MODULES FOR $\mathbb{Z}/p \times \mathbb{Z}/p$

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Abstract. We investigate various aspects of the modular representation theory of $\mathbb{Z}/p \times \mathbb{Z}/p$ with particular focus on modules of constant Jordan type. The special modules we consider and the constructions we introduce not only reveal some of the structure of $(\mathbb{Z}/p \times \mathbb{Z}/p)$-modules but also provide a guide to further study of the representation theory of finite group schemes.

0. Introduction

In this paper, we investigate finite dimensional kG-modules, where k is an infinite field of characteristic $p > 0$ and $G = \mathbb{Z}/p \times \mathbb{Z}/p$. Our objective is to introduce some new constructions and invariants for modules and to explore some of their properties. For $p > 2$, kG has wild representation type so that this elementary group algebra provides a good test for various constructions and techniques in modular representation theory. For example, the special case of $G = \mathbb{Z}/p \times \mathbb{Z}/p$ proved to be critical in the proof given in [7] that the generic and maximal Jordan types are well defined for a finite dimensional module over an arbitrary finite group scheme.

In joint work with Julia Pevtsova, we introduced a special class of modules for finite groups schemes, those of constant Jordan type. Most of the kG-modules we consider in this paper satisfy this property of constant Jordan type, and much of our effort is directed to understanding as much as possible for such modules. We introduce a more restrictive property, the "equal images property", satisfied by certain kG-modules and closed under taking quotients. We observe in Proposition 4.8 that there is a vast array of non-isomorphic indecomposable kG-modules which satisfy this equal images property, suggesting that kG-modules of constant Jordan type constitute a wild category for $p > 2$.

Our examples and constructions not only reveal interesting modules for kG but also suggest possible extensions from G to more general finite group schemes.

We begin with the equal images property and a particularly explicit class of kG-modules satisfying the property. These we call "W modules". W modules of the form $W_{n,2}$ have appeared in earlier work as "zig-zag" modules. Despite their seemingly special nature, the class of W modules provides a splitting of the Jordan type

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functor from the Grothendieck group of modules of constant Jordan type to $\mathbb{Z}^p$ (Theorem 3.2). Moreover, every kG-module with the equal images property is a quotient of a W module (Theorem 4.4).

In [6], the second author and Julia Pevtsova introduced vector bundles associated to finite dimensional modules for an infinitesimal group scheme which have constant j-type. This construction of vector bundles holds considerable promise in that it distinguishes various modules with the same local Jordan type and might exhibit explicit newly discovered vector bundles on quite elaborate singular varieties. In the special case of the infinitesimal group scheme $\mathbb{G}_{a(1)}^2$ with group algebra isomorphic to kG, this construction produces vector bundles on $\mathbb{P}^1$ which are accessible to computation. We give explicit determination of the first and second kernel bundles for W modules.

We introduce the construction of the "generic kernel" (as well as the "generic image") of a finite dimensional kG-module, a module which has the equal image property and thus is of constant Jordan type. This is essentially an extension of a construction introduced in [7] in order to prove that the "maximal Jordan type" of a kG-module is well defined, independent of the choice of generators of the augmentation ideal of kG. The generic kernel of a finite dimensional kG-module M can be characterized as the maximal submodule of M which has the equal images property (Proposition 6.8).

In Theorem 7.10 we exhibit an interesting filtration on kG-modules of constant rank which differs from the radical filtration. This filtration involves duality relating generic kernels and generic images, providing further insight into specific kG-modules. A cautionary example is provided in Example 8.4.

Corollary 10.3 identifies all cyclic kG-modules of constant Jordan type: they are all of the form kG/I^t, a quotient of kG by some power of the augmentation ideal. This result is surprisingly difficult to prove, and requires properties of the Wronskian of a collection of polynomials with coefficients in a field of characteristic $p > 0$.

We conclude this paper by identifying various vector bundles over $\mathbb{P}^1$ associated with certain classes of kG-modules. For this, we invoke a basic property of generic kernels and utilize duality to determine kernel and image bundles for cyclic kG-modules of constant Jordan type. We also determine kernel bundles for Heller shifts of the trivial module and certain kG-modules of constant rank associated to nilpotent cohomology classes.

We reiterate that unless specified otherwise p will denote a prime number, k will denote an infinite field of characteristic p, and G with denote the group $\mathbb{Z}/p \times \mathbb{Z}/p$.

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1. The equal images property

Although this paper concerns the representation theory of $\mathbb{Z}/p \times \mathbb{Z}/p$, this first section is valid for an elementary abelian p-group of arbitrary rank. Thus, in this section, we take $G = (\mathbb{Z}/p)^r$ for some $r \geq 2$. Our purpose in this section is to introduce the "equal images property" for a finite dimensional kG-module and explore some of its implications.

For notation, we let $kG = k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p)$. The radical of kG is the augmentation ideal $\mathrm{Rad}(kG)$ generated by $x_1, \dots, x_r$ where for each i, x_i is the class of t_i modulo the ideal $(t_1^p, \dots, t_r^p)$. Thus, $x_i^p = 0$ for all i.

We specialize to the case of an elementary abelian p-group the definition of a π-point of G introduced and exploited in [5] as well as the definition of a kG-module of constant Jordan type. The isomorphism type of a module M over $k[t]/(t^p)$ is determined entirely by its Jordan type, which is the sequence of sizes of the Jordan block of the action of t on M. Thus the Jordan type of M as a $k[t]/(t^p)$-module is a partition of the dimension of M. We write the Jordan type of M as $a_p[p]+\cdots+a_1[1]$, meaning that the Jordan form of the matrix of t on M has a_i blocks of size i.

Definition 1.1. Let G be an elementary abelian p-group. A π-point $\alpha_K : K[t]/t^p \to KG$ is a flat map of K-algebras for some field extension K/k. Two π-points, $\alpha_K : K[t]/t^p \to KG$ and $\beta_L : L[t]/t^p \to LG$ are equivalent if for any finite dimensional kG-module M, the restriction of $K \otimes M$ along α_K is a free KG-module if and only if the restriction of $L \otimes M$ along β_L is a free LG-module.

A finite dimensional kG-module M is said to have constant Jordan type if the Jordan type of $\alpha_K^*(M_K)$ is independent of π-point α_K. (Here, $M_K = K \otimes M$.)

The data of a K-algebra homomorphism $\alpha_K : K[t]/t^p \to KG$ is equivalent to a choice of

$$\ell_\alpha \;=\; \alpha_K(t) \in KG; \tag{1}$$

the condition that α_K be flat is equivalent to the condition that ℓ_α be an element of $\mathrm{Rad}(KG) \setminus \mathrm{Rad}^2(KG)$, an element in the radical of KG (since $t \in K[t]/t^p$ is nilpotent) but not in the square of the radical of KG.

The condition that the finite dimensional kG-module M has constant Jordan type can be reformulated in more classical terms as follows.

Proposition 1.2. *A finite dimensional kG-module M has constant Jordan type if and only if for some algebraic closure K/k and for all non-zero r-tuples $0 \neq \underline{a} = (a_1, \dots, a_r) \in K^r$, the Jordan type of $\sum_i a_i x_i$ acting on M_K is independent of $\underline{a}$.*

Proof. Clearly, if M has constant Jordan type, then that type is the Jordan type at every π-point $\alpha_{\underline{a}}$, $0 \neq \underline{a} = (a_1, \dots, a_r) \in K^r$ (i.e., is the type of every $\alpha_{\underline{a}}^*(M_K)$). Here, $\alpha_{\underline{a}} : K[t]/t^p \to KG$ sends t to $\sum a_i t_i$.

We recall that there is a partial ordering on Jordan types of π-points on M given by the dominance ordering on partitions of the dimension of M. We use [7, 2.7], which asserts the following two non-trivial properties of maximal and generic Jordan types. First, the Jordan type of M at any π-point whose equivalence class in $\Pi(G) = \mathbb{P}^{r-1}$ is a generic point is greater or equal to the Jordan type at any π-point of kG. Second, if $\alpha_K^*(M_K)$ achieves this maximal Jordan type (i.e., the generic Jordan type), then this maximal Jordan type is also the Jordan type for any $\beta_L^*(M_L)$ whenever α_K is equivalent to β_L.

To prove the converse, we now assume that $\alpha_{\underline{a}}^*(M_K)$ has Jordan type independent of $0 \neq \underline{a} \in K^r$ for some K/k algebraically closed. Then for an explicit choice of representative of the generic point of $\Pi(G)$ the Jordan type is the same as that of each $\alpha_{\underline{a}}^*(M_K)$ (using the fact that maximality of the Jordan type of M for points of $\mathbb{A}^r$ is an open condition). Thus, this common Jordan type is the maximal Jordan type of M among all π-points. Because maximality of the Jordan type of M is achieved by some representative of a point of $\Pi(G)$ if and only if it is achieved at every representative, maximality of the Jordan type of M is an open condition on the scheme of $\Pi(G)$ of equivalence classes of π-points. Because the points of $\Pi(G)$ represented by π-points of the form $\alpha_{\underline{a}}$ are dense, we conclude that maximality holds at every π-point of kG. In other words, M must have constant Jordan type. □

In much of what follows we take some care to be independent of the choice of coefficient fields. Some of the reasons for this are revealed in the following cautionary example.

Example 1.3. Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ and suppose that k is not algebraically closed. Write $kG = k[x,y]/(x^p, y^p)$. Let $f(z)$ be a polynomial of degree n with no root in k. Consider the 2n-dimensional kG-module with x, y acting as

$$x \to \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 0 \\ C_n & 0 \end{pmatrix}.$$

where I_n is the $n \times n$ identity matrix and C_n is the companion matrix of $f(z)$ (so that $f(z) = \mathrm{Det}(zI_n - C_n)$).

Then for any $0 \neq (a,b) \in k^2$, the image of $ax + by$ on M has dimension n, equal to $\mathrm{Rad}(M)$. This easily implies that the image of $ax + by$ on M is independent of $0 \neq (a,b) \in k^2$. Thus, the Jordan type of M at k-rational points of $\mathbb{A}^2$ is independent of the choice of k-rational point, and in fact equals the maximal Jordan type of M. However, if we take K/k to be finite field extension in which $f(x)$ has a root γ, then the rank of $\gamma x - y : M_K \to M_K$ has dimension less than n. Hence, M does not have constant Jordan type or constant rank.

We now introduce a property of certain kG-modules which we shall see implies the property of constant Jordan type.

Definition 1.4. A finite dimensional kG-module M is said to have **the equal images property** if

$$(\ell_\alpha(M_K))_\Omega \;=\; (\ell_\beta(M_L))_\Omega$$

for any two π-points $\alpha_K : K[t]/t^p \to KG$ and $\beta_L : L[t]/t^p \to LG$ and for any field extension Ω of both K, L. Here, as (1), $\ell_\alpha = \alpha_K(t)$.

The equal images property was called the "constant images property" in [1].

We shall need some equivalent conditions for the equal images property. The following lemma will be useful for this purpose. Here we make no assumptions on the field k other than the characteristic.

Lemma 1.5. *Let ℓ be any element which is in $\mathrm{Rad}(kG)$ but not in $\mathrm{Rad}^2(kG)$. We write $\ell = \sum_{i=1}^r a_i x_i + u$ for $a_1, \ldots, a_r$ in k and $u \in \mathrm{Rad}^2(kG)$. Then for any kG-module M it must be that $\ell M = \mathrm{Rad}(M)$ if and only if $(\sum_{i=1}^r a_i x_i)M = \mathrm{Rad}(M)$.*

Proof. Let $N = \mathrm{Rad}(M)$ and suppose that $\ell M = \mathrm{Rad}(M) = N$. Then $uN \subseteq \mathrm{Rad}(N)$ and we have that $N = \ell M = (\sum_{i=1}^r a_i x_i)M + \mathrm{Rad}(N)$. Then by Nakayama's Lemma [3], we have that $(\sum_{i=1}^r a_i x_i)M = N = \mathrm{Rad}(M)$. This proves one direction. The proof in the other direction is almost identical. $\square$

With the above lemma, we can give an equivalent definition for the equal images property.

Proposition 1.6. *Suppose that M is a kG-module. The following are equivalent.*

(1) *The module M has the equal images property.*
(2) *For any extension K of k, and any r-tuple $\underline{a} = (a_1, \ldots, a_r) \in K^r$ of elements of K, such that not all of the a_i's are zero, we have that $(\sum_{i=1}^r a_i x_i)M_K = \mathrm{Rad}(M_K)$.*

Proof. First suppose that M has the equal images property. Then for all i and j we have that $x_i M = x_j M$ and hence that $x_i M = \sum_{j=1}^r (x_j M) = \mathrm{Rad}(M)$. Now suppose that K is any extension of k and that $\alpha_K : K[t]/(t^p) \to kG$ is the π-point given by $\alpha_K(t) = \sum_{i=1}^r a_i x_i$. Then by the equal images property

$$\ell_\alpha = x_1 M_K = x_1(K \otimes M) = K \otimes x_1 M = K \otimes \mathrm{Rad}(M) = \mathrm{Rad}(M_K).$$

This shows that (1) implies (2). The reverse implication is straightforward with the help of Lemma 1.5. $\square$

A main point of the next proposition is that everything becomes much easier when the field of coefficients is algebraically closed.

Proposition 1.7. *Let M be a finite dimensional kG-module and let K be an algebraic closure of k. Then the following are equivalent:*

(1) *M has the equal images property.*
(2) *For all $\ell \in \mathrm{Rad}(KG) \setminus \mathrm{Rad}^2(KG)$, we have that $\ell(M_K) = \mathrm{Rad}(M_K)$.*
(3) *The submodule $\ell(M_K) \subset M_K$ is independent of $\ell \in \mathrm{Rad}(KG) \setminus \mathrm{Rad}^2(KG)$.*
(4) *For all $0 \neq \underline{a} \in K^r$, $Im\{\sum_{i=1}^r a_i t_i : M_K \to M_K\} = \mathrm{Rad}(M_K)$.*
(5) *The submodule $(\sum_{i=1}^r a_i t_i)(M_K) \subset M_K$ is independent of $0 \neq \underline{a} \in K^r$.*

Proof. The implication (1) $\Rightarrow$ (2) is a consequence of 1.5 and 1.6. Also, (2) $\Leftrightarrow$ (4), by 1.5. It is obvious that (2) $\Rightarrow$ (3) $\Rightarrow$ (5). If (5) holds, then we must have that $x_i M_K = x_j M_K$ for all i and j. Therefore, for any $0 \neq \underline{a} \in K^r$, we have that $\mathrm{Im}\,\{\sum_{i=1}^r a_i t_i : M_K \to M_K\} = x_1 M_K = \mathrm{Rad}(M_K)$. Hence, (5) $\Rightarrow$ (4). It remains only to show that (1) is implied by any of the other statements.

To prove that (2) $\Rightarrow$ (1), it suffices to assume (2) and consider some flat map $\beta_L : L[t]/t^p \to LG$ with L/K finitely generated, then prove that $\beta_L(M_L) = \mathrm{Rad}(M_L)$. We proceed by contradiction. Assume that $\ell_\beta(M_L) \neq \mathrm{Rad}(M_L)$, so that the dimension $\ell_\beta(M_L)$ is strictly less than the dimension of $\mathrm{Rad}(M_L)$. Choose a finitely generated K-subalgebra $A \subset L$ with field of fractions L such that there exists some flat A-algebra homomorphism $\beta_A : A[t]/t^p \to AG$ with the property that $\beta_L = L \otimes_A \beta_A$. For some (in fact, almost all) specializations $\phi : A \to K$ determining $\alpha_K \equiv K \otimes_A \beta_A : K[t]/t^p \to KG$, the dimension of $\ell_\alpha(M_K)$ (which equals the specialization of the Noetherian A-module $(\beta_A(t)(M_A)$ at ϕ) is strictly less than the dimension of $\mathrm{Rad}(M_K)$. This contradicts (2). $\square$

We give the following first example of a kG-modules with the equal images property.

Example 1.8. Let $I = \mathrm{Rad}(kG) \subset kG$ be the augmentation ideal. Then I^j has constant Jordan type for all j, $0 \leq j \leq rp - 1$ (and $I^{rp} = 0$). This is a consequence of the fact that I is invariant under all automorphisms of the group algebra and whenever u, v are two elements of $\mathrm{Rad}(kG)$ that are not in $\mathrm{Rad}^2(kG)$, then there is an automorphism that takes u to v. On the other hand, I^j has the equal images property if and only if $(r-1)p \leq j \leq rp - 1$. That is, for $j \leq (r-1)p$, it is straightforward to construct a monomial in $x_1, \dots, x_r$ that is a multiple of x_1 but not a multiple of x_2. (See Example 2.2 below.)

The preceding example suggests the following relationship between the constant Jordan type property and the equal images property.

Proposition 1.9. *Let M be a finite dimensional kG-module with the equal images property. Then $\mathrm{Rad}^s(M)$ also has the equal images property for all $s > 0$, and M has constant Jordan type.*

Proof. Let K/k be an algebraic closure, and suppose that ℓ_1 and ℓ_2 are two elements of $\mathrm{Rad}(KG)$ that are not in $\mathrm{Rad}^2(kG)$. Then

$$\ell_1 \ell_2 M = \ell_1 \mathrm{Rad}(M_K) = \ell_2 \ell_1 M_K = \ell_2 \mathrm{Rad}(M_K).$$

Because $\mathrm{Rad}(M_K) = \mathrm{Rad}(M)_K$, we have that $\mathrm{Rad}(M)$ has the equal images property by Proposition 1.7. Applying this again we get that $\mathrm{Rad}(\mathrm{Rad}(M)) = \mathrm{Rad}^2(M)$ has the equal images property and the first statement follows by a finite induction. Moreover, we see that for $\ell_1, \dots, \ell_i \in \mathrm{Rad}(KG) \setminus \mathrm{Rad}^2(KG)$, $\ell_i \cdots \ell_1(M_K) = \mathrm{Rad}^i(M_K) = \ell_1^i(M_K)$.

Now, the Jordan type of $\ell = \ell_1$ on M_K is determined by the ranks of the operators ℓ^i, $1 \le i < p$. Since M has the equal images property, the rank of ℓ^i is the dimension of $\operatorname{Rad}^i(M_K)$. Hence M has constant Jordan type by Proposition 1.2. $\square$

One pleasing aspect of the equal images property is its stability under taking quotients. The proof of the following proposition is essentially immediate, a consequence of the observation that if $M \to \overline{M}$ is a quotient map of kG-modules then the image of any $\alpha_K(t)$ on $\overline{M}$ is simply the quotient of the image of $\alpha_K(t)$ on M.

Proposition 1.10. *Let M be a kG-module with the equal images property and let $M \to \overline{M}$ be a quotient of kG-modules. Then $\overline{M}$ also has the equal images property.*

The above stability of the equal images property should be contrasted with the weaker property of constant Jordan type: any free kG-module has constant Jordan type, but most quotients do not.

2. W modules and the equal images property

In this and subsequent sections, G will denote the finite group $\mathbb{Z}/p \times \mathbb{Z}/p$ and k will denote a field of characteristic p. For notation, we let $kG = k[t_1, t_2]/(t_1^p, t_2^p)$, having radical generated by x and y which are the classes of t_1 and t_2.

As considered in [2, §2] (where the dual module is considered), the "zig-zag" modules

$$\begin{tikzcd}
\overset{\langle v_1\rangle}{\bullet} \arrow[dr, "y"] & & \overset{\langle v_2\rangle}{\bullet} \arrow[dl, "x"'] \arrow[dr, "y"] & & \cdots & & \overset{\langle v_n\rangle}{\bullet} \arrow[dl, "x"'] \\
& \bullet & & \bullet & \cdots & \bullet &
\end{tikzcd}$$

have constant Jordan type $(n-1)[2] + [1]$. The nodes in the diagram represent elements of a basis for the module. The arrows indicate the actions of x and y on these basis elements. We can write such a module (which we denote by $W_{n,2}$) as the kG-module generated by elements $\{v_1, \dots, v_n\}$ with relations generated by

$$xv_1 = 0 = x^2 v_n = yv_n; \quad x^2 v_i = 0 = yv_i - xv_{i+1}, \quad \text{for} \quad 1 \le i < n-1.$$

For $p \ge 3$, we have the following analogue of zig-zag modules (which we denote $W_{n,3}$):

$$\begin{tikzcd}
\overset{\langle v_1\rangle}{\bullet} \arrow[dr, "y"] & & \overset{\langle v_2\rangle}{\bullet} \arrow[dl, "x"'] \arrow[dr, "y"] & & \overset{\langle v_3\rangle}{\bullet} \arrow[dl, "x"'] & \cdots & \overset{\langle v_{n-1}\rangle}{\bullet} \arrow[dr, "y"] & & \overset{\langle v_n\rangle}{\bullet} \arrow[dl, "x"'] \\
& \bullet \arrow[dr, "y"] & & \bullet \arrow[dl, "x"'] \arrow[dr, "y"] & & \cdots & & \bullet \arrow[dl, "x"'] & \\
& & \bullet & & \bullet & \cdots & \bullet & &
\end{tikzcd}$$

As a kG-module, $W_{n,3}$ is generated by $\{v_1, \ldots, v_n\}$ with relations generated by

$$xv_1 = 0 = yv_n = x^3 v_n : \quad x^3 v_i = 0 = yv_i - xv_{i+1}, \quad \text{for} \quad 1 \leq i < n-1.$$

Clearly, the Jordan type of either x or y on $W_{n,3}$ is $(n-2)[3]+[2]+[1]$. As verified more generally in Proposition, 2.3, $W_{n,d}$ has the equal images property so that by Proposition 1.9, it too has constant Jordan type.

We proceed to investigate these "W" modules. Here is the general definition.

Definition 2.1. Let $n \geq d \geq 1$ and $d \leq p$. The W module $W_{n,d}$ is the kG-module generated by $\{v_1, \ldots, v_n\}$ with relations generated by

$$xv_1 = 0 = yv_n = x^d v_n; \quad x^d v_i = 0 = yv_i - xv_{i+1}, \quad \text{for} \quad 1 \leq i < n-1.$$

For $1 \leq n \leq d$, we set $W_{n,d}$ equal to $W_{n,n}$ as above.

Example 2.2. As an example, we note that high powers of the radical of kG are W modules. Specifically, the module $\mathrm{Rad}^{p+i-1}(kG) \cong W_{p-i,p-i}$ for $i = 0, \ldots, p-1$ as it has a set of generators

$$v_1 = x^{p-1}y^i, v_2 = x^{p-1}y^{i+1}, \ldots, v_{p-i} = x^i y^{p-1}$$

which satisfy precisely the relation of Definition 2.1, with $n = d = p - i$.

As defined, whether or not a kG-module M is a W module appears to depend upon our choice of generators x, y of $\mathrm{Rad}\, kG$. But we see in Corollary 4.7, which follows, that being a W module is independent of the choice of generators.

Proposition 2.3. *Suppose that n, d are positive integers such that $d \leq p$. Then $W_{n,d}$ has the equal images property. Thus, $W_{n,d}$ has constant Jordan type given by*

$$(n-d+1)[d] + [d-1] + \cdots + [1]. \tag{2}$$

Proof. By Proposition 1.7, it suffices to verify that $\mathrm{Im}\,\{ax + by : M_K \to M_K\} = \mathrm{Im}\,\{x : M_K \to M_K\}$ for every $0 \neq (a,b) \in K^2$, where K/k is an algebraic closure. So assume that $k = K$ is algebraically closed. Note that $\mathrm{Rad}(M)$ is generated by the elements $yv_1, \ldots, yv_{n-1}$, and we know that $yv_i = xv_{i+1}$. So, $xW_{n,d} = yW_{n,d} = \mathrm{Rad}(W_{n,d})$. Hence, the equal images property is a consequence of the observation that the k-subspace spanned by $(ax+by)v_1, \ldots, (ax+by)v_n$ is the same as the subspace spanned by $yv_1, \ldots, yv_{n-1}$.

By Proposition 1.9, $W_{n,d}$ thus has constant Jordan type. The Jordan type $(n-d+1)[d] + [d-1] + \cdots + [1]$ is determined by inspection. □

For use below, we record the following elementary properties of W modules.

Proposition 2.4. *Consider $W_{n,d}$ for some $n \geq d \geq 1$ and $d \leq p$. Then for any i, $1 \leq i \leq d$,*

$$W_{n,d}/\mathrm{Rad}^i(W_{n,d}) \;\cong\; W_{n,i}. \tag{3}$$

Furthermore, if $n_1, n_2, \ldots, n_m$, $d_1, d_2, \ldots, d_m$ are positive integers with $d_i \leq p$ for all i, then there exists a surjective homomorphism

$$\zeta : W_{n,d} \longrightarrow \oplus_{i=1}^m W_{n_i,d_i}$$

where $n = \sum n_i$ and $p \geq d \geq max\{d_i\}$.

Proof. The first assertion is evident from Definition 2.1.

To define a map satisfying the conditions of the second assertion we let $r_1 = 0$ and inductively let $r_i = r_{i-1} + n_{i-1}$ for $i = 2, \ldots, m$. Note that $r_m + n_m = n$. Now let ζ be given on the generators of $W_{n,d}$ as

$$\zeta(v_{r_i+j}) = v_j$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Note here that element v_{r_i+j} on the left side of the equation should be interpreted as the standard generators as in Definition 2.1 of the module $W_{n,d}$ and the element v_j on the right side of the equation is the standard generator of W_{n_i,d_i}. To check that ζ is a homomorphism, we need only note that the relations among the generators of $W_{n,d}$ are also satisfied by their images under ζ. In addition, ζ is surjective because every generator of the codomain is in the image of ζ. □

We record further natural maps of W modules which we will use in the next section.

Definition 2.5. We define natural maps of kG-modules for any $n \geq d > 1$ and $d \leq p$:

$$\iota_{n,d} : W_{n-1,d-1} \hookrightarrow W_{n,d} \tag{4}$$

defined by identifying $W_{n-1,d-1}$ with $\mathrm{Rad}(W_{n,d})$.

We define natural maps of kG-modules for any $n > d \geq 1$ and $d \leq p$:

$$\rho^l_{n,d},\ \rho^r_{n,d} : W_{n,d} \twoheadrightarrow W_{n-1,d} \tag{5}$$

where $\rho^l_{n,d}(v_i) = v_i, 1 \leq i < n;\ \rho^l_{n,d}(v_n) = 0 = \rho^r_{n,d}(v_1);\ \rho^r_{n,d}(v_i) = v_{i-1}, 1 < i \leq n.$

We conclude this section with a useful statement concerning the generation of the indecomposable W module $M = W_{n,d}$. Let $v_1, \ldots, v_n$ be generators for M satisfying the relations of Definition 2.1. Let (a, b) be any pair of elements of k, which are not both zero. Then the element

$$v_{a,b} = a^{n-1}v_1 - a^{n-2}bv_2 + a^{n-3}b^2v_3 + \cdots + (-1)^{n-1}b^{n-1}v_n$$

has the property that

$$(ax + by)v_{a,b} = 0. \tag{6}$$

This is a straightforward verification using the relations that $xv_1 = 0$, $yv_1 = xv_2$, and so forth.

Proposition 2.6. *Let $M = W_{n,d}$ for some $n \geq d \geq 1$. Suppose that the pairs $(a_1, b_1), \ldots, (a_n, b_n)$ in k^2 have the property that the classes $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ are distinct elements of $\mathbb{P}^1(k)$. Then*

$$M = \sum_{i=1}^{n} Ker\{(a_i x + b_i y) : M \to M\}. \tag{7}$$

Proof. For notational convenience, let $\mathcal{K}$ denote $\sum_{i=1}^{n} \mathrm{Ker}\{(a_i x + b_i y) : M \to M\}$. To prove the proposition, it suffices to show that each v_i lies in $\mathcal{K}$. To prove this, we show that the k-subspace V spanned by $v_1, \ldots, v_n$ coincides with the k-subspace $W \subseteq \mathcal{K}$ spanned by $v_{a_1,b_1}, \ldots, v_{a_n,b_n}$. Clearly, $W \subseteq V$, since each v_{a_i,b_i} is in V. Consequently, if we show that W has dimension n, then $W = V$, $V \subseteq \mathcal{K}$ and $M = \mathcal{K}$.

To prove that W has dimension n, it suffices to show that the elements v_{a_i,b_i}, $i = 1, \ldots, n$, are linearly independent. This is equivalent to the non-vanishing of the determinant

$$\mathrm{Det}\begin{pmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \ldots & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \ldots & b_2^{n-1} \\ & & & \\ & & \ldots & \\ a_n^{n-1} & a_n^{n-2}b_n & \ldots & b_n^{n-1} \end{pmatrix} \tag{8}$$

The above is a Vandermonde determinant (formally factor a_i^{n-1} out of the i^{th} row to put it in the form given in [9]). Formally, it is a polynomial in the variables $a_1, \ldots, a_n, b_1, \ldots, b_n$ and has value $\prod_{i<j}(a_j b_i - a_i b_j)$. Consequently, because the elements $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ we have the determinant is not zero. Hence the dimension of W is n and the proposition is proved. ☐

3. The Quillen exact category of W modules

We briefly recall for our context of W modules the Quillen exact category structure considered in [1]. We shall be interested in the case $G = \mathbb{Z}/p \times \mathbb{Z}/p$, but the following definitions apply to any finite group scheme.

Let

$$\mathcal{E}: \qquad 0 \to M_1 \to M_2 \to M_3 \to 0$$

be any short exact sequence of finite dimensional kG-modules. Then this sequence is said to be *locally split* if for any π-point $\alpha_K : K[t]/t^p \to KG$ the pull-back of $\mathcal{E}$ via α_K, $0 \to \alpha_K(M_{1,K}) \to \alpha_K(M_{2,K}) \to \alpha_K(M_{3,K}) \to 0$ is a split exact sequence. An *admissible monomorphism*, denoted $M_1 \hookrightarrow M_2$, is an injective map of kG-modules which fits into a locally split short exact sequence; an *admissible epimorphism*, denoted $M_2 \twoheadrightarrow M_3$, is a surjective map of kG-modules which fits into a locally split short exact sequence.

As in [1], it is straight-forward to verify that the following determines a sub exact category of the Quillen exact category of finite dimensional kG-modules with locally split short exact sequences.

Definition 3.1. We denote by $\mathcal{C}$ the Quillen exact category of modules of constant Jordan type as considered in [1]. We denote by $\mathcal{C}W$ the additive full sub-category generated by kG-modules of the form $W_{n,d}$. Then $\mathcal{C}W \subset \mathcal{C}$ inherits the structure of a Quillen exact category.

Theorem 3.2. *Let $\iota : \mathcal{C}W \to \mathcal{C}$ be the inclusion of Quillen exact categories and let $\iota_* : K_0(\mathcal{C}W) \to K_0(\mathcal{C})$ denote the induced map on Grothendieck groups. Then the composition*

$$\mathrm{JType} \circ \iota^* : K_0(\mathcal{C}W) \ \to \ \mathbb{Z}^p$$

is an isomorphism. Thus, ι^* gives a left splitting of the Jordan type homomorphism

$$\mathrm{JType} : K_0(\mathcal{C}) \ \to \ \mathbb{Z}^p.$$

The group $K_0(\mathcal{C}W)$ is minimally generated by the set of classes $\{[W_{n,p}] |\ n \geq 1\}$ as well as the set of classes $\{[W_{n,n}] |\ 1 \leq n \leq p\}$.

Proof. Surjectivity of $\mathrm{JType} \circ \iota^*$ is clear, for the images under $\mathrm{JType} \circ \iota^*$ of the classes of the modules

$$W_{p,p},\ W_{p-1,p} \equiv W_{p-1,p-1},\ \dots,\ W_{2,p},\ W_{1,p} = k$$

generate $\mathbb{Z}^p$.

To prove injectivity, we first observe the existence of a locally split exact sequence

$$0 \longrightarrow W_{n,d} \xrightarrow{\sigma} W_{n-1,d} \oplus W_{n-1,d} \xrightarrow{\tau} W_{n-2,d} \longrightarrow 0 \tag{9}$$

whenever $n-1 \geq d \geq 1$ and $d \leq p$. Here, $\sigma = (\rho^l_{n,d}, \rho^r_{n,d})$ and $\tau = (\rho^r_{n-1,d}, -\rho^l_{n-1,d})$, with $\rho^l_{n,d}, \rho^r_{n,d}$ as in (5). Note that each of these sequences is locally split because the middle term has precisely the same (constant) Jordan type as the direct sum of the two ends. The sequences in (9) provide the relations

$$[W_{n,d}] = 2[W_{n-1,d}] - [W_{n-2,d}] = 0; \quad n-1 \geq d \geq 1,\ d \leq p. \tag{10}$$

These relations easily enable us to write any $W_{n,d}$ as an integer linear combination of $W_{m,p} \equiv W_{m,m}$ with $m \leq d \leq p$. In particular, we conclude that $K_0(\mathcal{C}W)$ is generated by $\{W_{n,p};\ 1 \leq n \leq p\}$.

On the other hand, by (2) we see that

$$\mathrm{JType}(W_{n,p}) \ = \ [n] + [n-1] + \cdots + [1], \quad 1 \leq n \leq p. \tag{11}$$

Thus, $\{W_{n,p};\ 1 \leq n \leq p\}$ is also a linearly independent subset of $K_0(\mathcal{C}W)$. □

4. The ubiquity of W modules

Proposition 1.9 tells us that modules with the equal images property are modules of constant Jordan type. In Theorem 4.4, we verify that every module with the equal image property is a quotient of some W module.

Using the classification by A. Heller and I. Reiner [10] of kG-modules with radical square equal to 0, we identify those modules which also have the equal images property.

Proposition 4.1. *Assume that M is a kG-module having the equal images property and satisfying $\mathrm{Rad}^2(M) = \{0\}$. Then M is isomorphic to a direct sum of W modules. More precisely, there exist integers t and $n_1, \ldots, n_t$ such that*

$$M \;\cong\; W_{n_1,2} \oplus W_{n_1,2} \oplus \cdots \oplus W_{n_t,2}. \tag{12}$$

Proof. The indecomposable kG-modules M with $\mathrm{Rad}^2(M) = \{0\}$ have been completely classified in [10]. As pointed out in [1], using this classification it is an easy exercise to verify that if an indecomposable kG-modules M with $\mathrm{Rad}^2(M) = \{0\}$ has the equal images property then M is isomorphic to some $W_{n,2}$. □

From the above, we get some information on the Jordan types of modules with the equal images property.

Proposition 4.2. *Suppose that M has the equal images property and that the (constant) Jordan type of M is $a_p[p] + a_{p-1}[p-1] + \cdots + a_1[1]$. Then there exist some s such that $a_i = 0$ for $i > s$ and $a_i \neq 0$ for $i \leq s$.*

Proof. Let $n_i = \mathrm{Dim}(\mathrm{Rad}^{i-1}(M)) - \mathrm{Dim}(\mathrm{Rad}^i(M))$. Recall that $x^iM = \mathrm{Rad}^i(M)$ because of the equal images property. Hence, n_i is the number of Jordan blocks of the action of x on M that have size at least i. Let s be the least integer such that $n_s \neq 0$ or equivalently, $\mathrm{Rad}^{s-1}(M) \neq \{0\}$. Clearly, $a_i = 0$ for $i > s$, since the operation of x on M has no Jordan blocks of size larger than s. Moreover, $a_s = n_s$ and for $i < s$, we have that $a_i = n_i - n_{i-1}$.

Now by Propositions 1.9 and 1.10, the modules $N_i = \mathrm{Rad}^{i-1}(M)/\mathrm{Rad}^{i+1}(M)$ have the equal images property for all i. Hence by Proposition 4.1, each is a sum of W modules. It follows that

$$n_i \;=\; \mathrm{Dim}(N_i/\mathrm{Rad}(N_i)) \;>\; \mathrm{Dim}(\mathrm{Rad}(N_i)) \;=\; n_{i+1}$$

for all $i = 1, \ldots, s$. Consequently, $a_i \neq 0$ for $1 \leq i \leq s$. □

Our next proposition is the key to establishing that modules with the equal images property are quotients of direct sums of W modules.

Proposition 4.3. *Suppose that M has the equal images property and that m is an element of M. For any n sufficiently large, there exists a homomorphism $\psi : W_{n,p} \longrightarrow M$ such that for some s, $\psi(v_s) = m$ where $v_s \in W_{n,p}$ is a generator as in Definition 2.1.*

Proof. We first verify that for some t, there exist elements $m_1, m_2 \ldots, m_t$ such that

$$m_1 = m; \quad xm_i \ = ym_{i+1}, i = 1, \ldots, t-1; \quad xm_t = 0. \tag{13}$$

We proceed by induction on r where r is the least integer such that $\operatorname{Rad}^r(M) = \{0\}$. In the case that $r = 1$, M is a sum of trivial modules, so that we may assume that $t = 1$. If $r = 2$, then we use Proposition 4.1 and the explicit structure of $W_{n,2}$ to obtain $m_1, m_2 \ldots, m_t$ satisfying (13).

Proceeding inductively, we may assume that $m \notin \operatorname{Rad}(M)$. For if $m \in \operatorname{Rad}(M)$, we appeal to our induction hypothesis to find elements $m_1, m_2 \ldots, m_t$ in $\operatorname{Rad}(M)$ satisfying (13) since $\operatorname{Rad}(M)$ also has the equal images property. By Proposition 4.1, there exist integers m and $n_1, \ldots, n_m$ such that $M/\operatorname{Rad}^2(M) \cong W_{n_1,2} \oplus \cdots \oplus W_{n_m,2}$. Consequently, if n is the maximum of the integers $n_1, \ldots, n_m$, then we can find elements $m_1, \ldots, m_n$ such that $m_1 = m$ and for all $i = 1, \ldots, n-1$, $xm_i = ym_{i+1}$ and also that $xm_n \in \operatorname{Rad}^2(M)$. Then by induction we can find $m_{n+1}, \ldots m_t$ in $\operatorname{Rad}(M)$ for some t so that $xm_i = ym_{i+1}$ for $i = n, \ldots, t-1$ and that $xm_t = 0$.

Interchanging the roles of x and y, we conclude that there exist an integer $s \geq 1$ and elements $m_{-1}, \ldots, m_{-s}$ such that

$$m = m_{-1}; \quad ym_i = xm_{i-1}, i = -1, \ldots, -s; \quad ym_{-s} = 0. \tag{14}$$

Hence, we obtain a well defined kG-homomorphism

$$\psi : W_{s+t-1,p} \ \longrightarrow \ M; \quad \psi(v_i) = m_{-s-1+i}.$$

$\square$

Theorem 4.4. *Suppose that M is a kG-module. Then M has the equal images property if and only if there exists a positive integer n and a surjective homomorphism*

$$\psi : W_{n,d} \ \longrightarrow \ M,$$

where d satisfies $\operatorname{Rad}^d M = 0$.

Proof. If such a map ψ exists, then M has the equal images property by Lemma 1.10. Hence, we assume that M has the equal images property. Our objective is to construct the map ψ.

Let $m_1, \ldots, m_n$ be a collection of elements such that the cosets $m_i + \operatorname{Rad}(M)$ for $i = 1, \ldots n$ form a basis for $M/\operatorname{Rad}(M)$. By Proposition 4.3, for each i there is an integer n_i and a homomorphism $\psi_i : W_{2n_i,p} \longrightarrow M$ such that $\psi_i(v_{n_i}) = m_i$. Thus by Nakayama's Lemma we have that the sum of the maps

$$\Psi : \ \bigoplus_{i=1}^n \ W_{2n_i,p} \longrightarrow M$$

given by $\Psi(w_1, \ldots, w_n) = \sum \psi_i(w_i)$ is surjective. Moreover, each summand of this map clearly factors through $W_{2n_i,p}/\operatorname{Rad}^d W_{2n_i,p}$. The proof of the theorem is thus completed by an appeal to Proposition 2.4. $\square$

The following corollary of Theorem 4.4 will be key to our discussion in Section 6 of generic kernels.

Corollary 4.5. *Suppose that k is an infinite field and that $S \subset \mathbb{P}^1(k)$ is infinite. Let M be any kG-module having the equal images property. Then*

$$M = \sum_{\langle a,b\rangle \in S} Ker\{ax + by : M \to M\}.$$

Proof. By Theorem 4.4, there exist an integer n and a surjective homomorphism $\psi : W_{n,p} \to M$. To prove the theorem we need only note that for, any pair $(a, b) \neq (0, 0)$ in k^2,

$$\psi(\mathrm{Ker}\{ax + by : W_{n,p} \to W_{n,p}\}) \subseteq \mathrm{Ker}\{ax + by : M \to M\}.$$

Then by Lemma 2.6, we have that

$$\begin{aligned} M = \psi(W_{n,p}) &= \sum_{(a,b)\in S} \psi(\{m \in W_{n,p} | (ax + by)m = 0\}) \\ &\subseteq \sum_{(a,b)\in S} \{m \in M | (ax + by)m = 0\}, \end{aligned}$$

thus proving the theorem. ☐

If $M/\mathrm{Rad}^2(M)$ is indecomposable and if M has the equal images property, then we give in Proposition 2.9 a "much more efficient" surjection from a W module onto M than that described in the proof of Theorem 4.4..

Proposition 4.6. *Assume that M is a kG-module having the equal images property. If $M/\mathrm{Rad}^2(M)$ is indecomposable, then M is a quotient of $W_{n,p}$ for $n = \mathrm{Dim}\, M/\mathrm{Rad}(M)$.*

Proof. Because $M/\mathrm{Rad}^2(M)$ is indecomposable, Proposition 4.1 implies the existence of generators $a_1, \ldots, a_n$ with the property that

$$xa_n \equiv 0, \quad ya_1 \equiv 0, \quad \text{and} \quad xa_i \equiv ya_{i+1}$$

modulo $\mathrm{Rad}^2(M)$ where $i = 1, \ldots, n-1$. Moreover, the equal images property for M implies that

$$\mathrm{Rad}^2(M) = x^2M = xyM = y^2M.$$

We proceed to modify the set $\{a_1, \ldots, a_n\}$ of generators in order to obtain a new set of generators satisfying the conditions on $\{v_1, \ldots, v_n\}$ given in Definition 2.1. Our first step involves observing that ya_1 is in $\mathrm{Rad}^2(M)$; thus, $ya_1 = y^2w$ for some w. Replace a_1 by $a_1' = a_1 - yw$, so that $ya_1 = 0$. Since $xa_1' - ya_2 = y^2w'$ for some w', we may replace a_2 by $a_2' = a_2 + yw'$, so that $xa_1' = ya_2'$. Continuing, we obtain $\{a_1', \ldots, a_n'\}$ such that

$$ya_1' = 0, \quad \text{and} \quad xa_i' = ya_{i+1}'$$

for $i = 1, \ldots, n-1$.

There remains to modify this new generating set $\{a_1', \dots, a_n'\}$ to obtain a generating set which satisfies the above relations and further satisfies the relation that $xa_n' = 0$. Since xa_n' is in $\mathrm{Rad}^2(M)$, we may choose $\ell_1, \dots, \ell_{n-2} \in k$ such that

$$xa_n' \equiv x^2(\ell_1 a_1' + \cdots + \ell_{n-2}a_{n-2}') \quad \text{modulo } \mathrm{Rad}^3(M).$$

Observe that we do not need to add a term of the form $x^2\ell_{n-1}a_{n-1}'$, because $xa_{n-1}' = ya_n'$ so that $x^2a_{n-1}' = yxa_n'$ lies in $\mathrm{Rad}^3(M)$.

Set

$$b_1 = a_1',\ b_2 = a_2',\ b_3 = a_3' + x\ell_{n-2}a_1',\ \dots,\ b_i = a_i' + x\sum_{j=1}^{i-2} \ell_{n-i+j}a_j',\ \dots$$

A straight-forward calculation confirms that

$$yb_1 \equiv 0, \quad xb_n \equiv 0, \quad \text{and} \quad xb_i \equiv yb_{i+1} \quad \text{modulo } \mathrm{Rad}^3(M).$$

We continue this process, inductively obtaining the corresponding set of relation modulo the submodule $\mathrm{Rad}^j(M)$, stopping at $j = p$. □

Corollary 4.7. *Let $x', y' \in kG$ be a choice of generators of* $\mathrm{Rad}(kG)$*, let $n \geq d \geq 1$ with $d \leq p$, and let M be a kG-module generated by $\{w_1, \dots, w_n\}$ subject to the relations (and only the relations) generated by*

$$x'w_1 = 0 = y'w_n; \quad (x')^d w_i = 0; \quad y'w_i = x'w_{i+1},\ 1 \leq i < d.$$

Then M is isomorphic to $W_{n,d}$.

Proof. As argued in the proof of Proposition 2.3, M has the equal images property. Since M clearly satisfies the condition that $M/\mathrm{Rad}^2(M)$ is indecomposable, Proposition 4.6 implies that M is isomorphic to $W_{n,d}$. □

The following proposition verifies that there are a great many kG-modules of constant Jordan type. This is because any quotient of a W module must have the equal images property and thus also constant Jordan type.

Proposition 4.8. *Suppose that $M = W_{n,d}$ for some $n > 4$ and $3 \leq d \leq p$. Suppose that L and N are subspaces of* $\mathrm{Soc}(M)$ *(the socle of M) which in this case is equal to* $\mathrm{Rad}^{d-1}(M)$*. There exists an isomorphism $\varphi : M/L \xrightarrow{\sim} M/N$ of kG-modules only if $L = N$.*

Proof. Recall that $\mathrm{Soc}(M)$ is the sum of the irreducible submodules of V. Hence, $\mathrm{Soc}(M)$ is a trivial kG-module and $L,\ N$ are kG-submodules of M. If M is generated by $v_1, \dots, v_n$ as in Definition 2.1, then $\mathrm{Soc}(M)$ is the subspace of M generated by $y^{d-1}v_1, \dots, y^{d-1}v_{n-d+1}$ and both x and y annihilate the subspaces L and N.

We assume that $v_1, \dots, v_n$ are the generators of M with precisely the relations as in 2.1. Suppose that $\ell = ax - by$ for $(a, b) \neq (0, 0)$ in k^2. Multiplication by ℓ induces a surjective homomorphism

$$\ell :\ M/\mathrm{Rad}(M) \longrightarrow \mathrm{Rad}(M)/\mathrm{Rad}^2(M)$$

whose kernel has dimension one and is generated by the class of the element $m_\ell = a^{n-1}v_1 + a^{n-2}bv_2 + \cdots + b^{n-1}v_n$. In particular, we have that $\ell \cdot m_\ell = 0$. This is a straightforward verification using the generators and relations.

We claim that if $m \in M/L$, $m \notin \operatorname{Rad}(M/L)$, with $\ell m = 0$, then $m \equiv cm_\ell$ modulo $\operatorname{Rad}(M/L)$ for some non-zero $c \in k$. The reason is that $L \subseteq \operatorname{Rad}^{d-1}(M)$ for $d \geq 2$, and hence

$$M/\operatorname{Rad}(M) \simeq (M/L)/\operatorname{Rad}(M/L)$$

and

$$\operatorname{Rad}(M)/\operatorname{Rad}^2(M) \simeq \operatorname{Rad}(M/L)/\operatorname{Rad}^2(M/L)$$

where the isomorphisms are induced by the quotient map $M \to M/L$. The same claim holds for M/N.

We define $\theta = q_2^{-1}\varphi' q_1$, so that θ fits in the commutative diagram

$$\begin{tikzcd}
(M/L)/\operatorname{Rad}(M/L) \arrow[r, "\varphi'"] & (M/N)/\operatorname{Rad}(M/N) \\
M/\operatorname{Rad}(M) \arrow[u, "q_1"] \arrow[r, "\theta"] & M/\operatorname{Rad}(M) \arrow[u, "q_2"']
\end{tikzcd} \tag{15}$$

where φ' is the map induced by φ, and the two vertical arrows are isomorphisms induced by the natural quotient maps. The commutativity of (15) implies

$$\varphi(m_\ell + L) \equiv c_\ell m_\ell + N \quad \text{modulo } \operatorname{Rad}(M/N),$$

for some constant element c_ℓ. In particular, we get that

$$\theta(m_\ell + \operatorname{Rad}(M)) = c_\ell m_\ell + \operatorname{Rad}(M).$$

In other words, the class of m_ℓ is an eigenvector for the map of θ with eigenvalue c_ℓ.

Now because the field k is infinite, we may choose n distinct scalars $b_1, \ldots, b_n$ in k and form the elements $\ell_i = x + b_i y$. Then we have n elements

$$m_{\ell_i} = m_i = v_1 + b_i v_2 + b_i^2 v_3 + \cdots + b_i^{n-1} v_n.$$

Observe that the matrix of coefficients of this collection of elements has the form

$$\begin{pmatrix} 1 & b_1 & \ldots & b_1^{n-1} \\ 1 & b_2 & \ldots & b_2^{n-1} \\ & & & \\ & & & \\ 1 & b_n & \ldots & b_n^{n-1} \end{pmatrix}$$

which is a Vandermonde matrix. Its determinant is $\prod_{i \leq j}(b_i - b_j) \neq 0$ and we conclude that the set of classes of the elements $m_1, \ldots, m_n$ is a basis for $M/\operatorname{Rad}(M)$. As a consequence, the matrix of θ with respect to this basis is a diagonal matrix. We now show that it is a scalar matrix.

Suppose that $\ell = x + by$, where $b \in k$ is not equal to any of $b_1, b_2, \ldots, b_n$. Then there exist $a_1, \ldots, a_n$ such that

$$m_\ell \;=\; v_1 + bv_2 + \cdots + b^{n-1}v_n \;=\; a_1m_1 + \cdots + a_nm_n.$$

Note that $a_i \neq 0$ for all i. This is because, if $a_i = 0$, then substituting the row $\begin{pmatrix} 1 & b & \ldots & b^{n-1} \end{pmatrix}$ for the i^{th} row of the above matrix, we get a Vandermonde matrix with zero determinant which is not possible.

Applying θ to the above expression, we get that $c_\ell = c_{\ell_i}$ for every i. Hence, we have that for any v in $M/\operatorname{Rad}(M)$, $\theta(v) = cv$ where $c = c_\ell$ for any (and every) ℓ. In, particular, we have that $\varphi(m+L) \;\equiv\; cm + N \bmod \operatorname{Rad}(M/N)$ for any $m \in M$.

At the level of $\operatorname{Rad}^{d-1}(M/L)$, we have that

$$\varphi(y^{d-1}v_i + L) = cy^{d-1}v_i + N \qquad \text{for } i = 1, \ldots, n-d+1,$$

since $\operatorname{Rad}^d(M) = 0$. Hence, for any element $m \in L$, m can be expressed as a linear combination of the elements $y^{d-1}v_1, \ldots, y^{d-1}v_{n-d+1}$. It follows that $0 = \varphi(m+L) = cm + N$, and $m \in N$. Therefore $L = N$. $\square$

We end this section with the following question on the possibility of classifying modules of constant Jordan type. We remind the reader that if a module category has wild representation type, then it is generally considered that its objects can not be classified in any reasonable sense. That is, such a classification would imply the existence of a canonical form for pairs of $n \times n$ matrices for all n, which is thought not to exist (cf. [4]).

Question 4.9. *Does the full subcategory of the category of kG-modules consisting of all modules of constant Jordan type have wild representation type?*

5. Associated Bundles on $\mathbb{P}^1$

In [6], a construction is presented which associates to a finite dimensional kG-module M of constant Jordan type a family of algebraic vector bundles

$$\{\operatorname{Ker}\{\tilde{\Theta}^j_G, \mathcal{O}_{Proj(V(G))} \otimes M\}; 1 \leq j < p\}$$

on the projectivization of the k-scheme $V(G)$ of 1-parameter subgroups of G. Since the group algebra of $\mathbb{Z}/p\times\mathbb{Z}/p$ is isomorphic to the group algebra of the infinitesimal group scheme $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, we may view W modules for $\mathbb{Z}/p \times \mathbb{Z}/p$ as modules (of constant Jordan type) for $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$. For $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, $V(G)$ is simply affine 2-space $\mathbb{A}^2$ so that $\operatorname{Proj} V(G) \;\simeq\; \mathbb{P}^1$, the projective line.

In this section, we set $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ and set $\mathcal{O} = \mathcal{O}_{Proj(V(G))} = \mathbb{P}^1$. In the proposition below, we identify the bundles

$$\{\operatorname{Ker}\{\tilde{\Theta}^j_G, \mathcal{O} \otimes W_{n,d}\}, \quad j = 1, 2\}.$$

The general construction of Ker $\{\tilde{\Theta}^j_G, \mathcal{O}_{Proj(V(G)} \otimes M\}$ specialized to $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ can be described concretely as follows. Write $kG = k[x,y]/(x^p, y^p)$. Associated to a finite dimensional kG-module M consider the following endomorphism

$$k[s,t] \otimes M \longrightarrow k[s,t] \otimes M, \quad m \mapsto s \otimes x \cdot m + t \otimes y \cdot m.$$

We view this endomorphism as the endomorphism of the $k[s,t] \otimes kG$-module $k[s,t] \otimes M$ given by multiplication by

$$\Theta_G = s \otimes x + t \otimes y \in k[s,t] \otimes kG.$$

Because Θ_G is homogeneous of weight 1 (with respect to the grading on $k[s,t]$ assigning both s, t weight 1), we obtain a projectivization of our endomorphism

$$\tilde{\Theta}_G : \mathcal{O} \otimes M \longrightarrow \mathcal{O}[1] \otimes M,$$

a map of coherent $\mathcal{O}$-modules on the scheme $\mathbb{P}^1$.

Proposition 5.1. *(cf. [6, 5.1]) As above, let $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, and let M be a finite dimensional kG-module. Let $\alpha_{a,b} : K[u]/u^p \to KG$ be the (flat) map of K-algebras sending u to $ax + by$ for any $(a,b) \in \mathbb{A}^2_K - \{0\}$, where K/k is some field extension. Then the action of $\alpha_{a,b}(u)$ on $M_K = K \otimes M$ is given by restricting along $k[s,t] \to K$ via $s \mapsto a, t \mapsto b$ the action of multiplication by Θ_G on $k[s,t] \otimes M$.*

In particular, if for a given $j, 1 \leq j < p$, the rank of $\alpha^j_{a,b}, M \to M$ is independent of $(a,b) \neq 0$ (i.e., if M has constant j-type in the terminology of [6]), then the kernel of $\tilde{\Theta}^j_G$ on $\mathcal{O} \otimes M$ is an algebraic vector bundle on $\mathbb{P}^1$, sub-bundle of the trivial bundle $\mathcal{O} \otimes M$:

$$Ker\{\tilde{\Theta}^j_G, \mathcal{O} \otimes M\} \subset \mathcal{O} \otimes M;$$

moreover, the image of $\tilde{\Theta}^j_G$ on $\mathcal{O} \otimes M$ is an algebraic vector bundle on $\mathbb{P}^1$, a sub-bundle of $\mathcal{O}(j) \otimes M$:

$$Im\{\tilde{\Theta}^j_G, \mathcal{O} \otimes M\} \subset \mathcal{O}(j) \otimes M.$$

Remark 5.2. The infinitesimal group scheme $\mathbb{G}_{a(2)}$ also has group algebra isomorphic to the group algebra of $\mathbb{Z}/p \times \mathbb{Z}/p$. The affine scheme $V(\mathbb{G}_{a(2)})$ of 1-parameter subgroups can be identified with the spectrum of the graded polynomial algebra $k[s_0, s_1]$ with $deg(s_0) = 1, deg(s_1) = p$. The associated projective scheme is a weighted $\mathbb{P}^1$ isomorphic to $\mathbb{P}^1$ itself. This contrasts sharply with the rank 3 case: Proj $(V(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})) \simeq \mathbb{P}^2$ whereas Proj $(V(\mathbb{G}_{a(3)}))$ is the weighted projective space $\mathbb{P}(1, p, p^2)$ which is a singular variety.

The reader can readily check that the bundles on $\mathbb{P}^1 \simeq$ Proj $(V(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}))$ associated to a $\mathbb{Z}/p \times Z/p$-module M of constant Jordan type arising when we use the infinitesimal group scheme $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ are isomorphic to those arising when we use the infinitesimal group scheme $\mathbb{G}_{a(2)}$, once we identify Proj $(V(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}))$ with Proj $(V(\mathbb{G}_{a(2)})$.

We extend our notational shorthand of $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^1}$ by letting $\mathcal{O}(n)$ denote $\mathcal{O}_{\mathbb{P}^1}(n)$. The following proposition is essentially given by [6, 6.13].

Proposition 5.3. *As above, we use the isomorphism of the group algebra of* $(\mathbb{Z}/p)^2$ *with the group algebra of* kG, $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, *in order that we may view* $W_{n,2}$ *as a* kG*-module. Then*

$$Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,2}\} \;\simeq\; \mathcal{O}^{\oplus n-1} \oplus \mathcal{O}(-n+1).$$

Proof. Observe that

$$\mathrm{Im}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,2}\} \otimes \mathcal{O}(-1) \;\subset\; \mathcal{O} \otimes W_{n,2}$$

is a free module of rank $n-1$ on $\mathbb{P}^1$. So we may identify this image with $\mathcal{O} \otimes \mathrm{Rad}(W_{n-2})$. As shown in Proposition 6.13 of [6], the quotient of $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,2}\}$ modulo $\mathrm{Im}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,2}\} \otimes \mathcal{O}(-1)$ is $\mathcal{O}(-n+1)$. This immediately implies the assertion. Note here that the summand $\mathcal{O}(-n+1)$ is the kernel of the operator $\tilde{\Theta}_G$ on the subspace generated by $v_1, \ldots, v_n$ (in the notation of Definition 2.1). □

The extension of Proposition 5.3 to all $W_{n,d}$ now follows as we show in our next proposition.

Proposition 5.4. *As in Proposition 5.3, we view the* $\mathbb{Z}/p \times \mathbb{Z}/p$*-module* $W_{n,d}$ *as a* $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$*-module. Then*

$$Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\} \;\simeq \mathcal{O}^{\oplus n+1-d} \;\oplus\; \bigoplus_{i=1}^{d-1} \mathcal{O}(-n+i).$$

Proof. This is proved by induction on d using the embedding $\iota_{n,d} : W_{n-1,d-1} \hookrightarrow W_{n,d}$ of (4). By induction, $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-1,d-1}\}$ is isomorphic to $\mathcal{O}^{\oplus n+1-d} \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(-n+i)$ and embeds in $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\}$. Moreover, the summand $\mathcal{O}(-n+1)$ of $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,2}\}$ arising in Proposition 5.3 comes from the kernel of the operator θ_G on the first layer (i.e., the head) of $k[s,t] \otimes W_{n,2}$ and thus also embeds in $W_{n,d}$ with 0 intersection with $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-1,d-1}\}$. Thus, we have a natural injective map

$$\mathcal{O}(-n+1) \oplus \mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-1,d-1}\} \;\to\; \mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\}. \tag{16}$$

We readily verify that $\mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\}$ has rank n. That is, $\Theta_G : k[s,t] \otimes W_{n,d} \to k[s,t] \otimes W_{n,d}$ is seen to map the i-th layer of $W_{n,d}$ (which has rank $n+1-i$) onto the the $i-1$-st layer, except that the d-th layer necessarily maps to 0. Consequently, we conclude that (16) is an isomorphism by a comparison of dimensions. □

The computation of the kernel of $\tilde{\Theta}_G^2$ on $W_{n,d}$ now follows easily since Serre's computation of the cohomology of $\mathbb{P}^1$ (see[8, III.5.1]) implies that the extension mentioned below is split.

Proposition 5.5. *As above, we use the isomorphism of the group algebra of $\mathbb{Z}/p \times \mathbb{Z}/p$ with the group algebra of kG, $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, in order to view $W_{n,d}$ as a kG-module. Then*

$$Ker\{\tilde{\Theta}_G^2, \mathcal{O} \otimes W_{n,d}\} \simeq Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\} \oplus Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-1,d-1}\} \simeq$$

$$\mathcal{O}^{\oplus 2n+2-2d} \oplus \left(\bigoplus_{i=2}^{d-1} \mathcal{O}(-n+i)^{\oplus 2} \right) \oplus \mathcal{O}(-n+1).$$

Proof. The second asserted isomorphism follows immediately from Proposition 5.4. We investigate $\mathrm{Ker}\,\{\tilde{\Theta}_G^2, \mathcal{O} \otimes W_{n,d}\}$ by observing that it fits in a natural short exact sequence

(17)

$$0 \to \mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\} \to \mathrm{Ker}\,\{\tilde{\Theta}_G^2, \mathcal{O} \otimes W_{n,d}\} \to \mathrm{Ker}\,\{\tilde{\Theta}_G, Im\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n,d}\}\} \to 0$$

Because $\tilde{\Theta}_G$ induces a map on the associated graded of $\mathcal{O} \otimes W_{n,d}$ with respect to the socle filtration, we observe that

$$\mathrm{Ker}\,\{\tilde{\Theta}_G^2, \mathcal{O} \otimes W_{n,d}\} = \bigoplus_{i=1}^{d} \mathrm{Ker}\,\{\tilde{\Theta}_G^2 : W_{n-i,d-i} \to W_{n-i-2,d-i-2}/W_{n-i-3,d-i-3}\}$$

is the direct sum of kernels of $\tilde{\Theta}_G^2$ from the i-th layer to the $(i-2)^{nd}$ layer. Restricting the extension (17) to $K(i)$ we obtain extensions

$$(18) \quad 0 \to \mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-i,d-i}\} \to K(i) \to \mathrm{Ker}\,\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{n-i-1,d-i-1}\} \to 0.$$

The proposition now follows from the observation that the extensions (18) split, since the corresponding coherent Ext^1 groups vanish thanks to Serre's computation of $H^*(\mathbb{P}^1, \mathcal{O}(\ell))$. The one non-vanishing Ext^1-group which arises in this extension occurs in the case $n = 3$, with $Ext^1_{\mathcal{O}}(\mathcal{O}, \mathcal{O}(-3+1)) \neq 0$. For this one case, one easily observes that the copy of $\mathcal{O}$ in the quotient splits. □

6. Generic kernels

Throughout this section, we assume that the field k is infinite. We associate to any (finite dimensional) kG-module M a kG-submodule $\mathfrak{K}(M) \subset M$ which has constant Jordan type. Indeed, the characterization in Theorem 6.9 of $\mathfrak{K}(M) \subset M$ as the maximal submodule of M which has the equal images property implies that $M \mapsto \mathfrak{K}(M)$ is functorial. This is not immediately evident granted the choices we make in Definition 6.1.

We fix a choice of generators x, y of $\mathrm{Rad}\, kG$, so that $kG \simeq k[x,y]/x^p, y^p$. For any $\langle a, b \rangle \in \mathbb{P}^1(k)$, write

$${}_{\langle a,b \rangle}M \equiv \mathrm{Ker}\,\{ax + by : M \to M\}.$$

Note that ${}_{\langle a,b \rangle}M$ does not depend on the choice of the representing pair $(a, b) \in k^2$.

For any subset $S \subset \mathbb{P}^1(k)$, define the subspace

$$ {}_SM \;\equiv\; \sum_{\langle a,b\rangle \in S} {}_{\langle a,b\rangle}M $$

Since both x, y commute with each $ax+by \in S$, ${}_SM$ is a kG-submodules of M. Clearly, for $S' \subset S$, ${}_{S'}M \subset {}_SM$.

In the degenerate case in which $S = \emptyset$, we assume that ${}_SM = \{0\}$.

Definition 6.1. Let k be an infinite field and let M be a finite dimensional kG-module. We define the *generic kernel* of M to be

$$ \mathfrak{K}(M) \;\equiv\; \bigcap_{S \subset\, \mathbb{P}^1(k)\ cofinite} {}_SM. $$

Because M is finite dimensional, we may choose for a given M a cofinite $S \subset \mathbb{P}^1(k)$ such that ${}_SM = \mathfrak{K}(M)$. Observe that

$$ \mathfrak{K}(\mathfrak{K}(M)) \;=\; M, \tag{19} $$

since ${}_T({}_SM)) = {}_{S \cap T}M$.

We give the following elementary example. More examples follow later.

Example 6.2. Let $M = kG$, the free cyclic kG-module, with k infinite. Then

$$ \mathfrak{K}(M) \;=\; \mathrm{Rad}^{p-1}(kG) \;\simeq\; W_{p,p}. $$

To prove the equality $\mathfrak{K}(M) \;=\; \mathrm{Rad}^{p-1}(kG)$, we first verify that $\mathfrak{K}(M)$ is contained in $\mathrm{Rad}^{p-1}(kG)$. For this we note that kG has constant Jordan type $p[p]$ and hence any element of $\mathrm{Ker}\,\{ax+by : M \to M\}$ must be contained in $(ax+by)^{p-1}kG \subseteq \mathrm{Rad}^{p-1}(kG)$. Hence, $\mathfrak{K}(kG) \subseteq \mathrm{Rad}^{p-1}(kG)$ by the definition. On the other hand, for any $a, b \in k$, $(ax+by)^{p-1} \in \mathrm{Ker}\,\{ax+by : M \to M\}$. Hence, for *any* cofinite set $S \subseteq \mathbb{P}^1(k)$, the elements $(ax+by)^{p-1}$ with $ax+by$ in S generate all of $\mathrm{Rad}^{p-1}(kG)$. So, $\mathrm{Rad}^{p-1}(kG) \subseteq \mathfrak{K}(kG)$ by the definition of the generic kernel.

Note that it is an easy exercise to see that $\mathrm{Rad}^{p-1} kG \simeq W_{p,p}$ has the equal images property and thus is equal to its own generic kernel. Hence, we could also cite Proposition 6.8 which follows.

Proposition 6.3. *Suppose that the field k is algebraically closed. Let M be any kG-module. Then $\mathfrak{K}(M)$ has the equal images property.*

Proof. Using (19), we may and will assume that $M = \mathfrak{K}(M)$. By Proposition 1.7, it suffices to show for $0 \neq \underline{(a,b)} \in k^2$ that $\ell_{(a,b)} \cdot M = x \cdot M$. (In this notation, $\ell_{(1,0)}M \;=\; x \cdot M$.) Clearly, it suffices to assume that $b \neq 0$. Write $M = {}_SM$ where $S \subseteq \mathbb{P}^1(k)$ is cofinite and S contains neither $\langle 1,0\rangle$ (which corresponds to $x \in \mathrm{Rad}\,kG)$) nor $\langle a,b\rangle$. For any $\langle c,d\rangle \in S$, there exists elements $e, f \in k$ such that $cx + dy = ex + f(ax + by)$. Note here that e and f are both not zero because $\langle 1, 0\rangle \neq \langle c, d\rangle \neq \langle a, b\rangle$. Then

$$\ell_{(1,0)} \cdot {}_{\langle c,d\rangle}M \ = \ \ell_{(a,b)} \cdot {}_{\langle c,d\rangle}M. \tag{20}$$

The proposition now follows from the definition of the generic kernel by taking the sum indexed by $\langle c, d\rangle \in S$ on both sides of (20). □

The following proposition justifies our definition of generic kernel without passing to the algebraic closure of our base field k.

Proposition 6.4. *Let k be an infinite field and M a kG-module. Then for any field extension L/k,*

$$L \otimes \mathfrak{K}(M) \ = \ \mathfrak{K}(M_L).$$

Proof. Let $S \subset \mathbb{P}^1(k)$ be a cofinite subset such that $\mathfrak{K}(M) = {}_S M$. We first assume that the field L is algebraically closed. By Proposition 6.3, $\mathfrak{K}(M_L)$ has the equal images property so that we may apply Corollary 4.5 to $\mathfrak{K}(M_L)$ and the infinite subset $S \subset \mathbb{P}^1(k) \subset \mathbb{P}^1(L)$. We conclude that

$$\mathfrak{K}(M_L) \ = \ \sum_{\langle a,b\rangle \in S} \operatorname{Ker}\{ax + by : \mathfrak{K}(M_L) \to \mathfrak{K}(M_L)\} \ =$$

$$L \otimes \sum_{\langle a,b\rangle \in S} \operatorname{Ker}\{ax + by : \mathfrak{K}(M) \to \mathfrak{K}(M)\} \ = \ L \otimes \mathfrak{K}(M).$$

For an arbitrary field extension L/k, let $T \subset \mathbb{P}^1(L)$ be a cofinite subset such that $\mathfrak{K}(M_L) = {}_T(M_L)$. Since $S \cap T$ is also cofinite in $\mathbb{P}^1(k)$, we may assume that $S = S \cap T$. Thus,

$$L \otimes \mathfrak{K}(M) = L \otimes {}_S M = \sum_{\langle a,b\rangle \in S} L \otimes {}_{\langle a,b\rangle}M = \sum_{\langle a,b\rangle \in S} {}_{\langle a,b\rangle}(M_L) \ \subseteq \ \mathfrak{K}(M_L).$$

Let $\overline{L}$ be the algebraic closure of L. By the first proof of this proof, we have that

$$\operatorname{Dim}_k \mathfrak{K}(M) \ = \ \operatorname{Dim}_{\overline{L}} \mathfrak{K}(\overline{L} \otimes M) \ = \ \operatorname{Dim}_L(\mathfrak{K}(M_L)) \tag{21}$$

Since $L \otimes \mathfrak{K}(M) \subseteq \mathfrak{K}(M_L)$, (21) implies the asserted equality $L \otimes \mathfrak{K}(M) = \mathfrak{K}(M_L)$. □

We now extend Proposition 6.3 to arbitrary infinite fields.

Proposition 6.5. *Let k be an infinite field and M a finite dimensional kG-module. The generic kernel of M has the equal images property. In particular, $\mathfrak{K}(M)$ has constant Jordan type.*

Proof. Let Ω/k be an algebraic closure. By Proposition 6.3, $\mathfrak{K}(M_\Omega)$ has the equal images property. By the preceding Proposition 6.4, this become the assertion that $\Omega \otimes \mathfrak{K}(M)$ has the equal images property. By the equivalence of (1) and (2) of Proposition 1.7, this implies that $\mathfrak{K}(M)$ has the equal images property.

The second assertion follows immediately from the first and Proposition 1.9. □

We next verify that the maximal subset $S \subset \mathbb{P}^1(k)$ with the property that ${}_SM = \mathfrak{K}(M)$ has a natural characterization in terms of the action of kG on M. Observe that any infinite subset $S \subset \mathbb{P}^1(k)$ is dense in $\mathbb{P}^1$; in other words, any non-empty open subset $U \subset \mathbb{P}^1$ contains a point of S.

Proposition 6.6. *Let k be an infinite field and M a finite dimensional kG-module with generic kernel $\mathfrak{K}(M)$. For any $\ell_{(a,b)} = ax + by$, $(a,b) \neq 0$, $Ker\{\ell_{(a,b)} : M \to M\} \subset \mathfrak{K}(M)$ if and only if $Rank\, \ell_{(a,b)} : M \to M$ is maximal among ranks of $\ell_\alpha = \alpha_K(t) : M_K \to M_K$ as α_K ranges over all π-points of kG.*

Consequently, the cofinite subset

$$\{\langle a,b\rangle | Rank\, \ell_{(a,b)} : M \to M \text{ is maximal}\} \ \subset\ \mathbb{P}^1(k)$$

is the largest subset $S \subset \mathbb{P}^1(k)$ with the property that ${}_SM = \mathfrak{K}(M)$.

Proof. Since maximality of rank is an open condition, there is some $\langle c,d\rangle \in \mathbb{P}^1(k)$ such that $\mathrm{Rank}\, \ell_{\langle c,d\rangle} : M \to M$ is maximal and must satisfy the condition that $\mathrm{Ker}\{\ell_{\langle c,d\rangle} : M \to M\} \subset \mathfrak{K}(M)$. For an arbitrary $\langle a,b\rangle$ in $\mathbb{P}^1(k)$, the inclusion $\mathrm{Ker}\{\ell_{\langle a,b\rangle} : \mathfrak{K}(M) \to \mathfrak{K}(M)\} \hookrightarrow \mathrm{Ker}\{\ell : M \to M\}$ is the identity if and only if $\mathrm{Ker}\{\ell_{\langle a,b\rangle} : M \to M\} \subset \mathfrak{K}(M)$.

We now utilize the following chain

$$\begin{aligned} \mathrm{Dim}\,\mathrm{Ker}\{\ell_{\langle a,b\rangle} : M \to M\} \ &\geq\ \mathrm{Dim}\,\mathrm{Ker}\{\ell_{\langle c,d\rangle} : M \to M\} \\ &=\ \mathrm{Dim}\,\mathrm{Ker}\{\ell_{\langle c,d\rangle} : \mathfrak{K}(M) \to \mathfrak{K}(M)\} \\ &=\ \mathrm{Dim}\,\mathrm{Ker}\{\ell_{\langle a,b\rangle} : \mathfrak{K}(M) \to \mathfrak{K}(M)\}. \end{aligned}$$

In the above chain, the inequality is a consequence of the maximality of the rank of $\ell_{\langle c,d\rangle}$, the middle equality follows from the choice of $\ell_{\langle c,d\rangle}$ and the preceding observation, and the right equality is a consequence of the equal images property of $\mathfrak{K}(M)$. Thus, we conclude that the left inequality is an equality (and thus $\ell_{\langle a,b\rangle}$ has maximal rank) if and only if $\mathrm{Ker}\{\ell_{\langle a,b\rangle} : M \to M\} = \mathrm{Ker}\{\ell_{\langle a,b\rangle} : \mathfrak{K}(M) \to \mathfrak{K}(M)\}$. This happens if and only if $\mathrm{Ker}\{\ell_{\langle a,b\rangle} M \to M\} \subset \mathfrak{K}(M)$, by another application of the above observation.

If $\ell_{\langle a',b'\rangle} : M \to M$ does not have maximal rank and if $T \subset \mathbb{P}^1(k)$ contains $\langle a',b'\rangle$, then $\mathrm{Ker}\{\ell'_{\langle a',b'\rangle} : M \to M\} \subset {}_TM$, so that ${}_TM$ is not contained in $\mathfrak{K}(M)$. $\square$

As an immediate corollary of Proposition 6.6, we obtain the following characterization of a kG-module of constant rank.

Corollary 6.7. *Let k be an infinite field and M a finite dimensional kG-module. Then $\mathfrak{K}(M) = {}_{\mathbb{P}^1(k)}M$ if and only if the rank of $\ell_{(a,b)}$ on M is independent of $\langle a,b\rangle \in \mathbb{P}^1(k)$.*

Proposition 6.8. *Suppose that $\psi : M \to n$ is a homomorphism of kG-modules. Then $\psi(\mathfrak{K}(M)) \subseteq \mathfrak{K}(N)$. If $M = W_{n,d}$ for any n and d, then $\mathfrak{K}(M) = M$. If M is a KG-module having the equal images property then $\mathfrak{K}(M) = M$.*

Proof. The first statement is a consequence of the observation that $\psi(\mathrm{Ker}\,\{\ell : M \to M\}) \subseteq \mathrm{Ker}\,\{\ell : N \to N\}$. Hence, if $\mathfrak{K}(N) = {}_SN$ for S cofinite in $\mathbb{P}^1(k)$, then

$$\psi(\mathfrak{K}(M)) \subseteq \psi({}_SM) \subseteq {}_SN = \mathfrak{K}(N).$$

This proves the first statement. The last two statements follow from Proposition 2.6 and Corollary 4.5. ☐

The following theorem characterizes the generic kernel $\mathfrak{K}(M)$ of M.

Theorem 6.9. *Let M be a finite dimensional kG-module. The generic kernel $\mathfrak{K}(M)$ of M contains every submodule of M having the equal images property. Thus, $\mathfrak{K}(M)$ is the maximal submodule of M which has the equal images property. Moreover, $\mathfrak{K}(M)$ is the maximal submodule of M which can be written as the quotient of a W module.*

Proof. The first assertion is an immediate consequence of Propositions 6.3 and 6.8. The second now follows from Proposition 4.4. ☐

We obtain as an immediate corollary a proof of the statement that $\mathfrak{K}(M) \subset M$ is "intrinsic".

Corollary 6.10. *Retain the hypotheses and notation of Theorem 6.8. Then $\mathfrak{K}(M)$ does not depend upon the choice of $\{x, y\}$ generating* $\mathrm{Rad}(kG)$.

We explicitly determine generic kernels of the syzygy modules $\Omega^n(k)$ in Example 6.11 and "L_ζ-modules" in Example 6.12.

Example 6.11. A minimal projective resolution of the trivial module k for kG can be given as

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

where $P_n \simeq kG^{n+1}$ is a free kG-module with free basis $c_{n,0}, \dots, c_{n,n}$. The boundary maps are given by the formulae

$$\partial_{2n-1}(c_{2n-1,0}) = xc_{2n-2,0}, \quad \partial_{2n-1}(c_{2n-1,2i-1}) = x^{p-1}c_{2n-2,2i-1} + yc_{2n-2,2i-2},$$

$$\partial_{2n-1}(c_{2n-1,2i}) = xc_{2n-2,2i} - y^{p-1}c_{2n-2,2i-1}, \quad \text{and} \quad \partial_{2n-1}(c_{2n-1,2n-1}) = yc_{2n-2,2n-2},$$

and

$$\partial_{2n}(c_{2n,0}) = x^{p-1}c_{2n-1,0}, \quad \partial_{2n}(c_{2n,2i-1}) = xc_{2n-1,2i-1} - yc_{2n-1,2i-2},$$

$$\partial_{2n}(c_{2n,2i}) = x^{p-1}c_{2n-1,2i} + y^{p-1}c_{2n-1,2i-1}, \quad \text{and} \quad \partial_{2n}(c_{2n,2n}) = y^{p-1}c_{2n-1,2n-1},$$

This is a standard, well known calculation [3].

From the above we get that the module $\Omega^{2n-1}(k)$ has generators $a_1, b_1, a_2, \dots, a_n, b_n$ and relations

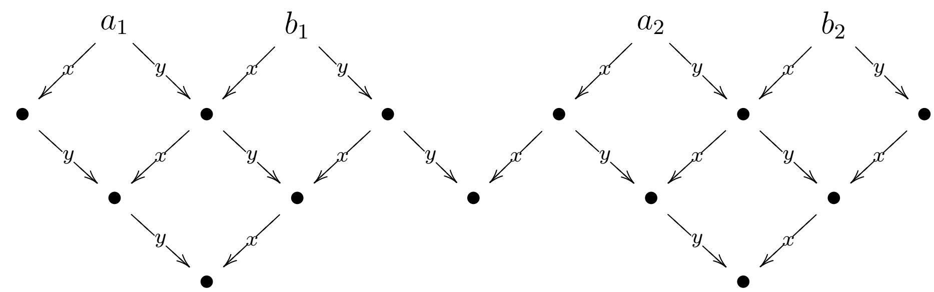
$$x^{p-1}a_1 = 0 = y^{p-1}b_n, \quad ya_i = xb_i, \quad \text{and} \quad x^{p-1}a_i = -y^{p-1}b_{i-1}.$$

That is, we take a_i to be the class of $c_{2n-1,2i}$ modulo the image of ∂_{2n} and b_i to be the class of $c_{2n-1,2i-1}$. The module $\Omega^{2n}(k)$ is generated by elements $a_1, b_1, a_2, \ldots, a_n, b_n, a_{n+1}$ with relations

$$xa_1 = 0 = ya_{n+1}, \quad xa_i = y^{p-1}b_{i-1}, \quad \text{and} \quad ya_i = -x^{p-1}b_i.$$

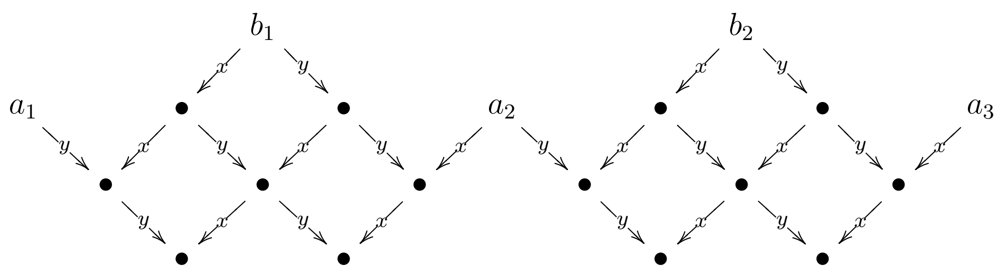
In the case that $p = 3$, we can draw diagrams which reveal the structure of $\Omega^3(k)$ and $\Omega^4(k)$ as follows:

$\Omega^3(k) =$



and

$\Omega^4(k) =$



With the above information, it can be calculated that $\mathfrak{K}(\Omega^{2n-1}(k))$ is the submodule of $\Omega^{2n-1}(k)$ generated by the elements

$$x^j y^{p-2-j} a_i, \;\; x^j y^{p-2-j} b_i \quad \text{for} \quad j = 0, \ldots, p-2 \quad \text{and} \quad i = 1, \ldots, n.$$

Likewise, we have that $\mathfrak{K}(\Omega^{2n-1}(k))$ is the submodule generated by the elements

$$a_1, \ldots, a_{n+1}, \qquad \text{and}$$

$$x^j y^{p-2-j} b_i, \quad \text{for} \quad j = 0, \ldots, p-2 \quad \text{and} \quad i = 1, \ldots, n.$$

The above conclusion is made using Theorem 6.9. That is, we need only look for the largest submodule having the equal images property. In the diagrams we look for the largest submodule having the "W" shape. This exercise is left to the reader.

Example 6.12. Similar arguments can be made for the L_ζ modules. For example, suppose that $\zeta \in \mathrm{H}^n(G, k)$ is a nilpotent element. Let L_ζ be the module which is the kernel of a cocycle

$$\zeta : \Omega^n(k) \to k,$$

representing the cohomology element ζ. Then $\mathfrak{K}(L_\zeta) = \mathfrak{K}(\Omega^n(k))$, the isomorphism being induced by the inclusion of L_ζ into $\Omega^n(k)$.

A justification for the above statement is the following. Suppose that $\ell = ax + by$ for $(a, b) \neq (0, 0)$ in k^2. If n is even, then the restriction $\Omega^n(k)_{\downarrow\langle\ell\rangle}$ of $\Omega^n(k)$ to the subalgebra generated by ℓ has the form $k \oplus k\langle\ell\rangle^s$ for some s. Here $k\langle\ell\rangle$ is the rank one free module over $k\langle\ell\rangle \cong k[t]/(t^p)$. Because ζ is nilpotent, we must have that the summand isomorphic to k lies in the kernel of the cocyle ζ, as otherwise the cocycle ζ would be left split. In particular,

$$\mathrm{Ker}\,\{\ell : \Omega^n(k) \to \Omega^n(k)\} \;\subseteq\; \mathrm{Ker}\,\{\zeta\}.$$

It is not difficult to see that the same happens also in the case that n is odd. Hence, from the definition of the generic kernel, we can see that $\mathfrak{K}(\Omega^n(k)) \subseteq \mathrm{Ker}\,\{\zeta\} = L_\zeta$, as asserted.

On the other hand, if ζ not nilpotent, then there exists an open subset S of $\mathbb{P}^1(k)$, such that for any $\langle a, b\rangle \in S$, the summand isomorphic to k in the decomposition of the restriction of $\Omega^n(k)$ to $\langle \ell_{(a,b)}\rangle$ is not in the kernel of ζ. As a result $\mathfrak{K}(L_\zeta)$ is a proper submodule of $\mathfrak{K}(\Omega^n(k))$. Again we should look for modules with the "W" shape. It can be proved in a similar way that for ζ not nilpotent, $\mathfrak{K}(L_\zeta) = x\mathfrak{K}(\Omega^n(k)) = \mathrm{Rad}(\mathfrak{K}(\Omega^n(k)))$.

An easy consequence of Proposition 6.8, is the functoriality of $M \mapsto \mathfrak{K}(M)$.

Proposition 6.13. *The association of a finite dimensional kG-module M to its generic kernel determines a functor*

$$\mathfrak{K} : mod(kG) \;\longrightarrow\; mod(kG).$$

Clearly, $\mathfrak{K}(-)$ preserves monomorphisms. However, as the following example shows, $\mathfrak{K}(-)$ is not left exact. (The augmentation map $kG \longrightarrow k$ is a simple example of an epimorphism with the property that the induced map on generic kernels is not an epimorphism.)

Example 6.14. Let $M = \mathrm{Rad}^{p-1}(kG)$ and let $L \subseteq M$ be the submodule generated by x^{p-1}. Then we have an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

where $N = M/L$ is the quotient. The modules M and N have the equal images property and hence they are equal to their own generic kernels. However, it is easy to check that the generic kernel of L is its socle which is isomorphic to k. So applying the functor $\mathfrak{K}$ we obtain the sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N$$

which is not exact at M.

7. Duality and filtrations

In this section we assume that the field k is infinite. Let kG be the group algebra of an elementary abelian group of rank 2, and fix generators x and y of the radical of kG. For M a finite dimensional kG-module, we let $M^{\#}$ denote its k-linear dual. For each $V \subset M$, we associate $V^{\perp} \subset M^{\#}$ defined as $V^{\perp} = \{f : M \to k | f(V) = 0\}$.

Lemma 7.1. *Let M be a finite dimensional $k[t]/t^p$-module and let $V \subset M$ be a submodule. Then the Jordan type of V is the same as that of $M^{\#}/V^{\perp}$.*

Proof. The natural pairing

$$V \otimes (M^{\#}/V^{\perp}) \;\to\; k, \quad v \otimes f \mapsto f(v)$$

is a perfect pairing of $k[t]/t^p$-modules. The lemma follows by observing that the $k[t]/t^p$-dual $W^{\#}$ of a $k[t]/t^p$-module W has the same Jordan type as that of W. □

We introduce the generic image $\mathfrak{I}(M)$ of a finite dimensional kG-module M, whose constructions and properties are analogous to those of the generic kernel $\mathfrak{K}(M)$ of M. We see in Proposition 7.4, $\mathfrak{K}(M)$ and $\mathfrak{I}(M)$ are related by duality.

For any subset $S \subset \mathbb{P}^1(k)$, define the subspace

$$M_S \;\equiv\; \bigcap_{\langle a,b\rangle \in S} \mathrm{Im}\,\{ax + by : M \to M\}$$

in analogy with ${}_SM$ of the last section. Here, as usual, the image of multiplication by $\ell = ax + by$ is independent of the representative of the class $\langle a, b\rangle \in \mathbb{P}^1$.

Since both x, y commute with each $ax + by \in S$, M_S is a kG-submodules of M. Clearly, for $S' \subset S$, $M_{S'} \supset M_S$.

Definition 7.2. We define the *generic image* of M to be

$$\mathfrak{I}(M) \;\equiv\; \sum_{S \subset \mathbb{P}^1(k)\ cofinite} M_S.$$

As with the generic kernel, because M is finite dimensional, there exist a subset S in $\mathbb{P}^1(k)$ which is cofinite and has the property that ${}_SM = \mathfrak{I}(M)$.

Example 7.3. Suppose that M is the submodule of kG generated by $m = y^{p-1}$. Then $(ax + by)m = axm$. Consequently, $\mathfrak{I}(M) = \mathrm{Rad}(kG)M = \mathrm{Rad}(M)$. On the other hand, the only subspace of M which is annihilated by $(ax + by)$ for $a \neq 0$ is generated by $x^{p-1}m$. Hence $\mathfrak{K}(M)$ is the submodule $x^{p-1}M$, which is contained in $\mathfrak{I}(M)$. This situation contrasts sharply with the results of Theorem 7.6 that follows. We note also that $\mathfrak{I}(M)$ does not have constant Jordan type in this example.

Proposition 7.4. *Let M be a finite $kG = k[x, y]/(x^p, y^p)$-module. Then*

$$\mathfrak{I}(M) \;\simeq\; \mathfrak{K}(M^{\#})^{\perp}, \quad \mathfrak{K}(M) \;\simeq\; \mathfrak{I}(M^{\#})^{\perp}. \tag{22}$$

It follows that $M^\#/\mathfrak{I}(M^\#)$ has constant Jordan type. In addition, $\mathfrak{I}(M)$ has constant Jordan type if and only if $M^\#/\mathfrak{K}(M^\#)$ does, in which case their Jordan types are equal.

Proof. For any linear map $\ell : M \to M$,

$$(\ell M)^\perp \;=\; \{g : M \to k | g \circ \ell = 0\} \;=\; \mathrm{Ker}\,\{\ell^\# : M^\# \to M^\#\} \tag{23}$$

For every $S \;\subset \mathbb{P}^1(k)$,

$$\left(\bigcap_{\langle a,b\rangle \in S} \mathrm{Im}\,\{ax + by : M \to M\} \right)^\perp \;=\; \sum_{\langle a,b\rangle \in S} \mathrm{Ker}\,\{(ax+by)^\# : M^\# \to M^\#\}. \tag{24}$$

Thus, we conclude that

$$\mathfrak{I}(M)^\perp \;=\; \mathfrak{K}(M^\#).$$

It follows that

$$M^\#/\mathfrak{I}(M^\#) \simeq \mathfrak{I}(M^\#)^\perp \;=\; \mathfrak{K}(M) \quad \text{and} \quad \mathfrak{I}(M) = \mathfrak{K}(M^\#)^\perp \simeq M^\#/\mathfrak{K}(M^\#).$$

The second statement is a consequence of the fact that $\mathfrak{K}(M)$ has the equal images property.

□

The next lemma is key to the proof of Theorem 7.6.

Lemma 7.5. *Let M be a kG-module of constant rank. Then for any $\langle a, b\rangle \in \mathbb{P}^1(k)$, the kernel of multiplication by $ax + by$ on $M/x\mathfrak{K}(M)$ is equal to $\mathfrak{K}(M)/x\mathfrak{K}(M)$. Hence, $(M/x\mathfrak{K}(M))^\# \subseteq M^\#$ has the equal images property, and thus is contained in $\mathfrak{K}(M^\#)$.*

Proof. Let $\overline{M}$ denote $M/x\mathfrak{K}(M)$. By Corollary 6.7, we know that $\mathfrak{K}(M) = \;_S(M)$ where $S = \mathbb{P}^1(k)$. Observe that

$$\mathfrak{K}(M)/x\mathfrak{K}(M) \;\subset\; \mathrm{Ker}\,\{ax + by : \overline{M} \to \overline{M}\},$$

since we know that $(ax + by)\mathfrak{K}(M) = x\mathfrak{K}(M)$.

To prove the reverse inclusion, suppose that $m + \mathfrak{K}(M)$ is in $\mathrm{Ker}\,\{ax + by : \overline{M} \to \overline{M}\}$ for some m in M. Then $(ax + by)m \in x\mathfrak{K}(M) = (ax + by)\mathfrak{K}(M)$. So $(ax + by)m = (ax + by)m'$ for some $m' \in \mathfrak{K}(M)$. But then $(ax + by)(m - m') = 0$ and thus $m - m' \in \mathrm{Ker}\,\{ax + by : M \to M\} \subseteq \mathfrak{K}(M)$. It follows that m is in $\mathfrak{K}(M)$, and hence

$$\mathrm{Ker}\,\{(ax + by) : \overline{M} \to \overline{M}\} \;\subset \mathfrak{K}(M)/x\mathfrak{K}(M).$$

The second assertion follows by dualization and an application of Proposition 6.8. □

Theorem 7.6. *If M is a kG-module of constant rank, then $\mathfrak{I}(M) = x\mathfrak{K}(M)$.*

Proof. Observe that $x\mathfrak{K}(M) \subseteq \mathfrak{I}(M)$, because $x\mathfrak{K}(M) = (ax + by)\mathfrak{K}(M)$ for any $\langle a, b\rangle \in \mathbb{P}(k)$. To prove $\mathfrak{I}(M) \subseteq x\mathfrak{K}(M)$, we apply Lemma 7.5. Namely, Proposition 7.4 and Lemma 7.5 imply that

$$\mathfrak{I}(M) \;=\; (\mathfrak{K}(M^{\#}))^{\perp} \;\subset\; ((M/x\mathfrak{K}(M))^{\#})^{\perp} \;=\; x\mathfrak{K}(M).$$

□

To formulate the next theorem, we find it convenient to introduce the dual notion of the equal images property of Definition 1.4. As before, if $\alpha_K : K[t]/t^p \to KG$, is a π-point, we let $\ell_\alpha = \alpha_K(t) \in KG$.

Definition 7.7. A finite dimensional kG-module M is said to have **the equal kernels property** if

$$(\mathrm{Ker}\,\{\ell_\alpha(M_K) : M_K \to M_K\})_\Omega \;=\; (\mathrm{Ker}\,\{\ell_\beta(M_L) : M_L \to M_L)_\Omega$$

for any two π-points $\alpha_K : K[t]/t^p \to KG,\ \beta_L : L[t]/t^p \to LG$ and any field extension Ω of both K, L.

As an example, notice that if M has constant rank, then $M/x\mathfrak{K}(M)$ has the equal kernels property by Lemma 7.5.

Remark 7.8. In Example 1.3, the module M fails to have the equal kernels property, even though the kernel of any operator of the form $ax + by$ on M is independent of the choice of $\langle a, b\rangle \in \mathbb{P}^1(k)$. That is, a field extension is necessary in order to expose the failure of the property. However, one can prove almost precise analogs of Propositions 1.6 and 1.7 for the equal kernels property. Here, image must be exchanged for kernel and radical for socle. So, for example, for the equal kernels property, the equation at the end of Proposition 1.6 would read

$$\mathrm{Ker}\,\{\sum_{i=1}^{r} a_i x_i : M_K \to M_K\} \;=\; \mathrm{Soc}(M_K).$$

The proofs can be constructed in a similar way. However, the results can also be verified using the following proposition.

Proposition 7.9. *A kG-module M has the equal kernels property if and only if $M^{\#}$ has the equal images property.*

Proof. Suppose that M has the equal kernels property. Let K be an extension of k and let $\alpha_K : K[t]/t^p \to KG$ be a π-point. Let $\ell = \alpha_K(t)$. Then, the dual statement of Equation 23 says that

$$\mathrm{Ker}\,\{\ell : M_K \to M_K\} = \{g : M_K^{\#} \to K | g \circ \ell^{\#} = 0\} = (\ell M_K^{\#})^{\perp}.$$

Because $\mathrm{Ker}\,\{\ell : M_K \to M_K\} = \mathrm{Ker}\,\{x : M_K \to M_K\}$, we have that

$$\ell M_K^{\#} \;=\; ((\ell M_K^{\#})^{\perp})^{\perp} \;=\; ((x M_K^{\#})^{\perp})^{\perp} \;=\; x M_K^{\#}.$$

Hence $M^\#$ has the equal images property. The proof in the other direction is similar. □

If M is a kG-module and $W \subset M$ is a submodule, then we denote by $x^{-1}W \subset M$ the kG-submodule consisting of elements $m \in M$ satisfying $xm \in W$.

We verify by inspection that for any submodule $W \subset M$ and any $i > 0$,

$$((x^i)^{-1}(W))^\perp \;=\; x^{i\#}(W^\perp), \tag{25}$$

Here, $x^{i\#} : M^\# \to M^\#$ sends f to $f \circ x^i$, where $x^i : M \to M$ is the action of $x^i \in kG$ on M. With this notation, we can prove the following.

Theorem 7.10. *Let M be a module of constant rank and let $W \subset M$ be its generic kernel. Then we have an increasing filtration of M,*

$$x^{p-1}(W) \subseteq x^{p-2}(W) \subseteq \cdots \subseteq x(W) \subseteq W \subseteq x^{-1}(W) \subseteq \cdots \subseteq x^{1-p}(W) \subseteq M$$

with the property that $x^i(W)$, for $i \geq 0$ has the equal images property and that $M/x^j(W)$ for $j < 0$ has the equal kernels property. Moreover, for any $\ell = ax + by$ with $\langle a, b\rangle \in \mathbb{P}^1(k)$ we have that $x^jW = \ell^jW$. Here, $x^j(W)$ denotes $(x^j)^{-1}(W)$ for $j < 0$.

Proof. First observe that all of the submodules in the theorem are stable under field extension. So there is no loss in generality by assuming that k is algebraically closed. Each of the $x^{-j}W$ is a submodule of M because kG is commutative.

Each of the modules $x^i(W)$ with $i > 0$ is a radical power of $W = \mathfrak{K}(M)$ and thus has the equal images property by Proposition 1.9.

Using Proposition 7.4, Theorem 7.6, and (25), we conclude that

$$((x^i)^{-1}(W))^\perp \;=\; x^{i\#}(W^\perp) \;=\; x^{i\#}(\mathfrak{I}(M^\#)) \;=\; (x^{i+1})^\#(\mathfrak{K}(M^\#)).$$

Hence, $((x^i)^{-1}(W))^\perp = (M/x^{-i}W)^\#$ has the equal images property. Thus, Proposition 7.9 implies that $M/x^{-i}(W)$ has the equal kernels property.

For $\ell = ax + by \neq 0$, we know that $x^jW = \ell^jW$ for $j \geq 0$ by the fact that W has the equal images property. For negative values of j the result is a consequence of the equal kernels property on the quotients. That is, for $j \leq 0$ we have that $x^{j-1}W = x^{-1}(x^jW)$ is precisely the inverse image under the quotient map $M \to M/(x^jM)$ of the kernel of x on $M/(x^jM)$. Because $M/(x^jM)$ has the equal kernels property, the kernel of multiplication by x is the same as the kernel of multiplication by ℓ. Therefore, $x^{-1}(x^jW) = \ell^{-1}(x^jW) = \ell^{-1}(\ell^jW) = \ell^{j-1}W$ as asserted. □

Question 7.11. *In Theorem 7.10, we know that the submodules $\mathfrak{K}(M)$, and $x^i\mathfrak{K}(M)$ have constant Jordan type for $i \geq 0$. Assuming that M has constant Jordan type, do the modules $x^{-i}\mathfrak{K}(M)$ also have constant Jordan type? If yes, then the characterization of the cyclic module of constant Jordan type becomes much easier.*

8. n^{th}-POWER GENERIC KERNELS

In this section, we assume that the field k is algebraically closed. This assumption is not necessary. That is, we could formulate Definition 8.1, which follows, along the lines of Definition 1.4 and then prove analogs of 1.6 and 1.7. But we will spare the reader the pain.

We fix a choice of generators x, y of $\operatorname{Rad} kG$, with $kG \simeq k[x,y]/x^p, y^p$.

Suppose that n is an integer with $1 \leq n \leq p-1$. For a subset $S \subset \mathbb{P}^1(k)$, define the submodule

$$ {}^n_S M \ \equiv \sum_{\langle a,b\rangle \in S} \operatorname{Ker}\{(ax+by)^n : M \to M\} $$

Since both x, y commute with each $ax+by \in S$, ${}_S M$ is a kG-submodules of M. Clearly, for $S' \subset S$, ${}^n_{S'}M \subset {}^n_S M$.

Definition 8.1. We define the n^{th}*-power generic kernel* of M to be

$$ \mathfrak{K}^n(M) \ = \bigcap_{S \subset \mathbb{P}^1 \ cofinite} {}^n_S M. $$

Example 8.2. For the free cyclic kG-module (which we denote kG), we have that

$$ \mathfrak{K}^n(kG) \ = \ \operatorname{Rad}^{p-n}(kG) \ = \ x^{-n+1}\mathfrak{K}(kG) $$

for any $1 \leq n < p$. The case $n = 1$ is in Example 6.2. The same sort of argument works also in this case. That is, any $\ell = ax + by$ has Jordan type $p[p]$ on kG and hence the kernel of ℓ^n on kG is in $\ell^{p-n}kG \subseteq \operatorname{Rad}^{p-n}(kG)$. So $\mathfrak{K}^n(kG) \subseteq \operatorname{Rad}^{p-n}(kG)$. To get the reverse inequality, we note that ℓ^{p-n} is in the kernel of multiplication by ℓ^n. Hence, some infinite subset (cofinite in $\mathbb{P}^1$) of elements of the form ℓ^{p-n} is in $\mathfrak{K}^n(kG)$. Some finite subset of these will generate $\operatorname{Rad}^{p-n}(kG)$.

Note that $\mathfrak{K}^1(M) = \mathfrak{K}(M)$ for any M. The n-power generic kernel enjoys many of the same properties as the generic kernel. For example. there must exist some non-empty open set $S_n \subseteq \mathbb{P}^1(k)$ such that $\mathfrak{K}^n(M) = {}^n_{S_n}M$. If we let $S = \cap S_n$, then S has the property that $\mathfrak{K}^n(M) = {}^n_S M$ for all n.

The following proposition provides a partial generalization of Example 8.2 to other kG-modules.

Proposition 8.3. *Suppose that M is a kG-module of constant rank. Then for any $n = 2, \ldots, p-1$ we have that*

$$ \mathfrak{K}^n(M) \ \subseteq \ x^{-n+1}\mathfrak{K}(M). $$

Proof. We prove the case that $n = 2$. The rest follows by similar arguments which we leave to the reader. Suppose that S is a subset of $\mathbb{P}^1$ such that

$$ \mathfrak{K}^2(M) = {}^2_S M = \sum_{\langle a,b\rangle \in S} \operatorname{Ker}\{(ax+by)^2 : M \to M\} $$

and

$$\mathfrak{K}(M) = {}_SM.$$

If $\langle a, b\rangle \in S$ and if $m \in \operatorname{Ker}\{(ax+by)^2 : M \to M\}$, then $(ax+by)^2m = 0$ and hence, $(ax+by)m \in \mathfrak{K}(M)$. So, $m \in (ax+by)^{-1}\mathfrak{K}(M) = x^{-1}\mathfrak{K}(M)$ by Theorem 7.10. It follows that $\mathfrak{K}^2(M) \subseteq x^{-1}M$. $\square$

On the other hand, the equality of Example 8.2 is not valid for general kG-modules as seen in the following example.

Example 8.4. Let M be the kG-module represented by the diagram

$$\begin{tikzcd}
 & & & a_1 \arrow[dl, "x"] \arrow[dr, "y"] & & & \\
b_1 \arrow[dr, "y"] & & b_2 \arrow[dl, "x"] & & b_3 \arrow[dr, "y"] & & b_4 \arrow[dl, "x"] \\
 & c_1 & & & & c_2 & .
\end{tikzcd}$$

That is, M has k-basis consisting of the elements $a_1, b_1, b_2, b_3, b_4, c_1, c_2$ and the matrices of x and y are given by

$$x \to \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0 \end{pmatrix}, \quad y \to \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}.$$

It is straightforward to see that M has constant Jordan type $[3] + [2] + 2[1]$. Note that for any $\langle a, b\rangle \in \mathbb{P}^1$, we have that $(ax+by)^2a_1 \neq 0$ and that $(ax+by)^2b_j = 0$ for $j = 1, \ldots, 4$. Consequently, $\mathfrak{K}^2(M)$ is the submodule N spanned by $b_1, \ldots, b_4, c_1, c_2$. Because N is the direct sum of two W modules, it is equal to its own generic kernel. Therefore we have that

$$\mathfrak{K}^2(M) \;=\; N \;=\; \mathfrak{K}(M) \;\neq\; x^{-1}\mathfrak{K}(M).$$

Motivated by Question 7.11, we ask the following.

Question 8.5. *If M has constant Jordan type, then does $\mathfrak{K}^n(M)$ also have constant Jordan type?*

9. Some preliminaries on cyclic modules

The purpose of this section is to develop some technical results that are essential for the proofs of the next section. The subject matter is something of a digression from the primary issues of the paper. The reader may wish to continue with the next section and return to this material only as needed.

Definition 9.1. Let $p_1(t), \ldots, p_r(t) \in k[t]$ be polynomials. For any $i > 0$, we denote by $p_j^{(i)}(t)$ the i-th derivative of $p_j(t)$. The **Wronskian** $W(p_1, \ldots, p_r)$ of $\{p_1(t), \ldots, p_r(t)\}$ is the following determinant:

$$W(p_1, \ldots, p_r) \;=\; \operatorname{Det} \begin{pmatrix} p_1(t) & p_2(t) & \cdots & p_r(t) \\ p_1^{(1)}(t) & p_2^{(1)}(t) & \cdots & p_r^{(1)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ p_1^{(r-1)}(t) & p_2^{(r-1)}(t) & \cdots & p_r^{(r-1)}(t) \end{pmatrix} \tag{26}$$

For k of characteristic 0, it is well known (cf. [11], pp. 524-525) that $W(p_1, \ldots, p_r) = 0$ if and only if $p_1(t), \ldots, p_r(t)$ are linearly dependent. Moreover, if one of the polynomials $p_i(t)$ has degree at least r, then $W(p_1, \ldots, p_r)$ is not a constant polynomial. This is because t would divide the leading term in every entry of some column. For a field of positive characteristic, we have the following.

Proposition 9.2. *Let k be a field of characteristic $p > 0$ and consider polynomials $p_1(t), \ldots, p_r(t) \in k[t]$. Then $W(p_1, \ldots, p_r) = 0$ if and only if $p_1(t), \ldots, p_r(t)$ are linearly dependent over $k(t^p) \subset k(t)$.*

Proof. We first assume that $p_1(t), \ldots, p_r(t)$ are linearly dependent over $k(t^p)$ so that we have a relation

$$0 \;=\; \sum_{i=1}^{r} e_i \cdot p_i(t) \quad \text{for} \quad e_i \in k(t^p).$$

By taking derivatives, and using the fact that the derivative of every e_i is zero, we obtain a system of equations

$$\begin{array}{rcl} e_1 p_1(t) + e_2 p_2(t) + \ldots + e_r p_r(t) & = & 0, \\ e_1 p_1^{(1)}(t) + e_2 p_2^{(1)}(t) + \ldots + e_r p_r^{(1)}(t) & = & 0, \\ & & \\ e_1 p_1^{(r-1)}(t) + e_2 p_2^{(r-1)}(t) + \ldots + e_r p_r^{(r-1)}(t) & = & 0. \end{array} \tag{27}$$

This implies that the columns $\underline{C}_1, \ldots, \underline{C}_r$ of the matrix

$$\begin{pmatrix} p_1(t) & p_2(t) & \cdots & p_r(t) \\ p_1^{(1)}(t) & p_2^{(1)}(t) & \cdots & p_r^{(1)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ p_1^{(r-1)}(t) & p_2^{(r-1)}(t) & \cdots & p_r^{(r-1)}(t) \end{pmatrix} \tag{28}$$

are linearly dependent over $k(t^p)$, and thus that the determinant of (28) (i.e., $W(p_1, \ldots, p_n)$) is 0.

Conversely, assume that $W(p_1, \ldots, p_r) = 0$. We proceed by induction on r, so that it suffices to also assume that $W(p_1, \ldots, p_{r-1}) \neq 0$. Without loss of generality, we may further assume that the right-most column $\underline{C}_r$ of (28) is a linear combination of the the first $r-1$ columns $\underline{C}_1, \ldots, \underline{C}_{r-1}$,

$$\underline{C}_r \; = \; \lambda_1 \underline{C}_1 + \cdots + \lambda_{n-1} \underline{C}_{r-1}, \quad \lambda_i \in k(t). \tag{29}$$

Subtracting the first derivative of the i^{th} row of (29) from the $(i+1)^{st}$-row for $i = 1, \ldots, r-2$, we conclude, using the product rule for derivatives, that

$$\lambda_1' p_1^{(i)}(t) + \cdots + \lambda_{r-1}' p_{r-1}^{(i)}(t) = 0, \quad 0 \leq i \leq r-2,$$

where λ_i' is the derivative of λ_i. Since $W(p_1, \ldots, p_{r-1}) \neq 0$, we see that $\lambda_i' = 0$, for all $1 \leq i \leq r-1$. Thus, the first row of (29) gives a linear dependence over $k(t^p)$ of $p_1, \ldots, p_r$. □

Corollary 9.3. *With notation as in Proposition 9.2, assume that the degree of each $p_i(t)$ is less than p. Then $W(p_1, \ldots, p_r) \neq 0$ if and only if $p_1(t), \ldots, p_r(t)$ are linearly independent over k. Moreover, if $deg(p_i(t)) \geq r$ for some i, then $W(p_1(t), \ldots, p_r(t))$ is not constant as a function of p.*

Proof. If $W(p_1, \ldots, p_r) \neq 0$, then Proposition 9.2 asserts that $p_1(t), \ldots, p_r(t)$ are linearly independent over $k(t^p)$ and thus also linearly independent over k.

Conversely, assume that $p_1(t), \ldots, p_r(t)$ are linearly independent over k. Since each $p_i(t)$ has degree less than p, we conclude that $r \leq p$. Using elementary transformations with constant coefficients on the system $\{p_1(t), \ldots, p_r(t)\}$, which do not change the value of the Wronskian $W(p_1, \ldots, p_r)$, we may assume that $deg(p_i(t)) \neq deg(p_j(t))$ for $i \neq j$. That is, if $deg(p_i(t)) = deg(p_j(t))$, then there is a scalar c such that $q_j(t) = p_j(t) - cp_i(t)$ has lower degree. Replacing $p_j(t)$ by $q_j(t)$ amounts to an elementary column operation which does not change the value of the Wronskian. As a result, we may assume that $p_1(t), \ldots, p_r(t)$ all have different degrees. Hence, they are linearly independent over $k(t^p)$. Hence by Proposition 9.2, the Wronskian is not zero.

To prove the final statement of the corollary, we note that by the above argument we may assume that no two of the polynomial $p_1(t), \ldots, p_r(t)$ have the same degree. Let $q_i(t)$ denote the leading term of $p_i(t)$. It is not difficult to see that the leading term of $W(q_1(t), \ldots, q_r(t))$ is the leading term of $W(p_1(t), \ldots, p_r(t))$. Because the elements $q_i(t)$ all have different degrees, they are independent over $k(t^p)$ and the Wronskian is not zero.

If for some i the degree of $q_i(t)$ is at least r, then every entry in the i^{th} column of matrix of the Wronskian $W(q_1(t), \ldots, q_r(t))$ is divisible by t. It follows that the leading term of $W(p_1(t), \ldots, p_r(t))$ is divisible by t and hence it is not constant. □

Proposition 9.4. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $k[t]_{\leq n}$ denote the k-vector space of polynomials of degree at most n for some $n < p$. Assume that $n < p$ if the characteristic p is positive. Fix an integer $j < n$, and suppose that $V \subset k[t]_{\leq n}$ is a proper subspace having the properties that*

(i.) *V contains a polynomial of degree $j+1$, and* (ii.) *for any $a \in k$, there exists some $f(t) \in V$ such that $(t-a)^j$ divides $f(t)$ (i.e., such that a is a root of multiplicity at least j of $f(t)$).*

Then there exists some $a \in k$ and some $g(t) \in W$ such that $(t-a)^{j+1}$ divides $g(t)$.

Proof. Observe that $\mathrm{Dim}\, k[t]_{\leq n} = n+1$. Let r denote $\mathrm{Dim}\, V$. Choose a basis $p_1(t), p_2(t), \ldots, p_r(t)$ of V with the property that

$$d_1 = deg(p_1(t)) > d_2 = deg(p_2(t)) > \cdots > d_r = deg(p_r(t)).$$

To emphasize its dependence on t, we let $D(t) = W(p_1(t), \ldots, p_r(t))$:

$$D(t) \;=\; \mathrm{Det}\begin{pmatrix} p_1(t) & p_2(t) & \cdots & p_r(t) \\ p_1^{(1)}(t) & p_2^{(1)}(t) & \cdots & p_r^{(1)}(t) \\ \ldots & \ldots & \cdots & \ldots \\ p_1^{(r-1)}(t) & p_2^{(r-1)}(t) & \cdots & p_r^{(r-1)}(t) \end{pmatrix} \tag{30}$$

be the Wronskian. By Corollary 9.3 in the case that $p > 0$, and Condition $(i.)$, $D(t)$ is not a constant polynomial. In particular, it is not zero.

We first verify that $\dim V = r \geq j+1$. By Condition $(ii.)$ for any element $a \in k$ there exists a non-zero polynomial $f(t) = \sum_{i=1}^{r} \mu_i p_i(t)$ in V such that $f(t)$ is is divisible by $(t-a)^j$. Consider

$$\begin{pmatrix} f(t) \\ f^{(1)}(t) \\ \ldots \\ f^{(r-1)}(t) \end{pmatrix} = \begin{pmatrix} p_1(t) & p_2(t) & \cdots & p_r(t) \\ p_1^{(1)}(t) & p_2^{(1)}(t) & \cdots & p_r^{(1)}(t) \\ \ldots & \ldots & \cdots & \ldots \\ p_1^{(r-1)}(t) & p_2^{(r-1)}(t) & \cdots & p_r^{(r-1)}(t) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_r \end{pmatrix}. \tag{31}$$

If $r < j+1$, then the column on the left of (31) vanishes when evaluated at $t = a$, so the fact that $f(t)$ is not zero implies that $D(a) = 0$ (i.e., a is a root of $D(t)$). As this happens for any a in k we have a contradiction to the fact that $D(t)$ can have only finitely many roots. Hence, we conclude that $r \geq j+1$.

We now proceed to show that some element of V has a root a of multiplicity $j+1$. This would prove the proposition. Suppose to the contrary that there is even a single element $\gamma \in k$ such that no multiple of $(t-\gamma)^{j+1}$ is in V. That is, suppose that V intersects trivially with the subspace U of all multiples of $(t-\gamma)^{j+1}$ in $k[t]_{\leq n}$. Then the dimension of V is at most equal to the dimension of $k[t]_{\leq n}$ minus the dimension of U. That is,

$$r = \mathrm{Dim}\, V \leq (n+1) - (n-(j+1)) = j+1.$$

Therefore, for the rest of the proof, we may assume that r = j+1.

Because the Wronskian $D(t)$ is not a constant polynomial and the field k is algebraically closed, $D(t)$ must have a root $a \in k$. Then the determinant of the matrix

$$\begin{pmatrix} p_1(a) & p_2(a) & \cdots & p_r(a) \\ p_1^{(1)}(a) & p_2^{(a)}(t) & \cdots & p_r^{(1)}(a) \\ \cdots & \cdots & \cdots & \cdots \\ p_1^{(r-1)}(a) & p_2^{(r-1)}(a) & \cdots & p_r^{(r-1)}(a) \end{pmatrix} \tag{32}$$

is zero, and we must have that its columns are linearly dependent vectors in k^r. Suppose that $\mu_1, \ldots, \mu_r$ are the coefficients of a dependence relation among the columns. Then we have that if $f(t) = \sum_{i=1}^r \mu_i p_i(t)$, then $f(a) = 0$ and moreover, $f^{(i)}(a) = 0$ for all $i = 1, \ldots, r-1$. Therefore $f(t) \in V$ and a is a root of $f(t)$ of multiplicity r. This proves the proposition. □

Corollary 9.5. *Let k be a field of characteristic $p \geq 0$. Let $k[x, y]_n$ denote the space of homogeneous polynomials of degree n in the variable x and y. Assume that $n < p$ if $p > 0$. Fix an integer $j < n$. Suppose that V is a subspace of $k[x, y]_n$ having the following properties.*

(i.) *The space V contains an element which is not divisible by y^{n-j-1}.*
(ii.) *For any $a \in k$, there exists an element $f(x, y)$ in V such that $f(x, y)$ is divisible by $(x - ay)^j$.*

Then for some $a \in k$, V contains an element $f(x, y)$ which is divisible by $(x-ay)^{j+1}$.

Proof. Let $\psi : k[x, y]_n \longrightarrow k[t]_{\leq n}$ be the linear transformation of vector spaces obtained by sending x to t and y to 1. Then under the isomorphism ψ, conditions ($i.$) and ($ii.$) of the corollary translate into conditions ($i.$) and ($ii.$) of Proposition 9.4. Moreover, the conclusion of Proposition 9.4 translates to the conclusion of the corollary. □

10. Cyclic modules of constant Jordan type

Our objective in this section is to show that any cyclic module of constant Jordan type is a quotient of the ring by the augmentation ideal of the ring. We break up the argument into two cases. For the first case, Theorem 10.1 completely settles the case of a field of characteristic 0 and goes part way to verifying this result for fields of positive characteristic.

Theorem 10.1. *Let k be a field of any characteristic. Consider a cyclic $k[x, y]$-module M which is finite dimensional and has constant Jordan type. Let $m \in M$ be a generator and let n be the least integer such that $x^{n+1}m = 0$. So $x^n \cdot m \neq 0$. If k has positive characteristic p, then assume that $n < p$. Then*

$$M \;\simeq\; k[x, y]/I^{n+1}$$

where $I = (x, y) \subset k[x, y]$ denotes the augmentation ideal.

Proof. Observe that the hypothesis that M has constant Jordan type implies that $(x+ay)^{n+1}m = 0$ for all a in k.. Hence, all monomials in x, y of degree $n+1$ kill M, so that M is a quotient of $k[x,y]/I^{n+1}$. Note here that even in the case that k is the prime field with p elements, the assumption that $n < p$ assures us that the elements $(x+ay)^{n+1}$ span the space of all homogeneous polynomials of degree $n+1$ in $k[x,y]$.

Without loss of generality, we may assume that the field k is algebraically closed. That is, suppose that K is the algebraic closure of k. Because M has constant Jordan type, so also does $K \otimes_k M$. Also, since $K \otimes_k -$ is an exact functor, a proof that $K \otimes_k M \simeq K[x,y]/(I')^{n+1}$ implies that $M \simeq k[x,y]/I^{n+1}$ where $I' = K \otimes I$ is the augmentation ideal of $K[x,y]$. Hence we assume that $k = K$.

Our strategy is to show that the Jordan type of M has $n+1$ blocks, one of size j for each $1 \leq j \leq n+1$, thereby showing that the induced map $k[x,y]/I^{n+1} \to M$ is an isomorphism. Note that there is exactly one block of size $j = n+1$, since the number of blocks of M of size $n+1$ equals $\mathrm{Dim}(x^n \cdot M)$. Yet, $x^n \cdot M = \langle x^n m \rangle$, and hence has dimension 1.

We assume that the Jordan type of M has exactly 1 block each of size $n+1, \ldots, j+1$. In other words, we assume that

$$\{m, \ldots, x^n m, ym, xym, \ldots, x^{n-1}ym, \ldots, y^{s-1}m, \ldots, y^{n-j}m, \ldots, x^j y^{n-j} m\}$$

are linearly independent. This condition is equivalent to the condition that

$$S = \{x^n m, x^{n-1}ym, \ldots, x^j y^{n-j} m\} \tag{33}$$

are linearly independent, so that m is not annihilated by any homogeneous polynomial of degree n divisible by x^j. Because M has constant Jordan type, m is not annihilated by any homogeneous polynomial divisible by $(x-ay)^j$ for any a in k.

Suppose that M does not have a block of size j. Then $x^{j-1}y^{n-j+1}m$ is a linear combination of the elements in S. That is, m is annihilated by some homogeneous polynomial of degree n divisible by x^{j-1}. We may repeat verbatim this discussion with x replaced by $x+ay$ for any $a \in k$. We conclude that m is annihilated by some homogeneous polynomial of degree n divisible by $(x+ay)^{j-1}$ for any $a \in k$.

Let V denote the vector space of all homogeneous polynomials $f(x,y)$ of degree n which annihilate m. By the above, we have that V satisfies both conditions ($i.$) and ($ii.$) of Corollary 9.5. Consequently, that corollary tells us that V contains a polynomial $f(x,y)$ which is divisible by $(x-ay)^j$ for some $a \in k$. But this is a contradiction. So there must also be a block of size j. $\square$

As seen in Corollary 10.3, the following theorem together with Theorem 10.1 easily implies the identification of all cyclic kG-modules of constant Jordan type.

Theorem 10.2. *Let k be a field of characteristic $p > 0$. Assume that M is a cyclic $k[x,y]/(x^p, y^p)$-module of constant Jordan type such that $x^{p-1} \cdot M \neq 0$. Set s equal* *to the number of Jordan blocks of M of size p. Then*

$$M \;\simeq\; A/I^{p+s-1}.$$

Proof. As in the proof of the last theorem, we can assume that the field k is algebraically closed. Let $A_x = k[x]/x^p \subset A$. The (constant) Jordan type of M is the type of M as an A_x-module. We readily verify that A/I^{p+s-1} has Jordan type $s[p] + [p-1] + \cdots + [s]$.

The number of blocks of length p of M as an A_x-module equals the dimension of $N = x^{p-1}M$. Note that N is a module over A_y of dimension s and is generated by $x^{p-1}m$. Consequently, $y^sN = 0$, and we must have also that $x^{p-1}y^sm = 0$. Because M has constant Jordan type, we conclude that $(x-\gamma y)^{p-1}y^tm = 0$ for any $\gamma \in k$, any $t \geq s$. This implies that

$$x^iy^jm = 0, \quad i+j \geq p-1+s. \tag{34}$$

Hence, $A \to M$ factors through the projection $A \twoheadrightarrow A/I^{p+s-1}$. In particular, to prove the theorem it suffices to prove that the Jordan type of M is the same as that of A/I^{p+s-1} as an A_x-module.

We proceed as follows. Suppose that the Jordan type of M includes s block of length p and also one each of blocks of length $p-1, p-2, \ldots, p-j+1$. That is suppose that $p-j$ is the largest integer such that there is no block of size $p-j$ for $p-j \geq s$. As an A_x-module, the blocks of size p can be assumed to be generated by $m, ym, \ldots, y^{s-1}m$, as discussed before. The block of size $s-i$ is generated by y^{s-1+i}, for $i = 1, \ldots, j-1$. Hence the sum of the A_x-socles of these blocks is spanned by the set of elements

$$S = \{x^{p-1}m, x^{p-1}ym, \ldots, x^{p-1}y^{s-1}m, x^{p-2}y^sm, \ldots, x^{p-j}y^{s+j-2}m\}$$

Because these elements lie in different blocks of M as an A_x-module, they must be k-linearly independent. Moreover, each is annihilated by multiplication by x. Because there is no block of size $p-j$ we must have that there exist scalars $a_0, \ldots, a_{s-2}, b_1, \ldots, b_j$ such that

$$\begin{aligned} x^{p-j-1}y^{s+j-1}m =& a_0x^{p-1}m + a_1x^{p-1}ym + \cdots + a_{s-2}x^{p-1}y^{s-2}m \\ &+ b_1x^{p-1}y^{s-1}m + b_2x^{p-2}y^sm + \cdots + b_jx^{p-j}y^{s+j-2}m. \end{aligned}$$

Multiplying by y, which annihilates all of the terms of degree $p+s-2$, we get that

$$a_0x^{p-1}ym + a_1x^{p-1}y^2m + \cdots + a_{s-2}x^{p-1}y^{s-1}m = 0$$

Hence the linear independence of the elements in the set S implies that $a_0 = \cdots = a_{s-2} = 0$. Consequently, there exists a homogeneous polynomial $f(x,y)$ of degree $p-1$ in $k[x,y]$ with the properties that

$$f(x,y)y^{s-1}m = 0,$$

and that $f(x,y)$ is divisible by x^{p-j-1}.

Let V be the k-space of all elements $f(x,y)$ in $k[x,y]_{p-1}$ such that $f(x,y)y^{s-1}m = 0$. We have shown that V contains an element which is divisible by x^{p-j-1}. We can repeat the same argument with x replaced by $x - ay$ for any $a \in k$. Hence, V must contain an element which is divisible by $(x-ay)^{p-j-1}$ for any a in k. This says that V satisfies condition $(ii.)$ of the hypothesis of Corollary 9.5.

Next we want to check that condition $(i.)$ of the hypothesis of 9.5 is satisfied at least for some choice of the variables. Suppose not. Then we must have that V contains every polynomial of the form $(x-ay)^{p-j-1}y^j$ for every a. That is, for every a, V must contain a non-zero polynomial which is divisible by both $(x-ay)^{p-j-1}$ and y^j. Because the elements are relatively prime, that polynomial must be a scalar multiple of the product of the two. Recalling the definition of the space V, we see that for every a we have that

$$(x-ay)^{p-j-1}y^{s+j-1}m = 0.$$

Because M has constant Jordan type we can make linear changes in the variables and repeat all of the same arguments. That is we can let $\overline{x} = ax+by$ and $\overline{y} = cx+dy$, as long as the vectors (a,b) and (c,d) are linearly independent in k^2. If condition $(i.)$ fails for $\overline{x}$ and $\overline{y}$, then we have that

$$(ax+by)^{p-j-1}(cx+dy)^{s+j-1}m = 0$$

Now the elements $(ax+by)^{p-j-1}(cx+dy)^{s+j-1}$ span I^{p-s-2}. Hence if condition $(i.)$ fails for all such choices of $\overline{x} = ax+by$ and $\overline{y} = cx+dy$ we must have that $I^{p-s-2}M = \{0\}$ which contradicts our hypothesis that there are s blocks of size p in the Jordan type of M.

From all of the above, we may assume that conditions $(i.)$ and $(ii.)$ of Corollary 9.5 are satisfied. Hence the corollary implies that some element $f(x,y)$ of V is divisible by $(x-ay)^{p-j}$ for some $a \in k$. Indeed, making the change of variables, replacing $x - ay$ by x, we may assume that there is a non-zero polynomial $g(x,y)$ of degree $j-1$ such that $g(x,y)x^{p-j-1}$ is contained in V. Hence,

$$g(x,y)y^{s-1}x^{p-j}m = 0.$$

This relation contradicts the linear independence of the set S, and hence, proves the theorem. □

Corollary 10.3. *Let k be a field of characteristic $p > 0$. Suppose that $A = k[x,y]/(x^p, y^p)$ be the group algebra of an elementary abelian p subgroup of rank 2. If $M = A \cdot m$ is a cyclic A-module having constant Jordan type, then $m \cong A/I^t$ for some t, where I is the augmentation ideal of A.*

Proof. If $x^{p-1}m = 0$, then we may regard M simply as a $k[x,y]$-module and invoke Theorem 10.1 to prove the result. Otherwise we apply Theorem 10.2. □

11. Further determination of bundles

In this final section, we determine the bundles on $\mathbb{P}^1$ associated to various kG-modules which we have considered.

Proposition 11.1. *Suppose that M is a module of constant rank. Then $\mathfrak{K}(M) = {}_{\mathbb{P}^1}M$. That is, if $\ell = ax + by$ with $(a,b) \neq (0,0)$ then $Ker\{\ell : M \to M\} \subseteq \mathfrak{K}(M)$.*

Consequently, if M is a module of constant rank, then

$$Ker\{\ell : M \to M\} = Ker\{\ell : \mathfrak{K}(M) \to \mathfrak{K}(M)\}.$$

Proof. The proof of the first assertion is a straightforward application of Proposition 6.6. The second assertion follows from the first, because one always has the inclusion $\operatorname{Ker}\{\ell : \mathfrak{K}(M) \to \mathfrak{K}(M)\} \subset \operatorname{Ker}\{\ell : M \to M\}$. □

Proposition 11.1 has the following "geometric formulation", for the operator $\tilde{\Theta}_G$ on $\mathcal{O} \otimes M$ (respectively, $\mathcal{O} \otimes \mathfrak{K}(M)$) has kernel above $\ell \in \mathbb{P}^1$ given by $\operatorname{Ker}\{\ell : M \to M\}$ (resp., $\operatorname{Ker}\{\ell : \mathfrak{K}(M) \to \mathfrak{K}(M)\}$).

Proposition 11.2. *Suppose that M is a module of constant rank. Then we have a coincidence of kernel bundles:*

$$Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes M\} \;=\; Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes \mathfrak{K}(M)\}.$$

We apply Proposition 11.2 to compute explicitly the kernel bundles for all cyclic kG-modules of constant Jordan type.

Proposition 11.3. *Let M be a cyclic kG-module of constant Jordan type, so that $M \simeq kG/I^t$ for some $, 1 \leq t \leq 2p$.*

1. *If $t \leq p$, then $Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes M\}$ is a free $\mathcal{O}$-module of rank t.*
2. *If $t > p$, then $\mathfrak{K}(M) = \mathfrak{K}(W_{p,t-p+1})$, so that*

$$Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes M\} \;\simeq\; Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{t-p+1,p})\}$$

is explicitly determined by Proposition 5.4.

Proof. If $t \leq p$, then the kernel of the endomorphism $\alpha_{(a,b)}(u)$ on M is the socle of M for any $\alpha_{a,b} : k[u]/u^p \to kG$ with $0 \neq (a,b)$. This immediately implies part (1).

By Proposition 11.2 and the fact that $W_{p,t-p+1}$ equals its own generic kernel, in order to verify part (3) it suffices to verify the equality $\mathfrak{K}(M) = \mathfrak{K}(W_{p,t-p+1})$. This is easily done by inspection. □

The generic kernels for the modules $\Omega^n(k)$ are calculated in Example 6.11. This information is used in the proof of the following.

Proposition 11.4. *Suppose that $n > 1$, then there are short exact sequences of vector bundles*

(35)

$$0 \to \; Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{p-1,p-1}\}^n \;\to\; Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes \Omega^{2n-1}(k)\} \;\to \mathcal{O}(-np+1) \;\to 0$$

and

$$(36) \quad 0 \to \ Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes W_{p,p-1}\}^n \ \to \ Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes \Omega^{2n}(k)\} \ \to \mathcal{O}(-np) \ \to 0$$

If $\zeta \in \mathrm{H}^n(G,k)$ is nilpotent, then the module L_ζ has constant rank, though not constant Jordan type. In this case we calculate that

$$Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes L_\zeta\} \simeq Ker\{\tilde{\Theta}_G, \mathcal{O} \otimes \Omega^{2n}(k)\}.$$

Proof. From 6.11, we observe that $\mathfrak{K}(\Omega^{2n-1}(k))/\operatorname{Rad}^2(\mathfrak{K}(\Omega^{2n-1}(k))) \cong W_{np,2}$. Thus, the kernel of $\tilde{\Theta}_G$ on the top (head) of $\mathcal{O} \otimes \mathfrak{K}(\Omega^{2n-1}(k))$ is isomorphic to $\mathcal{O}(-np+1)$ exactly as in 5.3. Consequently, by Proposition 11.2 we need only compute the kernel of $\tilde{\Theta}_G$ on $\mathcal{O} \otimes \operatorname{Rad}(\mathfrak{K}(\Omega^{2n-1}(k)))$. However, an easy computation reveals that $\operatorname{Rad}(\mathfrak{K}(\Omega^{2n-1}(k)))$ is isomorphic to a direct sum of n copies of $W_{p-1,p-1}$.

This verifies the first asserted equality. The verification of the second is similar.

We calculate that $\mathfrak{K}(\Omega^{2n}(k))/\operatorname{Rad}(\mathfrak{K}(\Omega^{2n}(k))) \cong W_{np+1,2}$ and hence, the contribution of the top of $\mathfrak{K}(\Omega^{2n}(k)))$ to the kernel of $\tilde{\Theta}_G$ is a copy of the bundle $\mathcal{O}(-np)$ as in 5.3. Finally, we observe that $\operatorname{Rad}(\mathfrak{K}(\Omega^{2n}(k)))$ is isomorphic to a direct sum of n copies of $W_{p,p-1}$ and we are done.

The last statement follows also from the calculation of Example 6.12. □

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