GENERIC AND MAXIMAL JORDAN TYPES

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Abstract. For a finite group scheme G over a field k of characteristic $p > 0$, we associate new invariants to a finite dimensional kG-module M. Namely, for each generic point of the projectivized cohomological variety $\operatorname{Proj} H^\bullet(G,k)$ we exhibit a "generic Jordan type" of M. In the very special case in which $G = E$ is an elementary abelian p-group, our construction specializes to the non-trivial observation that the Jordan type obtained by restricting M via a generic cyclic shifted subgroup does not depend upon a choice of generators for E. Furthermore, we construct the non-maximal support variety $\Gamma(G)_M$, a closed subset of $\operatorname{Proj} H^\bullet(G,k)$ which is proper even when the dimension of M is not divisible by p.

0. Introduction

Elementary abelian p-subgroups of a finite group G capture significant aspects of the cohomology and representation theory of G. For example, if k is a field of characteristic $p > 0$, then a theorem of D. Quillen [17] asserts that the Krull dimension of the cohomology algebra $H^\bullet(G,k)$ is equal to the maximum of the ranks of elementary abelian p-subgroups of G and a theorem of L. Chouinard [6] asserts that a kG-module is projective if and only if its restrictions to all elementary abelian p-subgroups of G is projective. The cohomology algebra $H^\bullet(E,k)$ of an elementary abelian p-group E is easily computed, whereas kE is of wild representation type provided that the rank of E is at least 2 (at least 3, for $p = 2$). Nonetheless, Chouinard's theorem and Quillen's geometric description of $\operatorname{Spec} H^\bullet(G,k)$ [17] provide the basis for interesting invariants of kG-modules, most notably the cohomological support variety $|G|_M$ of a kG-module M.

The investigation of the cohomology and representation theory of finite groups is naturally extended to other algebraic structures. Friedlander and B. Parshall developed a theory of support varieties for finite dimensional p-restricted Lie algebras g over a field k of characteristic $p > 0$ (e.g., [9]). For restricted Lie algebras, the role of the group algebra kG of the finite group G is played by the restricted enveloping algebra $u(g)$. Indeed, restricted Lie algebras lead one to more interesting geometrical structures than do finite groups, and seemingly lead to stronger results. For example, the theorem of G. Avrunin and L. Scott [1] identifying the cohomological support variety $|E|_M$ of a finite dimensional kE module M for an elementary abelian p-group E with the rank variety of J. Carlson [3] admits a formulation in the case of a restricted Lie algebras g in terms of closed subvarieties of the p-nilpotent cone of g (cf. [9], [14], [19]).

Finite groups and finite dimensional p-restricted Lie algebras are examples of finite group schemes. The representation theory of a finite group scheme is the

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study of kG-modules, where kG is the group algebra of G defined to be the k-linear dual of the representing Hopf algebra $k[G]$. A fundamental theorem of Friedlander and Suslin [13] is that the cohomology algebra $H^*(G,k)$ is finitely generated for any such finite group scheme.

If $\mathcal{G}$ is an affine algebraic group over k, then the r^{th} Frobenius kernel $G = \mathcal{G}_{(r)} \equiv ker\{F^r : \mathcal{G} \to \mathcal{G}^{(r)}\}$ is an infinitesimal group scheme (i.e. a connected finite group scheme). Such Frobenius kernels play an important role in the study of (rational) representations of $\mathcal{G}$: the family $\{\mathcal{G}_{(r)};\ r > 0\}$ faithfully captures the representation theory of $\mathcal{G}$. In [18], [19], Suslin, Friedlander, and C. Bendel extend to all infinitesimal group schemes G earlier work on the cohomology and support varieties of finite dimensional p-restricted Lie algebras (whose restricted enveloping algebras are the group algebras of infinitesimal group schemes of height 1).

In [10], [12], Friedlander and Pevtsova present a uniform approach to the study of the cohomology and related representation theory of all finite group schemes. This approach involves the use of π-points of G, which are finite flat maps of K-algebras $K[t]/t^p \to KG$ for field extensions K/k; these play the role of "cyclic shifted subgroups" in the case that G is an elementary abelian p-group and the role of 1-parameter subgroups in the case that G is an infinitesimal group scheme. In [12], the space $\Pi(G)$ of equivalence classes $[\alpha_K]$ of π-points $\alpha_K : K[t]/t^p \to KG$ of G is given a scheme structure without reference to cohomology such that $\Pi(G)$ is isomorphic as a scheme to $\operatorname{Proj} H^\bullet(G,k)$. In particular, there is a natural bijection between such equivalence classes of π-points and homogeneous prime ideals of $H^\bullet(G,k)$. The perspective of π-points leads to the considerations of this present paper.

As was first suggested in [8], this representation-theoretic interpretation of support varieties can be refined to determine invariants of kG-modules which provide more detailed information than that given by support varieties. Namely, for any π-point $\alpha_K : K[t]/t^p \to KG$, one can consider the Jordan type of $\alpha_K(t)$ as a nilpotent operator on M_K. As one quickly discovers, the answer may vary considerably depending on the representative α_K of an equivalence class of π-points in $\Pi(G)$. Even in the case of $G = \mathbb{Z}/p \times \mathbb{Z}/p$ the answer depends on the choice of generators of E which in turn determines the choice of representative of a π-point (cf. Example 2.3).

The following theorem (cf. Theorem 4.2) provides a new intrinsic invariant of modular representations of finite group schemes. In particular, in the case of an elementary abelian group, this invariant is independent of the choice of generators.

g-intro **Theorem 0.1.** *Let G be a finite group scheme, let M be a finite dimensional G-module and let α_K be a π-point of G which represents a generic point $[\alpha_K] \in \Pi(G)$. Then the Jordan type of $\alpha_K(t)$ viewed as a nilpotent operator on M_K depends only upon $[\alpha_K]$ and not the choice of α_K representing $[\alpha_K]$.*

For a given generic point $[\alpha_K] \in \Pi(G)$ and a given finite dimensional kG-module M, we call the Jordan type of $\alpha_K(t)$ on M_K the *generic Jordan type* of M (at $[\alpha_K]$). As we verify in Proposition 4.7, sending a module M to its generic Jordan type $[\alpha_K]^*(M_K)$ for generic $[\alpha_K] \in \Pi(G)$ determines a well defined tensor triangulated functor on stable module categories $[\alpha_K]^* : stmod(kG) \to stmod(K[t]/t^p)$. In other words, generic Jordan type is well-behaved with respect to short exact sequences, direct sums, tensor products, and Heller shifts.

We establish Theorem 0.1 by successively considering more general finite group schemes, beginning with elementary abelian p-groups and then abelian finite group schemes in §2 and then infinitesimal group schemes in §3 before treating the general case in §4. The following theorem (cf. Theorem 1.12) concerning commuting nilpotent matrices provides the key to our proof of Theorem 0.1.

Theorem 0.2. *Let k be an infinite field and let $\alpha, \alpha_1, \dots, \alpha_n$ be a set of commuting nilpotent $n \times n$ matrices such that the Jordan type of α is greater or equal to the Jordan type of any $\alpha + c_1\alpha_1 + \cdots + c_n\alpha_n$ for all $c_i \in k$. Then the Jordan type of α is greater or equal to the Jordan type of any polynomial in $\alpha, \alpha_1, \dots, \alpha_n$ without constant term.*

For a given kG-module M, some but not necessarily all of the generic Jordan types are maximal, but maximal Jordan types are also realized at non-generic points. The locus of non-maximal Jordan types for a given finite dimensional kG-module M provides us with a geometric invariant $\Gamma(G)_M$ which agrees with the support variety of M if and only if the maximal type of M is projective. (cf. Theorems 4.10, 5.2).

Theorem 0.3. *Let G be a finite group scheme over k and M a finite dimensional kG-module. Then whether or not $\alpha_K^*(M_K)$ is maximal depends only upon the equivalence class of α_K, $[\alpha_K] \in \Pi(G)$. The set of those equivalence classes of π-points $\alpha_K : K[t]/t^p \to KG$ such that $\alpha_K^*(M)$ is not of maximal type is a closed subvariety*

$$\Gamma(G)_M \ \subset \ \Pi(G)$$

of $\Pi(G) \simeq \mathrm{Proj}\, H^\bullet(G, k)$.

The prototype of our *non-maximal support variety* $\Gamma(G)_M$ was introduced by W. Wheeler ([21]) for elementary abelian p-groups. Our construction significantly strengthens Wheeler's result even in the case of an elementary abelian p-group E, since we do not rely upon a fixed set of generators for E.

The reader will find various examples throughout the paper intended to illustrate some of these new invariants and their behavior. More sophisticated examples which reflect deeper properties of the representation theory of G are considered in [4].

In §1, k will denote an infinite field and in subsequent sections k will denote an arbirary field of characteristic $p > 0$. If M is a k-vector space and K/k is a field extension, we use the notation M_K to denote $M \otimes_k K$. If M is a finite dimensional k vector space equipped with the structure of a $k[t]/t^p$-module, then the data of the block sizes of M as a $k[t]/t^p$-module (or, equivalently, of the Jordan form of t viewed as an endomorphism of M) will be called the *Jordan type* of M. In particular, the Jordan type of the $k[t]/t^p$-module M is the same as that of the $K[t]/t^p$-module M_K for any field extension K/k. We employ the notation $H^\bullet(G, k)$ for the commutative k-algebra given as the full cohomology algebra $H^*(G, k)$ of G if $\mathrm{char}\,(k) = 2$ and for the subalgebra of $H^*(G, k)$ generated by classes homogeneous of even degree if $p \neq 2$.

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1. Maximality for commuting nilpotent matrices

In this section, we consider a finite set of pair-wise commuting nilpotent matrices $\alpha_1, \ldots, \alpha_n$ with coefficients in an infinite field k. Theorem 1.12 establishes that maximality of the Jordan type among all polynomials in the α_i' s is realized by some linear combination of the α_i' s. This theorem is the essential step in our investigation in later sections of maximality of Jordan types associated to finite dimensional representations of finite group schemes.

Given two partitions $\underline{n} = n_1 \geq n_2 \ldots, \geq n_N$, and $\underline{m} = m_1 \geq m_2 \ldots, \geq m_N$ of N we say that $\underline{n}$ dominates $\underline{m}$, or $\underline{n} \geq \underline{m}$, if

$$\sum_{1 \leq j \leq k} n_j \geq \sum_{1 \leq j \leq k} m_j \quad \text{for } 1 \leq k \leq N$$

(see [7, 6.2.1]).

Definition 1.1. Let $\alpha \in M_N(k)$ be an $N \times N$ nilpotent matrix with coefficients in k. The Jordan type of α is the partition of N, $\underline{n} = n_1 \geq n_2 \geq \cdots \geq n_s$ with $\sum_j n_j = N$, such that the canonical Jordan form of α is a direct sum of indecomposable blocks of size n_j.

If $\alpha, \beta \in M_N(k)$ are two nilpotent matrices, we write $\alpha \geq \beta$ (and say that the Jordan type of α is greater or equal to the Jordan type of β) if the partition associated to α is greater or equal to the partition associated to β. The condition that $\alpha \geq \beta$ is equivalent to the condition that

ordering

$$\text{rank}\,(\alpha^s) \geq \text{rank}\,(\beta^s), \quad \forall s > 0, \tag{1.1.1}$$

(see [7, 6.2.2]) which in turn is equivalent to the condition that

$$dim_k \text{Ker}\,(\alpha^s) \leq dim_k \text{Ker}\,(\beta^s), \quad \forall s > 0.$$

We denote by $\sim$ the equivalence relation associated to this partial ordering:

$$\alpha \sim \beta \quad \text{iff} \quad \alpha \leq \beta \text{ and } \beta \leq \alpha.$$

Observe that $\alpha \sim \beta$ if and only if α and β have the same Jordan type if and only if they are conjugate (under the adjoint action of $GL_N(k)$) elements of $M_N(k)$.

The data of the Jordan type of a nilpotent matrix $\alpha \in M_N(k)$ is equivalent to each of the following:

- the isomorphism class of k^N as a $k[t]$-module, with t acting as α;
- the data of the ranks of $\alpha^s, \forall s > 0$;
- the data of the dimensions of the kernels of $\alpha^s, \forall s > 0$.

If M is an N-dimensional $k[t]$-module with t acting nilpotently, then we shall frequently refer to the "Jordan type of M" rather than the Jordan type of the matrix in $M_N(k)$ given by the action of t.

rkmax-equiv

Lemma 1.2. *Let $\alpha, \alpha_1, \cdots, \alpha_n \in M_N(k)$ be a family of commuting nilpotent matrices with coefficients in an infinite field k. The following conditions are equivalent:*

(1) *For all $\lambda_1, \cdots, \lambda_n \in k$,*

$$\textit{rank}\,(\alpha) \quad \geq \quad \textit{rank}\,(\alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n),$$

where $\alpha, \alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ are viewed as matrices in $M_N(k)$.

(2) *For every field extension K/k, all $\lambda_1, \cdots, \lambda_n \in K$,*

$$rank(\alpha) \quad \geq \quad rank(\alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n),$$

where $\alpha, \alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ are viewed as matrices in $M_N(K)$.

(3) *For $t_1, \ldots, t_n$ independent variables in the field $k(t_1, \ldots, t_n)$ of rational functions in n variables over k,*

$$rank(\alpha) = rank(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n).$$

Proof. For any $A \in M_N(k)$ and any pair of subsets $I, J \subset \{1, \ldots, N\}$ of cardinality $s \leq N$, denote by $A_{I,J}$ the associated minor, an $s \times s$ submatrix of A. The condition $\operatorname{rank} A \geq s$ is equivalent to the condition that there exist I and J such that $det(\mathcal{A}_{I,J}) \neq 0$. For any I and J as above, consider

$$d_{I,J}(t_1, \ldots, t_n) = det((\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n)_{I,J})$$

where $t_1, \ldots, t_n$ are independent variables in the field $k(t_1, \ldots, t_n)$. Let K/k be an arbitrary field extension. Then $d_{I,J}(t_1, \ldots, t_n) = 0$ if and only if $d_{I,J}(\lambda_1, \ldots, \lambda_n) = 0$ for almost all $\lambda_1, \ldots, \lambda_n \in K$. Thus,

$$\operatorname{rank}(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n) \geq \operatorname{rank}(\alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n) \quad \text{for all } \lambda_1, ..., \lambda_n \in K$$

$$\operatorname{rank}(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n) = \operatorname{rank}(\alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n) \quad \text{for almost all } \lambda_1, ..., \lambda_n \in K.$$

□

Since the argument of the proof of Lemma 1.2 applies equally to all powers of α, we immediately conclude the following proposition.

max-equiv **Proposition 1.3.** *Let $\alpha, \alpha_1, \cdots, \alpha_n \in M_N(k)$ be a family of commuting nilpotent matrices with coefficients in an infinite field k. Then the following conditions are equivalent:*

(1) *For all $\lambda_1, \cdots, \lambda_n \in k$,*

$$\alpha \quad \geq \quad \alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n.$$

(2) *For every field extensions K/k, all $\lambda_1, \cdots, \lambda_n \in K$,*

$$\alpha \quad \geq \quad \alpha + \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n.$$

(3) *For $t_1, \ldots, t_n$ independent variables in the field $k(t_1, \ldots, t_n)$ of rational functions in n variables over k,*

$$\alpha \sim \alpha + t_1\alpha_1 + \cdots + t_n\alpha_n$$

in $M_N(k(t_1, \ldots, t_n))$.

Definition 1.4. We say that α is *rank maximal* with respect to $\alpha_1, \ldots, \alpha_n$ provided that the equivalent conditions of Lemma 1.2 are satisfied for $\alpha, \alpha_1, \cdots, \alpha_n \in M_N(k)$.

Similarly, we say that α is *maximal* with respect to $\alpha_1, \ldots, \alpha_n$ if the equivalent conditions of Proposition 1.3 are satisfied for $\alpha, \alpha_1, \cdots, \alpha_n \in M_N(k)$.

existence **Remark 1.5.** Let $\alpha_1, \ldots, \alpha_n \in M_N(k)$ be a family of nilpotent commuting matrices. There exists $\alpha \in M_N(k)$ satisfying the conditions of Proposition 1.3. Indeed, let $s_1, \ldots, s_n$ be a set of indeterminants. Let

$$\alpha_{\underline{s}} = s_1\alpha_1 + \cdots + s_n\alpha_n \in M_N(k(s_1, \ldots, s_n)).$$

Observe that $\alpha_{\underline{s}} = s_1\alpha_1 + \cdots + s_n\alpha_n \sim (s_1+t_1)\alpha_1 + \cdots + (s_n+t_n)\alpha_n = \alpha_{\underline{s}} + t_1\alpha_1 + \cdots + t_n\alpha_n$ as elements of $M_N(k(s_1, \ldots, s_n, t_1, \ldots, t_n))$. Indeed, $(s_1 + t_1)\alpha_1 + \cdots + (s_n +$

$t_n)\alpha_n$ can be specialized to $s_1\alpha_1+\cdots+s_n\alpha_n$ by setting $t_i=0$. On the other hand, if one makes a change of variables $s_i'=s_i+t_i$, then $\alpha_s=(s_1'-t_1)\alpha_1+\cdots+(s_n'-t_n)\alpha_n$ can be specialized to $\alpha_{\underline{s}}+t_1\alpha_1+\cdots+t_n\alpha_n=s_1'\alpha_1+\cdots+s_n'\alpha_n$ by taking $t_i=0$. Thus, $\alpha_{\underline{s}}$ has the same Jordan type as $\alpha_{\underline{s}}+t_1\alpha_1+\cdots+t_n\alpha_n$. Proposition 1.3 implies that $\alpha_{\underline{s}}$ is maximal with respect to $\alpha_1,\ldots,\alpha_n$.

By upper semi-continuity of the Jordan type, there exists an open dense subset in $\mathbb{A}^n_k$ such that for any $(\lambda_1,\ldots,\lambda_n)$ from that subset we have $\alpha_\lambda=\lambda_1\alpha_1+\cdots+\lambda_n\alpha_n\sim\alpha_{\underline{s}}$. For any such λ, $\alpha=\alpha_\lambda$ satisfies the conditions of Proposition 1.3.

We proceed to investigate the rank-maximality condition in the special case of two commuting matrices. For any commuting nilpotent matrices $\alpha,\beta\in M_N(k)$ set

$$W_{\alpha,\beta}=\bigcup_{\mu\in k^*}\operatorname{Ker}(\alpha+\mu\beta)\subset k^m,$$

the subspace spanned by the union of the subspaces $\operatorname{Ker}(\alpha+\mu\beta)$. Since α,β commute, each $\operatorname{Ker}(\alpha+\mu\beta)$ is invariant under both α and β.

two **Lemma 1.6.** *Let $\alpha,\beta\in M_N(k)$ be a pair of commuting matrices and set $W=W_{\alpha,\beta}$. Then $\beta(W)=\alpha(W)\subset W$ and hence*

$$(\alpha+\lambda\beta)(W)\subset\alpha(W)\quad \textit{for all } \lambda\in k.$$

Proof. For any $\mu\in k^*$, the restriction of β to $\operatorname{Ker}(\alpha+\mu\beta)$ equals $-\mu^{-1}$ times the restriction of α to this Kernel and hence $\beta(\operatorname{Ker}(\alpha+\mu\beta))=\alpha(\operatorname{Ker}(\alpha+\mu\beta))$. Thus $\beta(W)=\alpha(W)$. The second claim is now obvious. □

Using Lemma 1.6, we obtain a useful criterion for when α is rank-maximal with respect to α,β.

image **Lemma 1.7.** *Let k be an infinite field, let $\alpha,\beta\in M_N(k)$ be two commuting nilpotent matrices, and set $W=W_{\alpha,\beta}$. Then the following conditions are equivalent*

- *α is rank-maximal with respect to β.*
- *$Ker(\alpha)\subset W$.*

Proof. Set $r=max_{\lambda\in k}\operatorname{rank}(\alpha+\lambda\beta)$. As seen in the proof of Lemma 1.3, for all but finitely many $\lambda\in k$ we have an equality $\operatorname{rank}(\alpha+\lambda\beta)=r$ and α is rank-maximal with respect to β if and only if $\operatorname{rank}(\alpha)=r$, i.e. $dim(\operatorname{Ker}(\alpha))=N-r$. By Lemma 1.6, $\operatorname{rank}(\alpha|_W)\geq\operatorname{rank}((\alpha+\lambda\beta)|_W)$ for all $\lambda\in k$ with equality holding for all but finitely many λ's. Thus for a "generic" $\lambda\in k$, we have the following chain of equalities and inequalities:

$$dim(\operatorname{Ker}\alpha)\geq dim((\operatorname{Ker}\alpha)\cap W)=dim(\operatorname{Ker}(\alpha|_W))$$

$$=dim(\operatorname{Ker}((\alpha+\lambda\beta)|_W)=dim(\operatorname{Ker}(\alpha+\lambda\beta))=N-r$$

where the last but one equality holds since $\operatorname{Ker}(\alpha+\lambda\beta)\subset W$ for all $\lambda\in k^*$ by definition of W. This chain shows that $dim(\operatorname{Ker}\alpha)=m-r$ if and only if $\operatorname{Ker}\alpha\subset W$. □

Lemma 1.7 will now enable us to prove the following key property of rank-maximality. In some sense, the property proved in Proposition 1.8 is the essential case of the general maximality result of Theorem 1.12.

two-max **Proposition 1.8.** *Let k be an infinite field and let $\alpha, \beta \in M_N(k)$ be two commuting nilpotent matrices with α rank-maximal with respect to β. Then for any nilpotent matrix $\gamma \in M_N(k)$ commuting with both α and β,*

$$rank(\alpha) = rank(\alpha + \beta\gamma).$$

Proof. Set $W = W_{\alpha,\beta}$. Since γ commutes with α and β, W is γ-invariant.

We first verify that $\alpha(W) = (\alpha + \beta\gamma)(W)$. To prove this formula it suffices to show that $\alpha(V_\mu) = (\alpha + \beta\gamma)(V_\mu)$, where $\mu \in k^*$ and $V = V_\mu = \mathrm{Ker}\,(\alpha + \mu\beta)$. This follows immediately from the following relations:

$$\alpha|_V = -\mu \cdot \beta|_V \quad (\alpha + \beta\gamma)|_V = -\mu \cdot \beta|_V \cdot (1 - \mu^{-1} \cdot \gamma|_V)$$

since the matrix $1 - \mu^{-1} \cdot \gamma|_V$ is invertible.

Next we show that $\mathrm{Ker}\,(\alpha + \beta\gamma) \subset W$. Let $v \in \mathrm{Ker}\,(\alpha + \beta\gamma)$ be an arbitrary vector. We are going to show that $\gamma^i(v) \in W$ for all $i \geq 0$ using descending induction on i. Since γ is nilpotent, this statement is trivial for i large enough. Assume now that $\gamma^{i+1}(v) \in W$. Then $\alpha(\gamma^i(v)) = -\beta\gamma^{i+1}(v) = \beta(-\gamma^{i+1}(v)) \in \beta(W) = \alpha(W)$. Thus there exists $w \in W$ such that $\alpha(\gamma^i(v) - w) = 0$. Hence $\gamma^i(v) \in W + \mathrm{Ker}\,\alpha = W$, since $\mathrm{Ker}\,\alpha \subset W$ according to Lemma 1.7.

Using $\alpha(W) = (\alpha + \beta\gamma)(W)$ and $\mathrm{Ker}\,(\alpha + \beta\gamma) \subset W$, we conclude that

$$\mathrm{rank}\,(\alpha + \beta\gamma) = N - dim(\mathrm{Ker}\,(\alpha + \beta\gamma)) = N - dim(\mathrm{Ker}\,((\alpha + \beta\gamma)|_W)) =$$
$$= N - dim\, W + dim\,(\alpha + \beta\gamma)(W) = N - dim\, W + dim\,\alpha(W) =$$
$$= N - dim\,\mathrm{Ker}\,\alpha|_W = N - dim\,\mathrm{Ker}\,\alpha = \mathrm{rank}\ \alpha.$$

□

The following theorem establishes that rank-maximality with respect to linear combinations of $\alpha_1, \ldots, \alpha_n$ implies rank-maximality with respect to all polynomials in $\alpha_1, \ldots, \alpha_n$. Our ultimate theorem, Theorem 1.12, refines this by replacing rank-maximality by maximality.

rk-thm **Theorem 1.9.** *Let $\alpha, \alpha_1, \cdots, \alpha_n, \beta_1, \ldots, \beta_n \in M_N(k)$ be a family of commuting nilpotent matrices, with k an infinite field. Assume that α is rank-maximal with respect to the family $\alpha_1, ..., \alpha_n$. Then α is also rank-maximal with respect to the family $\alpha_1, ..., \alpha_n, \alpha_1\beta_1, \ldots, \alpha_n\beta_n$.*

Moreover, rank $\alpha =$ rank$(\alpha + \alpha_1\beta_1 + \ldots \alpha_n\beta_n)$.

Proof. Our procedure is to expand the family of matrices with respect to which α is rank-maximal.

Our first (and key) step is to show that we can add $\alpha_1\beta_1$. We proceed to verify that

$$\mathrm{rank}\ \alpha \ = \ \mathrm{rank}\,(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n + t\alpha_1\beta_1)$$

for independent variables $t_1, ..., t_n, t$. Observe that $\alpha + t_1\alpha_1 + ... + t_n\alpha_n$ is obviously rank-maximal with respect to α_1, so that Proposition 1.8 thus implies that

$$\mathrm{rank}\,(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n + t\alpha_1\beta_1) = \mathrm{rank}\,(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n).$$

On the other hand,

$$\mathrm{rank}\,(\alpha + t_1\alpha_1 + \cdots + t_n\alpha_n) \ = \ \mathrm{rank}\ \alpha$$

in view of rank-maximality of α with respect to $\alpha_1, \ldots, \alpha_n$.

Iterating this procedure, we see that we can add all terms of the form $\alpha_i\beta_i$ to our family, thereby establishing the rank-maximality of α with respect to the family $\alpha_1, \dots, \alpha_n, \alpha_1\beta_1, \dots, \alpha_n\beta_n$.

To prove the rank-maximality of $\alpha + \alpha_1\beta_1 + \dots \alpha_n\beta_n$ and the equality rank $\alpha =$ rank $(\alpha + \alpha_1\beta_1 + \dots \alpha_n\beta_n)$, we first verify these assertions with $\alpha_1\beta_1$. The equality of ranks follows immediately from Proposition 1.8. There remains to show that for independent variables $t_1, \dots, t_n$ we have the equality

$$\operatorname{rank}(\alpha + \alpha_1\beta_1) = \operatorname{rank}(\alpha + \alpha_1\beta_1 + t_1\alpha_1 + \dots + t_n\alpha_n).$$

Since $\alpha + t_1\alpha_1 + \dots + t_n\alpha_n$ is maximal with respect to α_1, two applications of Proposition 1.8 imply the equalities

$$\operatorname{rank}(\alpha + t_1\alpha_1 + ... + t_n\alpha_n + \alpha_1\beta_1) = \operatorname{rank}(\alpha + t_1\alpha_1 + \dots + t_n\alpha_n) = \operatorname{rank}\alpha =$$

$$= \operatorname{rank}(\alpha + \alpha_1\beta_1).$$

Finally, we get the general case by iterating this procedure. □

To extend Theorem 1.9 to establish maximality of Jordan types, we employ the following auxilliary construction which reduces maximality to rank-maximality of associated matrices. Denote by $J_s \in M_s(k)$ the Jordan block of size s, i.e. J_s is a nilpotent matrix which acts on the standard basis of k^s according to the rule $J_s(e_1) = 0, J_s(e_i) = e_{i-1} \quad (2 \le i \le s)$.

Lemma 1.10. *For any finite dimensional k-vector space V and any $\alpha \in End_k(V)$, there is a natural isomorphism*

$$Ker(\alpha^s) \simeq Ker(\alpha \otimes 1_s + 1_V \otimes J_s).$$

Proof. Any $v \in V \otimes k^s$ can be written uniquely in the form $v_1 \otimes e_1 + ... + v_s \otimes e_s$ with appropriate $v_1, \dots, v_s \in V$. Moreover

$$(\alpha \otimes 1_s + 1_V \otimes J_s)(v_1 \otimes e_1 + \dots + v_s \otimes e_s) = \sum_{i=1}^{s-1}(\alpha(v_i) + v_{i+1}) \otimes e_i + \alpha(v_s) \otimes e_s.$$

Thus, equations defining Ker $(\alpha \otimes 1_s + 1_V \otimes J_s)$ look as follows:

$$\alpha(v_1) = -v_2,\ \alpha(v_2) = -v_3,\ \dots,\ \alpha(v_{s-1}) = -v_s,\ \alpha(v_s) = 0.$$

We conclude that v_1 determines all other components via the formulae

$$v_i = (-1)^{i-1}\alpha^{i-1}(v_1),$$

and the only equation for v_1 is $\alpha^s(v_1) = 0$, i.e. $v_1 \in \operatorname{Ker}\alpha^s$. □

multi-max **Corollary 1.11.** *Let $\alpha, \alpha_1, \dots, \alpha_n \in M_N(k)$ be a family of commuting nilpotent matrices with k an infinite field. Then the following conditions are equivalent.*

- *$rank(\alpha^s) \ge rank(\alpha + \lambda_1\alpha_1 + \dots + \lambda_n\alpha_n)^s \quad \forall\lambda_1, \dots, \lambda_n \in k$*
- *$\alpha \otimes 1_s + 1_N \otimes J_s$ is rank-maximal with respect to $\alpha_1 \otimes 1_s, \dots, \alpha_n \otimes 1_s$*

We now formulate the main theorem of this section, a result which will play a key role in later sections in our consideration of "local maximality" for finite dimensional modules for an arbitrary finite group scheme.

Jor-thm **Theorem 1.12.** *Let $\alpha, \alpha_1, \dots, \alpha_n, \beta_1, \dots \beta_n \in M_N(k)$ be a finite set of commuting nilpotent matrices with k an infinite field. Assume that α is maximal with respect to the family $\alpha_1, \dots, \alpha_n$. Then α is also maximal with respect to the family $\alpha_1, ..., \alpha_n, \alpha_1\beta_1, \dots, \alpha_n\beta_n$.*

Moreover,

$$\alpha \ \sim \ \alpha + \alpha_1\beta_1 + \cdots + \alpha_n\beta_n.$$

Proof. This follows immediately by combining Theorem 1.9 and Corollary 1.11. □

By taking β_i to be polynomials in α_j without constant term, we obtain the following special case of Theorem 1.12.

Jor-pol **Corollary 1.13.** *Assume that α is maximal with respect to a family $\alpha_1, \dots, \alpha_n \in M_N(k)$ of commuting nilpotent matrices. Then*

$$\alpha \geq \alpha + q(\alpha_1, \dots, \alpha_n)$$

for any polynomial $q \in k[X_1, \dots, X_n]$ with no non-zero constant term.

Moreover if $q \in k[X_1, \dots, X_n]$ with no non-zero constant or linear term, then

$$\alpha \ \sim \ \alpha + q(\alpha_1, \dots, \alpha_n).$$

Remark 1.14. In a few degenerate cases considered below, we shall (implicitly) have to include the possibility that $\alpha, \alpha_1, \dots, \alpha_n \in M_N(k)$ with k finite. Then we shall say α is maximal with respect to $\alpha_1, \dots, \alpha_n$ provided that this is the case when viewing these matrices in $M_N(K)$ for some infinite field extension K/k.

2. Maximal Jordan types for abelian finite group schemes

one

If $E = \mathbb{Z}/p^{\times r}$ is an elementary abelian p-group, then a choice of generators $g_1, \dots, g_r \in E$ determines an isomorphism $kE \simeq k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p)$ given by sending g_i to $t_i + 1$. A cyclic shifted subgroup of E with respect to the set of generators $\{g_1, \dots, g_r\}$ is a map of k-algebras specified by some $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in k^r$

$$\alpha_{\underline{\lambda}} : k[t]/t^p \to k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p), \quad t \mapsto \sum_{i=1}^{r} \lambda_i t_i.$$

A theorem of W. Wheeler [21] asserts that for any finite dimensional kE-module M, the subset of those $\underline{\lambda} \in k^r$ for which $\alpha_{\underline{\lambda}}^*(M)$ is not of maximal Jordan type is Zariski closed in $\mathbb{A}^r$.

In this section, we use Theorem 1.12 to verify that this maximal Jordan type is an invariant of the kE-module M independent of the choice of generators of kE. In particular, the intrinsic, basis-free nature of this maximal Jordan type tells us that this Jordan type is also maximal if we view the action of kE on M as the action of the restricted enveloping algebra $u(\mathcal{E})$ of a rank r abelian Lie algebra $\mathcal{E}$ with trivial p-restriction determined by an arbitrary choice of k-algebra isomorphism of $kE \simeq k\mathbb{G}_{a(1)}^{\times r} = u(\mathcal{E})$.

In Theorem 2.9, this intrinsic maximal Jordan type is shown to extend to the data of an arbitrary finite abelian group scheme over k acting upon a finite dimensional k-vector space.

In [10] (as corrected in [11]) and [12], the first two authors consider "π-points", an alternative to shifted subgroups of E that generalizes to any finite group scheme. We recall this alternative in the following definition.

Definition 2.1. Let G be a finite group scheme over k with group algebra kG. A π-point of G is a (left) flat map of K-algebras

$$\alpha_K : K[t]/t^p \to KG$$

which factors through the group algebra $KC_K \subset KG$ of some unipotent abelian subgroup scheme $C_K \subset G_K$ defined over some field extension K/k.

A π-point α_K is said to specialize to the π-point β_L if for every finite dimensional kG-module M the condition that $\beta_L^*(M_L)$ is free as an $L[t]/t^p$-module implies that $\alpha_K^*(M_K)$ is free as a $K[t]/t^p$-module.

If α_K specializes to β_L and β_L specializes to α_K, then we say that α_K is equivalent to β_L (denoted by $\alpha_K \sim \beta_L$). The equivalence class of the π-point α_K is denoted $[\alpha_K]$.

One of the fundamental properties of π-points is that there is a natural homeomorphism

homeo (2.1.1)
$$\Psi_G : \Pi(G) \xrightarrow{\sim} \operatorname{Proj} H^\bullet(G,k)$$

from the space $\Pi(G)$ of equivalence classes of π-points of G (with topology intrinsically determined by kG-modules) to $\operatorname{Proj} H^\bullet(G,k)$ with its Zariski topology, where $H^\bullet(G,k)$ is the commutative algebra of even dimensional cohomology of G for p odd and the commutative algebra of all cohomology of G for $p = 2$.

The following proposition makes explicit the equivalence relation $\alpha_K \sim \beta_L$ on π-points of an elementary abelian p-group E.

explicit **Proposition 2.2.** *Let $E = \mathbb{Z}/p^{\times r}$ be an elementary abelian p-group viewed as a finite group scheme over k. Choose some k-linear isomorphism*

$$kE \;\simeq\; k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p).$$

Let $\alpha_K : K[t]/t^p \to KE$, $\beta_L : L[t]/t^p \to LE$ be two π-points of kE. Then $\alpha_K \sim \beta_L$ if and only if there exist embeddings $K \subset \Omega$, $L \subset \Omega$ over k of K, L into some field extension Ω over k and some $0 \neq \omega \in \Omega$ such that

$$(\alpha_K \otimes_K \Omega)(t) - \omega(\beta_L \otimes_L \Omega)(t) \in \Omega[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p)$$

is a polynomial in the t_i's with no non-zero constant or linear term.

Proof. The projectivized rank variety of E can be identified with the projective space $\operatorname{Proj} I/I^2$ (see [2, II.4]) over an algebraically closed field Ω of characteristic p where I denotes the augmentation ideal of ΩE. Since Ω-rational points on the projectivized rank variety of E are precisely equivalence classes of π-points defined over Ω ([10, 2.6,2.9]), one concludes that two Ω-rational π-points $\alpha, \beta : \Omega[t]/t^p \to \Omega E$, are equivalent if and only if there exists is $0 \neq \omega \in \Omega$ such that $\alpha(t) - \omega\beta(t) \in I^2$.

To prove the "if" direction, we observe that non-equivalent π-points remain non-equivalent after base change. Thus, the argument of the preceding paragraph implies that the asserted condition is sufficient to imply $\alpha_K \sim \beta_L$.

Conversely, assume that α_K, β_L are equivalent as π-points of kE. Let Ω/k be an algebraically closed field containing both K and L. Let $\sigma : \Omega \to \Omega$ be a field automorphism of Ω over k, and consider the action on π-points

$$\sigma : \alpha_\Omega \;\mapsto\; \alpha_\Omega^\sigma$$

given by sending α_Ω to the Ω-algebra map which sends t to $(\alpha_\Omega(t))^\sigma$. If $\rho : kG \to \mathrm{End}_k(M)$ specifies a kG-module M, then $\rho(\alpha_\Omega^\sigma(t))$ when viewed as a matrix is simply the result of applying σ to the matrix entries of $\rho(\alpha_\Omega(t))$.

The assertion that $\alpha_\Omega, \beta_\Omega$ are equivalent as π-points of kE implies the existence of some automorphism σ of Ω/k with the property that $\alpha_\Omega, \beta_\Omega^\sigma$ are equivalent as π-points of ΩE. This is easily verified (as in [12, 4.6]) by viewing equivalence classes of $\alpha_\Omega, \beta_\Omega$ as Ω-rational points of $\mathrm{Proj}\, H^\bullet(G, \Omega)$ which map to the same point in $\mathrm{Proj}\, H^\bullet(G, k)$. Thus, another application of the argument in the first paragraph implies that we may find some $\omega \in \Omega$ such that $\alpha_\Omega(t) - \omega \cdot \beta_\Omega^\sigma(t) \in \Omega[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$ is a polynomial in the t_i's with no non-zero constant or linear term. Now twisting the embedding of $L \subset \Omega$ by the automorphism σ we obtain the desired relationship between α_K and β_L.

□

The following example reflects the "typical" situation in which equivalent π-points α, β of an elementary abelian group E lead to non-isomorphic $k[t]/t^p$-modules $\alpha^*(M), \beta^*(M)$.

depend **Example 2.3.** Assume $p > 2$. Let $E = \mathbb{Z}/p^{\times 2}$ be an elementary abelian group of rank 2, choose an identification of kE with $k[x, y]/(x^p, y^p)$, and let M be the kE-module (with respect to this identification of kE) defined as the quotient $k[x, y]/(x^2 - y, x^p)$ of kE. Let $\alpha : k[t]/t^p \to k[x, y]/(x^p, y^p)$ be defined by sending t to $x^2 - y$ and let $\beta : k[t]/t^p \to k[x, y]/(x^p, y^p)$ be defined by sending t to y. One readily checks that the $k[t]/t^p$-module $\alpha^*(M)$ is trivial whereas $\beta^*(M)$ consists of two Jordan blocks of sizes $\frac{p+1}{2}$ and $\frac{p-1}{2}$. Then $\alpha \sim \beta$ by Proposition 2.2, but $\alpha^*(M)$ is not isomorphic to $\beta^*(M)$.

The observation that the Jordan type can change when one replaces a cyclic shifted subgroup by another within the same equivalence class was made earlier by S. Kaptanoglu, who also observed that for $E = \mathbb{Z}/2^{\times 2}$ this phenomenon does not occur ([16]).

A major conclusion of our work is that if $\alpha_K^*(M_K)$ has "maximal Jordan type" for a finite dimensional kG-module M as defined below, then $\beta_L^*(M_K)$ has the same Jordan type whenever $\alpha_K \sim \beta_L$. We prove this for an arbitrary finite group scheme G, proceeding in incremental steps of increasing generality, beginning with the case of an elementary abelian p-group.

Definition 2.4. Let G be a finite group scheme over k and let M be a kG-module of dimension N. We say that a partition $\underline{n}$ of N is *a maximal Jordan type* for M if there exists some π-point $\alpha_K : K[t]/t^p \to KG$ such that the Jordan type of $\alpha_K^*(M_K)$ equals $\underline{n}$ and if there does not exist a π-point $\beta_L : L[t]/t^p \to LG$ such that the Jordan type of $\beta_L^*(M_L)$ is strictly greater than $\underline{n}$.

Furthermore, we say that a partition $\underline{n}$ of N is the *absolute maximal Jordan type* for M if there exists some π-point $\alpha_K : K[t]/t^p \to KG$ such that the Jordan type of $\alpha_K^*(M_K)$ equals $\underline{n}$ and if $\underline{n}$ is greater or equal to the Jordan type of $\beta_L^*(M_L)$ for every π-point β_L of G.

Remark 2.5. If $\Pi(G)$ is irreducible and if α_K is a generic π-point of G, then $\alpha_K^*(M_K)$ has absolute maximal Jordan type by Corollary 4.11. On the other hand, we shall see in Example 4.13 that a kG-module can have more than one maximal Jordan type if $\Pi(G)$ is reducible.

images **Remark 2.6.** Let M be a kG-module and $\alpha_K : K[t]/t^p \to KG$ be a π-point of G such that $\alpha_K^*(M_K)$ has a maximal Jordan type. Let $i : C_K \subset G_K$ be a subgroup scheme such that α_K factors as $i \circ \alpha'_K : K[t]/t^p \to KC_K \to KG$. Then $\alpha_K^*(M_K) = \alpha'^*_K(i^* M_K)$ is a maximal Jordan type for the restriction of M_K to C_K.

The following theorem is a re-interpretation of Theorem 1.12 in terms of π-points of an elementary abelian p-group $E = \mathbb{Z}/p^{\times r}$ viewed as a finite group scheme over k. We say that a π-point of a finite group scheme G is *generic* if its equivalence class is a generic point of $\Pi(G)$. One example of a generic π-point of $E = \mathbb{Z}/p^{\times r}$ is the map

generic
$$(2.6.1) \quad \alpha_{k(x_1,\dots,x_r)} : k(x_1,\dots,x_r)[t]/t^p \to k(x_1,\dots,x_r)E, \quad t \mapsto \sum_{i=1}^{r} x_i \cdot (g_i - 1)$$

where $\{g_1,\dots,g_r\}$ is a set of generators for E.

elem-ab **Theorem 2.7.** *Let E be an elementary abelian p-group of rank r, and let $\alpha_K : K[t]/t^p \to KE$ be any generic π-point of E. Then for any finite dimensional kE-module M, the Jordan type of $\alpha_K^*(M_K)$ is the absolute maximal Jordan type for M.*

Moreover, let M be a finite dimensional kE-module and α_K be a π-point of E such that the Jordan type of $\alpha_K^*(M_K)$ is the absolute maximal Jordan type for M. Then for any $\beta_L \sim \alpha_K$, the Jordan type of $\beta_L^*(M_L)$ equals that of $\alpha_K^*(M_K)$.

Proof. If $r = 1$, the assertion is trivial so that we may assume $r > 1$. Let $\{g_1,\dots,g_r\}$ be a set of generators for E, and let $\{t_i = g_i - 1\}_{1 \le i \le r}$ be the corresponding polynomial generators of kE. Let $\rho : kG \to \mathrm{End}_k(M)$ specify the kG-module structure of M. According to Remark 1.5, the π-point $\alpha_{k(x_1,\dots,x_r)}$ of (2.6.1) has the property that $\rho(\alpha_{k(x_1,\dots,x_r)}(t))$ is maximal with respect to the set $\{\rho(t_1),\dots,\rho(t_r)\}$. Theorem 1.12 thus implies that the Jordan type of $\rho(\alpha_{k(x_1,\dots,x_r)}(t))$ is greater or equal to the Jordan type of $\rho(\beta_K(t))$ for any π-point $\beta_K : K[t]/t^p \to KG$. Thus, $\alpha^*_{k(x_1,\dots,x_r)}(M_{k(x_1,\dots,x_r)})$ has absolute maximal Jordan type.

As mentioned above, $\alpha_{k(x_1,\dots,x_r)}$ is a generic π-point of E. Thus, to complete the proof of the theorem, it suffices to prove the second assertion. Let M be a given finite dimensional kE-module M with kE-structure specified by $\rho : kE \to \mathrm{End}_k(M)$. Assume that $\alpha_K^*(M_K)$ has maximal Jordan type so that $\rho(\alpha_K(t))$ is maximal with respect to the set $\{\rho(t_1),\dots,\rho(t_r)\}$. Since multiplying β_Ω by a scalar does not change the Jordan type of $\beta_\Omega^*(M_\Omega)$, Proposition 2.2 enables us to assume that $\alpha_\Omega(t) - \beta_\Omega(t) \in \Omega[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p)$ is a polynomial in the t_i's with no non-zero constant or linear term for some field extension Ω/k. Since extension of scalars evidently preserves the "maximality" property, we still have that $\rho(\alpha_\Omega(t))$ is maximal with respect to the set $\{\rho(t_1),\dots,\rho(t_r)\}$. Theorem 1.12 implies that $\rho(\alpha_\Omega(t))$ has the same Jordan type as $\rho(\beta_\Omega(t))$, thereby verifying that $\alpha_\Omega^*(M_\Omega)$ and $\beta_\Omega^*(M_\Omega)$ have the same Jordan type. Since extension of scalars preserves Jordan type, we conclude that $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$ have the same Jordan type. □

The following lemma enables us to frequently replace k by a conveniently chosen finite field extension F/k.

exten **Lemma 2.8.** *Let G be a finite group scheme defined over k, let α_K, β_L be π-points of G, and let F/k be a finite normal extension. Then $\alpha_K \sim \beta_L$ as π-points of G*

if and only if there exist embeddings of the composites $\widetilde{K} = FK$, $\widetilde{L} = FL$ *in some field extension* Ω/F *such that* $\alpha_{\widetilde{K}} \otimes_{\widetilde{K}} \Omega$, $\beta_{\widetilde{L}} \otimes_{\widetilde{L}} \Omega$ *are equivalent* π*-points of* G_F.

Moreover, for any finite field extension F/k*, we have that* α_K *is a generic* π*-point of* G *if and only if* $\alpha_{\widetilde{K}}$ *is a generic* π*-point of* G_F*, where* $\widetilde{K}$ *is again a composite of* F *and* K.

Proof. Observe that α_K, β_L are equivalent π-points of G if and only if their base changes $\alpha_{\widetilde{K}} : \tilde{K}[t]/t^p \to \widetilde{K}G$, $\beta_{\widetilde{L}} : \tilde{L}[t]/t^p \to \widetilde{L}G$ remain equivalent π-points of G. Thus, it suffices to assume that $\widetilde{K} = K, \widetilde{L} = L$, so that $F \subset K \cap L$.

We first consider the case in which F/k is separable and thus Galois. In this case, $H^\bullet(G, k) \to H^\bullet(G, F) = H^\bullet(G, k) \otimes_k F$ is a Galois map, so that

$$\Pi(G_F) \simeq \operatorname{Proj} H^\bullet(G_F, F) \to \operatorname{Proj} H^\bullet(G, k) \simeq \Pi(G)$$

is a finite Galois covering. In particular, the pre-image of any generic point of $\Pi(G)$ is a generic point of $\Pi(G_F)$ and any two points in the pre-image of a given point of $\Pi(G)$ (such as $[\alpha_K] = [\beta_L]$) are conjugate by an element of $Gal(F/k)$. Since the isomorphism $\Pi(G_F) \simeq \operatorname{Proj} H^\bullet(G_F, F)$ is compatible with the Galois action (see [12, 4.5]), we get $\alpha_K \sim \beta_L$ as π-points of G if and only if $\alpha_\Omega \sim \beta_\Omega^\sigma$ as π-points of G_F. Twisting the embedding of L into Ω by σ, we obtain the desired result.

More generally, F/k factors as $F/F^s/k$ where F^s/k is separable and F/F^s is purely inseparable. Now, the map $H^\bullet(G, F^s) \to H^\bullet(G, F)$ is a purely inseparable isogeny, so that

$$\Pi(G_F) \simeq \operatorname{Proj} H^\bullet(G_F, F) \to \operatorname{Proj} H^\bullet(G_{F^s}, F^s) \simeq \Pi(G_{F^s})$$

is a bijection. Thus, the general case follows. □

We now generalize Theorem 2.7 to an arbitrary finite abelian group scheme.

ab **Theorem 2.9.** *Let* C *be an abelian finite group scheme over* k *and let* $\alpha_K : K[t]/t^p \to KC$ *be any generic* π*-point of* C*. Then for any finite dimensional* kC*-module* M*,* $\alpha_K^*(M_K)$ *has absolute maximal Jordan type.*

Moreover, let M *be a finite dimensional* kC*-module and* α_K *be a* π*-point of* C *such that* $\alpha_K^*(M_K)$ *has maximal Jordan type. Then for any* $\beta_L \sim \alpha_K$*, the Jordan type of* $\beta_L^*(M_L)$ *equals that of* $\alpha_K^*(M_K)$.

Proof. Let C^D be the Cartier dual of C. By [12, 4.2], we may find some finite extension F/k such that $C_F^D = (C_F^D)^0 \times \pi_0(C_F^D)$, where $(C_F^D)^0$ is the connected component of C_F^D. Dualizing again, we obtain

$$C_F = (C_F^D)^D = ((C_F^D)^0)^D \times (\pi_0(C_F^D))^D$$

The first factor is coconnected which is equivalent to unipotent, and the second factor is dual to an etale group scheme, and, hence, is semi-simple. Lemma 2.8 enables us to assume that C itself is a product of a unipotent abelian finite group scheme and a semi-simple finite abelian group scheme.

Since any π-point of C necessarily factors (by definition) through a unipotent abelian subgroup scheme of C, we may assume that C is itself unipotent. Let $\bar{k}$ be the perfect closure of k. By [20, 14.4], the group algebra $\bar{k}C$ is of the form $\bar{k}[t_1, \ldots, t_r]/(t_1^{p^{e_1}}, \ldots, t_r^{p^{e_r}})$. Since every generator t_i is defined over some finite subextension k'/k, we can find a finite purely inseparable L/k such that $LC = L[t_1,\dots,t_r]/(t_1^{p^{e_1}},\dots,t_r^{p^{e_r}})$. Hence, applying Lemma 2.8 again, we may assume

$$kC \;\simeq\; k[t_1,\dots,t_r]/(t_1^{p^{e_1}},\dots,t_r^{p^{e_r}}).$$

Let $T_i = t_i^{p^{e_i-1}}$. Any π-point $\alpha_K : K[t]/t^p \to KC$ must send t to a polynomial in the t_i's, each monomial of which is divisible by some $T_j = t_j^{p^{e_j-1}}$ since $t^p = 0$. As seen in Remark 1.5, the π-point

$$\alpha_{k(x_1,\dots,x_r)} : k(x_1,\dots,x_r)[t]/t^p \to k(x_1,\dots,x_r)[t_1,\dots,t_r]/(t_1^{p^{e_1}},\dots,t_r^{p^{e_r}})$$

sending t to $\sum_i x_i T_i$ is a generic π-point of C; moreover, $\rho(\alpha_{k(x_1,\dots,x_r)}(t))$ is maximal with respect to $\{\rho(T_1),\dots,\rho(T_r)\}$ for any finite dimensional kC-module with structure specified by ρ by Theorem 2.7. Thus, $\alpha^*_{k(x_1,\dots,x_r)}(M_{k(x_1,\dots,x_r)})$ has absolute maximal Jordan type for any finite dimensional kC-module M by Theorem 1.12. Moreover, if M is a specified finite dimensional kC-module with structure specified by ρ and if $\alpha_K^*(M_K)$ has absolute maximal Jordan type and if $\beta_L \sim \alpha_K$, then as in the proof of Thereom 2.7 we conclude that Theorem 1.12 implies that $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$ have the same Jordan type. □

3. Generic and maximal Jordan types for infinitesimal group schemes

Recall that a finite group scheme G over k is said to be infinitesimal if its coordinate algebra $k[G]$ is a (Artinian) local k-algebra. The *height* of such an infinitesimal group scheme is the least integer r such that $f^{p^r} = 0$ for all f in the maximal ideal of $k[G]$. A *1-parameter subgroup* of an infinitesimal group scheme G of height $\leq r$ over some field extension K/k is a morphism $\mathbb{G}_{a(r),K} \to G_K$ of finite group schemes over K. If G is an infinitesimal group scheme of height $\leq r$, then the functor

$$\text{(f.g. commutative k-algebras)} \to (sets)$$

$$A \mapsto Hom_{grpsch/A}(\mathbb{G}_{a(r),A}, G_A)$$

is representable by an affine scheme denoted $V_r(G) = \operatorname{Spec} k[V_r(G)]$ ([18, 1.5]). Here, G_A is the group scheme over A given by base change, $G_A = G \times_{\operatorname{Spec} k} \operatorname{Spec} A$.

In particular, there is a universal 1-parameter subgroup

universal (3.0.1) $$u : \mathbb{G}_{a(r),k[V_r(G)]} \to G_{k[V_r(G)]}$$

associated to the identity map of $k[V_r(G)]$ for any infinitesimal group scheme G. In some ways, the existence of this universal 1-parameter subgroup makes the study of representations of infinitesimal group schemes more tractable than the representations of finite groups.

In order to relate 1-parameter subgroups to π-points, we use the map of group algebras

$$k\mathbb{G}_{a(1)} = k[t]/t^p \to k[u_0,\dots,u_{r-1}]/(u_0^p,\dots,u_{r-1}^p) = k\mathbb{G}_{a(r)}, \quad t \mapsto u_{r-1}$$

where u_i is the basis element dual to t^{p^i} (so that $u_i = \frac{1}{p^i!}\frac{d^{p^i}}{dt^{p^i}}$). This is a map of Hopf algebras if and only if $r = 1$ (in which case, we view this map as the identity). We employ the notation

$$\epsilon : k\mathbb{G}_{a(1)} \to k\mathbb{G}_{a(r)}$$

for this map of group algebras, and stay alert to the fact that ϵ is not necessarily a map of Hopf algebras. In this way, we obtain a map

$$\tilde{\Theta}_G : V_r(G)\backslash\{0\} \to \Pi(G)$$

which induces a homeomorphism (see [19, 5.2] and [12, 3.6]) of Zariski topological spaces

homeo1 (3.0.2) $$\Theta_G : \operatorname{Proj}(V_r(G)) \to \Pi(G).$$

This construction sends $x : \operatorname{Spec} k(x) \to V_r(G)$ to

$$\mu_{x*} \circ \epsilon : k(x)[t]/t^p \to k(x)\mathbb{G}_{a(r),k(x)} \to k(x)G_{k(x)} = k(x)G,$$

where $\mu_{x*} : k(x)\mathbb{G}_{a(r),k(x)} \to k(x)G_{k(x)}$ is the map on group algebras induced by the 1-parameter subgroup μ_x associated to the point $x \in V_r(G)$ with residue field $k(x)$. Indeed, the composition $\Psi_G \circ \Theta_G$ of the homeomorphisms (2.1.1) and (3.0.2) is induced by the isogeny exhibited in [19, 5.2].

1-parameter subgroups give us "natural" representatives for equivalence classes of π-points. As we now verify, for each finite dimensional kG-module M and each $[\alpha_K] \in \Pi(G)$, the Jordan type of $\alpha_K^*(M_K)$ is independent of the 1-parameter subgroup representing $[\alpha_K]$.

same **Proposition 3.1.** *Let G be an infinitesimal group scheme. For any π-point α_K, let $x \in V_r(G)\backslash\{0\}$ project onto $[\alpha_K] \in \Pi(G)$. Then*

$$\mu_{x*} \circ \epsilon : k(x)[t]/t^p \to k(x)G \ \sim \ \alpha_K : K[t]/t^p \to KG.$$

Moreover, if $\mu : \mathbb{G}_{a(r),K} \to G_K$, $\nu : \mathbb{G}_{a(r),L} \to G_L$ are non-zero 1-parameter subgroups of G which map to the same equivalence class of π-points in $\Pi(G)$, then the Jordan type of $(\mu_* \circ \epsilon)^*(M_K)$ is the same as the Jordan type of $(\nu_* \circ \epsilon)^*(M_L)$.

Proof. The first assertion is merely a recollection of the definition of Θ_G , granted that $\Theta_G(x) = [\alpha_K]$.

If $\mu : \mathbb{G}_{a(r),K} \to G_K$ is a 1-parameter subgroup for some field extension K/k and if $k(x)$ is the residue field at the point $x \in V_r(G)$ determined by the map $\operatorname{Spec} K \to V_r(G)$ corresponding to μ, then μ determines an extension $K/k(x)$ of fields over k and

(3.1.1) $$\mu = \mu_x \times_{\operatorname{Spec} k(x)} \operatorname{Spec} K.$$

In particular, $(\mu_* \circ \epsilon)^*(M_K)$ has the same Jordan type as $(\mu_{x*} \circ \epsilon)^*(M_{k(x)})$.

Assume now that $\mu : \mathbb{G}_{a(r),K} \to G_K$, $\nu : \mathbb{G}_{a(r),L} \to G_L$ are 1-parameter subgroups determining the same point of $\Pi(G)$. By the preceding paragraph, we may assume that $K = L$. Then μ, ν are related by the action $V_r(G) \times \mathbb{A}^1 \to V_r(G)$ reflected in the grading of the coordinate algebra $k[V_r(G)]$. As can be seen using [19, 1.12], the action of some $a \in \mathbb{A}^1(K)$ sends $\mu_* \circ \epsilon : K[t]/t^p \to KG$ to $a^{p^{r-1}}(\mu_* \circ \epsilon) : K[t]/t^p \to KG$. Thus, $(\mu_* \circ \epsilon)^*(M_K)$ has the same Jordan type as $(\nu_* \circ \epsilon)^*(M_K)$. □

We say that a point $\eta \in V_r(G)$ specializes to a point $\zeta \in V_r(G)$ if there exists a map $\psi : \operatorname{Spec} R \to V_r(G)$ where R is a local, integral k-algebra whose field of fractions has image η and whose residue field has image ζ. More generally, if $\mu : \mathbb{G}_{a(r),K} \to G$, $\nu : \mathbb{G}_{a(r),L} \to G$ are 1-parameter subgroups obtained by base change from η, ζ and if η specializes to ζ, then we say μ specializes to ν.

special **Proposition 3.2.** *Let G be an infinitesimal group scheme. Assume that the 1-parameter subgroup $\mu : \mathbb{G}_{a(r),K} \to G$ specializes to the 1-parameter subgroup $\nu : \mathbb{G}_{a(r),L} \to G$. Then for any finite dimensional kG-module M, the Jordan type of $(\mu_* \circ \epsilon)^*(M_K)$ is greater or equal to the Jordan type of $(\nu_* \circ \epsilon)^*(M_L)$.*

Proof. Let $\eta, \zeta \in V_r(G)$ be such that $\mu : \mathbb{G}_{a(r),K} \to KG$, $\nu : \mathbb{G}_{a(r),L} \to LG$ are obtained by base change from μ_η, μ_ζ. Let $\psi : \operatorname{Spec} R \to V_r(G)$ be a morphism with R a local, integral k-algebra whose field of fractions Ω has image η and whose residue field F has image ζ. This corresponds to an R-group scheme homomorphism $\psi : \mathbb{G}_{a(r),R} \to G_R$. By Proposition 3.1, if suffices to consider the $R[t]/t^p$-module obtained as the pull-back of M_R via $\psi \circ \epsilon$ and show that the Jordan form of the endomorphism t on $M_R \otimes_R \Omega$ is greater or equal to the Jordan form of the endomorphism t on $M_R \otimes_R F$. This in turn is equivalent to proving for all i, $1 \leq i < p$, that the rank of t^i on $M_R \otimes_R \Omega$ is greater or equal to the rank of t^i on $M_R \otimes_R F$. This "upper-semicontinuity" property follows easily from Nakayama's Lemma. □

A simple geometric argument shows that if $\eta \in V_r(G)$ projects onto $[\alpha_K]$, $\zeta \in V_r(G)$ projects onto $[\beta_L]$, and $[\beta_L]$ is a specialization of $[\alpha_K]$ as points of $\Pi(G)$, then η specializes to ζ as points of $V_r(G)$ as formulated prior to Proposition 3.2. . The following property of Jordan types if now an immediate corollary of Propositions 3.1 and 3.2.

Corollary 3.3. *Let G be an infinitesimal group scheme, and consider a π-point $\alpha_K : K[t]/t^p \to KG$ specializing to the π-point $\beta_L : L[t]/t^p \to LG$. Let $\mu : \mathbb{G}_{a(r),\Omega} \to G_\Omega$, $\nu : \mathbb{G}_{a(r),\Sigma} \to G_\Sigma$ be non-zero 1-parameter subgroups of G which project to $[\alpha_k], [\beta_L] \in \Pi(G)$. Then, for any finite dimensional kG-module M, the Jordan type of $(\mu_* \circ \epsilon)^*(M_\Omega)$ is greater or equal to the Jordan type of $(\nu_* \circ \epsilon)^*(M_\Sigma)$*

Using Theorem 2.9, we now establish the well-definedness of the generic Jordan type (i.e., independence of the choice of representing generic π-point) of a finite dimensional kG module for any infinitesimal group scheme G.

infin-gen **Theorem 3.4.** *Let G be an infinitesimal group scheme over k and let*

$$\alpha_K : K[t]/t^p \to KG, \quad \beta_L : L[t]/t^p \to LG$$

be equivalent generic π-points of G. Then for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ has the same Jordan type as $\beta_L^*(M_L)$.

Proof. Write $\alpha_K = i_C \circ \alpha'_K : K[t]/t^p \to KC_K \to KG$ where $C_K \subset G_K$ is an abelian subgroup scheme (which is necessarily unipotent). Choose a 1-parameter subgroup $\mu : \mathbb{G}_{a(r),K} \to C_K$ such that $[\mu_* \circ \epsilon] = [\alpha'_K] \in \Pi(C_K)$ and let $\eta : \mathbb{G}_{a(r),\Omega} \to C_\Omega$ be another 1-parameter subgroup such that $\eta_* \circ \epsilon$ is a generic π-point of C_K. Let M be a kG-module. By Theorem 2.9, $(\eta_* \circ \epsilon)^*(i_C^* M_\Omega)$ has maximal Jordan type among all $\gamma_F^*(i_C^* M_F)$ as γ_F ranges over all π-points of C_K. Since $(i_C \circ \eta \circ \epsilon)_* \sim (i_C \circ \mu \circ \epsilon)_*$ as π-points of G, Proposition 3.1 implies that $(\mu_* \circ \epsilon)^*(i_C^* M_K)$ has the same Jordan type as $(\eta_* \circ \epsilon)^*(i_C^* M_\Omega)$, and thus is maximal among all $\gamma_F^*(i_C^* M_F)$ as γ_F ranges over all π-points of C_K. Since $\mu_* \circ \epsilon \sim \alpha'_K$ as π-points of C_K, Theorem 2.9 implies that the Jordan type of $\alpha_K^*(M_K) = \alpha'_K(i_C^* M_K)$ is also the same as the Jordan type of $(\mu_* \circ \epsilon)^*(i_C^* M_K)$.

Consider $\beta_L : i_D \circ \beta' : L[t]/t^p \to LD_L \to LG$ equivalent to α_K (where $D_L \subset G_L$ is an abelian subgroup scheme) and let $\nu : \mathbb{G}_{a(r),L} \to G_L$ be a 1-parameter subgroup scheme such that $[\nu_* \circ \epsilon] = [\beta'_L] \in \Pi(D_L)$. The preceding argument tells us that the Jordan type of $\beta_L^*(M_L)$ is the same as the Jordan type of $(\nu_* \circ \epsilon)^*(i_D^* M_L)$. On the other hand, Proposition 3.1 implies that the Jordan type of $(\mu_* \circ \epsilon)^*(i_C^* M_K)$ equals that of $(\nu_* \circ \epsilon)^*(i_D^*(M_L)$ since $\mu_* \circ \epsilon \sim \alpha_K \sim \beta_L \sim \nu_* \circ \epsilon$. □

As seen implicitly in the proof of Theorem 2.9, $\Pi(C)$ is irreducible if C is an abelian finite group scheme. However, this is not true for a general finite group scheme, even in the case of a height 1 infinitesimal group scheme (i.e., a restricted Lie algebra). For example, the nullcone of a Borel subalgebra of a reductive Lie algebra is not necessarily irreducible for small primes (cf. [5]). If $\Pi(G)$ is not irreducible, then the Jordan type at a generic point need not be maximal, but Corollary 4.11 asserts that maximal Jordan types for a given finite dimensional module M are realized at generic points.

Of course, maximal Jordan types can be realized at π-points which are not generic. We see in the following theorem that if the Jordan type for a given kG-module is maximal at some π-point then this Jordan type is independent of the representative of that π-point.

infin-max

Theorem 3.5. *Let G be an infinitesimal group scheme over k and M a finite dimensional kG-module. Let $\beta_L : L[t]/t^p \to LG$ be a π-point of G with the property that the Jordan type of $\beta_L^*(M_L)$ is maximal. Then for any π-point $\alpha_K : K[t]/t^p \to KG$ which specializes to β_L the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$.*

Proof. Write $\alpha_K = i_C \circ \alpha'_K : K[t]/t^p \to KC_K \to KG$ and $\beta_L = i_C \circ \beta'_L : L[t]/t^p \to LD_L \to LG$ with $C_K \subset G_K, D_L \subset G_L$ abelian subgroup schemes. Let $\mu : \mathbb{G}_{a(r),K} \to C_K$, $\nu : \mathbb{G}_{a(r),L} \to D_L$ be 1-parameter subgroups such that $[\beta'_L] = [\nu_* \circ \epsilon] \in \Pi(D_L)$ and $[\alpha'_K] = [\mu_* \circ \epsilon] \in \Pi(C_K)$. The maximality of $\beta_L^*(M_L) = (\beta'_L)^*(i_D^* M_L)$ and the fact that $\beta'_L \sim \nu_* \circ \epsilon$ imply by Theorem 2.9 that the Jordan type of $\beta_L^*(M_L)$ equals that of $(\nu_* \circ \epsilon)^*(i_D^* M_L) = ((i_D \circ \nu)_* \circ \epsilon)^*(M_L)$. Because $\alpha_K \sim (i_C \circ \mu)_* \circ \epsilon$ specializes to $\beta_L \sim (i_D \circ \nu)_* \circ \epsilon$, Proposition 3.2 implies that $((i_C \circ \mu)_* \circ \epsilon)^*(M_K)$ is likewise maximal. Thus, applying Theorem 2.9 to $\alpha_K \sim (i_C \circ \mu)_* \circ \epsilon$, we conclude that $\alpha_K^*(M_K)$ is also maximal. □

4. Generic and maximal Jordan types for arbitrary finite group schemes

Let G be a finite group scheme of the form $G^o \rtimes \tau$, where the connected component $G^o \subset G$ is geometrically connected and where $\tau = \pi_0(G)$ is the (constant) group of connected components of G. For each elementary abelian p-subgroup $E \subset \tau$, define $\Pi_0((G^o)^E \times E) \subset \Pi((G^o)^E \times E)$ to be the subset of those π-points which do not admit a representative factoring through $(G^o)^E \times E'$ with E' a proper subgroup of E.

Recall the "Quillen decomposition" for $\Pi(G)$ by locally closed subspaces ([12, 4.13]):

disjoint

$$\coprod \Pi_0((G^o)^E \times E)/N_\tau(E) \simeq \Pi(G) \qquad (4.0.1)$$

where the disjoint union is indexed by conjugacy classes of elementary abelian p-subgroups of τ, and where $N_\tau(E)$ denotes the normalizer of $E \subset \tau$.

In order to consider the Jordan types $\alpha_K^*(M_K)$ for specific π-points representing $[\alpha_K] \in \Pi(G)$, we sharpen somewhat this decomposition by considering π-points up to the finer equivalence defining $\Pi(H)$ for subgroups H of the form $(G^o)^E \times E$.

specific

Proposition 4.1. *Let G be a finite group scheme of the form $G^o \rtimes \tau$, with $\tau = \pi_0(G)$ constant and G^o geometrically connected. Assume given $[\alpha_K] \in \Pi(G)$ and a representative*

$$\alpha_K = i \circ \alpha_K' : K[t]/t^p \to K((G^o)^E \times E) \to KG$$

of $[\alpha_K]$.

Then there exists a π-point $\alpha_K'' : K[t]/t^p \to K((G^o)^E \times E')$ equivalent to α_K' as a π-point of $(G^o)^E \times E$ with $E' \subset E$ such that $[\alpha_K] \in \Pi_0((G^o)^{E'} \times E')$.

Proof. Let $\alpha_K' : K[t]/t^p \to K((G^o)^E \times E)$ be given by

$$t \mapsto \quad \sum_{i=1}^m a_i f_i + \sum_{j=1}^r b_j s_j + p(\underline{f}, \underline{s}) \in K(G^o)^E \otimes_K KE,$$

where $\{f_i;\ 1 \le i \le n\}$ is a set of commuting nilpotent elements of $K((G^o)^E)$, where $s_j = g_j - 1$ with $\{g_1, \dots, g_r\} \subset E$ a set of generators for E, and where $p(\underline{f}, \underline{s})$ is a polynomial in the f_i, s_j without constant or linear term. Let $\alpha_{K,0}''$ be a π-point of G^0 defined by

$$\alpha_{K,0}''(t) = \sum_{i=1}^m a_i f_i$$

and $\alpha_{K,1}''$ be a π-point of τ defined by

$$\alpha_{K,1}''(t) = \sum_{j=1}^r b_j s_j$$

Finally, let

$$\alpha_K'' = \alpha_{K,0}'' + \alpha_{K,1}''$$

and let $E' \subset E$ equal the minimal subgroup such that $\sum_{j=1}^r b_j s_j \in KE'$. By [10, 2.2] and a simple base change argument, α_K'' is equivalent to α_K' as a π-point of $(G^o)^E \times E$.

To complete the proof, it suffices to show that α_K is not equivalent (as a π-point of G) to some π-point factoring through $(G^o)^{E''} \times E''$ for some strictly smaller subgroup $E'' \subset E'$. This is immediate if E' is trivial. Thus, we may assume that E' is non-trivial.

Suppose there exists a π-point $\gamma_L = i' \circ \gamma_L' : L[t]/t^p \to L((G^o)^{E''} \times E'') \to LG$ equivalent to α_K which factors through $(G^o)^{E''} \times E''$ with $E'' \subset E'$. We may assume that E'' is a minimal such subgroup for γ_L. By passing to a field extension if necessary, we may assume $L = K$. As for α_K, let $\gamma_K'' = \gamma_{K,0}'' + \gamma_{K,1}''$ be the "linear" part of γ_K' with respect to the same set of generators s_i of KE. If N is a (finite dimensional) τ-module, let p^*N denote the kG module with action given by projecting kG to $k\tau$. Observe that $(i \circ \alpha_K'')^*(M_K)$ is free if and only if $(i' \circ \gamma_K'')^*(M_K)$ is free since $i \circ \alpha_K'' \sim \alpha_K \sim \gamma_K \sim i' \circ \gamma_K''$. Since the action of G^0 on modules of the form p^*N is trivial, we conclude that $(i_E \circ \alpha_{K,1}'')^*(N_K) \simeq (i \circ \alpha_K'')^*(p^*N_K)$. Similarly, $(i_E'' \circ \gamma_{K,1}'')^*(N_K) \simeq (i \circ \gamma_K'')^*(p^*N_K)$. Since N can be taken to be an arbitrary τ-module, we conclude that $i_E \circ \alpha_{K,1}'' \sim i_E \circ \gamma_{K,1}''$ as π-points of τ, where $i_E : E \hookrightarrow \tau$ is the embedding of E into τ.

By the minimality assumption on E' for α''_K, we conclude that $[\alpha''_{K,1}] \in \Pi_0(E')$. Similarly, $[\gamma''_{K,1}] \in \Pi_0(E'')$. If E'' were a proper subgroup of E, then the "Quillen decomposition" for the finite group τ (see (4.0.1)) would imply that $\Pi_0(E')/N_\tau(E') \cap \Pi_0(E'')/N_\tau(E'') = \emptyset$ in $\Pi(\tau)$. This contradicts the fact that $i_E \circ \alpha''_{K,1} \sim i_E \circ \gamma''_{K,1}$, so that we conclude that $E'' = E'$ and thus that E' is minimal as required. □

The following theorem is the culmination of special cases proved earlier as Theorems 2.7, 2.9, and 3.4.

gen

Theorem 4.2. *Let G be a finite group scheme over k and let*

$$\alpha_K : K[t]/t^p \to KG, \quad \beta_L : L[t]/t^p \to LG$$

be equivalent generic π-points of G. Then for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ has the same Jordan type as $\beta_L^*(M_L)$.

Proof. As seen in [12, 4.2], we may find a finite field extension F/k such that G_F satisfies the hypotheses of Proposition 4.1 that G_F splits as a semi-direct product $G_F^o \rtimes \tau$, with $\tau = \pi_0(G_F)$ constant and G_F^o geometrically connected. Thus, appealing to Lemma 2.8, we may assume that G itself satisfies these conditions.

Let $[\alpha_K] \in \Pi(G)$ be a generic point and choose some representative $\alpha_K = i_C \circ \tilde{\alpha}_K : K[t]/t^p \to KC_K \to KG$ with $C_K \subset G_K$ a unipotent abelian subgroup scheme. Let $E \subset \pi_0(C_K)$ be the maximal elementary abelian p-subgroup and observe that $\Pi(C_K^o \times E) \to \Pi(C_K)$ is a homeomorphism. Thus, we may choose $\alpha'_K : K[t]/t^p \to K(C_K^o \times E) \subset K((G^o)^E \times E)$ whose composition with $K((G^o)^E \times E) \to KG$ represents $[\alpha_K]$. Since α_K is generic, α'_K is a generic π-point of $(G^o)^E \times E$. By Proposition 4.1, we can find $\alpha''_K : K[t]/t^p \to K((G^o)^E \times E')$ such that $[\alpha''_K] \in \Pi_0((G^o)^{E'} \times E')$ and $\alpha''_K \sim \alpha'_K$ as π-points of $(G^o)^E \times E$. The latter implies that α''_K is generic for $(G^o)^E \times E$. Since $k((G^o)^E \times E) \simeq k((G^o)^E \times \mathbb{G}_{a(1)}^{\times r})$ where r is the rank of E, we may apply Theorem 3.4 to $(G^o)^E \times E$. Thus,

$$\alpha_K^*(M_K) \simeq \alpha''^*_K(i'^* M_K),$$

where $i' : (G^o)^E \times E' \to G$. Repeating this argument for β_L, we conclude that

$$\beta_L^*(M_L) \simeq \beta''^*_L(j'^* M_L)$$

where $\beta''_L : L[t]/t^p \to L((G^0)^F \times F')$ is generic with $F' \subset F$, $[\beta''_L] \in \Pi_0((G^0)^{F'} \times F')$, and $j' \circ \beta''_L \sim \beta_L$. Here, $j' : (G^0)^F \times F' \hookrightarrow G$.

We therefore obtain $[\alpha''_K] \in \Pi_0((G^0)^{E'} \times E')$, $[\beta''_L] \in \Pi_0((G^o)^{F'} \times F')$ such that α''_K, β''_L become equivalent (generic) π-points of G after composing with the embeddings of the corresponding subgroup schemes into G. This implies that E' and F' are conjugate by an element of τ; otherwise

$$\Pi_0((G^0)^{E\prime} \times E')/N_\tau(E') \ \cap \ \Pi_0((G^0)^{F'} \times F')/N_\tau(F')$$

would be empty, which would contradict the Quillen decomposition (4.0.1). Thus, after replacing β''_L by a conjugate, which does not affect the Jordan type, we may assume that α''_K, β''_L are equivalent generic π-points of $(G^0)^{E'} \times E'$. Applying Theorem 3.4 once again (using the isomorphism $k((G^o)^{E'} \times E') \simeq k((G^o)^{E'} \times \mathbb{G}_{a(1)}^{\times r'})$ we conclude that the Jordan type of $\alpha''^*_K(i'^* M_K)$ equals that of $\beta''^*_L(j'^* M_L)$, and hence the Jordan type of $\alpha_K^*(M_K)$ equals that of $\beta_L^*(M_L)$. □

We emphasize the conclusion of Theorem 4.2 with the following definition. As we shall see in examples given at the end of this section, a given kG-module M can have different Jordan types at different (equivalence classes of) generic π-points; moreover, at some generic points, the Jordan type need not be maximal.

Definition 4.3. Let G be a finite group scheme and M be a kG-module. A generic Jordan type of M is the Jordan type of $\alpha_K^*(M_K)$ for a generic π-point α_K of G. We denote the isomorphism class of $\alpha^*(M_K)$ (as a $K[t]/t^p$-module) by

$$[\alpha_K]^*(M_K).$$

For the purposes of the following proposition we utilize the coproduct on the algebra $k[t]/t^p$ given by the formula $\nabla(t) = t \otimes 1 + 1 \otimes t$. We point out here that this is a temporary convention: as it will be shown in Corollary 4.5 and more generally in Remark 4.6 the Jordan type of a tensor product of finite dimensional $k[t]/t^p$-modules does not depend on the Hopf algebra structure given to $k[t]/t^p$.

Since a π-point is not necessary a Hopf algebra map and not even always equivalent to one, the following proposition is somewhat striking, and will not necessarily hold for a non-generic π-point.

unipotent **Proposition 4.4.** *Let C be a unipotent abelian finite group scheme of k and let $\alpha_K : K[t]/t^p \to KC$ be a generic π-point of C. Then for any pair of finite dimensional kC-modules M, N,*

$$\alpha_K^*(M_K \otimes_K N_K) \;\simeq\; \alpha_K^*(M_K) \otimes_K \alpha_K^*(N_K).$$

Proof. Because we may base change to the perfect closure of k without changing the conclusion, we may assume that k is perfect. Thus, we may assume that $kC \simeq k[t_1, \ldots, t_r]/(t_i^{p^{e_1}}, \ldots, t_r^{p^{e_r}})$ (see [20, 14.4]). Set $T_i = t_i^{p^{e_i-1}}$. Then any π-point $\beta : k[t]/t^p \to kC$ sends t to a polynomial in $\{t_i\}$ whose p-th power is trivial and thus each of whose monomials is divisible by some T_i; the flatness of β is equivalent to the conditon that some monomial constituting $\beta(t)$ is a non-zero multiple of some T_i.

Let $\rho_M : kC \to \mathrm{End}(M)$ be the map defined by the representation M of C, and similarly define ρ_N and $\rho_{M\otimes N}$. Since α_K is generic, Theorem 2.9 implies that $\rho_M(\alpha_K(t))$ is maximal among the images under ρ_M of all p-nilpotent elements of the commutative algebra kC, and similarly for N. Proposition 1.3 implies the equivalences

Mmax
$$\rho_M(\alpha_K(t)) \sim \rho_M(\alpha_K(t)) + s_1\rho_M(T_1) + \cdots + s_r\rho_M(T_r) \tag{4.4.1}$$

and

Nmax
$$\rho_N(\alpha_K(t)) \sim \rho_N(\alpha_K(t)) + s_{r+1}\rho_N(T_1) + \cdots + s_{2r}\rho_N(T_r) \tag{4.4.2}$$

where s_i are indeterminants. Therefore,

tensormax
$$\begin{aligned}\rho_M(\alpha_K(t)) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t)) \sim \rho_M(\alpha_K(t)) \otimes 1 + s_1\rho_M(T_1) \otimes 1 + \cdots +\\ s_r\rho_M(T_r) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t)) + 1 \otimes s_{r+1}\rho_N(T_1) + \cdots + 1 \otimes s_{2r}\rho_N(T_r)\end{aligned} \tag{4.4.3}$$

Indeed, if A is the matrix which makes the endomorphisms of 4.4.1 similar, and B is such a matrix for 4.4.2, then $A \otimes B$ makes the endomorphisms in 4.4.3 similar. Proposition 1.3 implies that $\rho_M(\alpha_K(t)) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t))$ is maximal with respect to $\{\rho_M(T_i) \otimes 1, 1 \otimes \rho_N(T_j); 1 \leq i, j \leq r\}$. Thus, Theorem 1.12 implies that

$\rho_M(\alpha_K(t)) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t))$ is maximal with respect to

rho (4.4.4)

$$\{\rho_M(q(\underline{t})) \otimes 1,\ 1 \otimes \rho_N(q'(\underline{t}));\quad \text{each monomial of } q, q' \text{ divisible by some } T_i\}.$$

Let $\nabla : kC \to kC \otimes kC$ be the coproduct on kC. Since $\nabla(\alpha_K(t)) - \alpha_K(t) \otimes 1 - 1 \otimes \alpha_K(t) \in I \otimes I$ (see [15, I.2.4]) and is p-nilpotent, this difference is a sum of terms each of which is either a product of the form $(q(\underline{t}) \otimes 1) \cdot (1 \otimes f(\underline{t}))$ with each monomial of $q(\underline{t})$ divisible by some T_i and with f having no constant term or a product of the form $(g(\underline{t}) \otimes 1) \cdot (1 \otimes q'(\underline{t}))$ with g having no constant term and each monomial of $q(\underline{t})$ divisible by some T_i. Theorem 1.12 and the maximality of $\rho_M(\alpha_K(t)) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t))$ with respect to (4.4.4) imply

$$\rho_M(\alpha_K(t)) \otimes 1 + 1 \otimes \rho_N(\alpha_K(t)) \ \sim\ ((\rho_M \otimes \rho_N) \circ \nabla)(\alpha_K(t)) = \rho_{M \otimes N}(\alpha_K(t)).$$

Thus, $\alpha_K^*(M) \otimes_K \alpha_K^*(N)$ and $\alpha_K^*(M_K \otimes_K N_K)$ have the same Jordan type.

□

The following is an interesting, though very special case of Proposition 4.4 in which $\alpha_K : K[t]/t^p \to KC$ is the identify $id : k[t]/t^p \to k[t]/t^p$.

indep **Corollary 4.5.** *Let C be a unipotent abelian finite group scheme with $kC = k[t]/t^p$ and let $M,\ N$ be finite dimensional kC-modules. Then the tensor product of $M,\ N$ as kC-modules is isomorphic as a $k[t]/t^p$-module with the tensor product $M \otimes N$ using the coproduct $\nabla(t) = t \otimes 1 + 1 \otimes t$.*

anyHopf **Remark 4.6.** The proof of Proposition 4.4 does not use the fact that kC is the Hopf algebra of a group scheme: namely, we do not need cocommutativity of the coproduct. The only fact about the coproduct which is needed is that $\nabla(\alpha_K(t)) - \alpha_K(t) \otimes 1 - 1 \otimes \alpha_K(t) \in I \otimes I$ which holds for any Hopf algebra (see, for example, ex.3 on p.19, [20]). With this in mind, we can strengthen the statement of the Corollary 4.5 as follows: Let $M,\ N$ be finite dimensional $k[t]/t^p$-modules, let $M \otimes N$ denote the $k[t]/t^p$-module determined by the coproduct $\nabla(t) = t \otimes 1 + 1 \otimes t$. Then for any other coproduct $\nabla' : k[t]/t^p \to k[t]/t^p \otimes k[t]/t^p$ associated to a Hopf algebra structure on $k[t]/t^p$, the resulting $k[t]/t^p$-module is isomorphic to $M \otimes N$.

Proposition 4.4 provides the key verification to enable us to prove the following pleasing properties of generic Jordan type.

gen-prop **Proposition 4.7.** *Let G be a finite group scheme over a field k and let $[\alpha_K] \in \Pi(G)$ be a generic point for some K/k. Let M, N be finite dimensional kG-modules.*

- $[\alpha_K]^*(M_K \oplus N_K) \simeq [\alpha_K]^*(M_K) \oplus [\alpha_K]^*(N_K)$.
- $[\alpha_K]^*(M_K \otimes N_K) \simeq [\alpha_K]^*(M_K) \otimes [\alpha_K]^*(N_K)$.
- $[\alpha_K]^*(\Omega(M_K)) = \Omega([\alpha_K]^*(M_K))$ *in the stable module category of finite dimensional $K[t]/t^p$-modules, $stmod(K[t]/t^p)$.*

Thus, $[\alpha_K]^* : (kG - modules) \to (K[t]/t^p - modules)$ induces a functor on tensor triangulated categories

$$\alpha_K^* : (stmod(kG)) \to (stmod(K[t]/t^p)).$$

Proof. The first property follows from the observation that α_K^* commutes with direct sums.

Let $\alpha_K : K[t]/t^p \to KG$ be a generic π-point and let $C_K \subset G_K$ be a unipotent abelian subgroup scheme through which α_K factors. Write $\alpha_K = i \circ \alpha'_K : K[t]/t^p \to KC_K \to KG$. If we replace α'_K by a generic π-point $\alpha''_L : L[t]/t^p \to C_L$ of C_K, the new composition $i \circ \alpha''_L$ represents the same generic point of $\Pi(G)$. Thus, $[\alpha_K]^* = [i \circ \alpha''_L]^*$. The second property now follows from the observation that i is a Hopf algebra map and that Proposition 4.4 applies to α''_L.

The third property follows from the exactness of α_K^* and the fact that α_K^* sends projectives to projectives.

□

The following example exhibits a finite dimensional kG-module M which has at least two distinct generic Jordan types for any finite group scheme G with $\Pi(G)$ reducible. We implicitly use Proposition 4.7 in the justification of this example.

different

Example 4.8. Let G be any finite group scheme with $\Pi(G)$ reducible; for example, G any finite group with at least two distinct conjugacy classes of maximal elementary abelian p-subgroups. Write $\Pi(G) = X \cup Y$ with X, Y proper closed subsets and choose generic points $[\alpha_K] \in X$, $[\beta_L] \in Y$. Extending scalars, if necessary, we can assume $L = K$. Choose $\zeta \in H^{2i}(G,k)$, $\xi \in H^{2j}(G,k)$ such that

$$\alpha_K^*(\zeta_K) = 0 \neq \beta_K^*(\zeta_K) \text{ and } \alpha_K^*(\xi_K) \neq 0 = \beta_K^*(\xi_K)$$

(Here, α_K^* is the map in cohomology $H^*(G,K) \to H^*(K[t]/t^p, K)$ induced by α_K.) Let L_ζ, L_ξ be the "Carlson modules" associated to ζ, ξ (cf. [2, II.5.9]). Then $\alpha_K^*(L_{\zeta,K})$ fits into the exact sequence of $\mathbb{Z}/p$ - modules

$$0 \to \alpha_K^*(L_{\zeta,K}) \to \Omega^{2i}K \to K \to 0$$

where the map $(\tilde{\zeta_K}) \downarrow_{\alpha_K} \colon \Omega^{2i}K \to K$ is given by ζ_K via the isomorphism $\underline{\mathrm{Hom}}(\Omega^{2i}K, K) = H^{2i}(G,K)$ and then via restriction via α_K. Since $\alpha_K^*(\zeta_K) = 0$, the map $(\tilde{\zeta_K}) \downarrow_{\alpha_K} \colon \Omega^{2i}K \to K$ factors through a projective module. Since $\Omega^{2i}K = K \oplus \text{ proj }$ as $\mathbb{Z}/p$ - modules, we conclude that the kernel $\alpha_K^*(L_{\zeta,K}) = K \oplus \Omega K \oplus \text{ proj }$. In other words, the Jordan type of $\alpha_K^*(L_{\zeta,K})$ is $[1]+[p-1]+[$ blocks of size $p]$. Arguing similarly, we get that $\beta_K^*(L_{\zeta,K})$ and $\alpha_K^*(L_{\xi,K})$ are projective whereas $\beta_K^*(L_{\xi,K})$ has Jordan form $[1] + [p-1] + [$ blocks of size $p]$.
Let $M = L_\zeta \oplus L_\xi^{\oplus 2}$. Then the generic Jordan types $[\alpha_K]^*(M_K)$, $[\beta_K]^*(M_K)$ are different. Indeed, it is enough to show that they stably different, i.e. different up to projective summands. The stable Jordan type of $[\alpha_K]^*(M_K)$ equals that of $[\alpha_K]^*(L_{\zeta,K})$ which is $[1] + [p-1]$, whereas the stable Jordan type of $[\beta_K]^*(M_K)$ equals that of $[\beta_K]^*(L_{\xi,K}^{\oplus 2})$ which is $2[1] + 2[p-1]$. In particular, M_K has absolute Jordan type $[\alpha_K]^*(M_K)$.

We give a familiar example involving finite groups in which there is more than one generic type, one of which dominates the others.

std-fp

Example 4.9. Consider the example of $G = GL(3, \mathbb{F}_p)$ with $p > 2$. The irreducible components of $\Pi(G)$ are indexed by the conjugacy classes of maximal elementary p-subgroups of G which are represented by subgroups of the unipotent subgroup $U(3, \mathbb{F}_p)$ of strictly upper triangular matrices. There are 3 conjugacy classes, represented by the following subgroups:

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\} \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\} \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\}$$

Let M be the standard 3-dimensional (rational) representation of G. Then the Jordan type of M indexed by the first of these maximal elementary abelian subgroups of G is a single block of size 3, whereas the Jordan types indexed by each of the other conjugacy classes of maximal elementary abelian subgroups are strictly smaller.

We now present the extension of Theorem 3.5 to arbitrary finite group schemes.

max **Theorem 4.10.** *Let G be a finite group scheme over k and M a finite dimensional kG-module. Let $\alpha_K : K[t]/t^p \to KG$ be such that $\alpha_K^*(M_K)$ has maximal Jordan type. Then for any π-point $\beta_L : L[t]/t^p \to LG$ which specializes to α_K, the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$; in particular, if $\alpha_K \sim \beta_L$, then $\alpha_K^*(M_K) \simeq \beta_L^*(M_L)$.*

Proof. We first consider the special case in which $\alpha_K \sim \beta_L$, and verify that $\beta_L^*(M_L)$ has the same (maximal) Jordan type as $\alpha_K^*(M_K)$. We proceed as in the proof of Theorem 4.2. Write $\alpha_K = i_C \circ \alpha'_K : K[t]/t^p \to KC_K \to KG$ with $C_K \subset G_K$ a unipotent abelian subgroup scheme. Observe that $\alpha_K^*(M_K)$ is maximal with respect to all $\gamma_\Omega^*(i_C^*(M_\Omega))$ as $\gamma_\Omega : \Omega[t]/t^p \to \Omega C_K$ runs through all π-points of C_K. By [10, 4.2], we may find some subgroup scheme $j : C_K^o \times E \subset C_K$ for some elementary abelian p-subgroup $E \subset \pi_0(C_K)$ and a π-point $\delta_\Omega : \Omega[t]/t^p \to \Omega(C_K^o \times E)$ such that $j \circ \delta_\Omega \sim \alpha'_K$ as π-points of C_K, extending scalars if necessary. The maximality of $\alpha^*(M_K) = \alpha_K'^*(i_C^*(M_K))$ implies that $(j \circ \delta_\Omega)^*(i_C^*(M_\Omega))$ has the same Jordan type by Theorem 2.9.

Let $\alpha''_\Omega = i_C \circ \delta_\Omega : \Omega[t]/t^p \to \Omega(C_K^o \times E) \to \Omega((G_K^o)^E \times E)$. By Proposition 4.1, we may find $\alpha'''_\Omega : \Omega[t]/t^p \to \Omega((G^0)^E \times E')$ equivalent to α''_Ω with the additional property that $[\alpha'''_\Omega] \in \Pi_0((G^0)^{E'} \times E')$ for some $E' \subset E$. Applying Theorem 3.5 to $(G^0)^E \times E \simeq (G^0)^E \times \mathbb{G}_{a(1)}^{\times r}$ $(r = rk(E))$, we conclude that $\alpha_\Omega'''^*(i^* M_\Omega)$ has the same maximal Jordan type as $\alpha_\Omega''^*(i^* M_\Omega) = (j \circ \delta_\Omega)^*(i_C^*(M_\Omega))$ where $i : (G_K^o)^E \times E \to G_K$ is the inclusion.

We now perform the same operations on β_L using the factorization $\beta_L = i_D \circ \beta'_L : L[t]/t^p \to LD_L \to LG$ through an abelian subgroup scheme $D_L \subset G_L$. For some elementary abelian subgroup $F \subset \pi_0(G)$ and some $F' \subset F$, we obtain $\beta'''_\Omega : \Omega[t]/t^p \to \Omega((G^o)^F \times F')$ with $[\beta'''_\Omega] \in \Pi_0((G^0)^{F'} \times F')$.

Using (4.0.1) as in the last paragraph of the proof of Theorem 4.2 we conclude that some $\pi_0(G)$-conjugate of β'''_Ω is equivalent to α'''_Ω as a π-point of $(G^0)^{E'} \times E'$. Thus, Theorem 3.5 implies that $\beta_\Omega'''^*(j^* M_\Omega)$ has the same maximal Jordan type as $\alpha_\Omega'''^*(i^* M_\Omega) \simeq \alpha_\Omega^*(M_\Omega)$, where $j : (G^o)^F \times F' \subset G$ is the inclusion. Consequently, the preceding argument given for α_K now "run backwards" and applied to β_L implies that $\beta_L^*(M_L)$ also has the same maximal Jordan type.

More generally, assume that β_L specializes to α_K as π-points of G and construct β'''_Ω, $[\beta'''_\Omega] \in \Pi_0((G^0)^{F'} \times F')$ as above. Let $j : (G^0)^{F'} \times F' \hookrightarrow G$ denote the embedding. Since the image of $\Pi((G^0)^{F'} \times F')$ in $\Pi(G)$ is closed, and since $j \circ \beta'''_L \sim \beta_L$ specializes to α_K, we can find $\gamma_\Omega : \Omega[t]/t^p \to \Omega((G^0)^{F'} \times F')$ such that $j \circ \beta'''_\Omega$ specializes to $j \circ \gamma_\Omega$, and $j \circ \gamma_\Omega \sim \alpha_K$ as π-points of G.

The Quillen decomposition (4.0.1) implies that β'''_Ω specializes to a $\pi_0(G)$-conjugate of γ_Ω as a π-point of $(G^0)^{F'} \times F'$. Since $j \circ \gamma_\Omega \sim \alpha_K$, the just proved special case (in which $\beta_L \sim \alpha_K$) implies that $\gamma_\Omega^*(j^*(M_\Omega))$ has the same maximal Jordan type. Thus, $\beta_\Omega'''^*(j^*(M_\Omega))$ has the same maximal Jordan type by Theorem 3.5 applied to

$(G^0)^{F'} \times F'$. Since $\beta_L \sim j \circ \beta'''_\Omega$, we conclude by another application of the case of equivalent π-points that $\beta_L^*(M_L)$ has the same maximal Jordan type. □

The following corollary follows immediately from Theorem 4.10.

irr **Corollary 4.11.** *Let G be a finite group scheme and M a finite dimensional kG-module. Then each maximal Jordan type of M can be realized as $\alpha_K^*(M_K)$ for some generic point $[\alpha_K] \in \Pi(G)$.*

In particular, if $\Pi(G)$ is irreducible, then for any finite dimensional kG-module M the Jordan type of $\alpha_K^*(M)$ is of absolute maximal type whenever α_K represents the generic point of $\Pi(G)$.

In order to consider examples obtained by induction from subgroup schemes, we shall employ the following well known property of support varieties. For lack of a good reference, we provide a simple proof of this result. Recall that if $H \subset G$ is a closed subgroup scheme and N an H-module, then the induced module $Ind_H^G(N)$ is the G-module with underlying vector space $(N \otimes k[G])^H$, with G action given by the left regular action of G of $k[G]$ and with H-invariants taken with respect to the given action on N and the right regular action on $k[G]$.

ind **Lemma 4.12.** *Let G be a finite group scheme and $i : H \subset G$ a closed subgroup scheme. Let N be a finite dimensional kH-module, and let $M = Ind_H^G(N)$. Then the natural map $i : \Pi(H) \to \Pi(G)$ satisfies the property*

$$\Pi(G)_M \ \subset \ i(\Pi(H)_N).$$

Proof. Observe that the action of $H^\bullet(G,k)$ on $\mathrm{Ext}_G^*(M,M)$ factors as the composition of the restriction $H^\bullet(G,k) \to H^\bullet(H,k)$ followed by the natural action of $H^\bullet(H,k)$ on $Ext_H^*(N, M\downarrow_H) = \mathrm{Ext}_G^*(M,M)$. Hence, a homogeneous prime ideal of $H^\bullet(G,k)$ contains the annihilator of $\mathrm{Ext}_G^*(M,M)$ (and hence lies in $\mathrm{Proj}\,|G|_M = \Pi(G)_M$) if and only if it is the pre-image of a homogeneous prime ideal of $H^\bullet(H,k)$ containing the annihilator of $\mathrm{Ext}_H^*(N, M\downarrow_H)$ and the latter are among the homogeneous prime ideals of $H^\bullet(H,k)$ containing the annihilator of $\mathrm{Ext}_H^*(N,N)$ (i.e., an element of $\mathrm{Proj}\,|H|_N = \Pi(H)_N$) since the action of $H^*(H,k)$ on $\mathrm{Ext}_H^*(N,M)$ factors through $\mathrm{Ext}_H^*(N,N)$. □

To complement Corollary 4.11, we give an example of a finite dimensional kG-module for which there is no absolute maximal type.

not-comparable **Example 4.13.** Let G be a finite p-group which has two conjugacy classes of elementary abelian subgroups, represented by E and E' respectively. Furthermore, we require E to be normal. Let $e = \#E$, $f = \frac{\#G}{\#E}$. Assume $p > 3$.

For example, take G to be the p-Sylow subgroup of the wreath product $\mathbb{Z}/p \wr S_p$, so that G is isomorphic to $(\mathbb{Z}/p)^p \rtimes \mathbb{Z}/p$. Then G has two non-conjugate elementary abelian p-subgroups: $E = (\mathbb{Z}/p)^{\times p}$ which is normal and $F = (\mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p)^{\mathbb{Z}/p} \times \mathbb{Z}/p$.

By Quillen stratification, $\Pi(G) = X \cup Y$ where $X = \Pi(E)/G$, $Y = \Pi(F)/N_G(F)$. Let $[\alpha_K] \in X$, $[\beta_L] \in Y$ be generic points. As usual, we may assume $L = K$ after scalar extension.

Let n be a positive integer, let $M = \mathrm{Ind}_E^G(\Omega_E k)$, and let $N = L_\xi^{\oplus n}$ where L_ξ is a Carlson module such that $[\beta_K] \in \Pi(G)_{L_\xi}$ but $[\alpha_K] \notin \Pi(G)_{L_\xi}$. This can be achieved by choosing $\xi \in H^{2N}(G,k)$ with the property that the restriction of ξ to $H^{2N}(F,k)$ is 0 and to $H^{2N}(E,k)$ is non-zero.

Since E is normal in G, the double coset formula implies that $M\downarrow_E \simeq (\Omega_E k)^{\oplus f}$. Since α_K is a π-point of E, the Jordan type of $\alpha_K^*(\Omega_E k)$ is

$$[p-1] + \frac{e-p}{p}[p]$$

and the dimension is $e-1$. Hence, the Jordan type of $\alpha_K^*(M_K)$ is

$$([p-1] + \frac{e-p}{p}[p])f$$

and the dimension is $(e-1)f$. Moreover, $\beta_K^*(M_K)$ is projective by Lemma 4.12.

Arguing as in Example 4.8, we get that $\alpha_K^*(N_K)$ is projective (of dimension $pn(\ell+1)$) where ℓ is the number of projective blocks in $\alpha_K^*(L_\xi)$. i.e. $\ell = \frac{\dim L_\xi - p}{p}$, and the Jordan type of $\beta_K^*(N_K)$ is

$$n\ell[p] + n[p-1] + n[1]$$

Thus, the generic Jordan types of $M \oplus N$ are given by
$\alpha_K^*((M\oplus N)_K) = \alpha_K^*(M_K) \oplus \alpha_K^*(N_K) \simeq ([p-1] + \frac{e-p}{p}[p])f + n(\ell+1)[p] =$

$$[p](\frac{e-p}{p}f + n(\ell+1)) + [p-1]f$$

$\beta_K^*((M\oplus N)_K) = \beta_K^*(M_K) \oplus \beta_K^*(N_K) \simeq \frac{(e-1)f}{p}[p] + n\ell[p] + n[p-1] + n[1] =$

$$[p](\frac{(e-1)f}{p} + n\ell) + n[p-1] + n[1]$$

Observe that for such two Jordan types to be incomparable it suffices for $\alpha_K^*((M\oplus N)_K)$ to have fewer blocks of size $[p]$ and fewer blocks altogether than $\beta_K^*((M\oplus N)_K)$. This amounts to two inequalities

$$\frac{(e-p)f}{p} + n(\ell+1) < \frac{(e-1)f}{p} + nl$$

and

$$\frac{(e-p)f}{p} + n(\ell+1) + f < \frac{(e-1)f}{p} + n\ell + 2n$$

This reduces to

last (4.13.1)
$$\frac{f}{p} < n < (p-1)\frac{f}{p}$$

Since $p > 3$ divides f, there exists n satisfying the inequalities 4.13.1. For such n, the corresponding module $M \oplus N$ has incomparable generic Jordan types at the points α_K and β_K.

The conclusion of Theorem 4.10 is not valid for generic Jordan types. Namely, as the following example shows, one can find a generic π-point α_K of a finite group scheme G, a π-point β which is a specialization of α and a G-module M such that the Jordan type of $\beta^*(M)$ is strictly greater than that of $\alpha_K^*(M_K)$. According to Corollary 4.11, in any such example $\Pi(G)$ must be reducible.

speclialization **Example 4.14.** We return to the example $G = (\mathbb{Z}/p)^p \rtimes \mathbb{Z}/p$ of Example 4.13 and retain the same notation $\Pi(G) = X \cup Y, E, F, e, f$. Let $M = \mathrm{Ind}_E^G k$. Then the double coset formula implies that $M\downarrow_E$ is trivial, so that $\Pi(G)_M = X$ by Lemma 4.12. Let $\{g_1, \ldots, g_p\}$ be generators of E, $t_i = g_i - 1$ and $K = k(s_1, \ldots, s_p)$.

Let $\alpha_K : K[t]/t^p \to KE \to KG$ be a generic π-point of G corresponding to the component X defined by

$$\alpha_K(t) = s_1t_1 + \cdots + s_pt_p.$$

The Jordan type of $\alpha_K^*(M_K)$ is trivial. Let γ be a k-rational π-point of G representing the point on the intersection $X \cap Y$ given by

$$\gamma(t) = t_1 + t_2 + \cdots + t_p.$$

Since E acts trivially on M, the Jordan type of $\gamma^*(M)$ is trivial.

Let $\beta : k[t]/t^p \to kF \to kG$ be a rational π-point of G such that $[\beta] \in Y$, $[\beta] \notin X$. Since $\Pi(G)_M \subset X$, we conclude that $\beta^*(M)$ is projective. Since both γ and β factor through kF, $\beta(t)$ and $\gamma(t)$ commute. Define a new π-point $\tilde{\gamma}$ by

$$\tilde{\gamma}(t) = \gamma(t) + \beta^2(t).$$

By Proposition 2.2, $[\gamma] = [\tilde{\gamma}]$ in $\Pi(G)$ and, hence, $\tilde{\gamma}$ is a specialization of α_K. On the other hand, since $\beta^*(M)$ is projective and $p > 2$, $\tilde{\gamma}^*(M)$ is non-trivial. Thus, the Jordan type of $\tilde{\gamma}^*(M)$ is greater than that of $\alpha_K^*(M_K)$.

As we see in the following example, the maximal Jordan type can sometimes be easily determined as a familiar invariant associated to a given kG-module M.

std-1 **Example 4.15.** Let $\mathcal{G}$ be a reductive algebraic group over k with Lie algebra g. Let $G = \mathcal{G}_{(1)}$, so that $kG \simeq u(g)$, the restricted enveloping algebra of $g = Lie(\mathcal{G})$. Then the maximal Jordan type of a finite dimensional kG-module M is the Jordan type of any regular nilpotent element $X \in g$ acting upon M provided that every nilpotent element of g is p-nilpotent.

Consider the special case $G = SL_{n(1)}$ and let M be the standard n-dimensional representation of SL_n restricted to G. We make no assumption on the size of p with respect to n (i.e., we do not assume that every nilpotent element of sl_n is p-nilpotent). Each p-nilpotent $X \in sl_n$ determines a 1-parameter subgroup $\mu_X : \mathbb{G}_{a(1)} \to G$. The Jordan type of $(\mu_X)^*(M)$ is merely the Jordan type of X itself. The maximal Jordan type is that given by the partition given by $\left[\frac{n}{p}\right]$ blocks of size p and one block of size $n - p\left[\frac{n}{p}\right]$.

More generally, let $G = SL_{n(r)}, r \geq 1$ with M once again the standard representation of SL_n restricted to G. Since any π-point of $SL_{n(1)}$ can be extended to a π-point of $SL_{n(r)}$, the maximal Jordan type of $M \downarrow_{SL_{n(1)}}$ is at most the maximal Jordan type of $M \downarrow_{SL_{n(r)}}$. On the other hand, the maximal Jordan type of $M \downarrow_{SL_{n(1)}}$ is maximal possible of any module of dimension n. Thus, the maximal Jordan type of M as $SL_{n(r)}$-module remains the partition with $\left[\frac{n}{p}\right]$ blocks of size p and one block of size $n - p\left[\frac{n}{p}\right]$.

The following most elementary example of an infinitesimal group scheme of height bigger than 1 gives a first indication of the behavior of generic Jordan type for infinitesimal group schemes.

twist **Example 4.16.** Let $G = \mathbb{G}_{a(r)}$ for some $r > s > 1$. Consider a finite dimensional $k\mathbb{G}_{(r-s)}$-module M and the pull-back $p^*M = M^{(s)}$ via the projection $p : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}/\mathbb{G}_{a(s)} \simeq \mathbb{G}_{a(r-s)}$ sending $u_i \in k\mathbb{G}_{a(r)}$ to $u_{i-s} \in k\mathbb{G}_{a(r-s)}$ for $i > s$ and 0 otherwise. Observe that the action of $u_i \in k\mathbb{G}_{a(r)}$ on $M^{(s)}$ can be identifed with the action of $u_{i-s} \in k\mathbb{G}_{a(r-s)}$ on M. We readily verify that the generic Jordan type of the $k\mathbb{G}_{(r-s)}$-module M is equal to the generic Jordan type of the $k\mathbb{G}_{a(r)}$-module $M^{(s)}$. Namely, the generic type of M is determined by the action of a generic linear combination of the generators $u_{i-s} \in k\mathbb{G}_{a(r-s)}$, and the generic type of $M^{(s)}$ is determined by the action of a generic linear combination of $u_i \in k\mathbb{G}_{a(r)}$. But since $u_i \in k\mathbb{G}_{a(r)}, i < s$ acts trivially on $M^{(s)}$ and $u_i \in k\mathbb{G}_{a(r)}, i \geq s$, acts as $u_{i-s} \in k\mathbb{G}_{a(r-s)}$ acts on M, these generic linear combinations clearly yield the same Jordan type.

5. Non-maximal support varieties

In this section we refine the construction of support varieties for modules over a finite group scheme. Cohomological support varieties were first introduced in the case of finite groups by J. Carlson [3], and subsequently extended by various authors, initially to restricted Lie algebras, then to infinitesimal group schemes, and finally to arbitrary finite group schemes. Carlson conjectured and G. Avrunin and L. Scott proved [1] that for elementary abelian p-groups the cohomological support variety of a module had a representation theoretic interpretation as the rank variety. This was subsequently generalized, with the ultimate formulation being the assertion of the existence of the naturally constructed homeomorphism (2.1.1) restricting to a homeomorphism of support varieties

$$\Pi(G)_M \;\simeq\; \mathrm{Proj}\,|G|_M$$

for any finite group scheme G and any finite dimensional kG-module M [12].

If the dimension of M is not divisible by p, then the support variety $\Pi(G)_M$ is necessarily equal to $\Pi(G)$. In this section, we introduce a refinement $\Gamma(G)_M \subset \Pi(G)$ of $\Pi(G)_M \subset \Pi(G)$ which does give information about a kG-module M even when the dimension of M is not divisible by p. On the other hand, if the (absolute) maximal Jordan type of M is projective, then the absolute maximal Jordan type of $\alpha_K^*(M_K)$ is the type of a projective $K[t]/t^p$-module so that the non-maximal support variety of M equals the support variety of M: $\Gamma(G)_M = \Pi(G)_M$.

def-non-max **Definition 5.1.** Let G be a finite group scheme over k and M a finite dimensional kG-module. We define

$$\Gamma(G)_M \;\subset\; \Pi(G)$$

to be the set of those equivalence classes $[\alpha_K] \in \Pi(G)$ such that the Jordan type of $\alpha_K^*(M)$ is not maximal for some choice $\alpha_K : K[t]/t^p \to KG$ of representative of $[\alpha_K]$.

Observe that Theorem 4.10 verifies that $[\alpha_K] \in \Gamma(G)_M$ if and only if for every choice of representative α_K of $[\alpha_K]$ the Jordan type of $\alpha_K^*(M)$ is not maximal.

non-max **Theorem 5.2.** *Let G be a finite group scheme over k and M a finite dimensional kG-module. Then $\Gamma(G)_M$ is a closed subspace of $\Pi(G)$, the* non-maximal support variety *of M.*

Proof. We first consider the special case in which G is infinitesimal of height r. Let A denote the coordinate algebra $k[V_r(G)]$ and consider the universal 1-paramter subgroup (3.0.1),

$$u : \mathbb{G}_{a(r),A} \to G_A,$$

which provides M_A with the rational $\mathbb{G}_{a(r),A}$-module structure. Let $X \subset V_r(G)$ be the subset of 1-parameter subgroups $\mu : \mathbb{G}_{a(r),K} \to G_K$ such that $(\mu_* \circ \epsilon)^*(M_K)$ does not have maximal Jordan type. Then X consists of those points $x \in V_r(G)$ at which $\epsilon^*(u^*(M_A) \otimes_A k(x))$ does not have maximal Jordan type.

Let $f_A : A[t]/t^p \to \operatorname{End}_A(M)$ be the composition

$$f_A : A[t]/t^p \xrightarrow{\epsilon} A\mathbb{G}_{a(r)} \xrightarrow{u_*} AG \xrightarrow{\rho_M} \operatorname{End}_A(M)$$

Then $x \in X$ if and only if $(f_A \otimes_A k(x))(t) \in \operatorname{End}_{k(x)}(M \otimes k(x))$ does not have maximal Jordan type. This is described by a set of equations on the ranks of powers of $f_A(t)$. Thus, X is the locus of points of $V_r(G) = \operatorname{Spec} A$ for which these equations admit a solution.

For a general finite group scheme G, we extend scalars to be able to assume that the Quillen decomposition (4.0.1) applies to G. Then $\Pi(G)$ is the finite union of the images of the closed maps $\Pi((G^o)^E \times E) \to \Pi(G)$ as $E \subset \tau = \pi_0(G)$ varies over conjugacy classes of elementary abelian p-subgroups of τ. Thus, for a given finite dimensional kG-module M, $\Gamma(G)_M$ consists of the union of the images of $\Pi((G^o)^E \times E)$ indexed by those conjugacy classes such that the Jordan type of M at the generic point of $(G^o)^E \times E$ is not maximal and the images of $\Gamma((G^o)^E \times E)_M \subset \Pi((G^o)^E \times E)$ indexed by those conjugacy class such that the Jordan type of M at the generic point of $(G^o)^E \times E$ is maximal. Consequently, it suffices to prove that each $\Gamma((G^o)^E \times E)_M \subset \Pi((G^o)^E \times E)$ is closed. Since $k((G^o)^E \times E)) \simeq k((G^o)^E \times \mathbb{G}_{a(1)}^{\times r})$ where r is the rank of E, this follows from the verification in the preceding paragraph for infinitesimal group schemes (applied to $(G^o)^E \times \mathbb{G}_{a(1)}^{\times r}$). □

contain **Proposition 5.3.** *Let G be a finite group scheme over k and M, N be finite dimensional kG-modules, and $f : H \to G$ be a flat map of finite group schemes Then*

(1) $\Gamma(G)_{M \oplus N} \subset \Gamma(G)_M \cup \Gamma(G)_N$. *Moreover, if $\Pi(G)$ is irreducible, then the equality holds.*
(2) $\Gamma(H)_{f^*M} \subset (f_*)^{-1}(\Gamma(G)_M)$ *where $f_* : \Pi(H) \to \Pi(G)$ is the map of schemes induced by f.*

Proof. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point with the property that $\alpha_K^*(M_K)$ is maximal for M and $\alpha_K^*(N_K)$ is maximal for N. Then the assertion that $\alpha_K^*((M \oplus N)_K)$ is maximal for $M \oplus N$ follows easily from the observation that $\alpha_K^*(-)$ commutes with direct sums. Namely, the condition that $\alpha_K^*(P_K)$ has maximal type for a kG-module P is a maximality condition on the ranks of powers of $\alpha_K(t) \in \operatorname{End}_K(P_K)$ which is clearly preserved under direct sums.

Assume $\Pi(G)$ is irreducible and let μ be a generic π-point of G. In this case any G-module has absolute maximal Jordan type at the π-point μ. Let $\alpha \in \Gamma(M) \cup \Gamma(N)$. Without loss of generality we may assume that $\alpha^*(M)$ is not maximal, i.e. $\alpha^*(M) < \mu^*(M)$. We have $\alpha^*(N) \leq \mu^*(N)$. The maximality condition on ranks which defines the partial ordering (1.1.1) together with the binomial formula imply that $\alpha^*(M \oplus N) < \mu^*(M \oplus N)$. Thus, $\alpha \in \Gamma(M \oplus N)$. We get $\Gamma(M) \cup \Gamma(N) \subset \Gamma(M \oplus N)$.

To show the second assertion observe that if $(f \circ \alpha_K)^*(M_K)$ is maximal among the π-points of G, then it is maximal among the π-points of the form $f \circ \beta$ where β is any π-point of H. Thus, $\alpha_K^*((f^*M)_K)$ is maximal for an H-module f^*M. This implies the required inclusion.

□

The following example demonstrates that the containment of statement (1) of Proposition 5.3 can not be sharpened to an equality.

Example 5.4. Let G be as in Example 4.8, $M_1 = L_\zeta$, $M_2 = L_\xi^{\oplus 2}$. Then $\Gamma(G)_{M_1} = \Pi(G)_{M_1}$, $\Gamma(G)_{M_2} = \Pi(G)_{M_2}$, so that $\Gamma(G)_{M_1} \cup \Gamma(G)_{M_2} = \Pi(G)$, whereas $\Gamma(G)_{M_1 \oplus M_2} \neq \Pi(G)$. Thus,

$$\Gamma(G)_{M_1 \oplus M_2} \; \neq \; \Gamma(G)_{M_1} \cup \Gamma(G)_{M_2}.$$

On the other hand, if both M and N have maximal Jordan types which are projective, then

$$\Gamma(G)_{M \oplus N} \; = \; \Pi(G)_{M \oplus N} \; = \; \Pi(G)_M \cup \Pi(G)_N \; = \; \Gamma(G)_M \cup \Gamma(G)_N.$$

The following examples involve the consideration of the standard n-dimensional representation of SL_n. For these examples, determination of the non-maximal support variety is particularly easy, but the behavior of these varieties is illustrative.

std **Example 5.5.** Let $G = SL_{n(1)}$ and let M be the standard n-dimensional representation of SL_n restricted to G. Recall that $\Pi(G_{(1)}) = \operatorname{Proj}(\mathcal{N}_p)$, the result of applying "proj" to the homogeneous closed subvariety $\mathcal{N}_p \subset s\ell_n$ of p-nilpotent matrices (cf. [9], [19]). By Example 4.15, the non-maximal support variety is homeomorphic to the subset of $\operatorname{Proj}(\mathcal{N}_p)$ consisting of equivalence classes of p-nilpotent matrices whose Jordan type is less than the type consisting of $\left[\frac{n}{p}\right]$ blocks of size p and one block of size $n - p\left[\frac{n}{p}\right]$. This agrees with $\Pi(G)_M \subset \Pi(G)$ if and only if n is divisible by p. For $p \geq n$, the non-maximal support variety is the complement inside the nullcone $\mathcal{N}$ (of all nilpotent matrices) of the open subset of regular nilpotent matrices.

Example 5.6. As in Example 4.9, let G be $GL(3, \mathbb{F}_p)$ with $p > 3$ and M be the standard 3-dimensional GL_n-module M restricted to $GL(n, \mathbb{F}_p)$. By the computation of Example 4.9, the non-maximal support variety is the union of two irreducible components.

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