_{j=1}^s a_j \beta_{x_j})^*(M)$ is free.

Proof. As in the proof of Proposition 4.3, the assertion follows by applying [3, Prop.8] to the actions of

$$\sum_{j=1}^{s} a_j \sum_{i>0}^{(p-1)p^r-1} (\rho \circ \phi_{x_j})_*((d/dt)^{(i)})), \quad \sum_{j=1}^{s} a_j \sum_{j=0}^{r-1} (\rho \circ \phi_{x_j})_*((d/dt)^{(p^j)})$$

in $End_k(M)$. $\square$

In earlier work [3, 4.6], J. Carlson, Z. Lin, and D. Nakano compared the support varieties of a rational G-module M when restricted to $G(\mathbb{F}_p)$ and $G_{(1)}$ for simple algebraic groups G (with a restriction on p) and a certain very restricted class of rational G-modules M which they denote $\mathcal{C}_p$. The following theorem encompasses all finite dimensional rational G-modules by replacing $G_{(1)}$ by $G_{(r)}$, keeps the same bound for $r = 1$ as in [3], and even in that case of $r = 1$ is somewhat more precise. The proof proceeds by reducing to the situation in Corollary 4.4.

Theorem 4.5. *Let G be a suitable affine algebraic group (see Definition 1.9). Assume that G is defined over $\mathbb{F}_p$ and consider some connected, smooth, closed algebraic subgroup $H \subset G$ also defined over $\mathbb{F}_p$ satisfying condition (S) (cf. Definition 1.11), for example take $H = G$. Let M be a finite dimensional rational H-module. If $(p-1)p^r \geq s_{\mathbb{F}_p}(M))$, then Ψ_H of Theorem 3.5 induces a homeomorphism*

$$\Psi_H : (\Pi(H(\mathbb{F}_p)))_M \stackrel{\sim}{\to} (\Pi(H_{(r)})_M / H(\mathbb{F}_p) \cap \Psi(\Pi(H(\mathbb{F}_p))).$$

Proof. Let $\alpha_K : K[u]/u^p \to KH(\mathbb{F}_p)$ be a π-point of $H(\mathbb{F}_p)$. Since the projectivity of $\alpha_K^*(M_K)$ depends only upon the equivalence class of α_K (by definition of the equivalence relation on π-points) and since any such π-point is equivalent to one which factors through some elementary abelian p-subgroup $E \subset H(\mathbb{F}_p)$ (see, for example, [11, 4.1]), we shall assume that α_K factors through some elementary abelian p-subgroup $E \subset H(\mathbb{F}_p)$. Moreover, by Theorem 1.7, $E = \underline{E}(\mathbb{F}_p)$ for some unipotent abelian algebraic subgroup $\underline{E} \subset H$ with embedding defined over $\mathbb{F}_p$.

Thus, $\Psi_{\underline{E}} : \Pi(E) \to \Pi(\underline{E}_{(r)})$ sends the equivalence class of α_K to a point of $\Pi(\underline{E}_{(r)})$ whose image in $\Pi(H_{(r)})$ equals $\Psi([\alpha_K])$.

The explicit description of $\Psi_{\underline{E}}$ in Theorem 3.5 tells us that we may represent $[\alpha_K]$ by a K-linear combination of π-points of the form $\alpha_x : k[u]/u^p \to kE$, $u \mapsto x-1$ and $\Psi([\alpha_K])$ by the corresponding K-linear combination of π-points $\beta_x = (\phi_x \circ \sigma)_* \circ \epsilon : k[u]/u^p \to k\mathbb{G}_{a(r)} \to k(\underline{E}_{(r)})$. Thus, the proof is completed by applying Corollary 4.4. □

The necessity of choosing r sufficiently large as in the statement of Theorem 4.5 is revealed by the following examples.

Example 4.6. In the following two examples, the homeomorphism of Theorem 4.5 fails for $r = 1$.

(1) Let N be a rational G-module which is projective as a $G(\mathbb{F}_p)$-module and let $M = N^{(1)}$ be the first Frobenius twist of N. Then $\Pi(G(\mathbb{F}_p))_M = \emptyset$, whereas $\Pi(G_{(1)})_M = \Pi(G)$.

(2) Let $G = \mathbb{G}_a$ and let M be the p-dimensional rational G module defined by $\rho^* : k[GL_p] \to k[t] : \ X_{i,i} \mapsto 1, X_{i,i+1} \mapsto \sum_{s=0}^{p-1} t^{p^s} \ if$ for $1 \leq i < p$, and $X_{i,j} \mapsto 0$ otherwise. Then the restriction of M to $G_{(1)}$ is projective, but the restriction of M to $G(\mathbb{F}_p)$ is trivial.

The isomorphism type of a $k[u]/(u^p)$-module M of dimension n is given by a partition of n into subsets of size $\leq p$. We denote the Jordan type of M (or isomorphism type of M as a $k[u]/u^p$-module) by JType(M), and write JType(M) $= \sum_{i=1}^p a_i[i]$; in other words, as a $k[u]/u^p$-module $M \simeq \bigoplus_{i=1}^p ([i])^{\oplus a_i}$ where $[i] = k[u]/u^i$. We shall compare Jordan types using the *dominance partial order*, the usual partial ordering of partitions. If $\underline{a} = \sum_{i=1}^p a_i[i]$ and $\underline{b} = \sum_{i=1}^p b_i[i]$ with $\sum_i a_i \cdot i = \sum_i b_i \cdot i = m$, then $\underline{a} \geq \underline{b}$ if and only if

$$\sum_{i=j+1}^{p} a_i(i-j) \ \geq \ \sum_{i=j+1}^{p} b_i(i-j), \quad \forall j, 1 \leq j < p. \tag{28}$$

Let M, N be $k[u]/u^p$-modules of the same dimension. Then the Jordan type of M is greater or equal to the Jordan type of N if and only if for every j, $1 \leq j < p$, the rank of u on M (which we call the *j-rank* of M and denote by j-Rank(M)) is greater than or equal to j-rank of N.

We recall terminology introduced in [11], [10].

Definition 4.7. Let G be a finite group scheme over k and M a kG-module.

(1) If $\alpha_K : K[u]/u^p \to KG$ is a π-point of G with the property that there does not exist another π-point $\beta_L : L[u]/u^p \to LG$ with JType($\beta_L^*(M_L)$ $>$ JType($\alpha_K^* M_K$), then $\alpha_K^*(M_K)$ is said to be of maximal Jordan type for M.

(2) If j is an integer with $1 \leq j < p$ and if $\alpha_K : K[u]/u^p \to KG$ is a π-point of G with the property that there does not exist another π-point $\beta_L : L[u]/u^p \to LG$ with j-Rank($\beta_L^*(M_L)$) $>$ j-Rank($\alpha_K^*(M_K)$), then $\alpha_K^*(M_K)$ is said to be of maximal j-rank for M.

Clearly, $\alpha_K^*(M_K)$ is of maximal Jordan type for M if and only if for every j, $1 \leq j < p$, $\alpha_K^*(M_K)$ is of maximal j-rank for M.

We used in the proof of Theorem 4.5 the fact that if $\alpha_K,\ \beta_L$ are equivalent π-points of a finite group scheme G and if M is a finite dimensional kG-module, then $\alpha_K^*(M_K)$ is free if and only if $\beta_L^*(M_L)$ is free. This independence of representative of the equivalence class of π-points is valid as well for maximal Jordan types and maximal j-ranks as we now recall.

Theorem 4.8. [11, 4.2, 4.10], [10, 3.6] *Let G be a finite group scheme, let M be a finite dimensional rational G-module and let j be a positive integer $< p$. If $\alpha_K^*(M_K)$ has maximal j-rank for M (respectively, maximal Jordan type) and if $[\alpha_K]$ lies in the closure of some $[\beta_L] \in \Pi(G)$, then*

$$\text{j-Rank}(\alpha_K^*(M_K)) = \text{j-Rank}(\beta_L^*(M_L)) \quad (\textit{resp.}\ \text{JType}(\alpha_K^*(M_K)) = \text{JType}(\beta_L^*(M_L))).$$

In particular, if $\alpha_K^*(M_K)$ has maximal j-rank for M and if $\beta_L \sim \alpha_K$, then j-Rank$(\alpha_K^*(M_K))$ *equals* j-Rank$(\beta_L^*(M_L))$.

We next recall refinements of the support variety $\Pi(G)_M$ for a finite group scheme and a finite dimensional kG-module M as introduced in [11], [10] justified by Theorem 4.8.

Definition 4.9. Let G be a finite group scheme over k and M be a finite dimensional kG-module. Then the *non-maximal j-rank variety* (resp., *non-maximal support variety*)

$$\Gamma^j(G)_M \ \subset\ \Pi(G) \quad (\text{r}\textit{esp.}\ \Gamma(G)_M \ \subset\ \Pi(G))$$

is defined to be the closed subset of those equivalence classes of π-points $\alpha_K : K[u]/u^p \to KG$ such that the j-rank (resp., Jordan type) of $\alpha_K^*(M_K)$ is strictly less than the j-rank (resp. Jordan type) of $\beta_L^*(M_L)$ for some π-point $\beta_L : L[u]/u^p \to LG$ of G.

Thus, if $\Pi(G)_M \neq \Pi(G)$, then $\Gamma^j(G)_M = \Gamma(G)_M = \Pi(G)_M$. On the other hand, if $\Pi(G)_M = \Pi(G)$ (as is always the case if p does not divide the dimension of M, for example), then $\Gamma^j(G)_M$ and $\Gamma(G)_M$ are strictly contained in $\Pi(G)_M$.

Example 4.10. Let G be a connected reductive algebraic group defined and split over k and let $M = H^0(G/B,\lambda)$ be the rational G-module obtained by inducing the 1-dimensional B-module k_λ from B to G for some dominant weight λ. Thus, $H^0(G/B,\lambda)$ is dual to the Weyl module $W(\lambda)$. If λ is a p-regular weight, then the dimension of M is not divisible by p, so that $\Pi(G)_M = \Pi(G)$ whereas $\Gamma^j(G)_M$ is a proper closed subset of $\Pi(G)$ for all $j,\ 1 \leq j < p$.

The following theorem is a considerable strengthening of Theorem 4.5 for it allows arbitrary Jordan types as maximal Jordan types, not simply those of the form $n[p]$ and it applies to j-rank which is finer than Jordan type. The role of Proposition 4.3 in the proof of Theorem 4.5 is now replaced by an appeal to Proposition 4.2.

Theorem 4.11. *Let G be a suitable affine algebraic group defined over $\mathbb{F}_p$ with $p \geq h(G)$, and let $H \subset G$ be a connected, smooth, closed algebraic subgroup also defined over $\mathbb{F}_p$ satisfying condition (S) (cf. Definition 1.11), for example take $H = G$. Let M be a finite dimensional rational H-module. Choose $r > 0$ so that $p^r \geq s_{\mathbb{F}_p}(M)$.*

Let $\alpha_K : K[u]/u^p \to KH(\mathbb{F}_p)$ be a π-point of $KH(\mathbb{F}_p)$ and let $\beta_L : L[u]/u^p \to LH_{(r)}$ represent $\Psi_H([\alpha]) \in \Pi(H_{(r)})/H(\mathbb{F}_p)$. If $\beta_L^*(M_L)$ has maximal j-rank for M as a $kH_{(r)}$-module for some $j,\ 1 \leq j < p$, then

$$\text{j-Rank}(\alpha_K^*(M_K)) \ = \ \text{j-Rank}(\beta_L^*(M_L)),$$

and $\alpha_K^*(M_K)$ has maximal j-rank for M as an $H(\mathbb{F}_p)$-module.

Proof. Observe that if β_L has maximal j-rank for M as an $H_{(r)}$-module, then so does the conjugate of β_L by any element $h \in H(\mathbb{F}_p)$ (since the conjugate M^h is isomorphic to M as an $H_{(r)}$-module). Moreover, if β_L represents $\Psi_H([\alpha_K])$ and if $\beta_L^*(M_L)$ has maximal j-rank for the restriction of M to $H_{(r)}$, then $\alpha_K^*(M_K)$ must have maximal j-rank for the restriction of M to $H(\mathbb{F}_p)$. (Otherwise, there would be a π-point $\alpha'_{K'}$ of $H(\mathbb{F}_p)$ such that $\alpha'^*_{K'}(M_{K'})$ has larger j-rank type so that a representative of $\Psi_H([\alpha_{K'}])$ would have larger j-rank than j-Rank$(\beta_L^*(M_L))$.) Consequently, by appealing to Theorem 4.8, we may replace replace α_K, β_L by equivalent π-points.

Thus, exactly as in the proof of Theorem 4.5, we may represent $[\alpha_K]$ by a K-linear combination of π-points of the form $\alpha_x : k[u]/u^p \to k\underline{E}(\mathbb{F}_p),\ u \mapsto x-1$ and $\Psi_H([\alpha_K])$ by the corresponding K-linear combination of π-points $\beta_x = (\phi_x \circ \sigma)_* \circ \epsilon : k[u]/u^p \to \underline{E}_{(r)}$. The proof is completed by applying Proposition 4.2 and [11, 1.13], the fundamental result underlying the proof of Theorem 4.8. □

Corollary 4.12. *With hypotheses and notation as in Theorem 4.11, the map Ψ_H of (21) induces injective maps*

$$\Gamma^j(H(\mathbb{F}_p))_M \hookrightarrow \Gamma^j(H_{(r)})_M/H(\mathbb{F}_p), \quad \Gamma(H(\mathbb{F}_p))_M \hookrightarrow \Gamma(H_{(r)})_M/H(\mathbb{F}_p)$$

for any j, $1 \leq j < p$.

Moreover, the images of these maps equal

$$(\Gamma^j(H_{(r)})_M/H(\mathbb{F}_p)) \cap \Psi_H(\Gamma^j(H(\mathbb{F}_p))), \quad (resp.\ (\Gamma(H_{(r)})_M/H(\mathbb{F}_p)) \cap \Psi_H(\Gamma(H(\mathbb{F}_p)))$$

inside $\Pi(H)/H(\mathbb{F}_p)$ if and only if the maximal j-rank (resp., Jordan type) of M as an $H(\mathbb{F}_p)$ module is also maximal for M as an $H_{(r)}$-module.

Proof. If $\alpha_K : K[u]/u^p \to KH(\mathbb{F}_p)$ is a π-point of $H(\mathbb{F}_p)$ at which M has non-maximal j-rank, then Theorem 4.11 asserts that at any π-point $\beta_L : L[u]/u^p \to H_{(r)}$ representing $\Psi_H([\alpha_K])$ the j-rank of M is non-maximal for $H_{(r)}$. Thus, Ψ_H restricts to $\Gamma^j(H(\mathbb{F}_p))_M \to \Gamma^j(H_{(r)})_M/H(\mathbb{F}_p)$, and this restriction is necessarily an embedding because Ψ_H is an embedding.

If $\alpha_K^*(M_K)$ has maximal j-rank for M as an $H(\mathbb{F}_p)$-module but $\beta_L^*(M_L)$ does not have maximal j-rank for M as an $H_{(r)}$-module for some π-point $\beta_L : L[u]/u^p \to H_{(r)}$ representing $\Psi_H([\alpha_K])$, then

$$\Psi_H([\alpha_K]) \in (\Gamma^j(H_{(r)})_M/H(\mathbb{F}_p)) \cap \Psi(H(\mathbb{F}_p)), \quad \Psi_H([\alpha_K]) \notin \Psi_H(\Gamma^j(H(\mathbb{F}_p))_M).$$

If every element of $(\Gamma^j(H_{(r)})_M/H(\mathbb{F}_p)) \cap \Psi_H(H(\mathbb{F}_p))$ lies in the image of $\Gamma^j(H(\mathbb{F}_p))_M$, then this necessarily says that each maximal j-rank of M as an $H(\mathbb{F}_p)$-module is a maximal j-rank for M as an $H_{(r)}$-module □

Example 4.13. Let G be a connected reductive algebraic group over an algebraically closed field k defined and split over $\mathbb{F}_p$, assume that $p \geq h$, and assume that every p-unipotent conjugacy class of G is defined over $\mathbb{F}_p$. For example, G could be of classical type. Then Lang's Theorem implies that every p-unipotent conjugacy class of G meets $G(\mathbb{F}_p)$ [13, 8.4]. Thus, for any finite dimensional rational G-module M with $p \geq s_{\mathbb{F}_q}(M)$, the maximal Jordan types of M as a $G(\mathbb{F}_p)$ module are also maximal for M as a $G_{(1)}$-module.

Moreover, since $\mathcal{N}_p(\mathfrak{g}_{\overline{k}})$ is irreducible, there is a unique maximal Jordan type for a given finite dimensional $kG_{(1)}$-module, namely the generic Jordan type. Thus, if

$p \geq s_{\mathbb{F}_q}(M)$, there is only one maximal Jordan type of the rational G-module M when restricted to $G(\mathbb{F}_p)$, even though the generic Jordan types of the restriction of M to $G(\mathbb{F}_p)$ may be different at different generic points of $\Pi(G(\mathbb{F}_p))$.

The following example suggest caution in trying to sharpen Theorem 4.11.

Example 4.14. Take $G = \mathbb{G}_a$ and consider the 2-dimensional rational G-module M determined by $\rho^* : k[GL_2] \to k[\mathbb{G}_a] = k[t]$ sending $X_{1,1}$ and $X_{2,2}$ to 1, sending $X_{2,1}$ to 0, and $X_{1,2}$ to $\sum_{s=0}^{p-1} t^{p^s}$. Then M restricted to $G(\mathbb{F}_p)$ is trivial, but M restricted to $\mathbb{G}_{(r)}$ is non-trivial for $1 \leq r < p$. Thus, $\Gamma(G(\mathbb{F}_p))_M = \emptyset$, but $\Gamma(G_{(r)})_M$ is non-empty and $\Psi : \Pi(G(\mathbb{F}_p)) \to \Pi(G_{(r)})$ is non-trivial.

A module M for a finite group scheme G is said to have constant j-rank for some j, $1 \leq j < p$ (respectively, constant Jordan type) if the j-rank (resp., Jordan type) of $\alpha_K^*(M_K)$ is the same for all π-points of G (see [2]). This condition on M is equivalent to the condition that $\Gamma^j(G)_M$ (resp., $\Gamma(G)_M$) be empty.

Corollary 4.15. *With hypotheses and notation as in Theorem 4.11, assume in addition that the restriction of M to $H_{(r)}$ has constant j-rank for some j, $1 \leq j < p$. Then the restriction of M to $H(\mathbb{F}_p)$ has constant Jordan type with the same Jordan type.*

We briefly consider the condition of Corollary 4.12 that a maximal j-rank of M as a $G(\mathbb{F}_p)$-module is also maximal for M as a $G_{(r)}$-module. We restrict our attention to groups G of classical type in order to apply Examples 1.12 and 2.3. In this case, every 1-parameter subgroup $\phi : \mathbb{G}_{a(r+1)} \to G_{(r+1)}$ admits a lifting to a 1-parameter subgroup of G of the form (10):

$$\tilde{\phi} \; = \; (\phi_{\exp(\alpha_0)}) \cdot (\phi_{\exp(\alpha_1)} \circ F) \cdots (\phi_{exp(\alpha_r)} \circ F^r) : : \mathbb{G}_{a(r+1)} \to G, \tag{29}$$

where $\alpha_0, \ldots \alpha_r$ are pair-wise commuting p-nilpotent elements of $\mathfrak{g} = Lie(G)$ with entries in k. We recall that every π-point of $G_{(r+1)}$ is equivalent to one of the form $\phi_* \circ \epsilon : k[u]/u^p \to \mathbb{G}_{a(r+1)} \to G_{(r+1)}$ associated to $\tilde{\phi}$ of the form (29). Thus, the maximal j-ranks of M as a $kG_{(r+1)}$-module occur among the j-ranks of $\phi_*(u_r)$ acting on M as $\tilde{\phi}$ ranges over 1-parameter subgroups of the form (29).

The following proposition shows that increasing r does not introduce new maximal Jordan types.

Proposition 4.16. *Let G denote be a direct product of general linear groups and simple algebraic groups of classical types over k, and assume that $p \geq h(G)$. Let M be a finite dimensional rational G-module and assume that $s(M) \leq p^r$ for some $r \geq 1$. Then the maximal j-ranks for M as a $kG_{(r+1)}$-module are the same as those for M as a $kG_{(r)}$-module.*

Proof. Let $\tilde{\phi} : \mathbb{G}_{a(r+1)} \to G$ be of the form (29). The condition that $s(M) \leq p^r$ implies that $\phi_{exp(\alpha_0)}(u_r)$ acts trivially on M. Since $\phi_{exp(\alpha_i)} \circ F^i$ commutes with $\phi_{exp(\alpha_j)} \circ F^j$ whenever α_i commutes with α_j, the action of $\phi(u_r)$ on M equals the action of $\phi'(u_r)$ on M, where

$$\phi' \; = \; (\phi_{\exp(\alpha_1)} \circ F) \cdots (\phi_{exp(\alpha_r)} \circ F^r) : \mathbb{G}_{a(r+1)} \to G_{r+1}.$$

Observe that Frobenius $F : \mathbb{G}_{a(r+1)} \to \mathbb{G}_{a(r+1)}$ induces

$$F_* : k\mathbb{G}_{(r+1)} \simeq k[u_0, \ldots, u_r] \; \to \; k[u_0, \ldots, u_r] \simeq k\mathbb{G}_{(r+1)}, \quad u_0 \mapsto 0; u_{j+1} \mapsto u_j. \tag{30}$$

Consequently, the action of $\phi'_*(u_r)$ on M equals the action of $\overline{\phi}_*(u_{r-1})$ on M, where

$$\overline{\phi} \;=\; (\phi_{\exp(\alpha_1)}) \cdots (\phi_{exp(\alpha_r)} \circ F^{r-1}) : \mathbb{G}_{a(r)} \to G_{(r)}.$$

□

We conclude this section with the following comparison of maximal Jordan types for irreducible SL_2-modules.

Example 4.17. Let $G = SL_2$. Then any finite dimensional rational G-module has constant Jordan type as an $SL_{2(1)}$-module by [2, 2.5]. Let $\phi_e : \mathbb{G}_a \to SL_2$ be the map sending t to the strictly upper triangular matrix with t in the $(1,2)$ position.

The Steinberg tensor product theorem tells us that any irreducible rational G-module M satisfying $s(M) \leq p^r$ is of the form

$$M \;=\; S(\lambda_0) \otimes S(\lambda_1)^{(1)} \otimes \cdots \otimes S(\lambda_{r-1})^{(r-1)} \tag{31}$$

for integers $\lambda_i,\ 0 \leq \lambda_i < p$. Using Proposition 4.16, we conclude that the maximal Jordan type of $S(\lambda_i)^{(i)}$ as a $kG_{(r)}$-module equals $[\lambda_i + 1]$ (i.e., a single block of size $\lambda_i + 1$) and this is the Jordan type of $\phi_{e*}(u_i)$ acting on M. Since $\phi_{e*}(u_j)$ acts trivially on $S(\lambda_i)^{(i)}$ for $i \neq j$, we conclude that this maximal Jordan type for $S(\lambda_i)^{(i)}$ is realized at the π-point $(\phi_e \circ \sigma)_* \circ \epsilon$.

Exactly as discussed in [10, 4.11], the tensor product formula for maximal Jordan types [2, 4.2] implies that the maximal Jordan type of M is the Jordan type of the tensor product $[\lambda_0 + 1] \otimes [\lambda_1 + 1] \otimes \cdots \otimes [\lambda_{r-1} + 1]$ of $k[u]/u^p$-modules (determned explicitly in [2, 10.3]). This maximal Jordan type for M also occurs at $(\phi_e \circ \sigma)_* \circ \epsilon$.

It would be interesting to know under what circumstances the maximal Jordan type as an $SL_{2(r)}$-module of an arbitrary (finite dimensional) rational SL_2-module M with $s(M) \leq p^r$ occurs at $(\phi_e \circ \sigma)_* \circ \epsilon$.

5. Extension to $G(\mathbb{F}_q),\ q = p^d$

In [6], we show how the Weil restriction functor enables one to extend techniques suitable for $G(\mathbb{F}_p)$ to $G(\mathbb{F}_q),\ q = p^d$, for algebraic groups G defined over $\mathbb{F}_q$. In this section, we indicate how this method applies to extend results of Section 4 from $G(\mathbb{F}_p)$ to $G(\mathbb{F}_q)$. The reader is referred to [16] for a helpful discussion of the Weil restriction functor applied to affine algebraic groups.

The following proposition is a extension of [6, 1.5].

Proposition 5.1. *Let k'/k be a finite field extension and G' an affine algebraic group over k'. Let $\mathcal{R}_{k'/k}G'$ be the Weil restriction of G', an affine algebraic group over k.*

Then the k-points of $\mathcal{R}_{k'/k}G'$ can be naturally identified with the k'-points of G'. Moreover, the r-th Frobenius kernel of $\mathcal{R}_{k'/k}G'$ can be identified with the Weil restriction of the r-th Frobenius kernel of G',

$$(\mathcal{R}_{k'/k}G')_{(r)} \;\simeq\; \mathcal{R}_{k'/k}(G'_{(r)}).$$

Proof. The proof of [6, 1.5] applies with only the minor change of replacing the (supported at the identity of G) 1-distributions $Dist_1(G')$ by $Distr_r(G')$. □

Remark 5.2. Let G' be a connected reductive algebraic group provided with the data of an $\mathbb{F}_q$-structure and assume that $p \geq h(G')$. Denote $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}G'$ by G. Observe that G is also a connected, reductive algebraic group over $\mathbb{F}_p$ which satisfies $p \geq h(G)$: namely, the base change from $\mathbb{F}_p$ to $\mathbb{F}_q$ of G splits as a product of copies of $Gal(\mathbb{F}_q/\mathbb{F}_p)$-conjugates of G' and thus has trivial unipotent radical. Hence, Theorems 4.5 and 4.11 apply to rational G-modules M, enabling a comparison of invariants for M restricted to $G(\mathbb{F}_p) = G'(\mathbb{F}_q)$ and to $G_{(r)}$.

In particular, one could take G' be the base change from $\mathbb{F}_p$ to $\mathbb{F}_q$ of a reductive algebraic group $\mathfrak{G}$ defined and split over $\mathbb{F}_p$. Then $G = \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}G'$ is a twisted form associated to $\mathfrak{G}$.

We leave elaboration to the interested reader.

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