COHOMOLOGY OF BIFUNCTORS

VINCENT FRANJOU* AND ERIC M. FRIEDLANDER**

Abstract. We initiate the study of the cohomology of (strict polynomial) bifunctors by introducing the foundational formalism, establishing numerous properties in analogy with the cohomology of functors, and providing computational techniques. Since one of the initial motivations for the study of functor cohomology was the determination of $H^*(GL(k), S^*(g\ell) \otimes \Lambda^*(g\ell))$, we keep this challenging example in mind as we achieve numerous computations which illustrate our methods.

Contents

0. Introduction

We fix a prime p, a base field k of characteristic p, and consider the category $\mathcal{V}$ of finite dimensional k-vector spaces and k-linear maps. The study of the cohomology of categories of functors from $\mathcal{V}$ to k-vector spaces has had numerous applications, including insight into the structure of modules for the Steenrod algebra [6] and

Date: March 27, 2006.
2000 *Mathematics Subject Classification.* 20J06, 18G40.
* partially supported by UMR 6629 CNRS/UN.
** partially supported by the NSF.

proof of finite generation of the cohomology of finite group schemes [8]. The computational power of functor cohomology arises as follows: the abelian category of strict polynomial functors $\mathcal{P}$ of bounded degree enjoys many pleasing properties which lead to various cohomological computations (cf. [9]); this cohomology for the category $\mathcal{P}$ is closely related to the cohomology for the abelian category of all functors $\mathcal{F}$ provided that our base field k is finite; for k finite, the cohomology of finite functors $F \in \mathcal{F}$ is equal to the stabilized cohomology of general linear groups with coefficients determined by F.

On the other hand, many natural coefficients modules for the general linear group are not given by functors but by bifunctors (contravariant in one variable, covariant in the other variable). Initially motivated by the quest to determine the group cohomology $H^*(GL(n,\mathbb{Z}/p^2),k)$, efforts have been made to compute the cohomology of $GL(n,k)$ with coefficients in symmetric and exterior powers of the adjoint representation $g\ell_n$ (cf. [4]). These coefficients are not given by functors but by bifunctors. In this paper, we provide computational tools and first computations towards the determination of the stable (with respect to n) values of $H^*(GL(n,k),S^d(g\ell))$ and $H^*(GL(n,k),\Lambda^d(g\ell))$.

Our first task is to formulate in terms of Ext groups in the category $\mathcal{P}^{op}\times\mathcal{P}$ of strict polynomial bifunctors the stable version of rational cohomology of the algebraic group GL with coefficients determined by the given bifunctor. In Theorem 1.5, we show that

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d(g\ell),T) \;\cong H^*_{rat}(GL_n,T(k^n,k^n))$$

where T is a strict polynomial bifunctor of homogeneous bidegree (d,d) with $n\geq d$. In the special case that T is of the form $A(g\ell)$ (for example, $S^d(g\ell)$), we write this as

$$H^*_{\mathcal{P}}(GL,A) \;\cong H^*_{rat}(GL_n,T(k^n,k^n)).$$

As for rational cohomology, the most relevant coefficients are given by beginning with a strict polynomial bifunctors T and applying the Frobenius twist operation (i.e., $I^{(1)}\circ(-)$) sufficiently often until the Ext-group of interest stabilizes. This "generic" strict polynomial bifunctor cohomology is our main target of computations.

In [8], the fundamental computation of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)},I^{(r)})$ is achieved, modeled on the computation of $\mathrm{Ext}^*_{\mathcal{F}}(I,I)$ in [6]. For bifunctor cohomology, the computation of

$$H^*_{\mathcal{P}}(GL,\otimes^{n(r)}) \equiv \mathrm{Ext}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{np^r},\mathcal{H}om(I^{(r)},I^{(r)})^{\otimes n}) \quad (0.0.1)$$

given in Theorem 1.8 plays an analogous role.

We prove various useful formal results concerning bifunctor cohomology. For example, in §2 we relate the strict polynomial bifunctor cohomology with coefficients in a functor of separable type (i.e., of the from $\mathcal{H}om(A,B)$ where A,B are strict polynomial functors) to Ext computations in the category $\mathcal{P}$. In §3, we establish a base change result (one of the important advantages of strict polynomial functors/bifunctors in contrast to "ususal" functors/bifunctors) and a twist stability theorem; both results follow from analogus results proved in [9] for Ext-groups in the category $\mathcal{P}$.

In §4, we consider bifunctor cohomology such as $H^*_{\mathcal{P}}(GL,S^d(g\ell^{(r)}))$ where d is less than p. This is in principle completely computable thanks to (1.8.1). However, computations for $p\leq d$ would appear to be much more difficult. In §5, we work out the case $p = 2 = d$, a computation which is quite involved. The applicability of the computation of §5 is extended in §6.

In the remaining two sections, we relate our computations of strict polynomial bifunctor cohomology to the cohomology of the finite groups $GL(n, k)$ where k is a finite field of characteristic p. In §7, we develop sufficient formalism for bifunctor cohomology to enable comparison of this bifunctor cohomology with both the cohomology of strict polynomial bifunctors and with group cohomology. Many explicit computations of group cohomology are presented in §8.

The first author gratefully acknowledges the hospitality of Northwestern University where this project was initiated. Both authors thank l'Institut Henri Poincaré for providing a working environment which facilitated the completion of this paper.

1. Rational cohomology of GL as Ext of strict polynomial bifunctors

We fix a prime p and a field k of characteristic p. Let $\mathcal{V}$ denote the category of finite dimensional k-vector spaces and k-linear maps. In this first section, we introduce the category of strict polynomial bifunctors and relate Ext-groups in this category with the rational cohomology of the (infinite) general linear group. The reader should be aware that our bifunctors are contravariant in the first variable, covariant in the second; bifunctors covariant in each variable were considered for example in [14].

In [8], the concept of a strict polynomial functor $T : \mathcal{V} \to \mathcal{V}$ was introduced, a modification of the usual notion of a functor from $T : \mathcal{V} \to \mathcal{V}$, formulated so that the action of $GL(W)$ on $T(W)$ is a rational representation for each $W \in \mathcal{V}$. Thus, T consists of the data of an association of $T(W) \in \mathcal{V}$ for each $W \in \mathcal{V}$, and a polynomial mapping $\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$ for each pair $V, W \in \mathcal{V}$. [By definition, a polynomial mapping $V \to W$ between finite dimensional k-vector spaces is an element in $S^*(V^{\#}) \otimes W$, an element in the tensor product of the symmetric algebra on the k-linear dual of V with W.] A strict polynomial functor is said to be homogeneous of degree d if for all pairs $V, W \in \mathcal{V}$ the polynomial mapping $\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W)$ is of degree d (i.e., is an element of $S^d(\mathrm{Hom}_k(V, W)^{\#}) \otimes \mathrm{Hom}_k(T(V), T(W))$.

The abelian category of strict polynomial functors of bounded degree, denoted $\mathcal{P}$, is a direct sum of subcategories $\mathcal{P}_d$ of strict polynomial functors homogeneous of degree d, $\mathcal{P} = \oplus_d \mathcal{P}_d$.

We recall a few examples of strict polynomial functors: the functors

$$\otimes^d : \mathcal{V} \to V, \quad S^d : \mathcal{V} \to \mathcal{V}, \quad \Gamma^d : \mathcal{V} \to \mathcal{V}, \quad \Lambda^d : \mathcal{V} \to \mathcal{V}$$

are each strict polynomial functors homogeneous of degree d. Here, the d-th symmetric power $S^d(V)$ of $V \in \mathcal{V}$ is the vector space of coinvariants of Σ_d acting on $\otimes^d(V) = V^{\otimes d}$, whereas the d-th divided power $\Gamma^d(V)$ is the vector space of invariants of Σ_d acting on $\otimes^d(V)$. For any $W \in \mathcal{V}$ and $T \in \mathcal{P}_d$, as shown in [8, 2.10] there are natural identifications

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d(\mathrm{Hom}(W, -), T) \;\cong\; T(W) \;\cong\; \mathrm{Hom}_{\mathcal{P}_d}(T, S^d(\mathrm{Hom}_k(W, -)). \tag{1.0.2}$$

Thus, P_W, I_W, defined by

$$P_W \;=:\; \Gamma^d(\mathrm{Hom}_k(W, -)), \quad I_W =: \; S^d(\mathrm{Hom}_k(W, -)),$$

are respectively projective and injective objects of $\mathcal{P}_d$. Moreover, P_W is a projective generator of $\mathcal{P}_d$ provided that $\dim_k W \geq d$, since the natural map

$$T(W) \otimes \Gamma^d(\mathrm{Hom}_k(W, -), -) \to T$$

is surjective if $\dim_k W \geq d$. Another important strict polynomial functor, homogeneous of degree p^r, is the functor

$$I^{(r)} : \mathcal{V} \to \mathcal{V}$$

with the property that the structure polynomial maps are identified with the p^r-th power polynomial map.

Definition 1.1. *We consider the category $\mathcal{P}^{op} \times \mathcal{P}$ of strict polynomial bifunctors of bounded degree (contravariant in the first variable, covariant in the second).*

Thus, a strict polynomial bifunctor T is a pair of functions, the first of which assigns to each pair $V, W \in \mathcal{V}$ some $T(V, W) \in \mathcal{V}$ and the second of which assigns a polynomial map

$$\mathrm{Hom}_k(V, V') \otimes \mathrm{Hom}_k(W, W') \to \mathrm{Hom}_k(T(V', W), T(V, W')) \tag{1.1.1}$$

satisfying the condition that $T(V, -), T(-, W^\#)^\# : \mathcal{V} \to \mathcal{V}$ are strict polynomial functors for each $V, W \in \mathcal{V}$ of uniformly bounded degree. If $T \in \mathcal{P}_d^{op} \times \mathcal{P}_e$, then we say that T is homogeneous of bidegree (d, e).

To emphasize the functorial nature of $T \in \mathcal{P}^{op} \times \mathcal{P}$, we shall use at times alternate notations

$$T \;\equiv\; T(-,-) \;\equiv T(-_1, -_2).$$

Observe that if T is a strict polynomial bifunctor, then for any $W \in \mathcal{V}$ the action

$$GL(W) \times T(W, W) \;\to\; T(W, W), \quad (g, x) \mapsto T(g^{-1}, g)(x) \tag{1.1.2}$$

is *rational*. Namely, for any commutative k-algebra A, this action extends to an action $GL(A \otimes_k W) \otimes A \otimes_k T(W, W) \to A \otimes_k T(W, W)$.

If $T \in \mathcal{P}^{op} \times \mathcal{P}$ and if $P \in \mathcal{P}$, then the composite $P \circ T$ is once again a strict polynomial bifunctor. If T_1, T_2 are strict polynomial functors and F a strict polynomial bifunctor, then the composite $F(T_1(-), T_2(-))$ is once again a strict polynomial bifunctor.

For two strict polynomial functors A_1 and A_2, let $\mathcal{H}om(A_1, A_2)$ denote the strict polynomial bifunctor defined by

$$\mathcal{H}om(A_1, A_2)(V, W) = \mathrm{Hom}_k(A_1(V), A_2(W)), \quad V, W \in \mathcal{V}.$$

We refer to such functors as functors of *separable type*. One can readily verify the natural identification

(1.1.3)

$$\mathrm{Hom}_{\mathcal{P}^{op} \times \mathcal{P}}(\mathcal{H}om(A_1, A_2), \mathcal{H}om(B_1, B_2)) \;\cong\; \mathrm{Hom}_{\mathcal{P}}(B_1, A_1) \otimes \mathrm{Hom}_{\mathcal{P}}(A_2, B_2)$$

Proposition 1.2. *The category of strict polynomial bifunctors of bounded degree, $\mathcal{P}^{op} \times \mathcal{P}$ admits a decomposition $\mathcal{P}^{op} \times \mathcal{P} \simeq \bigoplus_{d,e} (\mathcal{P}_d)^{op} \times \mathcal{P}_e$; in other words, every strict polynomial bifunctor of bounded degree can be written naturally as a direct sum of strict polynomial bifunctors of homogeneous bidegree.*

Any $T \in \mathcal{P}^{op} \times \mathcal{P}$ homogeneous of bidegree (d, e) admits a projective resolution by (projective strict polynomial) bifunctors of the form

$$P^{d,e}_{V,W}(-_1, -_2) = \mathcal{H}om(I_V(-_1), P_W(-_2)) = \Gamma^d \mathrm{Hom}_k(-_1, W) \boxtimes \Gamma^e \mathrm{Hom}_k(V, -_2)$$

and an injective resolution by (injective strict polynomial) bifunctors of the form

$$I^{d,e}_{V,W}(-_1,-_2) = \mathcal{H}om(P_V(-_1), I_W(-_2)) = S^d\mathrm{Hom}_k(-_1,V) \boxtimes S^e\mathrm{Hom}_k(W,-_2).$$

Here $\boxtimes$ *is the external tensor product producing a bifunctor from a pair of functors.*

Proof. Direct sum decomposition of $\mathcal{P}^{op}\times\mathcal{P}$ is proved exactly as decomposition of $\mathcal{P}$ is proved in [8]. Using 1.0.2, we easily obtain the isomorphism

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(P^{d,e}_{V,W},T) \simeq T(V,W) \simeq \mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T, I^{d,e}_{V,W}) \tag{1.2.1}$$

natural with respect to $T\in(\mathcal{P}_d)^{op}\times\mathcal{P}_e$. In particular, each $P^{d,e}_{V,W}$ is projective and each $I^{d,e}_{V,W}$ is injective in $(\mathcal{P}_d)^{op}\times\mathcal{P}_e$. As argued in [8, 2.10], for $T\in(\mathcal{P}_d)^{op}\times\mathcal{P}_e$ there is a natural surjective map

$$P^{d,e}_{V,W}\otimes T(V,W)\to T$$

and a natural injective map

$$T\to I^{d,e}_{V,W}\otimes T(V,W)$$

whenever $\dim_k V\geq d, \dim_k W\geq e$. □

The strict polynomial bifunctor (of bidegree $(1,1)$)

$$g\ell \;\equiv\; \mathcal{H}om(I,I) \;\equiv\; \mathcal{H}om(-,-) \;\in\mathcal{P}^{op}\times\mathcal{P}$$

plays a special role as we first see in the following proposition.

Proposition 1.3. *Let* T *be a strict polynomial bifunctor homogeneous of bidegree* (d,d) *and* $W\in\mathcal{V}$ *satisfy* $\dim_k(W)\geq d$. *Then there is a natural identification*

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) \;\cong\; \mathrm{H}^0(GL(W), T(W,W)), \tag{1.3.1}$$

where $T(W,W)$ *is given the rational* $GL(W)$*-module structure of (1.1.2).*

Moreover, for any $A_1,A_2\in\mathcal{P}_d$, *there is a natural identification*

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, \mathcal{H}om(A_1,A_2)) \;\cong\; \mathrm{Hom}_{\mathcal{P}}(A_1,A_2). \tag{1.3.2}$$

Proof. The identifications

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, I^{d,d}_{U,V}) \;\cong\; \Gamma^d\mathrm{Hom}_k(U,V) \;\cong\; \mathrm{Hom}_{\mathcal{P}}(\Gamma^d\mathrm{Hom}_k(U,-), S^d\mathrm{Hom}_k(V,-)).$$

follow immediately from (1.2.1) and (1.0.2); in particular, this establishes the validity of (1.3.2) for bifunctors T of the form $I^{d,d}_{U,V}$. By [8, 3.13], the right hand side can be identified with $H^0(GL(W),-)$ applied to $I^{d,d}_{U,V}(W,W)$ for any $W\in\mathcal{V}$ with $\dim_k W\geq d$. This establishes (1.3.1) for bifunctors T of the form $I^{d,d}_{U,V}$.

For a general strict polynomial bifunctor T homogeneous of bidegree (d,d), we apply Proposition 1.2 to obtain an injective resolution of T by injectives of the form $I^{d,d}_{U,V}$. Thus, (1.3.1) follows by the left exactness of $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell,-)$ and $H^0(GL(W),-)$ together with the above verification for functors of the form $I^{d,d}_{U,V}$. Applying (1.3.1) and [8, 3.13] once again, we conclude (1.3.2) for T in full generality. □

The existence of enough injectives (and/or projectives) in $\mathcal{P}^{op}\times\mathcal{P}$ enables us to define Ext-groups in the evident manner.

Definition 1.4. *For strict polynomial bifunctors T_1, T_2 of bounded degree, we define the* Ext*-groups* $\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(T_1, T_2)$ *as the derived functors of* $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T_1, -)$ *applied to T_2 (or, the derived functors of* $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(-, T_2)$ *applied to T_1),*

$$\mathrm{Ext}^i_{\mathcal{P}^{op}\times\mathcal{P}}(T_1, T_2) \;=\; \mathbf{R}^i\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T_1, T_2).$$

If A is a strict polynomial functor of degree d, then we employ the following notation:

$$\mathrm{H}^*_{\mathcal{P}}(GL, A) \;:= \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, A \circ g\ell), \tag{1.4.1}$$

The following theorem relates Ext-groups for bifunctors to the rational cohomology of the general linear group, thereby extending Proposition 1.3 to positive cohomological degrees.

Theorem 1.5. *Let T be a strict polynomial bifunctor homogenous of bidegree (d, d). If $\dim(W) \geq d$, then there is a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) \xrightarrow{\sim} \mathrm{H}^*_{rat}(GL(W), T(W, W)), \tag{1.5.1}$$

where $\mathrm{H}^i(GL(W), T(W, W))$ denotes the i-th rational cohomology group of the algebraic group $GL(W)$ with coefficients in the rational $GL(W)$-module $T(W, W)$.

Furthermore, if $F = \mathcal{H}om(A_1, A_2)$ is of separable type (with A_1, A_2 strict polynomial functors of degree d), then there is a natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, \mathcal{H}om(A_1, A_2)) \simeq \mathrm{Ext}^*_{\mathcal{P}}(A_1, A_2). \tag{1.5.2}$$

Proof. Since both $\mathrm{H}^*(GL(W), -(W, W))$ and $\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}((\Gamma^d g\ell, -)$ are cohomological δ-functors on $\mathcal{P}^{op} \times \mathcal{P}$ and since they agree in degree 0 by Proposition 1.3, to prove (1.5.1) it suffices to verify that

$$\mathrm{H}^i(GL(W), T(W, W)) = 0, \quad i > 0 \tag{1.5.3}$$

for T of the form $I^{d,d}_{U,V}$.

By [8, 3.13],

$$\mathrm{H}^*_{rat}(GL(W), I^{d,d}_{U,V}(W, W)) \simeq \mathrm{Ext}^i_{\mathcal{P}}(\Gamma^d\mathrm{Hom}_k(U, -), S^d\mathrm{Hom}_k(V, -))$$

whenever $\dim_k(W) \geq d$. Since $\Gamma^d(\mathrm{Hom}_k(U, -)$ is projective in $\mathcal{P}_d$ (and $S^d\mathrm{Hom}_k(V, -)$ is injective), we conclude these groups vanish in positive degrees. This establishes (1.5.1).

Now, (1.5.2) follows immediately from (1.5.1) and [8, 3.13] applied to $T = \mathcal{H}om(A_1, A_2)$. □

In Proposition 2.2, we give another construction of the natural isomorphism (1.5.2).

We shall frequently use the following computation, a fundamental result of [8].

Theorem 1.6. [8] *For any $r \geq 0$, the graded algebra with unit*

$$E_r \;:=\; \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$$

is the commutative k-algebra generated by classes

$$e_i \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(I^{(r)}, I^{(r)}), \quad 1 \leq i \leq r$$

subject to the relations $e_i^p = 0$.

We have the following adjunction isomorphism, proved exactly as in [9, 1.7.1].

Proposition 1.7. *Consider the "diagonal" functor*

$$-\circ D : (\mathcal{P}^{op})^{\times m} \times (\mathcal{P})^{\times n} \;\to\; (\mathcal{P}^{op})^{\times m} \times \mathcal{P}$$

sending a strict polynomial multi-functor T to the functor $T \circ D$ whose value on $(W_1, \dots, W_m, V)$ equals $T(W_1, \dots, W_m, V, \dots, V)$; also consider the "sum" functor

$$-\circ \bigoplus : (\mathcal{P}^{op})^{\times m} \times \mathcal{P} \;\to\; (\mathcal{P}^{op})^{\times m} \times (\mathcal{P})^{\times n}$$

sending a strict polynomial multi-functor S to the functor $S \circ \bigoplus$ whose value on $(W_1, \dots, W_m, V_1, \dots, V_n)$ equals $T(W_1, \dots, W_m, \oplus_i V_i)$. Then $-\circ D$, $-\circ \bigoplus$ are both exact, and $-\circ D$ is both left and right adjoint to $-\circ \bigoplus$. Consequently, there are natural identifications

$$\mathrm{Ext}^*_{(\mathcal{P}^{op})^{\times m}\times\mathcal{P}}(S, T\circ D) \;\cong\; \mathrm{Ext}^*_{(\mathcal{P}^{op})^{\times m}\times(\mathcal{P})^{\times n}}(S\circ \bigoplus, T)$$

$$\mathrm{Ext}^*_{(\mathcal{P}^{op})^{\times m}\times\mathcal{P}}(T\circ D, S) \;\cong\; \mathrm{Ext}^*_{(\mathcal{P}^{op})^{\times m}\times(\mathcal{P})^{\times n}}(T, S\circ \bigoplus).$$

Moreover, similar statements apply with $-\circ D$ and $-\circ \bigoplus$ applied to the contravariant variables of strict polynomial multi-functors.

Theorem 1.6 provides the basic ingredient in the computation of numerous functor cohomology groups (e.g.[9]). The following theorem is a first application of this theorem to bifunctor cohomology. Although this result has probably been "known" to experts, we know of no written proof (cf. [2]). The underlying principle of such a computation is to manipulate the bifunctors involved so that the Ext-computations reduce to computations of Ext-groups between external tensor products.

Theorem 1.8. *For any $n \geq 1$, we have a $\mathfrak{S}_n$-equivariant isomorphism*

$$\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{n(r)}) \simeq E_r^{\otimes n} \otimes k\mathfrak{S}_n, \tag{1.8.1}$$

where the action of $\mathfrak{S}_n$ on the right hand side is by permutation of the factors of $E_r^{\otimes n} = (\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}))^{\otimes n}$ and by conjugation on $k\mathfrak{S}_n$.

Proof. We identify $g\ell^{\otimes n}$ with $\mathcal{H}om(\otimes^n, \otimes^n)$. In this identification, the left action of a permutation σ on the left-hand side is taken to the composition $\sigma^{-1} \circ (-) \circ \sigma$.

We apply Proposition 1.5 to conclude the isomorphism

$$\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{n(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(\otimes^{n(r)}, \otimes^{n(r)}) \;=\; \mathrm{Ext}^*_{\mathcal{P}}(\otimes^{n(r)}, (I^{(r)} \boxtimes \cdots \boxtimes I^{(r)}) \circ D).$$

By Proposition 1.7, the right hand side is isomorphic to

$$\mathrm{Ext}^*_{\mathcal{P}\times\cdots\times\mathcal{P}}(\otimes^{n(r)} \circ \bigoplus, I^{(r)} \boxtimes \cdots \boxtimes I^{(r)}). \tag{1.8.2}$$

Expand $(V_1 \oplus \cdots \oplus V_n)^{\boxtimes^n} = (\otimes^n \circ \bigoplus)(V_1, \dots, V_n)$ as a direct sum of tensor products and observe that the only summands which are of degree 1 in each position are the $n!$ terms of the form $V_{\sigma(1)} \boxtimes \cdots \boxtimes V_{\sigma(n)}$ for some $\sigma \in \Sigma_n$. Thus, the Künneth theorem

$$\mathrm{Ext}^*_{\mathcal{P}\times\mathcal{P}}(P_1 \boxtimes Q_1, P_2 \boxtimes Q_2) \;\simeq\; \mathrm{Ext}^*_{\mathcal{P}}(P_1, Q_1) \otimes \mathrm{Ext}^*_{\mathcal{P}}(P_1, Q_1).$$

implies (1.8.1) as an additive isomorphism.

The left permutation action on $\otimes^{n(r)} \circ \bigoplus$ of (1.8.2) yields a right action which permutes the summands of this additive decomposition. We identify (1.8.1) as a right $\mathfrak{S}_n$-module as follows. For each permutation σ, we translate the summand

$$\mathrm{Ext}^*_{\mathcal{P}\times\cdots\times\mathcal{P}}(I^{(r)}_{\sigma(1)} \boxtimes \cdots \boxtimes I^{(r)}_{\sigma(n)}, I^{(r)} \boxtimes \cdots \boxtimes I^{(r)}).$$

indexed by σ back to the summand indexed by the identity using the isomorphism σ^*; in other words, we identify $x \otimes \sigma$ with $\sigma^*(x) \otimes 1$ In this way, the right action is given by:

$$\tau^*(x \otimes \sigma) = \tau^* \circ \sigma^*(x) = x \otimes \sigma\tau$$

which is simply the right action of $\mathfrak{S}_n$ on $k\mathfrak{S}_n$ in (1.8.1).

We now determine the left action with respect to this identification. It is enough to consider an elementary tensor $x = x_1 \otimes \cdots \otimes x_n \in E_r^{\otimes n} \otimes 1$ representing a class in cohomological degree s. Write each x_i as a Yoneda extension in $\mathcal{P}$

$$I^{(r)} = Q_i^0 \to Q_i^1 \to \cdots \to Q_i^{s_i} \to I^{(r)},$$

and choose the tensor product of these to represent x as a Yoneda extension:

$$I^{(r)} \boxtimes \cdots \boxtimes I^{(r)} = Q^0 \to Q^1 \to \cdots \to Q^s \to I^{(r)} \boxtimes \cdots \boxtimes I^{(r)},$$

so that Q^j is the sum

$$\bigoplus_{j_1+\cdots+j_n=j} Q_1^{j_1} \boxtimes \cdots \boxtimes Q_n^{j_n}.$$

Let σ be a permutation in $\mathfrak{S}_n$ and let us represent $x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$ by the tensor of the corresponding Yoneda extensions:

$$I^{(r)} \boxtimes \cdots \boxtimes I^{(r)} \to Q_\sigma^1 \to \cdots \to Q_\sigma^s \to I^{(r)} \boxtimes \cdots \boxtimes I^{(r)}.$$

By Proposition 1.7, the corresponding class in $\mathrm{Ext}^*_{\mathcal{P}}(\otimes^{n(r)}, \otimes^{n(r)})$ is obtained by precomposing with the diagonal D. The commutative diagram:

$$\begin{array}{ccccccccc}
I^{(r)} \otimes \cdots \otimes I^{(r)} & \longrightarrow & Q^1 \circ D & \longrightarrow & \cdots & \longrightarrow & Q^i \circ D & \longrightarrow & I^{(r)} \otimes \cdots \otimes I^{(r)} \\
\downarrow \sigma & & \downarrow \sigma & & & & \downarrow \sigma & & \downarrow \sigma \\
I^{(r)} \otimes \cdots \otimes I^{(r)} & \longrightarrow & Q^1_\sigma \circ D & \longrightarrow & \cdots & \longrightarrow & Q^i_\sigma \circ D & \longrightarrow & I^{(r)} \otimes \cdots \otimes I^{(r)}.
\end{array}$$

implies the relation:

$$\sigma_*(x_1 \otimes \cdots \otimes x_n) = \sigma^*(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes \sigma.$$

Thus, combining the above left action on the distinguished summand with the right $\mathfrak{S}_n$-action permuting the summands gives the asserted action. □

To apply Theorem 1.8, one must analyze the $\mathfrak{S}_n$-module

$$\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{n(r)}) \simeq E_r^{\otimes n} \otimes k\mathfrak{S}_n.$$

In the following example, we treat the case when $n = 3$.

Proposition 1.9. *We consider a field k of characteristic $p > 0$.*

- *The Poincaré series for $H^*_{\mathcal{P}}(GL, \otimes^{3(r)})$ equals*

$$6\left(\frac{1-t^{2p^r}}{1-t^2}\right)^3.$$

- *If $p > 3$, the Poincaré series for $H^*_{\mathcal{P}}(GL, \Lambda^{(3(r)})$ equals*

$$\left(\frac{1-t^{2p^r}}{1-t^2}\right)^3 - 2\left(\frac{1-t^{4p^r}}{1-t^4}\right)\left(\frac{1-t^{2p^r}}{1-t^2}\right) + 2\left(\frac{1-t^{6p^r}}{1-t^6}\right).$$

- *If $p > 3$, the Poincaré series for $H^*_{\mathcal{P}}(GL, S^{(3(r)})$ equals*

$$2\left(\frac{1-t^{4p^r}}{1-t^4}\right)\left(\frac{1-t^{2p^r}}{1-t^2}\right)+\left(\frac{1-t^{6p^r}}{1-t^6}\right).$$

Proof. We immediately verify that the Poincaré series for E_r equals $P_r(t) = \frac{1-t^{2p^r}}{1-t^2}$.

Index the isomorphism classes of transitive $\mathfrak{S}_3$-sets by their cardinality: S_1, the trivial $\mathfrak{S}_3$-set, S_2, S_3, $S_6 = \mathfrak{S}_3$. Every $\mathfrak{S}_3$-set is uniquely isomorphic to a disjoint union of these. Finite disjoint unions of copies of these four sets form a commutative semi-ring under disjoint union and cartesian product, with S_1 as unit. The product is given explicitly by:

$$S_1 \times S_i = S_i, \quad S_2 \times S_3 = S_6, \quad S_i \times S_i = iS_i, \quad S_i \times S_6 = iS_6.$$

For example, with respect to the conjugation action, $\mathfrak{S}_3 \simeq S_1 \coprod S_2 \coprod S_3$ corresponding to the partition of $\mathfrak{S}_3$ into conjugacy classes. Similarly, every finite-dimensional permutation module is isomorphic to a finite direct sum of $k[S_i]$, and every graded permutation module which is finite dimensional in each degree has a Poincaré series with coefficients in the above semi-ring.

For our purpose, the relevant example is the third tensor power, $E_r^{\otimes 3}$, of the graded vector space E_r with Poincaré series $P_r(t) = \sum_i \dim E_r^i t^i$. A direct computation verifies that the Poincaré series of $E_r^{\otimes 3}$ as a $\mathfrak{S}_3$-module equals

$$S_1 P_r(t^3) + S_3(P_r(t^2)P_r(t) - P_r(t^3)) + \frac{S_6}{6}(P_r(t)^3 - 3P_r(t^2)P_r(t) + 2P_r(t^3)).$$

(The subspace fixed by $\mathfrak{S}_3$ is spanned by elements of the form $x \otimes x \otimes x \in E_r^{\otimes 3}$, leading to the first summand; the cycles of length 3 are represented by elements of the form $x \otimes x \otimes y,\ x \neq y$, which leads to the second summand; the cycles of length 6 are represented by elements of the form $x \otimes y \otimes z$ with x, y, z distinct.) After multiplication using the product given explicitly above, one gets that the Poincaré series for $E_r^{\otimes 3} \otimes k\mathfrak{S}_3$ is equal to

$$S_1P_r(t^3)+S_2P_r(t^3)+S_3(4P_r(t^2)P_r(t)-3P_r(t^3))+S_6(P_r(t)^3-2P_r(t^2)P_r(t)+P_r(t^3)).$$

The Poincaré series of the graded vector spaces obtained by applying symmetrization and antisymmetrization functors (the functors $s^3(-)$) and $\lambda^3(-)$ of §4 below) to $E_r^{\otimes 3} \otimes k\mathfrak{S}_3$ is then obtained term by term from the corresponding result for the permutation modules $k[S_i]$. If p is not equal to 3, then $\lambda^3(k[S_1]) = \lambda^3(k[S_3]) = 0$, $\lambda^3(k[S_2]) = \lambda^3(k[S_6]) = k$, and $s^3(k[S_i]) = k,\ 1 \le i \le 6$.

As a result, the Poincaré series of the graded vector space $\lambda^3(E_r^{\otimes 3} \otimes k\mathfrak{S}_3)$ is equal to

$$P_r(t)^3 - 2P_r(t^2)P_r(t) + 2P_r(t^3)$$

whereas the Poincaré series for $s^3(E_r^{\otimes 3} \otimes k\mathfrak{S}_3)$ is equal to

$$2P_r(t^2)P_r(t) + P_r(t^3).$$

□

2. Cohomology of bifunctors of separable type

In Theorem 1.5, we established the isomorphism (1.5.2) between Ext-groups in the category $\mathcal{P}^{op} \times \mathcal{P}$ for bifunctors of separable type and Ext-groups in the category $\mathcal{P}$ of strict polynomial functors. The purpose of this section is to further exploit the special properties of such bifunctors of separable type. In particular, in

Proposition 2.6 we verify that a certain natural map is an isomorphism, a verification which will play a central role in the calculation of §5.

Begin with the following elementary lemma.

Lemma 2.1. *For $V, W \in \mathcal{V}$ and $T \in \mathcal{P}^{op} \times \mathcal{P}$, there is a natural identification*

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathcal{H}om(P_V, I_W)) \;\cong\; \mathrm{Hom}_{\mathcal{P}}(\mathrm{Hom}_{\mathcal{P}}(T(-_1,-_2), P_V(-_1)), I_W(-_2)).$$

Proof. By (1.2.1),

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathrm{Hom}_{\mathcal{P}}(P_V, I_W)) \;\cong\; T(V, W).$$

On the other hand,

$$\mathrm{Hom}_{\mathcal{P}}(T(-_1,-_2), P_V(-_1)) \;\cong\; T(V, -_2),$$

by (1.0.2); thus, another application of (1.0.2) implies

$$\mathrm{Hom}_{\mathcal{P}}(\mathrm{Hom}_{\mathcal{P}}(T(-_1,-_2), P_V(-_1)), I_W(-_2)) \;\cong\; T(V, W).$$

The composite of these natural identifications is readily seen to be the identification asserted in the statement of the lemma. □

The preceding lemma leads to the following spectral sequence, which provides an explanation and an extension of (1.5.2).

Proposition 2.2. *For two strict polynomial functors A_1 and A_2 homogeneous of degree d, we consider the bifunctor $\mathcal{H}om(A_1, A_2)$ of separable type. For any strict polynomial bifunctor T, there is a convergent spectral sequence of the form*

(2.2.1)
$$E_2^{s,t} \;=\; \mathrm{Ext}^s_{\mathcal{P}}(A_1(-_1), \mathrm{Ext}^t_{\mathcal{P}}(T(-_1,-_2), A_2(-_2))) \;\Rightarrow\; \mathrm{Ext}^{s+t}_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathcal{H}om(A_1, A_2)),$$

natural in A_1, A_2 and T.

If $T = \Gamma^d g\ell$, then this spectral sequence collapses to give the natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, \mathcal{H}om(A_1, A_2)) \simeq \mathrm{Ext}^*_{\mathcal{P}}(A_1, A_2).$$

Proof. If $P_\bullet \to A_1 \;\in \mathcal{P}$ is a projective resolution of A_1 and if $A_2 \to I^\bullet \;\in \mathcal{P}$ is an injective resolution of A_2, then the double complex $\mathcal{H}om(P_\bullet, I^\bullet)$ of injective bifunctors in $\mathcal{P}^{op} \times \mathcal{P}$ has total complex which is an injective resolution of $\mathcal{H}om(A_1, A_2)$. Thus, the cohomology of the total complex of the bicomplex $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathcal{H}om(P_\bullet, I^\bullet))$ equals $\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathcal{H}om(A_1, A_2))$. On the other hand, Lemma 2.1 identifies this bicomplex with

$$\mathrm{Hom}_{\mathcal{P}}(P_\bullet(-_1), \mathrm{Hom}_{\mathcal{P}}(T(-_1,-_2), I^\bullet(-_2)).$$

If we take the iterated cohomology of this bicomplex first with respect to the variable index of $I^\bullet$ we obtain

$$\mathrm{Hom}_{\mathcal{P}}(P_\bullet(-_1), \mathrm{Ext}^t_{\mathcal{P}}(T(-_1,-_2), A_2));$$

then taking cohomology with respect to the variable index of $P_\bullet$, we obtain the asserted E_2-term

$$\mathrm{Ext}^s_{\mathcal{P}}(A_1(-_1), \mathrm{Ext}^t_{\mathcal{P}}(T(-_1,-_2), A_2(-_2))).$$

Thus, (2.2.1) is one of the usual spectral sequences associated to the bicomplex $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(T, \mathcal{H}om(P_\bullet, I^\bullet))$.

For $T = \Gamma^d g\ell$, (1.3.2) provides the identification of bicomplexes

$$\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, \mathcal{H}om(P_\bullet, I^\bullet)) \;\cong\; \mathrm{Hom}_{\mathcal{P}}(P_\bullet, I^\bullet)$$

the cohomology of whose total complex equals $\mathrm{Ext}^*_{\mathcal{P}}(A_1, A_2)$; the constructed spectral sequence clearly collapses (since $\mathrm{Ext}^i(P_s, I^\bullet) = 0,\ i > 0$).

□

The following proposition, extending (1.1.3) to all cohomology degrees, can be viewed as a Künneth Theorem for bifunctor cohomology.

Proposition 2.3. *Let A_1, A_2, B_1, B_2 be strict polynomial functors of bounded degree. Then the spectral sequence (2.2.1) with $T = \mathcal{H}om(B_1, B_2)$ collapses to yield the natural identification*

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(B_1, B_2), \mathcal{H}om(A_1, A_2)) \ \cong\ \mathrm{Ext}^*_{\mathcal{P}}(A_1, B_1) \otimes \mathrm{Ext}^*_{\mathcal{P}}(B_2, A_2).$$

Proof. Consider the bicomplex $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(A_1, A_2), \mathcal{H}om(P_\bullet, I^\bullet))$ whose total complex computes $\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(A_1, A_2), \mathcal{H}om(B_1, B_2))$. By (1.1.3), this bicomplex can be identified with $\mathrm{Hom}_{\mathcal{P}}(P_\bullet, B_1) \otimes \mathrm{Hom}_{\mathcal{P}}(B_2, I^\bullet)$. The iterated cohomology of this bicomplex equals its total cohomology as well as equals $\mathrm{Ext}^*_{\mathcal{P}}(A_1, B_1) \otimes \mathrm{Ext}^*_{\mathcal{P}}(B_2, A_2)$ □

Remark 2.4. *The isomorphism of Proposition 2.3 is compatible with whatever Yoneda products makes sense. In particular, there is a ring isomorphism*

$$\mathrm{Ext}^*_{\mathcal{P}}(A_1, A_1)^{op} \otimes \mathrm{Ext}^*_{\mathcal{P}}(A_2, A_2) \xrightarrow{\cong} \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(A_1, A_2), \mathcal{H}om(A_1, A_2)).$$

The Yoneda product defines a right action of $\mathrm{Ext}^*_{\mathcal{P}}(A_1, A_1)$ and a left action of $\mathrm{Ext}^*_{\mathcal{P}}(A_2, A_2)$ on the E_2-term of the spectral sequence in Proposition 2.2, while the abutment of the spectral sequence is a module over the ring

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(A_1, A_2), \mathcal{H}om(A_1, A_2)).$$

Because of the way the spectral sequence of Proposition 2.2 is constructed, it is compatible with such Yoneda products, which makes it a $\mathrm{Ext}^*_{\mathcal{P}}(A_1, A_1)$-$\mathrm{Ext}^*_{\mathcal{P}}(A_2, A_2)$ bi-module spectral sequence.

Observe that we have a natural identification

$$gl^{(r)} =:\ I^{(r)} \circ gl \ =\ \mathcal{H}om(I^{(r)}, I^{(r)}) \tag{2.4.1}$$

for any $r \geq 0$. We use this identification implicitly in the construction of ∇ in the next lemma.

Lemma 2.5. *The map obtained by precomposing with $\mathcal{H}om(I, I)$,*

$$\nabla :\ \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \to \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(gl^{(r)}, gl^{(r)}) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})^{\otimes 2}$$

is a coproduct map on $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$, which, together with the Yoneda product, defines a graded Hopf algebra structure on $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$.

Proof. We readily identify ∇ with the map of [14, 3.7] which is induced by the natural map

$$\mathrm{Hom}_{\mathcal{P}}(I^{(r)}, I^{(r)}) \to \mathrm{Hom}_{\mathcal{P}\times\mathcal{P}}(I^{(r)} \circ \boxtimes, I^{(r)} \circ \boxtimes).$$

(Here, $\boxtimes \in \mathcal{P} \times \mathcal{P}$ sends (V, W) to $V \otimes W$.) This latter map is shown to provide a coproduct structure on $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ as asserted. □

Proposition 2.6. *Let $\gamma_r : \Gamma^{p^r} \to I^{(r)} \in \mathcal{P}$ be the canonical map sending $v^{\otimes r} \in \Gamma^{p^r}(V)$ to $v^{(r)} \in V^{(r)}$. Then the composition*

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \stackrel{\nabla}{\to} \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(g\ell^{(r)}, g\ell^{(r)}) \stackrel{\gamma_r^*}{\to} \mathrm{Ext}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^r} g\ell, g\ell^{(r)})$$

is an isomorphism.

Proof. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) & & \\
\downarrow \nabla & & \\
\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(g\ell^{(r)}, g\ell^{(r)}) & \xrightarrow{\gamma_r^*} & \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^r} g\ell, g\ell^{(r)}) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \otimes \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})) & & \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \\
& \searrow^{id\otimes\gamma_r^*} & \uparrow id\otimes\epsilon \\
& & \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \otimes \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{p^r}, I^{(r)})
\end{array}$$

where the left and right vertical isomorphisms are given by Proposition 2.3 and Proposition 2.2 respectively. Since $\Gamma^{p^r} \in \mathcal{P}$ is injective, the bottom vertical up arrow is an isomorphism, where ϵ is the projection onto $\mathrm{Ext}^0_{\mathcal{P}}(\Gamma^{p^r}, I^{(r)}) \cong k$. Using Lemma 2.5, we conclude that the composition

$$(id \otimes \epsilon) \circ (id \otimes \gamma_r^*) \circ \nabla$$

is an isomorphism.

To verify the commutativity of this diagram, we trace the map induced by $\gamma_r$on E_2-terms of the spectral sequences (2.2.1):

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, \mathrm{Ext}^*_{\mathcal{P}}(g\ell^{(r)}, I^{(r)})) \ \to \ \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{p^r}, I^{(r)})).$$

This is readily seen to be given by the lower diagonal map $id \otimes \gamma_r^*$ of the diagram, once one recognizes the collapsing of the two spectral sequences. □

3. Base change and twist stablity for bifunctor cohomology

If K/k is a field extension of our chosen base field k, then we let $\mathcal{V}_K$ denote the category of finite dimensional K-vector spaces and $\mathcal{P}_K$ the category of strict polynomial functors on $\mathcal{V}_K$ of bounded degree. We denote by $\mathcal{P}^{op} \times \mathcal{P}_K$ the category of strict polynomial bifunctors on $\mathcal{V}_K^{op} \times V_K$.

Following the construction in [14, 2.5], we define the base change $T_K \in \mathcal{P}^{op} \times \mathcal{P}_K$ of a strict polynomial bifunctor $T \in \mathcal{P}^{op} \times \mathcal{P}$ as follows. For any $V \in \mathcal{V}$, let V_K denote $V \otimes_k K$, the base change of V to K/k. Then we define

$$T_K(V', W') \ := \varinjlim_{\mathcal{V}_K(V')^{\sim} \times \mathcal{V}_K(W')} T(V, W)$$

where $\mathcal{V}_K(V')^{\sim}$ is the category whose objects are pairs (V, ϕ), $V \in \mathcal{V}$, $\phi : V' \to V \otimes_k K$ and whose maps $(V, \phi) \to (V_1, \phi_1)$ are K-linear maps $\theta : V_1 \otimes_k K \to V \otimes_k K$ such that $\phi = \theta \circ \phi_1$; and where $\mathcal{V}_K(W')$ is the category whose objects are pairs (W, ψ), $W \in \mathcal{V}$, $\psi : W \otimes_k K \to W'$ and whose maps $(W, \psi) \to (W_1, \psi_1)$ are

K-linear maps $\rho : W \otimes_k K \to W_1 \otimes_k K$ such that $\psi = \rho \circ \psi_1$. As in [14, 2.5], T_K is a well-defined strict polynomial bifunctor whose value on a pair of the form (V_K, W_K) equals $T(V,W)_K$.

Cohomological base change for strict polynomial bifunctors is formulated in the following proposition.

Proposition 3.1. *Let K/k be a field extension and $S, T \in \mathcal{P}^{op} \times \mathcal{P}$ be strict polynomial bifunctors of bounded degree. Then there is a natural isomorphism of graded K-vector spaces*

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S,T) \otimes_k K \xrightarrow{\sim} \mathrm{Ext}^*_{\mathcal{P}^{op}_K \times \mathcal{P}_K}(S_K, T_K).$$

Proof. As observed in [14, 2.6], sending T to T_K is exact. The verification that this base change preserves projective objects of the form

$$\Gamma^d \mathrm{Hom}_k(-_1, W) \boxtimes \Gamma^d(\mathrm{Hom}_k(V, -_2)$$

follows from the fact also verified in [14, 2.6] that the base change of $\Gamma^d\mathrm{Hom}_k(V,-) \in \mathcal{P}$ is naturally isomorphic to $\Gamma^d\mathrm{Hom}_K(V_K, -)$. Then the natural map

$$\mathcal{H}om_{\mathcal{P}^{op}\times\mathcal{P}}(S,T) \otimes_k K \xrightarrow{\sim} \mathcal{H}om_{\mathcal{P}^{op}_K \times \mathcal{P}_K}(S_K, T_K)$$

is an isomorphism for T of the form $\Gamma^d\mathrm{Hom}_k(-_1, W) \boxtimes \Gamma^d(\mathrm{Hom}_k(V, -_2)$. Since any T admits a resolution by such projective objects, the proposition follows. □

For any strict polynomial bifunctors T, the exactness of $I^{(1)} \circ (-)$ determines the *Frobenius twist* map

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S,T) \to \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(1)}, T^{(1)}). \tag{3.1.1}$$

Twist injectivity for strict polynomial bifunctor cohomology, given below, is essentially a reformulation of a theorem for rational cohomology proved by H. Andersen.

Proposition 3.2. *For any strict polynomial bifunctor $T \in \mathcal{P}^{op} \times \mathcal{P}$ of bidegree (d,d), the composition of (3.1.1) (taking $S = \Gamma^d g\ell$) and the natural map induced by $\Gamma^{pd} \to \Gamma^{d(1)}$ (dual to the p-th power map $S^{d(1)} \to S^{pd}$),*

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) \to \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{pd} g\ell, T^{(1)}). \tag{3.2.1}$$

is injective.

Proof. For any $W \in \mathcal{V}$, consider the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) & \xrightarrow{\cong} & H^0(GL(W), T(W,W)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{pd} g\ell, T^{(1)}) & \xrightarrow{\cong} & H^0(GL(W), T(W,W)^{(1)}), \end{array}$$

where the horizontal isomorphisms are those of (1.5.1), the left vertical map is the map induced by twist and $\Gamma^{pd} \to \Gamma^{d(1)}$, and the right vertical map is the Frobenius twist map. Tracing through the identification given in the proof of Proposition 1.3 for $T = I^{d,d}_{U,V}$, we easily conclude that this square commutes for such T. For general T, we consider the beginning of a resolution of T by injectives of this form, $T \to I^0 \to I^1$ and use the left exactness of $\mathrm{Hom}_B(\Gamma^d g\ell, -)$ and $H^0(GL(W), -)$. Since the right vertical map is evidently injective, so is the left vertical map; this verifies the asserted injectivity in cohomological degree 0.

For higher cohomological degree, embed T in some injective strict polynomial bifunctor J and denote the quotient J/T by $\overline{J}$. We identify (3.2.1) in cohomological degree 1 with the map on quotients induced by maps in cohomological degree 0, since

$$\mathrm{Ext}^1_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) \;=\; \mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, \overline{J})/\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, J)$$

and

$$H^1(GL, T(W,W)) \;=\; H^0(GL, \overline{J}(W,W))/H^0(GL, J(W,W)).$$

In cohomological degree $i > 1$, we identify (3.2.1) with the map in cohomological degree $i-1$ with T replaced by $\overline{J}$. Consequently, we conclude the commutativity of the square

$$\begin{array}{ccc} \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d g\ell, T) & \xrightarrow{\cong} & H^*(GL(W), T(W,W)) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{pd} g\ell, T^{(1)})) & \xrightarrow{\cong} & H^*(GL(W), T(W,W)^{(1)}) \end{array} \tag{3.2.2}$$

where the left vertical map is the map of the assertion, the right vertical map is the twist map in rational cohomology, and where the horizontal isomorphisms are those (1.5.1).

The injectivity of the left vertical map thus follows from Andersen's theorem asserting that the right vertical map is injective (cf. [1], [11, II.10.16]). □

We now prove *twist stability* for strict polynomial bifunctor cohomology, basically by reducing the assertion to the theorem of "strong twist stability" given in [9].

Theorem 3.3. *For any $S,\ T \in \mathcal{P}^{op}\times\mathcal{P}$ of bidegree (d,d), the twist map of (3.1.1) in cohomological degree s,*

$$\mathrm{Ext}^s_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r)}, T^{(r)}) \;\to\; \mathrm{Ext}^s_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r+1)} g\ell, T^{(r+1)}), \tag{3.3.1}$$

is an isomorphism provided that $r \geq log_p(\frac{s+1}{2})$.

Moreover, the twist map of (3.2.1) in cohomological degree s,

$$\mathrm{Ext}^s_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^r d} g\ell, T^{(r)}) \to \mathrm{Ext}^s_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^{r+1} d} g\ell, T^{(r+1)}), \tag{3.3.2}$$

is also an isomorphism provided that $r \geq log_p(\frac{s+1}{2})$.

Proof. We first assume that $S = \mathcal{H}om(A_1, A_2),\ T = \mathcal{H}om(B_1, B_2)$ are of separable type and identify (3.1.1) with

$$\mathrm{Ext}^*_{\mathcal{P}}(B_1, A_1) \otimes \mathrm{Ext}^*_{\mathcal{P}}(A_2, B_2) \;\to\; \mathrm{Ext}^*_{\mathcal{P}}(B_1^{(1)}, A_1^{(1)}) \otimes \mathrm{Ext}^*_{\mathcal{P}}(A_2^{(1)}, B_2^{(1)}). \tag{3.3.3}$$

Provided that $r \geq log_p(\frac{s+1}{2})$, (3.3.3) is an isomorphism in cohomological degree s by [9, 4.10].

More generally, assume that $S = \mathcal{H}om(A_1, A_2)$ is a bifunctor of separable type but that T is an arbitrary strict polynomial bifunctor homogeneous of degree (d,d). Choose a resolution of T, $T \to J^\bullet$, by bifunctors of separable type (each of which is strict polynomial of bidegree (d,d)) and compare the map induced by twist of hyperext spectral sequences obtained by applying $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r)}, -)$ and $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r+1)}, -)$ to double complexes which are injective resolutions of $J^{\bullet(r)}$

and $J^{\bullet(r+1)}$. The preceding special case implies that we have an isomorphism on E_1-terms of these spectral sequences,

$$E_1^{s,t} \stackrel{\sim}{\to} {}'E_1^{s,t}, \quad r \geq log_p \frac{t+1}{2}.$$

We identify the map on abutments,

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r)},T^{(r)}) \cong \mathbf{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(P_\bullet^{(r)},J^{\bullet(r)}) \ \to$$
$$\mathbf{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(P_\bullet^{(r+1)},J^{\bullet(r+1)}) \cong \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r+1)},T^{(r+1)}),$$

in cohomological degree s with (3.3.1), so that an easy spectral sequence argument completes the proof of the first assertion in this case.

Finally, we allow S to be an arbitrary strict polynomial bifunctor homogeneous of degree (d,d) and choose a resolution $P_\bullet \to S$ by bifunctors of separable type. Comparing hyperext spectral sequences as in the previous paragraph enables us to conclude the validity of the first assertion.

To prove the second assertion, we proceed in a similar fashion. We first observe that the commutativity of (3.2.2) in the special case that T is of separable type together with (1.5.2) enables us to apply the "strong twist stability theorem" of [9, 4.10] to conclude that (3.3.2) is an isomorphism whenever $r \geq log_p \frac{s+1}{2}$.

More generally, we choose a resolution $T \to J^\bullet$ of T by strict polynomial bifunctors of separable type. We consider the map of hyperext spectral sequences from

$$E_1^{s,t} = \mathrm{Ext}^t_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^r d}g\ell, J^{s(r)}) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^r d}g\ell, T^{(r)})$$

to

$${}'E_1^{s,t} = \mathrm{Ext}^t_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^{r+1} d}g\ell, J^{s(r+1)}) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{p^{r+1} d}g\ell, T^{(r+1)}).$$

Once again, an easy spectral sequence comparison theorem implies that

$$E_1^{s,t} \stackrel{\sim}{\to} {}'E_1^{s,t}, \quad r \geq log_p \frac{t+1}{2},$$

so that the induced map on abutments $E_\infty^s \to {}'E_\infty^s$ is an isomorphism provided that $r \geq log_p(\frac{s+1}{2})$. □

We restate Theorem 3.3 in the special case in which $T = A \circ g\ell$, using the notation of (1.4.1).

Corollary 3.4. *For any $A \in \mathcal{P}$,*

$$\mathrm{H}^s(\mathrm{GL}, A^{(r)}) \ \to \ \mathrm{H}^s(\mathrm{GL}, A^{(r+1)})$$

is an isomorphism provided that $r \geq log_p(\frac{s+1}{2})$.

4. Cohomology of summands of $gl^{(r)\otimes n}$, $n < p$

In this section, we consider summands of the bifunctor

$$g\ell^{\otimes n(r)} \ := \ \mathcal{H}om(I,I)^{\otimes n(r)}$$

for $n < p$. We are particularly interested in the n-th symmetric power $S^n(g\ell^{(r)})$ and the n-th exterior power $\Lambda^n(g\ell^{(r)})$, but the method of determining the cohomology of these summands applies more generally to summands given by idempotents in the group ring $k\mathfrak{S}_n$. Our hypothesis $n < p$ implies that this group ring is semi-simple. We refer the reader to [10] for further details.

Let $\lambda = (\lambda_1, \ldots, \lambda_j)$ be a partition of n and choose a Young diagram whose shape is given by λ (so that the first row of the Young diagram has λ_1 "boxes", the second has λ_2 boxes, etc). Let t_λ be an arbitrary choice of a Young tableau (i.e., an assignment of the numbers $\{1, \ldots, n\}$ to these boxes which is strictly increasing both to the left and downwards) with underlying Young diagram given by λ. Define $a_\lambda \in k\mathfrak{S}_n$ to be the sum (in $k\mathfrak{S}_n$) of the elements in $\mathfrak{S}_n$ which preserve the rows of this tableau, b_λ to be the alternating sum of the elements in $\mathfrak{S}_n$ which preserve the columns. Then some multiple of

$$c_\lambda := a_\lambda \cdot b_\lambda \in k\mathfrak{S}_n$$

is idempotent and

$$\{V_\lambda := k\mathfrak{S}_n \cdot c_\lambda; \quad \lambda \text{ a } \textit{partition of } n\}$$

constitutes a complete list of the irreducible representations of $\mathfrak{S}_n$ (where, as always in this section, we assume $p > n$).

In particular, if $\lambda = (n)$, then $V_{(n)} = k$ is the 1-dimensional trivial representation of $\mathfrak{S}_n$; if $\lambda = (1, \ldots, 1)$, then $V_{(1,\ldots,1)} = k_{sgn}$ is the 1-dimensional sign representation.

For any $k\mathfrak{S}_n$-module W and any partition λ of n, we define

$$s^\lambda(W) =: \ (k\mathfrak{S}_n \cdot c_\lambda) \otimes_{k\mathfrak{S}_n} W.$$

If $W = V^{\otimes n}$ for some finite dimensional vector space V and if $\mathfrak{S}_n$ acts on W by permuting the tensor factors, we introduce the notation

$$S^\lambda(V) := \ s^\lambda(V^{\otimes n});$$

in particular, $S^{(n)}(V) = S^n(V)$ and $s^{(1,1,\ldots,1)}(V) = \Lambda^n(V)$.

The following proposition provides an (implicit) description of cohomology of numerous bifunctors granted the explicit description of

$$\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes^n(r)}) = E_r^{\otimes n} \otimes k\mathfrak{S}_n$$

as a $\mathfrak{S}_n$-module given in Theorem 1.8.

Proposition 4.1. *Assume $p > n$ and let λ be a partition of n. Then*

$$\mathrm{H}^*_{\mathcal{P}}(GL, S^\lambda(g\ell^{(r)})) \ \cong s^\lambda(\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)})). \tag{4.1.1}$$

In particular,

$$\mathrm{H}^*_{\mathcal{P}}(GL, S^{n(r)}g\ell) \ \cong s^{(n)}(\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)})) \ \cong \ k \otimes_{k\mathfrak{S}_n} \mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)}),$$

$$\mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{n(r)}g\ell) \ \cong s^{(1,\ldots,1)}(\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)})) \ \cong \ k_{sgn} \otimes_{k\mathfrak{S}_n} \mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)}),$$

Proof. The fact that c_λ is a quasi-idempotent means that some non-zero multiple of c_λ, $e_\lambda = a_\lambda c_\lambda$, is idempotent. Thus,

$$k\mathfrak{S}_n \ \cong \ k\mathfrak{S}_n \cdot e_\lambda \oplus k\mathfrak{S}_n \cdot (1 - e_\lambda).$$

By functoriality,

$$s^\lambda(\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)})) =: \ \mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)}) \cdot c_\lambda \ = \ \mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes n(r)}) \cdot e_\lambda$$

equals

$$\mathrm{H}^*_{\mathcal{P}}(GL, (g\ell^{\otimes n(r)}) \cdot e_\lambda) \ = \ \mathrm{H}^*_{\mathcal{P}}(GL, (g\ell^{\otimes n(r)}) \cdot c_\lambda) \ = \ \mathrm{H}^*_{\mathcal{P}}(GL, S^\lambda(g\ell^{(r)})).$$

□

As a simple corollary of Proposition 4.1, we have the following vanishing of strict polynomial bifunctor cohomology for $r = 0$.

Corollary 4.2. *Assume $p > n$ and let λ be a partition of n. Then*

$$\mathrm{H}^s_{\mathcal{P}}(GL, S^\lambda(g\ell)) \;=\; 0, \quad s > 0.$$

Proof. This follows immediately from Proposition 4.1 and the observation that E_0 vanishes in positive degrees. □

We make explicit the case $n = 2$ of Proposition 4.1.

Corollary 4.3. *If $p > 2$, then $c_{(2)} = 1 + \tau,\ c_{(1,1)} = 1 - \tau \in k\mathfrak{S}_2$. Thus,*

$$\mathrm{H}^*_{\mathcal{P}}(GL, S^{2(r)}) \;\cong\; S^2(E_r) \oplus S^2(E_r), \quad \mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{2(r)}) \;\cong\; \Lambda^2(E_r) \oplus \Lambda^2(E_r).$$

Proof. Recall the triviality of the action of $\mathfrak{S}_2$ on the tensor factor $k\mathfrak{S}_2$ of

$$\mathrm{H}^*_{\mathcal{P}}(GL, g\ell^{\otimes^2(r)}) \cong E_r^{\otimes 2} \otimes k\mathfrak{S}_2.$$

Thus, $s_\lambda(\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)}))$ is naturally isomorphic to two copies of $s_\lambda(E_r^{\otimes 2})$, where the action of $\mathfrak{S}_2$ on $E_r^{\otimes 2}$ is permutation of tensor factors by Theorem 1.8. □

We derive one more simple corollary which follows immediately from Theorem 1.8, Proposition 4.1 and the fact that the graded group $E_r^{\otimes n}$ is 1-dimensional in degree 0.

Corollary 4.4. *Assume $p > n$ and let λ be a partition of n. Then*

$$\mathrm{H}^0_{\mathcal{P}}(GL, S^\lambda(g\ell^{(r)})) \;\cong s^\lambda(k\mathfrak{S}_n)$$

where $k\mathfrak{S}_n$ is a $\mathfrak{S}_n$-module via the conjugation action.

5. Computations for $S^2(gl^{(r)}),\ \Lambda^2(gl^{(r)})$, *and* $\Gamma^2(gl^{(r)})$ for $p = 2$

In this section, our base field k is an arbitrary field of characteristic 2 and r a non-negative integer. This section is dedicated to verifying the following computation.

Theorem 5.1. *Let k be a field of characteristic 2 and $r \geq 0$ a non-negative integer. Then*

- *The $\mathfrak{S}_2$-module $\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)})$ is isomorphic to $E_r^{\otimes 2} \otimes k\mathfrak{S}_2$ with $\mathfrak{S}_2$-action on $E_r^{\otimes 2}$ given by permuting the tensor factors and with $\mathfrak{S}_2$-action on $k\mathfrak{S}_2$ trivial; its Poincare series equals*

$$2\frac{(1 - t^{2^{r+1}})^2}{(1 - t^2)^2};$$

- *The vector space $\mathrm{H}^j(GL, S^{2(r)})$ is abstractly isomorphic to the vector space of coinvariants under the $\mathfrak{S}_2$-action of $\mathrm{H}^j(GL, \otimes^{2(r)})$; the Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(GL, S^{2(r)})$ is equal to:*

$$\frac{(1 - t^{2^{r+1}})^2}{(1 - t^2)^2} + \frac{1 - t^{2^{r+2}}}{1 - t^4};$$

- *The Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{2(r)})$ is equal to*

$$\frac{(1 - t^{2^{r+1}})^2}{(1 - t^2)^2} + t\frac{1 - t^{2^{r+2}}}{1 - t^4};$$

- *The Poincaré series of* $\mathrm{H}^*_{\mathcal{P}}(GL, \Gamma^{2(r)})$ *is equal to*

$$\frac{(1-t^{2^{r+1}})^2}{(1-t^2)^2} + t^2 \frac{1-t^{2^{r+2}}}{1-t^4}.$$

Remark 5.2. *It is interesting that our computations show that* $H^*_{\mathcal{P}}(GL, \Lambda^{2(r)})$ *can be non-zero in odd degrees (for* $p = 2$*), so that this is not given as* $\Lambda^2(E_r)$ *as one might hope.*

5.1. **Complexes and hypercohomology spectral sequences.** Adapting the techniques of [6], we utilize the de Rham and Koszul complexes of strict polynomial functors of degree 2, as well as the “Symmetric complex” which is special to characteristic 2:

$$\Omega, d: \qquad S^2 \xrightarrow{d^1} \otimes^2 \xrightarrow{d^2} \Lambda^2 \longrightarrow 0,$$

$$K, \kappa: \qquad \Lambda^2 \xrightarrow{\kappa^1} \otimes^2 \xrightarrow{\kappa^2} S^2 \longrightarrow 0,$$

$$S, \partial: \qquad S^2 \xrightarrow{d^1} \otimes^2 \xrightarrow{\kappa^2} S^2 \longrightarrow 0.$$

The Koszul complex is acyclic; the de Rham complex has cohomology $I^{(1)}$ in degree 0 and 1; the Symmetric complex has cohomology $I^{(1)}$ in degree 0.

The maps in these complexes belong to a commutative diagram:

(5.2.1)
$$\begin{array}{ccccc} & & S^2 & & \\ & \overset{\kappa^2}{\nearrow} & & \overset{d^1}{\searrow} & \\ \otimes^2 & & \xrightarrow{\quad 1+\tau \quad} & & \otimes^2 \\ & \underset{d^2}{\searrow} & & \underset{\kappa^1}{\nearrow} & \\ & & \Lambda^2 & & \end{array}.$$

Here, τ denotes the permutation of the two factors in $\otimes^2$, so that the composite from left to right is the norm map $1 + \tau$.

We shall employ four of the hypercohomology spectral sequences obtained by applying $\mathrm{Hom}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{2^{r+1}}\mathcal{H}om(I,I), -)$ to injective resolutions of the de Rham, Koszul, and Symmetric complexes. Namely,

- “1st de Rham” ${}_\Omega\mathrm{I}$: E_1 page ${}_\Omega\mathrm{I}_1$ has the form

$$\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) \overset{d^1_*}{\to} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) \overset{d^2_*}{\to} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \Lambda^{2(r)}).$$

- “2nd de Rham” ${}_\Omega\mathrm{II}$: E_2 page ${}_\Omega\mathrm{II}_2$ has the form

$$\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, I^{(r)}) \overset{d_2^{*,1}}{\to} \mathrm{H}^{*+2}_{\mathcal{P}}(\mathrm{GL}, I^{(r)}).$$

- “1st Koszul” ${}_K\mathrm{I}$: E_1 page ${}_K\mathrm{I}_1$ has the form

$$\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \Lambda^{2(r)}) \overset{\kappa^1_*}{\to} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) \overset{\kappa^2_*}{\to} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}).$$

- "1st Symmetric" ${}_S\mathrm{I}$: E_1 page ${}_S\mathrm{I}_1$ has the form

$$\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) \xrightarrow{d^1_*} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) \xrightarrow{\kappa^2_*} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}).$$

5.2. **Computation of** $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)})$**.** The special case $n = 2$ of Theorem 1.8 establishes the first assertion of Theorem 5.1

More explicitly, $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)})$ is isomorphic, as a graded representation of the symmetric group $\mathfrak{S}_2$, to the $\mathfrak{S}_2$-module $E_r^{\otimes 2} \otimes k[\mathfrak{S}_2]$, where the permutation τ exchanges the two tensor factor and $E_r = \mathrm{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})$ as in Theorem 1.6 . Note that because the symmetric group $\mathfrak{S}_2$ is commutative, the action by conjugation on $k[\mathfrak{S}_2]$ is trivial. So, for $0 \leq n < 2^{r+2} - 2$, $\mathrm{H}^n(\mathrm{GL}, \otimes^{2(r)})$ is zero in odd degrees, it is free over $\mathfrak{S}_2$ if $n \equiv 2$ $(mod\ 4)$, modulo 4, and for $n \equiv 0$ $(mod\ 4)$, it is a sum of a free $\mathfrak{S}_2$-module with two trivial $\mathfrak{S}_2$-modules generated by classes of the form $x \otimes x \otimes 1$ and $x \otimes x \otimes \tau$.

5.3. **Determination of the differential of** ${}_\Omega\mathrm{II}_2$**.** Recall that the 2nd de Rham hypercohomology spectral sequence has E_2-page with only two non-zero rows: ${}_\Omega\mathrm{II}_2^{*,1}, {}_\Omega\mathrm{II}_2^{*,0}$. Both these rows are given by $H^*(GL, I^{(r)}) \cong E_r$, which is 1 dimensional in non-negative even degrees less than 2^{r+1} and 0 otherwise. Thus, the following proposition completely determines this spectral sequence.

Proposition 5.3. *For any $n \geq 0$, the differential*

$$d_2^{n,1} : {}_\Omega\mathrm{II}_2^{n,1} = H^{n+2}(GL, I^{(r)}) \;\to\; H^n(GL, I^{(r)}) = {}_\Omega\mathrm{II}_2^{n+2,0}$$

is an isomorphism if $n \equiv 0$ $(mod\ 4)$ and is 0 when $n \equiv 2$ $(mod\ 4)$.

Consequently,

$${}_\Omega\mathrm{II}_\infty^{n,i} = \begin{cases} k & \textit{if } 0 \leq n \leq 2^{r+2} - 2, n \equiv 0 \ (mod\ 4); i = 0 \\ k & \textit{if } 0 \leq n \leq 2^{r+2} - 2, n \equiv 3 \ (mod\ 4); i = 1 \\ 0 & \textit{otherwise} \end{cases}$$

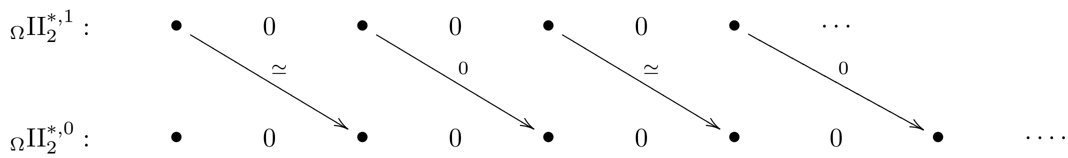
Proof. By general arguments on hypercohomology spectral sequences, the differential d_2 is left Yoneda product by a class e in $\mathrm{Ext}^2_{\mathcal{P}^{op}\times\mathcal{P}}(g\ell^{(r)}, g\ell^{(r)})$. The class e is obtained by precomposing with $g\ell$ the class $e_1 \in \mathrm{Ext}^2_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \mathrm{H}^2_{\mathcal{P}}(\mathrm{GL}, I^{(r)})$, where e_1 is constructed in [8] as the class multiplication by which gives the differential d_2 in a corresponding hypercohomology spectral sequence. Consequently, the isomorphism of Proposition 2.6

$$\gamma_r^* \circ \nabla : E_r = \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \xrightarrow{\cong} \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, I^{(r)})$$

sends the left Yoneda product with the class e_1 to the left Yoneda product with the class e. Since E_r is a divided power algebra on generators $e_1, \dots e_r$ of degrees $2 \cdot 2^0, \dots, 2 \cdot 2^{r-1}$ as recalled in Theorem 1.6, we conclude that the differential d_2: $\mathrm{H}^n(\mathrm{GL}, I^{(r)}) \to \mathrm{H}^{n+2}(\mathrm{GL}, I^{(r)})$ is an isomorphism when n is a multiple of 4, and it vanishes in other degrees. □

Perhaps the following diagram summarizing the assertion of Proposition 5.3 will help the reader when this proposition is applied below.

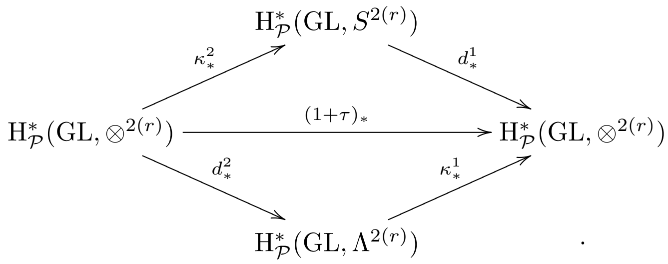
(5.3.1)



5.4. **Partial determination of ranks of $d^1_*, d^2_*, \kappa^1_*, \kappa^2_*$.** We proceed to consider the differentials of the first page of the first hypercohomology de Rham and Koszul spectral sequences.

The differential d_1 on the first page of the first hypercohomology spectral sequences are the maps induced by d and κ. Upon applying $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{2^{r+1}}, -)$ to (5.2.1), we obtain the commutative diagram:

(5.3.2)



The composite from left to right is induced by the norm map $1 + \tau$.

Proposition 5.4. *The columns $_\Omega I_2^{1,*}$, $_K I_2^{1,*}$, $_S I_2^{1,*}$ of the E_2-pages of the 1st Koszul, de Rham, and Symmetric spectral sequences have all entries equal to 0.*

Proof. Recall that the columns $_\Omega I_1^{1,*}$, $_K I_1^{1,*}$, $_S I_1^{1,*}$ are each given by $E_r^{\otimes 2} \otimes k\mathfrak{S}_2$ which vanishes in odd degree. Because the Koszul spectral sequence converges to 0, and because the Symmetric spectral sequence converges to E_{r+1} which vanishes in even degrees, we conclude that $_K I_2^{1,*} \ = \ _S I_2^{1,*} \ = \ 0$.

From the determination of $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)})$ made explicit in §5.2, it follows that

(5.4.1) $$\mathrm{rk}((1+\tau)_*) = \ker((1+\tau)_*) = \dim_n(E_r^{\otimes 2}), \quad n \equiv 2 \ (mod\ 4)$$

(5.4.2)
$$\mathrm{rk}((1+\tau)_*) = \dim_n(E_r^{\otimes 2}) - 1, \quad \ker((1+\tau)_*) = \dim_n(E_r^{\otimes 2}) + 1, \quad n \equiv 0 \ (mod\ 4)$$

for $0 \leq n < 2^{r+2} - 2$, where $\mathrm{rk}(-)$ denotes the dimension of the image, $\ker(-)$ denotes the dimension of the kernel, and $\dim_n(-)$ denotes the dimension of the homogeneous part of degree n.

Thus, in degrees $n \equiv 2 \ (mod\ 4)$, $_\Omega I_2^{1,n} = 0$. Since the de Rham spectral sequences have abutment which vanishes in degrees $n \equiv 2 \ (mod\ 4)$ by Proposition 5.3, we conclude that $_\Omega I_2^{1,n}$ also vanishes for $n \equiv 0 \ (mod\ 4)$.

□

Proposition 5.5. *We have the following equalities among the ranks of the d_1 differentials in the first hypercohomology spectral sequences:*

(5.5.1) $$\mathrm{rk}(\kappa^1_*) \ = \ \mathrm{rk}(d^1_*), \quad \mathrm{rk}(\kappa^2_*) \ = \ \mathrm{rk}(d^2_*).$$

Proof. Proposition 5.4 implies that each of the following sums equals $2 \cdot \dim(E_r^{\otimes 2})$:

$$\mathrm{rk}(\kappa_*^1) + \mathrm{rk}(\kappa_*^2) \; = \; \mathrm{rk}(d_*^1) + \mathrm{rk}(d_*^2) \; = \; \mathrm{rk}(d_*^1) + \mathrm{rk}(\kappa_*^2). \tag{5.5.2}$$

This immediately implies the asserted equalites. □

5.5. **The lower bound** $\mathrm{rk}(\kappa_*^2) \geq \dim(E_r)$**.** In order to determine the ranks of the differentials $\kappa_*^1, \kappa_*^2, d_*^1, d_*^2$ in our 1st hypercohomology spectral sequences in degrees $n \equiv 0 \ (mod\ 4)$, we need more information than that provided by Proposition 5.5. We shall establish a lower bound using the natural isomorphism (1.5.1):

$$\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, A) \; \cong H^*_{rat}(GL(W), A(\mathrm{Hom}_k(W,W)) \quad \text{provided } \dim_k(W) \geq d$$

for A a strict polynomial functor of degree d. We simplify the notation by setting

$$GL(W) := \; GL_n, \quad g\ell_n^{(r)} := \; I^{(r)}(\mathrm{Hom}_k(W,W)), \quad \text{if } \dim_k(W) = n. \tag{5.5.3}$$

Recall that an r-tuple $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1})$ of 2-nilpotent, pairwise commuting $n \times n$ matrices determines a *1-parameter subgroup*

$$exp_{\underline{\alpha}} : \mathbb{G}_{a(r)} \; \to \; GL_n,$$

a morphism of group schemes from the r^{th} Frobenius kernel of the additive group $\mathbb{G}_a$ to GL_n. As in [14], we use such 1-parameter subgroups to establish the non-triviality of a certain class in the proof of the following step in our computation.

Proposition 5.6. *We have the following inequality:*

$$\mathrm{rk}(\kappa_*^2) \; \geq \; \dim(E_r^{\otimes 2}).$$

More precisely, in degrees congruent to 2 modulo 4, this inequality is an equality and in degrees congruent to 0 modulo 4, this inequality can be refined to

$$\dim(E_r^{\otimes 2}) + 1 \; \geq \mathrm{rk}(\kappa_*^2) \; \geq \; \dim(E_r^{\otimes 2}).$$

Proof. We use the following evident inequalities:

$$\mathrm{rk}((1+\tau)_*) = \mathrm{rk}(d_*^1\kappa_*^2) \leq \mathrm{rk}(d_*^1) \leq \ker(d_*^2) \leq \ker((1+\tau)_*).$$
$$\mathrm{rk}((1+\tau)_*) = \mathrm{rk}(\kappa_*^1 d_*^2) \leq \mathrm{rk}(\kappa_*^1) \leq \ker(\kappa_*^2) \leq \ker((1+\tau)_*).$$

Thus, (5.4.1) implies that $\mathrm{rk}(\kappa_*^2) \; = \; \dim(E_r^{\otimes 2})$ in degrees congruent to 2 modulo 4 and that

$$\dim(E_r^{\otimes 2}) + 1 \geq \mathrm{rk}(\kappa_*^2) \; \geq \; \dim(E_r^{\otimes 2}) - 1$$

in degrees congruent to 0 modulo 4.

To prove the inequality $\mathrm{rk}(\kappa_*^2) \; \geq \; \dim(E_r^{\otimes 2})$ in degrees congruent to 0 modulo 4, it suffices to prove that the $\mathfrak{S}_2$-invariant class

$$e_r(j) \otimes e_r(j) \otimes 1 \; \in \; E_r^{\otimes 2} \otimes k\mathfrak{S}_2 \cong \mathrm{H}^{4j}(GL, \otimes^{2(r)})$$

maps non-trivially via κ_*^2 to $\mathrm{H}^{4j}(\mathrm{GL}, S^{2(r)})$ for each $j, 0 \leq j < 2^{r-1}$. Here $e_r(j)$ is the generator constructed in [8] (cf. [14, 3.1]) of the (1-dimensional) homogeneous summand of E_r in degree $2j$. We write $j = \sum_{i=0}^{r-1} j_i \cdot 2^i, \; 0 \leq j_i \leq 1$.

Consider the following commutative diagram

(5.6.1)
$$\begin{array}{ccccc}
\mathrm{H}^{2j}(\mathrm{GL}, I^{(r)})^{\otimes 2} & \xrightarrow{\mu} & \mathrm{H}^{4j}(\mathrm{GL}, \otimes^{(r)}) & \xrightarrow{(\kappa^2_*)} & \mathrm{H}^{4j}(\mathrm{GL}, S^{2(r)}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathrm{H}^{2j}(\mathrm{GL}_n, g\ell_n^{(r)})^{\otimes 2} & \xrightarrow{\mu} & \mathrm{H}^{4j}(\mathrm{GL}_n, g\ell_n^{(r)\otimes 2}) & \xrightarrow{(\kappa^2_*)} & \mathrm{H}^{4j}(\mathrm{GL}_n, S^{2(r)} g\ell_n) \\
\downarrow exp_{\underline{\alpha}^*} & & \downarrow exp_{\underline{\alpha}^*} & & \downarrow exp_{\underline{\alpha}^*} \\
\mathrm{H}^{2j}(\mathbb{G}_{a(r)}, g\ell_n^{(r)})^{\otimes 2} & \xrightarrow{\mu} & \mathrm{H}^{4j}(\mathbb{G}_{a(r)}, g\ell_n^{(r)\otimes 2}) & \xrightarrow{(\kappa^2_*)} & \mathrm{H}^{4j}(\mathbb{G}_{a(r)}, S^{2(r)} g\ell_n)
\end{array}$$

where $n \geq 2^{r+1}$, the upper vertical arrows are the isomorphisms of (5.5.3), μ is external cup product, and $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in M_n(k)^{\times r}$ is an r-tuple of pairwise commuting, square-zero matrices whose product $\times_i \alpha_k \in M_n(k)$ is non-zero.

By [14, 4.9], we conclude that $e_r(j) \otimes e_r(j) \in \mathrm{H}^{2j}(\mathrm{GL}, I^{(r)})^{\otimes 2}$ restricts to a non-zero element of the form

(5.6.2)
$$(x_1^{j_0 2^0} \cdot x_2^{j_1 2^1} \cdot \ldots \cdot x_r^{j_{r-1} 2^{r-1}} \otimes \alpha_0^{(r)j_0} \cdot \alpha_1^{(r)j_1} \cdot \ldots \cdot \alpha_{r-1}^{(r)j_{r-1}})^{\otimes 2} \in \mathrm{H}^{2j}(\mathbb{G}_{a(r)}, g\ell_n^{(r)})^{\otimes 2}.$$

Recall that
$$\mathrm{H}^*(\mathbb{G}_{a(r)}, g\ell_n^{(r)}) \cong \mathrm{H}^*(\mathbb{G}_{a(r)}, k) \otimes g\ell_n^{(r)}$$
and that $\mathrm{H}^*(\mathbb{G}_{a(r)}, k)$ is a polynomial algebra on generators λ_i with $\lambda_i^2 = x_i$. Thus, (5.6.2) imaps to a non-zero element of $\mathrm{H}^{4j}(\mathbb{G}_{a(r)}, S^{2(r)} g\ell_n) \cong \mathrm{H}^{4j}(\mathbb{G}_{a(r)}, k) \otimes S^{2(r)} g\ell_n$.

On the other hand, μ sends the external square of $e_j \in \mathrm{H}^{2j}(\mathrm{GL}, I^{(r)})^{\otimes 2}$ to $e_r(j) \otimes e_r(j) \otimes 1 \in \mathrm{H}^{4j}(\mathrm{GL}, \otimes^{(r)})$. Thus, a simple diagram chase implies that $\kappa^2_*(e_r(j) \otimes e_r(j) \otimes 1) \neq 0 \in \mathrm{H}^{4j}(\mathrm{GL}, S^{2(r)})$ as required. □

Remark 5.7. *The preceding technique of reducing to 1-parameter subgroups plays a key role in the identification given in* [14] *of the Hopf algebra structure recalled in Lemma 2.5.*

5.6. **1st hypercohomology spectral sequences, E_1-pages modulo 4.** As we shall see, we will be able to analyze the De Rham, Koszul, and Symmetric 1st hypercohomology spectral sequences by considering cohomology degrees modulo four. As seen in subsection 5.1, these have the following appearance. The ranks of various maps in these diagrams are indicated with notation above the arrows.

(5.7.1)
$$\begin{array}{ccccc}
\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) & \xrightarrow{d^1_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) & \xrightarrow{d^2_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \Lambda^{2(r)}) \\
s_{4n+3} & & 0 & & \lambda_{4n+3} \\
s_{4n+2} & \xrightarrow{=g_n} & 2g_n & \xrightarrow{=g_n} & \lambda_{4n+2} \\
s_{4n+1} & & 0 & & \lambda_{4n+1} \\
s_{4n} & \longrightarrow & 2f_n & \xrightarrow{\geq f_n} & \lambda_{4n}
\end{array}$$

(5.7.2)
$$
\begin{array}{ccccc}
\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \Lambda^{2(r)}) & \xrightarrow{\kappa^1_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) & \xrightarrow{\kappa^2_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) \\
\lambda_{4n+3} & & 0 & & s_{4n+3} \\
\lambda_{4n+2} & \xrightarrow{=g_n} & 2g_n & \xrightarrow{=g_n} & s_{4n+2} \\
\lambda_{4n+1} & & 0 & & s_{4n+1} \\
\lambda_{4n} & \longrightarrow & 2f_n & \xrightarrow{\geq f_n} & s_{4n}
\end{array}
$$

and

(5.7.3)
$$
\begin{array}{ccccc}
\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) & \xrightarrow{d^1_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{2(r)}) & \xrightarrow{\kappa^2_*} & \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{2(r)}) \\
s_{4n+3} & & 0 & & s_{4n+3} \\
s_{4n+2} & \xrightarrow{=g_n} & 2g_n & \xrightarrow{=g_n} & s_{4n+2} \\
s_{4n+1} & & 0 & & s_{4n+1} \\
s_{4n} & \longrightarrow & 2f_n & \xrightarrow{\geq f_n} & s_{4n}
\end{array}
$$

where n is some cohomological degree $< 2^{p^{r-1}}$, $2f_n = \dim_k \mathrm{H}^{4n}(GL, \otimes^{2(r)}), 2g_n = \dim_k \mathrm{H}^{4n+2}(GL, \otimes^{2(r)})$, where s_m, λ_m are the dimensions of the indicated cohomology groups (e.g., $s_{4n+3} = \mathrm{H}^{4n+3}(\mathrm{GL}, S^{2(r)})$), and where the equalities/inequalities above the arrows indicate the dimensions of the ranks of the maps. Recall that the inequalities in degrees congruent to 0 modulo 4 are given by Proposition 5.6.

We shall proceed to determine each s_i, λ_i as well as the ranks of each of the maps on these E_1-pages. Observe that this completely determines the spectral sequences for the abutments are readily determined by Proposition 5.3 in the case of the de Rham spectral sequence, is 0 in the case of the Koszul spectral sequence, and equals E_{r+1} in the case of the symmetric spectral sequence.

5.7. **Completion of analysis of page 1 of 1st spectral sequences.** We proceed by a sequence of elementary steps. We assume inductively (with respect to n) that

(1) the differential ${}_K d_2^{4n,0}$ of the Koszul spectral sequence is the 0-map
(2) the differential ${}_\Omega d_2^{4n,0}$ of the Koszul spectral sequence is the 0-map
(3) $\lambda_{4n-2} = g_{n-1}$.

Lemma 5.8. *We have the following equality:*

$$\lambda_{4n} \;=\; f_n.$$

Consequently,

$$\mathrm{rk}(\kappa^1_*) = f_n = \mathrm{rk}(\kappa^2_*).$$

Proof. The vanishing of the abutment of the Koszul spectral sequence and our inductive hypothesis that ${}_K d_2^{4n,0}$ vanishes imply that $\lambda_{4n} \to 2f_n$ is injective in the row $E_1^{4n,*}$ of the Koszul spectral sequence. Since the rank of $2f_n \to s_{4n}$ in this row is $\geq f_n$, we conclude that $\lambda_n \leq f_n$. On the other hand, from the de Rham spectral sequence we see that $\lambda_{4n} \geq f_n$.

This establishes that the rank of κ_*^2 equals f_n which immediately implies that the rank of κ_*^1 is also f_n. □

Lemma 5.9. *We have the following equality:*

$$s_{4n} \;=\; f_n + 1.$$

Proof. We employ the de Rham spectral sequence. Our inductive hypothesis asserts that ${}_\Omega d_2^{4n,0} : s_{4n} \to \lambda_{4n-1}$ is 0 and that $\lambda_{4n-2} = g_{n-1}$. Thus, Lemma 5.8 implies s_{4n} equals f_n plus the dimension of the abutment (i.e. E_{r+1}) in dimension $4n$. The lemma now follows from Proposition 5.3. □

Lemma 5.10. *We have the following equality:*

$$\lambda_{4n+1} \;=\; 1.$$

Proof. Since the abutment of the Koszul spectral sequence is 0, this follow immediately from Lemma 5.9 and the Koszul spectral sequence. □

Lemma 5.11. *We have the following equality:*

$$s_{4n+1} \;=\; 0.$$

Consequently, a 1-dimensional subspace of ${}_SI_1^{4n,2}$ survives to ${}_SI_\infty^{4n,2}$

Proof. By Proposition 5.3, the abutment of the de Rham spectral sequence vanishes in degree $4n+1$. Consequently, Lemma 5.8 applied to the de Rham spectral sequence implies the vanishing of s_{4n+1}. The second statement follows by applying the vanishing of s_{4n+1} and Lemma 5.9 to the symmetric spectral sequence. □

Lemma 5.12. *We have the following equality:*

$$s_{4n+2} \;=\; g_n \;=\; \lambda_{4n+2}.$$

Thus, our third induction hypothesis is verified for $n+1$.

Moreover, $\lambda_{4n+1} \in {}_DI_1^{4n,1}$ survives to ${}_DI_\infty^{4n,1}$

Proof. Since the abutment of the Symmetric spectral sequence is E_{r+1}, Lemmas 5.9 and 5.11 in conjunction with this spectral sequence imply $s_{4n+2} = g_n$. The vanishing of s_{4n+1} given by Lemma 5.11 and of the abutment of the Koszul spectral sequence implies that $\lambda_{4n+2} = g_n$.

Now, Proposition 5.3 and the equality $s_{4n+2} = g_n$ fed into the De Rham spectral sequence tells us that the one dimensional class of λ_{4n+1} established in Lemma 5.10 must survive. □

Lemma 5.13. *We have the following equality:*

$$\lambda_{4n+3} \;=\; 0.$$

Thus, our second induction hypothesis is verified for $n+1$.

Proof. Lemma 5.12 implies that the differential ${}_Kd_2^{4n+3,0}$ in the Koszul spectral sequence must be 0. Since the abutment of this spectral sequence vanishes, $\lambda_{4n+3} \;=\; 0$. □

Lemma 5.14. *We have the following equality:*

$$s_{4n+3} \;=\; 0.$$

Thus, our first induction hypothesis is verified for $n+1$.

Proof. Lemma 5.12 implies that the differential $d_2^{4n+3,0}$ in the Symmetric spectral sequence must be 0. Since the abutment of this spectral sequence vanishes in odd degrees, s_{4n+3} must vanish. □

5.8. **Completion of proof of Theorem 5.1.** The determination of the cohomology of $\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)})$ is given in Subsection 5.2. Its Poincaré series is verified by inspection.

We determine the Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(GL, S^{2(r)})$ from the determination of the values named in (5.7.3) given by Lemmas 5.9, 5.11, 5.12, and 5.14: this cohomology is half that of $\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)})$ plus an extra dimension in each cohomology degree $\equiv 0(mod\ 4)$ through degrees $\leq 2^{r+2}-4$. Similarly, the asserted Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{2(r)})$ follows from the determination of the values named in (5.7.2), given by Lemmas 5.8, 5.10, 5.12 and 5.13; this cohomology is half that of $\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)})$ plus an extra dimension in each cohomology degree $\equiv 1\ (mod\ 4)$ through degrees $\leq 2^{r+2}-3$.

To obtain the Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(GL, \Gamma^{2(r)})$, we use the dual of the (twisted, exact) Koszul complex:

$$0 \to\ \Gamma^{2(r)}\ \to\ \otimes^{2(r)}\ \to\ \Lambda^{2(r)}\ \to 0.$$

Since we know the ranks of the induced maps in cohomology in each degree (each rank equal to half the dimension of the cohomology of $\otimes^{2(r)}$ in that degree), the long exact sequence in cohomology implies that $\mathrm{H}^*_{\mathcal{P}}(GL, \Gamma^{2(r)})$ is half that of $\mathrm{H}^*_{\mathcal{P}}(GL, \otimes^{2(r)})$ plus an extra dimension in each cohomology degree $\equiv 2\ (mod\ 4)$ through degrees $\leq 2^{r+2}-2$.

6. Computations of cohomology for $A^d(g\ell^{(r)}) \otimes g\ell^{(r)}$

We recall that a (strict polynomial) functor $A \in \mathcal{P}_d$ is said to be of exponential type (cf. [9]) if it belongs to a sequence of functors $A^0, \ldots, A^d = A, A^{d+1}\ldots$ such that $A^n(V \oplus W) = \bigoplus_{i+j=n} A^i(V) \otimes A^j(W)$. Necessarily, $A^0 = 0$; we shall assume that $A^1 \;=\; I$ (i.e, the identity functor in $\mathcal{P}_1$).

Theorem 6.1. *Let* $A^d \in \mathcal{P}_d$ *be of exponential type for some* $d > 1$. *Then*
(6.1.1)

$$\mathrm{H}^*_{\mathcal{P}}(GL, A^d(g\ell^{(r)}) \otimes g\ell^{(r)}) \;\cong\; \Big(\mathrm{H}^*_{\mathcal{P}}(GL, A^d(g\ell^{(r)})) \oplus \mathrm{H}^*_{\mathcal{P}}(GL, A^{d-1}(g\ell^{(r)}) \otimes g\ell^{(r)})\Big) \otimes E_r.$$

Proof. We repeatedly use the adjunction isomorphisms of Proposition 1.7.

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{(d+1)p^r} g\ell, A^d(g\ell^{(r)}) \otimes g\ell^{(r)}) :=$$

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{(d+1)p^r}\mathcal{H}om(-_1,-_2), A^d(\mathcal{H}om(-_1,-_2)^{(r)} \otimes \mathcal{H}om(-_1,-_2)^{(r)}) =$$

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}\times\mathcal{P}^{op}}(\Gamma^{(d+1)p^r}\mathcal{H}om(-_1,-_2), A^d\mathcal{H}om(-_1,-_2)^{(r)} \otimes \mathcal{H}om(-_3,-_2)^{(r)} \circ D_{1,3})$$

which by adjunction equals

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}\times\mathcal{P}^{op}}(\Gamma^{(d+1)p^r}\mathcal{H}om(-_1+-_3,-_2), A^d\mathcal{H}om(-_1,-_2)^{(r)} \otimes \mathcal{H}om(-_3,-_2)^{(r)}.)$$

In order to compactify the notation, we replace $\mathrm{Ext}^*_{\mathcal{P}^{op}\times\cdots\times\mathcal{P}}(-,-)$ by $[-,-]$ and we replace $\mathcal{H}om$ by $\mathcal{H}$. Expanding out the exponential functor $\Gamma^{(d+1)p^r}(V\oplus W)$ and dropping terms whose degrees in some variable do not match in contravariant/covariant entries, we verify that the previous line equals in our new notation

$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2)\otimes\Gamma^{p^r}\mathcal{H}(-_3,-_2), A^d\mathcal{H}(-_1,-_2)^{(r)}\otimes\mathcal{H}(-_3,-_2)^{(r)}].$$

We apply adjunction again to get that this equals

$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2)\otimes\Gamma^{p^r}\mathcal{H}(-_3,-_4), A^d\mathcal{H}(-_1,-_2+-_4)^{(r)}\otimes\mathcal{H}(-_3,-_2+-_4)^{(r)}].$$

We now expand $A^d\mathcal{H}(-_1,-_2+-_4)^{(r)}\otimes\mathcal{H}(-_3,-_2+-_4)^{(r)}$ using the exponential property for A^* and the additivity of $\mathcal{H}om$. By once again dropping the terms whose degrees in some variable do not match in contravariant/covariant entries, we obtain

$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2)\otimes\Gamma^{p^r}\mathcal{H}(-_3,-_4), A^d\mathcal{H}(-_1,-_2)^{(r)}\otimes\mathcal{H}(-_3,-_4)^{(r)}]\oplus$$
$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2)\otimes\Gamma^{p^r}\mathcal{H}(-_3,-_4), A^{d-1}\mathcal{H}(-_1,-_2)^{(r)}\otimes\mathcal{H}(-_1,-_4)\otimes\mathcal{H}(-_3,-_2)].$$

After rearranging variables on the right hand side, we apply the Kúnneth theorem to obtain

$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2), A^d\mathcal{H}(-_1,-_2)^{(r)}]\otimes[\Gamma^{p^r}\mathcal{H}(-_3,-_4),\mathcal{H}(-_3,-_4)^{(r)}]\oplus$$
$$[\Gamma^{dp^r}\mathcal{H}(-_1,-_2), A^{d-1}\mathcal{H}(-_1,-_2)^{(r)}\otimes\mathcal{H}(-_1,-_2)^{(r)}]\otimes[\Gamma^{p^r}\mathcal{H}(-_3,-_4),\mathcal{H}(-_3,-_4)^{(r)}].$$

This is the computation asserted in (6.1.1), since $E_r=[\Gamma^{p^r}\mathcal{H}om(-,-),\mathcal{H}om(-,-)^{(r)}]$. □

Combining Theorems 5.1 and 6.1, we obtain the following explicit computations for k a field of characteristic 2.

Corollary 6.2. *If k is a field of characteristic 2 and $r\geq 0$ a non-negative integer, then*

$$\begin{aligned}
\mathrm{H}^*_{\mathcal{P}}(GL, S^{2(r)}\otimes I^{(r)}) &\cong (\mathrm{H}^*_{\mathcal{P}}(GL, S^{2(r)})\otimes E_r)\oplus(E_r^{\otimes 3}\oplus E_r^{\otimes 3})\\
\mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{2(r)}\otimes I^{(r)}) &\cong (\mathrm{H}^*_{\mathcal{P}}(GL, \Lambda^{2(r)})\otimes E_r)\oplus(E_r^{\otimes 3}\oplus E_r^{\otimes 3})\\
\mathrm{H}^*_{\mathcal{P}}(GL, \Gamma^{2(r)}\otimes I^{(r)}) &\cong (\mathrm{H}^*_{\mathcal{P}}(GL, \Gamma^{2(r)})\otimes E_r)\oplus(E_r^{\otimes 3}\oplus E_r^{\otimes 3}).
\end{aligned}$$

7. Comparison with cohomology of bifunctors

Throughout this section, our base field k will be assumed finite of order $q=p^e$ for some prime p. We now consider the category $\mathcal{F}$ of all functors from the category $\mathcal{V}$ of finite dimensional vector spaces over our base field k to the category of all k vector spaces, and the category $\mathcal{F}^{op}\times\mathcal{F}$ of bifunctors. We wish to study the induced map on cohomology of the forgetful functors

$$\mathcal{P}\ \to\ \mathcal{F},\qquad \mathcal{P}^{op}\times\mathcal{P}\ \to\ \mathcal{F}^{op}\times\mathcal{F}.$$

A significant aspect of the forgetful functor $\mathcal{P}\to\mathcal{F}$ is that the Frobenius twist $I^{(e)}\in\mathcal{P}$ maps to the identity functor in $\mathcal{F}$ and thus each $I^{(r)}\in\mathcal{P}$ becomes an invertible element (with respect to composition) in $\mathcal{F}$.

Recall that $F\in\mathcal{F}$ is said to be *finite* if it takes values in $\mathcal{V}$ (i.e., $F(V)$ is finite dimensional for every finite dimensional vector space V) and if it has finite

Eilenberg-MacLane degree (i.e., for some n, the n-th difference functor $\Delta^n F$ is 0). Any functor in $\mathcal{F}$ which is a strict polynomial functor (i.e., in the image of the forgetful functor $\mathcal{P} \to \mathcal{F}$) is finite. As shown in [13], [6], any finite functor has a resolution by projective objects in $\mathcal{F}$ which are finite direct sums of functors of the form $k[\mathrm{Hom}_k(W,-)]$ for some $W \in \mathcal{V}$ (defined by sending $V \in \mathcal{V}$ to the k-vector space on the underlying set of $\mathrm{Hom}_k(W,V)$).

For any $F \in \mathcal{F}$ and any $i, 1 \leq i \leq q-1$, we define $F^i \subset F$ to be the subfunctor whose value on $V \in \mathcal{V}$ consists of those elements $x \in F(V)$ such that $F(\mu) : F(V) \to F(V)$ maps x to $\mu^i x$ for any $\mu \in k^*$. If $\phi : F \to G$ is a map in $\mathcal{F}$, then the map ϕ restricts to a map $F^i \to G^i$ for each i. If F is a finite functor, then $F \cong \oplus_i F^i$. If $P \in \mathcal{P}_d$ is a strict polynomial functor of degree d, then its image under the forgetful functor is a finite functor of weight d (where d is taken modulo $q-1$).

We obtain injective objects in $\mathcal{F}$ by dualization of projectives: the injective $(k[\mathrm{Hom}_k((-)^\#, W)]) \in \mathcal{F}$ sends $V \in \mathcal{V}$ to the k-vector space on the underlying set of $\mathrm{Hom}_k(V^\#, W)$, where $V^\#$ is the linear dual of V. Thus, we have basic projective and injective objects in $\mathcal{F}^{op} \times \mathcal{F}$,

$$P_{W_1,W_2} = k[\mathrm{Hom}_k[(-), W_1)] \otimes k[\mathrm{Hom}_k(W_2, (-)]$$

$$I_{W_1,W_2} = k[\mathrm{Hom}_k(W_1, (-)^\#)] \otimes k[\mathrm{Hom}_k((-)^\#, W_2)].$$

The following somewhat ad hoc definition will be adequate for our purposes.

Definition 7.1. *A bifunctor $F \in \mathcal{F}$ is said to be finite if i.) $F(V,W)$ is finite dimensional for all $V, W \in \mathcal{V}$; ii.) F admits a resolution by projectives which are finite direct sums of basic projective bifunctors (i.e., of the form P_{W_1,W_2} with $W_1, W_2 \in \mathcal{V}$); and iii.) $\mathcal{F}$ admits a resolution injectives which are finite direct sums of basic injective bifunctors (i.e., of the form I_{W_1,W_2} with $W_1, W_2 \in \mathcal{V}$).*

For example, any bifunctor of the form $\mathcal{H}om(A_1, A_2)$ is finite whenever $A_1, A_2 \in \mathcal{F}$ are finite functors.

Proposition 7.2. *The forgetful functor $\Phi : \mathcal{P}^{op} \times \mathcal{P} \to \mathcal{F}^{op} \times \mathcal{F}$ sends a strict polynomial bifunctor of bounded degree to a finite bifunctor. In other words, any strict polynomial bifunctor T admits a resolution by projectives which are finite direct sums of basic projective bifunctors P_{W_1,W_2} and a resolution by injectives which are finite direct sums of basic injective biunctors I_{W_1,W_2}.*

Proof. We establish the existence of such a projective resolution for any strict polynomial bifunctor T; the argument for the existence of a corresponding injective resolution is similar.

We first assume that $T = \mathcal{H}om(A,B)$ is of separable type, where A, B are strict polynomial functors. Since every strict polynomial functor P is a finite functor (as observed in [9]), a resolution of A by finite direct sums of basic injective functors and a resolution of B by finite direct sums of basic projective functors determine (upon external tensor product) a resolution of T by finite direct sums of basic projective bifunctors.

A general strict polynomial bifunctor T admits a resolution by strict polynomial functors of separable type by Proposition 1.2. Thus, taking a resolution of each term in such a resolution, we obtain a bicomplex consisting of finite direct sums of basic projective bifunctors whose total complex gives us the required resolution of T. □

A special role is played by the bifunctor

$$k[g\ell] =: k[\mathrm{Hom}_k(-,-)] \in \mathcal{F}^{op} \times \mathcal{F}$$

as we first see in the following analogue of (1.5.2). This is proved in a manner exactly parallel to the proof of Proposition 2.2.

Proposition 7.3. *For two finite functors A_1 and A_2, we consider the bifunctor $\mathcal{H}om(A_1, A_2)$ of separable type. For any bifunctor F, there is a convergent spectral sequence of the form*

(7.3.1)
$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{F}}(A_1(-_1), \mathrm{Ext}^t_{\mathcal{F}}(F(-_1,-_2), A_2(-_2))) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{F}^{op}\times\mathcal{F}}(F, \mathcal{H}om(A_1, A_2)),$$

natural in A_1, A_2 and F.

If $F = k[g\ell]$, then this spectral sequence collapses to give the natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], \mathcal{H}om(A_1, A_2)) \cong \mathrm{Ext}^*_{\mathcal{F}}(A_1, A_2). \tag{7.3.2}$$

If $F = \mathcal{H}om(B_1, B_2)$ with B_1, B_2 also finite functors, then this spectral sequence collapses to give the natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(\mathcal{H}om(B_1, B_2), \mathcal{H}om(A_1, A_2)) \cong \mathrm{Ext}^*_{\mathcal{F}}(B_1, B_2) \otimes \mathrm{Ext}^*_{\mathcal{F}}(A_1, A_2). \tag{7.3.3}$$

Proof. We first verify by inspection that

$$\mathrm{Hom}_{\mathcal{F}^{op}\times\mathcal{F}}(P_{W_1,W_2}, F) \cong F(W_1, W_2) \cong \mathrm{Hom}_{\mathcal{F}^{op}\times\mathcal{F}}(F, I_{W_1,W_2}).$$

As in Lemma 2.1, this implies the isomorphism

$$\mathrm{Hom}_{\mathcal{F}^{op}\times\mathcal{F}}(F(-_1,-_2), k[\mathrm{Hom}_k(W_1, (-_1)^\#)] \otimes k[\mathrm{Hom}_k((-_2)^\#, W_2)]) \cong$$
$$\mathrm{Hom}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{F}}(F(-_1,-_2), k[\mathrm{Hom}_k(W_1, (-_1)^\#)]), k[\mathrm{Hom}_k((-_2)^\#, W_2)]).$$

Taking $F = k[g\ell]$, we proceed exactly as in the proof of Proposition 2.2 to establish the spectral sequence (7.3.1) and prove (7.3.2).

To prove (7.3.3), we argue exactly as in the proof of Proposition 2.3. □

Using a theorem of A. Suslin [9, A.1], we obtain the following interpretation of bifunctor cohomology. We introduce the notation

$$\mathrm{H}^*(GL(k), F) =: \mathrm{H}^*(GL(n,k), F(k^n, k^n)), \quad n >> 0$$

for a finite bifunctor F, where $\mathrm{H}^*(GL(n,k), F(k^n, k^n))$ denotes the usual group cohomology of the finite group $GL(n,k)$ with coefficients in the $GL(n,k)$-module $F(k^n, k^n)$.

Theorem 7.4. *Let $F \in \mathcal{F}^{op} \times \mathcal{F}$ be a finite bifunctor. Then the natural map*

$$\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], F) \to \mathrm{H}^*(GL(k), F)$$

is an isomorphism.

Moreover, for a given cohomological degree s,

$$\mathrm{Ext}^s_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], F) \cong \mathrm{H}^s(GL(n,k), F(k^n, k^n)), \quad n >> s.$$

Proof. For $n >> s$ and A_1, A_2 finite functors, a theorem of W. Dwyer [3] asserts that the natural map

$$\mathrm{Ext}^s_{GL(n,k)}(A_1(k^n), A_2(k^n)) \to \mathrm{Ext}^s_{GL(n+1,k)}(A_1(k^{n+1}), A_2(k^{n+1}))$$

is an isomorphism; we denote the stable value by $\mathrm{Ext}^s_{GL(k)}(A_1, A_2)$. Suslin's theorem [9, A.1] asserts that the natural map

$$\mathrm{Ext}^*_{\mathcal{F}}(A_1, A_2) \;\to\; \mathrm{Ext}^*_{GL(k)}(A_1, A_2)$$

is an isomorphism. (Although Suslin's theorem requires that k be finite as we also require in this section, this isomorphism has been generalized by A. Scorichenko to an arbitrary field; we refer the interested reader to [12] and [7].) This isomorphism, Dwyer's stability, and Proposition 7.3 imply the two assertions of the theorem in the special case in which F is of the form $\mathcal{H}om(A_1, A_2)$ with A_1, A_2 finite functors.

For a general finite bifunctor F, we first consider cohomological degree 0. Let $0 \to F \to I^0 \to I^1$ be exact in $\mathcal{F}^{op} \times \mathcal{F}$ with both I^0, I^1 injective objects in $\mathcal{F}^{op} \times \mathcal{F}$ which are directs sums of (finite) bifunctors of the form I_{W_1,W_2}. Then we conclude the natural isomorphism

$$\mathrm{Hom}^0_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], F) \;\to\; \mathrm{H}^0(GL(k), F)$$

by the left exactness of both $\mathrm{Hom}_{\mathcal{F}^{op}\times\mathcal{F}}$ and H^0 together with the special case verified above; the second assertion for cohomological degree 0 also follows from left exactness.

To complete the proof, we argue by induction on the cohomological degree and use a devissage argument. Namely, choose a short exact sequence $0 \to F \to I \to G \to 0$ in $\mathcal{F}^{op} \times \mathcal{F}$ with I a direct sum of (finite) bifunctors of the form I_{W_1,W_2}. Then the two assertions in cohomological degree $s-1$ for G imply the corresponding assertions in cohomological degree s for F. □

In parallel with the notation $H^*_{\mathcal{P}}(GL, A)$ introduced in (1.4.1), we employ the notation

$$H^*_{\mathcal{F}}(GL, A) =:\; \mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], A \circ g\ell) \;=\; \mathrm{H}^*(GL(k), A(g\ell))$$

for a bifunctor of the form $A \circ g\ell$ with A a functor of finite type.

To convey the similarities and differences of computations of cohomology in $\mathcal{P}^{op} \times \mathcal{P}$ and $\mathcal{F}^{op} \times \mathcal{F}$, we state and prove the analogue of Theorem 1.8.

Proposition 7.5. *Let $E_\infty =: \varinjlim_r E_r$, a divided power algebra on the infinite sequence of generators $e_1, \ldots, e_r, \ldots$ with the degree of e_i equal to $2p^{i-1}$. For any n, we have a $\mathfrak{S}_n$-equivariant isomorphism*

$$H^*_{\mathcal{F}}(GL, \otimes^n) \;\cong\; \varinjlim_r H^*(GL, \otimes^{n(r)}) \;\cong\; E_\infty^{\otimes n} \otimes k[\mathfrak{S}_n].$$

Proof. The adjunction isomorphisms of Proposition 1.7 are equally valid with $\mathcal{P}$ replaced by $\mathcal{F}$ (i.e., with strict polynomial multi-functors replaced by multi-functors).

To sort out

$$\mathrm{Ext}^*_{\mathcal{F}\times\cdots\times\mathcal{F}}(\otimes^n \circ \bigoplus, I \boxtimes \cdots \boxtimes I), \tag{7.5.1}$$

the degree argument can be replaced by a second adjunction comparing functors in $n-1$ and n variables. Consider the exact functors

$$\epsilon_0 : \mathcal{V}^{\times n-1} \;\rightleftarrows\; \mathcal{V}^{\times n} : \iota \tag{7.5.2}$$

which are left and right adjunct to each other, ϵ_0 given by $(V_1, \ldots, V_{n-1}) \mapsto (V_1, \ldots, V_{n-1}, 0)$ and ι given by $(V_1, \ldots, V_n) \mapsto (V_1, \ldots, V_{n-1})$. Thus, precomposition by ι and ϵ_0 determine left and right adjunct functors on functor categories

$$(-\circ\iota) : \mathcal{F}^{\times n-1} \;\rightleftarrows\; \mathcal{F}^{\times n} : (-\circ\epsilon_0). \tag{7.5.3}$$

Since $\boxtimes^n \circ \epsilon_0 = 0$, all the terms in the expansion of (7.5.1) with a missing variable are 0. This enables us to drop the same terms which were dropped for degree reasons in the proof of Theorem 1.8.

The Künneth Theorem remains valid for $\mathrm{Ext}^*_{\mathcal{F}}$ for finite functors. Thus, the proof of Theorem 1.8 applies to prove this proposition thanks to the fact proved in [8] that the natural map

$$\varinjlim_r \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r})\ \to\ \mathrm{Ext}^*_{\mathcal{F}}(I, I)$$

is an isomorphism. □

The following theorem extends the applicability of Proposition 7.5 with the important restriction that the degree of F is less than or equal to the cardinality of k. (An analysis of change of field as in [9] can be achieved to show that $H^*(GL(K), F_K)$ is a direct factor in $H^*(GL(k), F) \otimes_k K$ for an extension K/k of finite fields.)

Theorem 7.6. *Let S, T be strict polynomial bifunctors homogenous of degree $d \leq q$. Then there is a natural isomorphism*

$$\varinjlim_r \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(S^{(r)}, T^{(r)})\ \overset{\simeq}{\to}\ \mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(S, T).$$

Proof. For $S = \mathcal{H}om(B_1, B_2),\ T = \mathcal{H}om(A_1, A_2)$ of separable type, we employ the natural commutative square

$$\begin{array}{ccc} \varinjlim_r \mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\mathcal{H}om(B_1, B_2)^{(r)}, \mathcal{H}om(A_1, A_2)^{(r)}) & \longrightarrow & \mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(\mathcal{H}om(B_1, B_2), \mathcal{H}om(A_1, A_2)) \\ \downarrow & & \downarrow \\ \varinjlim_r \mathrm{Ext}^*_{\mathcal{P}}(A_1^{(r)}, B_2^{(r)}) \otimes \mathrm{Ext}^*_{\mathcal{P}}(B_2^{(r)}, A_2^{(r)}) & \longrightarrow & \mathrm{Ext}^*_{\mathcal{F}}(A_1, B_2) \otimes \mathrm{Ext}^*_{\mathcal{F}}(B_2, A_2) \end{array}$$

whose vertical arrows are the isomorphisms of Propositions 2.3 and 7.3. The upper horizontal arrow of the above square is an isomorphism by [9, 3.10] so that the lower horizontal arrow (the natural map in the statement of the theorem) is also an isomorphism.

For general strict polynomial bifunctors S, T, we consider resolutions $P_\bullet \to S$, $T \to J^\bullet$ of S, T by separable strict polynomial bifunctors. Then we identify the map of the theorem with the map on hyperext-groups

$$\varinjlim_r \mathbf{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(P_{\bullet(r)}, J^{\bullet(r)})\ \overset{\simeq}{\to}\ \mathbf{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(P_\bullet, J^\bullet).$$

The proof is completed by comparing spectral sequences for hyperext groups and employing the special case of the theorem proved above for strict polynomial bifunctors of separable type. □

8. Consequences for cohomology of discrete groups

In determining the K-theory of $\mathbb{Z}/p^2\mathbb{Z}$ in low degrees, Friedlander and L. Evens [4] used the Hochschild-Serre spectral sequence for the extension

$$1 \to g\ell(n, \mathbb{F}_p)\ \to\ GL(n, \mathbb{Z}/p^2\mathbb{Z})\ \to\ GL(n, \mathbb{F}_p) \to 1$$

which takes the form

$$E_2^{s,t} = H^s(GL(n, \mathbb{F}_2), S^t(g\ell(n, \mathbb{F}_2))) \Rightarrow H^{s+t}(GL(n, \mathbb{Z}/p^2\mathbb{Z}), \mathbb{F}_2). \tag{8.0.1}$$

for $\mathbb{Z}/2\mathbb{Z}$ and the form

(8.0.2)

$$E_2^{s,t} = H^s(GL(n,\mathbb{F}_p), \bigoplus_{t_1+2t_2=t} \Lambda^{t_1}(g\ell(n,\mathbb{F}_p))\otimes S^{t_2}(g\ell(n,\mathbb{F}_p))) \Rightarrow H^{s+t}(GL(n,\mathbb{Z}/p^2\mathbb{Z}),\mathbb{F}_p).$$

for p odd. We restate the calculations of [4] (which were formulated in homology with $g\ell$ replaced by the subfunctor of trace 0 matrices). In the computations which follow this statement, we extend each of these calculations to all cohomological degrees as well as weaken the restriction on the prime p.

Proposition 8.1. *(cf. [4]) Let k be a finite field of characteristic $p \geq 5$ and $n \geq 2$. Then*

$$\mathrm{H}^i(GL(n,k), g\ell(n,k)) = \begin{cases} k, & \text{if } i = 0,\ 2 \\ 0, & \text{if } i = 1,\ 3 \end{cases}$$

$$\mathrm{H}^i(GL(n,k), S^2 g\ell(n,k)) = \begin{cases} k^2 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

$$\mathrm{H}^i(GL(n,k), \Lambda^2 g\ell(n,k)) = \begin{cases} 0 & \text{if } i = 0,1 \\ k^2 & \text{if } i = 2 \end{cases}$$

$$\mathrm{H}^i(GL(n,k), \Lambda^3 g\ell(n,k)) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

Our results in previous sections enable us to provide considerable generalizations of these computations when stabilized (so that $GL(n,k)$ for $n \geq 2$ is replaced by $GL(n,k)$ for $n >> 0$). The isomorphism (8.2.2) of the following theorem gives an explicit computation in all degrees of the stabilizations of the very low degree cohomology computed in Proposition 8.1.

Theorem 8.2. *Let A be a strict polynomial functor of degree d and k a finite field of order $q = p^e$ for some prime p. Then*

$$\text{(8.2.1)} \qquad \mathrm{H}^s(GL(k), A(g\ell)) \;\cong\; \mathrm{H}^*_{\mathcal{P}}(GL, A^{(r)}) \quad \text{if } r \geq log_p(\frac{s+1}{2}),\ q \geq d.$$

This isomorphism enables explicit computations as follows:

- *If $p = d = 2$, then $\mathrm{H}^*(GL(k), A(g\ell))$ is given explicitly by Theorem 5.1.*
- *If $p > d$ and if λ is a partition of d, then*

$$\text{(8.2.2)} \qquad \mathrm{H}^s(GL(k), S^\lambda(g\ell)) \;\cong\; s_\lambda((E_r^{\otimes d} \otimes k\mathfrak{S}_d)^s) \quad \text{if } r \geq log_p(\frac{s+1}{2}).$$

Proof. Isomorphism (8.2.1) follows by applying Theorem 7.4 which relates $\mathrm{H}^*(GL,-)$ to $\mathrm{H}^*_{\mathcal{F}}(GL,-)$, Theorem 7.6 (requiring $d \leq q$) which relates $\mathrm{H}^*_{\mathcal{F}}(GL,-)$ to the "generic" cohomology $\varinjlim_r \mathrm{H}^*_{\mathcal{P}}(GL,(-)^{(r)})$, and Corollary 3.4 (requiring $r \geq log_p(\frac{s+1}{2})$) which enables one to avoid the colimit with respect to Frobenius twisting.

Isomorphism (8.2.2) follows immediately from isomorphism (8.2.1) and Proposition 4.1 (which requires $p > d$). □

Corollary 8.3. *Let k be a finite field of characteristic p. We have the following computations of the group cohomology of $GL(k)$ with coefficients as indicated:*

- *The Poincaré series for $H^*(GL(k), g\ell)$ equals*

$$\frac{1}{1-t^2}.$$

- *The Poincaré series for* $H^*(GL(k), g\ell^{\otimes 2})$ *equals*

$$2\left(\frac{1}{1-t^2}\right)^2.$$

- *The Poincaré series for* $H^*(GL(k), S^2(g\ell))$ *equals*

$$\frac{1}{(1-t^2)^2} + \frac{1}{1-t^4}.$$

- *If* $p = 2$*, then the Poincaré series for* $H^*(GL(k), \Lambda^2(g\ell))$ *equals*

$$\frac{1}{(1-t^2)^2} + \frac{t}{1-t^4}.$$

- *If* $p > 2$*, then the Poincaré series for* $H^*(GL(k), \Lambda^2(g\ell))$ *equals*

$$\frac{1}{(1-t^2)^2} - \frac{1}{1-t^4}.$$

- *If* $p = 2$*, then the Poincaré series for* $H^*(GL(k), \Gamma^2(g\ell))$ *equals*

$$\frac{1}{(1-t^2)^2} + \frac{t^2}{1-t^4}.$$

Proof. The asserted Poincaré series for $p = 2$ are obtained by taking the limits with respect to r of the Poincar'e series given in Theorem 5.1.

For $p > 2$, we apply Proposition 4.1. Observe that, in contrast with $p = 2$, for $p > 2$ there are split exact short exact sequence in cohomology

$$0 \to H^*(GL, \Lambda^{2(r)}) \to H^*(GL, \otimes^{2(r)}) \to H^*(GL, S^{2(r)}) \to 0.$$

□

The following proposition enables the computation of further terms in the spectral sequence (8.0.2).

Proposition 8.4. *Let* k *be a finite field with* $q = p^e$ *elements and consider* $A^d \in \mathcal{P}_d$ *of exponential type, with* $d \leq q$*. Then*

(8.4.1)

$$\mathrm{H}^*_{\mathcal{F}}(GL, A^d(g\ell) \otimes g\ell) \;\cong\; \left(\mathrm{H}^*_{\mathcal{F}}(GL, A^d(g\ell)) \oplus \mathrm{H}^*_{\mathcal{F}}(GL, A^{d-1}(g\ell) \otimes g\ell)\right) \otimes E_\infty.$$

Proof. The proof is very similar to that of Proposition 6.1; in particular, we repeatedly use adjunction isomorphisms.

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(k[g\ell], A^d(g\ell) \otimes g\ell) :&\\ &= \mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}}(k[\mathcal{H}om(-_1, -_2)], A^d(\mathcal{H}om(-_1, -_2) \otimes \mathcal{H}om(-_1, -_2))\\ &= \mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}\times\mathcal{F}^{op}}(k[\mathcal{H}om(-_1, -_2)], A^d\mathcal{H}om(-_1, -_2) \otimes \mathcal{H}om(-_3, -_2) \circ D_{1,3})\end{aligned}$$

which by adjunction equals

$$\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}\times}(k[\mathcal{H}om(-_1 + -_3, -_2)], A^d\mathcal{H}om(-_1, -_2) \otimes \mathcal{H}om(-_3, -_2)).$$

Since $k[V \oplus W] \cong k[V] \otimes k[W]$, the previous line equals:

$$\mathrm{Ext}^*_{\mathcal{F}^{op}\times\mathcal{F}\times\mathcal{F}^{op}}(k[\mathcal{H}om(-_1, -_2)] \otimes k[\mathcal{H}om(-_3, -_2)], A^d\mathcal{H}om(-_1, -_2) \otimes \mathcal{H}om(-_3, -_2)).$$

In order to compactify the notation, we replace $\mathrm{Ext}^*_{\mathcal{F}^{op}\times\cdots\times\mathcal{F}}(-,-)$ by $\{-,-\}$ and we replace $\mathcal{H}om$ by $\mathcal{H}$. We apply adjunction again to get that this equals in this notation

$$\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^d\mathcal{H}(-_1,-_2+-_4)\otimes\mathcal{H}(-_3,-_2+-_4)\}.$$

We now expand $A^d\mathcal{H}(-_1,-_2+-_4)\otimes\mathcal{H}(-_3,-_2+-_4)$ using the exponential property for A^* and the additivity of $\mathcal{H}om$. Typically, terms with missing variable will not contribute. In our case, for example,

$$\begin{aligned}&\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^d\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(-_3,-_2)\}\\ &=\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,0)], A^d\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(-_3,-_2)\}\\ &\qquad=\{k[\mathcal{H}(-_1,-_2)], A^d\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(-_3,-_2)\}\\ &\qquad=\{k[\mathcal{H}(-_1,-_2)], A^d\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(0,-_2)\}=0.\end{aligned}$$

Using the same techniques, and $A^1=I$, one shows that

$$\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^{d-1}\mathcal{H}(-_1,-_2)\otimes A^1\mathcal{H}(-_1,-_4)\otimes\mathcal{H}(-_3,-_4)\}=0.$$

Finally, terms

$$\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^{d-i}\mathcal{H}(-_1,-_2)\otimes A^i\mathcal{H}(-_1,-_4)\otimes\ldots\}$$

vanish when $i\geq 2$ because, when $d\leq q$, their degree (modulo $q-1$) in some variable do not match in contravariant/covariant entries. We obtain

$$\begin{aligned}&\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^d\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(-_3,-_4)\}\ +\\ &\{k[\mathcal{H}(-_1,-_2)]\otimes k[\mathcal{H}(-_3,-_4)], A^{d-1}\mathcal{H}(-_1,-_2)\otimes\mathcal{H}(-_1,-_4)\otimes\mathcal{H}(-_3,-_2)\}\end{aligned}$$

and conclude by rearranging variables as in Proposition 6.1. □

Remark 8.5. *Proposition 8.4 is also valid if we replace $g\ell$ with $g\ell^{(r)}$ for any $r\geq 0$. The proof is the same, except that many of the occurrences of $\mathcal{H}om(-,-)$ must be replaced by $\mathcal{H}om(-,-)^{(r)}$.*

We combine Proposition 8.4 with the computations of Corollary 8.3 to obtain further computations.

Corollary 8.6. *Let k be a finite field of characteristic p.*

- *The Poincaré series for $H^*(GL(k), S^2(g\ell)\otimes g\ell)$ equals*
$$\left(\frac{3}{(1-t^2)^2}+\frac{1}{1-t^4}\right)\cdot\frac{1}{1-t^2}.$$
- *If $p=2$, the Poincaré series for $H^*(GL(k),\Lambda^2(g\ell)\otimes g\ell)$ equals*
$$\left(\frac{3}{(1-t^2)^2}+\frac{t}{1-t^4}\right)\cdot\frac{1}{1-t^2}.$$
- *If $p>2$, the Poincaré series for $H^*(GL(k),\Lambda^2(g\ell)\otimes g\ell)$ equals*
$$\left(\frac{3}{(1-t^2)^2}-\frac{1}{1-t^4}\right)\cdot\frac{1}{1-t^2}.$$

We complete the computations begun in Proposition 8.1 with the following computation.

Proposition 8.7. *Let k be a finite field of characteristic $p>3$. Then*

- *The Poincaré series for* $H^*(GL(k), \Lambda^3(g\ell))$ *equals*

$$\frac{1+5t^6}{(1-t^2)(1-t^4)(1-t^6)}.$$

- *The Poincaré series for* $H^*(GL(k), S^3(g\ell))$ *equals*

$$\frac{3+2t^2+t^4}{(1-t^4)(1-t^6)}.$$

Proof. We use Proposition 4.1 in conjunction with Proposition 1.9 to compute $H^*_{\mathcal{P}}(GL, \Lambda^{3(r)})$ and $H^*_{\mathcal{P}}(GL, S^{3(r)})$. We apply Theorem 8.2 and take the limit as r goes to ∞ to obtain the assert Poincaré series. □

References

[1] H. Andersen, *The Frobenius morphism on the cohomology of homogeneous vector bundles on* G/B, Ann. of Math. 112 (1980), 113-121.

[2] S. Betley, Calculations in THH-theory, J. Algebra 180 (1996), 445-458.

[3] W. Dwyer, *Twisted homological stability for general linear groups*, Annals of Math. 111 1980 239-251.

[4] L. Evens, E. Friedlander, *On* $K_*(\mathbb{Z}/p^2Z)$ *and related homology groups*, Trans. A.M.S. 270 (1982) 1-46.

[5] V. Franjou et al., *Rational Representation, the Steenrod Algebra, and Functor Homology*, Panor. Synthéses 16, Soc. Math. France, Paris, 2003.

[6] V. Franjou, J. Lannes, and L. Schwartz, *Autour de la cohomologie de Maclane des corps finis*, Invent. Math. 115 (1994), 513-538.

[7] V. Franjou, T. Pirashvili, *Stable K theory is bifunctor homology (after Scorichenko)*, Rational Representation, the Steenrod Algebra, and Functor Homology, Panor. Synthéses 16, Soc. Math. France, Paris, 2003.

[8] E. Friedlander, A. Suslin, *Cohomology of finite group scheme over a field*, Invent. Math 127 (1997) 235-253.

[9] V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin, *General linear and functor cohomology over finite fields*, Annals of Math. 150 (1999), 663-728.

[10] W. Fulton, J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics 109, Springer-Verlag, 1991.

[11] J.C. Jantzen, *Representations of Algebraic groups*, Academic press, (1987).

[12] A. Scorichenko,

[13] L. Schwartz, *Unstable Modules over the Steenrod Algebra and Sullivan's Fixed Point Set Conjecture*, Univ. of Chicago Press, Chicago, 1994.

[14] A. Suslin, E. Friedlander, C. Bendel, *Infinitesimal 1-parameter subgroups and cohomology*, J. Amer. Math. Soc. 10 (1997) 693-728.

Laboratoire Jean Leray, Université de Nantes, BP 92208 - 44322 Nantes Cedex 3, France

E-mail address: vincent.franjou@univ-nantes.fr

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: eric@math.northwestern.edu