

II-SUPPORTS FOR MODULES FOR FINITE GROUP SCHEMES OVER A FIELD

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ABSTRACT. We introduce the space $\Pi(G)$ of equivalence classes of π -points of a finite group scheme G . The study of π -points can be viewed as the study of the representation theory of G in terms of “elementary subalgebras” of a very specific and simple form, or as an investigation of flat maps to the group algebra of G utilizing the representation theory of G . Our results extend to arbitrary finite group schemes G over arbitrary fields k of positive characteristic and to arbitrarily large G -modules the basic results about “cohomological support varieties” and their interpretation in terms of representation theory. In particular, we prove that the projectivity of any (possibly infinite dimensional) G -module can be detected by its restriction along π -points of G . We establish that $\Pi(G)$ is homeomorphic to $\text{Proj } H^\bullet(G, k)$, and using this homeomorphism we determine up to inseparable isogeny the best possible field of definition of an equivalence class of π -points. Unlike the cohomological invariant $M \mapsto \text{Proj } H^\bullet(G, k)$, the invariant $M \mapsto \Pi(G)_M$ satisfies good properties for all G -modules, thereby enabling us to determine the thick, tensor-ideal subcategories of the stable module category of finite dimensional kG -modules. Finally, using the stable module category of G , we provide $\Pi(G)$ with the structure of a ringed space which we show to be isomorphic to the scheme $\text{Proj } H^\bullet(G, k)$.

0. INTRODUCTION

In [15], the authors associated to any finite group scheme G over an algebraically closed field k of characteristic $p > 0$ a space $P(G)$ which they called the space of p -points of G . This space consists of equivalence classes of flat maps $k[t]/t^p \rightarrow kG$, with the equivalence relation determined in terms of the behaviour of restrictions of finite dimensional kG -modules. Furthermore, to a finite dimensional kG -module M , the authors associated a closed subspace $P(G)_M$. These invariants are generalizations of Carlson’s rank variety for an elementary abelian p -group E and the cohomological support variety for a finite dimensional kE -module M [8],[2]. The purpose of this paper is to pursue further the authors’ point of view, thereby extending earlier results to any finite group scheme G over an arbitrary field k of characteristic $p > 0$ and to an arbitrary kG -module M . We suggest that our construction of “generalized p -points” (which we call “ π -points”) is both more natural and more intrinsic than previous considerations which utilized a combination of cohomological and representation-theoretic invariants.

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The innovation which permits us to consider finite group schemes over an arbitrary field and their infinite dimensional (rational) representations is the consideration of equivalence classes of flat maps $K[t]/t^p \rightarrow KG_K$ for field extensions K/k . Our fundamental result is Theorem 3.6 which asserts that for an arbitrary finite group scheme over a field k there is a natural homeomorphism

$$\Psi_G : \Pi(G) \xrightarrow{\sim} \text{Proj}(H^\bullet(G, k))$$

relating the space $\Pi(G)$ of π -points of G to the projectivization of the affine scheme given by the cohomology algebra $H^\bullet(G, k)$. In other words, consideration of flat maps $K[t]/t^p \rightarrow KG_K$ for field extensions K/k enables us to capture the information encoded in the prime ideal spectrum of $H^\bullet(G, k)$ rather than simply that of the maximal ideal spectrum. Indeed, we verify in Theorem 4.2 a somewhat sharper result, in that we determine (up to a purely inseparable field extension of controlled p -th power degree) the minimal field of definition of such a π -point in terms of its image under Ψ_G .

The need to consider such field extensions K/k when one considers infinite dimensional kG -modules had been recognized earlier. Nevertheless, our results improve upon results found in the literature for infinite dimensional modules for various types of finite group schemes over an algebraically closed field [3], [5], [6], [17], [21], [22]. Perhaps the most important and difficult of these results is Theorem 5.3 which asserts that the projectivity of any (possibly infinite dimensional) module M for an arbitrary finite group scheme G can be detected “locally” in terms of the restrictions of M along the π -points of G . This was proved for finite groups in [6], for unipotent group schemes in [3] and for infinitesimal group schemes in [22]. This, together with the consideration of certain infinite dimensional modules introduced by Rickard in [25], provides us with the tools to analyze the tensor-ideal thick subcategories of the stable category of finite dimensional kG -modules.

Our consideration of the (projectivization) of the prime ideal spectrum rather than the maximal ideal spectrum of $H^\bullet(G, k)$ enables us to associate a good invariant (the Π -supports, $\Pi(G)_M \subset \Pi(G)$, of the kG -module M) to an arbitrary kG -module. This invariant $\Pi(G)_M$ is defined in module-theoretic terms, essentially as the “subset of those π -points at which M is not projective.” Although $\Pi(G)_M$ corresponds naturally to the cohomological support variety of M whenever M is finite dimensional, it does not have an evident cohomological interpretation for infinite dimensional kG -modules. The difference in behaviour of this invariant for finite dimensional and infinite dimensional kG -modules is evident in Corollary 6.7 which asserts that *every* subset of $\Pi(G)$ is of the form $\Pi(G)_M$ for some kG -module M . Our analysis is somewhat motivated by and fits with the point of view of Benson, Carlson, and Rickard [6].

We establish in Theorem 6.3 a bijection between the tensor-ideal thick subcategories of the triangulated category $\text{stmod}(G)$ of finite dimensional G -modules and subsets of $\Pi(G)$ closed under specialization. This theorem verifies the main conjecture of [17] (for ungraded Hopf algebras), a conjecture first formulated in [19] in the context of “axiomatic stable homotopy theory” and then considered in [17], [18]. As a corollary, we show that the lattice of thick, tensor-closed subcategories of the stable module category $\text{stmod}(G)$ is isomorphic to the the lattice of thick, tensor-closed of subcategories $D^{\text{perf}}(\text{Proj } H^\bullet(G, k))$, the full subcategory of the derived category of coherent sheaves on $\text{Proj } H^\bullet(G, k)$ consisting of perfect complexes.

Finally, Theorem 7.3 demonstrates how the scheme structure of $\text{Proj } H^\bullet(G, k)$ can be realized using $\{Pi(G)$ and the category $stmod(G)$.

We remark that the consideration of π -points suggests the formulation of finer invariants than $\Pi(G)_M$ which would provide more information about a kG -module M . We expect to discuss such finer invariants in a subsequent investigation.

Throughout this paper, p will be a prime number and all fields considered will be of characteristic p .

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1. RECOLLECTION OF COHOMOLOGICAL SUPPORT VARIETIES

Let G be a finite group scheme defined over a field k . Thus, G has a commutative coordinate algebra $k[G]$ which is finite dimensional over k and which has a coproduct induced by the group multiplication on G , providing $k[G]$ with the structure of a Hopf algebra over k . We denote by kG the k -linear dual of $k[G]$ and refer to kG as the group algebra of G . Thus, kG is a finite dimensional, co-commutative Hopf algebra over k .

Examples to keep in mind are that of a finite group π (so that $k\pi$ is the usual group algebra of π) and that of a finite dimensional, p -restricted Lie algebra g (so that the group algebra in this case can be identified with the restricted enveloping algebra of g). These are extreme cases: π is totally discrete (a finite, etale group scheme) and the group scheme $G_{(1)}$ associated to the (restricted) Lie algebra of an algebraic group over k is connected.

By definition, a G -module is a comodule for $k[G]$ (with its coproduct structure) or equivalently a module for kG . We denote the abelian category of G -modules by $Mod(G)$. If M is a kG -module, then we shall frequently consider the cohomology of G with coefficients in M ,

$$H^*(G, M) \equiv Ext_G^*(k, M)$$

as well as the $H^*(G, k)$ -algebra Ext -algebra $Ext_G^*(M, M)$. If $p = 2$, then $H^*(G, k)$ is itself a commutative k -algebra. If $p > 2$, then the even dimensional cohomology $H^\bullet(G, k)$ is a commutative k -algebra. We denote by

$$H^\bullet(G, k) = \begin{cases} H^*(G, k), & \text{if } p = 2, \\ H^{ev}(G, k) & \text{if } p > 2. \end{cases}$$

As shown in [16], the commutative k -algebra $H^\bullet(G, k)$ is finitely generated over k . Following Quillen [23], we consider the maximal ideal spectrum of $H^\bullet(G, k)$,

$$|G| \equiv \text{Specm } H^\bullet(G, k).$$

Following the work of Carlson [8] and others, for any finite dimensional kG -module M we consider

$$|G|_M = \text{Specm } H^\bullet(G, k) / \text{ann}_{H^\bullet(G, k)} Ext_G^*(M, M),$$

where the action of $H^\bullet(G, k)$ on $Ext_G^*(M, M)$ is via a natural ring homomorphism $H^\bullet(G, k) \rightarrow Ext_G^*(M, M)$ (so that this annihilator can be viewed more simply as the annihilator of $id_M \in Ext_G^0(M, M)$).

In this paper, we shall be interested in prime ideals which are not necessarily maximal. Indeed, this is the fundamental difference between this paper and [15]. We shall not give a special name for $\text{Spec } H^\bullet(G, k)$, the scheme of finite type over k whose points are the prime ideals of $H^\bullet(G, k)$ or to the scheme $\text{Spec } H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} Ext_G^*(M, M)$, refinements of $|G|$ and $|G|_M$ respectively.

We shall often change the base field k via a field extension K/k . We shall use the notations

$$G_K = G \times_{\text{Spec } k} \text{Spec } K, \quad M_K = M \otimes_k K$$

to indicate the base change of the group scheme G over k and the base change of the kG -module M (to a $KG = KG_K$ -module).

In [27, 28], a map of schemes

$$\Psi_G : V_r(G) \rightarrow \text{Spec } H^\bullet(G, k)$$

is exhibited for a finite, connected group scheme G over k and shown to be a homeomorphism. Here, $V_r(G)$ is the scheme of 1-parameter subgroups of G , a scheme representing a functor which makes no reference to cohomology. Moreover, this homeomorphism restricts to homeomorphisms

$$\Psi_G : V_r(G)_M \rightarrow \text{Spec } H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} Ext_G^*(M, M)$$

for any finite dimensional kG -module M , where once again $V_r(G)_M$ is defined without reference to cohomology. One of the primary objectives of this paper is to extend this correspondence to all finite group schemes; even for finite groups (other than elementary abelian p groups), such an extension has not been exhibited before.

2. π -POINTS OF G

We let G be a finite group scheme over a field k . In this section, we introduce our construction of the π -points of G and establish some of their basic properties. If $f : V \rightarrow W$ is a map of varieties or modules over k and if K/k is a field extension then we denote by $f_K = f \otimes 1_K : V_K \rightarrow W_K$ the evident base change of f . Given a map $\alpha : A \rightarrow B$ of algebras and a B -module M , we denote by $\alpha^*(M)$ the pull-back of M via α .

Our definition of π -point is an extension of our earlier definition of p -point, now allowing extensions of the base field k . This enables us to consider finite group schemes defined over a field k which is not algebraically closed. Moreover, even if the base field k is algebraically closed, it is typically necessary to consider more “generic” maps $K[t]/t^p \rightarrow KG$ than those defined over k when considering infinite dimensional kG -modules.

We remind the reader that the representation theory of $K[t]/t^p$ is particularly simple: a $K[t]/t^p$ -module is projective if and only if it is free; there are only finitely many indecomposable modules, one of dimension i for each i with $1 \leq i \leq p$.

Definition 2.1. Let G be a finite group scheme over k . A π -point of G (defined over a field extension K/k) is a flat map of K -algebras

$$\alpha_K : K[t]/t^p \rightarrow KG$$

(i.e., a K -linear ring homomorphism with respect to which KG is flat as a left $K[t]/t^p$ -module) which factors through the group algebra $KC_K \subset KG_K = KG$ of some abelian subgroup scheme $C_K \subset G_K$ of G_K .

If $\beta_L : L[t]/t^p \rightarrow LG$ is another π -point of G , then α_K is said to be a *specialization* of β_L , written $\beta_L \downarrow \alpha_K$, provided that for any finite dimensional kG -module M , $\alpha_K^*(M_K)$ being free implies that $\beta_L^*(M_L)$ is free.

Two π -points $\alpha_K : K[t]/t^p \rightarrow KG$, $\beta_L : L[t]/t^p \rightarrow LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.

Observe that the condition that a π -point $\alpha_K : K[t]/t^p \rightarrow KG$ factors through the group algebra of an abelian subgroup scheme $C_K \subset G_K$ is the only aspect of the definition of a π -point which uses the Hopf algebra structure of kG . We point out that the homeomorphism of Theorem 3.6 requires consideration of π -points α_K which factor through the group algebra of abelian subgroup schemes $C_K \subset G_K$ defined over field extensions K/k of positive transcendence degree even in the case in which $G = SL_{2(1)}$ (the first infinitesimal subgroup scheme of the algebraic group SL_2 , with group algebra the restricted enveloping algebra of sl_2).

Remark 2.2. Let $\alpha_K : K[t]/t^p \rightarrow KG$ be a π -point of G and M a finite dimensional kG -module of dimension m . Then $\alpha_K^*(M_K)$ is given by a p -nilpotent, $m \times m$ matrix with coefficients in K (specifying the action of t on M_K). Let $k \subset A \subset K$ be a finitely generated k -subalgebra of K with the property that α_K restricts to α_A (i.e., the matrix specifying the action of t is conjugate to a matrix T_α with coefficients in A). For any $\phi : A \rightarrow \bar{k}$, we define $\alpha_\phi : \bar{k}[t]/t^p \rightarrow \bar{k}G$ to be the base change of α_A via ϕ . Then $\alpha_K^*(M_K)$ is projective if and only if $T_\alpha \in M_m(K)$ consists of Jordan blocks each of which are of size p if and only if the rank of $T_\alpha \in M_m(K)$ is $\frac{p-1}{p} \cdot m$ which is the case if and only if there is some $\phi : A \rightarrow \bar{k}$ with the rank of $(T_\alpha)_\phi$ equal to $\frac{p-1}{p} \cdot m$ if and only if $\alpha_\phi^*(M_{\bar{k}})$ is projective for some $\phi : A \rightarrow \bar{k}$. In other words, each such α_ϕ is a specialization of α_K .

The following three examples involve sufficiently small finite group schemes G that their analysis is quite explicit. Nonetheless, the justification of the “genericity” assertions in these examples requires Theorem 3.6.

Example 2.3. Let G be the finite group $\mathbb{Z}/p \times \mathbb{Z}/p$, so that $kG \simeq k[x, y]/(x^p, y^p)$. A map $\alpha_K : K[t]/t^p \rightarrow KG$ is flat if and only if t is sent to a polynomial in x, y with non-vanishing linear term [15, 2.2]. Such a flat map α_K is equivalent to a flat map $\beta_K : K[t]/t^p \rightarrow KG$ if and only if $\alpha_K(t) - \beta_K(t)$ has no linear term [15, 2.2].

For example, a group homomorphism $\mathbb{Z}/p \rightarrow \mathbb{Z}/p \times \mathbb{Z}/p$ sending a generator σ of \mathbb{Z}/p to (ζ^i, ξ^j) where ζ, ξ are generators of \mathbb{Z}/p , induces a map of group algebras

$$k[\sigma]/(\sigma^p - 1) \rightarrow k[\zeta, \xi]/(\zeta^p - 1, \xi^p - 1); \quad \sigma \mapsto \zeta^i \xi^j.$$

Viewed as a map of algebras, this is equivalent to $\alpha : k[t]/t^p \rightarrow k[x, y]/(x^p, y^p)$ sending t to $ix + jy$ since the images of the nilpotent generator under the two maps differ by a polynomial in the generators of the augmentation ideal without linear term.

Thus, any equivalence class has a unique representative which is given by a linear polynomial in x and y . Let $K_0 = k(z, w)$, the field of fractions of the polynomial ring $k[z, w]$. Let $\eta_{K_0} : K_0[t]/t^p \rightarrow K_0[x, y]/(x^p, y^p)$ be the map that sends t to $zx + wy$. Then any flat map $\alpha : k[t]/t^p \rightarrow k[x, y]/(x^p, y^p)$ defined by sending t to a linear polynomial on x and y is a “specialization” of η_{K_0} in the sense that we

get α via specializing z, w to some elements of k . This is easily seen to imply that $\eta_{K_0} \downarrow \alpha$.

Indeed, we can be more efficient in defining a “generic” π -point for G , for we observe that any $\alpha : k[t]/t^p \rightarrow k[x, y]/(x^p, y^p)$ defined by sending t to a linear polynomial in x and y is a “specialization” of

$$\xi_{k(z)} : k(z)[t]/t^p \rightarrow k(z)[x, y]/(x^p, y^p), \quad t \mapsto zx + y.$$

Namely, the flat map

$$\phi_{a,b} : k[t]/t^p \rightarrow k[x, y]/(x^p, y^p), \quad t \mapsto ax + by$$

with $a, b \in k$ is a specialization of $\xi_{k(z)}$: if $b \neq 0$ (respectively, $a \neq 0$), then $\phi_{a,b}$ is equivalent to the specialization of $\xi_{k(z)}$ obtained by setting $z = \frac{a}{b}$ (resp., replacing $\xi_{k(z)}$ by the equivalent $\xi'_{k(z)} : k(z) \rightarrow k(z)[x, y]/(x^p, y^p), t \mapsto x + \frac{1}{z}y$ and setting $1/z = \frac{b}{a}$).

We give a direct proof of the fact that any π -point $\phi_{a,b}$ is a specialization of $\xi_{k(z)}$ in the sense of Definition 2.1 (which follows in much greater generality from Corollary 4.3, for example). We assume $b \neq 0$. Let M be a kE -module and suppose $\phi_{a,b}^*(M)$ is free. Write $\phi_{a,b}^*(M) = \bigoplus k[t]/t^p e_i$. Since $(ax + by)^{p-1}e_i \neq 0$ in M , we conclude that $(zx + y)^{p-1}e_i \neq 0$ in $M \otimes k(z)$. Therefore, $M \otimes k(z) \simeq \bigoplus k(z)[t]/t^p e_i$ and thus is free. In fact, we shall be able to conclude that any π -point α_K is a specialization of $\xi_{k(z)}$ in the sense of Definition 2.1.

Example 2.4. Let $E = (\mathbb{Z}/p)^{\oplus 3}$, $\text{char } k = 2$, $\langle g_1, g_2, g_3 \rangle$ be chosen generators of E . As in Example 2.3, any π -point of kE is a specialization of

$$\eta_{k(x,y)} : k(x, y)[t]/t^p \rightarrow k(x, y)E, \quad t \mapsto x(g_1 - 1) + y(g_2 - 1) + (g_3 - 1).$$

Let $M_{a,b,c}$ be a 4 dimensional kE -module indexed by the triple $a, b, c \in k$ with action of g_1, g_2, g_3 given by

$$g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \end{bmatrix} \quad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The computation of [4, II.5.8] together with the homeomorphism of Theorem 3.6 implies that

$$\alpha_{s,t,u} : k[t]/t^p \rightarrow kE, \quad t \mapsto s(g_1 - 1) + t(g_1 - 1) + u(g_3 - 1)$$

satisfies the condition that $\alpha_{s,t,u}^*(M_{a,b,c})$ is not projective if and only if $\langle s, t, u \rangle \in \mathbb{P}^2$ lies on the quadric $Q_{a,b,c}$ defined as the locus of the homogeneous polynomial $(x + ay)(x + by) = cz^2$. (In the terminology to be introduced in Definition 3.1, $\Pi(E)_{M_{a,b,c}} \subset \Pi(E)$ equals the quadric $Q_{a,b,c}$).

Thus, every π -point of kE for which the restriction of $M_{a,b,c}$ is not projective is a specialization of the π -point given as

$$\alpha_{K_0} : K_0[t]/t^p \rightarrow K_0E; \quad \alpha_{K_0}(t) = x(g_1 - 1) + y(g_2 - 1) + (g_3 - 1),$$

where $K_0 = \text{frac}\{k[x, y]/(x + ay)(x + by) - c\}$.

Example 2.5. Consider $G = (SL_2)_{(1)}$, the first infinitesimal kernel of the algebraic group SL_2 , and assume that $p > 2$ for simplicity. Then the group algebra kG can be identified with the restricted enveloping algebra of sl_2 , the (p -restricted) Lie algebra of 2×2 matrices of trace 0. We can explicitly describe kG as the (non-commutative) algebra given by

$$kG = k\{e, f, h\} / \langle e^p, f^p, h^p - h, he - eh - 2e, hf - fh + 2f, ef - fe - h \rangle.$$

Let K/k be a field extension. A choice of values $(E, F, H) \in K$, not all 0, for e, f, h determines a flat map

$$K[t]/t^p \rightarrow KG, \quad t \mapsto Ee + Ff + Hh.$$

If we let $x_{i,j}$ denote the natural coordinate functions on 2×2 matrices, then the variety of (p -) nilpotent elements is given by

$$N = \text{Spec } k[x_{1,1}, x_{1,2}, x_{2,1}] / x_{1,1}^2 + x_{1,2}x_{2,1}.$$

Let K_0 denote the field of fraction of N ,

$$K_0 = \text{frac}\{k[x_{1,1}, x_{1,2}, x_{2,1}] / x_{1,1}^2 + x_{1,2}x_{2,1}\}.$$

Then any flat map $K[t]/t^p \rightarrow KG$ is a specialization of the following “generic” flat map:

$$K_0[t]/t^p \rightarrow K_0G, \quad t \mapsto x_{1,2}e + x_{2,1}f + x_{1,1}h.$$

As in Example 2.3, we readily verify that we can more efficiently define this flat map as

$$\text{frac}\{k[x, y] / (1 + xy)\}[t]/t^p \rightarrow \text{frac}\{k[x, y] / (1 + xy)\}G, \quad t \mapsto ye + xf + h.$$

We next point out the following equivalent reformulations of the concept of equivalence. The proof of the following proposition follows immediately from the equality

$$(\alpha_E)^*(M_E) = (\alpha_K^*(M_K))_E$$

for any triple $E/K/k$ of field extensions and kG -module M and any π -point $\alpha_K : K[t]/t^p \rightarrow KG$.

Proposition 2.6. *Let G be a finite group scheme over a field k . Let $\alpha_K : K[t]/t^p \rightarrow KG$, $\beta_L : L[t]/t^p \rightarrow LG$ be π -points of G . Then the following conditions are equivalent:*

- (1) $\alpha_K \sim \beta_L$.
- (2) For some field extension E/k containing both K and L , $\alpha_E \sim \beta_E$.
- (3) For any field extension E/k containing both K and L , $\alpha_E \sim \beta_E$.

It is worth observing that the equivalence of α_E, β_E as π -points of G does not imply their equivalence as π -points of G_E (because for the latter one must test projectivity on all finite dimensional EG_E -modules and not simply those which arise from kG -modules). As we shall see, this can be reformulated as the observation that the space of π -points of G_E does not map injectively to the space of π -points of G . We discuss this further prior to Theorem 4.6.

The preceding proposition admits the following two immediate corollaries concerning the naturality properties of π -points.

Corollary 2.7. *Let $j : H \rightarrow G$ be a flat homomorphism of finite group schemes over a field k (i.e., assume with respect to the induced map $kH \rightarrow kG$ of group algebras that kG is flat as a left kH -module). Then composition with $j_* : kH \rightarrow kG$ sending the set of flat maps $\alpha_K : K[t]/t^p \rightarrow KH$ which factors through KC'_K for some abelian subgroup scheme $C'_K \subset H_K$ to the set of flat maps $\beta_L : L[t]/t^p \rightarrow LG$ which factors through LC_L for some abelian subgroup scheme $C_L \subset G_L$ induces a well defined map on equivalence classes.*

In other words, $j : H \rightarrow G$ induces a well defined map from the set of equivalence classes of π -points of H to the set of equivalence classes of π -points of G .

Corollary 2.8. *Let G be a finite group scheme over the field k , $L/K/k$ be field extensions. Then the natural inclusion of the set of flat maps $\alpha_E : E[t]/t^p \rightarrow EG$ which factors through EC_E for some abelian subgroup scheme $C_E \subset G_E$ and some field extension E/L into the set of flat maps $\beta_F : F[t]/t^p \rightarrow FG$ which factors through FC_F for some abelian subgroup scheme $C_F \subset G_F$ and some field extension F/K induces a well defined map on equivalence classes.*

In other words, the field extension L/K induces a well defined map from the set of equivalence classes of π -points of G_L to the set of equivalence classes of π -points of G_K .

The following construction of a finite dimensional kG -module L_ζ associated to a (homogeneous) element $\zeta \in H^\bullet(G, k)$ is due to J. Carlson [9]. We remind the reader of the construction of *Heller shifts* $\Omega^j(M)$ of a kG -module defined in terms of a minimum projective resolution of M (cf. [4]). The Heller shift $\Omega^j(M)$ is well defined, independent of choice of projective resolution, in the “stable category” $stmod(G)$ discussed in some detail in Section 6.

These *Carlson modules* L_ζ will be used frequently in what follows.

Proposition 2.9. *Let G be a finite group scheme over a field k and let $\alpha_K : K[t]/t^p \rightarrow KG$ be a π -point of G . Consider $\zeta \in H^{2i}(G, k)$ and let L_ζ be the kG -module defined by the short exact sequence*

$$0 \rightarrow L_\zeta \rightarrow \Omega^{2i}(k) \rightarrow k \rightarrow 0,$$

where the map $\Omega^{2i}(k) \rightarrow k$ represents $\zeta \in Hom_{stmod(G)}(\Omega^{2i}(k), k) = Ext_G^{2i}(k, k)$. Let $\ker\{\alpha_K^*\}$ denote the kernel of the algebra homomorphism $\alpha_K^* : H^\bullet(G_K, K) \rightarrow H^\bullet(K[t]/t^p, K)$.

Then $\zeta \in \ker\{\alpha_K^*\} \cap H^\bullet(G, k)$ if and only if $\alpha_K^*(L_{\zeta, K})$ is not projective as a $K[t]/t^p$ -module, where we use $L_{\zeta, K}$ to denote $(L_\zeta)_K$.

Proof. Since the Heller operators commute with field extensions, $L_{\zeta_K} = L_{\zeta, K}$ as KG -modules, where for clarity we have used $\zeta_K \in H^\bullet(G, K)$ to denote the image of $\zeta \in H^\bullet(G, k)$. We apply the flat map α_K to the short exact sequence of KG -modules to obtain a short exact sequence of $K[t]/t^p$ -modules:

$$0 \rightarrow \alpha_K^*(L_{\zeta, K}) \rightarrow \alpha_K^*(\Omega^{2i}(K)) \rightarrow K \rightarrow 0.$$

As argued in [15, 2.3], $\alpha_K^*(\zeta_K) \neq 0$ if and only if $\alpha_K^*(L_{\zeta, K}) = \alpha_K^*(L_{\zeta_K})$ is projective. \square

We now present our cohomological reformulation of specialization of general π -points of G .

Theorem 2.10. *Let G be a finite group scheme over k and α_K, β_L two π -points of G . Then $\beta_L \downarrow \alpha_K$ if and only if*

$$(2.10.1) \quad (\ker\{\beta_L^*\}) \cap H^\bullet(G, k) \subset (\ker\{\alpha_K^*\}) \cap H^\bullet(G, k).$$

Proof. We first show the “only if” part. Let α_K be a specialization of β_L . Let ζ be any homogeneous element in $(\ker\{\beta_L^*\}) \cap H^\bullet(G, k)$. By Proposition 2.9, $\beta_L^*(L_{\zeta, L})$ is not projective. Since $\beta_L \downarrow \alpha_K$, we conclude that $\alpha_K^*(L_{\zeta, K})$ is not projective. Applying 2.9 again, we get that $\zeta \in (\ker\{\alpha_K^*\}) \cap H^\bullet(G, k)$. Since the ideals under consideration are homogeneous, the asserted inclusion follows.

Conversely, suppose α_K is not a specialization of β_L . By Proposition 2.6, we can assume that both α_K and β_L are defined over the same algebraically closed field E/k . Clearly, if we enlarge the field, the intersections $(\ker\{\beta_L^*\}) \cap H^\bullet(G, k)$ and $(\ker\{\alpha_K^*\}) \cap H^\bullet(G, k)$ do not change, so that we may assume that $K = L = E$, with E algebraically closed.

Then, by Definition 2.1, there exists a finite dimensional kG -module M such that $\alpha_E^*(M_E)$ is projective but $\beta_E^*(M_E)$ is not. For a finite dimensional module, there is a natural isomorphism $Ext_{G_E}^*(M_E, M_E) \simeq Ext_G^*(M, M) \otimes_k E$. Furthermore, since tensoring with E is exact, we have $\text{ann}_{H^\bullet(G, k)}(Ext_G^*(M, M) \otimes_k E) = \text{ann}_{H^\bullet(G_E, E)}(Ext_{G_E}^*(M_E, M_E))$.

Theorem [15, 4.11] now implies that

$$(2.10.2) \quad \text{ann}_{H^\bullet(G, k)}(Ext_G^*(M, M) \otimes_k E) = \text{ann}_{H^\bullet(G_E, E)}(Ext_{G_E}^*(M_E, M_E)) \subset \ker\{\beta_E^*\},$$

and

$$(2.10.3) \quad \text{ann}_{H^\bullet(G, k)}(Ext_G^*(M, M) \otimes_k E) = \text{ann}_{H^\bullet(G_E, E)}(Ext_{G_E}^*(M_E, M_E)) \not\subset \ker\{\alpha_E^*\}.$$

Intersecting (2.10.2) with $H^*(G, k)$, we get

$$(2.10.4) \quad \text{ann}_{H^*(G, k)}(Ext_G^*(M, M)) \subset \ker\{\beta_E^*\} \cap H^*(G, k)$$

On the other hand, (2.10.3) implies that

$$(2.10.5) \quad \text{ann}_{H^*(G, k)}(Ext_G^*(M, M)) \not\subset \ker\{\alpha_E^*\} \cap H^*(G, k).$$

Indeed, if this inclusion did hold, then by tensoring with E and then applying the fact that $(\ker\{\alpha_E^*\} \cap H^*(G, k)) \otimes_k E \subset \ker\{\alpha_E^*\}$, we would get a contradiction to (2.10.3). (2.10.4) and (2.10.5) together imply that

$$(\ker\{\beta_E^*\}) \cap H^\bullet(G, k) \not\subset (\ker\{\alpha_E^*\}) \cap H^\bullet(G, k),$$

thereby proving the converse. □

As an immediate corollary, we add the following equivalent formulation of equivalence of π -points to those of Proposition 2.6 which will play a key role in the proof of our main theorem, Theorem 3.6.

Corollary 2.11. *Let G be a finite group scheme over k and α_K, β_L two π -points of G . Then $\beta_L \sim \alpha_K$ if and only if*

$$(\ker\{\beta_L^*\}) \cap H^\bullet(G, k) = (\ker\{\alpha_K^*\}) \cap H^\bullet(G, k).$$

3. THE HOMEOMORPHISM $\Psi_G : \Pi(G) \rightarrow \text{PROJ } H^\bullet(G, k)$

In this section, we show for an arbitrary finite group scheme G over an arbitrary field k of characteristic $p > 0$ that the prime ideal spectrum of the cohomology ring can be described in terms of π -points of G . This is a refinement of [15] which provides a representation theoretic interpretation of the maximal ideal spectrum of the cohomology ring of G provided that k is algebraically closed.

The bijectivity of Theorem 3.6 below in the special case in which the finite group scheme is an elementary abelian p -group E and k is algebraically closed is equivalent to the foundational result of J. Carlson identifying the (maximal ideal) spectrum of $H^\bullet(E, k)$ with the rank variety of “shifted subgroups” of E [8]; the fact that this bijection is a homeomorphism in this special case is equivalent to “Carlson’s Conjecture” proved by Avrunin and Scott [2]. In the special case in which G is connected, the homeomorphism of Theorem 3.6 is a weak form of the theorem of Suslin-Friedlander-Bendel which asserts that $\text{Spec } H^\bullet(G, k)$ is isogenous to the affine scheme of 1-parameter subgroups of G [27].

Definition 3.1. For any finite group scheme G over a field k , we denote by $\Pi(G)$ the set of equivalence classes of π -points of G ,

$$\Pi(G) \equiv \{[\alpha_K]; \alpha_K : K[t]/t^p \rightarrow KG \text{ is a } \pi\text{-point of } G\}.$$

For a finite dimensional kG -module M , we denote by

$$\Pi(G)_M \subset \Pi(G)$$

the subset of those equivalence classes $[\alpha_K]$ of π -points such that $\alpha_K^*(M_K)$ is not projective for any representative $\alpha_K : K[t]/t^p \rightarrow KG$ of the equivalence class $[\alpha_K]$. We say that $\Pi(G)_M$ is the Π -support of M .

Finally, we denote by

$$(3.1.1) \quad \Psi_G : \Pi(G) \rightarrow \text{Proj } H^\bullet(G, k)$$

the *injective* map sending an equivalence class $[\alpha_K]$ of π -points to the homogeneous prime ideal $\ker\{\alpha_K^*\} \cap H^\bullet(G, k)$.

The fact that Ψ_G is well defined and injective is immediately implied by Theorem 2.10.

Following Proposition 4.6, we shall employ the same definition of $\Pi(G)_M$ in the next section for M which are possibly infinite dimensional.

The following proposition, known as the “tensor product property”, is somewhat subtle because a π -point $\alpha : K[t]/t^p \rightarrow KG$ need not respect the coproduct structure and thereby need not commute with tensor products. This tensor product property is one of the most important properties of Π -supports. The corresponding statement for cohomological support varieties has no known proof using only cohomological methods.

Proposition 3.2. *Let G be a finite group scheme over a field k and let M, N be kG modules. Then*

$$\Pi(G)_{M \otimes N} = \Pi(G)_M \cap \Pi(G)_N.$$

Proof. For any π -point $\alpha : K[t]/t^p \rightarrow KG$ and any algebraically closed field extension E/k , $\alpha_K^*((M \otimes N)_K)$ is projective as a $K[t]/t^p$ -module if and only if $\alpha_E^*((M \otimes N)_E)$ is projective as a $E[t]/t^p$ -module. On the other hand, [15, 3.9] asserts that $\alpha_E^*((M \otimes N)_E)$ is projective if and only if either $\alpha_E^*(M_E)$ or $\alpha_E^*(N_E)$

is projective which is the case if and only if either $\alpha_K^*(M_K)$ or $\alpha_K^*(N_K)$ is projective. \square

We now provide a list of other properties of the association $M \mapsto \Pi(G)_M$ which follow naturally from our π -point of view. Namely, each of the properties can be checked one π -point at a time, thereby reducing the assertions to elementary properties of $K[t]/t^p$ -modules.

Proposition 3.3. *Let G be a finite group scheme over a field k and let M_1, M_2, M_3 be kG -modules. Then*

- (1) $\Pi(G)_k = \Pi(G)$.
- (2) If P is a projective kG -module, then $\Pi(G)_P = \emptyset$.
- (3) If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is exact, then

$$\Pi(G)_{M_i} \subset \Pi(G)_{M_j} \cup \Pi(G)_{M_k}$$

where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$.

- (4) $\Pi(G)_{M_1 \oplus M_2} = \Pi(G)_{M_1} \cup \Pi(G)_{M_2}$.

The topology we give to $\Pi(G)$ is the natural extension of that defined on the space $P(G)$ of abelian p -points for G over an algebraically closed field given in [15, 3.10]. Observe that the formulation of this topology is given without reference to cohomology, although the verification that our topology satisfies the defining axioms of a topology does involve cohomology.

Proposition 3.4. *Let G be a finite group scheme over a field k . The class of subsets of $\Pi(G)$,*

$$\{\Pi(G)_M \subset \Pi(G) : M \text{ finite dimensional } G\text{-module}\},$$

is the class of closed subsets of a (Noetherian) topology on $\Pi(G)$.

Moreover, we have the equality

$$\Pi(G)_M = \Psi_G^{-1}(\text{Proj}(H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M)))$$

for any finite dimensional kG -module M , where Ψ_G is the map of 3.1.1.

Proof. By Propositions 3.2 and 3.3, our class contains \emptyset , $\Pi(G)$ itself, and is closed under finite intersections and finite unions.

Observe that $\text{Proj } H^\bullet(G, k)$ is Noetherian and that each

$$\text{Proj}(H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M)) \subset \text{Proj } H^\bullet(G, k)$$

is closed. Therefore, to complete the verification that we have given $\Pi(G)$ a Noetherian topology, it suffices to verify the asserted equality. This is equivalent to the following assertion for any finite dimensional kG -module M and any π -point $\alpha_K : K[t]/t^p \rightarrow KG$: namely, $\alpha_K^*(M_K)$ is not projective if and only if $\ker\{\alpha_K^*\}$ contains $\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M)$. By base change from k to the algebraic closure of K , we may assume that k is algebraically closed and $K = k$. In this case, α_K is a p -point of G and the equality is verified (with $\Pi(G)_M \subset \Pi(G)$ replaced by $P(G)_M \subset P(G)$) in [15, 3.8]. \square

Remark 3.5. We call $\Pi(G)$ with this topology the space of π -points of G .

We now verify that our space $\Pi(G)$ is related by a naturally defined homeomorphism to $\text{Proj } H^\bullet(G, k)$.

Theorem 3.6. *Let G be a finite group scheme over a field k , let $\text{Proj} H^\bullet(G, k)$ denote the space of homogeneous ideals (excluding the augmentation ideal) of the graded, commutative algebra $H^\bullet(G, k)$ equipped with the Zariski topology, and let $\Pi(G)$ denote the set of π -points of G provided with the topology of Proposition 3.4. Then*

$$\Psi_G : \Pi(G) \rightarrow \text{Proj} H^\bullet(G, k), \quad [\alpha_K] \mapsto \ker\{\alpha_K^*\}$$

is a homeomorphism.

Moreover, if $j : H \rightarrow G$ is a flat homomorphism of finite group schemes over k , then the following square commutes:

$$(3.6.1) \quad \begin{array}{ccc} \Pi(H) & \xrightarrow{\Psi_H} & \text{Proj} H^\bullet(H, k) \\ \downarrow j^* & & \downarrow \\ \Pi(G) & \xrightarrow{\Psi_G} & \text{Proj} H^\bullet(G, k) \end{array}$$

In this square, the left vertical arrow is given by Corollary 2.7 and the right vertical arrow by the map $H^\bullet(H, k) \leftarrow H^\bullet(G, k)$ induced by $H \rightarrow G$.

Furthermore, if K/k is a field extension, then the following square commutes:

$$(3.6.2) \quad \begin{array}{ccc} \Pi(G_K) & \xrightarrow{\Psi_{G_K}} & \text{Proj} H^\bullet(G_K, K) \\ \downarrow & & \downarrow \\ \Pi(G) & \xrightarrow{\Psi_G} & \text{Proj} H^\bullet(G, k) \end{array}$$

In this square, the left vertical arrow is given by Corollary 2.8 and the right vertical arrow by the base change map $H^\bullet(G, k) \rightarrow H^\bullet(G_K, K)$.

Proof. The verifications of the commutativity of squares (3.6.1) and (3.6.2) are straight-forward, and we omit them.

The injectivity of Ψ_G is given by Theorem 2.10 (as stated in Definition 3.1). To prove surjectivity, we consider a point $x \in \text{Proj} H^\bullet(G, k)$ with residue field $k(x)$ and base change to the algebraic closure K of $k(x)$, so that x is the image of a K -rational point $\bar{x} \in \text{Proj} H^\bullet(G_K, K)$. The commutativity of square (3.6.2) enables us to replace k by K , and thus reduces us to showing the surjectivity of Ψ_G on k -rational points, with k algebraically closed. This is proved in [15, 4.8].

The equality in the statement of Proposition 3.4 implies that the bijective map Ψ_G sends a closed subset (which by definition is of the form $\Pi(G)_M$) of $\Pi(G)$ to a closed subset of $\text{Proj} H^\bullet(G, k)$, thereby establishing the continuity of $(\Psi_G)^{-1}$.

To complete the proof that Ψ_G is a homeomorphism, we show Ψ_G is continuous by verifying that every closed subset of $\text{Proj} H^\bullet(G, k)$ is of the form $\text{Proj}(H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M))$ for some finite dimensional kG -module M . This is equivalent to verifying that every closed conical subset of $\text{Spec} H^\bullet(G, k)$ is the variety of some (homogeneous) ideal of the form $\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M)$ for some finite dimensional kG -module M . We verify this by showing that if $I \subset H^\bullet(G, k)$ is a homogeneous ideal generated by finitely many elements $\zeta_i \in H^{2i}(G, k)$, then $V(I)$ equals $V(\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(M, M))$ where $M = \otimes_i L_{\zeta_i}$. (Here, L_ζ is the finite dimensional kG -module of Proposition 2.9 associated to a homogeneous element $\zeta \in H^\bullet(G, k)$.) This is an extension to prime ideal spectra of the

corresponding result for k -rational points with k algebraically closed which is proved in [9] for finite groups and in [28] for infinitesimal group schemes.

To verify that Ψ_G is continuous, we apply the surjectivity of Ψ_G and Proposition 2.9 to conclude for any homogeneous element $\zeta \in H^\bullet(G, k)$ that the zero locus of ζ in $\text{Proj } H^\bullet(G, k)$, $V(\zeta)$, equals the projectivization of the cohomological support scheme of L_ζ :

$$V(\zeta) = \text{Proj } V(\text{ann}_{H^\bullet(G, k)} \text{Ext}_G^*(L_\zeta, L_\zeta)).$$

Thus, Proposition 3.4 implies that

$$\Psi_G^{-1}(V(\zeta)) = \Pi(G)_{L_\zeta}.$$

Consequently, if I is generated by ζ_1, \dots, ζ_n , then Proposition 3.2 and the bijectivity of Ψ_G imply the equalities

$$V(I) = V(\zeta_1) \cap \dots \cap V(\zeta_n) = \Psi_G(\Pi(G)_{L_{\zeta_1}}) \cap \dots \cap \Psi_G(\Pi(G)_{L_{\zeta_n}}) = \Psi_G(\Pi(G)_M).$$

□

4. APPLICATIONS OF THE HOMEOMORPHISM Ψ

In this section, we give some first applications of Theorem 3.6. We begin by stating the following result established in the course of the proof of Theorem 3.6. This proposition tell us that we can “explicitly” realize any closed subset of $\Pi(G)$ as the Π -support of a module of the form $L_{\zeta_1} \otimes \dots \otimes L_{\zeta_n}$.

Proposition 4.1. *Let G be a finite group scheme over a field k and $I \subset H^\bullet(G, k)$ be a homogeneous ideal generated by homogeneous elements ζ_1, \dots, ζ_n . Then*

$$\Psi_G^{-1}(V(I)) = \Pi(G)_{L_{\zeta_1} \otimes \dots \otimes L_{\zeta_n}},$$

where $V(I) \subset \text{Proj } H^\bullet(G, k)$ is the zero locus of the homogeneous ideal I .

Let G be a finite group scheme over k and let A denote the coordinate algebra of G , $A = k[G]$. By definition, $\pi_0(G)$ is the spectrum of the maximal separable subalgebra of A . The projection $G \rightarrow \pi_0(G)$ admits a splitting if and only if the composition $G_{red} \rightarrow G \rightarrow \pi_0(G)$ is an isomorphism; i.e., if and only if A modulo its nilradical $N \subset A$ is a separable algebra. The two conditions that the projection $G_F \rightarrow \pi_0(G_F)$ split and that $\pi_0(G_F)$ be constant are equivalent to the condition that A_F/N_F is isomorphic to a product of copies of F , where $A_F = A \otimes_k F$ and $N_F \subset A_F$ is the nilradical of A_F . Since $A_{\bar{k}}/N_{\bar{k}}$ is isomorphic to a product of copies of \bar{k} (where \bar{k} is an algebraic closure of k) and since A is finite dimensional over k , we may therefore choose some F/k finite over k such that the projection $G_F \rightarrow \pi_0(G_F)$ splits (so that G_F is a semi-direct product $G_F^0 \rtimes \pi_0(G_F)$) and that $\pi_0(G_F)$ is a constant group scheme.

Utilizing Theorem 3.6, we obtain the following result concerning the field of definition of a representative of a π -point α_K . Since the map Ψ_G is defined in terms of the map α_K^* on cohomology determined by α_K , the connection between the field of definition of a representative of $[\alpha_K]$ and the residue field of an associated homogeneous prime ideal of the cohomology algebra is somewhat striking.

Theorem 4.2. *Let G be a finite group scheme over k and let F be a finite field extension F/k with the property that the projection $G_F \rightarrow \pi_0(G_F)$ splits and that $\pi_0(G_F)$ is a constant group scheme. Let r denote the height of the connected component of $G^0 \subset G$.*

For any π -point $\alpha_K : K[t]/t^p \rightarrow KG_K$ of G , let $k_{[\alpha]}$ denote the residue field of $\Psi_G([\alpha_K]) \in \text{Proj}H^\bullet(G, k)$. Then $[\alpha_K]$ is equivalent to some π -point $\beta_L : L[t]/t^p \rightarrow LG_L$ with L a purely inseparable extension of degree p^r of the composite $F \cdot k_{[\alpha]}$.

Proof. To prove the proposition we may replace G by G_F ; in other words, we may (and will) assume that $G \simeq G^0 \rtimes \pi_0(G)$ with $\pi_0(G)$ constant. We consider some π -point $\alpha_K : K[t]/t^p \rightarrow KG_K$ of G .

Let π denote the finite group $\pi_0(G)$. Suslin's detection theorem [26] asserts that modulo nilpotents any homogeneous element of $H^\bullet(G, k)$ has a non-zero restriction to some group homomorphism of the form $\mathbb{G}_{a(r)L} \times E \rightarrow G_L$ for some field extension L/k . Our condition on $\pi_0(G)$ implies that such a map must factor through some subgroup of G of the form $(G^0)^E \times E$, with $E \subset \pi$ an elementary abelian p -subgroup of π . Consequently, the natural map

$$H^\bullet(G, k) \rightarrow \bigoplus_{E \subset \pi} H^\bullet((G^0)^E \times E, k)$$

has nilpotent kernel, where the sum is indexed by conjugacy classes of elementary abelian p -subgroups of π . This implies that any point of $\text{Proj}H^\bullet(G, k)$ lies in the image of $\text{Proj}H^\bullet((G^0)^E \times E, k)$ for some elementary abelian p -subgroups $E \subset \pi$. The naturality of the homeomorphism Ψ_G of Theorem 3.6 (with respect to $(G^0)^E \times E \rightarrow G$) implies that $[\alpha_K]$ lies in the image of $\Pi((G^0)^E \times E)$ for such an elementary abelian p -group $E \subset \pi$.

Consequently, we may (and will) assume that $G \simeq G^0 \times E$ for some elementary abelian p -group E of rank s .

Assume first that G^0 is trivial, so that $G = E$. Then a choice of generators for E determines the rank variety $V_r(E)$ and we can identify $\text{Proj}(V_r(E))$ with $\Pi(E)$ – namely, each shifted cyclic subgroup of KE is a π -point of E , and we can represent any equivalence class of π -points by such a cyclic shifted subgroup. Then, the homeomorphism $\Psi_E : \Pi(E) \simeq \text{Proj}H^\bullet(E, k)$ refines to an isomorphism of schemes $\text{Proj}V_r(E) \simeq \text{Proj}H^\bullet(E, k)$. In particular, any π -point $\alpha_K : K[t]/t^p \rightarrow KE$ can be represented by a π -point defined over $k_{[\alpha]}$.

Assume now that $s = 0$, so that $G^0 = G$. Let $V(G^0)$ denote the scheme of 1-parameter subgroups of G^0 . By [28, 5.5], there is a natural k -algebra homomorphism

$$\psi : H^\bullet(G^0, k) \rightarrow k[V(G^0)]$$

which contains $k[V_r(G^0)]^{p^r}$. Thus, the bijective map $\Psi_{G^0} : V(G^0) \rightarrow \text{Spec}H^\bullet(G^0, k)$ induces a map on residue fields which is an isomorphism up to a purely inseparable extension of degree at most p^r . This clearly implies the same assertion for $\Psi_{G^0} : \text{Proj}V(G^0) \rightarrow \text{Proj}H^\bullet(G^0, k)$. We conclude that any π -point $\alpha_K : K[t]/t^p \rightarrow KG^0$ can be represented by a π -point defined over a purely inseparable extension of $k_{[\alpha]}$ of degree at most p^r .

We now complete the consideration of $G \simeq G^0 \times E$ by considering the case in which G^0 has height $r \geq 1$ and E has rank $s \geq 1$. Observe that every maximal abelian subgroup scheme $C_K \subset (G^0 \times E)_K$ is of the form $C_K^0 \times E$ where C_K^0 is a maximal abelian subgroup scheme of G_K^0 , and similarly every maximal abelian

subgroup scheme of $(G^0 \times \mathbb{G}_{a(1)}^s)_K$ is of the form $C_K^0 \times \mathbb{G}_{a(1)K}^s$ where once again C_K^0 is a maximal abelian subgroup scheme of G_K^0 . Thus, an isomorphism $kE \simeq k\mathbb{G}_{a(1)}^s$ determines a bijection between flat, K -linear maps of algebras $K[t]/t^p \rightarrow K(G^0 \times E)$ which factor through an abelian subgroup scheme of $(G^0 \times E)_K$ and flat, K -linear maps of algebras $K[t]/t^p \rightarrow K(G^0 \times \mathbb{G}_{a(1)}^s)$ which factor through an abelian subgroup scheme of $(G^0 \times \mathbb{G}_{a(1)}^s)_K$. Since this bijection together with the isomorphism $k(G^0 \times E) \simeq k(G^0 \times \mathbb{G}_{a(s)}^s)$ determines a homeomorphism between $\Pi(G^0 \times E)$ and $\Pi(G^0 \times \mathbb{G}_{a(1)}^s)$, we may apply the argument above for connected G to $G^0 \times \mathbb{G}_{a(1)}^s$ to conclude the result for $G = G^0 \times E$. \square

Essentially by definition, the condition (2.10.1):

$$(\ker\{\beta_L^*\}) \cap H^\bullet(G, k) \subset (\ker\{\alpha_K^*\}) \cap H^\bullet(G, k),$$

holds if and only if $(\ker\{\alpha_K^*\}) \cap H^\bullet(G, k)$ lies in the closure of $(\ker\{\beta_L^*\}) \cap H^\bullet(G, k)$ as points of $\text{Proj } H^\bullet(G, k)$. Thus, Theorems 2.10 and 3.6 imply the following topological interpretation of specialization of π -points.

Proposition 4.3. *Let G be a finite group scheme over k , and let α_K, β_L be π -points of G . Then $\beta_L \downarrow \alpha_K$ if and only if $\Psi_G(\alpha_K) \in \text{Proj } H^\bullet(G, k)$ lies in the closure of $\Psi_G(\beta_L)$.*

Consequently, the set of π -points of G which are specializations of a given π -point α_K form a closed subset $\overline{\{\alpha_K\}} \subset \Pi(G)$.

Proposition 4.4. *Let k/F be a field extension and $\sigma : k \rightarrow k$ a field automorphism over F . Assume that the finite group scheme G over k is defined over F , so that $G = G_F \times_{\text{Spec } F} \text{Spec } k$. Then there is a natural action of σ on $\Pi(G)$, $[\alpha] \mapsto [\alpha^\sigma]$, which commutes with the homeomorphism $\Psi_G : \Pi(G) \rightarrow \text{Proj } H^\bullet(G, k)$, where the action on the right is induced by the map*

$$\sigma \otimes 1 : H^\bullet(G, k) = k \otimes_F H^\bullet(G_F, F) \rightarrow k \otimes_F H^\bullet(G_F, F) = H^\bullet(G, k).$$

Moreover, if M is a kG -module defined over F , and $\alpha_K : k[t]/t^p \rightarrow KG_K$ is a π -point, then $(\alpha_K^\sigma)^(M_K)$ is projective if and only if $\alpha_K^*(M_K)$ is projective.*

Proof. Let $\alpha_K : K[t]/t^p \rightarrow KG$ be a π -point of G . By replacing K/k by a finite extension of K if necessary, we may assume that the automorphism σ of k/F extends to an automorphism $\tilde{\sigma} : K \rightarrow K$ over F .

Following [15, 5.5], we define $\alpha_K^{\tilde{\sigma}}$ to be the π -point obtained by precomposing with $\tilde{\sigma}^{-1} \otimes 1$ and postcomposing with $\tilde{\sigma} \otimes 1$:

$$(4.4.1) \quad K[t]/t^p = K \otimes_F F[t]/t^p \xrightarrow{\tilde{\sigma}^{-1} \otimes 1} K \otimes_F F[t]/t^p \xrightarrow{\alpha_K} K \otimes_F FG_F \xrightarrow{\tilde{\sigma} \otimes 1} K \otimes_F FG_F = KG$$

Recall that

$$\Psi_G([\alpha_K]) \equiv \ker\{\alpha_K^* : H^\bullet(G, K) \rightarrow H^\bullet(K[t]/t^p, K)\} \cap H^\bullet(G, k).$$

Since $(\tilde{\sigma}^{-1} \otimes 1)^* : H^\bullet(K[t]/t^p, K) \rightarrow H^\bullet(K[t]/t^p, K)$ is an isomorphism, we conclude that $\ker\{(\alpha_K^{\tilde{\sigma}})^*\}$ is the kernel of the composition

$$H^\bullet(G_K, K) \xrightarrow{(\tilde{\sigma} \otimes 1)^*} H^\bullet(G_K, K) \xrightarrow{\alpha_K^*} H^\bullet(K[t]/t^p, K).$$

Since $H^\bullet(G_K, K) \xrightarrow{(\tilde{\sigma} \otimes 1)^*} H^\bullet(G_K, K)$ restricts to $H^\bullet(G, k) \xrightarrow{(\sigma \otimes 1)^*} H^\bullet(G, k)$, we conclude that

$$\ker\{(\alpha_K^{\tilde{\sigma}})^*\} \cap H^\bullet(G, k)$$

equals

$$\sigma^*(\ker\{\alpha_K^*\} \cap H^\bullet(G, k)) = (\ker\{\alpha_K^*\} \cap H^\bullet(G, k))^\sigma,$$

where we denote by \mathcal{P}^σ the image of a homogeneous prime ideal $\mathcal{P} \subset H^\bullet(G, k)$ under the action of σ . Consequently, sending α_K to $\alpha_K^{\tilde{\sigma}}$ determines a well defined action on equivalence classes of π -points, does not depend upon the choice of extension $\tilde{\sigma}$ of σ , and commutes with Ψ_G .

Let M be a kG -module defined over F and write $M = k \otimes_F M_0$. Suppose that $\alpha_K^*(M_K)$ is free. To show that $(\alpha_K^{\tilde{\sigma}})^*(M_K)$ is free, it suffices to show that $H^1(K[t]/t^p, (\alpha_K^{\tilde{\sigma}})^*(M_K)) = 0$, i.e. that $\ker\{\alpha_K^{\tilde{\sigma}}(t)\} = \text{Im}\{(\alpha_K^{\tilde{\sigma}})^{p-1}(t)\}$, where we view $\alpha_K^{\tilde{\sigma}}(t)$ as an endomorphism of M_K . Let $\alpha_K(t) = \sum_i \alpha_i t_i$ where $\alpha_i \in K$ and t_i are generators of FG_F over F . The definition of $\alpha^{\tilde{\sigma}}$ given in (3.9.1) immediately implies that $\alpha_K^{\tilde{\sigma}}(t) = \sum_i \tilde{\sigma}(\alpha_i) t_i$. Let $m = \sum_j c_j \otimes m_j \in K \otimes_F M_0$ be in the kernel of $\alpha_K^{\tilde{\sigma}}(t)$. This amounts to the equation

$$(4.4.2) \quad \alpha_K^{\tilde{\sigma}}(t) \left(\sum_j c_j \otimes m_j \right) = \sum_{i,j} \tilde{\sigma}(\alpha_i) c_j \otimes t_i m_j = 0.$$

Applying $\tilde{\sigma}^{-1} \otimes 1$ to this equation, we obtain

$$\sum_{i,j} \alpha_i \tilde{\sigma}^{-1}(c_j) \otimes t_i m_j = \alpha_K(t) \left(\sum_j \tilde{\sigma}^{-1}(c_j) \otimes m_j \right) = 0.$$

Thus, $(\tilde{\sigma} \otimes 1)^{-1}(m) = \sum_j \tilde{\sigma}^{-1}(c_j) \otimes m_j \in \ker\{\alpha_K(t)\}$. Here, again, we view $\alpha_K(t)$ as an endomorphism of M_K . Since M_K is a free $K[t]/t^p$ -module with respect to the action of $\alpha_K(t)$, we get that $(\tilde{\sigma} \otimes 1)^{-1}(m) \in \text{Im}\{\alpha_K^{p-1}(t)\}$. Applying $\tilde{\sigma} \otimes 1$ to $(\tilde{\sigma} \otimes 1)^{-1}(m)$, we conclude that $m \in \text{Im}\{(\alpha_K^{\tilde{\sigma}})^{p-1}(t)\}$. Thus, the action of $\alpha_K^{\tilde{\sigma}}(t)$ on M is free. The invertibility of $\tilde{\sigma}$ enables the same proof to verify that $\alpha_K^*(M_K)$ is free as a $K[t]/t^p$ -module whenever $(\alpha_K^{\tilde{\sigma}})^*(M_K)$ is free. \square

Let $p \in \text{Proj } H^\bullet(G, k)$ be a closed point which is rational over a finite separable extension F/k but is not k -rational, and let $\tilde{p}, \tilde{q} \in \text{Proj } H^\bullet(G_F, F)$ be distinct points mapping to p . Choose π -points $\alpha_K : K[t]/t^p \rightarrow KG, \beta_L : L[t]/t^p \rightarrow LG$ with the property that $\Psi_{G_F}([\alpha_K]) = \tilde{p}, \Psi_{G_F}([\beta_L]) = \tilde{q}$. Then for every finite dimensional kG -module M , $\alpha_K^*(M_K)$ is projective if and only if $\beta_L^*(M_L)$ is projective; however, there exists a finite dimensional FG_F -module N such that $\alpha_K^*(N_K)$ is projective and $\beta^*(N_L)$ is not projective.

To further illustrate the behaviour of the map $\Pi(G_K) \rightarrow \Pi(G)$ of Corollary 2.8, we determine the pre-images of this map in the special case of Example 2.3.

Example 4.5. We adopt the notation and conventions of Example 2.3 and let $K = k(z)$, the field of fractions of the “generic” π -point of $G = \mathbb{Z}/p \times \mathbb{Z}/p$. As established in Example 2.3,

$$\xi_{k(z)} : k(z)[t]/t^p \rightarrow k(z)[x, y]/(x^p, y^p), \quad t \mapsto zx + y$$

represents the unique equivalence class of “generic” π -point of G . One readily observes that a π -point of G defined by $t \mapsto f(z)x + y$ with f any non-constant rational function f is equivalent to $\xi_{k(z)}$. However, points corresponding to distinct non-constant functions f are not equivalent as π -points of G_K (by [15, 2.2]). Thus the pre-image of the generic point of G under the map $\Pi(G_K) \rightarrow \Pi(G)$ has closed points in one-to-one correspondence with elements of $K^* - k^*$. On the other hand, a closed point of $\Pi(G)$ is represented by a flat map of the form

$$k[t]/t^p \rightarrow k[x, y]/(x^p, y^p), \quad t \mapsto ax + by$$

with at least one of $a, b \in k$ non-zero. The pre-image of such a point in $\Pi(G_K)$ consists of the a single element, the equivalence class of

$$K[t]/t^p \rightarrow K[x, y]/(x^p, y^p), \quad t \mapsto ax + by.$$

More generally, the pre-image of $\Pi(G_K) \rightarrow \Pi(G)$ above some $[\alpha_K] \in \Pi(G)$ is non-empty, and any point of this pre-image has closure in $\Pi(G_K)$ with dimension at most the transcendence degree of the residue field of $[\alpha_K]$ over k . This last statement can be verified using the the homeomorphism Ψ of Theorem3.6.

In view of this observation of the non-injectivity of the functorial map $\Pi(G_F) \rightarrow \Pi(G)$ for a field extension F/k , the following result is somewhat striking.

Theorem 4.6. *Let G be a finite group scheme over a field k . We say that two π -points $\alpha_K : K[t]/t^p \rightarrow KG$, $\beta_L : L[t]/t^p \rightarrow LG$ are strongly equivalent if for any (possibly infinite dimensional) kG -module M $\alpha_K^*(M_K)$ is projective if and only if $\beta_L^*(M_L)$ is projective.*

If $\alpha_K \sim \beta_L$, then α_K is strongly equivalent to β_L .

Proof. We first prove the statement in the special case when $L = K = k$, with k algebraically closed. Following notation introduced in [15], we will refer to π -points defined over the ground field k as p -points. We quote here the statement of [15, 2.2] which will be used extensively throughout the proof: let M be a k vector space and α, β and γ be pair-wise commuting endomorphisms of M such that α, β are p -nilpotent and γ is p^r -nilpotent for some $r \geq 1$. Then M is free as a $k[u]/u^p$ -module via the action of α if and only if M is free via the action of $\alpha + \beta\gamma$. We will refer to the replacement of a p -point α by a p -point $\alpha + \beta\gamma$ satisfying the conditions above as an “elementary operation” on p -points. Thus, [15, 2.2] implies that an elementary operation does not change the strong equivalence class of a p -point.

Let $\alpha : k[t]/t^p \rightarrow kC \rightarrow kG$ be a p -point of G , which factors through an abelian subgroup scheme C . Recall [15, 2.10], which asserts that a p -point of C is equivalent to one that factors through the co-connected component C^0 of C (i.e., the dual finite group scheme $(C^0)^\#$ is connected or equivalently, kC has no non-trivial idempotents). The proof of this result is given for not-necessarily finite dimensional C -modules, thereby enabling us to conclude that α is strongly equivalent to a p -point which factors through some co-connected abelian finite group scheme. We therefore assume that C is co-connected.

The structure theorem for connected finite group schemes [30, 14.4] implies that $kC \simeq k[t_1, t_2, \dots, t_n]/(t_1^{p^{i_1}}, \dots, t_n^{p^{i_n}})$. By [15, 4.11], the space of equivalence classes of p -points of C is homeomorphic to $\text{Proj } H^\bullet(C, k)$, which in turn is homeomorphic to \mathbb{P}_k^n . We conclude that two p -points of C are equivalent if and only if some non-zero scalar multiple of one coincides with the other modulo the ideal $(t_1, t_2, \dots, t_n)^2$.

But any such two p -points are connected by multiplication by a scalar and a series of elementary operations and, thus, are strongly equivalent by [15, 2.2]. We thereby conclude that equivalence implies strong equivalence for abelian finite group schemes. Now, [15, 4.2] implies that any p -point $\alpha : k[t]/t^p \rightarrow kC$ is equivalent and thus strongly equivalent to a p -point factoring through a quasi-elementary abelian subgroup scheme, i.e. a subgroup scheme isomorphic to $\mathbb{G}_{a(r)} \times E$ where E is an elementary abelian p -group.

We proceed now by assuming that α and β are equivalent p -points each of which factors through some quasi-elementary abelian subgroup scheme of G . Let G^0 be the connected component of G and $\pi = \pi_0(G)$ be the group of connected components. Corollary [15, 4.7] implies that α, β are conjugate by an element of π to equivalent p -points which factor through the same subgroup scheme $(G^0)^E \times E$, where $E \subset \pi$ is an elementary abelian subgroup of π . Since conjugation by elements of π does not change the strong equivalence class of a p -point, we are further reduced to the case in which G is of the special form $G' \times E$ with G' connected. Since $kE \simeq k\mathbb{G}_{a(r)}$, we may further assume that G itself is connected.

In this case, write α as the composition of some $\alpha_C : k[t]/t^p \rightarrow kC$ with C an abelian subgroup scheme of G and $kC \rightarrow kG$ induced by $C \subset G$. Then α_C is equivalent as a p -point of C to a composition of the form $i \circ \epsilon_r : k[t]/t^p \rightarrow k\mathbb{G}_{a(r)} \rightarrow kC$, where $\epsilon_r : k[t]/t^p \rightarrow k\mathbb{G}_{a(r)} \simeq k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$ is the algebra map sending t to u_{r-1} (and is thus a Hopf algebra map if and only if $r = 1$) and i is induced by some homomorphism of group schemes $\mathbb{G}_{a(r)} \rightarrow C$. Since equivalence implies strong equivalence for p -points of the abelian group scheme C , we conclude that α_C is strongly equivalent to $i \circ \epsilon_r$. Similarly, factor β in terms of $\beta_D : k[t]/t^p \rightarrow kD$, with β_D strongly equivalent as a p -point of D to a composition of the form $j \circ \epsilon'_s : k[t]/t^p \rightarrow k\mathbb{G}_{a(s)} \rightarrow kD$. If we assume that for both α, β these are the smallest possible factorizations, then the fact that $(j \circ \epsilon_r)^* = (i \circ \epsilon_s)^*$ implies that $r = s$. Thus, α and β are strongly equivalent to compositions of $\epsilon_r : k[t]/t^p \rightarrow k\mathbb{G}_{a(r)}$ with maps $k\mathbb{G}_{a(r)} \rightarrow kG$ induced by homomorphisms of group schemes (i.e., α, β are “1-parameter subgroups of G ”). Yet for p -points of this special form to be equivalent they must be equal by [15, 3.8].

Now, we deal with the general case of arbitrary π -points α_K, β_L . Let Ω/k be an algebraically closed field of transcendence degree at least the Krull dimension of $H^\bullet(G, k)$. In view of Remark 2.6 and the bijectivity of Ψ_G , we may assume $L = K = \Omega$. Corollary 2.11 implies that $(\ker\{\beta_\Omega^*\}) \cap H^\bullet(G, k) = (\ker\{\alpha_\Omega^*\}) \cap H^\bullet(G, k)$. We view $\ker\{\alpha_\Omega^*\}, \ker\{\beta_\Omega^*\}$ as geometric points of $\text{Spec } H^\bullet(G, k)$ mapping to the same scheme-theoretic point, $(\ker\{\beta_\Omega^*\}) \cap H^\bullet(G, k)$. From this point of view, we readily verify that there exists an automorphism over k , $\sigma : \Omega \rightarrow \Omega$, such that

$$\ker\{\beta_\Omega^*\} = (\ker\{\alpha_\Omega^*\})^\sigma.$$

Since $\Psi_{G_\Omega} : \Pi(G_\Omega) \rightarrow \text{Proj } H^\bullet(G_\Omega, \Omega)$ commutes with the action of σ by Proposition 4.4, we have the equality

$$(\ker\{\alpha_\Omega^*\})^\sigma = \ker\{(\alpha_\Omega^\sigma)^*\},$$

Since Ω is algebraically closed, it follows from Theorem [15, 4.11] that $\alpha_\Omega^\sigma \sim \beta_\Omega$ as p -points of G_Ω .

Thus, the special case verified above in which $L = K = k$ is algebraically closed implies that for any ΩG -module N , $(\alpha_\Omega^\sigma)^* N$ is projective if and only if $(\beta_\Omega)^* N$ is projective. On the other hand, Proposition 4.4 implies that for a kG -module

M , $(\alpha_\Omega^\sigma)^*(M_\Omega)$ is projective if and only if $\alpha_\Omega^*(M_\Omega)$ is projective. Hence, $\beta_\Omega^*(M_\Omega)$ is projective if and only if $\alpha_\Omega^*(M_\Omega)$ is projective for any kG -module M . In other words, α_K is strongly equivalent to β_L . □

In the next proposition, we give several characterizations of closed points of $\Pi(G)$. In particular, if k is algebraically closed, then the space $P(G)$ of p -points is exactly the subspace of closed points of $\Pi(G)$.

Proposition 4.7. *Let G be a finite group scheme over a field k . Then the following conditions are equivalent on a π -point $\alpha_K : K[t]/t^p \rightarrow KG$ of G .*

- (1) *The equivalence class $[\alpha_K]$ of α_K is a closed point of $\Pi(G)$.*
- (2) *Any specialization of α_K is equivalent to α_K .*
- (3) *α_K is equivalent to some π -point $\beta_F : F[t]/t^p \rightarrow FG$ with F/k finite. In particular, if k is algebraically closed, then the equivalence class of α_K , $[\alpha_K]$ is represented by a map of the form $\beta : k[t]/t^p \rightarrow kG$ and is thus a p -point in the sense of [15].*
- (4) *There exists some finite dimensional kG -module M such that whenever $\beta_L : L[t]/t^p \rightarrow LG$ is a π -point with $\beta_L^*(M_L)$ not projective then α_K is equivalent to β_L .*

Proof. Granted the topology on $\Pi(G)$ given in Proposition 3.4, a π -point α is a specialization of a π -point β if and only if α is in the closure of β . Thus, (1) and (2) are equivalent.

If $\alpha : K[t]/t^p \rightarrow KG$ is a π -point, then $\ker\{\alpha^*\} \in \text{Proj } H^\bullet(G, k)$ is defined over K . Consequently, (3) implies (1), for any point of $\text{Proj } H^\bullet(G, k)$ defined over an algebraic extension of k must be closed. Conversely, let \bar{k} denote the algebraic closure of k . Using Theorem 3.6, we see that any closed point of $\Pi(G)$ lies in the image of a closed point $\Pi(G_{\bar{k}})$ which corresponds (naturally and bijectively) to a rational point of $\text{Proj } H^\bullet(G_{\bar{k}}, \bar{k})$ which corresponds (naturally and bijectively) to a p -point of $G_{\bar{k}}$ by [15, 4.6]. Any such p -point $\alpha_{\bar{k}} : \bar{k}[t]/t^p \rightarrow \bar{k}G_{\bar{k}}$ is defined over some finite extension of k .

A π -point $[\alpha] \in \Pi(G)$ is closed if and only if there is a finitely generated kG -module with $\Pi(G)_M = \{[\alpha]\}$. This is a reformulation of the assertion that (1) is equivalent to (4). □

Example 4.8. The reader may find the following computation for $G = GL(3, \mathbb{F}_p)$ instructive, since there are distinct conjugacy classes of maximal elementary abelian p -groups in G . Consider the elements

$$e_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the subgroups generated by $(e_{12}, e_{13}), (e_{13}, e_{23}), (e_3, e_{13})$ represent the three distinct conjugacy classes of maximal elementary abelian p -groups in G .

Quillen’s “stratification theorem” [23] implies that $\text{Spec } H^\bullet(G, k)$ is the union of three irreducible surfaces, each the quotient of affine 2-space modulo a finite group, with common intersection an affine line modulo a finite group. Hence, Theorem 3.6 implies that $\Pi(G)$ is the 1-point union of 3 irreducible projective curves. In

particular, any π -point of G is a specialization of one of the following three “generic” π -points:

$$\alpha_{k(z)} : k(z)[t]/(t^p) \rightarrow k(z)G, \quad t \mapsto z(e_{12} - 1) + (e_{13} - 1),$$

$$\beta_{k(z)} : k(z)[t]/(t^p) \rightarrow k(z)G, \quad t \mapsto z(e_{23} - 1) + (e_{13} - 1),$$

and

$$\gamma_{k(z)} : k(z)[t]/(t^p) \rightarrow k(z)G, \quad t \mapsto z(e_3 - 1) + (e_{13} - 1).$$

We conclude this section with another interesting family of examples.

Example 4.9. Let F be a finite field of characteristic $\ell \neq p$ with the property that F contains all p -th roots of unity. Then Quillen determines $H^*(GL(n, F), k)$ in [24], establishing that

$$H^*(GL(n, F), k) = (H^*(T(n, F), k))^{\Sigma_n},$$

the invariants of the cohomology of the maximal torus $T(n, F) = (F^\times)^n$ under the permutation action of the symmetric group Σ_n [24]. Thus,

$$\text{Proj } H^\bullet(GL(n, F), k) = \mathbb{P}^{n-1},$$

$n - 1$ dimensional projective space over k .

Choose an element $1 \neq \mu \in F$ with the property that $\mu^p = 1$ and let $D_{i,i}(\mu) \in T(n, F)$ denote the diagonal matrix whose (i, i) -entry is μ and all of whose other diagonal entries equal 1. Let $K \equiv k(\lambda_1, \dots, \lambda_n)$ denote the pure transcendental field extension of transcendence degree n over k and consider

$$\alpha_K : K[t]/t^p \rightarrow KT(n, F), \quad t \mapsto \sum_{i=1}^n \lambda_i (D_{i,i}(\mu) - Id).$$

Then the composition of α_K with the map of group algebras induced by $i : T(n, F) \rightarrow GL(n, F)$ represents a generic π -point of $GL(n, F)$. The composition $i \circ \alpha_K$ can be represented more efficiently by the equivalent π -point

$$\beta_L : L[t]/t^p \rightarrow LGL(n, F), \quad t \mapsto \left(\sum_{i=1}^{n-1} \sigma_i \cdot (D_{i,i}(\mu) - Id) \right) + (D_{n,n}(\mu) - Id)$$

where

$$L = k\left(\frac{\sigma_1}{\sigma_n}, \dots, \frac{\sigma_{n-1}}{\sigma_n}\right)$$

and σ_i is the i -th elementary symmetric function in $\lambda_1, \dots, \lambda_n$ (invariant under Σ_n).

5. THE Π -SUPPORT OF AN ARBITRARY G -MODULE

One justification for considering the space $\Pi(G)$ of π -points of a finite group scheme G (rather than the simpler space $P(G)$ considered in [15]) is that this space serves as a useful invariant for kG -modules which are not necessarily finite dimensional. In particular, we shall verify in the next section (Corollary 6.7) that every subset of $\Pi(G)$ is the Π -support of some kG -module. Indeed, the consideration of non-closed points of $\Pi(G)$ when investigating infinite dimensional kG -modules is already foreshadowed in the work of Benson, Carlson, and Rickard (cf. [6]).

Theorem 4.6 allows us to extend the definition of the support to all, not necessarily finite dimensional, G -modules.

Definition 5.1. For a kG -module M , we define Π -support of M to be the subset

$$\Pi(G)_M \subset \Pi(G)$$

of those equivalence classes $[\alpha_K]$ of π -points such that $\alpha_K^*(M_K)$ is not projective for any representative $\alpha_K : K[t]/t^p \rightarrow KG$ of the equivalence class $[\alpha_K]$.

In view of Theorem 4.6, the properties of the π -support construction, $M \mapsto \Pi(G)_M$, stated in Propositions 3.2 and 3.3 extend to all kG -modules. The proofs of these properties for finite dimensional modules apply without change to infinite dimensional modules.

Proposition 5.2. *Let G be a finite group scheme over a field k and let M_1, M_2, M_3 be arbitrary kG -modules. Then*

- (1) $\Pi(G)_k = \Pi(G)$.
- (2) $\Pi(G)_{M_1 \otimes M_2} = \Pi(G)_{M_1} \cap \Pi(G)_{M_2}$.
- (3) $\Pi(G)_{M_1 \oplus M_2} = \Pi(G)_{M_1} \cup \Pi(G)_{M_2}$.
- (4) *If P is a projective kG -module, then $\Pi(G)_P = \emptyset$.*
- (5) *If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is exact, then*

$$\Pi(G)_{M_i} \subset \Pi(G)_{M_j} \cup \Pi(G)_{M_k}$$

where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$.

We next extend the ‘‘projectivity test’’ given by support varieties to arbitrary kG -modules. This theorem is a measure of the non-triviality of our Π -support construction. One can view this as a statement that local projectivity implies projectivity. This result, generalizing a sequence of results by many authors, has its origins in L. Chouinard’s proof [12] that projectivity of modules for a finite group G can be detected by restriction to elementary abelian p -subgroups $E \subset G$ and Dade’s investigation [13] of modules for elementary abelian p -groups leading to the concept due to Carlson [8] of shifted subgroups of the group algebra kE .

Theorem 5.3. *Let G be a finite group scheme over a field k and let M be any kG -module. Then M is projective if and only if for any π -point $\alpha_K : K\mathbb{Z}/p \rightarrow KG$, $\alpha_K^*M_K$ is projective.*

Proof. By base change if necessary to the algebraic closure \bar{k} of k , we may (and shall) assume that k is algebraically closed. The ‘‘only if’’ part is clear since π -points are flat maps.

Let G^0 denote the connected component of G , let $\pi = \pi_0(G)$ denote the discrete group of connected components of G , and let M be a kG -module such that α^*M_K is $K[t]/t^p$ -projective for every π -point $\alpha^* : K[t]/t^p \rightarrow KG$. Then we must prove that M is projective as a kG -module. To prove the projectivity of M , it suffices to prove for each irreducible kG -module S (necessarily finite dimensional) that $H^i(G, S \otimes M) = 0, i > 0$: this will then imply that $Ext_G^i(N, M) = 0, i > 0$ for all kG -modules N , which in turn implies that M is injective and thus also projective since kG is a Frobenius algebra by [14]. Since $S \otimes M$ necessarily satisfies $\alpha_K^*(S \otimes M)$ is projective since $\alpha_K^*(M)$ is projective for any π -point α_K , we may (and shall) simplify notation and replace $S \otimes M$ by M .

If $i : \tau \subset \pi$ is a subgroup and $G_\tau \subset G$ is the inverse image of τ with respect to the projection $G \rightarrow \pi$, then there is a natural transfer map $i_! : H^*(G_\tau, M|_{G_\tau}) \rightarrow H^*(G, M)$. A basic property of this transfer map guarantees that its composition

with the natural map $i^* : H^*(G, M) \rightarrow H^*(G_\tau, M|_{G_\tau})$, $i_! \circ i^*$, equals multiplication by $[\pi : \tau]$, the index of τ in π . Consequently, we may assume that π is a finite p -group (one may consult [3] for a careful presentation of the transfer map in this situation).

In the special case in which G is connected so that π is trivial, the projectivity of M is given by [22, 2.2]. We proceed by induction on the order of π and consider some surjective map $\pi \rightarrow \mathbb{Z}/p$. Let G^1 denote the kernel of the composition $G \rightarrow \pi \rightarrow \mathbb{Z}/p$. By induction, we may assume that M is projective when restricted to G^1 . Then the Lyndon-Hochschild-Serre spectral sequence for the extension $1 \rightarrow G^1 \rightarrow G \rightarrow \mathbb{Z}/p \rightarrow 1$ implies that

$$(5.3.1) \quad H^*(G, M) \simeq H^*(\mathbb{Z}/p, H^0(G^1, M)).$$

Thus, to prove the vanishing of $H^i(G, M)$, $i > 0$, it suffices to verify that $H^0(G^1, M)$ is projective as a \mathbb{Z}/p -module.

Assume to the contrary that $H^0(G^1, M)$ is not projective as a \mathbb{Z}/p -module. Then $H^*(\mathbb{Z}/p, H^0(G^1, M))$ is not torsion as a $H^\bullet(\mathbb{Z}/p, k)$ -module. The multiplicative structure of the Lyndon-Hochschild-Serre spectral sequence implies the compatibility of the pairing at the E_2 -level with the pairing of abutments; in particular, we conclude the compatibility of the pairing

$$\begin{aligned} (E_2^{*,0}(k) = H^*(\mathbb{Z}/p, H^0(G^1, k))) \otimes (E_2^{*,0}(M) = H^*(\mathbb{Z}/p, H^0(G^1, M))) \\ \rightarrow (E_2^{*,0}(M) = H^*(\mathbb{Z}/p, H^0(G^1, M))) \end{aligned}$$

with the pairing

$$H^\bullet(G, k) \otimes H^*(G, M) \rightarrow H^*(G, M).$$

Since the pairing at $E_2^{*,0}$ is that induced by the pairing

$$H^0(G^1, k) \otimes H^0(G^1, M) \rightarrow H^0(G^1, M),$$

the isomorphism (5.3.1) implies that $H^*(G, M)$ is not a torsion $H^\bullet(G, k)$ module.

We shall prove that homogeneous classes of positive degree in $H^\bullet(G, \Lambda)$ are nilpotent, where $\Lambda = \text{End}_k(M, M)$. Since the action of $H^*(G, k)$ on $H^*(G, M)$ factors through $H^*(G, \Lambda)$ (in other words, the action of $\text{Ext}_G^*(k, k)$ on $\text{Ext}_G^*(k, M)$ factors through $\text{Ext}_G^*(M, M) = H^*(G, \Lambda)$) and since $H^\bullet(G, k)$ is finitely generated, we may then conclude that the action of $H^\bullet(G, k)$ on $H^*(G, M)$ is nilpotent. This contradicts the preceding paragraph which asserts that $H^*(G, M)$ is not a torsion $H^\bullet(G, k)$ -module.

Let E be an elementary abelian p -subgroup of rank s in $\pi_0(G)$. Then as argued in the proof of Theorem 4.2, a choice of isomorphism of kE with $k\mathbb{G}_{a(1)}^s$ determines an isomorphism

$$\theta : k((G^0)^E \times \mathbb{G}_{a(1)}^s) \xrightarrow{\sim} k((G^0)^E \times E)$$

which induces a bijection between π -points. Applying [22], we conclude that our hypothesis on the kG -module M implies that θ^*M is projective as a $(G^0)^E \times \mathbb{G}_{a(1)}^s$ -module and thus that M is projective as a $(G^0)^E \times E$ -module. Since the projectivity of M implies that M is also injective as a $(G^0)^E \times E$ -module, we conclude that Λ is likewise injective (and thus projective) thanks to the adjunction isomorphism

$$\text{Hom}_{k((G^0)^E \times E)}(- \otimes M, M) \simeq \text{Hom}_{k((G^0)^E \times E)}(-, \text{End}_k(M, M)).$$

Now, let $\zeta \in H^\bullet(G, \Lambda)$ be some homogeneous class of positive degree. Every π -point $\alpha_K : K[t]/t^p \rightarrow KG_K$ is equivalent to some π -point α'_K which factors

through some $((G^0)^E \times E)_K \subset G_K$ and therefore has the property that $(\alpha'_K)^*(\Lambda_K)$ is projective. Since equivalence implies strong equivalence, we conclude for every π -point α_K of G that $\alpha_K^*(\Lambda_K)$ is projective and thus that the restriction map $\alpha_K^* : H^\bullet(G, \Lambda) \rightarrow H^\bullet(K[t]/t^p, \alpha_K^*(\Lambda_K))$ sends ζ to 0. By Suslin's Detection Theorem [26], we conclude that ζ is nilpotent as required. \square

As mentioned above, we shall see in Corollary 6.7 that *any* subset of $\Pi(G)$ is of the form $\Pi(G)_M$ whereas $\text{Proj } H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)} \text{Ext}^*(M, M) \subset \text{Proj } H^\bullet(G, k)$ is always closed. However, the equality

$$\Pi(G)_M = \Psi_G^{-1}(\text{Proj } H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)}(\text{Ext}_G^*(M, M)))$$

of Theorem 3.6 does admit the following partial generalization for arbitrary kG -modules.

Proposition 5.4. *Let G be a finite group scheme, M be a kG -module. Then*

$$\Psi_G(\Pi_M) \subset \text{Proj } H^\bullet(G, k)/\text{ann}_{H^\bullet(G, k)}(\text{Ext}_G^*(M, M)).$$

Proof. We must show that if $\alpha_K : K[t]/t^p \rightarrow KG$ is a π -point with the property that $\alpha_K^*(M_K)$ is not projective, then

$$\ker\{\alpha_K^*\} \cap H^\bullet(G, k) \supset \text{ann}_{H^\bullet(G, k)}(\text{Ext}_G^*(M, M)).$$

Since

$$\text{ann}_{H^\bullet(G, k)}(\text{Ext}_G^*(M, M)) \subset \text{ann}_{H^\bullet(G_K, K)}(\text{Ext}_{G_K}^*(M_K, M_K)) \cap H^\bullet(G, k),$$

we may assume $K = k$ and that k is algebraically closed. For notational simplicity, we write α for α_K .

Let $i : C \subset G$ be a subgroup scheme with the property that α factors as the composition of a p -point $\tilde{\alpha} : k[t]/t^p \rightarrow kC$ and the Hopf algebra map $kC \rightarrow kG$ induced by $C \subset G$. Thinking of Ext-groups in terms of extensions one sees easily that the square in the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccccc} H^\bullet(G, k) & \xrightarrow{i^*} & H^\bullet(C, k) & \xrightarrow{(\tilde{\alpha})^*} & H^\bullet(k[t]/t^p, k) \\ \otimes M \downarrow & & \otimes M \downarrow & & \\ \text{Ext}_G^*(M, M) & \longrightarrow & \text{Ext}_C^*(M, M) & & \end{array}$$

Thus, a simple diagram chase tell us that if the statement of the theorem is valid for $\tilde{\alpha}$, then it is valid for α .

Consequently, we may assume G is abelian. Moreover, by [15, 4.2], α is equivalent to some p -point $\alpha' : k[t]/t^p \rightarrow kG$ whose image lies in some quasi-elementary abelian subgroup $\mathcal{E} \subset G$ (i.e., a product of the form $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ for some $r, s \geq 0$). Using Proposition 2.9 exactly as in [15, 3.5], we conclude that $\ker\{\alpha^*\} = \ker\{\alpha'^*\}$. Thus, we may assume that $G = \mathcal{E}$ is quasi-elementary.

Let $\Lambda = \text{End}_k(M, M)$ with the usual \mathcal{E} -module structure : $g \bullet f(?) = gf(g^{-1}?)$ for $g \in \mathcal{E}$ and $f \in \text{End}_k(M, M)$. The corresponding $k\mathcal{E}$ -module structure is given as follows: for $u \in k\mathcal{E}$ with $\Delta(u) = \sum u' \otimes u''$ and $f \in \text{End}_k(M, M)$, we have $u \bullet f(?) = \sum u' f(s(u''))$, where s denotes the coinverse map. We proceed to show that $\alpha^*\Lambda$ is not projective as a $k[t]/t^p$ -module.

Let I be the augmentation ideal of the Hopf algebra $k\mathcal{E}$. Since $\alpha : k[t]/t^p \rightarrow k\mathcal{E}$ is flat, we conclude that $\alpha(t) \in I$ but $\alpha(t) \notin I^2$. The standard commutative diagrams

for the coproduct and counit maps (cf. [20, I.2.3]) imply that

$$\Delta(\alpha(t)) = 1 \otimes \alpha(t) + \alpha(t) \otimes 1 + \text{ terms from } I \otimes I.$$

We view elements from $k\mathcal{E} \otimes k\mathcal{E}$ as endomorphisms of Λ where the action of $u \otimes v \in k\mathcal{E} \otimes k\mathcal{E}$ on $f \in \text{End}_k(M, M)$ is given by $(u \otimes v) \bullet f(?) = uf(s(v)?)$. Note that if we precompose with $\Delta : k\mathcal{E} \rightarrow k\mathcal{E} \otimes k\mathcal{E}$, then this action is the action of $k\mathcal{E}$ on Λ explicitly described above. All endomorphisms of Λ coming from $k\mathcal{E} \otimes k\mathcal{E}$ commute and elements from $I \otimes k\mathcal{E}$ and $k\mathcal{E} \otimes I$ clearly yield p -nilpotent endomorphisms. Since any element from $I \otimes I$ can be written as a product of at least two elements of the form $u \otimes 1$ or $1 \otimes v$ with $u, v \in I$, we conclude that, as a Λ -endomorphism, $\Delta(\alpha(t)) - (1 \otimes \alpha(t) + \alpha(t) \otimes 1)$ is given by a polynomial without constant and linear terms on pairwise commuting p -nilpotent endomorphisms of Λ . Applying [15, 2.2], we conclude that $\alpha^*\Lambda$ is projective if and only if Λ is projective as a $k[t]/t^p$ -module with Λ equipped with the action of $k[t]/t^p$ defined via the action of $\alpha(t)$ on M with the coproduct Δ' on $k[t]/t^p$ given by $\Delta'(t) = 1 \otimes t + t \otimes 1$. With this new action of t , Λ becomes isomorphic, as a $k[t]/t^p$ -module, to $\text{End}_k(\alpha^*M, \alpha^*M)$. The latter is not projective, since α^*M is not projective. Hence, $\alpha^*\Lambda$ with the original action of t is not projective.

There is a canonical isomorphism $\text{Ext}_{\mathcal{E}}^*(M, M) \cong H^*(\mathcal{E}, \Lambda)$ compatible with the action of $H^*(\mathcal{E}, k)$. Thus, $\text{ann}_{H^\bullet(\mathcal{E}, k)}(\text{Ext}_{\mathcal{E}}^*(M, M)) = \text{ann}_{H^\bullet(\mathcal{E}, k)}(H^*(\mathcal{E}, \Lambda))$. Consider the following commutative diagram below, where vertical maps are induced by the natural \mathcal{E} -algebra map $k \rightarrow \Lambda$

$$\begin{array}{ccc} H^\bullet(\mathcal{E}, k) & \xrightarrow{\alpha^*} & H^\bullet(k[t]/t^p, k) \\ \downarrow & & \downarrow \\ H^*(\mathcal{E}, \Lambda) & \longrightarrow & H^*(k[t]/t^p, \alpha^*\Lambda). \end{array}$$

Since $\alpha^*\Lambda$ is not projective, the map $H^\bullet(k[t]/t^p, k) \rightarrow H^*(k[t]/t^p, \alpha^*\Lambda)$, induced by α , must be injective. Thus, $\text{ann}_{H^\bullet(\mathcal{E}, k)}(H^*(\mathcal{E}, \Lambda))$, which is the kernel of the left vertical arrow, is contained in the kernel of α^* . Since $H^\bullet(k[t]/t^p, k)$ is an integral domain, $\text{Ker } \alpha^*$ is a prime ideal, which allows us to conclude that $\sqrt{\text{ann}_{H^\bullet(\mathcal{E}, k)}(H^*(\mathcal{E}, \Lambda))} \subset \text{Ker } \alpha^*$. □

The following corollary is an elaboration of the ‘‘local projectivity test’’ (Theorem 5.3). Of course, we can not replace $\Pi(G)_M$ in Corollary 5.5 by $P(G)_M$ because any module M whose Π -support is non-empty but contains no p -points is not projective but satisfies $P(G)_M = \emptyset$.

Corollary 5.5. *Let G be a finite group scheme over a field k and M be a kG -module. The following are equivalent:*

- (1) M is projective,
- (2) $\Pi(G)_M = \emptyset$,
- (3) $\text{Proj } H^\bullet(G, k) / \text{ann}(\text{Ext}_G^*(M, M)) = \emptyset$

Proof. Theorem 5.3 implies the equivalence of (1) and (2), (1) clearly implies (3), and to finish the cycle we note that (3) implies (2) by Proposition 5.4. □

Π -supports satisfy the following functoriality properties with respect to change of finite group scheme.

Proposition 5.6. *Let $f : G' \rightarrow G$ be a flat map of finite group schemes over a field k . Then for any kG -module M ,*

$$\Pi(G')_{f^*M} = (f_*)^{-1}(\Pi(G)_M).$$

Let $\rho : \Pi(G_K) \rightarrow \Pi(G)$ be the map induced by a field extension K/k (as in Corollary 2.8). Then for any kG -module M ,

$$\Pi(G_K)_{M_K} = \rho^{-1}(\Pi(G)_M).$$

Furthermore, for any G_K -module N and any k -rational π -point $\alpha_k : k[t]/t^p \rightarrow kG$, $(K \otimes_k \alpha_k)^(N)$ is free if and only if $\alpha_k^*(N_{|G_k})$ is free.*

Proof. Let $\alpha : L[t]/t^p \rightarrow LG$ be a π -point of G . Then for a flat map $f : G' \rightarrow G$, $[\alpha] \in \Pi(G)_{f^*M}$ if and only if $\alpha^*(f^*M) = (f \circ \alpha)^*M$ is not projective if and only if $[\alpha] \in (f_*)^{-1}(\Pi(G')_M)$.

The second claim follows immediately from the fact that the map ρ is induced by the identity map on π -points of G defined over field extensions $L/K/k$. Namely, for such a π -point $\alpha_L : L[t]/t^p \rightarrow LG$ and a kG -module M , we have $[\alpha_L] \in \Pi(G_K)_{M_K}$ if and only if $\alpha_L^*(M_L)$ is not projective if and only if $[\alpha_L] \in \Pi(G)_M$.

For the last assertion, observe that $(K \otimes_k \alpha_k)(1 \otimes t) = \alpha_k(t)$ is a K -linear endomorphism of N . The freeness of N as either a $K \otimes_k k[t]/t^p$ or $k[t]/t^p$ -module is equivalent to the non-existence of some $n \in N$ with $tn = 0$ and n not divisible by t^{p-1} (using t to also denote $1 \otimes t$). □

The last assertion of Proposition 5.6 enables us to construct very explicit (but necessarily infinite dimensional) examples of G -modules with no closed points in their support.

Example 5.7. Take k to be algebraically closed and let K/k be a non-trivial field extension. Consider any finite group scheme G over k such that $\Pi(G)$ has dimension bigger than 0 and consider any K -rational point $[\alpha_K] \in \Pi(G_K)$ which maps to a non-closed point of $\Pi(G)$. Let N be a finite dimensional G_K -module with $\Pi(G_K)_N = \{[\alpha_K]\}$. Then the restriction of N to G , $N_{|G}$, is not projective but has the property that $\Pi(G)_{N_{|G}}$ contains no closed points of $\Pi(G)$.

One indication of the potential usefulness of the Π -support of a G -module M is that its dimension has a representation-theoretic interpretation. If M is finite dimensional, then the following proposition asserts that the closed subset $\Pi(G)_M$ has (Krull) dimension equal to the “complexity” of M (cf. [1]). If M is not finite dimensional, then $\Pi(G)_M \subset \Pi(G)$ need not be closed. Following [21], we define the *subset dimension* of $W \subset \Pi(G)$ as

$$\text{s. dim}(W) \stackrel{\text{def}}{=} \max_{s \in W} \dim(\bar{s}).$$

where \bar{s} denotes the closure of an arbitrary point $s \subset \Pi(G)$. As in [5], we define the complexity of an arbitrary kG -module M to be the smallest c such that M can be realized as a filtered colimit of finite-dimensional modules of complexity c .

Proposition 5.8. *Let G be a finite group scheme over a field k . Then for any kG -module M , the “subset dimension” of $\Pi(G)_M$ equals the complexity of M .*

Proof. This is proved exactly as in [21, 3.17], and we leave the transcription to the interested reader. □

6. TENSOR-IDEAL, THICK SUBCATEGORIES OF $stmod(G)$

In this section, we prove (in Theorem 6.3) the conjecture of Hovey, Palmieri, and Strickland [19] inspired by constructions of Benson, Carlson, and Rickard [7] for finite groups. In addition to the case of finite groups verified by [7], some special cases of Theorem 6.3 were proved by Hovey and Palmieri in [17], [18]. We also give an alternative description of the Π -support $\Pi(G)_M$ of a kG -module following a construction of Benson, Carlson, and Rickard for finite groups [6]. As we have throughout this paper, we work in the context of an arbitrary finite group scheme G over an arbitrary field k .

Let G be a finite group scheme over a field k and let $\text{Mod}(G)$ denote the abelian category of kG -modules. Recall that the stable module category $\text{StMod}(G)$ is the category whose objects are kG -modules, and whose group of homomorphisms between two kG -modules M, N is given by the following quotient:

$$\text{Hom}_G(M, N) / \{f : M \rightarrow N \text{ factoring through some projective}\}.$$

So defined, $\text{StMod}(G)$ is a triangulated category, with $M[1]$ represented by the cokernel of an embedding of M in an injective kG -module (i.e., $M[1] = \Omega^{-1}M$, where ΩM is the Heller shift of M , given as the kernel of a surjective map from a projective kG -module to M). Distinguished triangles come from exact sequences in $\text{Mod } G$.

We denote by $stmod(G) \subset \text{StMod}(G)$ the (triangulated) full subcategory of $\text{StMod}(G)$ whose objects are finite dimensional kG -modules. We shall say that kG -modules are *stably isomorphic* if they are isomorphic in $stmod(G)$.

We recall that a full subcategory \mathcal{C} of a triangulated category \mathcal{T} is said to be a *thick* subcategory if it is triangulated, closed under direct summands, and closed under arbitrary direct sums. Every thick subcategory of $stmod(G)$ is obtained by restricting some thick subcategory of $\text{StMod}(G)$ to its full subcategory of finite dimensional kG -modules. If \mathcal{T} has suitable (tensor) products (i.e., is symmetric monoidal), then a triangulated subcategory $\mathcal{C} \subset \mathcal{T}$ is said to be *tensor-ideal* if it is closed under taking tensor products with any element in \mathcal{T} .

Example 6.1. Let $C \subset \Pi(G)$ be a subset and let $\mathcal{C}_C \subset stmod(G)$ be the full subcategory of finite dimensional kG -modules M with $\Pi(G)_M \subset C$. Then Propositions 3.3 and 3.2 enable us to conclude that \mathcal{C}_C is a thick, tensor-ideal subcategory of $stmod(G)$.

Following Rickard [25], we associate to any thick, tensor-ideal subcategory $\mathcal{C} \subset stmod(G)$ (infinite dimensional) modules $E_{\mathcal{C}}, F_{\mathcal{C}}$ defined up to natural isomorphism with the following properties. Although these properties are stated for finite groups in [25] (cf. also [21] for connected finite group schemes), the proofs apply to any finite group scheme.

Proposition 6.2. *Let G be a finite group scheme over a field k . For each thick, tensor-ideal subcategory $\mathcal{C} \subset stmod(G)$ let $E_{\mathcal{C}}, F_{\mathcal{C}} \in StMod(G)$ denote the Rickard idempotents associated to \mathcal{C} as constructed in [25]. Then*

- (1) $E_{\mathcal{C}}, F_{\mathcal{C}}$ fit in a distinguished triangle in $StMod(G)$

$$E_{\mathcal{C}} \rightarrow k \rightarrow F_{\mathcal{C}} \rightarrow E_{\mathcal{C}}[1].$$

- (2) $E_{\mathcal{C}}$ is a filtered colimit of modules from \mathcal{C} and $F_{\mathcal{C}}$ is \mathcal{C} -local (i.e. there are no non-trivial maps $M \rightarrow F_{\mathcal{C}}$ in $StMod(G)$ whenever $M \in \mathcal{C}$).

- (3) For any $M \in \text{stmod}(G)$, $M \in \mathcal{C}$ if and only if M is stably isomorphic to $E_{\mathcal{C}} \otimes M$ if and only if $F_{\mathcal{C}} \otimes M$ is projective.
- (4) $E_{\mathcal{C}} \otimes E_{\mathcal{C}}$ is stably isomorphic to $E_{\mathcal{C}}$, $E_{\mathcal{C}} \otimes F_{\mathcal{C}}$ is projective, and $F_{\mathcal{C}} \otimes F_{\mathcal{C}}$ is stably isomorphic to $F_{\mathcal{C}}$.

A subset $W \subset \Pi(G)$ is *closed under specialization* if for any equivalence class of π -points $[\alpha] \in W$, W also contains the equivalence class of every specialization of α . Equivalently, W is closed under specialization if whenever a point lies in W then the closure of the point is contained in W . The following theorem gives a bijective correspondence between subsets of $\Pi(G)$ closed under specialization and thick tensor-ideal subcategories of $\text{stmod}(G)$. Since this correspondence clearly respects inclusions of subsets and subcategories, one could phrase the following theorem more elaborately in terms of lattices. This is the form in which Hovey-Palmieri-Strickland phrase their conjecture, which we now prove.

Observe that our proof of Theorem 6.3 requires in an essential way our consideration of arbitrary kG -modules and the properties given in Proposition 5.2.

Theorem 6.3. (Hovey-Palmieri-Strickland Conjecture) *Let G be a finite group scheme over a field k . Then there is a natural bijection between the subsets $W \subset \Pi(G)$ which are closed under specialization and the thick, tensor-ideal subcategories \mathcal{C} of $\text{stmod}(G)$.*

Namely, we associate to any subset $W \subset \Pi(G)$ the thick, tensor-ideal category $\mathcal{C}_W \subset \text{stmod}(G)$ of all finite dimensional modules M with $\Pi(G)_M \subset W$,

$$W \mapsto \mathcal{C}_W.$$

Moreover, we associate to any full subcategory $\mathcal{C} \subset \text{stmod}(G)$ the subset $W_{\mathcal{C}} \equiv \cup_{M \in \text{Obj}(\mathcal{C})} \Pi(G)_M$ closed under specialization,

$$\mathcal{C} \mapsto W_{\mathcal{C}}.$$

These constructions are mutually inverse when restricted to subsets $W \subset \Pi(G)$ closed under specialization and thick, tensor-ideal subcategories \mathcal{C} of $\text{stmod}(G)$.

Proof. For any $W \subset \Pi(G)$, $\mathcal{C}_W \subset \text{stmod}(G)$ is a thick, tensor-ideal category by Proposition 3.3 and Proposition 3.2. Moreover, if $\mathcal{C} \subset \text{stmod}(G)$ is a full subcategory, then the subset $\cup_{M \in \text{Obj}(\mathcal{C})} \Pi(G)_M \subset \Pi(G)$ is closed under specialization. We proceed to show that these correspondences are mutually inverse, using the Rickard idempotents of Proposition 6.2.

We first prove for any $W \subset \Pi(G)$ closed under specialization that $W = W_{\mathcal{C}_W}$. Essentially by definition, we have the containment $W_{\mathcal{C}_W} \subset W$ for any W . Conversely, any W closed under specialization is a (not necessarily finite) union of closed subsets, $W = \cup_i C_i$. By Corollary 4.1, we may find finite dimensional modules $M_{C_i} \in \mathcal{C}_W$ with $\Pi(G)_{M_{C_i}} = C_i$ so that $C_i \subset W_{\mathcal{C}_W}$, and thus $W = \cup_i C_i \subset W_{\mathcal{C}_W}$.

To complete the proof of the theorem, we show for any tensor-ideal thick subcategory $\mathcal{C} \subset \text{stmod}(G)$ that $\mathcal{C}_{W_{\mathcal{C}}} = \mathcal{C}$. Once again, one inclusion, namely $\mathcal{C} \subset \mathcal{C}_{W_{\mathcal{C}}}$, holds essentially by definition. The statement that $\mathcal{C}_{W_{\mathcal{C}}} \subset \mathcal{C}$ is equivalent to the statement that every finite dimensional kG -module M with $\Pi(G)_M \subset W_{\mathcal{C}}$ is contained in \mathcal{C} . We claim that

$$(6.3.1) \quad W_{\mathcal{C}} \subset \Pi(G)_{E_{\mathcal{C}}}.$$

Indeed, by Proposition 6.2.4, $M \otimes E_{\mathcal{C}}$ is stably isomorphic to M for any $M \in \mathcal{C}$. Thus, $\Pi(G)_M \cap \Pi(G)_{E_{\mathcal{C}}} = \Pi(G)_M$, which, in turn, implies that $\Pi(G)_M \subset \Pi_{E_{\mathcal{C}}}$. We conclude that 6.3.1 holds.

Thus, it suffices to consider a finite dimensional kG -module M satisfying $\Pi(G)_M \subset \Pi(G)_{E_{\mathcal{C}}}$ and verify that M is contained in \mathcal{C} . By Proposition 6.2.3,

$$\Pi(G)_{E_{\mathcal{C}}} \cap \Pi(G)_{F_{\mathcal{C}}} = \emptyset,$$

so that $\Pi(G)_M$ has empty intersection with $\Pi(G)_{F_{\mathcal{C}}}$. Thus, Theorem 5.3 in conjunction with the tensor product property (Proposition 3.2) implies that $M \otimes F_{\mathcal{C}}$ is projective. Consequently, Proposition 6.2.3 implies that $M \in \mathcal{C}$ as required. \square

As a corollary of Theorem 6.3 and a theorem of R. Thomason, we get the following suggestive bijection.

Corollary 6.4. *Let G be a finite group scheme over a field k of positive characteristic. Let $D^{perf}(\text{Proj}H^{\bullet}(G, k))$ be the full subcategory of perfect complexes in the derived category of coherent $\mathcal{O}_{\text{Proj}H^{\bullet}(G, k)}$ -modules, a tensor, triangulated category. Then there is an isomorphism between the lattice of thick, tensor-ideal subcategories of $\text{stmod}(G)$ and the lattice of thick, tensor-ideal subcategories of $D^{perf}(\text{Proj}H^{\bullet}(G, k))$.*

Proof. Theorem 6.3 establishes a bijection between the the lattice of thick, tensor-ideal subcategories of $\text{stmod}(G)$ and the lattice of subsets of $\Pi(G)$ which are closed under specialization whereas Thomason [29, 3.15] establishes a bijection between the latter lattice and the lattice of thick, tensor-ideal subcategories of $D^{perf}(\text{Proj}H^{\bullet}(G, k))$. \square

The ‘‘Rickard idempotents’’ of Proposition 6.2 enable us to realize *any* subset $S \subset \Pi(G)$ as the Π -support of some kG -module.

Definition 6.5. Let G be a finite group scheme over a field k of characteristic $p > 0$. For each equivalence class $[\alpha] \in \Pi(G)$, let $E_{[\alpha]}, F_{[\alpha]}$ be the Rickard idempotents associated to the thick, tensor-ideal subcategory $\mathcal{C}_{[\alpha]} \subset \text{stmod}(G)$ consisting of finite dimensional kG -modules whose Π -supports are contained in the closure of $[\alpha]$. Let $E_{\widetilde{[\alpha]}}, F_{\widetilde{[\alpha]}}$ be the Rickard idempotents associated to the thick, tensor-ideal subcategory $\mathcal{C}_{\widetilde{[\alpha]}} \subset \text{stmod}(G)$ consisting of finite dimensional kG -modules whose Π -supports are strictly contained in the closure of $[\alpha] \in \Pi(G)$ (i.e., do not contain $[\alpha]$). Finally, set

$$\kappa_{[\alpha]} \equiv E_{[\alpha]} \otimes F_{\widetilde{[\alpha]}}.$$

Proposition 6.6. *Let G be a finite group scheme over a field k , let $[\alpha] \in \Pi(G)$ be an equivalence class of π -points of G , and let $E_{[\alpha]}, F_{[\alpha]}, \kappa_{[\alpha]}$ be the kG -modules defined above. Then*

- (1) *The Π -support of $E_{[\alpha]}$ is the closure of $[\alpha] \in \Pi(G)$.*
- (2) *The Π -support of $F_{[\alpha]}$ is the complement in $\Pi(G)$ of the closure of $[\alpha]$.*
- (3) *The Π -support of $\kappa_{[\alpha]}$ equals $\{[\alpha]\}$.*

Proof. We first show for any closed under specialization subset $W \subset \Pi(G)$ with associated tensor-ideal thick subcategory $\mathcal{C} = \mathcal{C}_W$ that $\Pi(G)_{E_{\mathcal{C}}} = W$ and $\Pi(G)_{F_{\mathcal{C}}}$ is the complement of W .

Since W is closed under specialization, $W = \bigcup V_i$ where V_i are closed subsets of $\Pi(G)$. Let M_{V_i} be a finite dimensional kG -module with Π -support V_i . Since

$M_{V_i} \otimes E_C$ is stably isomorphic to M_{V_i} , the tensor product property implies the inclusion

$$V_i = \Pi(G)_{M_{V_i}} \subset \Pi(G)_{E_C}.$$

Thus, $W \subset \Pi(G)_{E_C}$. To prove the opposite inclusion, pick a π -point β which is not in W . Applying Proposition 6.2.2, we write $E = \operatorname{colim} M_i$ as a filtered colimit of finite dimensional modules M_i such that $\Pi(G)_{M_i} \subset W$. Since $\beta \notin W$, we conclude that $\beta^*(M_i)$ is projective for all M_i . Therefore, $\beta^*(E_C)$ is also projective. Thus, $[\beta] \notin \Pi(G)_{E_C}$ and the inclusion

$$\Pi(G)_{E_C} \subset W$$

follows.

Since $E_C \otimes F_C$ is projective, Proposition 3.2 implies that $\Pi(G)_{E_C} \cap \Pi(G)_{F_C} = \emptyset$ and, thus, $\Pi(G)_{F_C}$ is contained in the complement of W . On the other hand, Proposition 3.3 together with Proposition 6.2.1, imply the equality

$$\Pi(G)_{E_C} \cup \Pi(G)_{F_C} = \Pi(G).$$

Thus, $\Pi(G)_{F_C}$ is precisely the complement of W .

Now, (1) and (2) follow by applying the above to $W = \overline{[\alpha]}$, the closure $\{[\alpha]\} \subset \Pi(G)$. Applying the above argument to $W = \overline{[\alpha]} - [\alpha]$ in order to determine $\Pi(G)_{F_{\overline{[\alpha]}}}$ and using Proposition 3.2 again, we conclude (3). \square

The following is an immediate corollary of Proposition 6.6 together with the tensor product property.

Corollary 6.7. *Let G be a finite group scheme over a field k . Then for any subset $S \subset \Pi(G)$, there exists some kG -module M_S with Π -support equal to S ,*

$$\Pi(G)_{M_S} = S.$$

Namely, we may take

$$M_S = \bigoplus_{[\alpha] \in S} \kappa_{[\alpha]}.$$

Using κ -modules, one can provide an equivalent characterization of the Π -supports of a kG -module. This is an interpretation using π -points of the definition of Benson, Carlson, Rickard [6] of the support variety of an infinite dimensional module (for a finite group).

Proposition 6.8. *For any finite group scheme G over a field k and any equivalence class of π -points $[\alpha] \in \Pi(G)$,*

$$\Pi(G)_M = \{[\alpha] : \kappa_{[\alpha]} \otimes M \text{ is not projective}\}.$$

Proof. By Theorem 5.3, $\kappa_{[\alpha]} \otimes M$ is not projective if and only if the Π -support of $\kappa_{[\alpha]} \otimes M$ is non-empty which by Proposition 3.2 is the case if and only if the Π -supports of $\kappa_{[\alpha]}$ and M have non-empty intersection. Since $\Pi(G)_{\kappa_{[\alpha]}} = \{[\alpha]\}$ by Proposition 6.6.3, this is the case if and only if $[\alpha] \in \Pi(G)_M$. \square

Our final proposition verifies that the action on $\Pi(G)$ by an automorphism of k/F constructed in Proposition 3.9 naturally determines an action on $\Pi(G)_M$ provided that the kG -module M is obtained by base change from an FG_F -module. The existence of such an action is therefore an obstruction to descending the kG -module structure on M to an FG_F -module structure.

Proposition 6.9. *Let k/F be a field extension and $\sigma : k \rightarrow k$ a field automorphism over F . Assume that the finite group scheme G over k is defined over F , so that $G = G_F \times_{\text{Spec } F} \text{Spec } k$.*

- (1) *If M is a kG -module defined over F , then the action of σ stabilizes $\Pi(G)_M$.*
(2) *If k/F is a finite Galois extension with Galois group τ and if C is a subset of $\Pi(G)$ of the form $\Pi(G)_M$ for some kG -module M , then there exists an FG_F -module N with the property that $\Pi(G)_{N_k}$ is the closure of C under the action of τ . If C is closed, we may choose N to be finite dimensional.*

Proof. The first statement follows immediately from the second part of Proposition 4.4.

If V is a k -vector space and if $\sigma \in \tau$, we define a new k -vector space V^σ by

$$V^\sigma \equiv k \otimes_\sigma V,$$

where the tensor product $k \otimes_\sigma V$ is taken by viewing k as a k -module via σ . Equivalently, V coincides with V^σ as an abelian group but the action of k is twisted by σ^{-1} : $a \circ (1 \otimes_\sigma v) = a \otimes_\sigma v = 1 \otimes_\sigma \sigma^{-1}(a)v$. Since the group G is defined over F , the algebra $kG = k \otimes_F FG$ can be naturally identified with $kG^\sigma = k \otimes_\sigma k \otimes_F FG$ via the k -algebra isomorphism

$$(6.9.1) \quad kG = k \otimes_F FG \simeq k \otimes_\sigma k \otimes_F FG = kG^\sigma$$

$$a \otimes f \mapsto a \otimes 1 \otimes f.$$

For a kG -module M , the twisted module M^σ has a natural structure of a kG^σ -module: $kG^\sigma \otimes M^\sigma = (kG \otimes M)^\sigma \rightarrow M^\sigma$. We consider M^σ as a G -module via the algebra identification 6.9.1.

Let $C = \Pi(G)_M$ for some kG -module M . Let $\widetilde{M} = k \otimes_F (M \downarrow_{G_F})$. There is an isomorphism of kG -modules

$$(6.9.2) \quad \widetilde{M} \simeq \bigoplus_{\sigma \in \tau} M^\sigma,$$

given explicitly by

$$a \otimes m \mapsto (a \otimes_\sigma m)_{\sigma \in \tau}.$$

Indeed, one readily observes that $k \otimes_F k \rightarrow \bigoplus_{\sigma \in \tau} k^\sigma$ is a k -linear isomorphism: if $\{\alpha_\sigma\}_{\sigma \in \tau}$ is a basis of k over F , then the elements $(1 \otimes_\sigma \sigma'(\alpha_\sigma))_{\sigma \in \tau} \in \bigoplus_{\sigma \in \tau} k^\sigma$ indexed by $\sigma' \in \tau$, form a basis of $\bigoplus_{\sigma \in \tau} k^\sigma$ and are in the image of the map above. To verify isomorphism 6.9.2 for a general module M , we tensor $k \otimes_F k \simeq \bigoplus_{\sigma \in \tau} k^\sigma$ with M and observe that $k^\sigma \otimes_k M = k \otimes_\sigma k \otimes_k M = k \otimes_\sigma M = M^\sigma$.

We proceed to verify that

$$(\Pi(G)_M)^\sigma = \Pi(G)_{M^{\sigma^{-1}}},$$

i.e. that for a π -point $\alpha_K : K[t]/t^p \rightarrow KG_K$, $\alpha_K^*((M^{\sigma^{-1}})_K)$ is projective if and only if $(\alpha_K^\sigma)^*(M_K)$ is projective. By enlarging the field K if necessary, we assume that σ extends to an automorphism of K which we denote by $\tilde{\sigma}$. Let $\alpha_K(t) = \sum_i a_i t_i$ where $a_i \in K$ and t_i are generators of the algebra FG_F over F . Formula 4.4.2 implies that t acts on the $K[t]/t^p$ module $(\alpha_K^\sigma)^*(M_K)$ via $\alpha_K^\sigma(t) = \sum_i \tilde{\sigma}(a_i) t_i$. As

the action of $\alpha_K(t) = \sum_i a_i t_i$ on $M_K^{\sigma^{-1}} = (M_K)^{\sigma^{-1}}$ is the same as the action of $\sum_i \tilde{\sigma}(a_i) t_i$ on M_K , we conclude the desired equality $(\Pi(G)_M)^\sigma = \Pi(G)_{M^{\sigma^{-1}}}$. Thus, Isomorphism (6.9.2) implies that

$$\Pi(G)_{\tilde{M}} = \bigcup_{\sigma \in \tau} (\Pi(G)_M)^\sigma \equiv \tau(\Pi(G)_M).$$

Thus, we have shown for $N = M \downarrow_{G_F}$ that $\Pi(G)_{N_k} = \Pi(G)_{\tilde{M}}$ is the closure of $C = \Pi(G)_M$ with respect to the action of τ . By definition, if C is closed, then M can be chosen to be finite dimensional, and, therefore, N will also be finite dimensional. □

Corollary 6.7 implies that any subset of $\Pi(G)$ is realizable as a support set of some G -module M . If a subset is closed, then by definition it is realizable by a finite-dimensional module. Thus, the proposition above immediately implies the following “realization statement”.

Corollary 6.10. *Let k/F be a finite Galois field extension, and $C \subset \Pi(G)$ be a (closed) subset stable under the action of $\text{Gal}(k/F)$. Then there exists a (finite-dimensional) FG_F -module N such that $\Pi(G)_{N_k} = C$.*

7. REALIZATION OF THE SCHEME STRUCTURE FOR $\Pi(G)$

In this final section, we verify that we can endow the topological space $\Pi(G)$ with a sheaf of k -algebras determined by the stable module category $\text{stmod}(G)$ so that the associated ringed space is isomorphic to the scheme $\text{Proj } H^\bullet(G, k)$.

As usual, G will denote a finite group scheme over a field k of positive characteristic. We shall frequently make the identification

$$H^i(G, k) \simeq \text{Hom}_G(\Omega^i k, k) \simeq \text{Hom}_{\text{stmod}(G)}(\Omega^{i+j} k, \Omega^j k),$$

and we shall use the same notation α for a cohomology class in $H^i(G, k)$ and any G -map $\Omega^{i+j} \rightarrow \Omega^j k$ whose stable equivalence class represents this cohomology class.

Proposition 7.1. *Let $\zeta \in H^\bullet(G, k)$ be a homogeneous cohomology class of positive degree, let $W(\zeta)$ denote the closed subset $\Pi(G)_{L_\zeta} \subset \Pi(G)$, and let $\mathcal{C}_{W(\zeta)} \subset \text{stmod}(G) \equiv \mathcal{C}$ be the associated thick, tensor-ideal subcategory of the stable module category. Then there is a naturally defined map map of k -algebras*

$$(7.1.1) \quad \theta_\zeta : (H^\bullet(G, k)[1/\zeta])_0 \rightarrow \text{End}_{\mathcal{C}/\mathcal{C}_{W(\zeta)}}(k).$$

Here, $(H^\bullet(G, k)[1/\zeta])_0 \subset H^\bullet(G, k)[1/\zeta]$ is the subalgebra of elements of degree 0 in the graded algebra $H^\bullet(G, k)[1/\zeta]$.

Moreover, if $\eta \in H^\bullet(G, k)$ is another homogeneous cohomology class of positive degree, then θ_ζ and $\theta_{\zeta \cdot \eta}$ fit in a commutative square

$$(7.1.2) \quad \begin{array}{ccc} (H^\bullet(G, k)[1/\zeta])_0 & \xrightarrow{\theta_\zeta} & \text{End}_{\mathcal{C}/\mathcal{C}_{W(\zeta)}}(k) \\ \downarrow & & \downarrow \\ (H^\bullet(G, k)[1/\zeta \cdot \eta])_0 & \xrightarrow{\theta_{\zeta \cdot \eta}} & \text{End}_{\mathcal{C}/\mathcal{C}_{W(\zeta \cdot \eta)}}(k) \end{array}$$

whose vertical maps are the natural localization maps.

Proof. Let n denote the degree of ζ (so that n is even provided that $p \neq 2$). For $\alpha \in H^n(G, k)$, we define $\theta_\zeta(\alpha/\zeta^j)$ to be the endomorphism of k in the localized category $\mathcal{C}/\mathcal{C}_{W(\zeta)}$

$$\theta_\zeta(\alpha/\zeta^j) = k \xleftarrow{\zeta^j} \Omega^{jn} k \xrightarrow{\alpha} k.$$

To verify that θ_ζ is well defined, we must verify that if $\zeta^{m'} \cdot \beta = \zeta^m \cdot \alpha \in H^{n(m'+j)}(G, k) = H^{n(m+j)}(G, k)$, then $\theta_\zeta(\alpha/\zeta^{m+j}) = \theta_\zeta(\beta/\zeta^{m+j})$. This follows immediately from the equivalence relation describing morphisms in $\mathcal{C}/\mathcal{C}_{W(\zeta)}$ together with the existence of the commutative diagram in $stmod(G)$:

$$\begin{array}{ccccc} & & \Omega^{nj'} k & & \\ & \nearrow^{\zeta^{j'}} & \uparrow^{\zeta^{m'}} & \searrow^{\beta} & \\ k & \xleftarrow{\zeta^{j+m}} & \Omega^{n(j+m)} k & \xrightarrow{\zeta^m \alpha} & k \\ & \searrow_{\zeta^j} & \downarrow_{\zeta^m} & \nearrow_{\alpha} & \\ & & \Omega^{nj} k & & \end{array}$$

The commutativity of (7.1.2) follows immediately from the definition of θ_ζ .

The additivity of θ_ζ is evident. To show multiplicativity, we compare the diagram

$$\begin{array}{ccccc} k & \xleftarrow{\zeta^{j+j'}} & \Omega^{(j+j')n} k & \xrightarrow{\alpha\beta} & k \\ & \searrow_{\zeta^{j'}} & \swarrow_{\zeta^j} & \searrow_{\beta} & \nearrow_{\alpha} \\ & & \Omega^{j'n} k & & \Omega^{jn} k \end{array}$$

exhibiting composition in $(H^\bullet(G, k)[1/\zeta])_0$ with the diagram

$$\begin{array}{ccccc} \Omega^{(j+j')n} k & \xrightarrow{\beta} & \Omega^{jn} k & \xrightarrow{\alpha} & k \\ \downarrow_{\zeta^j} & & \downarrow_{\zeta^j} & & \\ k & \xleftarrow{\zeta^{j'}} & \Omega^{j'n} k & \xrightarrow{\beta} & k \end{array}$$

exhibiting composition in $End_{\mathcal{C}/\mathcal{C}_{W(\zeta)}}(k)$. □

The following proposition relies on a result of J. Carlson, P. Donovan, and W. Wheeler [10, 3.1] which is stated for finite groups but whose proof applies verbatim to any finite group scheme.

Proposition 7.2. *As in Proposition 7.1, let $\zeta \in H^n(G, k)$ be a homogeneous cohomology class of positive degree. Then the map θ_ζ of (7.1.1) is an isomorphism.*

Proof. By [10, 3.1], given any map $k \xleftarrow{s} M \xrightarrow{f} k$ in $End_{\mathcal{C}/\mathcal{C}_{W(\zeta)}}(k)$, we may find some $g : \Omega^{jn} k \rightarrow M$ and a commutative diagram

$$\begin{array}{ccccc}
 k & \xleftarrow{\zeta^j} & \Omega^{jn}k & \xrightarrow{\alpha} & k \\
 \parallel & & \downarrow g & & \parallel \\
 k & \xleftarrow{s} & M & \xrightarrow{f} & k
 \end{array}$$

Thus, θ_ζ is surjective.

To prove injectivity, suppose that $\theta_\zeta(\alpha/\zeta^j) = 0$; in other words, suppose we have a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega^{j'n}k & & \\
 & \delta & \uparrow & 0 & \\
 k & \xleftarrow{\xi} & X & \xrightarrow{\gamma} & k \\
 & \zeta^j & \downarrow h & \alpha & \\
 & & \Omega^{jn}k & &
 \end{array}$$

Then $\gamma = 0$. Using [10, 3.1] once again, we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega^{mn}k & & \\
 & \zeta^m & \downarrow g & 0 & \\
 & \xi & X & \downarrow h & 0 \\
 k & \xleftarrow{\zeta^j} & \Omega^{jn}k & \xrightarrow{\alpha} & k
 \end{array}$$

By possibly replacing m by some $m' > m$, we may assume that $m > j$. We conclude from the commutative diagram above that $\zeta^j \circ h \circ g = \zeta^m$. Reinterpreting $h \circ g$ as a cohomology class in degree $n(m - j)$ via the isomorphism $H^{mj}(G, k) \simeq \text{Hom}_{\text{stmod}(G)}(\Omega^{mn}k, \Omega^{mj}k)$, we may rewrite the equality above as $\zeta^j \cdot (h \circ g) = \zeta^m$. Using the diagram again, we obtain $\alpha \cdot (h \circ g) = 0$. Multiplying by ζ^j , we get $\alpha \cdot \zeta^m = 0$. Thus, $\alpha/\zeta^j = 0$ in $H^\bullet(G, k)[1/\zeta]$. \square

In some sense, the following theorem is the ultimate generalization and refinement of ‘‘Carlson’s Conjecture’’ which proposed the comparison of rank varieties and cohomological support varieties for kE -modules, where k was assumed to be algebraically closed of characteristic p and E an elementary abelian p -group.

Theorem 7.3. *Let G be a finite group scheme over a field k of positive characteristic. Consider the presheaf of commutative k -algebras $\Theta_{\Pi(G)}$ on the topological space $\Pi(G)$ defined on the complement $(\Pi(G) - W)$ of a closed subset $W \subset G$ by*

$$(\Pi(G) - W) \mapsto \text{End}_{\mathcal{C}/\mathcal{C}_W}(k)$$

and whose restriction maps are the evident localization maps. Then the presheaf Θ_G and its associated sheaf $\tilde{\Theta}_{\Pi(G)}$ take the same value on basic open subsets of the form $(\Pi(G) - W(\zeta))$, where $W(\zeta) = \Pi(G)_{L_\zeta} \subset \Pi(G)$,

$$(7.3.1) \quad \tilde{\Theta}_{\Pi(G)}(\Pi(G) - W(\zeta)) = \text{End}_{\mathcal{C}/\mathcal{C}_{W(\zeta)}}(k).$$

Moreover, the homeomorphism Ψ_G of Theorem 3.6 determines an isomorphism of ringed spaces

$$\Psi_G : (\Pi(G), \tilde{\Theta}_{\Pi(G)}) \xrightarrow{\sim} \text{Proj } H^\bullet(G, k).$$

Proof. As shown in the proof of Theorem 3.6 , $\Psi_G^{-1}(V(\zeta)) = \Pi(G)_{L_\zeta} = W(\zeta) \subset \Pi(G)$. The commutativity of (7.1.2) and the fact proved in Proposition 7.2 that the maps (7.1.1) are isomorphisms imply that these maps determine an isomorphism of presheaves

$$(7.3.2) \quad \Psi_G^*(\mathcal{O}_{\text{Proj } H^\bullet(G,k)}) \xrightarrow{\sim} \Theta_{\Pi(G)}$$

restricted to basic opens subsets. Consequently, (7.3.2) induces a map of associated sheaves which is an isomorphism on every stalk and hence an isomorphism

$$(7.3.3) \quad \Psi_G^*(\mathcal{O}_{\text{Proj } H^\bullet(G,k)}) \xrightarrow{\sim} \tilde{\Theta}_{\Pi(G)}.$$

In particular, (7.3.3) implies the isomorphisms

$$\Psi_G^*(\mathcal{O}_{\text{Proj } H^\bullet(G,k)})(\Pi(G) - W(\zeta)) \xrightarrow{\sim} \tilde{\Theta}_{\Pi(G)}(\Pi(G) - W(\zeta)) = \Theta_{\Pi(G)}(\Pi(G) - W(\zeta)).$$

Since Ψ_G is a homeomorphism, we conclude immediately that $\Psi_G : (\Pi(G), \Theta_{\Pi(G)}) \rightarrow \text{Proj } H^\bullet(G, k)$ is an isomorphism of ringed spaces. \square

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