WEIL RESTRICTION AND SUPPORT VARIETIES

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Abstract. This paper extends the comparison results for cohomological support varieties for Chevalley groups of the form $G(\mathbb{F}_p)$ established by J. Carlson, Z. Lin, and D. Nakano to Chevalley groups of the form $G(\mathbb{F}_{p^d})$. Other topics considered are complexity and non-maximal support varieties for modules. Techniques developed by J. Pevtsova and the author, together with the use of Weil restriction functors, are employed.

0. Introduction

Beginning with the foundational work of D. Quillen for finite groups [21], qualitative properties of the cohomology of various algebras have been analyzed and applied to representation theory. For a finite group, D. Quillen investigated the spectrum of the cohomology algebra $H^\bullet(G,k)$ of the group algebra of kG over an algebraically closed field k of characteristic p dividing the order of the group. Subsequently, J. Carlson introduced the support variety of a kG-module M [4], a subvariety of Quilen's cohomological variety. Many authors have continued to study these varieties for finite groups and related structures.

Consider a simple affine algebraic group G over k defined and split over $\mathbb{F}_p$ with p-restricted Lie algebra $Lie(G) = \mathfrak{g}$. In [15], Z. Lin and D. Nakano showed for a rational G-module M that the rate of growth of a projective resolution for M as a $kG(\mathbb{F}_p)$-module is dominated by one-half the rate of growth of a projective resolution for M viewed as module for the restricted enveloping algebra for $\mathfrak{g}$. This result is illuminated and clarified by recent work of J. Carlson, Z. Lin, and D. Nakano who gave a direct relation between the cohomological varieties of $G(\mathbb{F}_p)$ and $\mathfrak{g}$ [6]. For a restricted class of rational G-modules M, a comparison was also verified between the support variety of M for G and the support variety of M for $\mathfrak{g}$.

In this paper, we extend the results of [15] and [6], so that an appropriate generalization of the formulation of their results now applies to $G(F_q)$ where q is a prime power (with the same constraint on the prime, depending upon G, as required by [6]). The novelty of our approach is the consideration of the Weil restriction $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}G_{\mathbb{F}_q}$ of G, a simple group defined (but not split) over $\mathbb{F}_p$ whose group of $\mathbb{F}_p$-valued points equals the group $G(\mathbb{F}_q)$. The techniques we employ are those of J. Pevtsova and

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the author (e.g., [11]), enabling a somewhat simplified argument with a somewhat sharper result even in the case in which $q = p$ is a prime.

An intriguing aspect of our comparison is the differing dependence upon change of field from $\mathbb{F}_p$ to $\mathbb{F}_q$ of the group algebras $kG(\mathbb{F}_p)$, $kG(\mathbb{F}_q)$ and the restricted enveloping algebras $u(\mathfrak{g}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} k$, $u(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}\mathfrak{g}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_p} k$.

This paper is structured as follows. We begin by recalling the Weil restriction functor, especially for affine algebraic groups and p-restricted Lie algebras. In section 2, we recall the construction of log maps relating the p-unipotent variety of G with the p-nilpotent variety of $\mathfrak{g}$. This construction, originating in work of T.A. Springer, is an extension by J. Carlson, J. Lin, and D. Nakano [6] of a construction by G. Seitz ([22]). Theorem 3.5 states our basic comparison theorem, formulated in terms of the Π-point spaces of [12]; this is reformulated in terms of (projectivized) cohomological spectra in Corollary 3.6. In section 4, we show how the arguments of [6] apply to establish the comparison of complexities of a rational G-module viewed as a $kG(\mathbb{F}_q)$-module and as a $u(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}\mathfrak{g}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_p} k$-module. Finally, in the last section, we compare non-maximal support varieties for rational G-modules whose high weights are relatively small compared to p.

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1. Recollections of Weil restriction

We briefly recall the *Weil restriction* $\mathcal{R}_{k'/k}X'$ of a k'-scheme X'. This construction is particularly useful for our consideration of finite Chevalley groups $G(\mathbb{F}_q)$. Throughout this section, k will denote an arbitrary field of characteristic $p > 0$; this condition on the characteristic of k arises only when we consider the additional structure of a p-restricted Lie algebra on the Lie algebra $\mathfrak{g}$ of an algebraic group scheme over k.

Definition 1.1. Let k' be a finite field extension of k and let X' be an affine scheme of finite type over k'. Then the *Weil restriction* $\mathcal{R}_{k'/k}X'$ is the k-scheme whose A-valued points are given by

$$\mathcal{R}_{k'/k}X'(A) \;=\; X'(A \otimes_k k')$$

for any finitely generated commutative k-algebra A.

Thus,

$$\mathcal{R}_{k'/k}(-): \;(\text{affine } k'\text{-schemes}) \;\longrightarrow\; (\text{affine } k\text{-schemes})$$

is right adjoint to base change from k to k'.

For X' an affine k'-scheme with coordinate algebra $k'[X'] \;=\; k'[x_1,\ldots,x_n]/(f_1,\ldots,f_t)$, the coordinate algebra of the affine k-scheme $\mathcal{R}_{k'/k}X'$ is given as follows. Choose a basis $\{e_1, \ldots, e_d\}$ of k' over k. Then

$$k[\mathcal{R}_{k'/k}X'] \;=\; k[x_{1,1}, \ldots, x_{1,d}, \ldots, x_{n,1}, \ldots, x_{n,d}]/(f_{1,1}, \ldots, f_{t,d}), \tag{1}$$

where $f_{i,j}$ is the coefficient of e_j of the "vector relation" obtained by substituting $\sum_{s=1}^{d} x_{r,s}e_s$ for x_r in f_i. Observe that a product of the form $(\sum_{s=1}^{d} x_{r_1,s}e_s)(\sum_{s'=1}^{d} x_{r_2,s'}e_{s'})$ in such a vector relation requires formulas $e_i{\cdot}e_j = \sum_\ell a_{i,j,\ell}e_\ell$ giving the multiplication of k' with respect to the chosen basis over k.

Remark 1.2. Let X be a k-scheme, viewed as a functor on commutative k-algebras and let k'/k be a field extension. Then the restriction of the functor X to commutative k'-algebras is a k'-scheme, which we denote by $X \otimes_k k'$. In contrast to the relationship to the Weil restriction $\mathcal{R}_{k'/k}X'$ of the k'-scheme X', the k' points of X and of $X \otimes_k k'$ are equal.

The following basic properties of the Weil restriction are easily verified using Definition 1.1. (See appendices 2, 3 of [18].)

Proposition 1.3. *As in Definition 1.1, let k' be a finite field extension of k and let X' be an affine scheme of finite type over k'. Then there is a natural map of k' schemes*

$$X' \;\longrightarrow\; (\mathcal{R}_{k'/k}X') \otimes_k k'. \tag{2}$$

If k'/k is Galois, then (2) can be chosen to be the natural morphism

$$X' \;\longrightarrow\; \prod_{\sigma\in Gal(k'/k)} X'^{\sigma}, \quad x \mapsto \times_{\sigma\in Gal(k'/k)}\sigma(x). \tag{3}$$

Here, X'^{σ} is obtained by replacing the coefficients of the equations defining X' by their images under σ.

In particular, if X' is of the form $X \otimes_k k'$ for some k-scheme X and if k'/k is Galois, then (2) can be identified with the diagonal map

$$X' \;\longrightarrow\; X'^{d}, \quad x \mapsto \times_{\sigma\in Gal(k'/k)}\sigma(x), \tag{4}$$

where $d = [k' : k]$.

Of particular relevance to our present task is the consideration of the Weil restrictions of affine group schemes and their associated Lie algebras. The reader is referred to [19], where in particular it is proved that if G' is a simply connected, simple linear algebraic group over k', then $\mathcal{R}_{k'/k}G'$ is a simple k-group.

Definition 1.4. Let k'/k be a finite field extension and let $\mathfrak{g}'$ be a Lie algebra over k'. We define the Weil restriction $\mathcal{R}_{k'/k}\mathfrak{g}'$ to be the Lie algebra over k obtained by viewing the underlying k' vector space of $\mathfrak{g}'$ as a k-vector space and viewing the k' bilinear bracket on $\mathfrak{g}$ as a k-bilinear map on this underlying k-vector space. If $\mathfrak{g}'$ is p-restricted with p-operator $(-)^{[p]} : \mathfrak{g}' \to \mathfrak{g}'$, then the "same" operator on $\mathcal{R}_{k'/k}\mathfrak{g}'$ endows $\mathcal{R}_{k'/k}\mathfrak{g}'$ with the structure of a p-restricted Lie algebra over k.

We see below that applying $\mathcal{R}_{k'/k}(-)$ commutes with taking the Lie algebra of an affine k'-group scheme. If $\mathfrak{g}$ is a (p-restricted) Lie algebra over k, then we denote by $\mathfrak{g}_{k'}$ the (p-restricted) Lie algebra over k' given by base change as $\mathfrak{g} \otimes_k k'$.

Proposition 1.5. *Let k'/k be a finite extension, G' an affine group scheme of finite type over k', and $\mathfrak{g}'$ the p-restricted Lie algebra over k' of G'. Then $\mathcal{R}_{k'/k}G'$ is an affine group scheme over k, with p-restricted Lie algebra $\mathcal{R}_{k'/k}\mathfrak{g}'_k$ over k.*

Consequently, if $G' = G \otimes_k k'$ is defined over k, then $(\mathcal{R}_{k'/k}G') \otimes_k k'$ is naturally isomorphic to $G'^{\times d}$ as k'-groups and $(\mathcal{R}_{k'/k}\mathfrak{g}') \otimes_k k'$ is naturally isomorphic to $\mathfrak{g}_{k'}^{\oplus d}$ as p-restricted Lie algebras over k'.

Proof. We view the Lie algebra $\mathfrak{g}'$ of G' as the k' vector space of first order distributions on $k'[G']$ with support at the identity and without constant term, $Dist_1^+(G')$ (see the discussion of Section I.7.7 of [14]). Then the Lie bracket is given by the commutator of distributions associated to the coproduct $\nabla_{G'}$ of $k'[G']$. Identifying the resulting Lie algebra with left-invariant derivations, the p-operator $(-)^{[p]}$ becomes the p-th power of derivations.

The coproduct structure

$$\nabla_{G'} : \ k'[G'] \longrightarrow k'[G'] \otimes_{k'} k'[G']$$

determines by functoriality the coproduct structure

$$\nabla_{\mathcal{R}_{k'/k}G'} : \ k[\mathcal{R}_{k'/k}G'] \longrightarrow k[\mathcal{R}_{k'/k}G'] \otimes_k k[\mathcal{R}_{k'/k}G'].$$

The latter coproduct structure is made explicit using the description of the coordinate algebra $k[\mathcal{R}_{k'/k}G']$ as in (1). This explicit description gives an identification of $Dist_1^+(G')$ and $Dist_1^+(\mathcal{R}_{k'/k}G')$ as Lie algebras over k. Thus, we have an identification $Lie(\mathcal{R}_{k'/k}G')$ with $\mathcal{R}_{k'/k}\mathfrak{g}'_k$.

The second assertion follows from the first and Proposition 1.3. $\square$

To make this Weil restriction more concrete, we work out the case of the affine group scheme GL_n.

Example 1.6. Let k'/k be a Galois extension. A choice of basis $\{v_1, \ldots, v_d\}$ of k' over k determines an embedding of affine k-groups

$$\mathcal{R}_{k'/k}(GL_n \otimes k') \ \hookrightarrow \ GL_{nd}, \tag{5}$$

which for any k-algebra A sends the matrix

$$\alpha \in \mathcal{R}_{k'/k}(GL_n \otimes k')(A) = GL(n, A \otimes_k k')$$

to the matrix in $GL(nd, A)$ of the associated k-linear automorphism of $A \otimes_k k'$.

Choosing a $Gal(k'/k)$-invariant basis of k' over k, we identify the composition

$$GL_n \otimes_k k' \ \to \ \mathcal{R}_{k'/k}(GL_n \otimes k') \otimes_k k' \ \to \ GL_{nd} \otimes_k k'$$

as sending $\alpha \in GL(n, A \otimes_k k')$ to the matrix in $GL(nd, A \otimes_k k')$ defined by arranging $\sigma_1(\alpha), \ldots, \sigma_d(\alpha)$ as blocks along the diagonal.

Our goal is to interweave the Weil restriction functor with the comparison of (cohomological) support varieties of $G(\mathbb{F}_p)$ and $\mathfrak{g} = Lie(G)$ given in [6]. Since the cohomology of $G(\mathbb{F}_q)$ is in no sense obtained by base change from the cohomology of $G(\mathbb{F}_p)$ for $q = p^d$, the following base change property for the cohomology of (p-restricted) Lie algebras is perhaps surprising.

Proposition 1.7. *Let k' be a Galois extension of k of degree d. Let $\mathfrak{g}$ be a finite dimensional p-restricted Lie algebra over k and let $\mathfrak{g}'$ denote $\mathcal{R}_{k'/k}(\mathfrak{g} \otimes_k k')$. Then for any field extension L/k',*

$$\mathfrak{g}' \otimes_k L \;\cong\; (\mathfrak{g}_L^{\oplus d}) \,, \tag{6}$$

so that the restricted enveloping algebra $u(\mathfrak{g}' \otimes_k L) \simeq u(\mathfrak{g}') \otimes_k L$ is naturally isomorphic to $u(\mathfrak{g}_L^{\oplus d})$ as an L-algebra.

Consequently, for any restricted $\mathfrak{g}'$-modules M, N,

$$Ext^*_{u(\mathfrak{g}_L^{\oplus d})}(M \otimes_{k'} L, N \otimes_{k'} L) \;\simeq Ext^*_{u(\mathfrak{g}')}(M,N) \otimes_{k'} L. \tag{7}$$

Proof. By Proposition 1.5, $\mathfrak{g}' \otimes_k k' = \mathcal{R}_{k'/k}(\mathfrak{g} \otimes_k k') \otimes_k k'$ can be identified with $(\mathfrak{g} \otimes_k k')^{\oplus d} \;\simeq\; \mathfrak{g}_{k'}^{\oplus d}$. This immediately implies (6).

Furthermore, (6) implies the isomorphism $u(\mathfrak{g}' \otimes_{k'} L) \simeq u(\mathfrak{g}^{\oplus d} \otimes_k L)$. We establish (7) by recalling that taking Ext-groups commutes with (flat) base change:

$$Ext^*_{u(\mathfrak{g}')}(M,N) \otimes_{k'} L \;\simeq\; Ext^*_{u(\mathfrak{g}' \otimes_{k'} L)}(M \otimes_{k'} L, N \otimes_{k'} L).$$

□

2. log MAPS

Throughout this section, k will denote an algebraically closed field of characteristic $p > 0$ and G will denote a simple algebraic group over k, defined and split over over $\mathbb{F}_p$. Let $\mathfrak{g}$ denote the p-restricted Lie algebra of G. Write $B = T \cdot U$, a chosen split Borel subgroup of G. For a subset $J \subset \Pi$ of the simple roots of G, let U_J be the unipotent radical of the corresponding parabolic subgroup P_J. Let $\mathcal{U} \subset G$ be the closed subvariety of unipotent elements of G and let $\mathcal{N} \subset \mathfrak{g}$ be the closed subvariety of nilpotent elements of $\mathfrak{g}$. Furthermore, let $\mathcal{U}_1 \subset \mathcal{U}$ be the closed subvariety of p-unipotent elements of G, and similarly $\mathcal{N}_1 \subset \mathcal{N}$ be the closed subvariety of p-nilpotent elements of $\mathfrak{g}$.

For p *good* for G (i.e., p is not 2 if G is of type B, C, D, E, F, G and p is not 3 if G is of type E, F, G and p is not 5 if G is of type E_8) and assuming that the covering map $SL_{n+1} \to G$ is separable for G of type A_n, T. A. Springer established the existence of a G-equivariant isomorphism $\mathcal{U} \to \mathcal{N}$ which sends $U \subset \mathcal{U}$ to $\mathfrak{u} \equiv Lie(U) \subset \mathcal{N}$ (cf. [23], Thm 3.1; [2]). This logarithmic map was refined by G. Seitz as we recall in the following theorem.

Theorem 2.1. *([22, 5.3]) With notation as above, assume that $p \geq h(J)$, where $h(J) - 1$ is the nilpotent class of U_J. Then there is a unique isomorphism*

$$\log : U_J \xrightarrow{\sim} \mathfrak{u}_J \equiv Lie(U_J) \tag{8}$$

whose tangent map is the identity, which is compatible with P_J-conjugation, and which is compatible by conjugation with any automorphism of G normalizing P_J.

An important property of Seitz's logarithmic map log is that

$$\log(u \cdot v) \;=\; \log(u) + \log(v), \quad if \;[u,v] = 1 \tag{9}$$

Indeed, Seitz constructs the inverse isomorphism exp over $\mathbb{Z}_{(p)}$ using the classical exponential map in characteristic 0, so that this isomorphism is an isomorphism of group schemes provided that $\mathfrak{u}_{J,\mathbb{Z}_{(p)}}$ is equipped with the group structure given by the Hausdorff formula.

An important and useful property of the nilpotent cone $\mathcal{N}$ of a simple algebraic group G is that it is normal. Recall that $\mathcal{N}$ equals $\mathcal{N}_1$ if and only if $h(g) \leq p$ (where $h(G)$ denotes the Coxeter number of the simple algebraic group G). The normality of $\mathcal{N}_1$ for $h(G) > p$ has been proved in many cases: S. Donkin has proved normality for all orbit closures for G of type A [8] and J.F Thomsen [24] has established normality of $\mathcal{N}_1$ for certain other types.

Work of J. Carlson, Z. Lin, D. Nakano, B. Parshall, and D. Vella establishes that $\mathcal{N}_1 = G{\cdot}\mathfrak{u}_J$ for some $J \subset \Pi$ with $\mathfrak{u}_J \;=\; Lie(U_J)$ such that the nilpotency class of U_J is less than p (cf. [17], [7]). Using this result together with the hypothesis of normality and an investigation of centralizers in G of elements of U_J and $\mathfrak{u}_J$, J. Carlson, Z. Lin, and D. Nakano extend (8) to a G-equivariant isomorphism $\log : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{N}_1$. We recall this result in the following theorem.

Theorem 2.2. *([6], Thm 3) Let G be a simple algebraic group over k, defined and split over $\mathbb{F}_p$. Assume that $G_{sc} \to G$ is separable, that p is good for G, and that $\mathcal{N}_1$ is normal. Write $\mathcal{N}_1 = G \cdot \mathfrak{u}_J$ with $J \subset \Pi$. Then Seitz's logarithmic isomorphism* $\log : U_J \xrightarrow{\sim} \mathfrak{u}_J$ *extends uniquely to a G-equivariant isomorphism defined over $\mathbb{F}_p$*

$$\log : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{N}_1. \tag{10}$$

Applying the functoriality of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(-)$ with $q = p^d$, we extend the CLN/Seitz log map to algebraic k-groups of the form $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q}) \otimes_{\mathbb{F}_p} k$ with G as in Theorem 2.2.

Corollary 2.3. *Adopt the hypotheses and notation of Theorem 2.1. Then the Seitz logarithmic map (8) determines an $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(P_{J,\mathbb{F}_q})$-equivariant isomorphism*

$$\log_{/\mathbb{F}_q} \equiv \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\log) : \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(U_{J,\mathbb{F}_q}) \;\xrightarrow{\sim} \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{u}_{J,\mathbb{F}_q}). \tag{11}$$

Adopt the hypotheses and notation of Theorem 2.2. Then the log *map (10) determines an $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})$-equivariant isomorphism*

$$log_{/\mathbb{F}_q} \equiv \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\log) : \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{U}_{1,\mathbb{F}_q}) \;\xrightarrow{\sim} \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{N}_{1,\mathbb{F}_q}), \tag{12}$$

which identifies the p-unipotent subvariety of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})$ *with the p-nilpotent cone of* $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q})$.

Proof. The isomorphisms (11) and (12) follow immediately from the functoriality of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(-)$, granted Theorem 2.1, Proposition 1.5, and Theorem 2.2.

The base change to $k/\mathbb{F}_p$ of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{U}_{1,\mathbb{F}_q}) \subset \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})$ is isomorphic to the product embedding $\mathcal{U}_1^{\times d} \subset G^{\times d}$, so that this base change is the subvariety consisting of p-unipotent elements of $G^{\times d}$. Thus, $\mathcal{R}_{k'/k}(\mathcal{U}_1 \otimes_k k')$ is the p-unipotent subvariety of $\mathcal{R}_{k'/k}(G \otimes_k k')$. Similarly, $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{N}_{1,\mathbb{F}_q})$ is the p-nilpotent variety of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q})$. ☐

3. Comparison of cohomological varieties

As in the previous section, k shall denote an arbitrary field of characteristic $p > 0$. We consider an arbitrary power $q = p^d$ of p. Our main result is Corollary 3.6 which provides a natural embedding of the (projectivized) cohomological variety for $G(\mathbb{F}_q)$ into the cohomological variety of the restricted enveloping algebra $u(\mathfrak{g}_{\mathbb{F}_q}^{\oplus d})$ thereby sharpening (for $q = p$) and extending (to $d > 1$) work of J. Carlson, Z. Lin, and D. Nakano.

We shall find it convenient to formulate our comparison results in terms of Π-point spaces as recalled in the following definition. For a finite group scheme G over k, we denote by $H^\bullet(G, k)$ the commutative k-algebra $H^*(G, k)$ if $p = 2$ and $H^{ev}(G, k)$ if $p > 2$.

Definition 3.1. (see [12]) Let G be a finite group scheme over k; for example, G might be a finite group (a constant group scheme) or a Frobenius kernel of a group scheme of finite type over k. Let kG denote the group algebra of G, the k-linear dual of the coordinate algebra $k[G]$.

- A π-point of G is a left flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k which satisfies the condition that there exists some abelian unipotent subgroup scheme $C_K \subset G_K$ such that α_K factors through $KC_K \subset KG_K$.
- Two π-points α_K, β_L are said to be equivalent if the following condition is satisfied: for all finite dimensional kG-modules M, the $K[t]/t^p$-module $\alpha_K^*(M_K)$ is projective if and only if the $L[t]/t^p$-module $\beta_L^*(M_L)$ is projective.
- The set of equivalence classes of π-points, $\Pi(G)$, is provided a topology by declaring that a subset $Y \subset \Pi(G)$ is closed if and only if there exists some finite dimensional kG-module M such that

$$Y \;=\; \Pi(G)_M \;\equiv\; \{[\alpha_K] : \alpha_K^*(M_K) \text{ is not projective}\}.$$

As the following theorem recalls, Π-point spaces give an alternate formulation of (cohomological) support varieties.

Theorem 3.2. *([12], Thm 7.5) Let G be a finite group scheme over k. Then there is a natural isoomorphism of schemes*

$$\Phi_G : \operatorname{Proj} H^\bullet(G,k) \xrightarrow{\sim} \Pi(G),$$

where $\Pi(G)$ is equipped with a scheme structure defined in terms of endomorphisms of localizations of the stable module category of finite dimensional kG-modules. On the level of sets, a π-point represented by $\alpha_K : K[t]/t^p \to KG$ is sent via $\Phi_G^{-1}(-)$ to the homogeneous prime ideal $\mathcal{P} \subset H^\bullet(G,k)$ defined as the intersection of $H^\bullet(G,k)$ with the kernel of $\alpha_K^* : H^\bullet(G,K) \to H^\bullet(K[t]/t^p, K)$.

Moreover, for any finite dimensional kG-module M,

$$\Phi_G(\operatorname{Proj} H^\bullet(G,k)/Ann_{H^\bullet(G,k)}Ext^*_{kG}(M,M)) \;=\; \Pi(G)_M.$$

For notational convenience, we denote by $\underline{\mathfrak{g}}$ the finite group scheme of height 1 associated to a p-restricted Lie algebra $\mathfrak{g}$. Thus, if G is an algebraic group over k with p-restricted Lie algebra $\mathfrak{g}$, then $\underline{\mathfrak{g}} = G_{(1)}$, the first Frobenius kernel of G (defined as the kernel of the Frobenius map $F : G \to G^{(1)}$). The group algebra of the finite group scheme $\underline{\mathfrak{g}}$ is the restricted enveloping algebra of $\mathfrak{g}$, $u(\mathfrak{g})$.

We recall that π-points have convenient representatives for two special classes of finite group schemes, finite groups and group schemes of the form $\underline{\mathfrak{g}}$.

Proposition 3.3. *([12]) Let G be a finite group. For each $[\alpha_K] \in \Pi(G)$, there exists some elementary abelian p-subgroup $E \subset G$ with minimal set of generators $x_1,\dots,x_r$ and some non-zero r-tuple $\underline{a} = \{a_1,\dots,a_r\}$ of elements of K such that $[\alpha_K]$ is represented by the composition of the K-algebra homomorphism*

$$\alpha_{\underline{a}} : K[t]/t^p \to KE, \quad t \mapsto \sum_{i=1}^r a_i(x_i - 1), a_i \in K \tag{13}$$

and the inclusion $KE \subset KG$ of group algebras.

Let $\underline{\mathfrak{g}}$ be the height 1, infinitesimal group scheme associated to a finite dimensional p-restricted Lie algebra $\mathfrak{g}$ over k, and let $u(\mathfrak{g})$ denote the restricted enveloping algebra of $\mathfrak{g}$. For each $[\alpha_K] \in \Pi(\underline{\mathfrak{g}})$, there exists some p-nilpotent element $X \in \mathfrak{g}_K$ such that $[\alpha_K]$ is represented by the K-algebra homomorphism

$$\alpha_X : K[t[/t^p \to u(\mathfrak{g}_K), \quad t \mapsto X.$$

As in [11], Prop 5.8, for G a finite group we consider the continuous projection $\widetilde{\Pi(G)} \to \Pi(G)$, where $\widetilde{\Pi(G)}$ is the space of equivalence classes of flat maps $\alpha_K : K[t]/t^p \to KE$ for some K/k and some elementary abelian p-subgroup $E \subset G$. Two such maps $\alpha_K : K[t]/t^p \to KE$, $\beta_L : L[t]/t^p \to LE'$ are equivalent (i.e., determine the same point of $\widetilde{\Pi(G)}$) if there exists some field extension Ω of both K and L and some flat map $\gamma_\Omega : \Omega[t]/t^p \to \Omega(E \cap E')$ such that α_K, γ_Ω represent the same point of $\Pi(E)$ and β_L, γ_Ω represent the same point of $\Pi(E')$. By the Quillen stratification theorem (see Theorem 9.1.3 of [9]), if points in $\widetilde{\Pi(G)}$ represented by

$\alpha_K : K[t]/t^p \to KE$, $\beta_L : L[t]/t^p \to LE'$ are mapped to the same point of $\Pi(G)$, then there is some element $x \in G$ such that either x conjugates E into a subgroup of E' or x conjugates E' into a subgroup of E.

The proof we give of the following theorem is similar to that of Proposition 5.8 of [11].

Theorem 3.4. *Let G be a simple algebraic group over k defined and split over $\mathbb{F}_p$, assume p is good for G, and assume that $SL_{n+1} \to G$ is separable in the case that G is of type A_n. Let $J \subset \Pi$ be a subset of simple roots and assume that the nilpotent class of U_J is less than or equal to p. Then the isomorphism*

$$\log_{/\mathbb{F}_q} : \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(U_{J,\mathbb{F}_q}) \xrightarrow{\sim} \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{u}_{J,\mathbb{F}_q})$$

of (11) induces an embedding

$$\ell_{/\mathbb{F}_q} : \Pi(U_J(\mathbb{F}_q)) \ \hookrightarrow \ [\Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{u}_{J,\mathbb{F}_q}))]/U_j(\mathbb{F}_q).$$

Proof. We define

$$\tilde{\ell}_{/\mathbb{F}_q} : \Pi(\widetilde{U_J(\mathbb{F}_q)}) \to \Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{u}}_{J,\mathbb{F}_q}))], \quad \alpha_{\underline{a}} \mapsto \alpha_{X_{\underline{a}}} \tag{14}$$

as the map on equivalence classes induced by sending a π-point represented by $\alpha_{\underline{a}} : K[t]/t^p \to KE$, $t \mapsto \sum_{i=1}^r a_i(x_i - 1)$ for some elementary abelian p-subgroup $E \subset U_J(\mathbb{F}_q)$ and some choice of minimal set of generators $\{x_1, \ldots, x_r\}$ of E (as in Proposition 3.3) to

$$\alpha_{X_{\underline{a}}} : K[t]/t^p \to u(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{u}}_J)) \quad t \mapsto X_{\underline{a}} \ = \ \sum_{i=1}^r a_i \log_{/\mathbb{F}_q}(x_i).$$

We use the suggestive notation

$$\tilde{\ell}_{/\mathbb{F}_q} : [t \mapsto \sum_{i=1}^r a_i(x_i - 1)] \ \mapsto \ [t \mapsto \sum_{i=1}^r a_i \log_{/\mathbb{F}_q}(x_i)] \tag{15}$$

We proceed to show that $\tilde{\ell}_{/\mathbb{F}_q}$ is well-defined independent of choices made in the formulation (15). If $\beta_{\underline{b}} : L[t]/t^p \to LE'$ represents the same class in $\Pi(\widetilde{U_J(\mathbb{F}_q)})$ as $\alpha_{\underline{a}}$ for some elementary abelian p-subgroup $E' \subset U_J(\mathbb{F}_q)$, then both $\alpha_{\underline{a}}$ and $\beta_{\underline{b}}$ are equivalent (with respect to the equivalence relation defining $\widetilde{\Pi(G)})$ to maps $\Omega[t]/t^p \to \Omega(E \cap E')$ for some Ω extending both K and L.

For $E' \subset E$, we may begin with a choice of minimal generating set for E' and then extend this to a minimal generating set for E. Thus, to prove that $\tilde{\ell}_{/\mathbb{F}_q}$ is well-defined, it suffices to restrict $\tilde{\ell}_{/\mathbb{F}_q}$ to $\Pi(E)$ (which maps to $\Pi(\widetilde{U_J(\mathbb{F}_q)})$ and verify that $\tilde{\ell}_{/\mathbb{F}_q}$ does not depend upon the choice of minimal set of generators $x_1, \ldots, x_r$ of E. Extend (15) to a map $\mathcal{L}_{/\mathbb{F}_q}$ from the augmentation ideal of KE to the Lie algebra $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_p} K$ by

(16)

$$\mathcal{L}_{/\mathbb{F}_q} : rad(KE) \ \to \ \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_p} K, \qquad \sum_{I=(i_1,\dots,i_r)} a_I(x_I - 1) \mapsto \sum_I a_I \log_{/\mathbb{F}_q}(x_I),$$

where $x_I = \sum_{j=1}^r x_j^{i_j} \in E$ with $0 \leq i_j \leq p-1$ (with the group structure of E written multiplicatively). Observe that $\mathcal{L}_{/\mathbb{F}_q}$ does not depend upon the choice of minimal set of generators $\{x_1, \dots, x_r\}$ of E, for $\mathcal{L}_{/\mathbb{F}_q}$ involves a sum over all elements x_I of E.

Suppose now that $\{x_1, \dots, x_r\}$, $\{y_1, \dots, y_r\}$ are two sets of minimial sets of generators for E, and consider $\alpha_K : K[t]/t^p \to KE$, $\beta_L : L[t]/t^p \to LE$ written in the form (13) which represent the same point of $\Pi(E)$, $[\alpha_K] = [\beta_L] \in \Pi(E)$. We may replace K, L by a common field extension, thereby enabling us to assume that $K = L$. The equality $[\alpha_K] = [\beta_L]$ implies that we may adjust β_L by a non-zero scalar multiple in order to arrange that $\sum_{i=1}^r a_i(x_i - 1) - \sum_{j=1}^r b_j(y_j - 1) \in rad^2(KE)$ (cf. [12]), where $\alpha_K(t) = \sum_I a_i(x_i - 1)$ and $\beta_L(t) = \sum_j b_j(y_j - 1)$.

Thus it suffices to show that the K-linear map $\mathcal{L}_{/\mathbb{F}_q}$ as given in (16) vanishes on the basis $\{(x_1 - 1)^{i_1} \cdots (x_r - 1)^{i_r};\ \sum i_j > 1\}$ of $rad^2(KE)$. Observe that

$$\mathcal{L}_{/\mathbb{F}_q}((x_i - 1)(x_j - 1)) \ = \ \mathcal{L}_{/\mathbb{F}_q}((x_i x_j - 1) - (x_i - 1) - (x_j - 1)) \ =$$
$$\log_{/\mathbb{F}_q}(x_i x_j) - \log_{/\mathbb{F}_q}(x_i) - \log_{/\mathbb{F}_q}(x_j)$$

vanishes by (9). A simple induction argument now implies that

$$\mathcal{L}_{/\mathbb{F}_q}((x_1 - 1)^{i_1} \cdots (x_r - 1)^{i_r}) = 0, \quad \text{if } \sum i_j > 1.$$

The P_J-equivariance of log as given in Theorem 2.1 together with the definition of $\log_{\mathbb{F}_q}$ as $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\log)$ implies that

$$\tilde{\ell}_{/\mathbb{F}_q} : \Pi(\widetilde{U_J(\mathbb{F}_q)}) \to \Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{u}}_{J,\mathbb{F}_q}))$$

is $U_J(\mathbb{F}_q)$-equivariant. The well-definedness of the asserted map

$$\ell_{/\mathbb{F}_q} : \Pi(U_J(\mathbb{F}_q)) \ \longrightarrow \ [\Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{u}}))]/U_J(\mathbb{F}_q)$$

now follows from the Quillen stratification theorem [21] (as stated in Theorem 9.1.3 of [9], with projectivization in [12]). Namely, this theorem asserts that

$$\Pi(U_J(\mathbb{F}_q)) \ \simeq \ \varinjlim_{E \subset U_J(\mathbb{F}_q)} \Pi(E) \ \simeq \Pi(\widetilde{U_J(\mathbb{F}_q)})/U(\mathbb{F}_q).$$

The proof of the injectivity of $\tilde{\ell}_{/\mathbb{F}_q}$ (and thus of $\ell_{/\mathbb{F}_q}$) is a specialization of the proof of injectivity given below in the proof of Theorem 3.5. □

Using the Weil restriction of the CLN/Seitz map given in Corollary 2.3, we extend the applicability of the proof of Theorem 3.4 to prove the following. In the special case $q = p$, Theorem 3.5 keeps the reduced lower bound on p of [6] and the injectivity of [11].

Theorem 3.5. *Adopt the notation and hypotheses of Theorem 2.2: G is a simple algebraic group over k defined and split over $\mathbb{F}_p$, p is good for G, $\mathcal{N}_1$ is normal, and $SL_{n+1} \to G$ is separable in the case that G is of type A_n. Let $q = p^d$. Then the isomorphism*

$$\log_{/\mathbb{F}_q} : \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{U}_{1,\mathbb{F}_q}) \xrightarrow{\sim} \mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathcal{N}_{1,\mathbb{F}_q})$$

of (12) determines an embedding (of Zariski spaces)

$$\ell_{/\mathbb{F}_q} : \Pi(G(\mathbb{F}_q)) \;\hookrightarrow\; [\Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{g}}_{\mathbb{F}_q}))]/G(\mathbb{F}_q). \tag{17}$$

Proof. Exactly as for (14) in the proof of Theorem 3.4, we define

$$\tilde{\ell}_{/\mathbb{F}_q} : \Pi(\widetilde{G(\mathbb{F}_q)}) \;\to\; \Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q})) \tag{18}$$

using the construction of (15) and the $\log_{/\mathbb{F}_q}$ map of (12):

$$\tilde{\ell}_{/\mathbb{F}_q} : [t \mapsto \sum_{i=1}^{r} a_i(x_i - 1)] \;\mapsto\; [t \mapsto \sum_{i=1}^{r} a_i \log_{/\mathbb{F}_q}(x_i)] \tag{19}$$

The proof that this is well defined proceeds verbatim as in the proof of Theorem 3.4

To prove injectivity of $\tilde{\ell}_{/\mathbb{F}_q}$, we consider $\alpha_K : K[t]/t^p \to KE \to KG(\mathbb{F}_q)$ and $\beta_L : L[t]/t^p \to LE' \to LG(\mathbb{F}_q)$ with equal images under $\tilde{\ell}_{/\mathbb{F}_q}$. By replacing K, L by a common field extension, we may assume $K = L$. We choose $E,\ E'$ to be of minimal rank among representatives of α_K, β_L in $\Pi(\widetilde{G(\mathbb{F}_q)})$. Choose bases $\{x_i\},\ \{y_j\}$ for $E,\ E'$ with respect to which $\alpha_K = \alpha_{\underline{a}},\ \beta_L = \beta_{\underline{b}}$. The condition that $\log_{/\mathbb{F}_q}(\alpha_{\underline{a}}) = \log_{/\mathbb{F}_q}(\beta_{\underline{b}})$ implies that there is a non-zero $c \in K$ such that

$$\sum_{i=1}^{s} a_i \log_{/\mathbb{F}_q}(x_i) \;=\; c \sum_{j=1}^{t} b_j \log_{/\mathbb{F}_q}(y_j). \tag{20}$$

Since $\log_{/\mathbb{F}_q}$ restricts to an isomorphism of E and of E' onto their images, both $\{log_{/\mathbb{F}_q}(x_i)\},\ \{\log_{/\mathbb{F}_q}(y_j)\}$ are linearly independent subsets of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(g_{\mathbb{F}_q})$. Thus, (20) implies that $E = E'$ and that $c \in \mathbb{F}_p$. Then the fact that $\log_{/\mathbb{F}_q}$ restricts to an isomorphism of E onto its image implies that $\alpha_{\underline{a}}(t) = \beta_{\underline{b}}(t)$ and thus that $\alpha_{\underline{a}} = \beta_{\underline{b}}$. We thus conclude the injectivity of $\tilde{\ell}_{/\mathbb{F}_q}$.

The $R_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})$-equivariance of $\log_{/\mathbb{F}_q}$ given in (12) implies that $\tilde{\ell}_{/\mathbb{F}_q}$ given in (18) is $R_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})(\mathbb{F}_p) = G(\mathbb{F}_q)$-equivariant. By the Quillen stratification theorem [20], the map of $G(\mathbb{F}_q)$-coinvariants induced by (18) has the form

$$\ell_{/\mathbb{F}_q} : \Pi(G(\mathbb{F}_q)) \;\longrightarrow\; [\Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{g}}_{\mathbb{F}_q}))]/G(\mathbb{F}_q).$$

The injectivity of $\ell_{/\mathbb{F}_q}$ is implied by the injectivity of $\tilde{\ell}_{/\mathbb{F}_q}$ on $\Pi(\widetilde{G(\mathbb{F}_q)})$. □

Combining Theorems 3.2, 3.4 and 3.5, we obtain the following corollary concerning the (projectivized) cohomological variety associated to the Chevalley groups $G(\mathbb{F}_q)$. This corollary extends [6, Thm 5] and [11, 5.8] which are statements about $G(\mathbb{F}_p)$.

Corollary 3.6. *Assume the notation and hypotheses of Theorem 3.4. Let $\overline{k}$ denote some choice of algebraic closure of k. Then the isomorphism $log_{/\mathbb{F}_q}$ of (12) determines an embedding*

$$\ell_{/\mathbb{F}_q} : Proj(H^\bullet(U_J(\mathbb{F}_q),\overline{k})) \hookrightarrow [Proj(H^\bullet(u(\mathfrak{u}_{J,\mathbb{F}_q}^{\oplus d}),\overline{k}))]/U_J(\mathbb{F}_q), \tag{21}$$

where $u(\mathfrak{u}_{J,\mathbb{F}_q}^{\oplus d})$ is the restricted enveloping algebra of the p-restricted Lie algebra $\mathfrak{u}_{J,\mathbb{F}_q}^{\oplus d}$ over $\mathbb{F}_q$.

Assume the notation and hypotheses of Theorem 3.5. Then $\log_{/\mathbb{F}_q}$ of Corollary 2.3 determines an embedding

$$\ell_{/\mathbb{F}_q} : Proj(H^\bullet(G(\mathbb{F}_q),\overline{k})) \hookrightarrow [Proj(H^\bullet(u(\mathfrak{g}_{\mathbb{F}_q}^{\oplus d}),\overline{k}))]/G(\mathbb{F}_q), \tag{22}$$

where $u(\mathfrak{g}_{\mathbb{F}_q}^{\oplus p})$ is the restricted enveloping algebra of the p-restricted Lie algebra $\mathfrak{g}_{\mathbb{F}_q}^{\oplus d}$ over $\mathbb{F}_q$.

Proof. Theorem 3.4 in conjunction with Theorem 3.2 immediately gives the embedding

$$\ell_{/\mathbb{F}_q} : Proj(H^\bullet(U_J(\mathbb{F}_q),\overline{k})) \hookrightarrow [Proj(H^\bullet(u(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{u}_{J,\mathbb{F}_q})),\overline{k}))]/U_J(\mathbb{F}_q).$$

Thus, (21) follows by applying Proposition 1.7.

Similarly, Theorem 3.5 in conjunction with Theorem 3.2 immediately gives the embedding

$$\ell_{/\mathbb{F}_q} : Proj(H^\bullet(G(\mathbb{F}_q),\overline{k})) \hookrightarrow [Proj(H^\bullet(u(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q})),\overline{k}))]/G(\mathbb{F}_q).$$

Thus, (22) also follows by applying Proposition 1.7. ☐

As we now verify, the embedding (17) of Theorem 3.5 behaves well with respect to change of (finite) field.

Proposition 3.7. *Assume the notation and hypotheses of Theorem 3.5. Let $q = p^d$, $q' = (p^d)^e$. Then the embedding $\ell_{/\mathbb{F}_q}$ of Theorem 3.5 fits in a commutative diagram*

$$\begin{array}{ccccc} \Pi(G(\mathbb{F}_p)) & \longrightarrow & \Pi(G(\mathbb{F}_q)) & \longrightarrow & \Pi(G(\mathbb{F}_{q'})) \\ \downarrow \ell & & \downarrow \ell_{/\mathbb{F}_q} & & \downarrow \ell_{/\mathbb{F}_{q'}} \\ [\Pi(\underline{\mathfrak{g}})]/G(\mathbb{F}_p) & \longrightarrow & [\Pi(\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(\underline{\mathfrak{g}}_{\mathbb{F}_q}))]/G(\mathbb{F}_q) & \longrightarrow & [\Pi(\mathcal{R}_{\mathbb{F}_{q'}/\mathbb{F}_p}(\underline{\mathfrak{g}}_{\mathbb{F}_{q'}}))]/G(\mathbb{F}_{q'}) \end{array} . \tag{23}$$

Proof. The commutativity of the left square of (23) follows from the naturality of $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(-)$ and the formulation of $\log_{/\mathbb{F}_q}$ in Corollary 2.3 for $\mathcal{R}_{\mathbb{F}_q/\mathbb{F}_p}(G_{\mathbb{F}_q})$ as

$R_{\mathbb{F}_q/\mathbb{F}_p}(\log)$. The commutativity of the right square of (23) is verified by recognizing the right vertical map as obtained from the middle vertical map by applying $\mathcal{R}_{\mathbb{F}_{q'}/\mathbb{F}_q}(-)$. □

4. Comparison of complexity for rational G-modules

In this section, we extend the main result of Z. Lin and D. Nakano in [15] for Chevalley groups of the form $G(\mathbb{F}_p)$ so that it is applicable to $G(\mathbb{F}_q)$. We consider a reductive algebraic group G over k defined and split over $\mathbb{F}_p$ and a rational G-module M. Here, k is some algebraically closed field of characteristic p. In other words, M is a comodule for the coordinate algebra $k[G]$ equipped with a coproduct induced by the group structure of G. Theorem 4.3 establishes that complexity of M as a $G(\mathbb{F}_q)$-module is bounded above by half the complexity of M as a $u(\mathfrak{g}_k^{\oplus d})$-module).

We recall the following two propositions which lead quickly to our comparison of complexities. The first is a result of D. Quillen.

Proposition 4.1. *([20], Thm 1) Let G be a reductive algebraic group defined and split over $\mathbb{F}_p$, and let $J \subset \Pi$ be a chose subset of simple roots. Then there is a natural isomorphism*

$$\operatorname{gr} kU_J(\mathbb{F}_q) \;\simeq\; u(\mathfrak{u}_{J,\mathbb{F}_q} \otimes_{\mathbb{F}_p} k) \tag{24}$$

of finite dimensional k-algebras.

The second proposition presents the "May spectral sequence" of [16] in the context used by Z. Lin and D. Nakano.

Proposition 4.2. *(see [15], Thm 3.2) Let G be a reductive algebraic group defined and split over $\mathbb{F}_p$, and let $J \subset \Pi$ be a chose subset of simple roots. For any finite dimensional $kU_J(\mathbb{F}_q)$-module M, there is a (natural) first quadrant, convergent spectral sequence*

$$E_1^{s,t} = H^{s+t}(\operatorname{gr} kU_J(\mathbb{F}_q), \operatorname{gr} M) \Longrightarrow H^{s+t}(kU_J(\mathbb{F}_q), M). \tag{25}$$

Furthermore, if M is the restriction of a rational $T{\cdot}U_J$-module, then the $\operatorname{gr} U_J(\mathbb{F}_q)$-module $\operatorname{gr} M$ corresponds under the identification $\operatorname{gr} U_J(\mathbb{F}_q) \simeq \mathfrak{u}_{J,\mathbb{F}_q}$ to the restriction of M to $\underline{\mathfrak{u}}_{J,\mathbb{F}_q}$. In other words, this action is given by the restriction to $\underline{\mathfrak{u}}_{J,\mathbb{F}_q}$ of the rational action of of U_J on M, viewed as the action of a Lie algebra over $\mathbb{F}_p$.

Proof. The spectral sequence is established in [16]. The identification $\operatorname{gr} U_J(\mathbb{F}_q) \simeq \mathfrak{u}_{J,\mathbb{F}_q}$ is given in [15], Prop 2.2. The subsequent identification of the $\operatorname{gr} U_J(\mathbb{F}_q)$-module $\operatorname{gr} M$ is achieved in Theorem 3.2 of [15]. □

For a k-algebra A and a finite dimensional A-module M, the *complexity* $cx_A(M)$ is the smallest integer c such that there exists a projective resolution $P_\bullet$ of M as an A-module and an integer C such that $\dim_k P_n \leq Cn^{c-1}$.

For any finite group scheme G and any finite dimensional kG-module M, $cx_{kG}(M)$ equals the dimension of the support of $Ext^*_{kG}(M,M) = H^*(G, M^\# \otimes_k M)$ as an

$H^\bullet(G,k)$-module [1]. This is also the rate of growth of $\{n \mapsto Ext^*_{kG}(M,M)\}$. For G unipotent, this is also the rate of growth of $\{n \mapsto H^n(G,M)\}$. We denote by $|G|$ the Zariski space $SpecH^\bullet(G,k)$ and by $|G|_M \ \subset \ |G|_M$ the support of $Ext^*_{kG}(M,M)$. Recall the equality

$$\dim |G|_M \ = \ cx_{kG}(M). \tag{26}$$

The reader is referred to [3] for details, including proofs, of these properties of complexity.

Theorem 4.3. *Let G be a reductive algebraic group defined and split over $\mathbb{F}_p$, and let $J \subset \Pi$ be a chosen subset of simple roots. Let $q = p^d$.*

For any rational $T \cdot U_J$-module M

$$cx_{kU_J(\mathbb{F}_q)}(M) \ \le \ cx_{u(\mathfrak{u}_{J,\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)}(M), \tag{27}$$

where the action of $\mathfrak{u}_{J,\mathbb{F}_q} \otimes_{\mathbb{F}_p} k$ on M is the k-linear extension of the restriction of the rational action of U_J to $\mathfrak{u}_{J,\mathbb{F}_q}$ viewed as a Lie algebra over $\mathbb{F}_p$.

For any rational G-module M,

$$cx_{kG(\mathbb{F}_q)}(M) \ \le \ \frac{1}{2} cx_{u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)}(M), \tag{28}$$

where the action of $\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k$ on M is the k-linear extension of the restriction of the rational action of G to $\mathfrak{g}_{\mathbb{F}_q}$ viewed as a Lie algebra over $\mathbb{F}_p$.

Proof. By Propositions **??** and 4.1, the spectral sequence of (25) can be rewritten as

$$E_1^{s,t} = H^{s+t}(u(\mathfrak{u}_{J,\mathbb{F}_q} \otimes_{\mathbb{F}_q} k), \operatorname{gr} M) \Longrightarrow H^{s+t}(U_J(\mathbb{F}_q), M) \equiv H^{s+t}(kU_J(\mathbb{F}_q), M). \tag{29}$$

Thus, $H^\bullet(U_J(\mathbb{F}_q),k)$ is a subquotient of the (Noetherian) k-algebra $H^\bullet(u(\mathfrak{u}_{J,\mathbb{F}_q} \otimes_{\mathbb{F}_q} k),k)$; moreover, the $H^\bullet(U_J(\mathbb{F}_q),k)$-module $H^\bullet(U_J(\mathbb{F}_q),M)$ is a subquotient of the the finitely generated $H^\bullet(u(\mathfrak{u}_{\mathbb{F}_q} \otimes_{\mathbb{F}_q} k),k)$-module $H^\bullet(u(\mathfrak{u}_{\mathbb{F}_q} \otimes_{\mathbb{F}_q} k), \operatorname{gr} M)$. In particular, the rate of growth of $\{n \mapsto \dim H^n(U_J(\mathbb{F}_q),M)\}$ is less than or equal to that of $\{n \mapsto H^n(u(\mathfrak{u}_{\mathbb{F}_q} \otimes_{\mathbb{F}_q} k), \operatorname{gr} M)\}$. This implies (27).

If M is a rational G-module, then the cohomological support variety $|\underline{\mathfrak{g}}|_M$ is G stable as a subspace of $|\underline{\mathfrak{g}}| \ \subset \mathcal{N}_1$, and thus a union of finitely many G-orbits. (Here, $\mathcal{N}_1 \subset \mathfrak{g}$ is the p-nilpotent cone.) Thus, the dimension of $|\underline{\mathfrak{g}}|_M$ equals the largest dimension of a G-orbit contained in $|\underline{\mathfrak{g}}|_M$. Moreover, by Theorem 1.2 of [10], $|\underline{\mathfrak{g}}|_M = G \cdot |\underline{\mathfrak{b}}|_M$, where $\mathfrak{b} = Lie(B)$; and $|\underline{\mathfrak{u}}|_M = |\underline{\mathfrak{b}}|_M \subset |\underline{\mathfrak{g}}|_M$, for the p-nilpotent cone of $\mathfrak{u}$ equals that of $\mathfrak{b}$. As argued in Theorem 3.4 of [15], $dim\mathcal{O} = 2dim(\mathcal{O} \cap \mathfrak{u})$ for any G-orbit $\mathcal{O}$ in $\mathcal{N}$. Thus, (26) implies that

$$cx_{u(\mathfrak{u}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)}(M) \ = \ \frac{1}{2} cx_{u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)}(M). \tag{30}$$

The proof of (28) now follows by combining (27) and (30), plus observing that $U(\mathbb{F}_q) \subset G(\mathbb{F}_q)$ contains a p-Sylow subgroup, so that $cx_{kG(\mathbb{F}_q)}(M) \le cx_{kU(\mathbb{F}_q)}(M)$ for any finite dimensional $kG(\mathbb{F}_q)$-module M. □

Since a G-module M is projective if and only if its complexity is 0, we immediately conclude the following corollary.

Corollary 4.4. *Assume the hypotheses and notation of Theorem 4.3. For any rational G-module M, if M is projective as a $u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)$-module, then M is projective as a $kG(\mathbb{F}_q)$-module.*

5. Non-Maximal Support Varieties

Throughout this section, k will denote an arbitrary field of characteristic $p > 0$ and G will denote a simple algebraic group over k, defined and split over $\mathbb{F}_p$. We assume that the following condition on p (relative to G) is satisfied: p is good for G, $\mathcal{N}_1$ is normal, and $SL_{n+1} \to G$ is separable in the case that G is of type A_n.

We begin with the following observation. In the statement of Proposition 5.1, ρ denotes the sum of the fundamental dominant weights of G.

Proposition 5.1. *Let M be a rational G-module with the property that all high weights λ of M satisfy $\langle \lambda, \rho^\vee \rangle < p$. Let $E \subset G(\mathbb{F}_p)$ be an elementary abelian p-subgroup with minimal set of generators $x_1, \ldots, x_r$. Assume that $0 \neq \underline{a} = (a_1, \ldots, a_r) \in K^r$ satisfies the condition that $\alpha_{\underline{a}}^*(M_K)$ is maximal among Jordan types $\beta_L^*(M_L)$ as $\beta_L : L[t]/t^p \to LE$ ranges over all π-points of E.*

Let $X_{\underline{a}} \in \mathfrak{g}_K$ denote $\sum_{j=1}^r a_j log(x_j)$, and let $\alpha_{X_{\underline{a}}} : K[t]/t^p \to u(\mathfrak{g}_K)$ denote the associated π-point of $\underline{g}$. Then the Jordan type of $\alpha_{\underline{a}}^*(M_K)$ equals that of $\alpha_{X_{\underline{a}}}^*(M_K)$.

Proof. Let $X \in \mathcal{N}_1$ and let $u = \exp(X) = \sum_{i \geq 0} X^{(n)} \in \mathcal{U}_1$. (Here, $X^{(n)}$ is the n-th divided power distribution of X viewed as a linear distribution on $k[G]$.) Then the action of u on a rational G-module is given in terms of the action of prime power divided powers of X, since

$$u = \sum_{n=r_0+r_1p+\cdots+r_sp^s} X^{r_0} X^{(p)r_1} \cdots X^{(p^s)r_s}/r_0! \cdots r_s!. \tag{31}$$

The hypothesis on the high weights of M implies that $X^{(i)}$ acts trivially on M for every $i > 0$ (see secton 4.6 of [6]). Thus, the action of u on M is given by $\sum_{i=0}^{p-1} X^i/i!$.

Consequently, the action of the image of t on M_K under $\alpha_{\underline{a}} : K[t]/t^p \to KE \to kG(\mathbb{F}_p) \to End_K(M_K)$, namely the action of $\sum_{j=1}^r a_j(x_j - 1)$ on M_K, is given by the action of

$$\sum_{j=1}^r a_j \sum_{i=1}^{p-1} (log(x_j))^i/i!. \tag{32}$$

The condition on $\alpha_{\underline{a}}$ that the Jordan type of $\alpha_{\underline{a}}^*(M_K)$ is maximal implies by Theorem 2.7 of [13] that the Jordan type remains unchanged if we drop the non-linear terms of (32). In other words, the Jordan type of $\alpha_{\underline{a}}^*(M_K)$ equals that of $\alpha_{X_{\underline{a}}}^*(M_K)$. □

We recall from [13] that that the non-maximal support variety $\Gamma(H)_M$ of a finite group scheme H over k is the closed subvariety of $\Pi(H)$ consisting of those equivalence classes of π-points $\alpha_K : K[t]/t^p \to KH$ for which $\alpha_K^*(M_K)$ does not have maximal Jordan type. This is well defined, for $\alpha_K^*(M_K)$ has maximal Jordan type for M if and only if $\beta_L(M_L)$ has maximal Jordan type for every β_L equivalent to α_K.

By definition, $\Gamma(H)_M \subset \Pi(H)_M$, with equality if and only if $\Pi(H)_M \neq \Pi(H)$ which holds if and only if $\alpha_K^*(M_K)$ is projective for some π-point α_K. In particular, if the dimension of M is not divisible by p, then $\Pi(H)_M$ is independent of M (in fact, equal to $\Pi(H)$), whereas $\Gamma(H)_M \subset \Pi(G)$ will vary with M.

The following theorem should be compared to Corollary 2 of Theorem 6 of [6], which gives a partial comparison of the cohomological support varieties $|G(\mathbb{F}_p)|_M$ and $|\underline{\mathfrak{g}}|_M$ for rational G-modules with small weights.

Theorem 5.2. *Let M be a rational G-module with the property that all high weights λ of M satisfy $\langle \lambda, \rho^\vee \rangle < p$. Then the map of Theorem 3.5 in the special case $q = p$,*

$$\ell : \Pi(G(\mathbb{F}_p)) \longrightarrow \Pi(\underline{\mathfrak{g}})/G(\mathbb{F}_p),$$

satisfies the condition that if $\alpha_K^*(M_K)$ does not have maximal Jordan type (among the Jordan types of $\beta_L^*(M_L)$ as β_L ranges over π-points of $G(\mathbb{F}_p)$), then $\ell(\alpha_K)^*(M_K)$ is not maximal among Jordan types of M at π-points of $\underline{\mathfrak{g}}$.

In other words, ℓ induces an embedding

$$\Gamma(G(\mathbb{F}_p))_M \hookrightarrow (\Gamma(\underline{\mathfrak{g}})_M/G(\mathbb{F}_p)) \cap \ell(\Pi(G(\mathbb{F}_p)).$$

This embedding is an isomorphism if and only if the maximal Jordan type of M as a $u(\mathfrak{g})$-module occurs at a π-point $\alpha_K : K[t]/t^p \to u(\mathfrak{g})$ which factors through $kG(\mathbb{F}_p)$.

Proof. Consider a π-point of $G(\mathbb{F}_p)$ represented by some $\alpha_{\underline{a}} : K[t]/t^p \to KE$ for some maximal elementary abelian p-group $E \subset G(\mathbb{F}_p)$, some choice of minimal set of generators $x_1, \ldots, x_r$ of E, and some $0 \neq \underline{a} = (a_1, \ldots, a_r) \in K^r$. The map $\ell : \Pi(G(\mathbb{F}_p)) \hookrightarrow \Pi(\underline{\mathfrak{g}})/G(\mathbb{F}_p)$ of Theorem 3.5 sends the class $[\alpha_{\underline{a}}] \in \Pi(G(\mathbb{F}_p))$ to the class of $\alpha_{X_{\underline{a}}}$. As seen in the proof of Proposition 5.1, the action of $\alpha_{\underline{a}}(t) = \sum_{j=1}^r a_j(x_j - 1)$ on M_K is given as in (32) by the action of $\sum_{j=1}^r a_j \sum_{i=1}^{p-1} (log(x_j))^i/i!$ on M granted our condition on the weight of M. The action of $\ell(\alpha_{\underline{a}})(t) = \alpha_{X_{\underline{a}}}(t)$ on M is given by the action of $X_{\underline{a}} \equiv \sum_{j=1}^r a_j log(x_j)$ on M.

By Theorem 2.7 of [13], $\alpha_{\underline{a}}^*(M_K)$ has maximal Jordan type among all $\alpha_{\underline{b}}^*(M_L)$ as $\alpha_{\underline{b}} : L[t]/t^p \to LE$ varies among all $0 \neq \underline{b} = (b_1, \ldots, b_r) \in L^r$ for all field extensions L/k if and only if $\alpha_{X_{\underline{a}}}^*(M_K)$ has Jordan type greater or equal to that of $\alpha_{\underline{b}}^*(M_L)$ for all $0 \neq \underline{b} = (b_1, \ldots, b_r) \in L^r$ with L/k varying if and only if $\alpha_{X_{\underline{a}}}^*(M_K)$ has Jordan type greater or equal to that of all $\alpha_{X_{\underline{b}}}^*(M_L)$ for all $0 \neq \underline{b} = (b_1, \ldots, b_r) \in L^r$ with L/k varying. In other words, by Theorem 1.12 of [13], the maximality of the Jordan type of a p-nilpotent operator in $End_L(M_L)$ written as a polynomial in r pairwise commuting, p-nilpotent operators depends only upon the maximality of the operator associated to the linear part of the polynomial.

Consequently, if $\alpha^*_{\underline{a}}(M_K)$ does not have maximal Jordan type among the Jordan types of M at π-points of E, then $\alpha^*_{X_{\underline{a}}}(M_K)$ (with $\alpha_{X_{\underline{a}}}$ representing $\ell(\alpha_{\underline{a}})$) does not have maximal Jordan type among the Jordan types of M at π-points of $\underline{\mathfrak{g}}$.

The remaining assertions of the theorem now follow immediately. $\square$

We refer the reader to [5] for a discussion of modules of constant Jordan type for a finite group scheme H. Because the condition that M have constant Jordan type is equivalent to the condition that $\Gamma(H)_M$ be empty (by [5], Prop 3.6), Theorem 5.2 has the following immediate corollary.

Corollary 5.3. *Let G and M be as in Theorem 5.2. If M has constant Jordan type as a $u(\mathfrak{g})$-module, then M has constant Jordan type (with the same Jordan type) as a $kG(\mathbb{F}_p)$-module.*

Remark 5.4. Theorem 5.2 remains valid if with we replace "maximal Jordan type" by "maximal rank" or even "maximal j-rank", for some j with $1 \leq j < p$. Similarly, Corollary 5.3 remains valid if with we replace "constant Jordan type" by "constant rank" or even "constant j-rank", for some j with $1 \leq j < p$. The proof of these analogues is essentially the same once one recalls (from [13]) that the property of being of maximal rank or of maximal j-rank is independent of the representative of an equivalence class of π-points for any finite group scheme H and any kH-module M.

References

[1] G. Avrunin and L. Scott, *Quillen stratification for modules*, Invent Math. **66** (1982), 277 - 286.

[2] P. Bardsley and R.W. Richardson, *Étale slices for algebraic transformation groups in characteristic p*, Proc. London Math. Soc. **51** (1985), 295 - 317.

[3] D. Benson Representation and Cohomology, II. *Cohomology of groups and modules*, Cambridge University Press, 1991.

[4] J. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983), 104 - 143.

[5] J, Carlson, E. Friedlander, J. Pevtsova, J. Carlson, *Modules of Constant Jordan type*, *Journal fúr die reine und angewandte Mathematik* (2007), 1 - 44.

[6] J. Carlson, Z. Lin, and D. Nakano, *Support varieties for modules over Chevalley groups and classical Lie algebras*, Trans. AMS **360** (2008), 1870 - 1906.

[7] J. Carlson, Z. Lin, D. Nakano, and B. Parshall *The restricted nullcone*, Contemp. Math **325** (2003), 51 - 75.

[8] S. Donkin, *The normality of closures of conjugacy classes of matrices*, Invent. Math. **101** (1990), 717 - 736.

[9] L.Evens, The Cohomology of Groups, Oxford University Press, 1991.

[10] E. Friedlander, B. Parshall *Support varieties for restricted Lie algebras*, Invent. Math. **86** (1986), 553 - 562.

[11] E. Friedlander, J. Pevtsova, *Representation-theoretic support spaces for finite group schemes*, Amer. J. Math. **127** (2005), 379-420.

[12] E. Friedlander, J. Pevtsova, Π-*supports for modules for finite group schemes*, Duke Math. J. **139** (2007), 317 - 368.

[13] E. Friedlander, J. Pevtsova, A Suslin *Generic and Maximal Jordan types*, Invent. Math. **168** (2007), 485 - 522.

[14] J.C. Jantzen, Representations of Algebraic Groups, Academic Press, 1987.

[15] Z. Lin and D. Nakano, *Complexity for modules over finite Chevelley groups and classical Lie algebras*, Invent. Math. **138** (1999), 85 - 101.

[16] J.P. May, *The cohomology of Lie algebras and of Hopf algebras*, J. Alg **3** (1966), 123 - 146.

[17] D. Nakano, B. Parshall, and D. Vella, *Support varieties for algebraic groups*, J. Reine Angew. Math. **547** (2002), 15 - 47.

[18] J. Oesterlé, *Nombres de Tamagawa et groupes unipotents en charactéristique* p, Invent. Math. **78** (1984), 12 - 88.

[19] R. Pink, *On Weil restriction of reductive groups and a theorem of Prasad*, Math. Zeit. **248** (2004), 449 - 457.

[20] D. Quillen, *On the associated graded ring of a group ring*, J. Alg **10** (1968), 411 - 418.

[21] D. Quillen, *The spectrum of an equivariant ring: I, II*, Ann. Math. **94** (1971), 549 - 572, 573 - 602.

[22] G. Seitz, *Unipotent elements, tilting modules, and saturation*, Invent. Math. **141** (2000), 467 - 502.

[23] T.A. Springer and R. Steinberg, *Conjugacy classes*, *in* Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math **131** (1970), 166 - 266.

[24] J.F. Thomsen, *Normality of certain nilpotent varieties in positive characteristic*, J. Algebra **226** (2000), 865 - 874

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