Some Computations of Algebraic Cycle Homology

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(to Michael Artin, Friend and Teacher)

In Theorem 3.3 below, we compute the homology of the algebraic bivariant cycle complex $A_{m-1}(Speck, X)$ for a variety X of pure of dimension $m \geq 1$ over a perfect field k. In Theorem 4.4, we compute the mod-n homology of the complex $A_r(Spec\mathbf{C}, X)$ for a complex variety X. The definition of $A_r(Y, X)$ was introduced in a joint paper with Ofer Gabber in [F-G] to provide a "rational equivalence analogue" of bivariant morphic cohomology presented in a joint paper with Blaine Lawson in [F-L]. In particular, the homology of $A_r(Speck, X)$ is the "rational equivalence analogue" of Lawson homology.

In the special case of 0-cycles, the complex $A_0(Speck, X)$ is closely related to the Suslin complex $Sus_\bullet(X)$ of algebraic singular chains of the infinite symmetric product of X. Indeed, our computation in codimension 1 was inspired by S. Lichtenbaum's recent computation of the Suslin homology $H_*(Sus_\bullet(X))$ for the case that X is a curve [L]. Our computation of the mod-n homology of $A_r(Spec\mathbf{C}, X)$ is merely a rephrasing of the computation of the mod-n homology of a closely related complex for complex projective varieties achieved in [S-V] by A. Suslin and V. Voevodsky.

The paper is organized as follows. We begin by recalling the functor $Z_{X,r}(-)$ of continuous algebraic maps into the cycle space $Z_r(X)$. Section 2 then introduces the functor $R_X(-)$ of invertible rational functions with specialization, a functor which may have independent interest. After completing our computation of $H_*(A_{m-1}(Speck, X)$, we proceed in section 4 to interpret $Z_r(-)$ as terms of sheaves for Voevodsky's h-topology on X [V]. This enables us to compute $H_*(A_r(Spec\mathbf{C}, X), \mathbf{Z}/n)$ by applying the results of [S-V].

Thoughout this paper, we shall restrict attention to quasi-projective varieties X over a perfect field k of characteristic $p \geq 0$. Such a variety X is a reduced algebraic k-scheme of finite type which admits a locally closed embedding into some projective space over k. (As seen in [F-G;4.4], $A_r(Y, X)$ is independent up to natural isomorphism of this choice.) We shall let $X \subset \overline{X}$ denote some choice of projective closure of X. Moreover, we shall let $k(X)$ denote the ring of total quotients of X, by which we mean the product of the quotient fields of the irreducible components of X. An invertible rational function $f \in k(X)^*$ is an element in this product each of whose factors is non-zero.

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1. Equidimensional cycles.

The complexes $A_r(Y,X)$ are defined in terms of continuous algebraic maps into $Z_r(X)$, algebraic r-cycles on X. We begin by recalling this concept.

Definition 1.1. Let X be a variety and let $\mathcal{C}_r(\overline{X}) = \coprod C_{r,d}(\overline{X})$ denote the Chow monoid of r-cycles on on the projective closure $\overline{X}$ of X for some $r \geq 0$. For any variety Y, **a continuous algebraic map** $\psi : Y \to Z_r(X)$ is a set-theoretic function

$$Y(\overline{k}) \to Z_r(X)(\overline{k}) \simeq \mathcal{C}_r(\overline{X})^2(\overline{k})/R$$

induced by a correspondence (i.e., a closed subset) $C_\psi \subset Y \times \mathcal{C}_r(\overline{X})^2$. We denote by $Z_{X,r}(Y)$ the set of continuous algebraic maps from Y to $Z_r(X)$ with its natural abelian group structure.

In the above definition, the equivalence relation R on $\mathcal{C}_r(\overline{X})^2(\overline{k})$ consists of (Chow coordinates of) pairs of pairs of cycles $(Z_1, Z_2; W_1, W_2)$ on $\overline{X}_{\overline{k}}$ with the property that $Z_1 + W_2$, $Z_2 + W_1$ have equal restrictions to $X_{\overline{k}}$. As verified in [F-G;4.5], our definition of $Z_{X,r}(Y)$ is independent (up to natural isomorphism) of a choice of projective closure $X \subset \overline{X}$ for X. A continuous algebraic map $\psi : Y \to Z_r(X)$ determines a set-theoretic function

$$Y(\overline{K}) \to Z_r(X)(\overline{K}) \simeq \mathcal{C}_r(\overline{X})^2(\overline{K})/R$$

for any algebraically closed field extension $\overline{K}$ of k.

In the following example, we introduce notation which we shall frequently employ in our discussions below.

Example 1.2. Let X, Y be varieties with X projective and let $\mathcal{L}$ be a line bundle on $Y \times X$. For any global section $F \in \mathcal{L}(Y \times X)$, let $Z_F \subset Y \times X$ denote the codimension 1 cycle (with multiplicities) on $Y \times X$ defined by

$$Z_F = \sum_{\mathcal{P}} n_{\mathcal{P}} V_{\mathcal{P}}$$

where the sum is indexed by the height one primes $\mathcal{P}$ of $Y \times X$, where $n_{\mathcal{P}}$ is the length of the $O_{X,\mathcal{P}}$ -module $O_{X,\mathcal{P}}/(f)$ with (f) a local equation for F, and where $V_{\mathcal{P}}$ is the irreducible subvariety with generic point $\mathcal{P}$.

If F is such that Z_F is flat over Y and if X is purely m-dimensional, then F naturally determines a morphism $h_F : Y \to \mathcal{C}_{m-1}(X)$ (indeed, h_F factors through a morphism to the Hilbert scheme of codimension 1 ideals on X). Since Z_F is flat over Y, $h_F(y)$ is the Chow point of the cycle associated to the scheme-theoretic fibre $(Z_F)_y$ [F;1.3]. If F, G are both global sections of $\mathcal{L}$ such that Z_F, Z_G are flat over Y, then h_F, h_G determine

$$\psi_{F/G} : Y \to Z_{m-1}(X).$$

Whereas morphisms from a normal variety Y to a Chow variety $C_{r,d}(X)$ of a projective variety X correspond (bijectively) to effective cycles on $Y \times X$ equidimensional of relative dimension r over Y, elements of $Z_{X,r}(Y)$ are somewhat more subtle even if Y, X are smooth as the following example reveals.

Example 1.3. Let X denote the result of blowing-up some rational point of P^2, which we view as the origin of $A^2 \subset P^2$. Let $Y = A^2$, and choose a non-constant map $Y \to PGL(3)$ sending the origin to the identity and whose differential at the origin is 0. Then the graph Γ_g of the composition $g : Y \to PGL(3) \times P^2 \to P^2$ agrees with the graph Γ_i of the inclusion $i : Y \subset P^2$ to first order at the origin. The difference $\Gamma_g - \Gamma_i$ is the cycle associated to a continuous algebraic map $\psi : Y \to Z_0(X)$ which does not arise from a morphism $Y \to \mathcal{C}_0(X)^2$. Namely, the restrictions of g, i to $Y - \{0\}$ determine a morphism $h_{g/i} : Y - \{0\} \to C_{0,1}(X)^2$. We may take $C_\psi \subset Y \times C_{0,1}(X)^2$ to be the closure of the graph of $h_{g/i}$.

In the next proposition, we give a somewhat more explicit description of continuous algebraic maps $\psi : Y \to Z_r(X)$. In the proof of this proposition, we show that there is a natural minimal correspondence Γ_ψ representing ψ, a fact which we shall exploit in our interpretation of $Z_{X,r}(-)$ as a sheaf in the h-topology. Recall that a dominant morphism $Y' \to Y$ is said to be generically radiciel if the associated extension of rings of total quotients $k(Y) \to k(Y')$ is purely inseparable.

Proposition 1.4. Let $\psi : Y \to Z_r(X)$ be a continuous algebraic map. Then $C_\psi \subset Y \times \mathcal{C}_r(\overline{X})^2$ can be chosen to be generically radiciel (as well as proper and surjective) over Y. Moreover, such a ψ naturally determines a cycle on $Y^j \times X$ for some $j \geq 0$, where $Y^j \to Y$ denotes the finite, radiciel (endo-) morphism of Y given by the j-th iterate of the (geometric) Frobenius map.

Proof. Since $\mathcal{C}_r(\overline{X})^2$ is projective, $C_\psi \to Y$ is necessarily proper; since every geometric point is in the domain of ψ, $C_\psi \to Y$ is necessarily surjective.

Let C_ψ be a correspondence representing ψ. We define the "saturation" $\tilde{C}_\psi = \coprod(\tilde{C}_\psi)_{d,e}$ of C_ψ as follows: if $(C_\psi)_{d,e} = C_\psi \cap [Y \times \mathcal{C}_{r,d}(\overline{X}) \times \mathcal{C}_{r,e}(\overline{X})]$, define $(\tilde{C}_\psi)_{d,e}$ to be the union over $f < d, e$ of the projections to $Y \times \mathcal{C}_{r,d-f}(\overline{X}) \times \mathcal{C}_{r,e-f}(\overline{X})$ of the preimage of $(C_\psi)_{d,e}$ via the addition map

$$Y \times \mathcal{C}_{r,d-f}(\overline{X}) \times \mathcal{C}_{r,e-f}(\overline{X}) \times \mathcal{C}_{r,f}(\overline{X}) \to Y \times \mathcal{C}_{r,d}(\overline{X}) \times \mathcal{C}_{r,e}(\overline{X}).$$

So defined, $\tilde{C}_\psi$ also represents ψ. Namely, a geometric point

$$(\underline{y}, \underline{z}, \underline{w}) : Spec\overline{k} \to Y \times \mathcal{C}_r(\overline{X})^2$$

is a geometric point of $\tilde{C}_\psi$ if and only if there exists some geometric point $\underline{u} : Spec\overline{k} \to \mathcal{C}_r(\overline{X})$ such that $(\underline{y}, \underline{z} + \underline{u}, \underline{w} + \underline{u})$ is a geometric point of C_ψ.

Consider a (not necessarily closed) point $y \in Y$, a geometric point $\underline{y} : Spec\overline{K} \to Y$ above y, and a representative $(\underline{z}, \underline{w}) : Spec\overline{K} \to \mathcal{C}_{r,d}(\overline{X}) \times \mathcal{C}_{r,e}(\overline{X})$ of $\psi(\underline{y})$. We say that $(\underline{z}, \underline{w})$ is a minimal representative for $\psi(\underline{y})$ if there does not exist some geometric point

$$(\underline{a}, \underline{b}, \underline{c}) : Spec\overline{K} \to \mathcal{C}_{r,e-f}(\overline{X}) \times \mathcal{C}_{r,e-f}(\overline{X}) \times \mathcal{C}_{r,f}(\overline{X})$$

which maps to $(\underline{z}, \underline{w})$ via the addition map for any $f > 0$.

We define $\Gamma_\psi \subset \tilde{C}_\psi$ to be the closure of the finitely many points (indexed by the irreducible components of Y) $(\eta, \gamma, \delta) \in \tilde{C}_\psi$ with the property that η is a generic point of Y, $\underline{\eta}$ is some geometric point above η, and there exists some minimal representative for $\psi(\underline{\eta})$, $(\underline{\gamma}, \underline{\delta})$, above (γ, δ). Since $\Gamma_\psi \subset \tilde{C}_\psi$ and since $pr_1 : \Gamma_\psi \to Y$ is surjective, Γ_ψ also represents ψ. Moreover, if C'_ψ is another correspondence representing ψ and if $\Gamma'_\psi \subset \tilde{C}'_\psi$ is constructed as was $\Gamma_\psi \subset \tilde{C}_\psi$, then $\Gamma''_\psi \subset \tilde{C}'_\psi \times_Y \tilde{C}_\psi$ maps bijectively onto both $\Gamma_\psi, \Gamma'_\psi$ and therefore identifies the closed subvarieties $\Gamma_\psi, \Gamma'_\psi$ of $Y \times \mathcal{C}_r(\overline{X})^2$.

To verify that $\Gamma_\psi \to Y$ is generically radiciel, it suffices to verify that the projection $\Gamma_\psi \to Y$ is generically 1-1 on geometric points. Consider a geometric point $(\underline{\eta}, \underline{\gamma}, \underline{\delta})$ of Γ_ψ with $\underline{\eta}$ a geometric point of Y above a generic point η of Y. Let g be any element of the Galois group $Gal(\overline{L}, L)$ of the the function field L of some irreducible component of Γ_ψ containing $(\underline{\eta}, \underline{\gamma}, \underline{\delta})$. By minimality, if g fixes $\underline{\eta}$ then g must also fix $(\underline{\gamma}, \underline{\delta})$. Moreover, there can be no non-trivial specializations of $(\underline{\eta}, \underline{\gamma}, \underline{\delta})$ of the form $(\underline{\eta}, \underline{\gamma}', \underline{\delta}')$, for minimality implies that such a specialization would determine a distinct image of $\underline{\eta}$ under ψ. Thus, $(\underline{\eta}, \underline{\gamma}, \underline{\delta})$ is the unique point of Γ_ψ lying above $\underline{\eta}$.

Finally, since $\Gamma_\psi \to Y$ is generically radiciel, we may choose j sufficiently large that the field of fractions of the generic point of Γ_ψ lying above η is a subfield of $k(\eta)^{1/p^j}$ for each generic point η of Y. Since k is perfect, $k(\eta)^{1/p^j}$ is the residue field of a (unique, generic) point η^j of Y^j lying above $\eta \in Y$. Define $\Gamma^j_\psi \subset Y^j \times \mathcal{C}_r(\overline{X})^2$ to be the (reduced) subvariety with support $\Gamma_\psi \times_Y Y^j$. Then we readily verify that the projection $Y^j \times X \to Y \times X$ maps Γ^j_ψ bijectively onto Γ_ψ and that $pr_1 : \Gamma^j_\psi \to Y^j$ is generically birational. In particular, Γ^j_ψ is the closure of the graph of a morphism $V \to \mathcal{C}_r(\overline{X})^2$ for some dense open $V \subset Y^j$. This morphism determines a cycle on $V \times \overline{X}$ equidimensional over V whose closure is a cycle on $Y^j \times \overline{X}$ whose restriction is a cycle on $Y^j \times X$. Clearly, this last cycle does not depend on the choice of dense open $V \subset Y^j$.

2. Invertible rational functions with specialization.

An immediate consequence of Proposition 1.4 is the existence of a map (natural with respect to X)

$$Z_{X,m-1}(Y) \to Lim_j\{A_{m+n-1}(Y^j \times X)\}$$

for varieties X, Y of pure dimension $m, n \geq 1$, where $A_k(W)$ denotes the group of rational equivalence classes of cycles of dimension k on a variety W (in the sense of [Fu]) and Lim_j denotes the direct limit indexed by j. In order to understand the kernel of this map, we introduce the following definition. (We adopt the convention that if T is a not necessarily reduced k-scheme, then $k(T)^*$ denotes the multiplicative group of elements of the product of the residue fields at the generic points of T each of whose factors in non-zero.)

Definition 2.1. For varieties Y, X, we define

$$R_X(Y) \subset Lim_j\{k(Y^j \times X)^*\}$$

to consist of invertible rational functions f on some $Y^j \times X$ which can be realized as follows. There should exist a blowing-up (i.e., a proper, surjective, and biratonal morphism) $p : Y' \to Y^j$, some line bundle $\mathcal{L}$ on $Y' \times \overline{X}$, and non-zero global sections $F, G \in \mathcal{L}(Y' \times \overline{X})$ such that

a.) $f = F/G \in k(Y^j \times X)^* = k(Y' \times X)^*$

b.) the zero loci $Z_F, Z_G \subset Y' \times \overline{X}$ of F, G are both equidimensional over Y'

c.) for every geometric point $\underline{y}$ of Y, $F_E/G_E \in k(E \times X)^*$ lies in the image of $k(X_{\underline{y}})^*$, where $E = Y'_{\underline{y}}$ is the geometric fibre of $Y' \to Y$ over $\underline{y}$, $\mathcal{L}_E$ is the restriction of $\mathcal{L}$ to $E \times X$, and F_E, G_E are the restrictions of F, G to $\mathcal{L}_E$.

Using [R-G;§5.2], we see that $R_X(Y) \subset Lim_j\{k(Y^j \times X)^*\}$ is unchanged if we replace b.) by

b′.) the zero loci $Z_F, Z_G \subset Y' \times \overline{X}$ of F, G are both flat over Y'.

Namely, if $Z_F, Z_G \subset Y' \times \overline{X}$ are equidimensional over Y', there exists a projective morphism $g : Y'' \to Y'$ together with a dense open subset $U \subset Y'$ with $g^{-1}(U) \subset Y''$ mapping isomorphically to U such that the proper transforms of Z_F, Z_G under g (i.e., the closures in $Y'' \times \overline{X}$ of the restrictions of Z_F, Z_G to $g^{-1}(U) \times \overline{X} \simeq U \times \overline{X}$) are flat over Y''. Moreover, these proper transforms equal $Z_{F'}, Z_{G'}$, where $F', G' \in g^*\mathcal{L}(Y'' \times \overline{X})$ are the images of F, G.

The equidimensionality of Z_F, Z_G over Y' is equivalent to the non-vanishing of $F_{\underline{y}'}, G_{\underline{y}'}$ for all points $y' \in Y'$. Thus, (2.1.b) implies that $F_E/G_E \in k(E \times X)^*$. An immediate consequence of the above definition is the fact that $R_X(Y) = R_U(Y)$ for any dense open subset $U \subset X$.

The preceding definition of $R_X(Y)$ is formulated in geometric language in order to easily relate it to $Z_X(Y)$. The following proposition provides a more algebraic version of the condition on a non-zero rational function to lie in $R_X(Y)$.

Proposition 2.2. For varieties Y, X, an invertible rational function $f \in Lim_j\{k(Y^j \times X)^*\}$ lies in $R_X(Y)$ if and only if there exists a blowing-up $p : Y' \to Y^j$, some affine open covering $\{V_i\}$ of Y', and some affine open subsets $U_i = SpecA_i \subset V_i \times X$ dense in each fibre of $pr_1 : U_i \to V_i$ such that the restriction of f to each $k(U_i)$ is a regular function $f'_i \in A_i$ with the following property

(*) for every geometric point $\underline{y}$ of Y and every V_i admitting a lifting of $\underline{y}$, the restriction of f'_i to $Y'_{\underline{y}} \times X$ is an invertible rational function $f'_{\underline{y},i} \in k(Y'_{\underline{y}} \times X)^*$ which lies in the image of $k(X_{\underline{y}})^*$.

Proof. Suppose $f \in R_X(Y)$ is given by the data of Definition 2.1. Pulling back this data to the normalization of Y', we may assume that Y' is normal. Furthermore, to obtain $\{f'_i\}$ satisfying (*) we may replace X by a dense affine open subset and thus assume X is affine. Since Z_G is flat over Y', its complement U_i in $V_i \times X$ maps surjectively to V_i. We define $f'_i = F/G \in \mathcal{L}(U_i) \otimes \mathcal{L}^{-1}(U_i) = O_X(U_i)$. Then the restriction of f'_i to $k(Y'_{\underline{y}} \times X)$ equals F_E/G_E, so that condition (*) follows immediately from (2.1.c).

Conversely, consider $f \in k(Y^j \times X)$, $p' : Y'' \to Y^j$, $\{V'_i\}$ an affine open covering of Y'', and $U'_i = SpecA'_i \subset V'_i \times X$ such that the restrictions $f''_i \in A'_i$ of f satisfy (*).

Choose some line bundle $\mathcal{L}'$ on $Y'' \times \overline{X}$ and global sections $F', G' \in \mathcal{L}'(Y'' \times \overline{X})$ such that $F'/G' = f$. Let $p : Y' \to Y''$ be some blowing-up such that the proper transforms of $Z_{p^*(F')}, Z_{p^*(G')}$ are flat over Y'. These proper transforms are the global sections F, G of a line bundle $\mathcal{L}$ on $Y' \times \overline{X}$. Set $V_i = V_i' \times_{Y''} Y' \subset Y'$, $U_i = SpecA_i = U_i' \times_{Y''} Y'$, and $f_i' \in A_i$ equal to the image of $f_i'' \in A_i'$. Then the restriction of f_i'' under $k(Y''_{\underline{y}} \times X) \to k(Y'_{\underline{y}} \times X)$ equals the restriction of F/G to $k(Y'_{\underline{y}} \times X)$, for both are the restrictions of f to some open subset of $Y' \times X$ meeting $Y'_y \times X$ in a dense open subset. Thus, (*) for $\{f_i''\}$ implies (2.1.c) for F, G.

Using either the conditions (a), (b$'$), (c) of Definition 2.1 or the condition (*) of Proposition 2.2, we easily verify that $R_X(-)$ is functorial: any morphism $g : V \to Y$ induces a homomorphism

$$g^* : R_X(Y) \to R_X(V).$$

We view $R_X(-)$ as the functor of "invertible rational functions with specialization" as justified in the following proposition.

Proposition 2.3. Consider $f \in R_X(Y)$. Then for any geometric point $\underline{y} \in Y$, there is an invertible rational function $f_{\underline{y}} \in k(X_{\underline{y}})^*$, the **specialization of** f at $\underline{y}$, satisfying the following properties:

i.) if $Y' \to Y^j$, $\mathcal{L}$ on $Y' \times \overline{X}$, and $F, G \in \mathcal{L}(Y' \times \overline{X})$ are data for f as in (2.1), then $f_{\underline{y}} = F_{\underline{y}'}/G_{\underline{y}'}$ for any geometric point $\underline{y}'$ of Y' lifting $\underline{y}$.

ii.) if $\{f_i'\}$ are as in (*) of (2.2), then $f_{\underline{y}} = (f_i')_{\underline{y}'}$ whenever $\underline{y}'$ of V_i lifts $\underline{y}$.

Proof. By (2.1.c), we may use (i.) for a chosen set of defining data for $f \in R_X(Y)$ to define $f_{\underline{y}}$ depending only on $\underline{y}$ and not upon the choice of $\underline{y}'$ of Y' lifting $\underline{y}$. As discussed in the proof of Proposition 2.2, any data as in (ii.) is refined by data of similar form arising from data as in (i.). Thus, it suffices to show that $f_{\underline{y}}$ defined as in (i.) does not depend upon the choice of defining data $Y' \to Y^j$, $\mathcal{L}$ on $Y' \times \overline{X}$, and $F, G \in \mathcal{L}(Y' \times \overline{X})$. Given another choice of defining data $Y_1 \to Y^j$, $\mathcal{L}_1$ on $Y_1 \times \overline{X}$, and $F_1, G_1 \in \mathcal{L}_1(Y_1 \times \overline{X})$ define Y'' to be the fibre product of Y' and Y_1 over Y, define $\mathcal{L}''$ to be $\mathcal{L} \otimes \mathcal{L}_1$, and observe that $Z_{F \otimes F_1}, Z_{G \otimes F_1}, Z_{F \otimes G_1}$ are each flat over Y''. Then for any geometric point $\underline{y}''$ of Y'' mapping to the geometric points $\underline{y}'$ of Y', $\underline{y}_1$ of Y_1,

$$F_{\underline{y}'}/G_{\underline{y}'} = (F \otimes F_1)_{\underline{y}''}/(G \otimes F_1)_{\underline{y}''} = (F \otimes F_1)_{\underline{y}''}/(F \otimes G_1)_{\underline{y}''} = (F_1)_{\underline{y}_1}/(G_1)_{\underline{y}_1}.$$

The relationship between $R_X(Y)$ and $Z_{X,m-1}(Y)$ is given by the following theorem.

Theorem 2.4. For varieties Y, X with X of pure dimension $m \geq 1$, there exists a natural cycle map

$$cyc : R_X(Y) \to Z_{m-1,X}(Y)$$

with the property that if $\psi : Y \to Z_{m-1}(X)$ equals $cyc(f)$ for some $f \in Lim_j\{k(Y^j \times X)\}$ and if $\underline{y} \in Y(\overline{k})$, then $\psi(\underline{y})$ is the Chow coordinate of the principal divisor $(f_{\underline{y}})$ of $f_{\underline{y}} \in k(X_{\underline{y}})$.

Moreover, if Y is normal of pure dimension n, then

$$R_X(Y) \to Z_{X,m-1}(Y) \to Lim_j\{A_{m+n-1}(Y^j \times X)\}$$

is exact.

Proof. To define $cyc : R_X(Y) \to Z_{m-1,\underline{X}}(Y)$, consider some $f \in R_X(Y)$ given as $f = F/G$ with F, G global sections of $\mathcal{L}$ on $Y' \times \overline{X}$ whose zero loci are flat over Y', a blowing-up of Y^j. As in Example 1.2, Z_F, Z_G determine a morphism $h = h_{F/G} : Y' \to \mathcal{C}_{m-1}(\overline{X})^2$ sending a geometric point $\underline{y}$ to the Chow points of the cycles $(Z_F)_{\underline{y}} = Z_{F_{\underline{y}}}$, $(Z_G)_{\underline{y}} = Z_{G_{\underline{y}}}$. Denote by $C_h \subset Y' \times \mathcal{C}_{m-1}(\overline{X})^2$ the associated graph of h. Condition c.) of (2.1) implies that this graph determines a well defined function $\psi : Y(\overline{k}) \to Z_{m-1}(X)(\overline{k})$. Since $f_{\underline{y}} = F_{\underline{y}}/G_{\underline{y}}$ and since $(f_{\underline{y}}) = Z_{F_{\underline{y}}} - Z_{G_{\underline{y}}}$, we conclude that $\psi(\underline{y})$ is the Chow coordinate of $(f_{\underline{y}})$. In particular, we conclude that ψ does not depend upon the choice of data for f.

To verify that the composition $R_X(Y) \to Lim_j\{A_{m+m-1}(Y^j \times \overline{X})\}$ is trivial for Y of pure dimension n, we show that this composition sends f to its associated principal divisor (f) on $Y^j \times \overline{X}$. Let $U \subset Y'$ be an open subset mapping isomorphically onto its image in Y^j and let $h_U : U \to \mathcal{C}_{m-1}(\overline{X})^2$ be determined by the restrictions of Z_F, Z_G to $U \times \overline{X}$. Then the composition $R_X(Y) \to Z_{X,m-1}(Y) \to Lim_j\{A^1(Y^j \times \overline{X})\}$ sends f to the closure of the cycle on $U \times \overline{X}$ determined by h_U. On the other hand, since Z_F, Z_G dominate Y', each component of (f) dominates Y^j and so (f) is the closure of (f_U), the divisor of f viewed as a rational function on $U \times \overline{X}$. Since f_U equals F/G as a rational function on $U \times \overline{X}$, we immediately conclude that the difference of the cycles on $U \times \overline{X}$ determined by h_U is preciesly (f_U).

We now assume that Y is normal and proceed to prove the asserted exactness. Consider some $\psi : Y \to Z_{m-1}(X)$ whose associated cycle on $Y^j \times X$ is the divisor (f) of some rational function $f \in k(Y^j \times X)$. As argued above, there exists some open $U \subset Y^j$ such that the restriction $(f)_U$ of (f) to $U \times \overline{X}$ is given by a morphism $h_U : U \to \mathcal{C}_{m-1}(\overline{X})^2$; moreover, $(f)_U$ equals (f_U), the divisor associated to f viewed as a rational function on $U \times \overline{X}$.

We define $p : Y' \to Y^j$ to be some blowing-up such that $f \in k(Y' \times X)$ is of the form F/G, where F, G are global sections of a line bundle $\mathcal{L}$ on $Y' \times \overline{X}$ with the property that their zero loci are flat over Y'. Then Z_F, Z_G determine $h_{F/G} : Y' \to \mathcal{C}_{m-1}(\overline{X})^2$ whose restriction to U agrees with h_U and therefore must necessarily "descend to" ψ in view of the fact that the projection of the graph of $h_{F/G}$ to $Y \times \mathcal{C}_{m-1}(\overline{X})^2$ must equal C_ψ.

To verify that $f \in R_X(Y)$, we consider a geometric point $\underline{y}$ of Y and let $(\underline{y}) \to Y$ denote the inclusion of the spectrum of the strict hensel local ring $O_{Y,\underline{y}}$ of Y at $\overline{y}$. Let T denote $Y' \times_Y (\underline{y})$ and let $F_T, G_T \in \mathcal{L}(T \times X)$ be the restrictions of F, G. Since $O_{Y,\underline{y}}$ is an algebra over its residue field $k(\underline{y})$, we obtain a natural map $T \to \underline{y}$ which determines $T \times X \to X_{\underline{y}}$. Let $\underline{y}' \to T$ be a geometric point lifting $\underline{y}$ and let $F_{\underline{y}}, G_{\underline{y}} \in \mathcal{L}(T \times X)$ denote the images of F_T, G_T under the composition

$$\mathcal{L}(T \times X) \to \mathcal{L}(X_{\underline{y}'}) \simeq \mathcal{L}(X_{\underline{y}}) \to \mathcal{L}(T \times X).$$

Let

$$g_T = F_T/G_T \cdot G_{\underline{y}}/F_{\underline{y}} \in k(T \times X)^*.$$

Since the restriction of g_T to $k(X_{\underline{y}'})$ equals 1, the divisor $(g_T) \subset T \times X$ of g_T does not meet $X_{\underline{y}'}$. Our hypothesis that $h_{F/G} : Y' \to \mathcal{C}_{m-1}(\overline{X})^2$ descends to the well defined function $\psi : Y \to Z_{m-1}(X)$ implies that (g_T) does not meet $E \times X$, where $E \subset T$ is the geometric fibre $Y'_{\underline{y}}$. Since $E \subset T$ is the closed fibre of the proper map $T \to (\underline{y})$, we conclude that (g_T) is empty. Since $T \times X'$ is normal where $X' \subset X$ is the complement of the singular locus of X , we conclude as in Lemma 3.2 below that g_T is a globally defined regular function on $T \times X'$ with globally defined inverse. Thus, g_T restricts to

$$g = F_E/G_E \cdot G_{\underline{y}}/F_{\underline{y}} \in O(E \times X')^*.$$

Since Y^j is normal and p is proper and birational, the Zariski Connectedness Theorem implies that E is connected. Consequently, g lies in the image of the inclusion $O(X'_{\underline{y}})^* \to O(E \times X')^*$ (induced by the projection $E \times X' \to X'_{\underline{y}}$). Since g equals 1 when restricted to $X'_{\underline{y}'} \subset E \times X'$, we conclude that $g = 1$. Therefore, F_E/G_E equals $F_{\underline{y}}/G_{\underline{y}}$, whereas the latter lies in the image of $k(X_{\underline{y}})^*$. We conclude that $f \in k(Y^j \times X)$ satisfies condition (c.) of (2.1) and thus lies in $R_X(Y)$.

3. Acyclicity and the computation.

The key step in our computation of the homology of the complex $A_{m-1}(Speck, X)$ is the proof in Proposition 3.1 of the acyclicity of the complex associated to the functor $R_X(-)$. The formulation of such complexes goes back to an early definition of algebraic K-theory by M. Karoubi and O. Villamajor [K-V], and has subsequently been used by Bloch, Suslin, and others.

We recall that the algebraic singular n-simplex $\Delta[n]$ is defined to be $Speck[t_1, \ldots, t_n] = Speck[T_0, \ldots, T_n]/\sum T_i - 1$. There are natural (linear) face and degeneracy maps between these algebraic simplices, so that any contravariant, abelian group valued functor Φ on k-varieties determines a simplicial abelian group $\Phi(\Delta[*])$ whose abelian group of n-simplices is $\Phi(\Delta[n])$. We naturally associate a normalized chain complex $\tilde{\Phi}(\Delta[*])$ to such a simplicial abelian group which has the property

$$\pi_n(\Phi(\Delta[*])) = H_n(\tilde{\Phi}(\Delta[*])).$$

The complex $A_r(Y, X)$ is defined by

$$A_r(Y, X) = \tilde{Z}_{X,r}(Y \times \Delta[*]).$$

We say that the simplicial abelian group $\Phi(\Delta[*])$ is *acyclic* if all of its homotopy groups vanish (or, equivalently, if the homology of its normalized chain complex is 0).

The following proof of acyclicity is based upon a suggestion of Ofer Gaber.

Proposition 3.1. For any variety X, $R_X(\Delta[*])$ is acyclic.

Proof. It suffices to show that any finite subcomplex of $R_X(\Delta[*])$ is contained in a contractible subcomplex. Consider some finite complex $K \subset R_X(\Delta[*])$ with simplices $f_\alpha \in R_X(\Delta[n]) \cap k(Y^j \times X)^*$ for some sufficiently large j. Choose an affine open $U = SpecA$ of X such that there exist $P_\alpha, Q_\alpha \in A[t_1^{1/p^j}, \ldots, t_n^{1/p^j}]$ with $f_\alpha = P_\alpha/Q_\alpha$. Choose data for f_α as in Proposition 2.2: $Y'_\alpha \to Y^j$, affine covering $\{V_{\alpha,i}\}$ of Y'_α, affine open subsets $U_{\alpha,i} = SpecA_{\alpha,i} \subset V_{\alpha,i} \times X$, and $\{f'_{\alpha,i}\} \in A_{\alpha,i}$. Let $V \subset k(X)$ be the finite dimensional vector space over k spanned by the coefficients of the $\{f'_{\alpha,i}\}$ and choose $\rho \in A$ to be some non-zero rational function on X regular on U which is not the quotient of two non-zero elements of $V \otimes_k \overline{k}$.

Associate to $f_\alpha(t_1^{1/p^j}, \ldots, t_n^{1/p^j}) \in R_X(\Delta[n])$ the rational function

$$g_\alpha(t_1^{1/p^j}, \ldots, t_{n+1}^{1/p^j}) = \rho \cdot t_1^{1/p^j} + f_\alpha(t_2^{1/p^j}, \ldots, t_{n+1}^{1/p^j}) \cdot (1 - t_1^{1/p^j}).$$

We claim that g_α is an element of $R_X(\Delta[n+1])$, with data $\Delta[1]^j \times Y'_\alpha$, affine covering $\{\Delta[1]^j \times V_{\alpha,i}\}$ of $\Delta[1]^j \times Y'_\alpha$, affine open subsets $\Delta[1]^j \times U_{\alpha,i} \subset \Delta[1]^j \times Y'_\alpha \times X$, and

$$\{g'_{\alpha,i} = \rho \cdot t_1^{1/p^j} + f'_{\alpha,i} \cdot (1 - t_1^{1/p^j})\}.$$

Then our choice of ρ guarantees that $g'_{\alpha,i} \in k(X_{\underline{y}'})^*$ for every geometric point $\underline{y}'$ of $\Delta[1]^j \times V_{\alpha,i}$. Hence, condition (*) of (2.2) for $\{g'_{\alpha,i}\}$ follows from that condition for $\{f'_\alpha\}$ interpreted using (2.3.ii.).

To conclude the proof, we observe that the subcomplex of $R_X(\Delta[*])$ generated by the simplices g_α is the cone with vertex ρ on the complex K and is thereby a contractible complex containing K.

As pointed out to us by V. Voevodsky, a non-zero, non-invertible rational function can have empty divisor. (For example, the divisor of $\frac{y-x}{y+x} \in Spec(k[x,y]/y^2 - x^3 - x^2)$ is empty.) The following lemma introduces notation for such rational functions and recalls that this difficulty does not arise for normal varieties.

Lemma 3.2. We define $U_X(Y)$ for varieties Y, X with X of pure dimension $m \geq 1$ by

$$U_X(Y) = \ker\{cyc : R_X(Y) \to Z_{X,m-1}(Y)\}.$$

i.) $U_X(Y)$ is the subgroup of $Lim_j\{k(Y^j \times X)^*\}$ consisting of those invertible rational functions whose divisor (f) is empty.

ii.) If X is normal, then $U(X) \equiv U_X(Speck)$ is the group $O(X)^*$ of units of the ring $O(X)$ of global functions on X.

iii.) $U_X(\Delta[n]) = U(X)$ for any $n \geq 0$.

Proof. Since any rational function $f \in k(Y^j \times X)^*$ whose divisor is empty is necessarily an element of $R_X(Y)$, (i.) follows immediately from the definition of $cyc : R_X(Y) \to Z_{X,m-1}(Y)$. Moreover, (ii.) follows directly from the following well known fact (cf. [M;Thm3.8]): any noetherian normal domain A is the intersection $A = \cap A_{\mathcal{P}}$ of its localizations at all height one prime ideals $\mathcal{P}$. To verify (iii.), we may argue by induction to reduce to the case that $n = 1$. Replacing $t^{1/j}$ by s, we may assume $j = 0$. Consider some non-constant $f \in k(X)(t)$; we proceed to exhibit some geometric point of $X \times \Delta[1]$ at which f is regular and vanishes, thereby verifying that (f) is non-empty. Let $U = \underline{SpecA}$ be an affine open of X such that $f = p(t)/q(t)$ with $p(t), q(t) \in A[t]$; let $A \to A' \subset \overline{k(X)}$ be an algebra of finite type such that there exists some $\alpha \in A'$ with $p(\alpha) = 0, q(\alpha) \neq 0$. Then any homomorphism $\theta : A' \to \overline{k}$ sending t, α to the same element of $\overline{k}$ is a geometric point at which f is regular and vanishes.

The results of the following compuation agree with E. Nart's computation [N] of Bloch's higher Chow groups [B] in codimension one for an integral variety X normal over a perfect field k:

$$H_i(A_{m-1}(Speck, X)) = CH^1(X, i).$$

We remind the reader of our standing hypothesis that our ground field k is perfect.

Theorem 3.3. Let X be a variety of pure dimension $m \geq 1$. Then there exists an exact sequence of simplicial abelian groups

$$0 \to U(X) \to R_X(\Delta[*]) \to Z_{X,m-1}(\Delta[*]) \to A_{m-1}(X) \to 0.$$

Consequently,

$$H_0(A_{m-1}(Speck, X)) = \pi_0(Z_{X,m-1}(\Delta[*])) = A_{m-1}(X)$$

$$H_1(A_{m-1}(Speck, X)) = \pi_1(Z_{X,m-1}(\Delta[*])) = U(X)$$

$$H_i(A_{m-1}(Speck, X)) = \pi_i(Z_{X,m-1}(\Delta[*])) = 0 \quad , \quad i > 0.$$

Proof. Lemma 3.2 identifies $U_X(\Delta[*])$ as the kernel of $cyc : R_X Z_{X,m-1}(\Delta[*])$ and verifies that $U_X(\Delta[*])$ is the constant simplicial group $U(X)$. Since each of the projections $\Delta[n]^j \times X \to X$ is isomorphic to $\Delta[n] \times X \to X$ and the latter induces an isomorphism on Chow groups $A_{m-1}(X) \to A_{m+n-1}(\Delta[n] \times X)$ (cf. [Fu;3.3]), we conclude that $Lim_j\{A_{m+*-1}(\Delta[*] \times X)\}$ is the constant simplicial abelian group $A_{m-1}(X)$. Consequently, Theorem 2.4 verifies the exactness of the asserted exact sequence at $Z_{X,m-1}(\Delta[*])$. To complete the proof of the asserted exactness, we merely observe the surjectivity of

$$Z_{X,m-1}(\Delta[0]) = Z_{m-1}(X) \to A_{m-1}(X) = Lim_j\{A_{m+*-1}(\Delta[*] \times X)\}$$

The computation now follows by breaking up this exact sequence into two short exact sequences of simplicial abelian groups each of which induces a long exact sequence in homology groups of associated normalized chain complexes. Namely, the acyclicity of $R_X(\Delta[*])$ proved in Proposition 3.1 and the fact that the constant simplicial abelian groups $A_{m-1}(X)$, $U(X)$ are equal to their 0-th homology groups immediately imply the the asserted values of

$$H_i(A_{m-1}(Speck, X)) = \pi_i(Z_{X,m-1}(\Delta[*])).$$

4. Sheaves for the h-topology.

In [V], V. Voevodsky introduced a Grothendieck topology for schemes whose coverings $\{p_i : X_i \to X\}$ are finite families of morphisms of finite type such that $\coprod p_i : \coprod X_i \to X$ is a universal topological epimorphism for the Zariski topology. In this section, we briefly investigate the functor $Z_{X,r}(-)$ from the point of view of sheaves for this **h-topology** on the category of schemes of finite type over k. (The functors Φ we consider can be viewed as functors on non-reduced algebraic k-schemes by sending such a scheme Y to $\Phi(Y_{red})$).

Proposition 4.1. Let X be a projective variety. Then the contravariant functor sending a variety Y to $Z_{X,r}(Y)$ is the sheafification for the h-topology of the presheaf (i.e., contravariant functor)

$$Y \mapsto mor(Y, \mathcal{C}_r(X))^+$$

which sends Y to the group completion of the abelian monoid of morphisms from Y to the Chow monoid $\mathcal{C}_r(X)$.

Proof. Let $\{p_i : V_i \to Y\}$ be a covering for the h-topology, so that each of the finitely many $p_i : V_i \to Y$ is a morphism of finite type and $\coprod p_i : \coprod V_i \to Y$ is a universal topological epimorphism. Consider the sequence of abelian groups

$$Z_{X,r}(Y) \to \prod Z_{X,r}(V_i) \to \prod Z_{X,r}(V_{i,j})$$

where $V_{i,j}$ denotes the fibre product of V_i and V_j over Y. Clearly, the composite is 0 and the left arrow is injective (for $\coprod V_i \to Y$ is surjective). To verify the sheaf axiom, we may (and shall) assume that each V_i is irreducible and dominates an irreducible component Y_j of Y ([V;3.1.3]).

Consider a compatible family $\{\psi_i\} \in \prod Z_{X,r}(V_i)$ and represent each $\psi_i : V_i \to Z_r(X)$ by $\Gamma_i \equiv \Gamma_{\psi_i} \subset V_i \times \mathcal{C}_r(X)^2$, the minimal correspondence representing ψ_i as in proof of Proposition 1.4. Then Γ_i is the closure in $V_i \times \mathcal{C}_r(X)^2$ of some point $(\nu_i, \gamma_i, \delta_i)$. Let ω_j be a generic point of Y and observe that if $\nu_i, \nu_{i'}$ are generic points of $V_i, V_{i'}$ mapping to ω_j then the compatibility condition on $\{\psi_i\}$ together with the minimality condition on $\{\Gamma_i\}$ imply that $(\gamma_i, \delta_i) = (\gamma_{i'}, \delta_{i'})$; let this common value be (γ_j, δ_j). We define $\Gamma \subset Y \times \mathcal{C}_r(X)^2$ to be the closure of the finite set $\{(\omega_j, \gamma_j, \delta_j)\}$.

So defined, $\Gamma \to Y$ is generically one-to-one, proper, and surjective. Since a point of $V_i \times \mathcal{C}_r(X)^2$ lies in Γ_i if and only if this point is a specialization of $(\nu_i, \gamma_i, \delta_i)$, $\Gamma_i = V_i \times_Y \Gamma$. The fact that each $\Gamma_i \subset V_i \times \mathcal{C}_r(X)^2$ corresponds to the graph of a set-theoretic function on geometric points together with the compatibility condition on the $\{\psi_i\}$ implies that Γ likewise corresponds to the graph of a set-theoretic function on geometric points. Thus, Γ represents $\psi \in Z_{X,r}(Y)$ restricting to $\{\psi_i\}$, thereby verifying the sheaf axiom for $Z_{X,r}(-)$.

Observe that the natural transformation

$$\mathrm{mor}(-, \mathcal{C}_r(X))^+ \to Z_{X,r}(-)$$

is injective, for if $f, g : Y \to \mathcal{C}_r(X)$ are distinct morphisms then their "difference" $h_{f/g} : Y \to Z_r(X)$ is not 0 (in fact, non-zero generically). On the other hand, given any

$\psi \in Z_{X,r}(Y)$, we consider $\Gamma_\psi \subset Y \times \mathcal{C}_r(X)^2$ as in the proof of Proposition 1.3. Then, $pr_1 : \Gamma_\psi \to Y$ is proper, surjective, and generically radiciel; in particular, this map is a universal topological epimorphism. The restriction of ψ to $Z_{X,r}(\Gamma_\psi)$ lies in the image of $\mathrm{mor}(\Gamma_\psi, \mathcal{C}_r(X))^+$, whereas the injectivity of $\mathrm{mor}(-,{}_r(X))^+ \to Z_{X,r}(-)$ implies that this restriction has equal images in $\mathrm{mor}(\Gamma_\psi \times_Y \Gamma_\psi, \mathcal{C}_r(X))^+$. We conclude that ψ lies in the image of the Y-sections of the h-sheaf associated to the presheaf $\mathrm{mor}(-, \mathcal{C}_r(X))^+$.

Proposition 4.2. Let X be a projective variety, $W \subset X$ be a closed subvariety, and $U \subset X$ the open complement of W. Then

$$0 \to Z_{W,r}(-) \to Z_{X,r}(-) \to Z_{U,r}(-) \to 0$$

is a short exact sequence of abelian sheaves in the h-topology.

Proof. The proof of Proposition 4.1 that $Z_{X,r}(-)$ is a sheaf for the h-topology applies essentially verbatim to show that $Z_{U,r}(-)$ is also a sheaf for the h-topology. The natural map

$$Z_{X,r}(Y)/Z_{W,r}(Y) \to Z_{U,r}(Y)$$

is clearly injective. The argument at the end of the proof of Proposition 4.1 verifies the surjectivity of this map on the level of sheaves in the h-topology.

Remark 4.3 For the purposes of generalization, one should interpret Theorem 3.3 in terms of an exact sequence of sheaves for the h-topology. From this point of view, $cyc : R_X(-) \to Z_{X,m-1}(-)$ is a homomorphism of h-sheaves whose kernel and cokernel are "homotopy-invariant."

We recall that the Lawson homology groups of a complex quasi-projective variety X with projective closure $X \subset \overline{X}$ can be defined as the homotopy groups of of the topological abelian group $Z_r(X)(\mathbf{C})$,

$$L_rH_{2r+i}(X) = \pi_i(Z_r(X)(\mathbf{C})).$$

(The topological group structure on $Z_r(X)(\mathbf{C})$ is that induced as a quotient of $\mathcal{C}_r(\overline{X})^2(\mathbf{C})$ given the analytic topology).

Observe that any $\psi : \Delta[n] \to Z_r(X)$ naturally determines a continuous map $\psi_{\mathbf{C}} : \Delta[n](\mathbf{C}) \to Z_r(X)(\mathbf{C})$, so that we obtain a natural homomorphism of simplicial abelian groups

$$Z_{X,r}(\Delta[*]) \to Sing.Z_r(X)(\mathbf{C}).$$

As in [F-G], let $\tilde{Z}_r(X)$ denote the normalized chain complex associated to $Sing.Z_r(X)(\mathbf{C})$, so that

$$L_rH_{2r+i}(X) = H_i(\tilde{Z}_r(X)).$$

We conclude this paper with the following theorem whose sketched proof is merely a rephrasing of the main arguments of [S-V].

Theorem 4.4. If X is a complex quasi-projective variety, then for any $n > 0$ the above map of simplicial abelian groups

$$Z_{X,r}(\Delta[*]) \to Sing.Z_r(X)(\mathbf{C})$$

induces an isomorphism:

$$\pi_*(Z_{X,r}(\Delta[*]), \mathbf{Z}/n) \equiv H_*(A_r(Spec\mathbf{C}, X) \otimes \mathbf{Z}/n) \simeq H_*(\tilde{Z}_r(X) \otimes \mathbf{Z}/n).$$

Sketch of proof, summarizing the arguments of [S-V]. Since the Chow monoid $\mathcal{C}_r(X)$ has the cancellation property, $Z_{X,r}(Y)$ is torsion free for any Y; consequently, $A_r(Spec\mathbf{C}, X)$ is also torsion free. We conclude that the assertion of the theorem is equivalent to the statement that the natural map $Z_{X,r}(\Delta[*]) \to Sing.Z_r(X)(\mathbf{C})$ induces an isomorphism of (hyper-) ext-groups

$$Ext^*(Sing.Z_r(X)(\mathbf{C}), \mathbf{Z}/n) \simeq Ext^*(Z_{X,r}(\Delta[*]), \mathbf{Z}/n)$$

for any $n > 0$, where we have identified the simplicial abelian groups with their associated chain complexes.

Let $\mathcal{F}$ denote the h-sheaf $Z_{X,r}(-)$ and let $\mathcal{F}_*$ denote the complex of h-sheaves with $\mathcal{F}_q(-) = \mathcal{F}(- \times \Delta[q]) = Z_{X,r}(- \times \Delta[q])$. The "homotopy invariance" of ext-groups of h-sheaves mod-n enables us to conclude that the spectral sequence

$$E_1^{p,q} = Ext^p_{h-sh}(\mathcal{F}_q, \mathcal{Z}/n) \Rightarrow Ext^{p+q}_{h-sh}(\mathcal{F}_*, \mathbf{Z}/n)$$

degenerates at E_2. This implies that

$$Ext^*_{h-sh}(\mathcal{F}, \mathbf{Z}/n) \simeq Ext^*_{h-sh}(\mathcal{F}_*, \mathbf{Z}/n)$$

(where Ext_{h-sh} denotes ext in the abelian category of abelian h-sheaves).

The second spectral sequence for these hyper-ext groups has the form

$${}'E_2^{p,q} = Ext^p_{h-sh}(\mathcal{H}_q(\mathcal{F}_*)\tilde{}, \mathbf{Z}/n) \Rightarrow Ext^{p+q}_{h-sh}(\mathcal{F}_*, \mathbf{Z}/n)$$

where $\mathcal{H}_q(\mathcal{F}_*)\tilde{}$ is the h-sheaf associated to the presheaf sending Y to $H_q(\mathcal{F}(Y \times \Delta[*]))$. The key result [S-V;4.1] implies that

$$Ext^p_{h-sh}(\mathcal{H}_q(\mathcal{F}_*)\tilde{}, \mathbf{Z}/n) = Ext^p(H_q(\mathcal{F}(\Delta[*]), \mathbf{Z}/n).$$

(This is a form of "Suslin rigidity" for the presheaves $\mathcal{H}_q(\mathcal{F}_*)$.) Consequently, we conclude

$$Ext^*_{h-sh}(\mathcal{F}, \mathbf{Z}/n) \simeq Ext^*_{h-sh}(\mathcal{F}_*, \mathbf{Z}/n) \simeq Ext^*(\mathcal{F}(\Delta[*]), \mathbf{Z}/n).$$

As discussed in [S-V], a similar argument applies to abelian sheaves on the category of topological spaces admitting a triangulation as well as to abelian sheaves on the etale site. We define $\mathcal{F}^{top}(T)$ to be $Hom_{cont}(T, Z_r(X)(\mathbf{C}))$, whereas we define $\mathcal{F}^{et}$ to be the sheaf on the etale site associated to the preseheaf sending W to $mor(W,\mathcal{C}_r(X))^+$. As sketched in the preceding paragraph, we conclude that

$$Ext^*_{top-sh}(\mathcal{F}^{top},\mathbf{Z}/n) \simeq Ext^*_{top-sh}(\mathcal{F}^{top}_*,\mathbf{Z}/n) \simeq Ext^*(\mathcal{F}^{top}(\Delta[*]),\mathbf{Z}/n)$$

$$Ext^*_{et-sh}(\mathcal{F}^{et},\mathbf{Z}/n) \simeq Ext^*_{et-sh}(\mathcal{F}^{et}_*,\mathbf{Z}/n) \simeq Ext^*(\mathcal{F}^{et}(\Delta[*]),\mathbf{Z}/n).$$

Let j (respectively, j') denote the morphism from the h-site (resp., topological site) to the etale site. One verifes that

$$\mathcal{F} = j^*\mathcal{F}^{et} \quad , \quad \mathcal{F}^{top} = j'^*\mathcal{F}^{et}.$$

We thus obtain the following commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(\Delta[*]) & \rightarrow & j_*\mathcal{F}_* & \leftarrow & j_*\mathcal{F} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}^{et}(\Delta[*]) & \rightarrow & \mathcal{F}^{et}_* & \leftarrow & \mathcal{F}^{et} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}^{top}(\Delta[*]) & \rightarrow & j'_*\mathcal{F}^{top}_* & \leftarrow & j'_*\mathcal{F}^{top} \end{array} .$$

As discussed above, the horizontal arrows of this diagram induce an isomorphism of ext-groups with $\mathbf{Z}/n$- coefficients. The fact that the right vertical arrows induce an isomorphism of ext-groups with $\mathbf{Z}/n$-coefficients is a fundamental comparison theorem relating the h-topology with the etale and analytic topologies for complex varieties (cf. [S-V;10.11],[V]). We thus conclude that the left-vertical arrows induce the isomorphism

$$Ext^*(Sing.Z_r(X)(\mathbf{C}),\mathbf{Z}/n) = Ext^*(\mathcal{F}^{top}(\Delta[*]),\mathbf{Z}/n)$$

$$\simeq Ext^*(\mathcal{F}(\Delta[*]),\mathbf{Z}/n) = Ext^*(Z_{X,r}(\Delta[*])(\mathbf{C}),\mathbf{Z}/n).$$

As mentioned by Suslin and Voevodsky, validity of resolution of singularities would suggest that some analogue of this theorem should be valid for algebraically closed fields of positive characteristic.

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