MODULES OF CONSTANT JORDAN TYPE

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Abstract. We introduce the class of modules of constant Jordan type for a finite group scheme G over a field k of characteristic $p > 0$. This class is closed under taking direct sums, tensor products, duals, Heller shifts and direct summands, and includes endotrivial modules. It contains all modules in an Auslander-Reiten component which has at least one module in the class. Highly non-trivial examples are constructed using cohomological techniques. We offer conjectures suggesting that there are strong conditions on a partition to be the Jordan type associated to a module of constant Jordan type.

Contents

0. Introduction

In [17] and [19], the second and third authors have introduced a seemingly naive approach to the study of representations of finite groups and related structures on vector spaces over a field k of characteristic $p > 0$. The basic idea is to restrict representations to certain subalgebras ("π-points") isomorphic to the group algebra of $\mathbb{Z}/p\mathbb{Z}$, for we completely understand the representation theory of the algebra

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$k\mathbb{Z}/p\mathbb{Z}$ in terms of partitions (or "Jordan types"). The simplicity of this approach enables the authors to consider representation theory in a very general context (of a finite group scheme G over an arbitrary field of characteristic $p > 0$) and prove both global results about the stable module category and explicit results for specific examples. The naivety of the approach is somewhat misleading for underlying many theorems are somewhat difficult cohomological results, especially results giving finite generation and detection modulo nilpotence on subalgebras of special form.

In a recent paper [20], the second and third authors in collaboration with Andrei Suslin have adopted this naive point of view to formulate and investigate new invariants for such representations. The authors introduce "maximal" and "generic" Jordan types for a given representation whose existence even in the very special case of the finite group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is highly non-trivial.

Indeed, this special example $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (for $p > 2$) is challenging from a representation-theoretic point of view for its group algebra has wild representation type. With such "wildness" in mind, it is natural to investigate classes of representations of G with special properties. That is the purpose of this present paper, in which we investigate modules of constant Jordan type. Although the formulation of this concept requires the approach of "π-points" and our study utilizes many of the techniques of the papers mentioned above, the resulting class of modules appears to be a most natural one to study for those considering the modular representation theory of finite groups, p-restricted Lie algebras, and other finite group schemes. We remark here on two aspects of this class of modules of constant Jordan type: it includes the much-studied class of endotrivial modules and also includes many other modules even in the special case of an elementary abelian p-group; the classification of such modules of constant Jordan type appears to be very difficult, sufficiently difficult that even in the special case $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ we can only speculate on what Jordan types are realized.

In Definition 1.5, we introduce the concept of a kG-module M of constant Jordan type, a finite dimensional module with the property that $\alpha_K^*(M_K)$ has Jordan type independent of the π-point $\alpha_K : K[t]/t^p \longrightarrow KG$ with K/k an arbitrary field extension. As verified in Theorem 5.6, a kG-module is an endotrivial module if and only if it has constant Jordan type of a very special form. For certain explicit finite group schemes, various examples of modules of constant Jordan type can be constructed directly as we show in §2. Much of our effort in the first half of this paper is dedicated to showing that the class of modules of constant Jordan type is closed under various natural operations: Heller shifts (cf. Proposition 1.8), direct sums (cf. Proposition 1.8), direct summands (cf. Theorem 3.7), linear duals (cf. Proposition 5.2), and tensor products (cf. Corollary 4.3).

To establish these results, we continue the study initiated in [20] of the condition that a π-point α_K be maximal for a given kG-module M. This development is a second aim of the paper of some independent interest. In §3, we formulate the natural relationship of strict specialization of π-points (closely related to the relationship of specialization of equivalence classes of π-points considered in [20]). In §4, we investigate the surprisingly subtle behavior of the condition of maximality of π-points with respect to the tensor product of two given kG-modules. The relevance of maximality of π-points for a kG-module M is emphasized by Proposition 3.6 which asserts that the kG-module M has constant Jordan type if and only if the non-maximal support variety of M, $\Gamma(G)_M$, is empty.

In the second half of this paper, we give several methods of constructing modules of constant Jordan type. One is provided in Proposition 6.1 and a second in Theorem 6.6 (as well as Proposition 6.7). Much preliminary effort is required for us to establish in Theorem 6.13 that our second method provides examples which can not be realized by the first. Indeed, the example provided by this theorem shows how subtle is the behavior of the class of modules of constant Jordan type with respect to extensions. A third method of construction using the Auslander-Reiten theory of almost split sequences is detailed in §8: Theorem 8.5 establishes that any module in the same Auslander-Reiten component as a module of constant Jordan type is also of constant Jordan type, whereas Theorem 8.6 constructs kG-modules of constant Jordan type $n[1] + (proj)$ provided that G satisfies a mild cohomological property.

As one indication of the combinatorics involved in the existence of modules of constant Jordan type, we show in Theorem 7.1 that our techniques give a new proof of a special case of Macaulay's Generalized Principal Ideal Theorem.

We have made little progress in classifying those partitions which are realizable as the Jordan type of modules of constant Jordan type. For example, for a rank 2 elementary abelian p-group E, we conjecture but can not prove that no partition of type $n[p] + 1[2]$ is the Jordan type of kE-module of constant Jordan type. In §9, we mention numerous questions and conjectures on the constraints of a Jordan type associated to a module of constant Jordan type.

Throughout this paper, k will denote an arbitrary field of finite characteristic and $p > 0$ will denote the characteristic of k. Without explicit mention to the contrary, kG-modules are assumed to be finitely generated. We let $M_n(k)$ denote the algebra of $n \times n$ matrices over k.

We thank Valery Alexeev for enabling us to extend Theorem 7.1 to the case of characteristic 0 and Victor Ostrik for his suggestion of using tilting modules to establish the formulas of the appendix. We are grateful to David Eisenbud for the reference to Theorem 7.1 and to Karin Erdmann for drawing our attention to possible connections with the Auslander-Reiten theory.

1. Constant Jordan Type

In this first section, we introduce modules of constant Jordan type and investigate some of the basic properties of this class of modules.

Recall that a finite group scheme G (over k) is a group scheme over k whose coordinate algebra $k[G]$ is finite dimensional over k. We denote the linear dual of $k[G]$ by kG and call this the group algebra of G. A (rational) G-module is a comodule for $k[G]$ or equivalently a kG-module.

We remind the reader that the isomorphism class of a finite dimensional $k[t]/t^p$-module M of dimension n (over k) is given by a partition of n into blocks of size $\leq p$. Equivalently, if we let $\rho_M : k[t]/t^p \longrightarrow M_n(k)$ be the representation associated to M, then the isomorphism type of M is specified by the conjugacy class of the element $\rho_M(t) \in M_n(k)$ with p-th power 0. We shall often denote the isomorphism type of M by $a_p[p] + \cdots + a_1[1]$, where a_i denotes the number of blocks of size i in the partition of n associated to M.

We call the isomorphism type of a finite dimensional $k[t]/t^p$-module M the *Jordan type* of M. For any finite dimensional $k[t]/t^p$-module M, the *stable Jordan type* of M is the "stable equivalence" class of Jordan types, where two Jordan types $a_p[p]+\cdots+a_1[1]$ and $b_p[p]+\cdots+b_1[1]$ are stably equivalent if $a_i = b_i$, for all $i < p$.

We may view a Jordan type $a_p[p] + \cdots + a_1[1]$ as a *partition* of $n = \sum_{i=1}^p ia_i$. If $\sum ia_i = \sum ib_i$, then we say that the Jordan type of $\underline{a} = a_p[p] + \cdots + a_1[1]$ is greater or equal to the Jordan type of $\underline{b} = b_p[p] + \cdots + b_1[1]$ (denoted $\underline{a} \geq \underline{b}$) provided that

$$\sum_{i=j}^{p} ia_i \;\geq\; \sum_{i=j}^{p} ib_i, \quad 1 \leq j \leq p. \tag{1}$$

If $\underline{a} \geq \underline{b}$ and if $\sum_{i=j}^p ia_i > \sum_{i=j}^p ib_i$ for some j, then we write $\underline{a} > \underline{b}$. Note that this is the usual dominance ordering on partitions.

Remark 1.1. Let $M,\ N$ be $k[t]/t^p$-modules of dimension n given by

$$\rho_M,\ \rho_N : k[t]/t^p \longrightarrow M_n(k).$$

Then the Jordan type $\underline{a}$ of M is greater or equal to (respectively, greater than) the Jordan type of $\underline{b}$ of N if and only if for every $j, 1 \leq j < p$, the rank of ρ_M^j is greater or equal to the the rank of ρ_N^j (resp., and strictly greater for some j).

Definition 1.2. A π-point for a finite group scheme G is a left flat map of K-algebras $\alpha_K : K[t]/t^p \longrightarrow KG$, for some field extension K/k, which factors through the group algebra $KC_K \subset KG_K$ of some unipotent abelian subgroup scheme $C_K \subset G_K$. If M is a finite dimensional kG-module, the Jordan type of the π-point α_K on M is the isomorphism class of the $K[t]/t^p$-module $\alpha_K^*(M_K)$ (where $M_K = K \otimes_k M$). We emphasize here that $\alpha_K^*(M_K)$ denotes the restriction of M_K to a $K[t]/t^p$-module along the map α_K. We say that the Jordan type of $\alpha_K^*(M_K)$ is the Jordan type of α_K on M.

Definition 1.3. Let $\alpha_K : K[t]/t^p \longrightarrow KG$, $\beta_L : L[t]/t^p \longrightarrow LG$ be π-points of G. Then α_K is said to be a *specialization* of β_L (written $\beta_L \downarrow \alpha_K$) if for every finite dimensional kG-module M the $K[t]/t^p$-module $\alpha_K^*(M_K)$ is projective whenever the $L[t]/t^p$-module $\beta_L^*(M_L)$ is projective. We say that α_K, β_L are *equivalent* and write $\alpha_K \sim \beta_L$ provided that $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.

The following theorem summarizes the close relationship between the set of equivalence classes of π-points of G and the cohomology $\mathrm{H}^\bullet(G,k)$. Here, $\mathrm{H}^\bullet(G,k) = \mathrm{H}^*(G,k)$, the cohomology algebra of G provided that $p=2$; for $p>2$, $\mathrm{H}^\bullet(G,k) \subset \mathrm{H}^*(G,k)$ denotes the commutative subalgebra of even dimensional classes.

Theorem 1.4. *(cf.* [19, 3.6]*) The set of equivalence classes of π-points of a finite group scheme G, written $\Pi(G)$, admits a scheme structure determined by the stable module category,* stmod(kG)*. With this structure, $\Pi(G)$ is isomorphic to the scheme $Proj\,\mathrm{H}^\bullet(G,k)$.*

In particular, the closed subsets of $\Pi(G)$ are of the form $\Pi(G)_M$ where M is a finite dimensional kG-module and $\Pi(G)_M$ is the subset of those equivalence classes of π-points $\alpha_K : K[t]/t^p \longrightarrow KG$ such that $\alpha_K^*(M_K)$ is not projective.

We now introduce modules of constant Jordan type, whose study is the primary object of interest in this paper.

Definition 1.5. The finite dimensional kG-module M is said to be of constant Jordan type if the Jordan type of $\alpha_K^*(M_K)$ is independent of the choice of π-point $\alpha_K : K[t]/t^p \longrightarrow KG$.

Remark 1.6. The Jordan type of $\alpha_K^*(M_K)$ for a finite dimensional kG-module M at a π-point $\alpha_K : K[t]/t^p \longrightarrow KG$ typically depends not only upon the equivalence class $[\alpha_K] \in \Pi(G)$ but also upon the representative of this equivalence class. However, there are some exceptions. The central conclusion of [20] is that in either of the following two situations, the Jordan type of $\alpha_K^*(M_K)$ does not change if we replace α_K by some β_L with $\alpha_K \sim \beta_L$:

(1) If $[\alpha_K] \in \Pi(G)$ is a generic point; otherwise said, if α_K is a generic π-point.
(2) If for the given finite dimensional kG-module M, there does not exist any π-point β_L such that the Jordan type of $\beta_L^*(M_L)$ is strictly greater than the Jordan type of $\alpha_K^*(M_K)$. In this situation we say that α_K has maximal Jordan type on M.

We recall that the non-maximal support variety, $\Gamma(G)_M \subset \Pi(G)$ associated to a finite dimensional kG-module M is defined to be the (closed) subspace of those points $x \in \Pi(G)$ with the property that for some (and thus any) representative α_K of x the Jordan type of $\alpha_K^*(M_K)$ is not maximal for M, or equivalently, α_K does not have maximal Jordan type on M [20, 5.1].

Remark 1.6(2) immediately leads us to the following equivalent formulation of the property of constant Jordan type.

Proposition 1.7. *A finite dimensional kG-module is of constant Jordan type $a_p[p]+\cdots+a_1[1]$ if and only if for each equivalence class $[\alpha_K] \in \Pi(G)$ there exists some representative $\alpha_K : K[t]/t^p \longrightarrow KG$ with the property $\alpha_K^*(M_K)$ has type $a_p[p]+\cdots+a_1[1]$.*

Since any π-point $\alpha_K : K[t]/t^p \longrightarrow KG$ has the property that α_K^* commutes with direct sums and (modulo projectives) Heller shifts, we conclude the following.

Proposition 1.8. *Let G be an arbitrary finite group scheme.*

- *The trivial kG-module k has constant Jordan type.*
- *A finite dimensional projective kG-module has constant Jordan type. If kG is not semi-simple, then the Jordan type of a kG-projective module P is equal to $\frac{\mathrm{Dim}_k P}{p}[p]$.*
- *If $\Omega^i(k)$ denotes the i-th Heller shift of k for some $i \in \mathbb{Z}$, then $\Omega^i(k)$ has constant Jordan type equal to $n[p]+1[1]$ for some $n \geq 0$ if i is even and equal to $m[p]+1[p-1]$ for some $m \geq 0$ if i is odd.*
- *If M has constant Jordan type, then $\Omega^i(M)$ also has constant Jordan type for any $i \in \mathbb{Z}$*
- *If M, M' are kG-modules of constant Jordan type, then $M \oplus M'$ also has constant Jordan type.*

The preceding proposition will be supplemented in subsequent sections by propositions asserting that other familiar operations on modules of constant Jordan type yield modules of constant Jordan type: taking a direct summand (by Theorem 3.7), taking the tensor product (by Corollary 4.3), and taking $\mathrm{Hom}_k(-,-)$ (by 5.4).

We make explicit the following elementary functoriality property.

Proposition 1.9. *If $f : H \longrightarrow G$ is a flat map of finite group schemes and if M is a kG-module of constant Jordan type, then $f^*(M)$ is a kH-module of the same constant Jordan type.*

Proof. If $\alpha_K : K[t]/t^p \longrightarrow KH$ is a π-point of H, then $f \circ \alpha_K$ is a π-point of G and the Jordan type of $\alpha_K^*((f^*M))_K)$ equals that of $(f \circ \alpha_K)^*(M_K)$. □

Remark 1.10. To verify whether or not a given finite dimensional kG-module M has constant Jordan type it suffices to check that the Jordan type of $\alpha_K^*(M_K)$ does not vary as $[\alpha_K] \in \Pi(G)$ ranges over closed points of $\Pi(G)$. Thus, it suffices to consider α_K with K/k finite. In particular, if k is algebraically closed, it suffices to consider k-rational points of $\Pi(G)$. The collection of these points is denoted by $P(G)$, and was investigated extensively in [17].

Because a kG-module M has constant Jordan type if and only if its base change M_K has constant Jordan type as a KG-module for any field extension K/k, one could replace k by its algebraic closure and consider only k-rational points for the algebraically closed field k.

2. Examples of modules of constant Jordan type

We give examples of modules of constant Jordan type in special situations. Perhaps it is worth remarking that these examples are quite different from endotrivial modules considered in §5.

Proposition 2.1. *Let E be an elementary abelian p-group and let $I \subset kE$ be the augmentation ideal of the group algebra kE. Then, for any $n \geq m$, I^m/I^n is a kE-module of constant Jordan type.*

Proof. Let $\beta : k[t]/t^p \longrightarrow kE \simeq k[t_1, \ldots, t_r](t_1^p, \ldots, t_r^p)$ be the π-point defined by $\beta(t) = t_1$ and let $\alpha_K : K[t]/t^p \longrightarrow KE$ be an arbitrary π-point. Since α_K is flat and $\alpha_K(t)$ has p-th power 0, $\alpha_K(t)$ is a polynomial in $t_1, \ldots, t_r$ with constant term 0 and non-vanishing linear term. Consequently, we may choose an automorphism $\theta_\alpha : KE \longrightarrow KE$ which sends t_1 to $\alpha_K(t)$, so that $\alpha_K = \theta_\alpha \circ \beta$. The automorphism θ_α necessarily sends any power I^m of the augmentation ideal isomorphically onto itself, so that

$$\theta_\alpha^{-1} : \alpha_K^*(I^m/I^n) \ = \ (\theta_\alpha \circ \beta)^*((I^m/I^n)_K) \ \simeq \ (\beta^*(I^m/I^n))_K.$$

In other words, the Jordan type of $\alpha_K^*(I^m/I^n)$ does not depend upon the choice of π-point α_K. □

Example 2.2. As an elementary, specific case of Proposition 2.1, we consider kE/I^2 where E is an elementary abelian p-group of rank r. This is a kE-module of dimension $r+1$ and can be represented explicitly as follows. Give the kE-module structure on k^{r+1} by defining the generators $\{t_1, t_2, \ldots, t_r\}$ to act by multiplication by $\{e_{1,2}, e_{1,3}, \ldots, e_{1,r+1}\}$, pairwise commuting elementary matrices of size $(r+1)\times(r+1)$ with p-th power 0. The constant Jordan type of the module kE/I^2 is $1[2]+(r-1)[1]$.

Remark 2.3. By Proposition 5.2, the dual $(kE/I^2)^\#$ of kE/I^2 is also a module of constant Jordan type of the same Jordan type as kE/I^2. In the special case of $r = 2$, $(kE/I^2, (kE/I^2)^\#)$ constitute the example produced years ago by Jens Jantzen of two non-isomorphic modules with the same "local Jordan type."

Example 2.4. We give a somewhat more interesting example to show how the residue characteristic p plays a role. Consider the elementary abelian p-group of rank 2, $kE = k[x, y]/(x^p, y^p)$. Define the kE-module W of dimension 13 generated by v_1, v_2, v_3, v_4 and spanned as a k-vector space by

$$\{v_1, v_2, v_3, v_4, x(v_1), x(v_2), x(v_3), x(v_4), x^2(v_1), x^2(v_2), x^2(v_3), y(v_1), yx(v_1)\}$$

with

$$x(v_i) = y(v_{i+1}),\ y^2(v_1) = x^2(v_4) = x^3(v_i) = 0.$$

We represent this module with the following diagram

$$
\begin{tikzcd}[column sep=small]
 & \overset{\langle v_1\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & & \overset{\langle v_2\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & & \overset{\langle v_3\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & & \overset{\langle v_4\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\
 & \bullet & & \bullet & & \bullet & & \bullet &
\end{tikzcd}
\tag{2}
$$

The vertices correspond to k-linear space generators, and the arrows indicate the action of the generators x and y of the group algebra. A vertex with no out-coming arrows corresponds to a trivial 1-dimensional submodule.

If $p = 5$, then $W \simeq I^3/I^6$, and W has constant Jordan type $3[3] + 2[2]$. More generally, the Jordan type of the kE-module I^{p-2}/I^{p+1} is $(p-2)[3] + 2[2]$.

If $p > 5$, then W does not have constant Jordan type. Namely, the Jordan type for both x and y on W is $3[3] + 2[2]$, whereas the Jordan type of $x + y$ is $4[3] + 1[1]$. The Jordan blocks of size 3 for the action of $x + y$ are generated by v_1, v_2, v_3 and v_4 and the trivial block is generated by $x(v_1 - v_2 + v_3 - v_4) + y(v_4)$.

Proposition 2.5. *Let sl_2 denote the p-restricted Lie algebra of 2×2 matrices of trace 0 and let $u(sl_2)$ denote the restricted enveloping algebra of sl_2, the group algebra of the height 1 infinitesimal group scheme $G = SL_{2(1)}$. If the finite dimensional $u(sl_2)$-module M is the restriction of a rational SL_2-module, then M has constant Jordan type. On the other hand, other $u(sl_2)$-modules typically do not have constant Jordan type.*

Proof. Let $\rho : SL_2 \longrightarrow GL_m$ determine a rational SL_2-module M of dimension m. Then the induced map of p-restricted Lie algebras $d\rho : sl_2 \longrightarrow gl_m$ defines the associated $u(sl_2)$-module structure on M. For any field extension K/k and any $x \in SL_2(K)$, the rational $SL_{2,K}$-module $M_K^{\rho(x)}$ given by $Ad(\rho(x)) \circ \rho$ is isomorphic to M_K and thus the associated $u(sl_{2,K})$-module $M_K^{\rho(x)}$ given by $d(Ad(\rho(x) \circ \rho)$ is isomorphic to M_K.

Recall that the space of k-rational points of $\Pi(G)$ can be identified with the space of k-rational lines of the nilpotent variety $\mathcal{N}(sl_2)$. Indeed, each equivalence class of π-points is represented by some $\alpha_K : K[t]/t^p \longrightarrow u(sl_{2,K})$ sending t to a nilpotent matrix of $sl_2(K)$. Moreover, the Jordan type of a given finite dimensional $u(sl_2)$-module does not depend upon the choice of such a representative of a given point of $\Pi(G)$ by [20, 3.1]. The action of $SL_2(K)$ on $\Pi(G)_K$ sending α_K to $d\rho(Ad(x)) \circ \alpha_K = Ad(\rho(x)) \circ \alpha_K$ corresponds to the natural action of SL_2 on $\mathcal{N}(sl_2)$. Consequently, the transitivity of this action together with Proposition 1.7 implies that the rational SL_2-module M has constant Jordan type as a $u(sl_2)$-module.

On the other hand, any $u(sl_2)$-module M whose Π-support space $\Pi(G)_M$ is a non-empty proper subset of the 1-dimensional variety $\Pi(G)$ has Jordan type of M at a generic π-point of $\Pi(G)$ equal to $\frac{m}{p}[p]$ (where $m = \operatorname{Dim} M$) and strictly smaller Jordan type at any π-point representing a point in $\Pi(G)_M$. Such M abound, since every finite subset of $\Pi(G)$ is of the form $\Pi(G)_M$ for some finite dimensional kG-module M. For example, let $b \subset sl_2$ be the Lie subalgebra of lower triangular matrices and consider the $u(sl_2)$-module $M = u(sl_2) \otimes_{u(b)} k$, the module obtained from the trivial $u(b)$-module by coinduction. Then M is free when restricted to the 1-dimensional subalgebra of strictly upper triangular matrices and trivial when restricted to the 1-dimensional subalgebra of strictly lower triangular matrices. $\square$

We conclude this section of explicit examples with a family of modules $V_n, n > 0$ which are modules of constant Jordan type $n[2]+1[1]$ regardless of the prime p. The module V_n can be represented by the following diagram:

$$\begin{tikzcd}
& \overset{\langle v_1\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & & \overset{\langle v_2\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & & \cdots & & \overset{\langle v_n\rangle}{\bullet} \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet & & \bullet & & \bullet & \cdots & \bullet & & \bullet
\end{tikzcd}$$

The verification of the assertion that V_n does indeed have constant Jordan type is a simple computation.

In the following proposition we describe the modules V_n explicitly in terms of generators and relations.

Proposition 2.6. *Consider a rank 2 elementary abelian p-group, so that* $kE = k[x,y]/(x^p, y^p)$. *Consider the kE-module* V_n *of dimension* $2n+1$ *generated by* $v_1, \ldots, v_n$ *spanned as a k-vector space by* $\{v_1, \ldots, v_n, x(v_1), \ldots, x(v_n), y(v_1)\}$ *with* $x(v_i) = y(v_{i+1}),\ x^2(v_i) = xy(v_i) = y^2(v_i) = 0$. *Then* V_n *has constant Jordan type* $n[2]+1[1]$.

3. Specialization, $\Gamma(G)_M$, and constant Jordan type

In this section, we introduce a strict form of specialization of π-points which was considered briefly in [20, 3.2] for infinitesimal group schemes. We also recall the non-maximal support variety $\Gamma(G)_M$ of a finite dimensional kG-module M. We use both strict specialization and the non-maximal support variety to further explore the class of modules of constant Jordan type.

The condition recalled in Definition 1.2 that the π-point α_K specializes to the π-point β_L is equivalent to the algebro-geometric condition that the point $[\alpha_K] \in \Pi(G)$ specializes to $[\beta_L]$ (cf. [19]). For some purposes, this notion of specialization is not sufficiently strong. Namely, if $\Pi(G)$ is reducible, then there exist finite dimensional kG-modules M and π-points α_K, β_L such that α_K specializes to β_L but the Jordan type of $\alpha_K^*(M_K)$ is smaller than the Jordan type of $\beta_L^*(M_L)$ (see [19, 4.14]).

The following definition introduces a stricter definition of specialization which is a natural extension of [20, 3.2] from infinitesimal group schemes to arbitrary finite group schemes.

Definition 3.1. Let G be a finite group scheme and α_K, β_L be π-points of G. We say that α_K strictly specializes to β_L (and write $\alpha_K \downdownarrows \beta_L$), if there exists a commutative local domain R over k with field of fractions K and residue field L, together with a map of R-algebras $\nu_R : R[t]/t^p \longrightarrow RG$ such that $\nu_R \otimes_R K = \alpha_K,\ \nu_R \otimes_R L = \beta_L$.

Theorem 3.2. *Let R be a commutative local domain with field of fractions K and residue field L. Let $A \in M_N(R)$ be an $N \times N$ matrix which has p-th power 0 and coefficients in R. Then the Jordan type of $A \otimes_R K \in M_N(K)$ is greater or equal to the Jordan type of $A \otimes_R L \in M_N(L)$.*

Consequently, if α_K, β_L are π-points of a finite group scheme G with $\alpha_K \downdownarrows \beta_L$ and if M is a finite dimensional kG-module, then $\alpha_K^*(M_K)$ has Jordan type greater or equal to that of $\beta_L^*(M_L)$. In particular, $\alpha_K \downdownarrows \beta_L$ implies $\alpha_K \downarrow \beta_L$.

Proof. Let $M[i]$ denote the cokernel of $A^i : R^N \longrightarrow R^N$ for some $i < p$. Let $m_1, \dots, m_s \in M[i]$ be chosen so that $\overline{m}_1, \dots, \overline{m}_s \in M[i] \otimes_R L$ is a basis, where $\overline{m}_i = m_i \otimes 1_L$. By Nakayama's Lemma, $m_1, \dots, m_s$ generate $M[i]$ as an R-module and thus their images in $M[i] \otimes_R K$ span. Observe that $M[i] \otimes_R K$ is the cokernel of $(A \otimes_R K)^i : K^N \longrightarrow K^N$ and that $M[i] \otimes_R L$ is the cokernel of the homomorphism $(A \otimes_R L)^i : L^N \longrightarrow L^N$. This implies that the rank of $(A \otimes_R K)^i$ is greater or equal to the rank of $(A \otimes_R L)^i$ for any $i < p$ so that the Jordan type of $A \otimes_R K \in M_N(K)$ is greater or equal to the Jordan type of $A \otimes_R L \in M_N(L)$.

The second statement is a special case of the first. The last statement follows from the observation that if $\beta_L^*(M_L)$ is projective then its Jordan type is the maximal possible Jordan type on M and hence $\alpha_K^*(M_K)$ must also be projective assuming that $\alpha_K \downdownarrows \beta_L$. $\square$

The next theorem verifies that specialization of points in $\Pi(G)$ can be represented by strict specialization.

Theorem 3.3. *Let G be a finite group scheme and let $\alpha_K,\ \beta_L$ be π-points of G with $\alpha_K \downarrow \beta_L$ be π-points of G. Then there exist π-points $\alpha'_{K'} \sim \alpha_K$ and $\beta'_{L'} \sim \beta_L$ such that $\alpha'_{K'} \downdownarrows \beta'_{L'}$.*

Proof. Using [19, 4.13], we can choose an elementary abelian p-group $E \subset \pi_0(G)$ such that $[\alpha_K], [\beta_L]$ are in the image of the closed map $\Pi((G^0)^E \times E) \longrightarrow \Pi(G)$. Observe that the group algebra of $(G^0)^E \times E$ is isomorphic to the group algebra of some infinitesimal group scheme H. Because the relationship of strict specialization given in Definition 3.1 and the definition of specialization given in Definition 1.2 involve only the group algebra of the given finite group scheme, we may replace G by H. Thus, we assume G is an infinitesimal group scheme.

Let $\operatorname{Spec} A \subset \Pi(G)$ be an irreducible affine open subset containing both $[\alpha_K]$ and $[\beta_L]$. By replacing A by A_{red}, we may assume that A is a domain. Let R denote the local A-algebra defined as the localization at the prime corresponding to $[\alpha_K]$ of the quotient of A by the prime corresponding to $[\beta_L]$. Set K' to be the field of fractions of R and L' to be the residue field of R.

Let r denote the height of the infinitesimal group scheme G and let $V_r(G)$ be the scheme of 1-parameter subgroups of G. Recall that the natural morphism $\Theta_G : V_r(G)\backslash\{0\} \longrightarrow \Pi(G)$, which is determined by sending a 1-parameter subgroup $\mu : \mathbb{G}_{a(r)_K} \longrightarrow G_K$ to the π-point $\mu_* \circ \epsilon : K[t]/t^p \longrightarrow K\mathbb{G}_{a(r)} \longrightarrow KG$, where ϵ has the property that its composition with the map on group algebras induced by the projection $\mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}/\mathbb{G}_{a(r-1)}$, is an isomorphism. Because Θ_G together with the isogeny $H^\bullet(G,k) \to k[V_r(G)]$ of [23, 5.2] induces the isomorphism $\operatorname{Proj}(V_r(G))_{red} \longrightarrow \Pi(G)_{red}$ of Theorem 1.4, we conclude that the given morphism $\operatorname{Spec} R \longrightarrow \Pi(G)_{red}$ lifts to a morphism

$$\operatorname{Spec} R \longrightarrow (V_r(G))_{red} \subset V_r(G),$$

which corresponds to a morphism of R-group schemes $\nu : \mathbb{G}_{a(r),R} \longrightarrow G_R$.

We define ν_R to be the map of R-algebras given as the composition

$$R[t]/t^p \xrightarrow{\epsilon} R(\mathbb{G}_{a(r),R}) \xrightarrow{\nu_*} RG.$$

By construction, $\nu_R \otimes_R K'$ is a π-point of G in the equivalence class $[\alpha_K]$ (the image of $\operatorname{Spec} K'$) and $\nu_R \otimes_R L'$ is a π-point of G in the equivalence class $[\beta_L]$. $\square$

The proof of the theorem in the title of [7] applies with only minor notational change to prove the same assertion, given below, for an arbitrary finite group scheme.

Theorem 3.4. *(cf. [7]) Let G be a finite group scheme and M a finite dimensional indecomposable kG-module. Then $\Pi(G)_M$ is connected. In particular, $\Pi(G) = \Pi(G)_k$ is connected.*

Theorems 3.3 and 3.4 enable the following equivalent formulation of the condition that a module has constant Jordan type.

Proposition 3.5. *Let G be a finite group scheme and let M be a kG-module. Then M has constant Jordan type if and only if α_K and β_L have the same Jordan type on M whenever $\alpha_K,\ \beta_L$ are π-points of G satisfying $\alpha_K \downarrow\downarrow \beta_L$.*

Proof. If M has constant Jordan type and $\alpha_K,\ \beta_L$ are π-points of G, then the Jordan types of α_K and and β_L on M are necessarily equal.

To prove the converse, we recall that if $x \in \Pi(G)$ is a point such that M has maximal Jordan type at some representative of x, then the Jordan type on M is the same for every representative of x (see Remark 1.6). Let $x \in \Pi(G)$ be a point which has maximal Jordan type on M and let $y \in \Pi(G)$ be an arbitrary point. Using the fact that $\Pi(G)$ is connected (by Theorem 3.4) and Noetherian, choose a chain of points $x = x_0, x_1, \ldots, x_n = y \in \Pi(G)$ such that there exist morphisms $\operatorname{Spec} R_i \longrightarrow \Pi(G)_{red}$ sending $\{\operatorname{Spec} K_i, \operatorname{Spec} L_i\}$ to the (unordered) pair $\{x_{i-1}, x_i\}$ where R_i a commutative local domain with field of fractions K_i and residue field L_i.

Our hypothesis in conjunction with Theorem 3.3 implies that there are representing π-points α_{K_i}, α_{L_i} of x_i and x_{i-1} at which M has the same Jordan type. Consequently, M has the same maximal Jordan type at each representative of each x_i. Thus, the Jordan type of M at any representative of $x_n = y$ equals this same maximal Jordan type; in other words, M has constant Jordan type. □

We give a useful characterization of modules M of constant Jordan type in terms of their non-maximal support varieties $\Gamma(G)_M$ recalled in Remark 1.6.

Proposition 3.6. *Let M be a finite dimensional kG-module. Then M has constant Jordan type if and only if $\Gamma(G)_M = \emptyset$.*

Proof. If M has constant Jordan type, then clearly $\Gamma(G)_M = \emptyset$. Conversely, let $\Gamma(G)_M = \emptyset$, and let α_K, β_L be any two π-points of G satisfying $\alpha_K \downdownarrows \beta_L$. By Theorem 3.2, $\alpha_K^*(M_K)$ has Jordan type greater or equal to that of $\beta_L^*(M_L)$. Since $\Gamma(G)_M = \emptyset$, we must have that the Jordan types of $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$ are the same. Hence, Proposition 3.5 implies that M has constant Jordan type. □

The following "closure property" of modules of constant Jordan type is somewhat striking.

Theorem 3.7. *Let M be a kG-module of constant Jordan type. Then any direct summand of M also has constant Jordan type.*

Proof. Write $M = M' \oplus M''$. Let α_K, β_L be two π-points such that $\alpha_K \downdownarrows \beta_L$. By Theorem 3.2, we have

$$\text{(3)}\quad \mathrm{JType}(\alpha_K^*(M'_K)) \geq \mathrm{JType}(\beta_L^*(M'_L)) \text{ and } \mathrm{JType}(\alpha_K^*(M''_K)) \geq \mathrm{JType}(\beta_L^*(M''_L)).$$

Hence,

$$\mathrm{JType}(\alpha_K^*(M'_K)) \oplus JType(\alpha_K^*(M''_K)) = JType(\alpha_K^*(M)) = \mathrm{JType}(\beta_L^*(M_L)) =$$

$$= \mathrm{JType}(\beta_L^*(M'_L)) \oplus \mathrm{JType}(\beta_L^*(M''_L)),$$

where JType is the Jordan type of the $K[t]/t^p$-module. Therefore, both inequalities in (3) must be equalities. Since this holds for any pair $\alpha_K \downdownarrows \beta_L$, the statement now follows immediately from Proposition 3.5. □

4. Behavior with respect to tensor products

We begin with the following "order preserving" property of Jordan types. The proof given below uses an explicit description of the tensor product of indecomposable $k[t]/t^p$-modules which is presented in the Appendix.

Proposition 4.1. *Let M, N and L be $k[t]/t^p$-modules with Jordan types $\underline{a} = a_p[p] + \cdots + a_1[1]$, $\underline{b} = b_p[p] + \cdots + b_1[1]$ and $\underline{c} = c_p[p] + \cdots + c_1[1]$ respectively such that $\operatorname{Dim} M = \operatorname{Dim} N$. Let $\underline{a} \otimes \underline{c}$, $\underline{b} \otimes \underline{c}$ denote the Jordan types of $M \otimes L$, $N \otimes L$ respectively.*

If $\underline{a} \geq \underline{b}$, then $\underline{a} \otimes \underline{c} \geq \underline{b} \otimes \underline{c}$.

If $\underline{a} > \underline{b}$ and if $c_i \neq 0$ for some $i < p$, then $\underline{a} \otimes \underline{c} > \underline{b} \otimes \underline{c}$.

Proof. For a $k[t]/t^p$-module M of dimension m, we denote the representation afforded by M by $\rho_M : k[t]/t^p \longrightarrow \operatorname{End}_k(A) \simeq M_m(k)$. The dominance condition (1) on Jordan types $\underline{a} \geq \underline{b}$ of M and N both of dimension m can be formulated as the condition that

$$\operatorname{Rk} \rho_M(t)^i \geq \operatorname{Rk} \rho_N(t)^i, \quad 1 \leq i < p$$

or equivalently that

$$\operatorname{Dim} \operatorname{Ker} \{\rho_M(t)^i\} \leq \operatorname{Dim} \operatorname{Ker} \{\rho_N(t)^i\}, \quad 1 \leq i < p. \tag{4}$$

For $\underline{a} > \underline{b}$, the additional condition is that $\operatorname{Rk} \rho_M(t)^i > \operatorname{Rk} \rho_N(t)^i$ for some i.

If $L \simeq L_1 \oplus L_2$, so that $\underline{c} = \underline{c}_1 + \underline{c}_2$, then we immediately conclude that

$$\underline{a} \otimes \underline{c} \;=\; \underline{a} \otimes \underline{c}_1 \;+\; \underline{a} \otimes \underline{c}_2.$$

Thus, we may assume that L is indecomposable, say $L = [\ell]$.

Denote by $J_s \in M_s(k)$ the Jordan block of size s. Then $J_s = \rho_{[s]}(t)$ once we choose an appropriate basis. Since t acts on $M \otimes [\ell]$ as $t \otimes 1 + 1 \otimes t$, we get the formula

$$\rho_{M \otimes [\ell]}(t) = \rho_M(t) \otimes \operatorname{Id}_{[s]} + \operatorname{Id}_M \otimes J_s$$

Lemma 1.10 of [20] implies

$$\operatorname{Ker} \{\rho_{M \otimes [\ell]}(t)\} = \operatorname{Ker} \{\rho_M(t)^\ell\} \tag{5}$$

Applying the same argument to $M \otimes [\ell]$, we obtain

$$\operatorname{Ker} \{\rho_{M \otimes [\ell]}(t)^s\} = \operatorname{Ker} \{\rho_{M \otimes [\ell] \otimes [s]}(t)\} \tag{6}$$

Using notation introduced in the appendix, we write $[\ell] \otimes [s] = \bigoplus_j C^j_{\ell s}[j]$. With this notation,

$$\operatorname{Dim} \operatorname{Ker} \{\rho_{M \otimes [\ell]}(t)^s\} = \operatorname{Dim} \operatorname{Ker} \{\rho_{M \otimes (\bigoplus C^j_{\ell s}[j])}(t)\} = \operatorname{Dim} \operatorname{Ker} \{\rho_{\bigoplus C^j_{\ell s}(M \otimes [j])}(t)\}$$

$$= \sum C^j_{\ell s} \operatorname{Dim} \operatorname{Ker} \{\rho_{M \otimes [j]}(t)\} = \sum C^j_{\ell s} \operatorname{Dim} \operatorname{Ker} \{\rho_M(t)^j\}.$$

Consequently,

$$\operatorname{Dim} \operatorname{Ker} \{\rho_{M \otimes [\ell]}(t)^s\} = \sum C^j_{\ell s} \operatorname{Dim} \operatorname{Ker} \{\rho_M(t)^j\}$$

and

$$\mathrm{Dim\,Ker}\,\{\rho_{N\otimes[\ell]}(t)^s\} = \sum C^j_{\ell s}\,\mathrm{Dim\,Ker}\,\{\rho_N(t)^j\}$$

Hence, (4) implies that

$$\mathrm{Dim\,Ker}\,\{\rho_{M\otimes[\ell]}(t)^s\} \ \le \mathrm{Dim\,Ker}\,\{\rho_{N\otimes[\ell]}(t)^s\}$$

for all s. Therefore, $\underline{a}\otimes\underline{c} \ \ge \ \underline{b}\otimes\underline{c}$.

Now assume $\underline{a} \ > \ \underline{b}$ and that some $c_i \neq 0, i < p$. Then, we may assume that $L = [\ell]$ with $\ell < p$. Choose j such that $\mathrm{Dim\,Ker}\,\{\rho_M(t)^j\} < \mathrm{Dim\,Ker}\,\{\rho_N(t)^j\}$. The formula 10.1 for the coefficients $C^j_{\ell s}$ implies that there exists some s such that $C^j_{\ell s} = 1$. That is, if $j \ge \ell$, then take $s = j - \ell + 1$, and if $j < \ell$, take $s = \ell - j + 1$. For such s we get the strict inequality

$$\mathrm{Dim\,Ker}\,\{\rho_{M\otimes[\ell]}(t)^s\} \ < \ \mathrm{Dim\,Ker}\,\{\rho_{N\otimes[\ell]}(t)^s\}$$

Therefore, $\underline{a}\otimes\underline{c} \ > \ \underline{b}\otimes\underline{c}$.

□

Proposition 4.1 enables us to establish various tensor product properties of maximal Jordan type. We should point out that there are some subtleties which must be carefully considered. For example if M and N are kG-modules and α_K is a π-point whose image is not a sub-Hopf algebra of KG, then it is not always true that $\alpha_K^*(M_K\otimes N_K) \ \simeq \ \alpha_K^*(M_K)\otimes\alpha_K^*(N_K)$. Hence, Proposition 4.1 can not be applied directly. The somewhat surprising Examples 4.5 and 4.6 indicate the subtle relationship between tensor products and maximal Jordan types. In view of these examples, we take some care in the proofs of the following properties.

Theorem 4.2. *Let G be a finite group scheme, and consider two finite dimensional kG-modules M, N and a π-point α_K of G.*

(1) *If α_K has maximal Jordan type on both M and N, then*

$$\alpha_K^*(M_K\otimes N_K) \ \simeq \ \alpha_K^*(M_K)\otimes\alpha_K^*(N_K).$$

(2) *If α_K has maximal Jordan type on $M\otimes N$, then*

$$\alpha_K^*(M_K\otimes N_K) \ \simeq \ \alpha_K^*(M_K)\otimes\alpha_K^*(N_K),$$

(3) *If $\Pi(G)$ is irreducible and if α_K has maximal Jordan type on both M and N, then α_K has maximal Jordan type on $M\otimes N$ and that Jordan type is equal to the Jordan type of $\alpha_K^*(M_K)\otimes\alpha_K^*(N_K)$.*

Proof. Let $i : C_K \subset G_K$ be a unipotent abelian subgroup scheme through which α_K factors. Since $i : KC_K \longrightarrow KG_K$ is a map of Hopf algebras, i^* commutes with tensor products. Observe that the maximality of α_K on M, N or $M\otimes N$ as kG-modules implies the maximality of α_K on M_K, N_K and $M_K\otimes_K N_K$ as KC_K-modules. Thus, statement (1) follows from [20, 4.4] applied to C_K.

To prove (2), we choose a generic π-point η_Ω of C_K, $\eta_\Omega : \Omega[t]/t^p \longrightarrow \Omega C_K$. We have

$$\eta_\Omega^*(M_\Omega) \otimes_\Omega \eta_\Omega^*(N_\Omega) \ \simeq \ \eta_\Omega^*(M_\Omega \otimes_\Omega N_\Omega)$$

where the isomorphism follows from [20, 4.4] since η_Ω is generic. Because of the maximality of α_K, this module has the same Jordan type as $\alpha_K^*(M_K \otimes_K N_K)$. Since $\Pi(C_K)$ is irreducible, the module M_K has absolute maximal Jordan type at the generic π-point η_Ω ([20, 4.11]). Hence, the Jordan types of $\alpha_K^*(M_K)$ and $\alpha_K^*(N_K)$ are comparable to those of $\eta_\Omega^*(M_\Omega)$ and $\eta_\Omega^*(N_\Omega)$. In the special case that either $\alpha_K^*(M_K)$ or $\alpha_K^*(N_K)$ is projective, then either $\eta_\Omega^*(M_\Omega)$ or $\eta_\Omega^*(N_\Omega)$ is projective. Hence, the modules $\eta_\Omega^*(M_\Omega) \otimes_\Omega \eta_\Omega^*(N_\Omega)$ and $\alpha_K^*(M_K) \otimes_K \alpha_K^*(N_K)$ have the same Jordan type, both tensor products being projective. If neither $\alpha_K^*(M_K)$ nor $\alpha_K^*(N_K)$ is projective, and if either $\mathrm{JType}(\eta_\Omega^*(M_\Omega)) \gneqq \mathrm{JType}(\alpha_K^*(M_K))$ or $\mathrm{JType}(\eta_\Omega^*(N_\Omega)) \gneqq \mathrm{JType}(\alpha_K^*(N_K))$, then Proposition 4.1 implies that $\mathrm{JType}(\eta_\Omega^*(M_\Omega \otimes_\Omega N_\Omega)) \gneqq \mathrm{JType}(\alpha_K^*(M_K \otimes N_K))$ contradicting the maximality of the Jordan type of $\alpha_K^*(M_K \otimes N_K)$. Thus, we have that $\alpha_K^*(M_K \otimes_K N_K)$ has the same Jordan type as $\eta_\Omega^*(M_\Omega) \otimes_\Omega \eta_\Omega^*(N_\Omega)$ which has the same Jordan type as $\alpha_K^*(M_K) \otimes_K \alpha_K^*(N_K)$.

To prove (3) we assume that $\Pi(G)$ is irreducible. Let $\eta_\Omega : \Omega[t]/t^p \longrightarrow \Omega G$ be a generic π-point of G. Applying [20, 4.7] to η_Ω, we get

$$\eta_\Omega^*(M_\Omega) \otimes_\Omega \eta_\Omega^*(N_\Omega) \ \simeq \ \eta_\Omega^*(M_\Omega \otimes_\Omega N_\Omega)$$

Since $\Pi(G)$ is irreducible, the absolute maximal Jordan type of any finite dimensional kG-module is realized at η_Ω. Hence, the maximality assumption on α_K on M and N implies that $\alpha_K^*(M_K)$ has the same Jordan type as $\eta_\Omega^*(M_\Omega)$ and $\alpha_K^*(N_K)$ has the same Jordan type as $\eta_\Omega^*(N_\Omega)$. We conclude that

$$\alpha_K^*(M_K \otimes N_K) \ \simeq \ \alpha_K^*(M_K) \otimes \alpha_K^*(N_K)$$

has the same Jordan type as

$$\eta_\Omega^*(M_\Omega) \otimes_\Omega \eta_\Omega^*(N_\Omega) \ \simeq \ \eta_\Omega^*(M_\Omega \otimes_\Omega N_\Omega)$$

where the first isomorphism follows from statement (1). Hence, the Jordan type of α_K on $M \otimes N$ is maximal. □

An easy corollary of Theorem 4.2 is the following assertion that the tensor product of modules of constant Jordan type is again of constant Jordan type.

Corollary 4.3. *Let G be a finite group scheme and let M, N be finite dimensional kG-modules. If M and N have constant Jordan type, then $M \otimes N$ also has constant Jordan type.*

Proof. We merely observe that α_K has maximal Jordan type on both M and N for any π-point $\alpha_K : K[t]/t^p \longrightarrow KG$ whenever M, N are of constant Jordan type. Thus, the corollary follows from the first statement of Theorem 4.2. □

The following consequence of Theorem 4.2 will be used to prove Proposition 4.7 .

Corollary 4.4. *Let G be a finite group scheme such that $\Pi(G)$ is irreducible and let $\alpha_K : K[t]/t^p \longrightarrow KG$ be a π-point of G. Let M be a kG-module such that $\alpha_K^*(M_K)$ is not projective and let N be another kG-module such that α_K does not have maximal Jordan type on N. Then α_K does not have maximal Jordan type on $M \otimes N$.*

Proof. If the Jordan type of $\alpha_K^*(M_K \otimes_K N_K)$ were maximal, then the second statement of Theorem 4.2 would imply that this maximal type is the same as that of $\alpha_K^*(M_K) \otimes_K \alpha_K^*(N_K)$. However, the hypotheses on $\alpha_K^*(M_K)$, $\alpha_K^*(N_K)$ together with Proposition 4.1 would then lead to an immediate contradiction. $\square$

We give two examples to illustrate that naturally formulated improvements of Theorem 4.2 are not valid. Our first example illustrates that the maximality of both M and N at a given π-point is not sufficient to imply the maximality of $M \otimes N$ at that π-point. Namely, we construct a module W and a π-point β_L such that β_L has maximal Jordan type on W but not on $W \otimes W$ (even though we have $\beta_L^*(W_L \otimes W_L) \simeq \beta_L^*(W_L) \otimes_L \beta_L^*(W_L)$ by Theorem 4.2). As usual, this anomaly comes from the fact that the ordering on Jordan types is not total.

Example 4.5. Let G be a finite p-group which has two conjugacy classes of maximal elementary abelian subgroups, represented by E and E' respectively. Furthermore, we require E to be normal. Let $e = |E|$, $f = \frac{|G|}{|E|}$. Assume that $p > 3$.

For example, take G to be the p-Sylow subgroup of the wreath product $\mathbb{Z}/p \wr S_p$, so that G is isomorphic to $(\mathbb{Z}/p)^p \rtimes \mathbb{Z}/p$. Then G has two non-conjugate maximal elementary abelian p-subgroups: $E = (\mathbb{Z}/p)^p$ which is normal and

$$F = (\mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p)^{\mathbb{Z}/p} \times \mathbb{Z}/p \cong (\mathbb{Z}/p)^2.$$

By Quillen stratification, $\Pi(G) = X \cup Y$ where $X = \Pi(E)/G$, $Y = \Pi(F)/N_G(F)$. Let $[\alpha_K] \in X$, $[\beta_L] \in Y$ be generic points.

Choose a homogeneous cohomology class $\xi \in \mathrm{H}^\bullet(G, k)$ with the property that ξ vanishes on $[\beta_L]$ but does not vanish on $[\alpha_K]$, and let L_ξ be Carlson module which has the property that the support of L_ξ is the zero locus of ξ. Set $M = \mathrm{Ind}_E^G(\Omega_E^1(k))$, set $N = L_\xi^{\oplus n}$ for some positive integer n which is to be determined, and set $W = M \oplus N$. It was shown in Example [20, 4.13], that if we pick n to satisfy the inequality

$$\frac{f}{p} < n < (p-1)\frac{f}{p} \tag{7}$$

then the Jordan types $\alpha_K^*(W_K)$ and $\beta_L^*(W_L)$ are maximal, incomparable generic Jordan types of W.

Let $d = \operatorname{Dim} W$. We proceed to deduce a condition on n which would ensure the inequality

$$\mathrm{JType}(\alpha_K^*(W_K^{\otimes 2})) > \mathrm{JType}(\beta_L^*(W_L^{\otimes 2}))$$

By the Appendix, $[p-1]\otimes[p-1] = (p-2)[p]+1[1]$. Since $\alpha_K^*(W_K) = f[p-1]+m[p]$ for some m (see [20, 4.13]), and has dimension d, we get

$$\alpha_K^*(W_K^{\otimes 2}) \simeq (\alpha_K^*(W_K))^{\otimes 2} \simeq (\frac{d^2-f^2}{p})[p] + f^2[1].$$

Similarly, applying the decomposition of $\beta_L^*(W_L)$ obtained in [20, 4.13], we get

$$\beta_L^*(W_L^{\otimes 2}) \simeq (\beta_L^*(W_L))^{\otimes 2} \simeq (\frac{d^2-2n^2p}{p})[p] + 2n^2[p-1] + 2n^2[1].$$

In order for the Jordan type of $\alpha_K^*(W_K^{\otimes 2})$ to dominate that of $\beta_L^*(W_L^{\otimes 2})$ we need to choose n such that $\alpha_K^*(W_K^{\otimes 2})$ has more blocks of size p than $\beta_L^*(W_L^{\otimes 2})$ and fewer blocks altogether. In other words, we need for the following inequalities to hold:

$$\frac{d^2-f^2}{p} > \frac{d^2-2n^2p}{p}$$

comparing the number of blocks of size p, and

$$\frac{d^2-f^2}{p} + f^2 < \frac{d^2-2n^2p}{p} + 4n^2$$

comparing the overall number of blocks. Simplifying, we get

$$f^2 < 2n^2p,$$

$$(p-1)f^2 < 2n^2p.$$

Observe that the second inequality implies the first and simplifies to

$$\sqrt{\frac{p-1}{2p}}f < n.$$

Since p is greater than 3 and divides f, it is possible to choose n to satisfy

$$\sqrt{\frac{p-1}{2p}}f < n < \frac{p-1}{p}f.$$

Since such n automatically satisfies the inequality (7), we conclude that W has maximal non-comparable types at $[\alpha_K], [\beta_L]$ but that the Jordan type of $\alpha_K^*(W_K \otimes_K W_K)$ is strictly greater than that of $\beta_L^*(W_L \otimes_L W_L)$. Thus, β_L has maximal Jordan type on W but not on $W \otimes W$.

Our second example shows that the maximal Jordan type of $M \otimes N$ can occur at a π-point at which one of M, N does not have maximal Jordan type and neither has projective type. This phenomenon can only occur if $\Pi(G)$ is reducible.

Example 4.6. Let G be a finite group with exactly two conjugacy classes of maximal elementary abelian p-groups (e.g., the p-Sylow subgroup of the wreath product $\mathbb{Z}/p \wr S_p$ as in Example 4.5) and write $\Pi(G) = X \cup Y$ with X, Y irreducible closed subsets. Let $[\alpha_K]$, $[\beta_L]$, L_ξ be as in Example 4.5. Choose another homogeneous cohomology class $\zeta \in \mathrm{H}^\bullet(G,k)$ with the property that ζ vanishes on $[\alpha_K]$ but does not vanish on $[\beta_L]$. Let L_ζ be the corresponding Carlson module so that the support of L_ζ is the zero locus of ζ. Let $M = L_\zeta \oplus L_\xi^{\oplus 2}$, so that M has maximal Jordan type $[\mathrm{proj}]+1[p-1]+1[1]$ at α_K, and Jordan type $[\mathrm{proj}]+2[p-1]+2[1]$ at β_L. Similarly, let $N = L_\zeta^{\oplus 2} \oplus L_\xi$, so that N has Jordan type $[\mathrm{proj}] + 2[p-1] + 2[1]$ at α_K and maximal Jordan type $[\mathrm{proj}]+1[p-1]+1[1]$ at β_L. Then $M \otimes N$ has maximal Jordan type at both α_K and β_L.

Corollary 4.4 enables us to prove the following property of the non-maximal support variety.

Proposition 4.7. *Assume $\Pi(G)$ is irreducible and let M, N be kG-modules. Then*
$$\Gamma(G)_{M\otimes N} \;=\; (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

Proof. If $\alpha_K : K[t]/t^p \longrightarrow KG$ is a π-point such that either $\alpha_K^*(M_K)$ or $\alpha_K^*(N_K)$ is projective, then $\alpha_K^*(M_K) \otimes_K \alpha_K^*(N_K)$ is projective and thus of maximum type. Consequently, statement (1) of Theorem 4.2 implies that $\alpha_K^*(M_K \otimes_K N_K)$ is also projective and thus also maximum. For such α_K, $[\alpha_K] \notin \Gamma(G)_{M\otimes N}$. In other words, $\Gamma(G)_{M\otimes N} \;\subset\; \Pi(G)_M \cap \Pi(G)_N$.

Now suppose that α_K has maximal Jordan type on both M and N. The statement (3) of Theorem 4.2 implies that α_K has maximal Jordan type on $M \otimes N$. Hence, $\Gamma(G)_{M\otimes N} \;\subset\; \Gamma(G)_M \cap \Gamma(G)_N$. We have established the inclusion
$$\Gamma(G)_{M\otimes N} \;\subset\; (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

On the other hand, assume that neither $\alpha_K^*(M_K)$ nor $\alpha_K^*(N_K)$ is projective, and that the Jordan type of either $\alpha_K^*(M_K)$ or $\alpha_K^*(N_K)$ is not maximal. In other words, we assume that $[\alpha_K] \in (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N)$. Corollary 4.4 implies that α_K does not have maximal Jordan type on $M \otimes N$. Hence, $[\alpha_K] \in \Gamma(G)_{M\otimes N}$. $\square$

Remark 4.8. The previous examples show the necessity of the hypothesis of irreducibility in Proposition 4.7. Example 4.6 contradicts the inclusion
$$\Gamma(G)_{M\otimes N} \;\supset\; (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N),$$
whereas Example 4.5 contradicts the inclusion
$$\Gamma(G)_{M\otimes N} \;\subset\; (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

Next we offer a suggestive characterization of modules of constant Jordan type for those finite group schemes G with $\Pi(G)$ irreducible.

Proposition 4.9. *If $\Pi(G)$ is irreducible, then a non-projective kG-module M has constant Jordan type if and only if for every finite dimensional kG-module N the tensor product $M \otimes N$ has the property that $\Gamma(G)_N = \Gamma(G)_{M\otimes N}$.*

Proof. First assume that M does not have constant Jordan type. If $N = k$ (the trivial module), then $\Gamma(G)_k$ is empty whereas $\Gamma(G)_{k\otimes M}$ is not empty.

Conversely, assume that M has constant Jordan type and M is not projective. Let $\alpha_K : K[t]/t^p \longrightarrow KG$ be a π-point such that N has maximal Jordan type at α_K. By Theorem 4.2, the Jordan type of $\alpha_K^*(M_K\otimes N_K) \simeq \alpha_K^*(M_K)\otimes\alpha_K^*(N_K)$ is maximal for $M \otimes N$. Hence, we get the inclusion $\Gamma(G)_N \supset \Gamma(G)_{M\otimes N}$. To prove the opposite inclusion, assume $[\alpha_K] \in \Gamma(G)_N$. Since M_K is not projective, Corollary 4.4 implies that α_K does not have maximal Jordan type on $M \otimes N$. Hence, $[\alpha_K] \in \Gamma(G)_{M\otimes N}$, and we have established the opposite inclusion. □

5. Endotrivial modules

We recall the definition of an endotrivial module, classically formulated for finite groups but admitting a natural extension to all finite group schemes. Endotrivial modules were introduced by Dade [11], who showed that for an abelian p-group, the only endotrivial kG-modules have the form $\Omega^n(k) \oplus P$ where P is a projective module. The endotrivial modules are the building blocks for the endopermutation modules which for many groups are the sources of the simple modules and are also a part of the Picard group of self equivalences of the stable module category. See [8] for references. A classification of the endotrivial modules for finite p-groups was completed in [9].

Definition 5.1. Let G be a finite group scheme over k. A kG-module M is an endotrivial module provided $\mathrm{End}_k(M)$ is stably isomorphic as a kG-module to the trivial module. In other words, M is endotrivial provided that there exists a kG-projective module P and a kG-isomorphism

$$\mathrm{Hom}_k(M, M) \;\cong\; k \oplus P.$$

As can readily be verified (for example, using formula (28) of the Appendix) an indecomposable $k[t]/t^p$-module is endotrivial if and only if it is stably isomorphic to either $1[1]$ or $1[p-1]$, the trivial module k and the Heller shift $\Omega^1(k)$ of k. More generally, Theorem 5.6 below implies that for any finite group scheme the Heller shifts $\Omega^i(k)$ of the trivial module are endotrivial modules. As mentioned earlier, for elementary abelian p-groups, these are the only indecomposable endotrivial modules. On the other hand, there do exist sporadic examples of finite groups admitting other endotrivial modules. For example, if G is a dihedral group of order 8, then $\mathrm{Rad}(kG)/\mathrm{Rad}^4(kG)$ is the direct sum of two modules of dimension three that are endotrivial (see [8]).

As we show in Theorem 5.6 below, every endotrivial module is a module of constant Jordan type. As seen in section 2 and as well in the next section, there exist many examples of modules of constant Jordan type which are not direct sums of endotrivial modules.

If M, N are kG-modules, then we may identify $\mathrm{Hom}_k(M,N)$ as a kG-module with $M^\# \otimes N$, where $M^\# = \mathrm{Hom}_k(M,k)$. For our purposes, it suffices to analyze $M^\#$ and then apply Section 4 in order to investigate $\mathrm{Hom}_k(M,N)$.

Let G be a finite group scheme and (as usual) let kG denote the group algebra of G. Denote by S the antipode of the Hopf algebra kG. If $\rho_M : kG \longrightarrow \mathrm{End}_k(M) = M^\# \otimes M$ is a finite dimensional representation of G determining the kG-module M, then

$$\rho_{M^\#} = \phi \circ \rho_M \circ S : \quad kG \longrightarrow kG \longrightarrow M^\# \otimes M \longrightarrow M \otimes M^\# , \tag{8}$$

where ϕ exchanges factors (and thus is the transpose from the point of view of matrices).

Observe that the dual $[i]^\#$ of the indecomposable $k[t]/t^p$-module $[i]$ is indecomposable, and thus isomorphic to $[i]$ as can be seen by comparing dimensions. The following proposition enables us to work with the generic and maximal Jordan types of $\mathrm{Hom}_k(M,N)$ for finite dimensional kG-modules $M,\ N$.

Proposition 5.2. *Let G be a finite group scheme, M be a finite dimensional kG-module, and $\alpha_K : K[t]/t^p \longrightarrow KG$ be a π-point of G. Assume that α_K has maximal Jordan type on M. Then*

$$\alpha_K^*(M_K^\#) \ \simeq\ \alpha_K^*(M_K) \ \simeq (\alpha_K^*(M_K))^\#.$$

Moreover, α_K has maximal Jordan type on M if and only if it has maximal Jordan type on $M^\#$.

Proof. Let $C_K \subset G_K$ be a unipotent abelian group scheme through which α_K factors, so that α_K has maximal Jordan type for M_K as a KC_K-module. The restriction of the KG-module $M_K^\#$ to KC_K is the dual of the restriction of M_K, since $KC_K \longrightarrow KG_K$ is a map of Hopf algebras. Thus, to prove the first asserted isomorphism, it suffices to assume G is a unipotent abelian finite group scheme.

By [25, 14.4], $kG \simeq k[T_1,\dots,T_n]/(T_i^{p^{e_i}})$. Let $t_i = T^{p^{e_i-1}}$. Let $I = (T_1,\dots,T_n)$ be the augmentation ideal of kG, and let $I^{(p)}$ be the ideal generated by $(t_1,\dots,t_n)$. Let $\nabla : kG \to kG \otimes kG$ be the coproduct in kG. Recall that $\nabla(\alpha_K(t)) - 1 \otimes \alpha_K(t) - \alpha_K(t) \otimes 1 \in I \otimes I$ (cf. [22, I.2.4(1)]). Since $\alpha_K(t)$ has p-th power 0, we can refine this further, concluding that

$$\nabla(\alpha_K(t)) - 1 \otimes \alpha_K(t) - \alpha_K(t) \otimes 1 \in I \otimes I^{(p)} + I^{(p)} \otimes I. \tag{9}$$

Let $\mu : kG \otimes kG \to kG$ be the multiplication map in kG. We have $(\mu \circ (S \otimes \mathrm{Id}))(\nabla(\mathrm{\alpha_K(t)}) = \epsilon(\mathrm{\alpha_K(t)}) = 0$ by one of the Hopf algebra axioms (see [22, I.2.3]) where $\epsilon : kG \to k$ is the counit map, and S is the antipode. Hence, applying $\mu \circ (S \otimes \mathrm{Id})$ to (9), we get

$$\alpha_K(t) + S(\alpha_K(t)) \subset I \cdot I^{(p)} + I^{(p)} \cdot I.$$

Therefore, we can apply [20, 1.12] to obtain that $\alpha_K(t)$ and $S(\alpha_K(t))$ have the same (maximal) Jordan type. Hence, upon composing (8) with α_K, we conclude that $\alpha_K^*(M_K^{\#}) \simeq \alpha_K^*(M_K)$.

The second isomorphism follows immediately from the observation that $[i]^{\#} = [i]$. □

Since sending M to $M^{\#}$ is idempotent, we get the following corollary.

Corollary 5.3. *Let G be a finite group scheme, and let M be a finite dimensional kG-module. Then*

$$\Gamma(G)_M = \Gamma(G)_{M^{\#}}.$$

Another corollary follows immediately from Corollary 4.3, Proposition 5.2 and the isomorphism $\mathrm{Hom}_k(M, N) \simeq M^{\#} \otimes N$.

Corollary 5.4. *Let G be a finite group scheme, and let M, N be finite dimensional kG-modules of constant Jordan types $\underline{a} = a_p[p] + \cdots + a_1[1]$, $\underline{b} = b_p[p] + \cdots + b_1[1]$, respectively. Then* $\mathrm{Hom}_k(M, N)$ *has constant Jordan type $\underline{a} \otimes \underline{b}$ (given explicitly by the formula (28)).*

Corollary 5.5. *Let G be a finite group scheme, and consider two finite dimensional kG-modules M, N, and a π-point α_K of G. If α_K has maximal Jordan type on* $\mathrm{Hom}_K(M_K, N_K)$*, then*

$$\alpha_K^*(\mathrm{Hom}_K(M_K, N_K)) \simeq \mathrm{Hom}_K(\alpha_K^*(M_K), \alpha_K^*(N_K))$$

Proof. Let $i : C_K \hookrightarrow G_K$ be a unipotent abelian group scheme through which α_K factors, so that α_K is maximal on $\mathrm{Hom}_K(M_K, N_K)$ as a KC_K-module. Since $i : C_K \hookrightarrow G_K$ is a map of Hopf algebras, it commutes with Hom. Hence, we may assume that G is a unipotent abelian group scheme. In particular, $\Pi(G)$ is irreducible.

Since α_K has maximal Jordan type on $\mathrm{Hom}_K(M_K, N_K) \simeq M_K^{\#} \otimes N_K$, Theorem 4.2(2) implies that

$$\alpha_K^*((M_K)^{\#} \otimes N_K) \simeq \alpha_K^*((M_K)^{\#}) \otimes \alpha_K^*(N_K).$$

If $\alpha_K^*((M_K)^{\#}) \otimes \alpha_K^*(N_K))$ is projective, then either $\alpha_K^*(N_K)$ or $\alpha_K^*((M_K)^{\#})$ is projective. Since projectivity of $\alpha_K^*((M_K)^{\#})$ implies projectivity of $\alpha_K^*(M_K)^{\#}$, we conclude that in this case $\mathrm{Hom}_K(\alpha_K^*(M_K), \alpha_K^*(N_K))$ is projective.

Assume that $\alpha_K^*((M_K)^{\#}) \otimes \alpha_K^*(N_K)$ is not projective. In this case neither $\alpha_K^*(N_K)$ nor $\alpha_K^*((M_K)^{\#})$ is projective. Since $\Pi(G)$ is irreducible, Corollary 4.4 implies that α_K has maximal Jordan types on both N and $M^{\#}$. Hence, $\alpha_K^*((M_K)^{\#}) \simeq (\alpha_K^*(M_K))^{\#}$. Therefore, $\mathrm{Hom}_K(\alpha_K^*(M_K), \alpha_K^*(N_K)) \simeq (\alpha_K^*(M_K))^{\#} \otimes \alpha_K^*(N_K) \simeq \alpha_K^*((M_K)^{\#}) \otimes \alpha_K^*(N_K) \simeq \alpha_K^*(\mathrm{Hom}_K(M_K, N_K))$. □

We now conclude that endotrivial modules are modules of constant Jordan type. The second statement of the theorem provides a “local” criterion of endotriviality, similar to the projectivity criterion given by the Dade’s lemma (see [11]).

Theorem 5.6. *Let G be a finite group scheme, and let M be a finite dimensional kG-module.*

(1) *If M is endotrivial, then M has constant Jordan type of the form either $m[p]+1[1]$ or $m[p]+1[p-1]$ for some $m \geq 0$, and thus $\alpha_K^*(M_K)$ is endotrivial for every π-point α_K of G.*
(2) *Conversely, if $\alpha_K^*(M_K)$ is endotrivial for each π-point α_K of G (and hence of the form either $m[p]+1[1]$ or $m[p]+1[p-1]$), then M is endotrivial.*

Proof. Observe that any endotrivial module must have dimension whose square is congruent to 1 modulo p and thus must have dimension congruent to either 1 or $p-1$ modulo p. To prove the first assertion, we assume that M is endotrivial, so that $\mathrm{End}_k(M) = k \oplus (\mathrm{proj})$. Thus, $\alpha_K^*(\mathrm{End}_K(M))$ has Jordan type $m[p]+1[1]$ at each π-point α_K. In particular, every π-point α_K has maximal Jordan type on $\mathrm{End}_k(M)$. By Corollary 5.5, $\alpha_K^*(\mathrm{End}_K(M_K)) \simeq \mathrm{End}_K(\alpha_K^*(M_K))$. Hence, $\alpha_K^*(M_K)$ is an endotrivial $K[t]/t^p$-module. The statement now follows from the fact that the only such modules are of the form $m[p]+1[1]$ or $m[p]+1[p-1]$ for some $m \geq 0$.

To prove the converse, we assume that $\alpha_K^*(M_K)$ is an endotrivial $K[t]/t^p$-module for each π-point $\alpha_K : K[t]/t^p \longrightarrow KG$. Thus, for each α_K, $\alpha_K^*(M_K)$ has Jordan type either $m[p]+1[1]$ or $m[p]+1[p-1]$ for some $m \geq 0$. Since the dimension of M can not be congruent to both 1 and $p-1$ modulo p, we conclude that M has constant Jordan type. Consider the short exact sequence

$$0 \longrightarrow X \longrightarrow \mathrm{End}_k(M) \xrightarrow{\mathrm{Tr}} k \longrightarrow 0, \tag{10}$$

where Tr is the trace map. By Corollary 5.5, $\alpha_K^*(\mathrm{End}_K(M_K)) \simeq \mathrm{End}_K(\alpha_K^*(M_K))$. Moreover, because the dimension of $\mathrm{End}_k(M)$ is relatively prime to p, the trace map of (10) splits. Pulling back the split short exact sequence (10) via α_K, we conclude that

$$\alpha_K^*(X) \simeq \ker\{\mathrm{End}_K(\, \alpha_K^*(M_K)) \longrightarrow k\ \} \simeq m[p]$$

is projective for all π-points α_K and thus X is projective ([19, 5.4]). Hence, M is endotrivial. $\square$

6. Constructing modules of constant Jordan type

In this section, we consider two different methods of constructing modules of constant Jordan type. Proposition 6.1 presents the observation that an extension of modules of constant Jordan type has “total module” also of constant Jordan type if the extension splits when pulled back along any π-point. This observation fits well with the Auslander-Reiten theory of almost split exact sequences as we discuss in

§8. Proposition 6.6 presents a method of constructing extensions of constant Jordan type whose pull-backs along π-points are not split.

Proposition 6.1. *Suppose that G is a finite group scheme over k. Let M and N be kG-modules of constant Jordan type, and suppose that*

$$0 \longrightarrow M \longrightarrow B \longrightarrow N \longrightarrow 0 \tag{11}$$

is an exact sequence. Let $\zeta \in Ext^1_{kG}(N, M)$ be the class of (11). If for every π-point $\alpha_K : K[t]/t^p \longrightarrow KG$ the restriction $\alpha_K^*(\zeta)$ is zero, then B has constant Jordan type. Moreover, if the Jordan types of M and N are $\sum_{i=1}^p m_i[i]$ and $\sum_{i=1}^p n_i[i]$, then the Jordan type of B is $\sum_{i=1}^p (m_i + n_i)[i]$.

Proof. Because the cohomology class vanishes under restriction to a π-point α_K, we have that the restriction of (11) along α_K splits and $\alpha_K^*(B_K) \simeq \alpha_K^*(M_K) \oplus \alpha_K^*(N_K)$. The result is now obvious. □

Proposition 6.1 does not always produce "new" examples of modules of constant Jordan type as we observe in the special case $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Example 6.2. Let G be $\mathbb{Z}/2 \times \mathbb{Z}/2$. As presented in [2], [21], there is a complete classification of the kG-modules. Using this classification, we observe that the only indecomposable kG-modules of constant Jordan type are of the form $\Omega^n(k)$ for some n.

Namely, it is shown in [21] that all of the indecomposable kG-modules of odd dimension are of the form $\Omega^n(k)$ for some n. On the other hand, the non-projective indecomposable modules of even dimension are all isomorphic to L_ζ for $\zeta \in \mathrm{H}^n(G, k)$. The support varieties of these even dimensional modules are proper non-trivial subvarieties of $\Pi(G)$, so that none of these have constant Jordan type.

It is instructive to look more closely to see why Proposition 6.1 does not determine other modules of constant Jordan type in this example. Observe that $\mathrm{H}^*(G, k) \cong \mathrm{Ext}^*_{kG}(k, k)$ is a polynomial ring in two variables having no element whose restriction to every π-point vanishes. Hence the only possible application of Proposition 6.1 would be in a situation where $N \cong \Omega^n(k)$, $M \cong \Omega^m(k)$ and $n < m$. Then (11) represents an element of negative Tate cohomology, ζ of $\hat{\mathrm{H}}^{n-m+1}(G, k)$. By Proposition 6.3 which follows, ζ restricts to zero at every π-point, but the middle term of this non-split short exact sequence splits as $\Omega^{n+a}(k) \oplus \Omega^{n+b} \oplus (\mathrm{proj})$ where a and b are nonnegative integers such that $a + b = m - n$ (cf. the proof of Theorem 6.13).

The preceding example is special since $\mathbb{Z}/2 \times \mathbb{Z}/2$ has tame representation type. For more general groups, the observation of Proposition 6.1 in conjunction with the following proposition does give new examples.

Proposition 6.3. *Let G be a finite group scheme with the property that every π-point factors by way of a flat map through a unipotent abelian group scheme whose* *cohomology has Krull dimension at least 2. Let $\zeta \in \widehat{\mathrm{H}}^n(G,k)$ for $n < 0$ be an element in negative Tate cohomology of G corresponding to a short exact sequence of the form*

$$0 \longrightarrow k \longrightarrow E \longrightarrow \Omega^{n-1}(k) \longrightarrow 0. \tag{12}$$

Then for any π-point $\alpha_K : K[t]/t^p \longrightarrow KG$, the pull-back of (12) along α_K is split (i.e. $\alpha_K^*(\zeta_K) = 0$).

Proof. Recall that every element ζ of $\widehat{\mathrm{H}}^n(G,k)$ is represented by a homomorphism $\zeta' : k \longrightarrow \Omega^{-n}(k)$. In order to prove the lemma, it is sufficient to show that for any π-point $\alpha_K : K[t]/(t^p) : \longrightarrow KG$, the restriction of ζ' factors through a projective $K[t]/(t^p)$-module. We proceed to establish such a factorization using our knowledge of the modules $\Omega^{-n}(K)$ for $n < 0$. By Theorem 1.4, equivalent π-points induce the same map in cohomology. Thus, it suffices to prove the statement for some representative α_K of each $[\alpha_k] \in \Pi(G)$.

Our hypothesis immediately allows us to replace G by some abelian unipotent group scheme (defined over K) whose cohomology has Krull dimension at least 2. After possibly passing to some finite extension of K, [25, 14.1] enables us to conclude that $KG \simeq K[T_1, \ldots, T_r]/(T_1^{p^{e_1}}, \ldots, T_r^{p^{e_r}})$, where $r \geq 2$. Because neither our hypothesis nor our conclusion depends upon the coalgebra structure on KG, we may assume that G is an abelian p-group. Let $t_i = T_i^{p^{e_i - 1}}$ and let $E \subset G$ be the (unique) elementary abelian p-group with $KE = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \subset KG$. By [17, 4.1], any K-rational π-point of KG has a representative factoring through KE. Thus, we may assume that $G = E$ is an elementary abelian p-group of rank at least 2. Changing the generators of KE, we may further assume $t_1 = \alpha_K(t)$; moreover, it suffices to assume that E has rank 2, for if the proposition is valid for an elementary abelian subgroup of G of rank 2, then it is valid for G itself. Thus, we are reduced to the case that KG is isomorphic to $K[u,v]/(u^p, v^p)$ with $u = \alpha_K(t)$.

The structure of a minimal KG-projective resolution of K is well known [10]. For example, suppose that $m = -n > 0$ is even. Then $\Omega^m(K)$ is a submodule of a projective module $P = KG^m$. A set of generators $a_1, \ldots, a_m$ for P can be chosen so that $\Omega^m(K)$ is the submodule generated by the elements

$$u^{p-1}a_1, \;\; va_1 - ua_2, \;\; v^{p-1}a_2 - u^{p-1}a_3, \;\; va_3 - ua_4, \ldots, \;\; va_{m-1} - ua_m, \;\; v^{p-1}a_m.$$

That is, P is the $(m-1)^{st}$ term of the projective resolution and $\Omega^m(K)$ is the kernel of the boundary map. Every KG-fixed point of P, and thus also of $\Omega^m(K)$ is a linear combination of the elements $u^{p-1}v^{p-1}a_i$. Moreover, for i odd,

$$u^{p-1}v^{p-1}a_i \;=\; u^{p-1}v^{p-2}(va_i - ua_{i+1})$$

and for i even

$$u^{p-1}v^{p-1}a_i \;=\; u^{p-1}(v^{p-1}a_i - u^{p-1}a_{i+1}).$$

Consequently, every KG-fixed point of $\Omega^m(K)$ has the form $u^{p-1}z$ for some $z \in \Omega^m(K)$. Thus, as a map of $K[t]/t^p$-modules (via $\alpha_K : K[t]/t^p \longrightarrow KG$), ζ'_K factors as the composition of $K \longrightarrow K[t]/t^p$ sending 1 to t^{p-1} and $K[t]/t^p \longrightarrow \Omega^m(K)$ sending 1 to z. Hence, $\alpha_K^*(\zeta')$ factors through the rank 1 projective module $K[t]/t^p$ as required. □

The following property is an immediate corollary of Proposition 6.3.

Corollary 6.4. *Let E be an elementary abelian p-group of rank at least 2, and let*

$$\theta : \Omega^m(k) \longrightarrow \Omega^n(k)$$

be a homomorphism. If the restriction of θ along some π-point $\alpha_K : K[t]/t^p \to KE$ does not factor through a projective $K[t]/t^p$-module, then $m \geq n$.

The following proposition shows that there are limitations on the Jordan types which can be realized as extensions.

Proposition 6.5. *Let G be a finite group scheme over a field k of characteristic $p > 2$ with the property that every π-point factors by way of a flat map through a unipotent abelian group scheme whose cohomology has Krull dimension at least 2. Then there does not exist a kG-module M of constant Jordan type $n[p] + 1[2]$ which is an extension of a kG-module of constant Jordan type $m[p] + 1[1]$ and one of constant Jordan type $(n - m)[p] + 1[1]$.*

Proof. As argued above in the proof of Proposition 6.3, we may assume that G is an elementary abelian p-group of rank 2. Consider a short exact sequence of the form

$$E : \qquad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{13}$$

in which both L and N have stable constant Jordan type $1[1]$. By Theorem 5.6, L and N are endotrivial modules, so that $L \cong \Omega^a(k)$, $N \cong \Omega^b(k)$ for some integers a and b. Since $p > 2$, both a and b must be even integers.

Thus the class of (13) is a cohomology class ζ in

$$\operatorname{Ext}^1_{kG}(\Omega^b(k), \Omega^a(k)) \cong \widehat{\mathrm{H}}^{b-a+1}(G, k).$$

If M has constant Jordan type $n[p] + 1[2]$, then the restriction of (13) along any π-point α_K does not split; equivalently, the restricted class $\alpha_K^*(\zeta)$ does not vanish. However, the only such cohomology classes ζ are scalar multiples of the identity in degree $b - a + 1 = 0$. Since both a and b are even integers, such classes ζ can not occur. □

We now proceed to describe a second method of constructing modules of stable constant Jordan type $n[1]$ that cannot be created by the methods of Proposition 6.1. The construction can be summarized as follows: if a certain map represented by a matrix with coefficients in $\widehat{\mathrm{H}}^*(G, k)$ has the maximal possible rank when restricted to any π-point of G, then it has a kernel of constant Jordan type.

Theorem 6.6. *Let G be a finite group scheme, and choose integers $m > n$, m_i and n_j such that all m_i and n_j are even if $p > 2$. Choose cohomology classes $\zeta_{i,j} \in \widehat{\mathrm{H}}^{m_j - n_i}(G,k)$ and let $\hat{\zeta}_{i,j} : \Omega^{m_j}(k) \to \Omega^{n_i}(k)$ represent $\zeta_{i,j}$. We consider an exact sequence*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\varphi = (\hat{\zeta}_{i,j})} N \longrightarrow 0$$

of kG-modules where

$$M \;\cong \sum_{j=1}^{m} \Omega^{m_j}(k) \oplus (\mathrm{proj}) \quad \text{and} \quad N \cong \sum_{i=1}^{n} \Omega^{n_i}(k).$$

Assume that for every π-point $\alpha_K : K[t]/t^p \longrightarrow KG$, the restriction of the matrix of cohomology elements $(\alpha_K^*(\zeta_{i,j})) \in M_{n,m}(\widehat{\mathrm{H}}^*(K[t]/t^p, K))$ has rank n. Then the module L has stable constant Jordan type $(m-n)[1]$.

Proof. Let $\alpha_K : K[t]/t^p \longrightarrow KG$ be a π-point. For any s, which is even if $p > 2$, the restriction of the module $\Omega^s(k)$ along α_K has the form $\alpha_K^*(\Omega^s(K)) \cong K \oplus (\mathrm{proj})$. Moreover, if we have a map $\zeta : \Omega^s(k) \longrightarrow \Omega^t(k)$ whose cohomology class when restricted along α_K is not zero, then the composition

$$K \xrightarrow{\iota} \alpha_K^*(\Omega^s(K)) \xrightarrow{\alpha_K^*(\zeta_K)} \alpha_K(\Omega^t(K)) \xrightarrow{\rho} K,$$

where ι and ρ are the split inclusion and projection maps, is an isomorphism. Thus our hypothesis asserts that the composition

$$K^m \xrightarrow{\iota} \alpha_K^*(M_K) \xrightarrow{\varphi_K} \alpha_K^*(N_K) \xrightarrow{\rho} K^n$$

is a split surjection. It follows that the kernel of φ has stable Jordan type $(m-n)[1]$ on L. As this happens for any α_K, we are done. $\square$

Theorem 6.6 is stated for even dimensional cohomology classes (for $p > 2$), yet as we observe in the following proposition a very similar argument can also be applied to odd dimensional classes to yield additional modules of constant Jordan type.

Proposition 6.7. *Let G be a finite group scheme, and choose positive integers $m > n$, $m_1, \ldots m_m$ all odd, $n_1, \ldots, n_n$ all even. Assume $p > 2$. Choose cohomology classes $\zeta_{i,j} \in \widehat{\mathrm{H}}^{m_j - n_i}(G,k)$ and let $\hat{\zeta}_{i,j} : \Omega^{m_j}(k) \to \Omega^{n_i}(k)$ represent $\zeta_{i,j}$. We consider an exact sequence*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\varphi = (\hat{\zeta}_{i,j})} N \longrightarrow 0$$

of kG-modules where

$$M \;\cong \sum_{j=1}^{m} \Omega^{m_j}(k) \oplus (\text{proj}) \quad \textit{and} \quad N \cong \sum_{i=1}^{n} \Omega^{n_i}(k).$$

Assume that for every π-point $\alpha_K : K[t]/(t^p) \longrightarrow KG$, the restriction of the matrix of cohomology elements $\alpha_K^*(\zeta_{i,j}) \in M_{n,m}(\widehat{\mathrm{H}}^*(K[t]/t^p, K))$ has rank n. Then the module L has stable constant Jordan type $(m-n)[p-1] + n[p-2]$.

Proof. Our hypothesis implies that the restriction of $(\hat{\zeta}_{i,j})$ via any α_K is a map of $K[t]/t^p$-modules which remains surjective after free summands are dropped. We write such a map of $K[t]/t^p$-modules symbolically as a map of their (stable) Jordan types $m[p-1] \twoheadrightarrow n[1]$. Such a surjective map of $K[t]/t^p$-modules with indicated stable Jordan type necessarily has kernel with Jordan type $(m-n)[p-1]+n[p-2]$. □

In general, it is very easy to construct examples for which Theorem 6.6 and Proposition 6.7 are relevant. These modules are multi-parametrized versions of important modules introduced and studied by the first author.

Example 6.8. Let $\xi_1 \ldots, \xi_r$ be homogeneous elements in $\mathrm{H}^\bullet(G,k)$ such that the radical of the ideal generated by the ξ_i's is the augmentation ideal of $\mathrm{H}^\bullet(G,k)$. Alternatively, for p odd, let $\xi_1 \ldots, \xi_r$ be homogeneous elements in $\mathrm{H}^*(G,k)$ of odd degree whose Bocksteins generate an ideal of $\mathrm{H}^\bullet(G,k)$ whose radical is the augmentation ideal. For each i, let $\hat{\xi}_i : \Omega^{n_i}(k) \longrightarrow k$ be a cocycle representing ξ_i, where n_i is the degree of ξ_i. Define

$$\varphi \;:\; \bigoplus_{i=1}^{r} \Omega_i^n(k) \to k$$

by the formula $\varphi(a_1, \ldots, a_r) = \hat{\xi}_1(a_1) + \cdots + \hat{\xi}_r(a_r)$. Then

$$L_{\xi_1,\ldots,\xi_r} \;\equiv\; \operatorname{Ker}\varphi \tag{14}$$

has constant Jordan type.

In Theorem 6.13 below, we give an explicit example of such an $L_{\xi_1,\ldots,\xi_r}$ which can not be constructed using the technique of Proposition 6.1. The detailed verification of this example will occupy the remainder of this section, and involves the following four lemmas.

Recall that if $E \simeq (\mathbb{Z}/p)^{\times r}$ is an elementary abelian p-group of rank r, then

$$\mathrm{H}^*(E,k) \cong \begin{cases} k[\zeta_1,\ldots,\zeta_r,\eta_1,\ldots,\eta_r]/(\eta_1^2,\ldots,\eta_r^2) & \text{if} \quad p>2 \\ k[\eta_1,\ldots,\eta_r] & \text{if} \quad p=2. \end{cases} \tag{15}$$

where each η_j has degree one and each ζ_i has degree 2.

Lemma 6.9. *Let E be an elementary abelian p-group of rank r, for $r > 1$. For $n > 0$,*

(1) $\operatorname{Dim} \mathrm{H}^n(E,k) = \binom{n+r-1}{r-1}$.
(2) $\operatorname{Dim} P_n = \binom{n+r-1}{r-1} \cdot p^r$, *where P_n is the n-th term of a minimal kE-projective resolution of k.*
(3) $\operatorname{Dim} \Omega^n(k) = p^r \cdot a_{r,n} + (-1)^n$, $n > 0$ *where*

$$a_{r,n} = \binom{n+r-2}{r-1} - \binom{n+r-3}{r-1} + \cdots + (-1)^{n-1}\binom{r-1}{r-1}.$$

(4) *If $r = 2$, then* $\operatorname{Dim}(\Omega^{2n}(k)) = p^2 n + 1$.
(5) *If $r = 3$, then* $\operatorname{Dim}(\Omega^{2n}(k)) = p^3 n(n+1) + 1$

Finally, if $n < 0$, then $\operatorname{Dim}(\Omega^n(k)) = \operatorname{Dim}(\Omega^{-n}(k))$.

Proof. The computation of $\operatorname{Dim} \mathrm{H}^n(E,k)$ is a straightforward and familiar computation.

In a minimal projective kE-resolution of k,

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0,$$

each P_n is a direct sum of copies of kG, the number of copies equal to the dimension of $\mathrm{H}^n(E,k)$.

Using the vanishing of the Euler characteristic of an exact sequence, we conclude the asserted formula for the dimension of $\Omega^n(k)$.

The assertions for $r = 2, 3$ are special cases. $\square$

Lemma 6.10. *Let E be an elementary abelian p-group of rank 3, n be a negative integer, and $Q \longrightarrow \Omega^{2n}(k)$ be the projective cover of the $2n^{\text{th}}$ Heller shift of k. Then* $\operatorname{Dim} Q = p^3(2n^2 - n)$.

Proof. Taking duals we have an injection $\Omega^{-2n}(k) \longrightarrow Q^*$, where Q^* is the injective hull of $\Omega^{-2n}(k)$. Hence, $Q^* \simeq P_{-2n-1}$, the $(-2n-1)^{\text{st}}$ term of the minimal projective resolution of k. As seen in Lemma 6.9(2), the dimension of P_{-2n-1} is $p^3\binom{-2n-1+2}{2} = p^3(2n^2 - n)$. $\square$

Lemma 6.11. *Suppose that E is an elementary abelian p-group of rank at least 2. Let*

$$\theta : \Omega^m(k) \longrightarrow \Omega^n(k)$$

be a homomorphism for some nonnegative integers m and n, which are both even in case $p > 2$. Assume that for some π-point $\alpha_K : K[t]/(t^p) \longrightarrow KE$ the restriction along α of θ does not factor through a projective $K[t]/(t^p)$-module. Then θ is surjective.

Proof. By Corollary 6.4, we have $m-n \geq 0$. Let $\widehat{\theta}$ denote the cohomology class of θ in

$$\mathrm{Ext}_{kE}^{m-n}(k,k) \cong \mathrm{H}^{m-n}(E,k).$$

The condition on the restriction of θ along α together with the fact that $m-n$ is even if $p>2$ implies that $\widehat{\theta}$ is a non-nilpotent element of $\mathrm{H}^{m-n}(E,k)$. Since $\mathrm{Hom}_{kE}(\Omega^n(k),k) \cong \mathrm{H}^n(E,k)$, it follows that θ induces an injective map

$$\theta' : \mathrm{Hom}_{kE}(\Omega^n(k),k) \longrightarrow \mathrm{Hom}_{kE}(\Omega^m(k),k)\ .$$

So θ must be surjective. $\square$

Lemma 6.12. *Suppose that E is an elementary abelian p-group of rank 2. Let $\xi_1 \in \mathrm{H}^{2m}(E,k)$, $\xi_2 \in \mathrm{H}^{2n}(E,k)$, and assume that the radical of the ideal generated by ξ_1, ξ_2 is the augmentation ideal of $\mathrm{H}^\bullet(E,k)$. Consider the exact sequence*

$$0 \longrightarrow L_{\xi_1,\xi_2} \longrightarrow \Omega^{2m}(k) \oplus \Omega^{2n}(k) \xrightarrow{\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}} k \longrightarrow 0.$$

Then $L_{\xi_1,\xi_2} \simeq \Omega^{2m+2n}(k)$.

Proof. The condition on ξ_1, ξ_2 is equivalent to the condition that the matrix (ξ_1,ξ_2) has rank 1 when restricted to any π-point of G. Hence, by Theorem 6.6, L_{ξ_1,ξ_2} has stable constant Jordan type 1[1], and it is an endotrivial module. By Lemma 6.9, the dimension of L_{ξ_1,ξ_2} is $p^2(m+n)+1$, and hence $L_{\xi_1,\xi_2} \simeq \Omega^{\pm(2m+2n)}(k)$.

Now suppose that $L_{\xi_1,\xi_2} \simeq \Omega^{-(2m+2n)}(k)$. Then the sequence represents a non-zero element $\gamma \in \mathrm{H}^{2m+2n+1}(E,k)$ which has the property that $\alpha_K^*(\gamma_K)$ is zero for any π-point $\alpha_K : K[t]/t^p \longrightarrow KE$. However, because $2m+2n+1$ is both positive and odd, there is no such element. Therefore, we must have that $L_{\xi_1,\xi_2} \simeq \Omega^{2m+2n}(k)$ as desired. $\square$

As we show in the following proposition, there exist examples of modules of constant Jordan types constructed as in Proposition 6.6 which are not middle terms of extensions of endotrivial modules as in Proposition 6.1.

Theorem 6.13. *Let G be an elementary abelian p-group of rank 3 and consider the kG-module $L = L_{\zeta_1,\zeta_2,\zeta_3}$ as in (14), where $\{\zeta_i\}_{i=1,2,3} \subset \mathrm{H}^2(G,k)$ form a system of generators of $\mathrm{H}^\bullet(G,k)_{red}$. Then there does not exist a projective kG-modules P such that $M = L \oplus P$ fits in a short exact sequence of the form*

$$0 \longrightarrow L' \longrightarrow M \longrightarrow L'' \longrightarrow 0 \tag{16}$$

with both L', L'' endotrivial modules.

Proof. Observe that M has stable Jordan type 2[1], so that in any short exact sequence of the form (16) both L', L'' must have stable Jordan type 1[1]. We assume that such a short exact sequence exists and proceed to obtain a contradiction. First, by eliminating projective summands we may reduce to the case when both L' and L'' are projective-free. Since E is a p-group, we have $P \cong (kE)^t$ for some $t \geq 0$. Since L', L'' have stable constant Jordan type $1[1]$, they are endotrivial by Theorem 5.6. If $p > 2$, then we immediately conclude $L' \cong \Omega^{2n_1}(k)$, $L'' \cong \Omega^{2n_2}(k)$ for some integers n_1, n_2 since L', L'' have stable Jordan type $1[1]$. If $p = 2$, then $\operatorname{Dim} L' + \operatorname{Dim} L'' = \operatorname{Dim} L + \operatorname{Dim} P \equiv 2 \pmod{\operatorname{Dim} kE}$. Since $\operatorname{Dim} \Omega^n(k) \equiv (-1)^n \pmod{\operatorname{Dim} kE}$, we get that L', L'' must be even syzygies in the case $p = 2$ as well. Thus, the sequence becomes

$$0 \longrightarrow \Omega^{2n_1}(k) \longrightarrow L \oplus (kE)^t \longrightarrow \Omega^{2n_2}(k) \longrightarrow 0 \tag{17}$$

Consider the subgroup $F = \langle g_1, g_2 \rangle \subseteq E$. The restriction of ζ_3 to F vanishes, so that

$$L_{|F} \;\cong\; \Omega^2(k) \oplus L_{|F,\zeta_1\zeta_2} \oplus (\text{proj}),$$

where $L_{|F,\zeta_1\zeta_2}$ is constructed as in (14) with respect to the group F. By Lemma 6.12, $L_{|F,\zeta_1\zeta_2} \;\cong\; \Omega^4(k)$ as a kF-module. Consequently,

$$L_{|F} \;\cong\; \Omega^2(k) \oplus \Omega^4(k) \oplus (\text{proj})\,.$$

Restricting (17) to F and eliminating the projective summands at the ends, we obtain an exact sequence of kF-modules

$$0 \longrightarrow \Omega^{2n_1}(k) \longrightarrow \Omega^2(k) \oplus \Omega^4(k) \oplus (\text{proj}) \longrightarrow \Omega^{2n_2}(k) \longrightarrow 0 \tag{18}$$

By performing a shift and eliminating excess projectives, we get the sequence of kF-modules

$$0 \longrightarrow \Omega^{2n_1-2n_2}(k) \longrightarrow \Omega^{4-2n_2}(k) \oplus \Omega^{2-2n_2}(k) \oplus (\text{proj}) \xrightarrow{\theta} k \longrightarrow 0 \tag{19}$$

where θ restricts to θ_1 on $\Omega^{4-2n_2}(k)$ and to θ_2 on $\Omega^{2-2n_2}(k)$. Since the kernel of θ has stable Jordan type $1[1]$, at least one of θ_1, θ_2 does not factor through a projective module when restricted to any π-point of F. Hence, Proposition 6.3 implies that $n_2 \leq 2$. By the same argument applied to the other end of Sequence (19), we have that $2n_1 - 2n_2$ cannot be less than $2 - 2n_2$ or that $n_1 \geq 1$. We further observe that if there were a non-trivial projective summand in the middle term of (19), then θ restricted to the projective summand would be surjective, and, hence, the kernel would have stable Jordan type different from $1[1]$. It follows that there is no projective summand in the middle term of (19). Hence, we get $p^3(|4-2n_2| + |2-2n_2|) + 2 = \operatorname{Dim}(\Omega^{4-2n_2}(k) \oplus \Omega^{2-2n_2}(k)) = \operatorname{Dim} \Omega^{2n_1-2n_2}(k) + 1 = p^3(|2n_1 - 2n_2|) + 2$ by Lemma 6.9. Hence,

$$|2 - n_2| + |1 - n_2| = |n_1 - n_2| \tag{20}$$

We conclude that $n_1 + n_2 = 3$ when $n_2 \leq 1$. Moreover, n_1 and n_2 must have different parity.

We consider two cases:

(I) $n_2 \geq 0$, that is $n_2 = 2, 1, 0$, and
(II) $n_2 < 0$.

Case I. The strategy here is to first show that $t = 0$, and then get a contradiction by a dimension count. (Recall that t is the rank of the free summand of the middle term of (17).)

Step I.1. We show that $t = 0$. The sequence (17) represents a cohomology class in $\widehat{\mathrm{H}}^{2n_2-2n_1+1}(E,k)$ which we denote by η. Restriction of (17) to any π-point of E has the form

$$0 \longrightarrow n[p] + 1[1] \longrightarrow (n+m)[p] + 2[1] \longrightarrow m[p] + 1[1] \longrightarrow 0$$

This sequence of $\mathbb{Z}/p$-modules is necessarily split. Hence, η vanishes upon restriction to any π-point of E. Moreover, if the sequence (17) is split then there is no projective summand in the middle term. Thus, we may assume that η is not zero.

Applying $\mathrm{Hom}_{kE}(-,k)$ to the short exact sequence (17), we obtain a long exact sequence

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^{-1}_{kE}(\Omega^{2n_1}(k),k) \xrightarrow{\delta} \widehat{\mathrm{Ext}}^{0}_{kE}(\Omega^{2n_2}(k),k) \longrightarrow \widehat{\mathrm{Ext}}^{0}_{kE}(L,k) \longrightarrow \cdots.$$

which is equivalent to

$$\cdots \longrightarrow \widehat{\mathrm{H}}^{2n_1-1}(E,k) \xrightarrow{\cdot\eta} \widehat{\mathrm{H}}^{2n_2}(E,k) \longrightarrow \widehat{\mathrm{Ext}}^{0}_{kE}(L,k) \longrightarrow \cdots. \tag{21}$$

The rank of the free summand $(kE)^t$ in the middle term of the sequence (17) equals the dimension of the image of the connecting homomorphism δ, which is multiplication by η.

We consider the three possible values for $n_2 \in \{0,1,2\}$ separately.

• $n_2 = 2$. Then the sequence (18) becomes

$$0 \longrightarrow \Omega^{2n_1}(k) \xrightarrow{i} \Omega^2(k) \oplus \Omega^4(k) \oplus (\mathrm{proj}) \xrightarrow{\theta} \Omega^4(k) \longrightarrow 0$$

By Proposition 6.3, θ_1 factors through a projective module when restricted to any π-point of E. Hence, the restriction of θ_2 to any π-point does not factor through a projective module. Therefore, $\theta_2 : \Omega^4(k) \to \Omega^4(k)$ is surjective by Lemma 6.11. Hence, θ_2 is an isomorphism. We conclude that the sequence splits and, thus, $n_1 = 1$. Hence, η is a non-trivial cohomology class in $\mathrm{H}^{2n_2-2n_1+1}(E,k) = \mathrm{H}^3(E,k)$ which must vanish on restriction along every π-point. If $p = 2$ then there is no such class. If $p > 2$, then up to scalar multiple, the cohomology class is $\eta_1\eta_2\eta_3$ where η_1, η_2, η_3 are the nilpotent generators in degree 1 of $\mathrm{H}^*(E,k)$. Thus, the multiplication by η of any odd-dimensional class in zero. Hence, $t = 0$.

• $n_2 = 1$. In this case $n_1 = 2$, and the sequence (17) takes the form

$$0 \longrightarrow \Omega^4(k) \xrightarrow{i} L \oplus (kE)^t \xrightarrow{\theta} \Omega^2(k) \longrightarrow 0$$

Therefore, η is a non-trivial cohomology class in $\widehat{\mathrm{H}}^{-1}(E,k)$. As $\widehat{\mathrm{H}}^{-1}(E,k)$ is 1-dimensional, η is a multiple of any non-zero element in $\widehat{\mathrm{H}}^{-1}(E,k)$. On the other hand, an almost split sequence $0 \longrightarrow \Omega^4(k) \longrightarrow M \longrightarrow \Omega^2(k) \longrightarrow 0$ represents such cohomology class. By [13], the middle term of this almost split sequence is indecomposable. Hence, $t = 0$ in this case.

• $n_2 = 0$. In this case, $n_1 = 3$ and the image of the connecting homomorphism δ belongs to $\widehat{\mathrm{H}}^{2n_2}(E,k) = \mathrm{H}^0(E,k)$. Hence, we have $t = 1$ or $t = 0$. Suppose $t = 1$. Then the sequence (17) has the form

$$0 \longrightarrow \Omega^{2n_1}(k) \longrightarrow L \oplus kE \longrightarrow k \longrightarrow 0$$

The dimension of L is $3\,\mathrm{Dim}(\Omega^2(k)) - 1$ which, by Lemma 6.9, is equal to $3(2p^3 + 1) - 1 = 6p^3 + 2$. Hence, the dimension of the middle term of the sequence is at most $7p^3 + 2$ On the other hand, by the same lemma, $\mathrm{Dim}\,\Omega^{2n_1}(k) = n_1(n_1 + 1)p^3 + 1 = 12p^3 + 1$. which presents us with a contradiction. Hence, $t = 0$ in this case.

Step I.2. We now compare the dimensions of the terms of the sequence (17) in which we take $t = 0$. Note that n_1 and n_2 are both non-negative in the cases that we are considering. We get

$$6p^3 + 2 = \mathrm{Dim}\,L = \mathrm{Dim}\,\Omega^{2n_1}(k) + \mathrm{Dim}\,\Omega^{2n_2}(k) = n_1(n_1+1)p^3 + 1 + n_2(n_2+1)p^3 + 1$$

using Lemma 6.9. Hence, we get the following equation on n_1, n_2:

$$n_1^2 + n_2^2 + n_1 + n_2 = 6$$

The only integer solutions are $(2, 0)$ and $(0, 2)$. This is impossible since (n_1, n_2) must have the opposite parity by (20). Hence, we obtain a contradiction as desired, and the case $n_2 \geq 0$ is finished.

Case II. We now consider the case of negative n_2. The projective summand $(kE)^t$ in the middle term of the sequence (17) can not be any bigger than a projective cover of $\Omega^{2n_2}(k)$ as otherwise, kernel of the map $(kE)^t \longrightarrow \Omega^{2n_2}(k)$ has a projective submodule which is then a direct summand of the right hand term of (17). By Lemma 6.10, $t \leq 2n_2^2 - n_2$. Hence,

$$\begin{aligned}\mathrm{Dim}\,L &\geq \mathrm{Dim}\,\Omega^{2n_2}(k) + \mathrm{Dim}\,\Omega^{2n_1}(k) - p^3(2n_2^2 - n_2)\\ &= p^3(n_2^2 - n_2) + p^3(n_1^2 + n_1) + 2 - p^3(2n_2^2 - n_2)\end{aligned}$$

Simplifying, and using the fact that $n_2 + n_1 = 3$, we get

$$\begin{aligned}\operatorname{Dim} L &\geq p^3(n_1^2 - n_2^2 + 3 - n_2) + 2 \\ &= p^3(3(n_1 - n_2) + 3 - n_2) + 2 = p^3(3n_1 + 3 - 4n_2) + 2\end{aligned}$$

Since $n_1 \geq 1$, and $n_2 < 0$, we conclude that $\operatorname{Dim} L \geq 10p^3 + 2$. On the other hand, $\operatorname{Dim} L = 6p^3 + 2$ by Lemma 6.9, and we have a contradiction. □

7. Constraint on ranks

A consequence of our constructive techniques is a new proof of a special case of Macaulay's Generalized Principle Ideal Theorem. The point is that if the coefficients are in a field of finite characteristic, then we can represent homogeneous elements of a multivariable polynomial ring as elements in the cohomology ring of an elementary abelian p-group. We can represent a matrix of such elements as a map of modules over the group algebra. Specifically we have the following

Theorem 7.1. *(See Exercise 10.9 of* [12]*) Suppose that k is an infinite field. Fix integers n, r and $d_1, \dots, d_{r+1}$, with $n \geq 3$, $r \geq 2$, and $d_i > 0$ for all i. Let $P = k[x_1, \dots, x_n]$ be a polynomial ring in n variables. Let $A = A(x_1, \dots, x_n)$ be a $(r+1)\times r$ matrix with the property that every entry in column i of A is a homogeneous polynomial in P of degree d_i for all $i = 1, \dots, r+1$. Then there is some point $\alpha \in k^n \backslash \{0\}$ such that $A(\alpha)$ has rank less than r. Equivalently, the determinants of the $r \times r$ minors of A (which are elements of P) have a common non-trivial zero.*

Proof. Assume first that the characteristic of k is $p > 0$, as in the rest of the paper. Let E denote an elementary abelian p-group of rank n. As recalled in (15), $\mathrm{H}^*(E, k)$ contains a polynomial subring $Q \cong k[\zeta_1, \dots, \zeta_n]$, where the elements ζ_i are in degree 2 if $p > 2$ and in degree 1 if $p = 2$. For the purposes of the argument, we assume that $p > 2$. The proof in the even characteristic case is very similar.

Let $A = (a_{i,j})$ where for each i and j, $a_{i,j} = a_{i,j}(x_1, \dots, x_n)$ is a homogeneous polynomial of degree d_j. Then, $a_{i,j}(\zeta_1, \dots, \zeta_n)$ is an element in $\mathrm{H}^{2d_j}(E, k)$. Moreover, such an element is uniquely represented by a cocycle

$$a'_{i,j}(\zeta_1, \dots, \zeta_n): \quad \Omega^{2d_j}(k) \longrightarrow k.$$

Now we let A' be the map

$$A': \quad \bigoplus_{t=1}^{r+1} \Omega^{2d_t}(k) \longrightarrow k^r$$

whose matrix is $A' = (a'_{i,j}(\zeta_1, \dots, \zeta_n))$.

We proceed to prove the theorem by contradiction, observing that if $A(\alpha)$ had rank r for all $\alpha \in k^n \backslash \{0\}$, then, because k is infinite (so that the k-rational points of

$\Pi(E)$ are dense), the kernel L of A' would be a module of constant Jordan type (with stable Jordan type $1[1]$) as in Theorem 6.6. By Theorem 5.6, L is an endotrivial module. Hence, $L \cong \Omega^{2m}(k) + (\mathrm{proj})$ for some m. Since $\bigoplus_{t=1}^{r+1} \Omega^{2d_i}(k)$ does not have projective summands, we conclude that $L \cong \Omega^{2m}(k)$.

As in the proof of Theorem 6.13, we can use a dimension argument to ascertain the value of m. Let $H \subset E$ be an elementary abelian subgroup of rank 2. Restricting to H and eliminating the projective summand in L and in the domain of A', we get that the dimension of the projective-free part of $L\downarrow_H$ is precisely $p^2 \sum_{i=1}^{r+1} d_i + 1$ by Lemma 6.9(4). Consequently, by the same lemma $m = \sum_{i=1}^{r+1} d_i$.

Now let $H' \subset E$ be an elementary abelian p-subgroup of rank 3, and let L' be the projective-free part of the restriction of L to H'. Since $L' \simeq \Omega^{2m}(k)$ as H'-modules, Lemma 6.9(5) implies that

$$\operatorname{Dim} L' = p^3 m(m+1) + 1 = p^3 \Big(\sum_{i=1}^{r+1} d_i\Big)\Big(\sum_{i=1}^{r+1} d_i + 1\Big) + 1 \tag{22}$$

On the other hand, L' is the projective-free part of the kernel of the map A' restricted to H'. Applying Lemma 6.9(4) to compute the dimension of the H'-module $\bigoplus_{t=1}^{r+1} \Omega^{2d_i}(k)$ we get

$$\operatorname{Dim} L' = p^3 \sum_{t=1}^{r+1} d_i(d_i+1) + 1. \tag{23}$$

As all $d_i > 0$, the formula (22) clearly yields a greater value than (23). Thus, we get a contradiction.

Now, we consider a field of characteristic 0, still denoted k. Let $R \subset k$ be the ring finitely generated over $\mathbb{Z}$ by the coefficients of the (homogeneous polynomial) entries of A. The $r+1$ determinants of the $r \times r$ minors of A define a closed subscheme Z of the projective scheme $\mathbb{P}_R^{n-1}$. By the preceding argument for fields of positive characteristic, Z intersects each geometric fiber of $\mathbb{P}_R^{n-1} \longrightarrow \operatorname{Spec} R$ at a point of $\operatorname{Spec} R$ with positive residue characteristic.

Observe that the geometric points of $\operatorname{Spec} R$ whose residue characteristics are positive are dense in $\operatorname{Spec} R$. Hence, the image of $Z \subset \mathbb{P}_R^{n-1}$ in $\operatorname{Spec} R$ is both closed and dense and thus all of $\operatorname{Spec} R$. We conclude that Z intersects every geometric fiber of $\mathbb{P}_R^{n-1} \longrightarrow \operatorname{Spec} R$, including that given by $R \longrightarrow k$. $\square$

8. Auslander Reiten Components

In this section, we use Auslander-Reiten theory of almost split sequences to generate indecomposable modules of constant Jordan type. We refer the reader to [1] or [3, I.4] for basic facts on almost split sequences and Auslander-Reiten quivers.

We fix the following notation which differs from the standard in the case of finite groups: Let $A \subset B$ be rings, and M be a left A-module. Then

$$\mathrm{Coind}_A^B M = B \otimes_A M$$

Let G be a finite group scheme, and $\alpha_K : K[t]/t^p \longrightarrow KG$ be a π-point. We denote by $K\langle \alpha_K(t)\rangle$ the subalgebra of $KG = KG_K$ generated by $\alpha_K(t)$.

Lemma 8.1. *Let G be a finite group scheme and let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G. Let N be a finite dimensional $K[t]/t^p$-module which is not projective and set $M = \mathrm{Coind}_{K\langle\alpha_K(t)\rangle}^{KG} N$. Then*

$$\Pi(G_K)_M \ = \ \{[\alpha_K]\} \ \subset \ \Pi(G_K).$$

Proof. If $\Pi(G_K)_M$ were empty, then M would be projective, and, hence, injective. On the other hand, the Eckmann-Shapiro Lemma (cf. [3, I.2.8.4]) would enable us to then conclude that $\mathrm{Ext}^{*>0}_{G_K}(M,K) = \mathrm{Ext}^{*>0}_{K\langle\alpha_K(t)\rangle}(N,K) = 0$ which would contradict our assumption that N is not projective. Thus, $\Pi(G_K)_M$ is non-empty.

Let $U_K \subset G_K$ be a unipotent abelian subgroup scheme through which α_K factors. The proof of [20, 4.12] which is stated for induction rather than coinduction implies that

$$\Pi(G_K)_M \ \subset \ \mathrm{im}\{\Pi(U_K) \to \Pi(G_K)\}$$

so that we may assume that $G_K = U_K$ is a unipotent abelian finite group scheme over K.

After possibly replacing K by some purely inseparable extension which does not change the space $\Pi(G_K)_M$, we may by [25, 14.4] assume that KG_K is isomorphic (as an algebra) to $K[T_1,\ldots,T_n]/(T_1^{p^{e_1}},\ldots,T_n^{p^{e_n}})$ for suitable choice of $n, e_1, \ldots, e_n$. Let $t_i = T_i^{p^{e_i-1}}$, and recall that any π-point $\beta_L : L[t]/t^p \to LG_K$ must send t to a sum of monomials in $T_1,\ldots,T_n$ at least one of which is a non-linear scalar multiple of some t_i and each of which are divisible by some (possibly varying) t_i. By a change of generators, we may arrange that $\alpha_K(t) = t_1 + p(T)$ where each monomial of the polynomial $p(T)$ is a non-scalar multiple of some t_i.

In order to verify that $[\beta_L] \notin \Pi(G_K)_M$ for $[\beta_L] \neq [\alpha_K]$, we may choose a representative β_L of $[\beta_L]$ which is linear in the t_i's. Assuming $[\beta_L] \neq [\alpha_K]$, we may change generators once again so that $\alpha_K(t)$ retains the form $t_1 + p(T)$ as above and $\beta_L(t) = t_2$. The condition that $[\beta_L] \notin \Pi(G_K)_M$ is equivalent to the condition that $\beta_L^*(M_L)$ is free. Clearly, it suffices to assume that N is indecomposable of the form $K[t]/t^i, i < p$. Then $M_L \ \cong \ L[T_1,\ldots,T_n]/(t_1^p,\ldots,t_n^p,(t_1+p(T))^i)$ which is free over

$L\langle t_2\rangle$ with a monomial basis $\{T_1^{j_1}, T_2^{j_2}, \ldots, T_n^{j_n}\}$, where $0 \leq j_1 < ip^{e_1-1}, 0 \leq j_2 < p^{e_2-1}, 0 \leq j_3 < p^{e_3} \ldots 0 \leq j_n < p^{e_n}$. □

Recall that for a finite-dimensional kG-module M the *transpose* of M (in the sense of Auslander-Reiten), denoted $\mathrm{Tr\,M}$, can be defined as

$$\mathrm{Tr\,M} = \Omega^{-2}(\mathrm{M}^{\#})$$

(see [1, IV]). Since kG is a finite-dimensional Hopf algebra, the coinverse map gives an isomorphism between $(kG)^{op}$ and kG. Hence, the standard duality functor $\mathrm{D} : \mathrm{kG} \longrightarrow kG^{op}$ (see [1, II.3]) can be identified with taking linear duals. We conclude that the *translation* functor $\tau = \mathrm{D} \circ \mathrm{Tr} : \mathrm{kG} \to \mathrm{kG}$ is given by

$$\tau M = (\Omega^{-2}(M^{\#}))^{\#} \tag{24}$$

Further recall that if M is an indecomposable kG-module, then the almost split sequence of kG-modules with right-most term M (unique up to isomorphism) is of the form

$$0 \to \tau M \to B \to M \to 0.$$

We require the following elementary lemma concerning the base change of an almost split sequences of kG-modules.

Lemma 8.2. *Let*

$$\mathcal{E} : 0 \longrightarrow N \longrightarrow B \longrightarrow M \longrightarrow 0$$

be an almost split sequence of kG-modules and let K/k be a field extension. Then

$$\mathcal{E}_K : 0 \longrightarrow N_K \longrightarrow B_K \longrightarrow M_K \longrightarrow 0.$$

is a direct sum of almost split sequences of KG_K-modules.

Proof. Write $M_K \cong \oplus M_K^i$ as a direct sum of indecomposable KG_K-modules and let

$$\mathcal{E}_K^i : 0 \longrightarrow N_K^i \longrightarrow B_K^i \longrightarrow M_K^i \longrightarrow 0.$$

be the almost split sequence of KG_K-modules with right-most term M_K^i. Because τ commutes with direct sums, $N_K \cong \oplus N_K^i$. Thus, it suffices to show for each i that there is a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_K^i \arrow[r] & B_K^i \arrow[r] & M_K^i \arrow[r] & 0
\end{tikzcd} \tag{25}$$

whose left and right vertical arrows are given by base change followed by summand projection.

To establish the map (25), we consider the following commutative square

$$
\begin{tikzcd}
\mathrm{Hom}_{kG}(M,M)^{\#} \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{kG}(M,N) \arrow[d] \\
\mathrm{Hom}_{KG_K}(M^i_K,M^i_K)^{\#} \arrow[r, "\cong"] & \mathrm{Ext}^1_{KG_K}(M^i_K,N^i_K)
\end{tikzcd}
\tag{26}
$$

whose horizontal arrows arise from the natural duality relating $\mathrm{Hom}_\Lambda(M,-)$ and $\mathrm{Ext}^1_\Lambda(-,\tau M)$ (cf. [3, 4.12]) and whose vertical arrows are once again induced by base change followed by summand projection. Any non-zero element in the socle of the module $\mathrm{Hom}_{kG}(M,M)^{\#}$ for the local ring $\mathrm{End}_{kG}(M)$ is mapped via the upper horizontal arrow of (26) to the class of the almost split sequence $\mathcal{E}$. On the other hand, the left vertical arrow of (26) sends such an element to a non-zero element of the socle of the module $\mathrm{Hom}_{KG_K}(M^i_K,M^i_K)^{\#}$ for the local ring $\mathrm{End}_{KG_K}(M^i_K)$, and this is mapped by the lower horizontal arrow of (26) to the class of the almost split sequence $\mathcal{E}^i_K$. Thus, we conclude the existence of the map (25). $\square$

The following proposition is in some sense an extension of Lemma 6.3.

Proposition 8.3. *Let G be a finite group scheme such that the dimension of $\Pi(G)$ is at least 1, and let M be an indecomposable non-projective kG-module of constant Jordan type. Consider the almost split sequence of kG-modules*

$$\mathcal{E} : 0 \longrightarrow N \longrightarrow B \longrightarrow M \longrightarrow 0.$$

Then for any π-point $\alpha_K : K[t]/t^p \longrightarrow KG$, $\alpha_K^*(\mathcal{E}_K)$ is a split short exact sequence of $K[t]/t^p$-modules.

Proof. Write $M_K \cong \oplus M^i_K$ as a direct sum of indecomposable KG_K-modules as in the proof of Lemma 8.2. By Theorem 3.7, each M^i_K is a module of constant Jordan type. Let

$$\mathcal{E}^i_K : 0 \longrightarrow N^i_K \longrightarrow B^i_K \longrightarrow M^i_K \longrightarrow 0.$$

be the almost split sequence of KG_K-modules with right-most term M^i_K. Since $\mathcal{E}_K$ is a direct sum of almost split sequences $\mathcal{E}^i_K$ by Lemma 8.2, it suffices to prove that $\alpha_K^*(\mathcal{E}^i_K)$ is split for each i. Hence, we may assume that M_K is neither projective nor decomposable, and that $\mathcal{E}_K : 0 \longrightarrow N_K \longrightarrow B_K \longrightarrow M_K \longrightarrow 0$ is an almost split sequence of KG_K-modules.

Let $\tilde{M}_K = \mathrm{Coind}^{\mathrm{KG}}_{\mathrm{K}\langle\alpha_\mathrm{K}(\mathrm{t})\rangle}(\alpha^*_\mathrm{K}(\mathrm{M_K}))$. We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{KG_K}(\tilde{M}_K,B_K) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{KG_K}(\tilde{M}_K,M_K) \arrow[d, "\cong"] \\
\mathrm{Hom}_{K\langle\alpha_K(t)\rangle}(\alpha_K^*(M_K),\alpha_K^*(B_K)) \arrow[r] & \mathrm{Hom}_{K\langle\alpha_K(t)\rangle}(\alpha_K^*(M_K),\alpha_K^*(M_K))
\end{tikzcd}
\tag{27}
$$

where the vertical arrows are isomorphisms by the Eckmann-Shapiro Lemma. If $\alpha_K^*(\mathcal{E}_K)$ were not split, then the lower horizontal arrow of (27) and thus also the upper horizontal arrow of (27) would not be surjective. On the other hand, the defining property of almost split sequences would then imply that $\tilde{M}_K$ must have M_K as a direct summand. If so, then $\Pi(G_K)_{M_K} \subset \Pi(G_K)_{\tilde{M}_K}$. Since M_K is a non-projective module of constant Jordan type, we have $\Pi(G_K)_{M_K} = \Pi(G_K)$, and, hence, the support of M_K has dimension at least 1. Since $\Pi(G_K)_{\tilde{M}_K}$ consists of only 1 point by Lemma 8.1, we obtain a contradiction. $\square$

The following lemma is a straight-forward consequence of Proposition 1.8 and Corollary 5.4.

Lemma 8.4. *Let G be a finite group scheme, and M be a finite-dimensional kG-module. Then M is a module of constant Jordan type if and only if τM is a module of constant Jordan type, where τ is given in (24). Moreover, if M is a module of constant Jordan type then the stable Jordan types of M and τM are the same.*

Proof. Recall $\alpha_K^*(-)$ commutes with Heller shifts for any π-point α_K, and that the second Heller shift $\Omega^2(-)$ of a $K[t]/t^p$-module preserves its stable Jordan type. Since $\alpha_K^*(-)$ also commutes up to isomorphism with taking the linear dual of a kG-module M provided that the π-point $\alpha_K : K[t/t^p \to KG$ is maximal on M, we conclude that if M is a module of constant Jordan type then τM also has constant Jordan type with the same stable Jordan type. Moreover, the operator τ has an inverse in the stable category. Namely, the inverse sends M to $(\Omega^2(M^\#))^\#$. If M is indecomposable and not projective, then the inverse operator sends τM to M. Then the same argument as above implies that if τM has constant Jordan type then so does M. $\square$

The following theorem asserts that whether or not an indecomposable kG-module M has constant Jordan type is a function of the connected components of the stable Auslander-Reiten quiver of kG containing M.

Theorem 8.5. *Let G be a finite group scheme, and let M be an indecomposable non-projective module of constant Jordan type. Let Θ be a component of the stable Auslander-Reiten quiver of kG containing the vertex $[M]$. Then for any $[N] \in \Theta$, the module N has constant Jordan type.*

Proof. We first consider the case when $\operatorname{Dim}\Pi(G) = 0$. By Theorem 3.4, $\Pi(G)$ consists of one point. Hence, any module is tautologically a module of constant Jordan type.

We may therefore assume that $\operatorname{Dim}\Pi(G) \geq 1$. Let $[N]$ be any successor of $[M]$ in the stable quiver component Θ. Then there exists an almost split sequence

$$\mathcal{E} : 0 \longrightarrow \tau M \longrightarrow B \longrightarrow M \longrightarrow 0$$

such that N is a direct summand of B. By Lemma 8.4, τM is a module of constant Jordan type. Let $\alpha_K : K[t]/t^p \to KG_K$ be a π-point. By Proposition 8.3, $\alpha_K^*(\mathcal{E}_K)$ splits. Thus, $\alpha_K^*(B_K) = \alpha^*(\tau M_K) \oplus \alpha^*(M_K)$. We conclude that B is a module of constant Jordan type, so that Theorem 3.7 implies that N has constant Jordan type.

Now let $[N]$ be any predecessor of $[M]$, i.e. there is an arrow $[N] \to [M]$. By [1, V.1.12] and [1, V.5.3], there is an arrow $[\tau M] \to [N]$. Applying the argument above to τM and N, we conclude that N has constant Jordan type.

Since Θ is connected, the argument is finished by induction. $\square$

To prove the following "realization of constant types" result, we appeal to the work of K. Erdmann [13] in the case of finite groups and that of R. Farnsteiner [15], [16] for arbitrary finite group schemes. Namely, a result of [14] (see also [16]) following earlier work of Webb [26] asserts that if kG has wild representation type then the Auslander-Reiten component of the trivial module has a very restricted form. Results of Erdmann and Farnsteiner assert that under hypotheses specified in the theorem below, the Auslander-Reiten component of the trivial module must have type A_∞.

Theorem 8.6. *Let G be a finite group scheme (over k algebraically closed) satisfying one of the following conditions: either G is a finite group which has p-rank at least 2 and whose Sylow p-subgroup is not dihedral or semi-dihedral or $\Pi(G)$ has dimension at least 2. Then for any n there exists an indecomposable module of stable constant Jordan type $n[1]$.*

Proof. By [13, 2] in the case of finite groups and [16, 3.3] for arbitrary finite group schemes, our assumptions imply that the stable Auslander-Reiten component of the trivial module must have the tree class A_∞.

Let V_n be an indecomposable module representing the n^{th} node of the tree containing the trivial module. The bottom node has label 0. By Proposition 8.5, V_n has constant Jordan type for every n. Let $\underline{a}_n$ be the stable Jordan type of V_n. Proposition 8.3 implies that the middle term B_n of the almost split sequence

$$0 \longrightarrow \tau V_n \longrightarrow B_n \longrightarrow V_n \longrightarrow 0$$

has stable constant Jordan type $2\underline{a}_n$.

Since the tree class is A_∞, V_1 must be the only non-projective indecomposable summand of the middle term of the almost split sequence for V_0:

$$0 \longrightarrow \tau V_0 \longrightarrow B_0 \longrightarrow V_0 \longrightarrow 0$$

Hence, $\underline{a}_1 = 2\underline{a}_0$. The middle term of the almost split sequence for V_1

$$0 \longrightarrow \tau V_1 \longrightarrow B_1 \longrightarrow V_1 \longrightarrow 0$$

has two indecomposable non-projective summands, one of which is isomorphic to τV_0. Hence, the other summand has stable constant Jordan type $2\underline{a}_1 - \underline{a}_0 = 3\underline{a}_0$.

Proceeding by induction, we see that the module which represents the n^{th} node in this A_∞ tree has stable constant Jordan type $\underline{a}_n = n\underline{a}_0$.

We immediately conclude that k must be at the bottom node since the stable Jordan type of k is $1[1]$. Hence, $\underline{a}_0 = 1[1]$. Therefore, V_n is an indecomposable module of stable constant Jordan type $n[1]$. □

9. Questions and Conjectures

We offer a few broad questions as well as specific conjectures which provide challenges for further investigation.

Question 9.1. *For a given finite group scheme G, what Jordan types are realized as the (generic) Jordan type of finite dimensional kG-modules with constant Jordan type?*

Certainly, there are constraints as the following examples illustrate.

Example 9.2. Let G be a quasi-elementary abelian group scheme, $G = \mathbb{G}_{a(s)} \times E$ with E and elementary abelian p-group of rank r and $p > 2$. Then any kG-module M of constant Jordan type of stable type $1[1]$ is endotrivial, and hence of the form $\Omega^i(k)$ [11]. The Jordan type of such a module has the form $1[1] + mp^{r+s-1}[p]$ for some $m \geq 0$ (cf. Lemma 6.9).

Example 9.3. We verify that there does not exist a finite dimensional kE-module of constant Jordan type $[2]+[p]$ for $p > 3$ for E an elementary abelian p-group of rank 2, which the reader can see as very limited evidence for Conjecture 9.5. Suppose V is such a kE-module and write $kE = k[x,y]/(x^p, y^p)$. Consider the k-vector space basis $u, xu, v, xv, x^2v, \ldots, x^{p-1}v$ for V.

We will show that some linear combination $y - bx$ satisfies $(y - bx)^{p-1}V = 0$, so that the Jordan form associated to $y - bx$ has no block of size p. Observe that $y(v)$ written in our given basis has coefficient 0 for v because y is nilpotent and only v in our basis satisfies $x^{p-1}v \neq 0$. Second, suppose that $y(v)$ has coefficient b for xv and consider $y - bx$. Then once again $(y - bx)(v)$ has coefficient 0 for v and by construction coefficient 0 for xv. Let us replace y by $y - bx$, so that $y(v) \in span\{u, xu, x^2v, \ldots, x^{p-1}v\}$.

If we apply y to the given basis, the only basis element with the property that y applied to it can have non-zero coefficient for u is u itself, since any of the other basis elements would have to be annihilated by x^2 and thus in the image of x. Since y is nilpotent, we conclude that $y(u) \in span\{xu, x^2v, \ldots, x^{p-1}v\}$ so that $y^2(u) \in span\{x^4v, \ldots, x^{p-1}v\}$.

Thus, y^3u and y^3v are both contained in x^4V, and we conclude that $y^{p-1}(x^iv) = 0 = y^{p-1}(x^iu)$.

The challenge of Question 9.1 seems more interesting if we work stably, so that we identify two Jordan types $\underline{a} = a_p[p] + \cdots + a_1[1],\ \underline{b} = b_p[p] + \cdots + b_1[1]$ provided that $a_i = b_i,\ i \neq p$.

Question 9.4. *For which finite group schemes G is every stable Jordan type the (generic) Jordan type of a finite dimensional kG-module of constant Jordan type?*

The following is a specific conjecture would be a step towards answering the previous question.

Conjecture 9.5. *Let E be an elementary abelian p-group of rank ≥ 2, with $p > 3$. Then there does not exist a finite dimensional kE-module of stable constant Jordan type* $[2]$.

Andrei Suslin has formulated the following intriguing question whose affirmative answer would in particular verify the preceding conjecture.

Question 9.6. *Let E be an elementary abelian p-group of rank 2, with $p > 3$. Let M be a kE-module with constant Jordan type $\sum_i a_i[i]$ and let i be an integer, $1 < i < p$. Is it the case that if $a_i \neq 0$, then either $a_{i+1} \neq 0$ or $a_{i-1} \neq 0$?*

Of course, if Conjecture 9.5 is valid, then it follows that there is no module of stable constant Jordan type $[2]$ for any finite group scheme G containing a quasi-elementary subgroup scheme $H = \mathbb{G}_{a(r)} \times E$ such that the rank s of E plus r is greater or equal to 2.

We can make many other "non-existence conjectures" such as the following. We recall that for E an elementary abelian p-group of rank $n \geq 2$ there exists a kE-module whose stable type is constant of type $1[2] + (n-1)[1]$

Conjecture 9.7. *Let E be an elementary abelian p-group of rank $n \geq 2$, with $p > 3$. There does not exist a kE-module whose stable type is constant of type $1[2] + j[1]$ with $j \leq n - 2$.*

We next formulate questions of a more qualitative nature.

Definition 9.8. Let G be a finite group scheme over a field k (of characteristic $p > 0$). We denote by $\mathcal{L} = \mathbb{N}^p$ the (additive) lattice of Jordan types over k. We denote by

$$\mathcal{R}(G) \subset \mathcal{L}$$

the sublattice of those Jordan types which can be realized as the (generic) Jordan types of kG-modules of constant Jordan type.

Question 9.9. *For which finite group schemes G is $\mathcal{L}/\mathcal{R}(G)$ finite? Among such finite group schemes, how does the invariant $\mathcal{L}/\mathcal{R}(G)$ behave?*

For those finite group schemes G for which $\mathcal{L}/\mathcal{R}(G)$ is infinite, can we give some interpretation of the rank of this quotient in more familiar terms?

In view of our discussion involving Auslander-Reiten almost split sequences, it seems of considerable interest to consider $\mathcal{I}(G)$ as defined below.

Definition 9.10. Let G be a finite group scheme over a field k (of characteristic $p > 0$) and let $\overline{\mathcal{R}}(G) \subset \overline{\mathcal{L}} = \mathbb{N}^{p-1}$ denote the subset of those stable Jordan types realizable as the (generic) stable Jordan type of a finite dimensional kG-module of constant Jordan type. Further, let us denote by $\mathcal{I}(G) \subset \overline{\mathcal{R}}(G) \subset \overline{\mathcal{L}} = \mathbb{N}^{p-1}$ the subset of stable Jordan types which are the (generic) Jordan types of *indecomposable* kG-modules of constant Jordan type.

Question 9.11. *For which G is $\mathcal{I}(G)$ closed under addition?*

Remark 9.12. If G is the Klein four group, $G \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, then the only non-projective indecomposable modules of constant Jordan type are Heller shifts of the trivial module k. Thus, for this choice of G, $\mathcal{I}(G)$ is not closed under addition.

One is tempted to ask many questions concerning how the realizability of modules of constant type behaves with respect to change of finite group scheme. We ask one such question.

Question 9.13. *For which $H \subset G$ does restriction induce a bijection from the set of Jordan types realized as Jordan types of kG-modules of constant Jordan type to the set of Jordan types realized as Jordan types of kH-modules of constant Jordan type?*

10. APPENDIX: Decomposition of tensor products of $k[t]/t^p$-modules

The purpose of this appendix is to establish a closed form for tensor products of $k[t]/t^p$-modules, presumably implicit in [24]. Here, we view $k[t]/t^p$ as a self-dual Hopf algebra; in other words, as the restricted enveloping algebra of the 1-dimensional p-restricted Lie algebra (with trivial p-restriction operator). Thus, the coproduct is given by the formula $t \mapsto 1 \otimes t + t \otimes 1$. Our technique will be to reduce this to a formula for the tensor product of tilting modules, a formula known in much greater generality.

The result of the appendix can be summarized in the following formula. We assume $j \geq i$, and for convenience we write $[i]$ for the indecomposable $k[t]/t^p$-module which has dimension i and Jordan type $1[i]$.

$$
[i] \otimes [j] = \begin{cases} \displaystyle\sum_{\substack{\ell = j-i+1 \\ \ell \equiv i+j+1 \bmod 2}}^{\ell=i+j-1} [\ell] & \text{if } j+i \le p \\ \\ (j+i-p)[\mathrm{p}] + \displaystyle\sum_{\substack{\ell = j-i+1 \\ \ell \equiv i+j+1 \bmod 2}}^{\ell=2\mathrm{p}-1-\mathrm{i}-\mathrm{j}} [\ell] & \text{if } j+i > p \end{cases} \tag{28}
$$

$$
= \begin{cases} [\mathrm{j}-\mathrm{i}+1] + [\mathrm{j}-\mathrm{i}+3] \dots [\mathrm{j}+\mathrm{i}-3] + [\mathrm{j}+\mathrm{i}-1] & \text{if } j+i \le p \\ \\ [\mathrm{j}-\mathrm{i}+1] + \cdots + [2\mathrm{p}-1-\mathrm{i}-\mathrm{j}] + (\mathrm{j}+\mathrm{i}-\mathrm{p})[\mathrm{p}] & \text{if } j+i > p. \end{cases}
$$

Let $V(\lambda)$ denote the Weyl module of highest weight λ for the algebraic group SL_2. Let e, h, f be the standard generators for sl_2. Viewing $V(\lambda)$ as a module for the distribution algebra of SL_2 we can restrict $V(\lambda)$ to $u(sl_2) \simeq Dist_1(SL_2)$ and then further to $k[t]/t^p \simeq u(\langle e \rangle) \subset u(sl_2)$ where $\langle e \rangle$ is a 1-dimensional p-restricted Lie algebra generated by the element e. For $\lambda \le p-1$, the restriction of $V(\lambda)$ to $u(sl_2)$ is a simple sl_2-module of highest weight λ (see [22, II.3]). The sl_2-representation theory implies that

$$
V(\lambda) \downarrow_{u(\langle e \rangle)} \simeq [\lambda + 1] \tag{29}
$$

as $k[t]/t^p$-modules. It is known that the tensor product of two Weyl modules with weights in the restricted range splits as a direct sum of Weyl modules, at least modulo a projective summand. Thus, we can effectively replace calculating the decomposition of $k[t]/t^p$ modules $[i] \otimes [j]$ into indecomposable modules with calculating the decomposition of SL_2-modules $V(i-1) \otimes V(j-1)$ into the direct sum of Weyl modules. We now implement this strategy in the following proof of the formula 28.

Proposition 10.1. *Let C^ℓ_{ij} be the structure constants in the decomposition of the tensor product $[i] \otimes [j]$ of $k[t]/t^p$-modules into indecomposable modules:*

$$
[i] \otimes [j] = \sum_{1 \le \ell \le p} C^\ell_{ij} [l]
$$

Then

I. $C^p_{ij} = \begin{cases} i+j-p & \text{if } i+j > p \\ 0 & \text{if } i+j \le p, \end{cases}$

and for $1 \le \ell \le p-1$,

$$II.\ C^{\ell}_{ij} = \begin{cases} 1 & \textit{if } \ |j-i|+1 \le \ell \le \min\{i+j-1, 2p-1-i-j\} \\ & \qquad \textit{and } \ell \equiv i+j+1 \ \textit{mod}\ 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If either $[i]$ or $[j]$ is projective, then the statement is clear. Thus, we may assume that $i, j < p$.

Suppose that $0 \le \lambda, \mu \le p-2$. Then the Weyl modules $V(\lambda)$, $V(\mu)$ are tilting modules of highest weights λ, μ respectively [22, II.E.1]. By [22, II.E.2, and II.E.7] tensor product of tilting modules decomposes as a direct sum of indecomposable tilting modules. By [22, II.E.8], an indecomposable SL_2-tilting module $T(\lambda)$ of highest weight λ restricts to a projective sl_2-module unless it is isomorphic to the Weyl module of highest weight λ where $0 \le \lambda \le p-2$. These observations imply that there is a decomposition

$$(V(\lambda) \otimes V(\mu)) \downarrow_{sl_2} = \sum_{0 \le \nu \le p-2} B^{\nu}_{\lambda,\mu} V(\nu) \downarrow_{sl_2} + (\mathrm{proj})$$

for some multiplicities $B^{\nu}_{\lambda,\mu}$. Isomorphism (29) implies that

$$C^{\ell}_{ij} = B^{\ell-1}_{i-1,j-1}$$

for $1 \le i, j, \ell \le p-1$. Thus, it suffices to compute multiplicities $B^{\nu}_{\lambda,\mu}$.

Assume $0 \le \nu \le p-2$. By [22, E.12], the multiplicity of $V(\nu)$ as a direct summand of $V(\lambda) \otimes V(\mu)$ equals to

$$B^{\nu}_{\lambda,\mu} = \sum_{w \in W_p} (-1)^{\ell(w)} \dim V(\lambda)_{w\cdot\nu-\mu} \tag{30}$$

where $W_p = \mathbb{Z}/2 \ltimes 2p\mathbb{Z}$ is the affine Weyl group, $\ell(w)$ is the length function and $w \cdot \nu = w(\nu+1) - 1$ is the standard "dot" action.

The weights of $V(\lambda)$ are $\langle \lambda, \lambda-2, \ldots, -\lambda \rangle$ and all weight spaces are 1-dimensional. Assume that $\lambda \ge \nu$. To compute the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$ we first determine all $w \in W_p$ such that $\mathrm{Dim}\, V(\lambda)_{w\cdot\nu-\mu}$ can be non-trivial. Since the weights of $V(\lambda)$ are between $-\lambda$ and λ, we get

$$-\lambda \le w \cdot \nu - \mu \le \lambda$$

Because $0 \le \lambda, \mu \le p-2$, this condition implies

$$-(p-2) \le w \cdot \nu \le 2p-4$$

Since we also have $0 \le \nu \le p-2$, we conclude that there are only three elements of W_p for which non-zero multiplicity can occur. Specifically these elements are:

(a) Identity element: $w_0 = id$; $w_0 \cdot \nu = \nu$
(b) Simple reflection: $w_1 = s_\alpha$; $w_1 \cdot \nu = -\nu - 2$
(c) Simple reflection followed by the translation by $2p$: $w_2 \cdot \nu = 2(p-1) - \nu$

We record the respective values of the length function here: $\ell(w_0) = 0$, $\ell(w_1) = 1$, $\ell(w_2) = 1$. Since $w_i(\nu) \equiv \nu \bmod 2$ for any i, we assume $\nu \equiv \lambda + \mu \bmod 2$.

CASE I. Assume $\lambda + \mu < p$. Then $w_2 \cdot \nu - \mu = 2(p-1) - \nu - \mu > \lambda$. Thus, we only need to consider w_0 and w_1.

For (a): w_0, we have $V(\lambda)_{w_0 \cdot \nu - \mu} = 1$ for all ν such that $V(\lambda)$ has a weight $\nu - \mu$, i.e.

$$-\lambda + \mu \leq \nu \leq \lambda + \mu$$

and $V(\lambda)_{w_0 \cdot \nu - \mu} = 0$ otherwise. Since we assumed $\mu \leq \lambda$, we have $V(\lambda)_{w_0 \cdot \nu - \mu} = 1$ if and only if $\nu \leq \lambda + \mu$.

For (b): w_1, it is the case that $V(\lambda)_{w_1 \cdot \nu - \mu} = 1$ if an only if the space $V(\lambda)$ has a weight $w_1 \cdot \nu - \mu = -\nu - 2 - \mu$, i.e.

$$-\lambda + \mu \leq -\nu - 2 \leq \lambda + \mu.$$

Simplifying, we get $\nu \leq \lambda - \mu - 2$.

Since $\ell(w_0) = 0$, $\ell(w_1) = 1$, the first case yields the following result.

$$B^{\nu}_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda - \mu \leq \nu \leq \lambda + \mu \\ 0 & \text{otherwise.} \end{cases}$$

CASE II. Assume $\lambda + \mu \geq p$.

For (a): w_0, we argue as in Case I to get that the multiplicity 1 for all ν.

For (b): w_1, by the same argument as in Case I, we get multiplicity 1 whenever $\nu \leq \lambda - \mu - 2$.

For (c): w_2, we have that $V(\lambda)_{w_2 \cdot \nu - \mu} = 1$ whenever $V(\lambda)$ has a weight $w_2 \cdot \nu - \mu = 2(p-1) - \nu - \mu$. This translates to

$$2(p-1) - \lambda - \mu \leq \nu.$$

Inserting this information into formula (30), we see that in this case

$$B^{\nu}_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda - \mu \leq \nu \leq 2(p-2) - \lambda - \mu \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the final result for multiplicities $B^{\nu}_{\lambda\mu}$ is the following.

$$B^{\nu}_{\lambda\mu} = \begin{cases} 1 & \lambda - \mu \leq \nu \leq \lambda + \mu, \nu \equiv \lambda + \mu \bmod 2 \text{ and } \lambda + \mu < p \\ 1 & \lambda - \mu \leq \nu \leq 2(p-2) - \lambda - \mu, \nu \equiv \lambda + \mu \bmod 2 \\ & \quad \text{and } \lambda + \mu \geq p \\ 0 & \text{otherwise.} \end{cases}$$

Since $C^\ell_{ij} = B^{\ell-1}_{i-1,j-1}$, the formula above immediately produces the multiplicities C^ℓ_{ij} for $\ell < p$.

$$C^\ell_{ij} = \begin{cases} 1 & \text{if } j-i+1 \leq \ell \leq \min\{i+j-1, 2p-1-i-j\}, \\ & \quad\quad \text{and } \ell \equiv i+j+1 \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the multiplicity C^p_{ij} is computed by comparing dimensions. We get
$C^p_{ij} = \begin{cases} i+j-p & \text{if } i+j > p \\ 0 & \text{if } i+j \leq p \end{cases}$ □

We conclude the appendix with the analogous formula for the tensor product of indecomposable kC_p-modules where C_p is a cyclic group of order p. A subtlety here is that even though the module categories for kC_p and the algebra $k[t]/t^p$ with the coproduct $t \mapsto 1 \otimes t + t \otimes 1$ are equivalent, the tensor product structure comes from two different coproducts. Nonetheless, the tensor multiplicities turn out to be the same.

Corollary 10.2. *Let $[i]$, $1 \leq i \leq p$, be indecomposable kC_p-modules. Then*

$$[i] \otimes [j] = \sum_{1 \leq \ell \leq p} C^\ell_{ij}[l]$$

where C^ℓ_{ij} are as determined in Proposition 10.1.

Proof. By [20, 4.5] the tensor product of any two kC_p-modules M, N is isomorphic as $k[t]/t^p$-module to the tensor product $M \otimes N$ using the coproduct $t \mapsto 1 \otimes t + t \otimes 1$. The statement now follows from 10.1. □

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