LIPSCHITZ COCYCLES AND POINCARÉ DUALITY

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Abstract. Geometric measure theory enables one to view cohomology as equivalence classes of graphs of multi-valued Lipschitz maps to spheres. This geometric point of view gives a new formulation of cohomology, relative cohomology, and cohomology with supports as homotopy groups of spaces of Lipschitz cocycles. Using the graphing construction of the first author and H. Blaine Lawson, this leads to a formulation and proof of weak equivalences whose associated map on homotopy groups is a form of Alexander duality for a compact subpolyhedron of a compact, oriented smooth manifold.

0. Introduction

In [6], the first author and H. Blaine Lawson established for a compact oriented n-dimensional pseudo-manifold A which is smoothable outside a subcomplex of codimension ≥ 2 and a cohomology class $\alpha \in H^j(A,\mathbb{Z})$ that $\alpha \cap [A] \in H_{n-j}(A)$ can be represented by the geometric measure-theoretic slice of the graph of a multi-valued Lipschitz map from A to S^j. The purpose of this paper is to extend the constructions of [6] to a compactifiable pseudo-manifold $A - A_\infty$ and then use this extension to prove a form of Alexander duality whenever $A - A_\infty$ is smooth. Indeed, we prove a stronger result which is the geometric measure-theoretic analogue of the Friedlander-Lawson [5] and Friedlander-Voevodsky [7] duality theorems for smooth complex algebraic varieties. Namely, as seen in Corollary 5.5, the graphing construction for Lipschitz maps determines a weak equivalence relating the topological abelian groups of Lipschitz cocycles on $A - A_\infty$ and an appropriate group of rectifiable currents on A with boundary in closed tubular neighborhood of A_∞.

Our arguments involve a mixture of elementary simplicial topology and geometric measure theory. The fundamental construction Γ^{top} involves the graphing of a compact oriented n-dimensional pseudo-manifold A equipped with a triangulation. Such a space is a Lipschitz neighborhood retract, admitting a good formulation of currents. Moreover, the triangulation enables us to work cell-by-cell, enabling local arguments and consideration of subcomplexes and their complements.

To summarize in more detail, Section 1 introduces the open and closed subsets of polyhedra which we shall employ, and discusses various spaces of Lipschitz maps with target a symmetric product of a sphere. In Section 2, we define the Lipschitz cocyle space $\mathcal{Z}^m(A)$ of codimension m cocycles on a finite polyhedron A, the relative cocyle space $\mathcal{Z}^m(A,C)$ for a closed subset $C \subset A$, the cocycle space $\mathcal{Z}^m(A - A_\infty)$ of codimension m Lipschitz cocycles on the complement of a closed subpolyhedron $A_\infty \subset A$, and the space $\mathcal{Z}^m_{A_\infty}(A)$ of codimension m Lipschitz cocycles on A with

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support in A_∞. As seen in Section 3, the homotopy groups of these cycle spaces satisfy the expected properties of singular cohomology.

The key construction, basically that of [6], is the construction of a rectifiable current, the geometric graph $\Gamma(f)$, of a Lipschitz map on $A - O_\Delta(A_\infty)$, the complement in a compact, oriented pseudo-manifold A of an open neighborhood of a subpolyhedron $A_\infty \subset A$. In Section 4, this is shown to determine a continuous map from *good* Lipschitz cocycles of codimension m to integral cycles (i.e., rectifiable currents with 0 boundary) on $A_+ \wedge S^m$ modulo integral cycles on $D_\Delta(A_\infty)_+ \wedge S^m$. With this formalism in place, we formulate and prove in Section 5 a refinement (in the sense of a map of spaces) of Alexander duality.

In particular, the special case in which A_∞ is empty is the following (yielding Poincaré Duality upon taking homotopy groups).

pd-intro **Theorem 0.1.** *Let A be a smooth compact oriented manifold of dimension n. Then the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m(A) \ \to \ \mathcal{Z}_n(A_+ \wedge S^m)$$

is a weak equivalence.

As seen in Corollary 5.5, we prove a similar statement for $A - A_\infty$, thereby obtaining a form of Alexander duality.

In the final section, we show how the Thom class and the Thom isomorphism admit a natural formulation in terms of Lipschitz cocycles. We conclude by conjecturing a space-level compatibility of the Thom isomorphism and the Gysin map constructed in terms of our duality map.

We thank Blaine Lawson for sharing with us his geometric insight into Poincaré duality.

1. Polyhedra, rectifiable currents, and Lipschitz Maps

sec:maps

We consider a compact polyhedron A, a cell complex which is the geometric realization of a finite simplicial complex. We shall typically consider a (piece-wise linear) triangulation Δ on A associated to some choice of structure of a finite simplicial complex, and then consider refinements of such a triangulation. By abuse of notation, we shall refer to a compact polyhedron together with a given (finite, piece-wise linear) triangulation as a *simplicial complex* as well. We denote by $\Delta(k)$ the (finite) set of open k-simplices of Δ, each homeomorphic to an open k-disk. Observe that the d-fold symmetric product $SP^d(A)$ of A is again a compact polyhedron.

If A is a compact polyhedron, then a simplicial structure on A determines an embedding of A in a Euclidean space $\mathbb{R}^N$, where N denotes the number of vertices of the simplicial structure. A tubular neighborhood of $A \subset \mathbb{R}^N$ provides A with the structure of a Lipschitz neighborhood retract of $\mathbb{R}^N$. Such an embedding provides A with a piecewise smooth Riemannian metric (compatible with the triangulation on A given by the simplicial structure). The class of Lipschitz functions associated to such a metric on A is in fact independent of the choice of such a metric.

Definition 1.1. Let A be a compact polyhedron, a Lipschitz neighborhood retract with Lipschitz retraction $U \to A$ of some tubular neighborhood U of A in a Euclidean space. A rectifiable k-current on A is an element in the closure (with respect to the mass norm) of the space of Lipschitz polyhedral k-chains on A. We denote by $\mathcal{I}_k(A)$ the space of rectifiable k-currents on A with rectifiable boundary (i.e., **integral k-currents**) equipped with the flat norm topology. We denote by $\mathcal{Z}_k(A) \subset \mathcal{I}_k(A)$ the subspace of rectifiable k-currents with 0-boundary (i.e. **integral k-cycles**).

We recall the following theorem of F. Almgren [1] (and restated in [6, 1.2]).

Alm **Theorem 1.2.** *Let $C \subset A$ be a closed subspace, with both A, C compact, local Lipschitz neighborhood retracts in Euclidean spaces. Then there is a natural isomorphism*

$$\mathcal{A} : \pi_j\{\mathcal{Z}_r(A,B)/\mathcal{I}_r(C)\} \xrightarrow{\sim} H_{r+j}(A,C).$$

Here, $\mathcal{I}_r(C)$ denotes the integral r-currents on B with the flat norm topology and $\mathcal{Z}_r(A,C)$ denotes the integral r-currents on A whose boundary has support in C, also provided with the flat norm topology.

Moreover, $\mathcal{Z}_r(A)/\mathcal{Z}_r(C)$ is a closed subspace of $\mathcal{Z}_r(A,C)/\mathcal{I}_r(C)$ with discrete quotient, thereby determining the short exact sequence

$$0 \to \mathcal{Z}_r(A)/\mathcal{Z}_r(C) \;\to\; \mathcal{Z}_r(A,C)/\mathcal{I}_r(C) \;\to\; \ker\{H_{r-1}(C) \to H_{r-1}(A)\} \to 0.$$

We shall work with non-compact spaces of the form $A - A_\infty$, where A is a compact polyhedron and $A_\infty \subset A$ is a (closed) subcomplex with respect to some finite triangulation of A. We shall refer to such a space $A - A_\infty$ as a **compactifiable polyhedron**.

nbhds **Definition 1.3.** Let A be a finite polyhedron. Equip A with a (finite, piece-wise linear) triangulation Δ and let $A_\infty \subset A$ be a closed subpolyhedron that is a subcomplex for the triangulation Δ. Embed A in Euclidean space $\mathbb{R}^s$, where s is the number of vertices of A, so that each vertex is a distance 1 along the corresponding axis of $\mathbb{R}^s$.

We define

$$\begin{aligned} D_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) \leq \frac{1}{4}\} \\ O_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) < \frac{1}{4}\} \\ S_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) = \frac{1}{4}\} \end{aligned}$$

Note that $D_\Delta(A_\infty)$, $S_\Delta(A_\infty)$ and $A - O_\Delta(A_\infty)$ are all closed subcomplexes of A for some suitable subdivision of the triangulation Δ. It is useful to observe that if Δ' refines Δ then $O_{\Delta'}(A_\infty) \subset O_\Delta(A_\infty)$.

The following proposition constructs a flow from $A - A_\infty$ to the closed subpolyhedron $A - O_\Delta(A_\infty)$.

flow **Proposition 1.4.** *Let $U = A - A_\infty$ be a compactifiable polyhedron.*

(1) *There is a homotopy $H : U \times I \to U$ relating the identity of U to a retraction $U \to A - O_\Delta(A_\infty)$ and restricting to $H_| : (A - O_\Delta(A_\infty)) \times I \to A - O_\Delta(A_\infty)$.*
(2) *There is a deformation retraction $F : D_\Delta(A_\infty) \times I \to D_\Delta(A_\infty)$ which is a deformation retraction to the subpolyhedron A_∞.*

Proof. Observe that the closure $\overline{\sigma}$ of an open simplex of A meets $D_\Delta(A_\infty)$ if and only if $\overline{\sigma}$ meets A_∞. Let $Y \subset A$ denote the union of those closed simplices of A meeting A_∞, and triangulate Y using the first barycentric subdivision of the given triangulation of A. We define the "link" $L \subset A$ of $A_\infty \subset A$ to be the sub-simplicial complex of Y with vertices the barycenters of simplices of A whose closures do not intersect A_∞. Let $\tilde{L} \subset Y$ consist of those points $y \in Y;\ d(y, L) \leq \frac{1}{4}$. Then we employ a continuous map

$$F : Y \times I \ \rightarrow \ Y$$

satisfying

- $F(y, 0) = y,\ y \in Y$
- $F(x, t) = x,\ x \in A_\infty$ *and* $t \in I$
- $F(y, t) = y,\ y \in \tilde{L}$ *and* $t \in I$
- $F(-, t)$ is a homeomorphism for $t \neq 1$
- $F(D_\Delta(A_\infty) \times \{1\})) \subset A_\infty$
- $F(D_\Delta(A_\infty) \times I) \subset D_\Delta(A_\infty)$.

Any such map F restricted to $D_\Delta(A_\infty)$ is a deformation retraction to A_∞. Moreover, , we obtain a deformation retraction $F' : (Y - A_\infty) \times I \rightarrow Y - A_\infty$ of $Y - A_\infty$ to $Y - D_\Delta(A_\infty)$ by setting F'_t equal to the inverse of the restriction to $Y - A_\infty$ of F_{1-t}. We define $H : U \times I \rightarrow I$ to be this retraction on $(Y - A_\infty) \times I$ and the identity flow on $(A - Y) \times I$. □

We shall use the following elementary lemma from homotopy theory.

lem:retract **Lemma 1.5.** *Let $Y \subset X$ be an inclusion of a subspace of a topological space X. Suppose there is a homotopy $H : X \times I \rightarrow X$ such that*

- $H(-, 0) = id_X$
- $H(X \times \{1\}) \subset Y$
- $H(Y \times \{t\}) \subset Y$ *for all* t.

Then the inclusion $Y \rightarrow X$ is a homotopy equivalence with homotopy inverse $H(-, 1)$.

Applying Lemma 1.5 to the homotopy of Proposition 1.4 (1), we immediately obtain the following corollary.

Corollary 1.6. *Let $U = A - A_\infty$ be a compactifiable polyhedron and let Δ be some finite piece-wise linear triangulation of A such that A_∞ is a subcomplex. Then the embedding*

$$A' = (A - O_\Delta(A_\infty)) \ \subset \ U$$

is a weak equivalence and A' is a compact polyhedron.

We recall that a continuous map $f : A \rightarrow B$ of metric spaces is said to be Lipschitz with Lipschitz constant K if for all pairs of points a, a' in A the following inequality is satisfied:

$$d_B(f(a), f(a')) \ \leq \ K \cdot d_A(a, a').$$

def:lipmaps **Definition 1.7.** If A, B are metric spaces, then we define

$$Map_{Lip}(A, B)$$

to be the set of Lipschitz maps from A to B with topology of convergence with bounded Lipschitz constant. In other words, the sequence $\{f_n\}$ converges to $f : A \rightarrow B$ in this topology if it is uniformly convergent and there is a $K > 0$ that serves as Lipschitz constant for all the f_n.

rem:lipindependence **Remark 1.8.** If A and B are compact polyhedra equipped with a piecewise smooth metric via embeddings as Lipschitz neighborhood retracts, then the subset

$$Map_{Lip}(A, B) \hookrightarrow Map_{cont}(A, B)$$

together with its topology is independent of the choice of embedding. On the strength of this observation we will refer to its elements as *Lipschitz maps* from A to B without reference to the specific piecewise smooth metric chosen.

Observe that if B is a compact polyhedron, then so is its d-fold symmetric power $SP^d(B)$ for any $d > 0$.

In [6, 1.5], the embedding $Map_{Lip}(A, SP^d(S^m)) \hookrightarrow Map_{cont}(A, SP^d(S^m))$ is shown to be a weak homotopy equivalence. We extend this result by allowing A to be a compactifiable polyhedron.

cont **Proposition 1.9.** *Let A, B be compact polyhedra, and $d > 0$. Retain the hypotheses and notation of Definition 1.3. Then each of the maps of the following chain is a weak homotopy equivalence*

$$Map_{cont}(A - A_\infty, SP^d(B)) \ \rightarrow \ Map_{cont}(A - O_\Delta(A_\infty), SP^d(B)) \leftarrow$$
$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B)).$$

Proof. The homotopy $H : U \times I \to U$ of Proposition 1.4 implies that the first map is a homotopy equivalence by Lemma 1.5; [6, 1.5] verifies that the second map is a weak equivalence. □

inv **Lemma 1.10.** *If Δ' is a refinement of the triangulation Δ of A, then for any $d > 0$ the natural restriction map*

$$Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(B)) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B))$$

is a Serre fibration and a weak equivalence.

Proof. Let $\Lambda^j[n] \subset \Delta[n]$ denote the inclusion of the union of all faces of the n-simplex $\Delta[n]$ except the j-th face into $\Delta[n]$. For each $n \geq 0$ and each j, $0 \leq j \leq n$, we use the structure of $A - O_\Delta(A_\infty) \subset A - O_{\Delta'}(A_\infty)$ (as a simplicial embedding of finite complexes) to exhibit a strong deformation retraction of $(A - O_{\Delta'}(A_\infty)) \times \Delta[n]$ to

$$((A - O_{\Delta'}(A_\infty)) \times \Lambda^j[n]) \cup_{((A - O_\Delta(A_\infty)) \times \Lambda^j[n])} ((A - O_\Delta(A_\infty)) \times \Delta[n])$$

which is a Lipschitz map with Lipschitz constant 1. This implies the Serre lifting property for $Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(S^m)) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$. The fact that this map is a weak equivalence follows from Proposition 1.9 and the fact that both $A - O_\Delta(A_\infty)$, $A - O_{\Delta'}(A_\infty)$ are homotopy equivalent to $A - A_\infty$. □

Definition 1.11. Let $\mathcal{T}(A)$ denote the category of finite,piece-wise linear triangulations of the polyhedron A, with one triangulation Δ' mapping to another Δ provided that Δ' is a refinement of Δ. For any closed subpolyehdron $A_\infty \subset A$ which is a subcomplex with respect to a triangulation Δ, any finite polyhedron B and any $d > 0$, we define

bound
$$(1) \quad Map^b_{Lip}(A - A_\infty, SP^d(B)) \equiv \varprojlim_{\Delta' \in \mathcal{T}(A)/\Delta} Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(B)).$$

rem:funct **Remark 1.12.** Let A, A', B be finite polyhedra and let $A_\infty \subset A$, $A'_\infty \subset A'$ be subpolyhedra. Then a Lipschitz map $f : A \to A'$ with the property that $f^{-1}(A'_\infty) \subset A_\infty$ induces a continuous map

funct
$$f^* : Map^b_{Lip}(A - A_\infty, SP^d(B)) \longrightarrow Map^b_{Lip}(A - A_\infty, SP^d(B)). \tag{2}$$

In particular, there is a natural restriction map

$$Map^b_{Lip}(A - A_\infty, SP^d(B)) \to Map^b_{Lip}(A - A'_\infty, SP^d(B))$$

whenever $A_\infty \subset A'_\infty$ is an inclusion of closed subcomplexes for some finite triangulation. This implies that the assignment

$$U \mapsto Map^b_{Lip}(U, SP^d(B))$$

is a contravariant functor on the category of open subsets of A whose complement is a closed subcomplex for some sufficiently fine finite triangulation of A.

proj-inj **Corollary 1.13.** *The natural projection and inclusion maps*

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B)) \twoheadleftarrow Map^b_{Lip}(A - A_\infty, SP^d(B)) \hookrightarrow Map_{cont}(A - A_\infty, SP^d(B))$$

are weak equivalences for any finite, piece-wise linear triangulation Δ of A such that A_∞ inherits the structure of a subcomplex.

Proof. The fact that the projection is a weak equivalence follows from Lemma 1.10 and the standard fact that the inverse limit of a tower of maps each of which is a Serre fibration and a weak equivalence projects via a Serre fibration and a weak equivalence to each term in the tower. The fact that the inclusion is a weak equivalence follows from Proposition 1.9. □

sec:cocycles

2. Lipschitz cocycle spaces

Recall that $(\coprod_d SP^d(S^m))^+$ is a model for the generalized Eilenberg-MacLane space $K(\mathbb{Z}, m) \times K(\mathbb{Z}, 0)$, so that

$$\pi_i(Map_{cont}(A, (\coprod_d SP^d(S^m))^+) = \begin{cases} H^m(A) \oplus H^0(A) & \text{if } i = 0 \\ H^{m-i}(A) & \text{if } i > 0. \end{cases}$$

where $H^i(A) = H^i(A, \mathbb{Z})$ denotes singular cohomology with $\mathbb{Z}$ coefficients. Since A is compact,

$$(\coprod_d Map_{cont}(A, SP^d(S^m))^+ \cong Map_{cont}(A, (\coprod_d SP^d(S^m))^+).$$

Thus,

$$\pi_i \ker\{(\coprod_d Map_{cont}(A, SP^d(S^m))^+ \to H^0(A)\} \cong H^i(A), \quad i \geq 0.$$

This motivates the following definition of Lipschitz cocycle spaces. We set

lip-map
$$Map_{Lip}(A, SP^\infty(S^m)^+) = \ker\{(\coprod_{d \geq 0} Map_{Lip}(A, SP^d(S^m)))^+ \to H^0(A)\}. \tag{3}$$

def:compcocycle

Definition 2.1. Let A be a compact polyhedron and $C \subset A$ a closed subset that is a subcomplex with respect to some triangulation of A. Following [6] we define the topological abelian group $\mathcal{Z}^m(A)$ of **Lipschitz m-cocycles** on A (topological Abelian group $\mathcal{Z}^m(A,C)$ of relative Lipschitz cocycles, respectively) as

$$\mathcal{Z}^m(A) = Map_{Lip}(A, SP^\infty(S^m)^+)$$

and

$$\mathcal{Z}^m(A,C) = \ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(C)\}$$

Note that by Remark 1.8, these groups are well-defined independent of the choice of a realizations of A and C as Lipschitz neighborhood retracts.

cohomology

Proposition 2.2. *Let A be a compact polyhedron and $C \subset A$ a closed subcomplex with respect to some triangulation. Then there are isomorphisms*

$$\begin{aligned} \pi_i \mathcal{Z}^m(A) &\cong H^{m-i}(A) \\ \pi_i \mathcal{Z}^m(A,C) &\cong H^{m-i}(A,C) \end{aligned}$$

where $H^*(A)$ denotes the singular cohomology of A with $\mathbb{Z}$ coefficients.

These isomorphisms are natural for Lipschitz maps of (pairs of) compact polyhedra. Since every continuous map of compact polyhedra is homotopic to a Lipschitz map, these isomorphisms are, in fact, natural on the homotopy category of compact polyhedra.

Proof. The proposition follows from the special case of $A_\infty = \emptyset$ of Proposition 1.9 (i.e., from [6, 1.5]) and the above representation of cohomology of A in terms of $Map_{cont}(A, SP^d(S^m))$. $\square$

A key theorem which enables us to consider Lipschitz cocycles on compactifiable finite polyhedra (rather than relative groups as we do for geometric cycle spaces) is the following important theorem of Kirszbraun.

Kirz

Theorem 2.3. *(Kirszbraun's theorem; cf.* [2, 2.10.43]*) Let $S \subset \mathbb{R}^m$ be an arbitrary subset of $\mathbb{R}^m$ and consider $f : S \to \mathbb{R}^n$, a Lipschitz map with Lipschitz constant K. Then there exists an extension $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^n$ which is also a Lipschitz map with Lipschitz constant K.*

In particular, $Map_{Lip}(A, SP^\infty(S^m)^+) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^\infty(S^m)^+)$ is surjective by Kirszbraun's Theorem.

Definition 2.4. Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a (closed) subcomplex with respect to some finite, piece-wise linear triangulation of A. We set

$$\mathcal{Z}^m(A - A_\infty) \equiv Map^b_{Lip}(A - A_\infty, SP^\infty(S^m)^+),$$

where the right-hand side of is defined to be

$$\ker\{(\coprod_{d\geq 0} \varprojlim_{\Delta'/\Delta} Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(S^m)))^+ \to H^0(A - A_\infty)\}.$$

Corollary 1.13 has the following reassuring corollary.

exc1

Corollary 2.5. *Let U be provided with open embeddings $U \subset A$, $U \subset A'$. Assume that A, A' admit finite triangulations such that $A - U \subset A$, $A' - U \subset A'$ are subpolyhedra. Then $\mathcal{Z}^m(U)$ determined by the compactification $U \subset A$ is weakly* *equivalent to the corresponding topological group determined by the compactification* $U \subset A'$.

Moreover, for any finite triangulation Δ *of* A *such that* $A_\infty \subset A$ *is a subcomplex, there is a natural homomorphism*

$$\mathcal{Z}^m(U) \ \longrightarrow \ \mathcal{Z}^m(A - O_\Delta(A_\infty))$$

which is a weak equivalence. In particular,

$$\pi \mathcal{Z}^m(A - A_\infty) \ \cong \ H^{m-i}(A - A_\infty).$$

Remark 2.6. Although $\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_\Delta(A_\infty))$ is surjective for a given triangulation Δ of A, it would appear that $\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - A_\infty)$ is not surjective.

supp **Definition 2.7.** Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty, \ C \ \subset A$ be (closed) subcomplexes. Set $C_\infty = C \cap A_\infty$. Then we define the space of **Lipschitz cocycles** on A with support in C to be the topological abelian group

$$\mathcal{Z}^m_C(A) \ \equiv \ \varprojlim_{\Delta'/\Delta} \ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(C))\}$$

where the inverse limit is taken over all triangulations Δ' refining Δ as above.

More generally, we define

locc
$$\mathcal{Z}^m_{C-C_\infty}(A - A_\infty) \ \equiv \ \varprojlim_{\Delta'/\Delta} \ker\{\mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(A - O_\Delta(A_\infty \cup C))\}. \tag{4}$$

Because all the transition maps in the inverse system defining $\mathcal{Z}^m_{C-C_\infty}(A - A_\infty)$ are Serre fibrations and weak equivalences, this inverse limit is weakly equivalent to $\ker\{\mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(A - O_\Delta(A_\infty \cup C))\}$ for any finite, piece-wise linear triangulation Δ of A for which $A_\infty, C \ \subset A$ are subcomplexes. Thus,

$$\pi_i \mathcal{Z}^m_{C-C_\infty}(A - A_\infty) \ \cong \ H^{m-i}_{C-C_\infty}(A - A_\infty),$$

the cohomology of $A - A_\infty$ with supports in $C - C_\infty$.

sec:properties

3. Properties of Cocycle Spaces

In this section, we verify a few of the expected properties of cocycle spaces: multiplicative structure in Proposition 3.1, localization in Proposition 3.2, Mayer-Vietoris in Proposition 3.3, excision in Proposition 3.6, and transfer in Proposition 3.7.

mult **Proposition 3.1.** *(Multiplicative structure): Smash product of spheres,* $S^m \times S^{m\prime} \longrightarrow S^{m+m'}$ *induces a natural multiplicative structure on the (graded) integral cocycle spaces* $\mathcal{Z}^*(A - A_\infty)$, $\mathcal{Z}^*_C(A - A_\infty)$ *leading to graded commutative, associative product structures on their homotopy groups.*

Proof. First, observe that the smash product $S^m \times S^{m\prime} \longrightarrow S^{m+m'}$ induces Lipschitz maps

pair
$$SP^d(S^m) \times SP^e(S^{m'}) \longrightarrow SP^{de}(S^{m+m'}). \tag{5}$$

Thus, given Lipschitz maps $f : A \to SP^d(S^m), \ g : A \to SP^e(S^{m'})$, we obtain the Lipschitz map $f \wedge g : A \to SP^{de}(S^{m+m'})$. This determines a pairing of monoids

$$\coprod_{d \geq 0} Map_{Lip}(A, SP^d(S^m)) \times \coprod_{e \geq 0} Map_{Lip}(A, SP^e(S^{m'})) \longrightarrow$$

$$\longrightarrow \coprod_{f\geq 0} Map_{Lip}(A, SP^f(S^{m+m'})).$$

The pairings (5) induce the usual product structure $K(\mathbb{Z}, m) \times K(\mathbb{Z}, m') \to K(\mathbb{Z}, m+m')$ which in turn induces the cup product in cohomology. Thus, Proposition 2.2 implies that the pairings on homotopy groups of Lipschitz cocycles spaces is graded commutative and associative. □

`localization` **Proposition 3.2.** *(Localization) Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a (closed) subcomplex. Then the natural triple of topological abelian groups*

`triple`
$$\mathcal{Z}^m_{A_\infty}(A) \ \to \ \mathcal{Z}^m(A) \ \to \ \mathcal{Z}^m(A - A_\infty) \tag{6}$$

is a fibration sequence.

Proof. For each Δ for which A_∞ is a subcomplex of A, the short exact sequence

$$\ker(\mathcal{Z}^m(A) \ \to \ \mathcal{Z}^m(A - O_\Delta(A))) \ \to \ \mathcal{Z}^m(A) \ \to \ \mathcal{Z}^m(A - O_\Delta(A))$$

is a fibration sequence by [8]. As argued in the proof of Corollary 1.13, this fibration sequence is weakly homotopy equivalent to (6). □

`MV` **Proposition 3.3.** *(Mayer-Vietoris) Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a closed subcomplex. Let $D \subset A$ be a compact subpolyhedron containing $O_\Delta(A_\infty)$. Then there is a natural short exact sequence of topological abelian groups*

$$\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_\Delta(A_\infty)) \times \mathcal{Z}^m(D) \to Z^m(D - O_\Delta(A_\infty))$$

which determines the following homotopy Cartesian square

`square`
$$\begin{tikzcd} \mathcal{Z}^m(A) \arrow[r] \arrow[d] & \mathcal{Z}^m(D) \arrow[d] \\ \mathcal{Z}^m(A - A_\infty) \arrow[r] & Z^m(D - A_\infty) \end{tikzcd} \ . \tag{7}$$

Proof. Observe that for each Δ for which A_∞, C are subcomplexes of A, the short exact sequence

$$\mathcal{Z}^m(A) \ \to \ \mathcal{Z}^m(A - O_\Delta(A_\infty)) \times \mathcal{Z}^m(D) \ \to \ \mathcal{Z}^m(D - O_\Delta(A_\infty))$$

is a fibration sequence by [8]. Arguing once again as in the proof of Corollary 1.13 in order to pass to the limit over open tubular neighborhoods of A_∞, we conclude that

$$\mathcal{Z}^m(A) \ \to \ \mathcal{Z}^m(A - A_\infty) \times \mathcal{Z}^m(D) \ \to \ \mathcal{Z}^m(D - A_\infty)$$

is a fibration sequence. This implies that (7) is homotopy Cartesian. □

`cor:openMV` **Corollary 3.4.** *Let A be a compact polyhedron and let $A_\infty \subset A$ and $B_\infty \subset A$ be closed subcomplexes with respect to some finite triangulation Δ of A. Assume there is a finite subdivision Δ' of Δ such that $O_{\Delta'}(A_\infty) \cap O_{\Delta'}(B_\infty) = \emptyset$. Then there is a natural fibration sequence of topological abelian groups*

$$\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - A_\infty) \times \mathcal{Z}^m(A - B_\infty) \to \mathcal{Z}^m(A - (A_\infty \cup B_\infty))$$

which determines the following homotopy Cartesian square

$$\begin{tikzcd}
\mathcal{Z}^m(A) \arrow[r] \arrow[d] & \mathcal{Z}^m(A - B_\infty) \arrow[d] \\
\mathcal{Z}^m(A - A_\infty) \arrow[r] & \mathcal{Z}^m(A - (A_\infty \cup B_\infty)
\end{tikzcd}.$$

Here the cocycle spaces are taken with respect to the common compactification A, as indicated by the notation.

Proof. This is an immediate consequence of Theorem 3.3 and Corollary 2.5, once we know the various restiction maps are well-defined. This necessary functoriality property is supplied by the discussion in remark 1.12. $\square$

cor:supportsMV **Corollary 3.5.** *Let A be a compact polyhedron and let $A_\infty \subset A$ and $B_\infty \subset A$ be closed subcomplexes with respect to some finite triangulation Δ of A. Then there is a fibration sequence of topological abelian groups*

$$\mathcal{Z}^m_{A_\infty \cap B_\infty}(A) \to \mathcal{Z}^m_{A_\infty}(A) \times \mathcal{Z}^m_{B_\infty}(A) \to \mathcal{Z}^m_{A_\infty \cup B_\infty}(A).$$

Proof. For any refinement Δ' of Δ, the sequence

$$\begin{tikzcd}
\ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty \cap B_\infty))\} \arrow[d] \\
\ker\{\mathcal{Z}^m(A) \times \mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty)) \times \mathcal{Z}^m(A - O_{\Delta'}(B_\infty))\} \arrow[d] \\
\ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty \cup B_\infty))\}
\end{tikzcd} \tag{8}$$

is a fibration sequence by the 3×3 lemma and Mayer-Vietoris 3.3 for the closed cover $A - O_{\Delta'}(A_\infty \cap B_\infty) = (A - O_{\Delta'}(A_\infty)) \cup (A - O_{\Delta'}(B_\infty))$. $\square$

excision **Proposition 3.6.** *(Excision) Let Δ be a finite triangulation of A and let A_∞, D be closed subpolyhedra such that $O_\Delta(A_\infty) \subset D$. Then the restriction map*

exc
$$\mathcal{Z}^m_{A_\infty}(A) \ \longrightarrow \ \mathcal{Z}^m_{A_\infty}(D) \tag{9}$$

is a weak equivalence.

Proof. This follows immediately from Proposition 3.2 applied to the vertical maps of the homotopy Cartesian square (7). $\square$

transfer **Proposition 3.7.** *(Transfer) Let A, B be finite polyhedra related by a continuous $g : A \to SP^e(B)$ with associated ramified covering map $p : B \to A$. Assume that $A_\infty \subset A$ is a nowhere dense closed subpolyhedron with the property that $p : B - B_\infty \to A - A_\infty$ is a covering space map, where $B_\infty = p^{-1}(A_\infty)$. Then p induces a transfer map $p_! : \mathcal{Z}^m(B) \to \mathcal{Z}^m(A)$.*

Moreover, the restriction of $p_!$ to $\mathcal{Z}^m(B - B_\infty)$ has image in $\mathcal{Z}^m(A - A_\infty)$ and satisfies

$$p^* \circ p_! = e(-) : \ \mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(B - B_\infty) \to \mathcal{Z}^m(A - A_\infty),$$

where $e(-)$ is the e-th power map of the topological abelian group $\mathcal{Z}^m(A - A_\infty)$.

Proof. The map g induces maps $g^{(d)} : SP^d(A) \to SP^{de}(B)$ in the obvious manner, and each of these is Lipschitz. These maps determine a map of abelian monoids

$$\coprod_f Map_{Lip}(B, SP^f(S^m)) \to \coprod_f Map_{Lip}(A, SP^d(S^m)$$

whose group completion is the asserted map $p_! : \mathcal{Z}^m(B) \to \mathcal{Z}^m(A)$.

Choose a triangulation $\tilde{\Delta}$ of B with the property that $p(\tilde{\Delta}) = \Delta$ is a triangulation of A such that with respect to $\tilde{\Delta}$ (respectively, Δ) $B_\infty \subset B$ (resp, $A_\infty \subset A$) is a subpolyhedron. Then the restriction of p to $B - O_{\tilde{\Delta}}(B_\infty)$ is a covering space map to $A - O_\Delta(A_\infty)$. We see by inspection that the composition

$$\coprod_f Map_{Lip}(B - O_{\tilde{\Delta}}(B_\infty), SP^f(S^m)) \to \coprod_f Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m)$$

is mutliplication by e. Thus, the second assertion of the proposition follows from Corollary 1.13. $\square$

sec:graph

4. The Graphing Construction Γ^{top}

The purpose of this section is to establish a continuous graphing map

$$\Gamma^{top} : \mathcal{Z}^m(A - A_\infty) \ \longrightarrow \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \varinjlim_{\Delta} \ \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m),$$

where the finite polyhedron A is a compact pseudo-manifold of dimension n. (For notational convenience, we employ the abbreviation

wt

$$\widetilde{Z_r(A)}/Z_r(B) \ \equiv \ \mathcal{Z}_r(A, B)/\mathcal{I}_r(B), \tag{10}$$

the extension of $\ker\{H_{r-1}(B) \to H_{r-1}(A)\}$ by $\mathcal{Z}_r(A)/\mathcal{Z}_r(B)$ given in Theorem 1.2.) Our construction extends that of [6] in the case $A_\infty = \emptyset$, and refines the construction there by avoiding the use of the not-everywhere-defined Federer slice construction.

The condition imposed on a compact polyhedron A to be a compact oriented pseudo-manifold of dimension n implies that A has an orientation given by a fundamental class $0 \neq [A] \in H_n(A)$.

We repeat the definition of pseudo-manifold given in [6]. Since our definition requires a "resolution of the singularities, all of which are in codimension ≥ 2", this is somewhat stronger than that found elsewhere in the literature.

Definition 4.1. Let A be a compact connected polyhedron. A is said to be a **compact oriented pseudo-manifold** of dimension n if A admits a triangulation Δ satisfying:

- Every simplex of Δ is contained in the closure of some n-simplex $\tau \in \Delta(n)$.
- For some smooth closed oriented n-manifold M equipped with a smooth triangulation, there exists a polyhedral map $p : M \to A$ restricting to a homeomorphism $M - M' \to A - sk_{n-2}A$, where $M' \subset M$ is a subcomplex of dimension $\leq n - 2$.

If A_∞ is a nowhere dense subpolyhedron of the compact oriented pseudo-manifold A of dimension n, then its complement $A - A_\infty$ is said to be a **compactifiable oriented pseudo-manifold** of dimension n.

Example 4.2. The underlying analytic space $A = X^{an}$ of any connected complex quasi-projective variety X of complex dimension k is a compactifiable oriented pseudo-manifold of dimension $2k$.

If X is a projective variety of dimension n over $\mathbb{R}$ whose underlying analytic space X^{an} is compact and connected, then X^{an} is an compact oriented pseudo -manifold provided that X is smooth in codimension 1.

We proceed to construct the graph of a Lipschitz cocycle $f : A - O_\Delta(A_\infty) \to SP^d(S^m)$. As constructed in [6, 2.4], the geometric graph of f is the rectifiable current

geom
$$\Gamma(f) \ \equiv \ \sum_{\sigma \in \Delta'} \Gamma_\sigma \ \in \ \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m), \tag{11}$$

where the sum is indexed by (open) simplices of $A - O_\Delta(A_\infty)$ in a triangulation Δ' refining Δ with the property that $A - O_\Delta(A_\infty)$ is a subcomplex, and Γ_σ is the push-forward of the simplex σ (viewed as a rectifiable current on $A - O_\Delta(A_\infty)$) to the graph. Observe that this construction requires A to be provided with an orientation which is then inherited in a compatible way by each open simplex $\sigma \in \Delta'$.

There are two awkward aspects of this definition: even for A compact, $\Gamma(f)$ might not be a cycle and the function $f \ \mapsto \Gamma(f)$ might not be continuous from Lipschitz cocycles to Lipschitz cycles. These difficulties are overcome in [6] by restricting attention to the dense subset of "good" Lipschitz cocycles as we now recall.

Definition 4.3. [6, 3.2] Let B be a finite polyhedron. Choose a compact neighborhood U of $SP^d(S^m) \subset \mathbb{R}^N$ and a Lipschitz retraction $\pi : U \to SP^d(S^m)$ such that $\pi^{-1}(\Sigma)$ is a subcomplex of codimension ≥ 1, where $\Sigma \subset SP^d(S^m)$ is the singular set. A Lipschitz map $f : B \to SP^d(S^m)$ is said to be **good** if f is of the form $f = \pi \circ \tilde{f}$ where $\tilde{f}$ when restricted to each open simplex of B is smooth and transverse to every open simplex of $\pi^{-1}(\Sigma) \subset U$.

The following lemma is an immediate consequence of [6, 3.4] and Kirszbraun's Theorem (Theorem 2.3).

dense **Lemma 4.4.** *Let A be a compact oriented pseudo-manifold of dimension n, Δ a triangulation of A, and $A_\infty \subset A$ a subpolyhedron, and Δ' a refinement of Δ such that $A - O_\Delta(A_\infty)$ is a subcomplex with respect to Δ'. Then the subspace*

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \ \subset \ Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$$

of maps $f : A - O_\Delta(A_\infty) \to SP^d(S^m)$ which admit an extension to a good (with respect to Δ') Lipschitz map $\tilde{f} : A \to SP^d(S^m)$ is dense.

Arguing as in [6, 3.6], we obtain the following graphing construction. We remind the reader that the smash product $T_+ \wedge S^m$ of a (non-pointed) space T and the pointed m-sphere (with base point chosen to be ∞ when we view S^m as the 1-point compactification of $\mathbb{R}^m$ for $m > 0$ is given by

$$T_+ \wedge S^m \ \equiv \ (T \times S)^m)/(T \times \{\infty\}).$$

Theorem 4.5. *Let A be a compact, oriented pseudo-manifold of dimension n, Δ a (piece-wise linear) triangulation of A, and $A_\infty, C \ \subset A$ a subpolyhedron. There is a uniquely defined continuous extension*

gtop
$$\Gamma^{top} : \mathcal{Z}^m(A - O_\Delta(A_\infty)) \ \longrightarrow \ \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m), \tag{12}$$

of the geometric graph construction (11) sending $f \in Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good}$ *to the projection of* $\Gamma(f) \in \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m)$.

As Δ *varies over finer triangulations of* A, *these maps determine the continuous graphing map*

open
$$\Gamma^{top} : \mathcal{Z}^m(A - A_\infty) \longrightarrow \mathcal{Z}_n(\widetilde{A \wedge S^m}) / \varinjlim_{\Delta} \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m). \tag{13}$$

Proof. As constructed in (11), $\Gamma(f)$ is a rectifiable current on $A - O_\Delta(A_\infty)$ for any $f \in Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good}$. The continuity of this graphing construction

$$\Gamma : Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \longrightarrow \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m)$$

is given by [6, 3.5].

As verified in [6, 3.6], $\Gamma(\tilde{f})$ is an integral n-cycle on $A \times S^m$ for any good Lipschitz map $\tilde{f} : A \to SP^d(S^m)$. Since the restrictions of $\Gamma(f)$ and $\Gamma(\tilde{f})$ agree on any open inside $A - O_\Delta(A_\infty)$, we see that the boundary of $\Gamma(f)$ is supported on $D_\Delta(A_\infty) \times S^m$.

The push-forward of (rectifiable) currents via the proper Lipschitz map $(A - O_\Delta(A_\infty)) \times S^m \to (A - O_\Delta(A_\infty))_+ \wedge S^m$ is that given by Federer in [2, 4.1.7]. Thus, for each $d \geq 0$, we obtain continuous graphing maps

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \to \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m).$$

Using Lemma 4.4, we extend this graphing construction to Γ^{top} on $Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$ exactly as in the proof of [6, 3.6], and then use the universal property of group completion to obtain the asserted map (12).

Finally, the graphing map Γ^{top} on $\mathcal{Z}^m(A - A_\infty)$ of (13) is the (inverse) limit indexed by triangulations Δ of these maps ☐

Remark 4.6. Observe that the map in homotopy induced by (13) has the form

$$H^{m-j}(A - A_\infty) \longrightarrow H^{BM}_{n-m+j}((A - A_\infty)_+ \wedge S^n),$$

where $H^{BM}_*(-)$ denotes Borel-Moore (singular) homology.

As we see in the following proposition, Γ^{top} is compatible with localization. Recall the definition of Lipschitz cocycles $\mathcal{Z}^m_{C - C_\infty}(A - A_\infty)$ given in Definition 4.

loc
Proposition 4.7. *(compatibility with localization) Choose a finite triangulation* Δ *of* A *such that the compact subpolyhedra* A_∞, $C \subset A$ *are subcomplexes such that* $D_\Delta(A_\infty) \cap C = \emptyset$. *Then* Γ^{top} *of (12) restricts to a continuous homomorphism on relative Lipschitz cocycles*

rel
$$\Gamma^{top} : \mathcal{Z}^m_C(A - O_\Delta(A_\infty)) \longrightarrow \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m) \tag{14}$$

which fits in a map of fibration sequences

rel1
$$\begin{tikzcd}
\mathcal{Z}^m_{C-C_\infty}(A - O_\Delta(A_\infty)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_\infty)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_\infty \coprod C)) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \mathcal{Z}_n(D_A(A_\infty \coprod C)_+ \wedge S^m).
\end{tikzcd} \tag{15}$$

Proof. The fact that Γ^{top} restricts to (14) on relative Lipschitz cocycles follows from the observation that if $\tilde{f},\ \tilde{f}' \in Map^{good}_{Lip}(A, SP^d(S^m))$ have equal restrictions to $A - O_\Delta(A_\infty \coprod C)$, then $\Gamma(f) - \Gamma(g)$ has support on $D_\Delta(A_\infty \coprod C) \times S^m$. Thus, the upper square commutes by construction. The naturality of (13) implies the commutativity of the lower square of (14). Both columns are fibration sequence because they are short exact sequences of topological groups (cf. [8]). $\square$

Similarly, we see that Γ^{top} is compatible with Mayer-Vietoris.

MV-comp **Proposition 4.8.** *(compatibility with Mayer-Vietoris) Let Δ be a finite triangulation of A, and consider closed subpolyhedra $A_1,\ A_2 \subset A$ with $A_1 \cap A_2 = A_{1,2}$ and $A_{1+2} = A_1 \cup A_2$. Then Γ^{top} determines a map of* **Mayer-Vietoris** *fibration sequences*

$$\begin{tikzcd}
\mathcal{Z}^m(A - O_\Delta(A_{1,2})) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(\widetilde{A_+ \times S^m})/\mathcal{Z}_n(D_\Delta(A_{1,2})_+ \wedge S^m) \arrow[d] \\
\times_{i=1,2}\mathcal{Z}^m(A - O_\Delta(A_i)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \times_{i=1,2}\mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_i)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_{1+2})) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_{1+2})_+ \wedge S^m)
\end{tikzcd}$$

Proof. The vertical columns are short exact sequences of topological groups. The squares commute by construction. $\square$

cor:supportsMV-comp **Corollary 4.9.** *Let A be a compact pseudomanifold with finite triangulation Δ, and consider closed subpolyhedra $A_1,\ A_2 \subset A$ with $A_1 \cap A_2 = A_{1,2}$ and $A_{1+2} = A_1 \cup A_2$. Then Γ^{top} determines a map of* **Mayer-Vietoris** *fibration sequences*

$$\begin{tikzcd}
\mathcal{Z}^m_{A_{12}}(A) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(D_\Delta(A_{12})_+ \wedge S^m) \arrow[d] \\
\times_{i=1,2}\mathcal{Z}^m_{A_i}(A) \arrow[r, "\Gamma^{top}"] \arrow[d] & \times_{i=1,2}\mathcal{Z}_n(D_\Delta(A_i)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m_{A_{1+2}}(A) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(D_\Delta(A_{1+2})_+ \wedge S^m)
\end{tikzcd}$$

Proof. The left column is a fibration sequence by Corollary 3.5. The right column is a short exact sequence of topological groups. The diagram commutes by construction. $\square$

sec:duality

5. Poincaré duality

We will prove a version of Alexander (or, Poincaré-Lefschetz) duality, showing that the graphing construction of Section 4 provides a weak equivalence

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \ \rightarrow\ \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

where A is a (compact, or compactifiable) oriented manifold of dimension n and $C \subset A$ is a closed subpolyhedron that is a subcomplex with respect to some finite triangulation of A. As in other proofs of duality, the basic case that needs to be checked by hand occurs when C is a point, or more generally, a (sufficiently small) simplex.

duality:point

Example 5.1. Let A be a smooth compact oriented manifold of dimension n with a triangulation Δ with the property that for every closed simplex B of Δ, $D_\Delta(B)$ is contained in a Euclidean neighborhood, and let $C \subset A$ be a (closed) simplex. Let $m \geq 0$. Then the graphing map

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \ \to \ \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Indeed, using Proposition 2.2 to identify $\pi_i(\mathcal{Z}^m_C(A))$ with $H^{m-i}(A, A-C)$ and Theorem 1.2 to identify $\pi_i(\mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m))$ with $H_{n+i}(D_\Delta(C) \times S^m, D_\Delta(C) \times \infty))$, we conclude that both groups are zero unless $m = n$ and $i = 0$, when the homomorphism $\Gamma^{top}_\#$ induced by Γ^{top} is of the form $\mathbb{Z} \to \mathbb{Z}$. The generator of the source here is the collapsing map

$$A \to A/(A - O_\Delta(C)) \equiv S^n$$

defining the orientation, minus the constant map to ∞. This is a difference of good Lipschitz maps, and the resulting class in homology is the class given by the difference of the collapsing map and the constant map on the boundary, i.e., it is a generator. This shows that $\Gamma^{top}_\#$ is onto, and hence an isomorphism, as asserted.

Now a bootstrap argument allows to realize Poincaré - Lefschetz duality as the map in homotopy of a map of topological abelian groups.

duality:lefschetz

Theorem 5.2. *Let A be a smooth compact oriented manifold which admits a triangulation and let $C \subset A$ be a compact subspace that is also a subcomplex for some finite triangulation of A. Then for a sufficiently fine triangulation Δ of A such that C is a subcomplex the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \ \to \ \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Proof. We proceed by induction of the dimension d of C and the number r of simplices of C. If $d < 0$, then there is nothing to prove. If $r = 1$, then C is a point, and we are done by Example 5.1. Now suppose $d \geq 0$, $r > 1$, and we have proved the assertion for all closed subcomplexes C' with at most $r-1$ simplices or of dimension less than d. Let K be a subcomplex of C with $r-1$ simplices, and let B be the closed simplex such that $C = K \cup B$. There are two cases.

First Case: B is disjoint from K. Then the inductive step follows from the obvious fact that $\mathcal{Z}^m_C(A) \equiv \mathcal{Z}^m_K(A) \times \mathcal{Z}^m_B(A)$.

Second Case: B is not disjoint from K. Then $B \cap K = \partial B$. Since the dimension of ∂B is less than d, the inductive hypothesis implies that our assertion holds for the subcomplexes B, K and ∂B. Now the Mayer-Vietoris sequence of Corollary 3.5 and the compatibility of graphing and Mayever-Vietoris given in Corollary 4.9 complete the inductive step. □

Remark 5.3. While it may look as if neither orientability nor smoothness is ever used in the proof of Theorem 5.2, note that the assumptions are needed for the graphing construction to work, as discussed in Section 4.

As a corollary, we obtain Poincaré Duality in the form stated as Theorem 0.1 in the introduction.

`duality:poincare` **Corollary 5.4.** *Let A be a smooth compact oriented manifold of dimension n. Then the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m(A) \;\to\; \mathcal{Z}_n(A_+ \wedge S^m)$$

is a weak equivalence.

Proof. Choose a sufficiently fine triangulation, and let $C = A$ in Theorem 5.2. $\square$

`duality:open` **Corollary 5.5.** *Let A be a smooth oriented compact manifold, and let $C \subset A$ be a compact subspace that is a subcomplex with respect to some finite triangulation. Then the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m(A - C) \longrightarrow \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \varinjlim_{\Delta} \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Proof. This follows easily from the preceeding results and the compatibility of graphing with localization proved in Proposition 4.7 (taking $A_\infty = \emptyset$ in (15)). $\square$

`def:gysin` **Definition 5.6.** Assume A is a compact, oriented n-manifold and $i : A_\infty \subset A$ is a closed, compact, oriented submanifold of codimension e. For any $m \geq e$, we refer to the homotopy class of maps

`gysin` (16)

$$i^! = (\Gamma^{top})^{-1} \circ (i_* \wedge \Sigma^e) \circ \Gamma^{top} : \mathcal{Z}^{m-e}(A_\infty) \to \mathcal{Z}_{n-m}((A_\infty)_+ \wedge S^{n-e}) \to \mathcal{Z}_n(A_+ \wedge S^m) \to \mathcal{Z}^m(A)$$

as the **Gysin map**.

Remark 5.7. The map on homotopy groups induced by the Gysin map,

$$i^! : H^{m-j-e}(A_\infty) = \pi_j(\mathcal{Z}^{m-e}(A_\infty)) \;\to\; \pi_j(\mathcal{Z}^m(A)) = H^{m-j}(A)$$

(isomorphic to $H_{n+j-m}(A_\infty) \to H_{n+j-m}(A)$), is the Poincaré dual of $i^* : H^\bullet(A) \to H^\bullet(A_\infty)$.

`sec:thom`

6. Thom Classes and Thom Isomorphism

In this last section we formulate the Thom isomorphism in our context and relate it to the Gysin map of Definition 5.6.

`disk` **Proposition 6.1.** *Let A be a compact, oriented pseudo-manifold of dimension n and let $C \subset A$ be a closed submanifold of codimension e, smoothly embedded in the smooth locus of A. Let Δ be a triangulation of A such that C is a subpolyhedron. Then the retraction $p : D_\Delta(C) \to C$ is an oriented disk bundle, called the* **oriented normal disk bundle** *of $C \subset A$. In other words, C admits an open covering $\{U_i\}$ such that each restriction $p_{|U_i} : p^{-1}(U_i) \to U_i$ is homeomorphic to the product projection $D^e \times U_i \to U_i$ and such that $p_{|U_i}, p_{|U_j}$ are related by a continuous maps from $U_i \cap U_j$ to the group of continuous, orientation-preserving, origin-preserving homeomorphisms of the disk D^e.*

Let B be a compact polyhedron and suppose $\mathbf{D} \to B$ is an oriented disk bundle of rank e over B, given a structure of a compact polyhedron such that B is embedded as a subpolyhedron via the zero section and such that the associated sphere bundle $\mathbf{S} \to B$ is also a subpolyhedron.

def:thomclass **Definition 6.2.** A (geometric) *Thom class* of $\mathbf{D} \to B$ of an oriented disk bundle over a compact polyhedron B is an element $t \in \mathcal{Z}^e(\mathbf{D}, \mathbf{S})$ such that the restriction t_x of t to any fiber $(\mathbf{D}_x, \mathbf{S}_x)$ over a point $x \in B$ defines the given orientation of the disk $\mathbf{D}_x$, that is, t_x is in the same connected component of $\mathcal{Z}^e(\{x\})$ as the difference of the collapsing map $(\mathbf{D}_x, \mathbf{S}_x) \to (S^e, \infty)$ and the constant map to ∞. We write $T(\mathbf{D})$ for the subspace in $\mathcal{Z}^e(\mathbf{D}, \mathbf{S})$ of all geometric Thom classes.

rem:trivialthom **Remark 6.3.** Let B be a contractible polyhedron and $(\mathbf{D}, \mathbf{S}) = B \times (D^e, S^{e-1})$ be the trivial disk and sphere bundles over B, with a choice of orientation. Then there is an obvious geometric Thom class defined as the difference of the projection to (D^e, S^{e-1}) followed by the collapsing map to S^e and the constant map to $\infty \in S^e$. Since $\mathcal{Z}^e(B \times D^e, B \times S^{e-1})$ is homotopy equivalent to $\mathbb{Z}$, the space $T(\mathbf{D})$ is contractible.

thm:thomclass **Theorem 6.4.** *For any compact polyhedron B and oriented disk bundle $(\mathbf{D}, \mathbf{S})$ as above, the space of geometric Thom classes $T(\mathbf{D})$ is contractible, and in particular, non-empty.*

Proof. By induction of the dimension of B. It follows immediately from Remark 6.3 that the assertion is true if the dimension of B is zero.

Suppose now that $\dim(B) = n$, and the assertion has been proved for all polyhedra of dimension less than n. Let $B^{(n-1)}$ be the $(n-1)$-skeleton of B, B_n the disjoint union of the n-simplices of B, and let ∂B_n be the union of the boundaries. Moreover, let $\mathbf{D}^{(n-1)}$, $\mathbf{D}_n$ and $\partial \mathbf{D}_n$ be the respective restrictions of the oriented disk bundle $\mathbf{D}$, and similarly for the sphere bundle $\mathbf{S}$. We may choose embeddings of $\mathbf{D}$, and of the union $\coprod_n \mathbf{D}_n$ as neighborhood retracts such that all the attaching maps and characteristic maps are Lipschitz continuous. In this way, we obtain a commutative square

Tsquare (17)
$$\begin{tikzcd} T(\mathbf{D}) \arrow[r] \arrow[d] & T(\mathbf{D}^{(n-1)}) \arrow[d] \\ T(\mathbf{D}_n) \arrow[r] & T(\partial \mathbf{D}_n) \end{tikzcd}$$

and the assertion follows from:
Claim: The square (17) is homotopy cartesian.

Indeed, it is clear that this is a cartesian square; therefore it suffices to show that one of the maps, say, $T(\mathbf{D}_n) \to T(\partial \mathbf{D}_n)$, is a (Serre) fibration. Since T obviously transforms finite disjoint unions into finite products, we may assume that there is only one n-simplex. Further, we can choose a Lipschitz continuous trivialization of the disk bundle $\mathbf{D}_n \to B_n$. Hence, we only need to prove that the map $T(\Delta^n \times D^e) \to T(\partial \Delta^n \times D^e)$ induced by restriction is a Serre fibration. Take a commutative diagram

fibsquare (18)
$$\begin{tikzcd} D^k \arrow[r, "f"] \arrow[d] & T(\Delta^n \times D^e) \arrow[d] \\ D^k \times I \arrow[r, "g"] & T(\partial \Delta^n \times D^e) \end{tikzcd}$$

Recall that an element $\phi \in T(\Delta^n \times D^e)$ is simply a difference of two Lipschitz continuous maps ϕ_+ and ϕ_- from $\Delta^n \times D^e$ to some symmetric powers of S^e that become equal on the sphere bundle $\Delta^n \times S^{e-1}$. The maps f and g in the square (18) have compact source and can therefore be taken to correspond to continuous families of Lipschitz maps $G_+ - G_-$ and $F_+ - F_-$ where the $+$-maps have target in some symmetric power $SP^r(S^e)$ and the $-$-maps have target in some symmetric power $SP^s(S^e)$, and such that the restrictions of G_+ and F_+ (respectively, G_- and F_-) to $D^k \times \partial\Delta^n \times D^e$ coincide.

Now we can glue G_+ and F_+ (respectively, G_- and F_-) along $D^k \times \partial\Delta^n \times D^e$ to obtain a continuous family of Lipschitz maps

$$\Phi_+ = G_+ \cup F_+ : D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \to SP^r(S^e)$$

and similarly a continuous family of Lipschitz maps

$$\Phi_- = G_- \cup F_- : D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \to SP^s(S^e)$$

with the property that $\Phi_+ - \Phi_-$ is everywhere locally on $D^k \times I \times \partial\Delta^n \cup D^k \times \Delta^n$ the orientation of the trivial e-disk bundle.

The inclusion (which is the identity on the fibers of the e-disk bundle)

$$D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \hookrightarrow D^k \times I \times \Delta^n \times D^e$$

is Lipschitz homeomorphic to an inclusion $D^{n+k} \times D^e \hookrightarrow D^{n+k} \times I \times D^e$ that is the identity on the fibers of the (trivial) e-disk bundle. Along the latter inclusion, we can extend Φ_+ and Φ_- constantly along the I-direction to families of Lipschitz maps $\tilde{\Phi}_+$ and $\tilde{\Phi}_-$. Clearly, on each fiber of the e-disk bundle, $\tilde{\Phi}_+ - \tilde{\Phi}_-$ defines the orientation of the disk D^e. That is, we have constructed a lift $\tilde{\Phi} : D^k \times I \to T(\Delta^n \times D^e)$ of the diagram (18), as needed. □

Now that we have defined the Thom classes, we can easily prove the Thom isomorphism theorem.

thomclass **Theorem 6.5.** *Let B be a compact polyhedron, $\mathbf{D} \to B$ an oriented disk bundle and $t \in T(\mathbf{D})$ a geometric Thom class. Then multiplication by t defines a weak equivalence, independent up to homotopy of the choice of t.*

$$t : \mathcal{Z}^i(B) \to \mathcal{Z}^{e+i}(\mathbf{D}, \mathbf{S})$$

Proof. In the case that the disk bundle is trivial, multiplication by t is the suspension isomorphism. The general case follows using Mayer-Vietoris and the 5-Lemma. □

The following Thom Isomorphism theorem follows easily from Theorem 6.5.

tau **Theorem 6.6.** *(Thom isomorphism) Let A be a compact polyhedron and $C \subset A$ a closed subpolyhedron of constant codimension $e > 0$ admitting an oriented normal disk bundle in A (as in Proposition 6.1, for example). Then any geometric Thom class $t(C \subset A, \Delta)$ determines a class $\tau_C \in Z^e_C(A)$. Moreover, multiplication by such a class*

$$\tau_C : \mathcal{Z}^i(C) \to \mathcal{Z}^{e+i}_C(A)$$

is a weak equivalence, independent up to homotopy of the choice of τ_C.

Proof. The class $t_C = t(C \subset A, \Delta)$ lies in the kernel of $\mathcal{Z}^e(D_\Delta(C)) \to \mathcal{Z}^e(S_\Delta(C))$. Hence, we may extend t_C by 0 on $A - O_\Delta(C)$, obtaining the class τ_C in $\mathcal{Z}^e(A)$ which vanishes off $O_\Delta(C)$; in other words,

$$\tau_C \ \in \ \ker\{\mathcal{Z}^e(A) \to \mathcal{Z}^e(A - O_\Delta(C)\} \ = \ \mathcal{Z}_C(A).$$

We now apply the excision property of Proposition 3.6 to identify the Thom isomorphism of Theorem 6.5 with multiplication by τ_C. □

We conclude with the following theorem relating the Gysin map of Definition 5.6 and multiplication by the Thom class of the normal disk bundle of a closed embedding of compact smooth manifolds.

Theorem 6.7. *Let A be a compact, connected pseudo-manifold of dimension n and let $C \subset A$ be a smooth closed embedding of codimension $e \geq 1$ of the smooth manifold C missing the singular locus of A. Then the following square is commutative in the homotopy category*

compat

$$\begin{tikzcd} \mathcal{Z}^{m-e}(C) \arrow[r, "\tau_C"] \arrow[d, "\Gamma^{top}"] & \mathcal{Z}^m_C(A) \arrow[d, "p_* \circ \Gamma^{top}"] \\ \mathcal{Z}_{n-e}(C_+ \wedge S^{m-e}) \arrow[r, "\Sigma^e"] & \mathcal{Z}_n(C_+ \wedge S^m). \end{tikzcd} \tag{19}$$

Here the right vertical map is the composition $p_* \circ \Gamma^{top}$ of the pushforward along the retraction $D_\Delta(C)_+ \to C_+$ with the graphing map $\Gamma^{top} : \mathcal{Z}^m_C(A) \to Z_n(D_\Delta(C)_+ \wedge S^m)$.

Proof. By localization, we may as well assume that A is a manifold. Employing the tubular neighborhood theorem and using localization once more, we may in fact assume that A is of the form $\mathbb{P}(E \oplus \mathbf{1})$ for some oriented vector bundle E (namely, the normal bundle) and that the emebedding $C \to A$ is the inclusion $i : C = \mathbb{P}(\mathbf{1}) \to \mathbb{P}(E \oplus \mathbf{1})$. Let $\pi : A \to C$ the projection of the projective space bundle. Note also that all groups in the diagram are contractible if $e > m$, so we may as well assume that $m \geq e$.

Our proof will be in two steps. First, we will prove the assertion under the assumption that the normal bundle E is trivial. This case is essentially obivous, once we untangle the definition of the various maps involved. Then, we will repeat the matching argument from the proof of Theorem 6.4 in our context, imposing the additional condition in each step that diagram (19) commute. Some care is required to ensure that the graphing map is defined on each piece.

Step 1: Assume that the vector bundle $E \to C$ is the trivial rank e bundle, choose a triangulation Δ with the property that i and π are simplicial maps and let $\mathbf{D} = D_\Delta(C)$ (resp., $\mathbf{S} = D_\Delta(C) - O_\Delta(C)$) be the associated disk bundle and sphere bundle, respectively. By Kirszbraun's theorem, the natural map of topological abelian groups

$$\mathcal{Z}^m(\mathbf{D}, \mathbf{S}) \to \mathcal{Z}^m(A, A - O_\Delta(C))$$

is an isomorphism for all m. Moreover, the argument of the proof of Proposition 4.7 applies to give the graphing map

$$\Gamma^{top} : \mathcal{Z}^m(\mathbf{D}, \mathbf{S}) \to \mathcal{Z}_n(\mathbf{D}_+ \wedge S^m).$$

Let t be the "obvious" geometric Thom class of Remark 6.3 for the oriented disk bundle $\mathbf{D}$. We need to show that the following diagram commutes for all $m \geq e$:

compat:trivial

$$\begin{tikzcd} \mathcal{Z}^{m-e}(C) \arrow[r, "t"] \arrow[d, "\Gamma^{top}"] & \mathcal{Z}^m(\mathbf{D}, \mathbf{S}) \arrow[d, "\pi_* \circ \Gamma^{top}"] \\ \mathcal{Z}_{n-e}(C_+ \wedge S^{m-e}) \arrow[r, "\Sigma^e"] & \mathcal{Z}_n(C_+ \wedge S^m). \end{tikzcd} \tag{20}$$

In fact, let $f : C \to SP^r(S^{m-e})$ be a good Lipschitz map. Then the product $f \cup t \in \mathcal{Z}^m(\mathbf{D}, \mathbf{S})$ is represented by the difference of

$$F_+ : \mathbf{D} = C \times D^e \to SP^r(S^m)$$

and

$$F_- : \mathbf{D} = C \times D^e \to SP^r(S^m)$$

where $F_+(c,x) = f(c) \wedge \overline{x}$ with $\overline{x} \in S^e$ the image of x under the collapsing map, and $F_-(c,x)$ is simply r times the constant map to the point. Note that F_+ is the smash product of a good Lipschitz map and a simplicial map; hence the graph $\Gamma^{top}(F_+)$ is obtained as the actual geometric graph of this multi-valued map. Trivially, the same is true for the constant map.

Now the current $\mathfrak{c} = \pi_*(\Gamma^{top}(F_+) - \Gamma^{top}(F_-))$ acts as follows: let ω be a differential form on $C \times S^{m-e} \times S^e$ (which we view as a "differential form" on $C_+ \wedge S^m$; note we can use actual differential forms, since $C \times S^{m-e} \times S^e$ is, in fact, a smooth manifold). Use a normalized product measure

$$\mu = \mu_1 \times \mu_2 \times \mu_3 \times \mu_4$$

on $C \times D^e \times S^{m-e} \times S^e$. Then

$$\int_{\mathfrak{c}} \omega d(\mu_1 \times \mu_3 \times \mu_4) \tag{21}$$

$$= \int_{\Gamma^{top}(F_+)} \pi^* \omega d\mu - \int_{\Gamma^{top}(F_-)} \pi^* \omega d\mu \tag{22}$$

$$= \int_{\Gamma^{top}(F_+)} \pi^* \omega d\mu - r \int_{C \times D^e \times \{\infty\} \times \{\infty\}} \pi^* \omega d\mu. \tag{23}$$

Observe that the F_--term vanishes, since the corresponding current is supported in a set of dimension $n < n+m-(m-e)$, which has measure 0. The remaining term is obtained by first integrating $\pi^*\omega$ over the set of points of the form $(c, x, f(c), \overline{x})$ (where $f(c)$ is any of the multiple values of f at c). This can be realized by first integrating $\pi^*\omega$ along the fibers $S^e = \{(x, \overline{x}) | x \in D^e\} \subset D^e \times S^e$, and then integrating the resulting form $\int_{S^e} \pi^*\omega$ over the graph of f. In turn, that is equivalent to integrating ω itself first along the fibers S^e, and then over the graph of f.

On the other hand, for any current $\mathfrak{a}$ on $C \times S^{m-e}$, the current $\Sigma^e(\mathfrak{a})$ is obtained by integrating a form ω on $C \times S^{m-e} \times S^e$ first along the fibers S^e and then integrating the resulting form $\int_{S^e} \omega$ over the current $\mathfrak{a}$.

That is, diagram (20) is in fact a strictly commutative diagram of spaces.

Step 2: To be done.

□

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