REPRESENTATION-THEORETIC SUPPORT SPACES FOR FINITE GROUP SCHEMES

ERIC M. FRIEDLANDER* AND JULIA PEVTSOVA**

Abstract. We introduce the space $P(G)$ of abelian p-points of a finite group scheme over an algebraically closed field of characteristic $p > 0$. We construct a homeomorphism $\Psi_G : P(G) \to \mathrm{Proj}\,|G|$ from $P(G)$ to the projectivization of the cohomology variety for any finite group G. For an elementary abelian p-group (respectively, an infinitesimal group scheme), $P(G)$ can be identified with the projectivization of the variety of cyclic shifted subgroups (resp., variety of 1-parameter subgroups). For a finite dimensional G-module M, Ψ_G restricts to a homeomorphism $P(G)_M \to \mathrm{Proj}\,|G|_M$, thereby giving a representation-theoretic interpretation of the cohomological support variety.

0. Introduction

Even though the cohomology groups $H^i(G,k)$ of a finite group are typically difficult to compute, D. Quillen in his seminal papers [17] gave a general description of the maximal ideal spectrum $|G|$ of the commutative k-algebra $H^{ev}(G,k)$ in terms of the elementary abelian p-groups of G, where k is a field of characteristic p. Following work of J. Alperin and L. Evens [1], J. Carlson extended Quillen's work by studying the cohomological support variety $|G|_M \subset |G|$ of a finite dimensional kG-modules M for a finite group G [5]. One important aspect of this invariant $|G|_M$ of the G-module M is that it satisfies "Carlson's Conjecture", proved by G. Avrunin and L. Scott in [2]: for an elementary abelian p-group E and a finite dimensional E-module M, $|E|_M$ admits a description without recourse to cohomology, one in terms of shifted subgroups of E.

The first author and B. Parshall initiated an analogous theory for p-restricted Lie algebras and their restricted modules (e.g., [12]). Work of Friedlander-Parshall [12] and J. Jantzen [14] provided a description of the cohomological support variety of a restricted g-module M with the role of elementary abelian subgroups of a finite group being played by p-nilpotent elements in the p-restricted Lie algebra g. This work was refined and generalized to all infinitesimal group schemes (i.e., group schemes whose coordinate algebra is local) by C. Bendel, A. Suslin, and the first author [20]. In this generality, the role of elementary abelian subgroups of a finite group is played by 1-parameter subgroup schemes of a given infinitesimal group scheme.

There are many differences between the cohomological behavior of finite groups and infinitesimal group schemes. For example, if $H \subset G$ is an inclusion of infinitesimal group schemes then the induced map on cohomological support varieties $|H| \to |G|$ is always injective, whereas this is rarely the case for an inclusion of

* partially supported by the NSF and NSA.
** supported by the NSF under agreement No. DMS-9729992.

finite groups. As another example, if the G-module M is obtained by induction of the H-module N for some $H \subset G$, then the support $|G|_M$ is simply the image of $|H|_M$ in the context of finite groups but has no simple description in the context of infinitesimal group schemes. Nevertheless, it seems of considerable interest to find similarities and parallels between the two theories, developing a theory which applies to all finite group schemes and specializing to these two contexts.

The purpose of this paper is to introduce a common generalization for all finite group schemes of shifted subgroups (for elementary abelian p-groups) and 1-parameter subgroups (for infinitesimal group schemes) which serves to give a purely representation theoretic interpretation of cohomological support varieties. We achieve this by introducing "abelian p-points" of finite group schemes, which are certain maps of algebras $k\mathbb{Z}/p \to kG$, where kG is the dual to the coordinate algebra of the finite group scheme G. We provide the set $P(G)$ of abelian p-points of G with an intrinsic topology, and we introduce a closed subspace $P(G)_M \subset P(G)$ for each finite dimensional G-module M. We then verify that there is a homeomorphism $\Psi_G : \operatorname{Proj}|G| \simeq P(G)$ which restricts to a homeomorphism $\operatorname{Proj}|G|_M \simeq P(G)_M$ for every finite dimensional G-module M. Thus, our space $P(G)_M$ is an extension to all finite group schemes of Carlson's construction of the rank variety of an elementary abelian p-group for which the analogue of Carlson's conjecture remains valid. As shown in Theorem 5.6, the construction $M \mapsto P(G)_M$ has many good properties; in particular, it satisfies tensor product property: $P(G)_M \cap P(G)_N = P(G)_{M\otimes N}$.

Although we use the geometric language of varieties and schemes, the more algebraically inclined reader might keep in mind the natural equivalence of categories between finite group schemes (over k) and finite dimensional, co-commutative Hopf algebras, together with the natural equivalence for a given finite group scheme G relating G-modules to modules for the corresponding Hopf algebra.

We conclude this introduction with a brief description of the contents of our paper. In §1, we summarize known results concerning support varieties for finite groups and infinitesimal group schemes, results which we extend in the remainder of the paper. Finite abelian group schemes play a central role in our constructions and analysis of $P(G)$, so that we consider these in detail in §2. $P(G)$, the space of abelian p-points of G, is presented in §3, where we give the natural construction $\Psi_G : P(G) \to \operatorname{Proj}|G|$. §3 also establishes that Ψ_G is a bijection whenever G is either constant or connected. The fact that Ψ_G is a homeomorphism is proved in §4; for many special cases, our proof is self-contained but for a general finite group scheme, we rely on a recent result of A. Suslin (Theorem 4.10 below). Finally in §5, we reinterpret $P(G)$ in cohomological terms, prove a generalization of the "Quillen stratification" of the cohomological support variety of a finite group scheme, and give a reinterpretation of a result of J. Carlson, Z. Lin, and D. Nakano [7] relating the support varieties of $G(\mathbb{F}_p)$ and $G_{(1)}$ for a reductive algebraic group G.

Throughout this paper, p will be a fixed prime and k an algebraically closed field of characteristic p. The group schemes we consider will be group schemes over such a field k, algebraically closed of characteristic $p > 0$.

The authors thank Andrei Suslin for various useful conversations, and thank IHES, IAS for their hospitality.

1. Recollections for finite group schemes

In this section, we recall various definitions and results concerning finite group schemes and their cohomology. We begin with the definition of a finite group scheme.

Definition 1.1. An affine group scheme G over a field k is a representable functor $G : (k - alg) \to (grps)$ from the category of finitely generated commutative k-algebras to the category of groups. We denote the representing (commutative) algebra by $k[G]$ and call it the *coordinate algebra of* G. If $k[G]$ is finite dimensional, then we say that G is a finite group scheme; we denote the k-linear dual algebra of $k[G]$ by kG and call it the *group algebra* of G.

For example if π is a finite group, then π determines a "constant functor" whose value on each finitely generated, commutative, connected k-algebra is the group π. The coordinate algebra of π, $k[\pi]$, is a product of copies of k indexed by the elements of π, $k^{\times|\pi|}$. We call the finite group scheme so associated to a finite group a *constant finite group scheme*.

Another important example arises from a finite dimensional p-restricted Lie algebra g over k. The restricted universal enveloping algebra $V(g)$ is a possibly non-commutative k-algebra of dimension $p^{\dim g}$. Given a coproduct specified by requiring the elements of $g \subset V(g)$ to be primitive, $V(g)$ becomes a co-commutative, finite dimensional Hopf algebra whose coordinate algebra $V(g)^{\#}$ (the k-linear dual of $V(g)$) is local.

More generally, we say that a finite group scheme G is *infinitesimal* if the coordinate algebra $k[G]$ of G is a local algebra. Important examples of such infinitesimal group schemes arise from algebraic groups as follows. Let G be an affine algebraic group over k (so that G is represented by a finitely generated, commutative, reduced k-algebra $k[G]$). Consider the Frobenius map

$$F : G \to G^{(1)}$$

where $G^{(1)}$ is the base change of G via the Frobenius map (i.e., p-th power map) on k. Then the kernel of the r-th iterate of F,

$$G_{(r)} = \ker\{F^r : G \to G^{(r)}\}$$

is an infinitesimal group scheme, called the r-th Frobenius kernel of G (the reader may consult [13] for a precise definition). Taking $r = 1$, we find that $k[G_{(1)}]$ is the dual algebra of $V(g)$ where g is the Lie algebra of G.

Any finite group scheme G over k is a semi-direct product of its connected component G^0 (which is necessarily an infinitesimal group scheme) and its group of connected components $\pi = \pi_0(G)$ (which is necessarily constant). This is readily verified by observing that G_{red}, the reduced closed subscheme associated to G, maps isomorphically onto π.

If G is a group scheme over k, then a G-module M is a k-vector space equipped with the structure of a (left) comodule over the coordinate algebra $k[G]$ of G. If G is a finite group scheme, then such a G-module can be equivalently described as a (left) module over the group algebra kG of G. The category of G-modules, Mod_G, is an abelian category with enough injectives; moreover, if G is a finite group scheme, then every injective module of Mod_G is projective and every projective module is injective (cf. [9]). We let $mod_G \subset Mod_G$ denote the full subcategory of finitely generated G-modules. If G is a finite group scheme, then mod_G has enough injectives and projectives.

For any pair of G-modules M, N, we use standard homological algebra to define $\mathrm{Ext}^i_G(M,N)$, the i-th right derived functor of $\mathrm{Hom}_{Mod_G}(M,-)$ applied to N. As usual, we denote $\mathrm{Ext}^*_G(k,M)$ by $H^*(G,M)$.

Remark 1.2. It is sometimes convenient to consider the stable category $Stmod_G$, an additive category whose objects are the same as the objects of Mod_G and whose maps $\mathrm{Hom}_{Stmod_G}(M,N)$ are defined to be the quotient of $\mathrm{Hom}_{Mod_G}(M,N)$ by the subgroup of those maps from M to N which factor through a projective G-module. Similarly, the stable category $stmod_G$ is the full subcategory of $Stmod_G$ with objects the finitely generated G-modules. Then $Stmod_G$ and $stmod_G$ admit the structure of triangulated categories with distinguished triangles given by short exact sequences in Mod_G and mod_G and with shift operator $[n]$ applied to some G-module M given by the n-th Heller operator applied to M,

$$M[n] \ = \ \Omega^{-n}(M).$$

Thus, for $n > 0$, $M[n]$ can be represented as the quotient of the map $I^{n-2} \to I^{n-1}$ in some injective resolution $M \to I^\bullet$ of M; and $M[-n]$ can be represented as the kernel of the map $P_{n-1} \to P_{n-2}$ in some projective resolution $P_\bullet \to M$ of M. (For $n = 1$, we set $P_{n-2} = M = I^{n-2}$.) For G-modules M, N, we have natural identifications

$$\mathrm{Ext}^n_G(M,N) \simeq \mathrm{Hom}_{Stmod_G}(M[-n],N) \simeq \mathrm{Hom}_{Stmod_G}(M,N[n]).$$

We recall that if G is a finite group scheme, then $H^*(G,k)$ is a finitely generated algebra over k and $H^*(G,M)$ is a finitely generated module over $H^*(G,k)$ [13, 1.1]. Recall that for $p > 2$, $H^*(G,k)$ is only graded commutative so that to apply standard tools from commutative algebra we work with its even dimensional part. Throughout the paper we shall use the following notation:

$$H^\bullet(G,k) = \begin{cases} \bigoplus\limits_{n \geq 0} H^{2n}(G,k) & p > 2 \\ H^*(G,k) & p = 2 \end{cases}$$

Definition 1.3. Let G be a finite group scheme. The *cohomological variety* of G, denoted $|G|$, is the maximal ideal spectrum of the finitely generated commutative k-algebra $H^\bullet(G,k)$.

For any finitely generated G-module M, we denote by

$$|G|_M \ \subset \ |G|$$

the closed conical subset of those maximal ideals containing the annihilator of $\mathrm{Ext}^*_G(M,M)$. Thus, $|G|_M$ is the set of closed points of the "support" of the $H^\bullet(G,k)$-module $\mathrm{Ext}^*_G(M,M)$ in the sense of commutative algebra. So defined, $|G|_M$ is called the *(cohomological) support variety* of M.

Remark 1.4. The support variety of M has a natural scheme structure. However, we shall ignore this finer structure, for we do not have a natural scheme structure on our space of p-points $P(G)$ or on the subspace $P(G)_M \subset P(G)$ other than in the special case in which G is an infinitesimal group scheme.

Support varieties for finite groups (cf. [4], [6]) and infinitesimal group schemes (cf. [20]) satisfy many pleasing properties as indicated in the following theorem.

Recall that the complexity $\mathrm{cx}(M)$ of a G-module M is the growth of a minimal projective resolution of M as a G-module: the least integer r such that there exists some number c and a resolution $\cdots \to P_1 \to P_0 \to k$ of M by G-projectives with $\dim P_n \leq c \cdot n^{r-1}$.

Theorem 1.5. *Let G be a finite group or an infinitesimal group scheme and M be a finite dimensional G-module. The support variety $|G|_M$ satisfies the following properties:*

1. $|G|_k = |G|$.
2. *If $M_1 \to M_2 \to M_3 \to M_1[1]$ is a distinguished triangle in $stmod_G$, then $|G|_{M_2} \subset |G|_{M_1} \cup |G|_{M_3}$.*
3. $|G|_{M \oplus N} = |G|_M \cup |G|_N$.
4. $|G|_{M \otimes N} = |G|_M \cap |G|_N$.
5. *$|G|_M = \{0\}$ if and only if M is projective.*
6. *Let $i : H \subset G$ be a closed embedding of finite group schemes. Then $|H|_M = (i_*)^{-1}(|G|_M)$*
7. $dim|G|_M = cx(M)$.

Remark 1.6. In fact, the same properties hold for any finite group scheme as will be shown in Theorem 5.6.

Remark 1.7. One might ask whether the Hopf algebra structure on kG plays a necessary role in the formulation of $|G|_M$. One can define the algebra structure on $H^*(G,k)$ using the Yoneda product of extensions, thereby not utilizing the presence of the coordinate algebra $k[G]$. However, $H^*(G,k)$ acts upon $End^*_G(M,M)$ by tensoring a self extension of M by a self extension of k, and we use the coalgebra structure on kG to give the tensor product (over k) of two G-modules the structure of a G-module.

It is interesting to observe that if k is the only irreducible G-module, then $|G|_M \subset |G|$ can be described as the subset of those maximal ideals containing the annihilator of $H^*(G,M)$. Since the action of $H^*(G,k)$ on $H^*(G,M)$ can be described in terms of the Yoneda product, we see that in this case $|G|_M$ does not depend upon the coalgebra structure on kG. (Methods used to prove this claim for finite groups in [4, II.5.7] equally apply to any finite group scheme.) In particular, since $k\mathbb{Z}/p \simeq k\mathbb{G}_{a(1)}$, upon identifying the module categories of $\mathbb{Z}/p$ and $\mathbb{G}_{a(1)}$ we can equate $|\mathbb{Z}/p|_M$ with $|\mathbb{G}_{a(1)}|_M$ for any finite dimensional $k\mathbb{Z}/p$-module M.

Much of this paper will be concerned with providing a construction of "support varieties" for arbitrary finite group schemes which are representation-theoretic as opposed to cohomological. We conclude this section by recalling such constructions for elementary abelian p-groups and for infinitesimal group schemes.

Let E be an elementary abelian p group, $E \simeq (\mathbb{Z}/p)^r$ for some $r > 0$. Choose $x_1, \ldots, x_r \in I$ determining a basis for I/I^2, where I denotes the augmentation ideal of kE. Let further $\alpha_1, \alpha_2, \ldots, \alpha_s$ be a collection of $s \leq r$ k-linearly independent combinations of the x_i's. Then $\{1+\alpha_1, \ldots, 1+\alpha_s\}$ generate an elementary abelian subgroup of the multiplicative group of kE, called a *shifted subgroup* of E. Note that the group algebra of a shifted subgroup is a subalgebra of kE. In particular, a *cyclic shifted subgroup* of E is a subgroup of order p of kE generated by $1+\alpha$, where $\alpha = \Sigma_i a_i x_i$ is a non-trivial k-linear combination of the x_i's. We shall denote such a shifted cyclic subgroup by $\langle 1+\alpha \rangle$. Observe that even though $\langle 1+\alpha\rangle$ is not an "honest" subgroup of E, the map of augmented algebras $k\langle 1+\alpha\rangle \to kE$

induces a well-defined map in cohomology. We shall use this observation extensively throughout the paper.

The k-vector space $V(E) \subset I$ spanned by the x_i's is called the *rank variety* of E and can be considered as a parameter space for shifted subgroups of E via the correspondence $\alpha = \Sigma_i a_i x_i \mapsto \langle 1+\alpha \rangle$, with zero added. The following theorem reformulated more geometrically to be more suitable for our purposes, was conjectured by J. Carlson [5] and proved by G. Avrunin and L. Scott [2].

Theorem 1.8. *Let E be an elementary abelian p-group of rank r. Then there is an identification of $|E|$ with the rank variety $V(E)$,*

$$\Psi_E : V(E) \xrightarrow{\sim} |E|,$$

sending 0 to 0 and satisfying the property that the induced map on associated projective varieties $Proj(V(E) \xrightarrow{\sim} Proj|E|$ sends any non-zero multiple of a non-trivial $\alpha = \sum_1^r a_i x_i \in kE$ to $\ker\{H^\bullet(E,k) \to H^\bullet(\langle 1+\alpha \rangle, k)\}$, where the map in cohomology is induced by the embedding of group algebras $k\langle 1+\alpha\rangle \hookrightarrow kE$.

Moreover, Ψ_E has the following property for any finite dimensional E-module M: $\Psi_E^{-1}(|E|_M)$ equals the subset consisting of 0 and those cyclic shifted subgroups $\langle 1+\alpha\rangle$ such that the restriction of M to $k\langle 1+\alpha\rangle$ is not free as a $k\langle 1+\alpha\rangle \simeq k\mathbb{Z}/p$-module.

Let G be an infinitesimal group scheme of height $\leq r$ (i.e., the p^rth power of any element of the augmentation ideal of $k[G]$ is trivial). A *1-parameter subgroup of G* is a map $\mathbb{G}_{a(r)} \to G$ of group schemes, where $\mathbb{G}_{a(r)}$ is the r-th Frobeinus kernel of the additive group $\mathbb{G}_a$. Let $v_0, \ldots, v_{p^r-1}$ be a basis for $k\mathbb{G}_{a(r)}$ dual to the standard basis $T^0, T^1, \ldots, T^{p^r-1}$ of $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$. Denote $v_{p^i} = u_i$, so that

$$k\mathbb{G}_{a(r)} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

We shall have frequent need of the algebra map

$$\epsilon : k\mathbb{Z}/p \simeq k[u_{r-1}]/u_{r-1}^p \hookrightarrow k\mathbb{G}_{a(r)}.$$

The following theorem was proved by Suslin-Friedlander-Bendel in [20], following work by Friedlander-Parshall and J. Jantzen in the special case of restricted Lie algebras (i.e., infinitesimal group schemes of height 1).

Theorem 1.9. *Let G be an infinitesimal group scheme of height $\leq r$. Then the functor on finitely generated commutative k-algebras sending an algebra A to the set of all morphisms of group schemes over A, $\mathbb{G}_{a(r),A} \to G_A$, is representable by an affine scheme whose k points are the 1-parameter subgroups of G. We denote the set of closed points of this scheme by $V(G)$. There is natural homeomorphism*

$$\Psi_G : V(G) \xrightarrow{\sim} |G|$$

with the property for any finite dimensional G-module M that $\Psi^{-1}(|G|_M)$ equals $V(G)_M$, the subset consisting of those 1-parameter subgroups $\alpha : \mathbb{G}_{a(r)} \to G$ for which the pull-back of M via α is not free over the subalgebra $k[u_{r-1}]/u_{r-1}^p \hookrightarrow k\mathbb{G}_{a(r)}$.

2. p-POINTS FOR FINITE ABELIAN GROUP SCHEMES

Finite abelian group schemes play a central role in our study of arbitrary finite group schemes. In this section, we define the set $P(C)$ of equivalence classes of "p-points" of a finite abelian group scheme C; in the next section, we shall define an abelian p-point of an arbitrary finite group scheme as a p-point of some abelian subgroup scheme and we shall consider the set $P(G)$ of equivalence classes of abelian p-points of G.

The main result of this section is the existence of a natural bijection

$$\Psi_C : P(C) \xrightarrow{\sim} \operatorname{Proj}|C|$$

between the equivalence classes of p-points of C and maximal (non-trivial) homogeneous ideals of the graded, commutative algebra $H^\bullet(C, k)$.

For a finite abelian group scheme C we denote by $C^\#$ the finite abelian group scheme whose coordinate algebra is the group algebra of C. The correspondence $C \leftrightarrow C^\#$ is known as Cartier duality for finite abelian group schemes. A group scheme G is said to be *connected* if its coordinate algebra $k[G]$ is a connected algebra (i.e., $k[G]$ has no non-trivial idempotents); an infinitesimal group scheme is a finite group scheme which is connected (or, equivalently, local). A finite abelian group scheme C will be called *co-connected* if $C^\#$ is connected (i.e., if the group algebra kC of C is connected).

The reader may consult [21],[8] for a classification of such abelian finite group schemes. For our purposes, the following elementary analysis is useful.

Any abelian finite group scheme over a perfect field splits as a direct product of groups of four types (cf.[21]):

1), connected with connected dual (e.g. $\mathbb{G}_{a(r)}$);

2), constant with connected dual (e.g. $\mathbb{Z}/p$);

3), connected with constant dual (e.g. μ_p); and

4), constant with constant dual (e.g. $\mathbb{Z}/p'$ with $(p', p) = 1$).

Witt vectors provide interesting examples of finite abelian group schemes of type (1). Consider the functor

$$\mathcal{W}_n : (comm\, k - alg) \to (groups)$$

defined by sending a commutative k-algebra A to $W_n(A)^+$, the additive group of the ring of Witt vectors of length n of A. Then, for $n, r \geq 1$, the r-th Frobenius kernel $\mathcal{W}_{n(r)}$ of $\mathcal{W}_n$ is a connected, co-connected finite abelian group scheme of height r. For example, $\mathbb{G}_{a(r)} \simeq \mathcal{W}_{1(r)}$, $\mathbb{G}^\#_{a(r)} \simeq \mathcal{W}_{r(1)}$.

We recall the following result concerning the structure of the coordinate algebra of connected finite group schemes and thus of the group algebras of finite abelian group schemes of type (1).

Theorem 2.1. [21, 14.4] *Let G be an infinitesimal group scheme. Then the coordinate algebra of G is isomorphic as a k-algebra to an algebra of the following form:*

$$k[G] \;\simeq\; k[t_1, \ldots, t_n]/(t_1^{p^{r_1}}, \ldots, t_n^{p^{r_n}}).$$

The classification of finite abelian groups easily implies that the group algebra of a finite abelian group scheme of type (2) has the same form as that exhibited in Theorem 2.1, a tensor product of algebras of the form $k[t]/t^{p^e}$.

The group algebras of finite abelian group schemes of type (3) and (4) are products of the ground field k and thus semi-simple.

We will need the following straightforward generalization of Lemma 6.4 in [20] (cf. also [5, 6.4]).

Proposition 2.2. *Let M be a k-vector space and α, β, γ be pairwise commuting M-endomorphisms. Assume further that α, β are p-nilpotent and γ is p^r-nilpotent (where r can be greater than 1) in $End_k(M)$. Then M is projective as a $k[u]/u^p$-module where the action of u is given by α if and only if it is projective as a $k[u]/u^p$-module where the action of u is given by $\alpha + \beta\gamma$.*

Proof. Suppose M is projective as a $k[\alpha]$-module. We use the notation $H^i(\alpha, M)$ for the cohomology of $k[u]/u^p$ with coefficients in M, where u acts on M via α. To verify that M is projective as a $k[\alpha+\beta\gamma]$-module it suffices to check that

$$\ker(\alpha+\beta\gamma)/Im(\alpha+\beta\gamma)^{p-1} = H^2(\alpha+\beta\gamma, M)$$

is trivial. Since the action of γ is nilpotent, it further suffices to show that the homomorphism induced by γ

$$\gamma : H^2(\alpha+\beta\gamma, M) \to H^2(\alpha+\beta\gamma, M)$$

is injective. Consider $x \in \ker(\alpha+\beta\gamma)$ with the property that there exists some y so that

$$\gamma x = (\alpha+\beta\gamma)^{p-1}y. \tag{2.2.1}$$

Multiplying (2.2.1) by $-\beta$ and using the hypothesis $(\alpha+\beta\gamma)x = 0$, we get

$$\alpha x = -\beta\gamma x = -\beta(\alpha+\beta\gamma)^{p-1}y = \alpha\Big(\frac{-\beta(\alpha+\beta\gamma)^{p-1}}{\alpha}\Big)y$$

where the quotient makes sense since $\beta^p = 0$. Since M is projective over $k[\alpha]$, we have $\ker\,\alpha = Im\,\alpha^{p-1}$ and, hence,

$$x + \frac{\beta(\alpha+\beta\gamma)^{p-1}}{\alpha}y = \alpha^{p-1}z. \tag{2.2.2}$$

Multiplying by γ and using (2.2.1) again, we get

$$(\alpha+\beta\gamma)^{p-1}y = -\frac{\beta\gamma(\alpha+\beta\gamma)^{p-1}}{\alpha}y + \alpha^{p-1}\gamma z \tag{2.2.3}$$

For any b with $b^p = 0$,

$$\frac{b}{a}(a+b)^{p-1} + (a+b)^{p-1} = a^{p-1}. \tag{2.2.4}$$

Applying (2.2.4) to $a = \alpha$ and $b = \beta\gamma$, we simplify (2.2.3) to

$$\alpha^{p-1}y = \alpha^{p-1}\gamma z.$$

Using projectivity of M over $k[\alpha]$ again, we conclude that $y-\gamma z \in \ker(\alpha^{p-1}) = \operatorname{Im}(\alpha)$. Hence, $y = \gamma z + \alpha t$. Substituting this into (2.2.2) and applying (2.2.4) once again, we get

$$x = -\frac{\beta(\alpha+\beta\gamma)^{p-1}}{\alpha} y + \alpha^{p-1} z = -\frac{\beta(\alpha+\beta\gamma)^{p-1}}{\alpha}(\gamma z + \alpha t) + \alpha^{p-1} z =$$

$$(\alpha+\beta\gamma)^{p-1}(-\beta t) + (-\frac{\beta\gamma(\alpha+\beta\gamma)^{p-1}}{\alpha} + \alpha^{p-1})z = (\alpha+\beta\gamma)^{p-1}(-\beta t) + (\alpha+\beta\gamma)^{p-1} z,$$

and, hence, x is zero in $H^2(\alpha+\beta\gamma, M)$. The statement follows. □

In the following proposition we use a construction of J.Carlson ([6]) to provide the first link between the cohomological and representation-theoretic properties of flat maps $\alpha : k\mathbb{Z}/p \to kG$ which we will study throughout the paper. For a G-module M, we denote by $\alpha^*(M)$ the pull-back of M to $k\mathbb{Z}/p$ via α. By "flat" map of agebras we mean *left* flat, i.e. a map $\alpha : k\mathbb{Z}/p \to kG$ is flat if $\alpha^*(kG)$ is flat as a *left* $k\mathbb{Z}/p$-module.

Proposition 2.3. *Let G be a finite group scheme and $\alpha, \beta : k\mathbb{Z}/p \to kG$ be flat maps of k-algebras. If $\ker\{\alpha^* : H^\bullet(G,k) \to H^\bullet(k\mathbb{Z}/p,k)\}$ is not contained in $\ker\{\beta^* : H^\bullet(G,k) \to H^\bullet(k\mathbb{Z}/p,k)\}$, then there exists a finite dimensional G-module M such that $\alpha^*(M)$ is not projective as a $k\mathbb{Z}/p$-module but $\beta^*(M)$ is projective as a $k\mathbb{Z}/p$-module.*

Proof. For convenience, we will assume $p > 2$. For $p = 2$ one simply has to consider homogeneous cohomology class in any, not necessarily even, degree. Let $\zeta \in H^{2n}(G,k)$ be a non-zero, positive dimensional cohomology class. We represent such a cohomology class by a map $\zeta : \Omega^{2n}k \to k$ (cf. Remark 1.2) and denote by L_ζ the G-submodule $\ker\{\zeta\} \subset \Omega^{2n}k$. Applying the exact functor α^* to the short exact sequence $0 \to L_\zeta \to \Omega^{2n}k \stackrel{\zeta}{\to} k \to 0$, we get a short exact sequence of $k\mathbb{Z}/p$-modules

$$0 \to \alpha^* L_\zeta \to \alpha^*(\Omega^{2n}k) \stackrel{\alpha^*(\zeta)}{\to} k \to 0.$$

Since α is flat, pull-back via α takes projectives to projectives. Therefore, α^* commutes with Ω^{2n} in the stable category. Observe that $\Omega^{2n}k$ is always isomorphic to k in the stable category of $k\mathbb{Z}/p$-modules. We conclude that $\alpha^*(\zeta)$ is non-trivial if and only if it is a stable isomorphism if and only if $\alpha^* L_\zeta$ is projective.

Assume now that $\zeta \in \ker\{\alpha^*\}$, $\zeta \notin \ker\{\beta^*\}$. Then $\alpha^*(L_\zeta)$ is not projective, whereas $\beta^*(L_\zeta)$ is projective by the preceding discussion. □

Our approach to the study of finite group schemes G focuses on the group algebra kG of G; in the special case that G is abelian, we shall not use the co-algebra structure. As defined in 2.5, a "p-point" of a finite abelian group scheme C is simply a flat map of algebras $\alpha : k\mathbb{Z}/p \to kC$. We begin by verifying that the condition of flatness on such a map α has a useful cohomological interpretation.

We shall often find it convenient to identify $k\mathbb{Z}/p$ with the truncated polynomial algebra $k[u]/u^p$. Such an identification amounts to a choice of generator $\sigma \in \mathbb{Z}/p$ determining the isomorphism $k[u]/u^p \xrightarrow{\sim} k\mathbb{Z}/p$ sending u to $\sigma - 1$.

Proposition 2.4. *Let C be an abelian finite group scheme and let $\alpha : k\mathbb{Z}/p \to kC$ be a flat map of k-algebras. Then*

$$\alpha^* : H^\bullet(C,k) \to H^\bullet(\mathbb{Z}/p,k)$$

is surjective.

Conversely, if G is a finite group scheme and $\alpha : k\mathbb{Z}/p \to kG$ is a map of k-algebras such that $\alpha^* : H^\bullet(G,k) \to H^\bullet(\mathbb{Z}/p,k)$ is non-zero in some positive degree, then α is necessarily flat.

Proof. We recall that α^* is indeed a map of algebras even if α is not a map of Hopf algebras (cf. 1.7). Since any abelian finite group scheme is a direct product of a co-connected group scheme and a group scheme which is cohomologically trivial, we may assume that C is co-connected. Applying Theorem 2.1 to $C^\#$, we conclude that $kC \simeq k[C^\#]$ is isomorphic to a group algebra $k[t_1,\dots,t_n]/(t_1^{p^{r_1}},\dots,t_n^{p^{r_n}})$ of some abelian p-group A. Let E be the maximal elementary abelian subgroup of A. Observe that $kE \simeq k[t_1^{p^{r_1-1}},\dots,t_n^{p^{r_n-1}}]/(t_1^{p^{r_1}},\dots,t_n^{p^{r_n}})$. Fix a generator u of the augmentation ideal of $k\mathbb{Z}/p \simeq k[u]/u^p$. We decompose α as

$$\alpha = \alpha' + \alpha'',$$

where α' factors through kE and $\alpha''(u)$ does not contain any monomials from kE. Since α is p-nilpotent, α'' must be p-nilpotent. Applying Proposition 2.2, we conclude that for any C-module M, $\alpha^*(M)$ is projective if and only if $(\alpha')^*(M)$ is projective. In particular, taking $M = kC$ we conclude that α' is flat because α is flat. We further observe that the embedding $E \subset A$ induces an isomorphism in cohomology. Thus, to prove surjectivity of $\alpha^* : H^\bullet(C,k) \to H^\bullet(\mathbb{Z}/p,k)$ it suffices to show surjectivity of $(\alpha')^* : H^\bullet(E,k) \to H^\bullet(\mathbb{Z}/p,k)$.

Applying Proposition 2.2 again, we may assume that α' sends the generator u of $k\mathbb{Z}/p$ to a non-trivial linear combination of $t_1^{p_1^{r_1-1}},\dots t_n^{p_n^{r_n-1}}$. By choosing different generators of $kE \simeq k[t_1^{p^{r_1-1}},\dots,t_n^{p^{r_n-1}}]/(t_1^{p^{r_1}},\dots,t_n^{p^{r_n}})$, we may further assume that α' sends u to $t_1^{p^{r_1-1}}$. Such a map α corresponds to a map of finite groups $\mathbb{Z}/p \to E = (\mathbb{Z}/p)^n$, which is an embedding of the first factor. This map splits and, therefore, the induced map in cohomology is surjective.

Conversely, we readily verify that α induces a natural map $\alpha^* : H^\bullet(G,M) \to H^\bullet(\mathbb{Z}/p,\alpha^*(M))$ for any G-module M. We consider the following commutative diagram of k-algebras

$$\begin{array}{ccc} H^\bullet(G,k) & \xrightarrow{\alpha^*} & H^\bullet(\mathbb{Z}/p,k) \\ \downarrow & & \downarrow \\ H^\bullet(G,kG) & \xrightarrow{\alpha^*} & H^\bullet(\mathbb{Z}/p,\alpha^*(kG)) \end{array} \tag{2.4.1}$$

Since kG is free as a kG-module, $H^{*>0}(G,kG) = 0$ and, thus, the composition $H^\bullet(G,k) \xrightarrow{\alpha^*} H^\bullet(\mathbb{Z}/p,k) \to H^\bullet(\mathbb{Z}/p,\alpha^*(kG))$ is trivial in positive degrees. Since the image of α^* contains some power of a generator t of the polynomial ring $H^\bullet(\mathbb{Z}/p,k)$, we conclude that t acts nilpotently on $H^\bullet(\mathbb{Z}/p,\alpha^*(kG))$. On the other hand, multiplication by t induces a periodicity isomorphism. Thus, $\alpha^*(kG)$ must be acyclic in positive degrees as a $\mathbb{Z}/p$-module. This in turn implies that $\alpha^*(kG)$ is free and hence flat as a $k\mathbb{Z}/p$-module. □

Definition 2.5. A p-point of a finite dimensional commutative k-algebra $\mathcal{A}$ is a flat map of k-algebras

$$\alpha : k\mathbb{Z}/p \to \mathcal{A}.$$

Two p-points $\alpha, \beta : k\mathbb{Z}/p \to \mathcal{A}$ are said to be equivalent if for any finite dimensional $\mathcal{A}$-module M

$$\alpha^*(M) \text{ is free } \Leftrightarrow \beta^*(M) \text{ is free.}$$

A p-point of an abelian finite group scheme C is a p-point of kC.

We denote by $P(C)$ the set of equivalence classes of p-points of C

$$P(C) = \mathrm{Hom}_{k-alg,flat}(k\mathbb{Z}/p, kC)/\sim .$$

We give two important examples of p-points. In the case of elementary abelian p-groups considered in Example 2.6, a "cyclic shifted subgroup" naturally determines an equivalence class of p-points, but to obtain a representing p-point a choice must be made. In the case of a finite connected abelian group scheme C as considered in Example 2.7, a natural choice of representative of an equivalence class of p-points is determined by a 1-parameter subgroup of C.

Let $A = \oplus_{n\geq 0} A_n$ be a commutative, graded k-algebra, so that $V = \mathrm{Spec}\, A$ is a conical affine variety. We denote by $\mathrm{Proj}\, V$ the set of homogeneous ideals of A which are maximal among those homogeneous ideals other than the augmentation ideal (i.e., "maximal non-trivial homogeneous ideals") and give it the Zariski topology. If $\phi : A \to k$ is a map of k-algebras with $A = \oplus A_n$ graded as above, and $t \in k^*$, we define $t \cdot \ker\phi \in \mathrm{Spec}\, A$ as the kernel of the map sending $\alpha_i \in A_i$ to $t^i\phi(a_i)$. Then $\mathrm{Proj}\, V$ can be identified with the quotient of the non-zero maximal ideals of A modulo this action of k^*.

Example 2.6. Let E be an elementary abelian p-group of rank r and choose $\{x_1, \ldots, x_r\} \subset I$ determining a basis of I/I^2, where I denotes the augmentation ideal of kE. Then the cyclic shifted subgroup (with respect to this choice) generated by $\Sigma_i a_i x_i$ determines a p-point of E, given by $\alpha : k[u]/u^p \to kE$ sending u to $\Sigma_i \alpha_i x_i$. We thus get a map from Carlson's rank variety $V(E)$ (cf. Theorem 1.8) to $P(E)$. Viewing $V(E) \simeq I/I^2$ as an affine space associated to the affine scheme $\mathrm{Spec}\, S^*((I/I^2)^\#)$, we identify $\mathrm{Proj}\, V(E)$ with the quotient of $V(E) - \{0\}$ by the action of k^*. Since homothetic cyclic shifted subgroups clearly determine equivalent p-points of E, we get a well-defined map

$$\Theta_E : \mathrm{Proj}\, V(E) \to P(E).$$

In fact, Proposition 2.9 will imply that Θ_E does not depend on the choice of basis $\{x_1, \ldots, x_r\}$. Using Proposition 2.2 we easily conclude that $\alpha : k[u]/u^p \to kE$ is flat if and only if $\alpha(u)$ when written as a polynomial in $\{x_1, \cdots, x_r\}$ has some non-zero linear term.

Example 2.7. Let C be a finite, connected abelian group scheme of height $\leq r$. Then for any 1-parameter subgroup $\alpha : \mathbb{G}_{a(r)} \to C$, the composition $\alpha \circ \epsilon : k\mathbb{Z}/p \to kC$ (cf. Theorem 1.9) is a p-point of C. As in the previous example, this determines a well-defined map

$$\Theta_C : \mathrm{Proj}\, V(C) \to P(C).$$

The following proposition introduces a natural map from equivalence classes of p-points to support varieties. We shall say that a map $H \to G$ of finite group schemes is a *flat map* if the induced map on group algebras $kH \to kG$ is flat where we view kG as a left kH-module.

Proposition 2.8. *If C is a finite abelian group scheme, then sending a p-point $\alpha : k\mathbb{Z}/p \to kC$ to $\ker\{\alpha^*\}$ determines a well-defined map of sets*

$$\Psi_C : P(C) \to Proj|C|$$

which is natural with respect to a flat map $C \to C'$ of finite abelian group schemes.

Proof. By definition, a p-point $\alpha : k\mathbb{Z}/p \to kC$ of a finite abelian group scheme C is flat and by Proposition 2.4 its induced map $\alpha^* : H^\bullet(C,k) \to H^\bullet(k\mathbb{Z}/p,k)$ is non-trivial. Consider p-points $\alpha, \beta : k\mathbb{Z}/p \to kC$. Recall that the maximal non-trivial homogeneous ideals of a commutative, graded ring A^* such as $H^\bullet(G,k)$ are in natural 1-1 correspondence with the kernels of non-trivial graded maps $A^* \to k[x]$. Thus, $\ker\{\alpha^*\}, \ker\{\beta^*\} \in \mathrm{Proj}\,|C|$. If $\alpha, \beta : k\mathbb{Z}/p \to kC$ satisfy $\ker\{\alpha^*\} \neq \ker\{\beta^*\}$, then α is not equivalent to β by Proposition 2.3. Thus, Ψ_C is well defined. The asserted naturality of Ψ_C is immediate. □

The infinitesimal group schemes $\mathbb{G}_{a(r)}$ play a special role among all infinitesimal group schemes, just as elementary abelian groups play a special role among all finite groups. For this reason, the computation given by the following proposition will play a central role in our later discussions.

Proposition 2.9. *For any $r \geq 1$, the maps*

$$\Theta_{\mathbb{G}_{a(r)}} : Proj\,V(\mathbb{G}_{a(r)}) \to P(\mathbb{G}_{a(r)}), \quad \Psi_{\mathbb{G}_{a(r)}} : P(\mathbb{G}_{a(r)}) \to Proj\,|\mathbb{G}_{a(r)}|$$

are bijections.

Similarly, for any elementary abelian p-group E, the maps

$$\Theta_E : Proj\,V(E) \to P(E), \quad \Psi_E : P(E) \to Proj\,|E|$$

are again bijections.

Proof. As established in [20], the composition

$$\mathrm{Proj}\,V(\mathbb{G}_{a(r)}) \stackrel{\Theta_{\mathbb{G}_{a(r)}}}{\to} P(\mathbb{G}_{a(r)}) \stackrel{\Psi_{\mathbb{G}_{a(r)}}}{\to} \mathrm{Proj}\,|\mathbb{G}_{a(r)}|$$

is a homeomorphism. Therefore, it suffices to show that Θ is surjective. By Lemma 2.2, any p-point of $\mathbb{G}_{a(r)}$ is equivalent to a non-trivial linear combination of generators $u_0, \dots u_{r-1}$ of the group algebra $k\mathbb{G}_{a(r)}$. On the other hand, $\mathrm{Proj}\,V(\mathbb{G}_{a(r)}) \simeq \mathbb{P}^{r-1}$ (cf. [19, 1.10]) and $\Theta_{\mathbb{G}_{a(r)}}$ is injective since $\Psi_{\mathbb{G}_{a(r)}} \circ \Theta_{\mathbb{G}_{a(r)}}$ is injective. Thus, $\Theta_{\mathbb{G}_{a(r)}}$ is surjective.

Let r denote the rank of E. Observe that $kE \simeq k\mathbb{G}_{a(r)}$. This immediately implies that $\Psi_E : P(E) \to \mathrm{Proj}\,|E|$ is bijective. As in the case of $\mathbb{G}_{a(r)}$, we have that the composition

$$\mathrm{Proj}\,V(E) \stackrel{\Theta_E}{\to} P(E) \stackrel{\Psi_E}{\to} \mathrm{Proj}\,|E|$$

is a homeomorphism ([5]). □

Lemma 2.10. *Let C be a finite abelian group scheme and write C as a product $C = C_1 \times C_2$ with $C_1^\#$ connected and $C_2^\#$ constant. Then the projection $p_1 : C_1 \times C_2 \twoheadrightarrow C_1$ induces a bijection*

$$p_{1,*} : P(C) \stackrel{\sim}{\to} P(C_1).$$

Proof. Since kC splits as a k-algebra, the factors are direct summands of kC as a kC-module, where the action of kC on kC_i is given via the projection map p_i. Thus, the projection $p_1 : kC \to kC_1$ is flat and, therefore, $p_{1,*} : P(C) \to P(C_1)$ is well-defined. Denote by i_1 the embedding $C_1 \hookrightarrow C$. Then $i_{1,*}$ induces a well-defined map $P(C_1) \to P(C)$ such that the composition

$$P(C_1) \stackrel{i_{1,*}}{\to} P(C) \stackrel{p_{1,*}}{\to} P(C_1)$$

is the identity map. Thus, it suffices to show that $p_{1,*}$ is injective. Semi-simplicity of kC_2 further implies that any C_2 - module splits as a direct sum of irreducible modules, where C_2 acts on any irreducible module via multiplication by a scalar. Such an action, pulled back via a map $\alpha : k\mathbb{Z}/p \to k(C_1 \times C_2)$ always becomes trivial. Thus, for any C-module M we have an isomorphism of $\mathbb{Z}/p$-modules

$$\alpha^*(M) \simeq (\alpha^* \circ p_1^*)(p_{1,*}M).$$

We conclude that two p-points $\alpha, \beta : k\mathbb{Z}/p \to kC$ are equivalent only if $p_1 \circ \alpha, p_1 \circ \beta : k\mathbb{Z}/p \to kC \to kC_1$ are equivalent. Thus, $p_{1,*} : P(C) \to P(C_1)$ is injective. □

Theorem 2.1 and Lemma 2.10 in conjunction with Proposition 2.9 enable us to establish that the map Ψ_C of Proposition 2.8 is a bijection for all finite abelian group schemes C. In Theorem 4.11 below, we establish such a bijection for all finite group schemes.

Theorem 2.11. *For any finite abelian group scheme C, the map*

$$\Psi_C : P(C) \to \mathit{Proj}\,|C|$$

of Proposition 2.8 is bijective.

Proof. Let $C = C_1 \times C_2 \times C_3 \times C_4$, where the splitting corresponds to the four types of finite abelian group schemes. Both C_3 and C_4 are semi-simple and, thus, cohomologically trivial. This implies that

$$\operatorname{Proj}|C| = \operatorname{Proj}|C_1 \times C_2|.$$

By Lemma 2.10 we also have

$$P(C) = P(C_1 \times C_2).$$

We may thus assume that C is co-connected. By the classification of finite abelian groups and Theorem 2.1, the group algebra of C is isomorphic to a tensor product of algebras of the form $k[T]/T^{p^r}$,

$$\theta_C : k[T_1, \dots T_n]/(T_1^{p^{r_1}}, \dots T_n^{p^{r_n}}) \stackrel{\sim}{\to} kC.$$

Let $\mathcal{A}$ be the subalgebra of kC generated by $\{\theta_C(T_1^{p^{r_1-1}}), \dots, \theta_C(T_n^{p^{r_n-1}})\}$. We use the notation $P(\mathcal{A})$ for the set of equivalence classes of p-points of $\mathcal{A}$, and $|\mathcal{A}|$ for the maximal ideal spectrum of the cohomology ring $H^\bullet(\mathcal{A}, k)$. Since we may identify $\mathcal{A} \to kC$ with the map on group algebras of the embedding of the subgroup of elements of order p in the finite abelian group $\prod_{i=1}^n \mathbb{Z}/p^{r_i}\mathbb{Z}$, we conclude that $\mathcal{A} \to kC$ induces an isomorphism $H^\bullet(C,k)_{red} \simeq H^\bullet(\mathcal{A},k)_{red}$ and thus a bijection $\operatorname{Proj}|\mathcal{A}| \simeq \operatorname{Proj}|C|$.

We consider the following commutative square

$$
\begin{array}{ccc}
P(\mathcal{A}) & \longrightarrow & P(C) \\
\downarrow{\scriptstyle \Psi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Psi_C} \\
\operatorname{Proj}|\mathcal{A}| & \xrightarrow{\sim} & \operatorname{Proj}|C|
\end{array} \tag{2.11.1}
$$

whose left vertical map is an isomorphism by Proposition 2.9.

To complete the proof that Ψ_C is a bijection, we show that the upper horizontal map of the above square is surjective. For this we verify that any p-point $\alpha : k\mathbb{Z}/p \to kC$ is equivalent to a p-point of $\mathcal{A}$. Observe that the monomials on the set $\{\theta_C(T_1), \ldots, \theta_C(T_n)\}$ constitute a basis of kC as a k-vector space, whereas monomials on the set $\{\theta_C(T_1^{p^{r_1-1}}), \ldots, \theta_C(T_n^{p^{r_n-1}})\}$ constitute a basis for $\mathcal{A}$.

Let u be a generator of $k[u]/u^p \simeq k\mathbb{Z}/p$ and let $\alpha : k[u]/u^p \to kC$ be a p-point of C. Write $\alpha(u)$ as a linear combination of these monomials. Since $\alpha(u)$ is p-nilpotent, each monomial is also p-nilpotent. Thus, each monomial has a factor of the form $\theta_C(T_i)^{p^{r_i-1}}$. Lemma 2.2 implies that subtracting any monomial which is a product of $\theta_C(T_i)^{p^{r_i-1}}$ and anything else of positive degree does not change the equivalence class of α. We conclude that α is equivalent to a p-point which sends u to a linear combination of $\{\theta_C(T_1^{p^{r_1-1}}), \ldots, \theta_C(T_n^{p^{r_n-1}})\}$ and is thus in the image of a p-point of $\mathcal{A}$. □

Theorem 2.11 enables us to show that the maps Θ in Examples 2.6 and 2.7 are bijections.

Corollary 2.12. *Let C be either the constant group scheme associated to a finite elementary abelian p-group or a finite abelian connected group scheme. Then the natural maps*

$$\Theta_C : \mathit{Proj}\, V(C) \to P(C)$$

exhibited in Example 2.6 and Example 2.7 are bijections.

Proof. One knows that the composition $\operatorname{Proj} V(C) \to P(C) \to \operatorname{Proj}|C|$ is a bijection: in the case of an elementary abelian p-group this follows easily from the well know determination of the cohomology of such a finite group; for a connected finite abelian group scheme this is proved in [20]. Thus, the corollary follows from Theorem 2.11. □

We conclude this section with the following corollary of Theorem 2.11.

Corollary 2.13. *Let $\alpha, \beta : k\mathbb{Z}/p \to kC$ be two p-points of a finite abelian group scheme C. If α is not equivalent to β, then there exists a finite dimensional C-module M such that $\alpha^*(M)$ is projective and $\beta^*(M)$ is not projective as $k\mathbb{Z}/p$-modules.*

Proof. By Theorem 2.11, the assumption that α is not equivalent to β implies that

$$\ker\{\alpha^* : H^\bullet(C,k) \to H^\bullet(\mathbb{Z}/p,k)\} \neq \ker\{\beta^* : H^\bullet(C,k) \to H^\bullet(\mathbb{Z}/p,k)\}.$$

This implies that $\ker\{\beta^*\} \not\subset \ker\{\alpha^*\}$ since both $\ker\{\alpha^*\}, \ker\{\beta^*\}$ are maximal non-trivial homogeneous ideals of kC. The statement now follows from Proposition 2.3. □

3. The space $P(G)$ of abelian p-points

In this section, we introduce our space $P(G)$ of abelian p-points of G and prove various properties (summarized in Theorem 3.12). As we see below, this provides a uniform representation-theoretic interpretation of (projectivized) cohomological support varieties of both finite groups and infinitesimal group schemes.

Motivation for our definition of $P(G)$ for a non necessarily abelian finite group scheme comes from the following example.

Example 3.1. Let G be a finite group (i.e., a constant finite group scheme). Then Quillen's stratification for the cohomology algebra $H^\bullet(G,k)$ asserts that the natural map

$$\varinjlim_{\{E\to G\}} |E| \to |G|.$$

is a homeomorphism, where the indexing category for the colimit is the category whose objects are elementary abelian subgroups of G and whose maps are compositions of group inclusions and maps induced by conjugations by elements of G.

It is useful to keep in mind the fact that any subgroup scheme $H \subset G$ of a finite group scheme is automatically closed (cf. [21, 15.3]).

Definition 3.2. Let G be a finite group scheme. An abelian p-point of G is a p-point of some abelian subgroup scheme $C \subset G$; in other words, a map $\alpha : k\mathbb{Z}/p \to kG$ of k-algebras given as the composition of some flat map $k\mathbb{Z}/p \to kC$ followed by the map of group algebras $kC \to kG$ induced by the embedding $C \subset G$.

We say that two abelian p-points $\alpha, \beta : k\mathbb{Z}/p \to kG$ are equivalent and write $\alpha \sim \beta$ provided that $\alpha^*(M)$ is free as a $\mathbb{Z}/p$-module if and only if $\beta^*(M)$ is free as a $\mathbb{Z}/p$-module for all finite dimensional G-modules M.

We denote by $P(G)$ the set of equivalence classes of abelian p-points of G,

$$P(G) = Hom^{ab}_{k-alg,flat}(k\mathbb{Z}/p, kG)/\sim.$$

For any finite dimensional G-module M, we denote by $P(G)_M$ the subset of $P(G)$ represented by abelian p-points $\alpha : k\mathbb{Z}/p \to kG$ such that $\alpha^*(M)$ is not free as a $\mathbb{Z}/p$-module.

Remark 3.3. The map of algebras $kH \to kG$ induced by a closed embedding of finite group schemes $H \subset G$ is always flat, where we view kG as a (left) kH-module using left multiplication of H on G (cf. [15, I.3]). Thus, an abelian p-point $\alpha : kZ/p \to kG$ of a finite group scheme is necessarily flat.

Lemma 3.4. *Let G be a finite group scheme and $\alpha : k\mathbb{Z}/p \to kG$ be an abelian p-point of G. Then $\alpha^* : H^\bullet(G,k) \to H^\bullet(\mathbb{Z}/p,k)$ is non-trivial (i.e.,* $\ker\{\alpha^*\}$ *does not equal the augmentation ideal of $H^\bullet(G,k)$).*

Proof. By definition, α factors as $i \circ \alpha' : k\mathbb{Z}/p \to kC \to kG$, where $i : kC \to kG$ is the map on group algebras associated to some abelian subgroup scheme $C \subset G$. By a theorem of Friedlander-Suslin (cf. [13]), $H^\bullet(G, \mathrm{Ind}_C^G k)$ is a finite module over the algebra $H^\bullet(G,k)$ and, thus, the map $i^* : H^\bullet(G,k) \to H^\bullet(C,k) \simeq H^\bullet(G, \mathrm{Ind}_C^G k)$ is finite. By Proposition 2.4, ${\alpha'}^*$ is surjective, so that $\alpha^* = {\alpha'}^* \circ i^*$ is also a finite map and thus non-trivial. □

Lemma 3.4 enables us to show that the construction of Proposition 2.8 applies to an arbitrary finite group scheme.

Proposition 3.5. *Let G be a finite group scheme. Then sending an abelian p-point $\alpha : k\mathbb{Z}/p \to kG$ to $\ker\{\alpha^* : H^\bullet(G,k) \to H^\bullet(\mathbb{Z}/p,k)\}$ determines a well defined map of sets*

$$\Psi_G : P(G) \to Proj\,|G|, \quad \alpha \mapsto \ker\{\alpha^*\}.$$

This construction is natural with respect to flat maps $H \to G$ of finite group schemes.

Proof. By Lemma 3.4, $\ker\{\alpha^*\}$ determines a point of $\mathrm{Proj}\,|G|$ for any abelian p-point $\alpha : k\mathbb{Z}/p \to kG$. By Proposition 2.3, this determines a well defined set map on equivalence classes of p-points: if $\ker\{\alpha^*\} \neq \ker\{\beta^*\}$, then α is not equivalent to β. The asserted naturality is immediate from the definitions. □

As we see in the next proposition, Ψ_G enables us to equate the representation-theoretic $P(G)$ with the (projectivized) cohomological variety $\mathrm{Proj}\,|G|$ for any finite group G.

Proposition 3.6. *Let G be a constant finite group scheme and let $\{E \to G\}$ denote the indexing category whose objects are the elementary abelian p-subgroups of G (viewed as a finite group) and whose maps are compositions of group inclusions and maps induced by conjugations by elements of G. There is a natural bijection*

$$\varinjlim \Psi_E : \varinjlim_{\{E\to G\}} P(E) \overset{\sim}{\to} P(G),$$

which determines a bijection

$$\Psi_G : P(G) \overset{\sim}{\to} Proj\,|G|,$$

extending the bijection of Proposition 2.9 for elementary abelian p-groups.

Moreover, if M is any finite dimensional G-module, then $\Psi_G^{-1}(Proj\,|G|_M) = P(G)_M$.

Proof. Consider the following diagram:

(3.6.1)
$$\begin{array}{ccc} \varinjlim_{\{E\to G\}} P(E) & \longrightarrow & P(G) \\ \downarrow{\scriptstyle \varinjlim \Psi_E} & & \downarrow{\scriptstyle \Psi_G} \\ \varinjlim_{\{E\to G\}} \mathrm{Proj}\,|E| & \longrightarrow & \mathrm{Proj}\,|G| \end{array}$$

By Proposition 2.8, $\varinjlim_{E\to G} \Psi_E$ is well defined. By Proposition 2.9, Ψ_E and thus $\varinjlim_{E\to G} \Psi_E$ are homeomorphisms. Proposition 3.5 implies that the right vertical and upper horizontal arrows are well-defined and the diagram is commutative.

Let A be any abelian subgroup of G, A_p be its Sylow subgroup, and E_A be the subgroup of elements of A whose p-th power is the identity. By Proposition 2.10, $P(A) = P(A_p)$. Applying Proposition 2.2, one checks immediately that the map $P(E_A) \to P(A_p)$ induced by the embedding $E_A \subset A_p$ in bijective. Thus, any abelian p-point of G is equivalent to a p-point which factors through some elementary abelian p-subgroup. We conclude that the upper horizontal map is surjective. By "Quillen stratification" (i.e., Example 3.1), the lower horizontal map is also a homeomorphism. Thus, all maps in 3.6.1 are homeomorphisms.

To prove the last assertion, we must show for any finite dimensional G-module M and any abelian p-point $\alpha : k\mathbb{Z}/p \to kG$ that $\ker\{\alpha^*\}$ (the image $\Psi_G(\alpha)$ of the equivalence class of α) lies in $\operatorname{Proj}|G|_M$ if and only if $\alpha^*(M)$ is not free. Choose some elementary abelian p-group $E \subset G$ such that $\ker\{\alpha^*\} \in \operatorname{Proj}|G|$ lies in the image of $\operatorname{Proj}|E| \to \operatorname{Proj}|G|$. Since Ψ_G (as well as Ψ_E) is a bijection, α is equivalent to $i \circ \alpha' : k\mathbb{Z}/p \to kE \to kG$, where i denotes the embedding $E \subset G$. Theorem 1.8 implies that $(\alpha')^*(M)$ is free if and only if $\ker\{\alpha'^*\} \in \operatorname{Proj}|E|_M$. Since $|E|_M = (i_*)^{-1}|G|_M$ by Theorem 1.5.6, the statement follows. □

The following is essentially an immediate corollary of Proposition 3.6.

Corollary 3.7. *Let G be a constant finite group scheme and α, β be two abelian p-points which factor through elementary abelian subgroups. Then $\alpha \sim \beta$ if and only there exists some elementary abelian p-subgroup $E \subset G$ such that α, β are G-conjugate to equivalent p-points $\alpha', \beta' : k\mathbb{Z}/p \to kE$ of E (i.e. α' (respectively, β') is composition of alpha (respectively, β) with an automorphism of kG induced by conjugation by an element of G).*

We next verify for an infinitesimal group scheme G that $P(G)$ once again gives a natural "representation-theoretic" model for the projectivization of the cohomological support variety $|G|$ of G.

Proposition 3.8. *Let G be an infinitesimal group scheme of height $\leq r$ and let $V(G)$ be the variety of k points of the scheme $V(G)$ of 1-parameter subgroups of G (as in Theorem 1.9). Then there are bijections*

$$Proj\, V(G) \stackrel{\Theta_G}{\to} P(G) \stackrel{\Psi_G}{\to} Proj\,|G|$$

extending the bijections of Theorem 2.11 and Corollary 2.12 for G abelian.

Furthermore, for any finite dimensional G-module M,

$$\Psi_G^{-1}(Proj\,|G|_M) = P(G)_M, \quad \Theta_G^{-1}(P(G)_M) = Proj\, V(G)_M.$$

Proof. By Theorem[19, 1.5], an embedding of infinitesimal group schemes $H \subset G$ induces an embedding of schemes $V(H) \hookrightarrow V(G)$. Thus, the naturality of Θ together with the surjectivity of Θ for abelian group schemes (cf. 2.12) implies the surjectivity of Θ_G.

Injectivity of Θ_G uses the construction given in the proof of Proposition 2.3. In more detail, let s, t be two non-homothetic 1-parameter subgroups of G. We denote by the same letters the corresponding points of $\operatorname{Proj} V(G)$. By Theorem [20, 7.5]), there exists a G-module M_s such that $\operatorname{Proj}(V(G)_{M_s}) = \{s\}$, and, hence, the pull-back of M via $\Theta_G(s) \sim s \circ \epsilon$ is not free. On the other hand, since $\{t\} \notin \operatorname{Proj}(V(G)_{M_s})$, we have that $(\Theta_G(t))^*M$ is free. Thus, $\Theta_G(s) \not\sim \Theta_G(t)$.

Essentially by definition, $\Theta_G^{-1}(P(G)_M) = \operatorname{Proj} V(G)_M$. As verified in [20, 6.3.1], $\Psi_G \circ \Theta_G$ is a bijection which sends $\operatorname{Proj} V(G)_M$ to $\operatorname{Proj}|G|_M$. We conclude that $\Psi^{-1}(\operatorname{Proj}|G|_M) = P(G)_M$ and $\Theta_G^{-1}(P(G)_M) = \operatorname{Proj} V(G)_M$.

□

We now proceed to give $P(G)$ a natural topology. We first require the following lemma. This lemma is not immediate: since an abelian p-point $\alpha : k\mathbb{Z}/p \to kG$ is not necessarily a map of Hopf algebras (i.e., might not preserve the coproduct), $\alpha^* : mod_G \to mod_{\mathbb{Z}/p}$ does not necessarily commute with tensor products.

Lemma 3.9. *Let G be a finite group scheme, let M, N be finite dimensional G-modules, and let $\alpha : k\mathbb{Z}/p \to kG$ be an abelian p-point of G. Then $\alpha^*(M \otimes N)$ is free if and only if either $\alpha^*(M)$ or $\alpha^*(N)$ is free.*

Proof. By definition of an abelian p-point, we can assume that G itself is abelian. Lemma 2.10 allows us to reduce further to the case G is co-connected. Denote by ∇ the coproduct on G, and by I the augmentation ideal of kG. Since G is assumed to be a co-connected finite abelian group scheme, the ideal I is nilpotent. The action of α on $M \otimes N$ is given by $\alpha \bullet (m \otimes n) = (\nabla\alpha)(m \otimes n)$. Observe that

$$\nabla\alpha = 1 \otimes \alpha + \alpha \otimes 1 + \text{ terms from } I \otimes I.$$

(This is evidently true for generators of I and, thus, for any element in I.) Since any p-nilpotent element from $I \otimes I$ can be represented as a sum of products of at least two nilpotent elements of $kG \otimes kG$ out of which at least one is p-nilpotent, Proposition 2.2 implies that $\alpha^*(M \otimes N)$ is free if and only if $(\alpha \otimes 1 + 1 \otimes \alpha)^*(M \otimes N)$ is free. The latter is isomorphic to $\alpha^*(M) \otimes \alpha^*(N)$ as a $\mathbb{G}_{a(1)}$-module (we use identification $k\mathbb{Z}/p \simeq k\mathbb{G}_{a(1)}$), and is free if and only if either $\alpha^*(M)$ or $\alpha^*(N)$ is free. The statement follows. □

We can now give $P(G)$ the added structure of a topological space, a structure which is "purely representation-theoretic".

Proposition 3.10. *Let G be a finite group scheme. Then the class of subsets*

$$\{P(G)_M \subset P(G) : M \text{ finite dimensional } G\text{−module}\}$$

is closed under finite unions and arbitrary intersections. We call $P(G)$ with the Noetherian topology whose closed subsets are subsets of the form $P(G)_M$ the space of abelian p-points *of G.*

Proof. We get the empty set for $M = kG$ and the entire space $P(G)$ for $M = k$. Since

$$P(G)_M \cup P(G)_N = P(G)_{M \oplus N} \subset P(G),$$

we see that the class of subsets of $P(G)$ of the form $P(G)_M$ is closed under unions. On the other hand, Lemma 3.9 implies the equality

$$P(G)_M \cap P(G)_N = P(G)_{M \otimes N} \subset P(G),$$

so that the class of such subsets is also closed under finite intersections. To prove that this class of subsets is closed under arbitrary intersections, it suffices to show that any strictly decreasing chain of such subsets is finite and thus that $P(G)$ is Noetherian. Observe that any abelian subgroup scheme of G factors through a subgroup scheme of the form $(G^0)^A \times A$, where $A \subset \pi_0(G)$ is an abelian p-subgroup of the group of connected components of G, and $(G^0)^A$ is the group scheme of the invariants of the connected component of G under the action of A. Thus, $P(G)$ is covered by the union of $P((G^0)^A \times A)$ where A runs through the finite set of all abelian p-subgroups of $\pi_0(G)$. Note that for any $r > 0$, $k\mathbb{Z}/p^r \simeq k\mathbb{G}^{\#}_{a(r)}$. Since A is isomorphic to a product of cyclic p-groups, we conclude that $k((G^0)^A \times A)$ is isomorphic to a group algebra of some infinitesimal group scheme. Proposition 3.8 implies that $P((G^0)^E \times E)$ is Noetherian. We conclude that $P(G)$ is Noetherian. □

Corollary 3.11 (of the proof)**.** *Let G be a finite group scheme and M be a finite dimensional G-module. The association of the space $P(G)_M$ to M satisfies properties (1)-(4) of Theorem 1.5.*

Proof. The proof of Proposition 3.10 verifies properties (1),(3) and (4), and the property for distinguished triangles follows from the observation that if any two modules in a short exact sequences of $\mathbb{Z}/p$ modules are free, then the third one is also free. □

The following theorem is mostly a summary of previous propositions.

Theorem 3.12. *Assume that the finite group scheme is either constant or connected. Then the map $\Psi_G : P(G) \to Proj|G|$ of Proposition 3.5 is a homeomorphism provided that $P(G)$ is given the topology of Proposition 3.10 and $Proj|G|$ is given the Zariski topology.*

Proof. By Propositions 3.6, 3.8, and 3.5, it suffices to verify that every closed subspace of $\mathrm{Proj}\,|G|$ is of the form $\mathrm{Proj}\,|G|_M$. For G finite, this is a theorem of J. Carlson [6] proved using the argument given in the proof of Proposition 3.5. A similar argument is applied in [20, 7.6] to prove this statement for an infinitesimal group scheme G. □

4. $\Psi_G : P(G) \to \mathrm{Proj}\,|G|$ is a homeomorphism

The objective of this section is to prove that the map Ψ_G of Proposition 3.5 is a homeomorphism. We first prove in Theorem 4.6 that Ψ_G is always injective. Then, in Theorem 4.8, we prove that the surjectivity of Ψ_G follows for a given finite group scheme G provided that one can show that for any cohomology class $\zeta \in H^\bullet(G,k)$ which is not nilpotent there exists some field extension K/k and some quasi-elementary subgroup scheme $i : \mathcal{E}_K \subset G_K$ such that $0 \neq i^*(\zeta_K) \in H^\bullet(\mathcal{E}_K, K)$. The proof is completed by appealing to a recent theorem of A. Suslin which verifies such a detection theorem for an arbitrary finite group scheme, thereby extending earlier results achieved by numerous authors.

The following lemma describes abelian p-points of direct products of finite group schemes.

Lemma 4.1. *Let $G = G_1 \times G_2$ be a product of finite group schemes. Any equivalence class of abelian p-points of G has a representative of the form $\alpha_1 \otimes c_1 + c_2 \otimes \alpha_2$, where α_1,α_2 are abelian p-points of G_1, G_2 respectively and $c_1, c_2 = 0$ or 1.*

Proof. Let α be an abelian p-point of G which factors through an abelian subgroup $A_1 \times A_2 \subset G_1 \times G_2$. Denote by p_i, $i = 1, 2$, the projection $A_1 \times A_2 \twoheadrightarrow A_i$. By Proposition 2.10, we can assume that both A_1 and A_2 are co-connected. Denote by u a generator of the augmentation ideal of $k\mathbb{Z}/p$. Observe that $k(A_1 \times A_2) \simeq kA_1 \otimes kA_2$. We write $\alpha(u)$ as a sum of simple tensors on the generators of kA_1 and kA_2. Since $\alpha(u)$ is p-nilpotent, each simple tensor must be p-nilpotent, and, therefore, at least one of the factors in each simple tensor is p-nilpotent. Proposition 2.2 now implies that for any $A_1 \times A_2$-module M, M is projective restricted to α if and only if it is projective restricted to $p_1(\alpha) \otimes 1 + 1 \otimes p_2(\alpha)$. Let $c_i = 1$ if $p_i(\alpha) : k\mathbb{Z}/p \to kA_i$ is flat and 0 otherwise. Since α is flat, at least one of c_1 and c_2 are non-zero. Setting α_i to be $p_i(\alpha)$ if $c_i = 1$ and to be any abelian p-point otherwise, we conclude that α is equivalent to $\alpha_1 \otimes c_1 + c_2 \otimes \alpha_2$. Indeed, if $c_1 = c_2 = 1$, this is clear. Suppose $c_1 = 0$. Then $p_1(\alpha)$ is not a flat map. This, in turn, implies that the pull-back of kA_1 via $p_1(\alpha)$ is not projective. Appealing to Proposition2.2 once again, we conclude that $p_1(\alpha)(u)$ belongs to the square of the augmentation ideal of kA_1. Thus, $p_1(\alpha)\otimes 1 + 1\otimes p_2(\alpha)$ is equivalent to $1\otimes p(\alpha_2)$ as claimed.

□

A subgroup scheme $\mathcal{E}\subset G$ of a finite group scheme is said to be *quasi-elementary* provided that $\mathcal{E}$ is isomorphic to a product of the form $\mathbb{G}_{a(r)}\times(\mathbb{Z}/p)^s$ for some $r,s\geq 0$. Quasi-elementary subgroup schemes are sufficient to "capture" the abelian p-points of G as we see in the following proposition.

Proposition 4.2. *Let C be a finite abelian group scheme. Then for any p-point $\alpha : k\mathbb{Z}/p\to kC$ of C, there exists a p-point $\beta : k\mathbb{Z}/p\to k\mathcal{E}$ of some quasi-elementary subgroup scheme $\mathcal{E}\subset C$ such that $i_*\circ\beta : k\mathbb{Z}/p\to k\mathcal{E}\to kC$ is equivalent to α. (Here, $i_* : k\mathcal{E}\to kC$ is the map of group algebras induced by $i:\mathcal{E}\subset C$.)*

Proof. By Lemma 2.10, we may assume that C is co-connected. Write $C = C^0\times C'$, where $C^0\subset C$ is the component of the identity and $C' = \pi_0(C)$. Let $E'\subset C'$ be the subgroup of C' consisting of elements whose p-th power is the identity. Then E' is an elementary abelian p-group and $|E'|\to|C'|$ is a bijection. Using results established in the earlier sections and the Kunneth formula, we get a sequence of bijections

$$P(C)\xrightarrow{\sim}\operatorname{Proj}(|C^0\times C'|)\xrightarrow{\sim}\operatorname{Proj}(|C^0|\times|C'|)\xrightarrow{\sim}\operatorname{Proj}(V(C^0)\times V(E')). \tag{4.2.1}$$

Let α be a p-point of C. Under the identification above, we associate to α the class of some point (α^0,α') in $\operatorname{Proj}(V(C^0)\times V(E'))$ where α^0 is a 1-parameter subgroup of C^0 and α' is a cyclic shifted subgroup of C' and where we view $\operatorname{Proj} X$ of a conical affine scheme X as a scheme of coinvariants of X with respect to the natural action of $\mathbb{G}_m$. Since α is flat, at least one of α^0 and α' is non-zero. A direct computation shows that the p-point of C defined as $\alpha^0\circ\epsilon\otimes 1 + 1\otimes\alpha'$ is identified with the class of (α^0,α') in $\operatorname{Proj}(V(C^0)\times V(E'))$ under 4.2.1. Thus, α and $\alpha^0\circ\epsilon\otimes 1+1\otimes\alpha'$ are in the same equivalence class. Since the latter factors through a quasi-elementary abelian subgroup by construction, the statement follows.

□

We shall say that a map $\phi : A\to B$ of commutative k-algebras is *essentially surjective* if for every element $b\in B$ there exists some $n>0$ such that b^n lies in the image of A. Thus, if $\phi : A\to B$ is essentially surjective, then the induced map on prime ideal spectra $\operatorname{Spec} B\to\operatorname{Spec} A$ is injective.

To establish injectivity of the map Ψ_G, we first prove two facts about essential surjectivity of certain maps in cohomology. These results are versions of Lemma 1.9 of [13] and the Main Lemma of [10].

Let E be an elementary abelian group scheme. Denote by $\sigma_{\mathbf{E}}\in H^\bullet(E,k)$ the cohomology class of E defined as

$$\sigma_E = \prod_{0\neq\xi\in H^1(E,\mathbb{F}_p)}\beta(\xi),$$

where β denotes the Bockstein homomorphism. Recall that 0-locus of σ_E equals $\bigcup\limits_{E'<E} i_{E,E',*}|E'|$, where E' runs over all proper subgroups of E and $i_{E,E',*}$ denotes the map on support varieties induced by the embedding $i : E'\hookrightarrow E$.

Let $G = G^0\rtimes E$ be a finite group scheme with group of connected components the elementary abelian p-group E, and denote the projection $G\to E$ by p. We denote by σ_G the pull-back of σ_E to G,

$$\sigma_G = p^*(\sigma_E) \in H^\bullet(G, k).$$

The statement and proof of the following proposition is essentially a translation into our context of the discussion given in [11, ch.9] in the context of finite groups. To reassure the reader, we give details of the proof.

Proposition 4.3. *Let G be a finite group scheme with group of connected components π, let $E \subset \pi$ be an elementary abelian p-subgroup, and let $G' \subset G$ be the subgroup scheme with the same connected component G^0 as G and with group of connected components E. Let $\xi \in H^\bullet(G', k)$ be a homogeneous cohomology class invariant under the normalizer $N_\pi(E)$ of E in π and let $|N_\pi(E)/E| = lp^e$, where $(l, p) = 1$. Then there exists $\eta \in H^\bullet(G, k)$ such that the restriction of η to $H^\bullet(G', k)$ equals $(\sigma_{G'}\xi)^{p^e}$.*

Proof. Fix a set of double coset representatives of $E\backslash\pi/E$: $h_1, \dots h_m$. Let further $h_{i_j}, 1 \leq j \leq n_i$ be a set of left coset representatives of π/E such that $h_{i_1} = h_i$ and h_{i_j} for a given i are in the double coset corresponding to h_i. We denote by n the order of π/E. i.e. $n = n_1 + \cdots + n_m$. The choice of left coset representatives defines an embedding $G \hookrightarrow G' \wr \Sigma_n$ in the following way. Let A be a connected commutative Noetherian k-algebra. For $g \in G(A)$ we have $gh_{i_j} = h_{s(i_j)}g'_{i_j}$, where $s \in \Sigma_n$, $g'_{i_j} \in G'(A)$. We define

$$g \mapsto (g'_{1_1}, \dots, g'_{m_{n_m}}, s) \in (G' \wr \Sigma_n)(A).$$

We extend this map naturally to all finitely generated commutative k-algebras, thereby obtaining an embedding of group schemes $G \hookrightarrow G' \wr \Sigma_n$. Note that if $g \in G'(A)$, then h_{i_j} and $h_{s(i_j)}$ are in the same double coset and, thus, s splits as a product $s_1 \times \cdots \times s_m \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$ where each s_i is a permutation on $\{h_{i_j}\}_{1 \leq j \leq n_i}$, defined by $gh_{i_j} = h_{i_{s_i(j)}}g'_{i_j}$.

Let $k \to I^\bullet$ be a cobar resolution of k as a trivial G'-module. We introduce the action of Σ_n by permutations with an appropriate sign on $(I^\bullet)^{\otimes n}$, $s \circ (\otimes x_i) = \pm \otimes x_{s^{-1}(i)}$ (cf. [11, ch.9]). In even dimensions the sign is always $+1$. The action of Σ_n is compatible with the natural action of G' on $(I^\bullet)^{\otimes n}$ and, thus, makes $(I^\bullet)^{\otimes n}$ into an acyclic complex over $G' \wr \Sigma_n$. We get a commutative diagram

(4.3.1)

$$\begin{array}{ccccccc}
H^*(((I^\bullet)^{\otimes n})^{G' \wr \Sigma_n}) & \longrightarrow & H^*(G' \wr \Sigma_n, k) & \longrightarrow & H^*(G, k) & \longrightarrow & H^*(G', k) \\
& \searrow & & & & \nearrow & \\
& & H^*(((I^\bullet)^{\otimes n})^{G'}) & & & &
\end{array}$$

Let $x \in (I^{2f})^{G'}$ be a cocycle representing ξ. Let further $\sigma \in (I^*)^{G'}$ be a cocycle representing $\sigma_{G'}$. Consider the element $\nu = (1 + \sigma x)^{\otimes n} \in (I^\bullet)^{\otimes n}$ (the latter has a structure of a d.g.a.). Clearly, $\nu \in (((I^\bullet)^{G'})^{\otimes n})^{\Sigma_n} = ((I^\bullet)^{\otimes n})^{G' \wr \Sigma_n}$ and, thus, defines a cohomology class in $H^*(G, k)$ under the map above. We claim that this cohomology class further restricts to $(1 + \sigma_{G'}\xi)^{|N_\pi(E)|}$ in $H^\bullet(G', k)$. Assuming this claim, we finish the proof of the proposition. Let $\mu \in H^*(G, k)$ be an element whose image under the restriction map $H^*(G, k) \to H^*(G', k)$ equals $(1 + \sigma_{G'}\xi)^{lp^e}$ where $(l, p) = 1$. Then $(1 + \sigma_{G'}\xi)^{lp^e} = (1 + (\sigma_{G'}\xi)^{p^e})^l =$

$1+l(\sigma_{G'}\xi)^{p^e}+\{$ elements of higher degree $\}$. Taking the homogeneous part of μ of degree $p^e(\deg(\xi)+\deg(\sigma_{G'}))$, we get that $l(\sigma_{G'}\xi)^{p^e}$, and hence $(\sigma_{G'}\xi)^{p^e}$, is in the image of the restriction map, as needed.

We now proceed to prove the claim. As a G'-complex, $(I^\bullet)^{\otimes n}$ can be decomposed as $(I^\bullet)^{n_1}\otimes(I^\bullet)^{n_2}\otimes\cdots\otimes(I^\bullet)^{n_m}$ where G' preserves each tensor factor. We then get an isomorphism on cohomology via the Kunneth formula:

$$H^*(((I^\bullet)^{\otimes n})^{G'}) \simeq H^*(((I^\bullet)^{\otimes n_1})^{G'}) \otimes \dots H^*(((I^\bullet)^{\otimes n_m})^{G'}). \tag{4.3.2}$$

We will determine the image of ν in each factor separately and take their product. We consider two cases: $h_i\in N_\pi(E)$ and $h_i\notin N_\pi(E)$.

Case I. $h_i\in N_\pi(E)$. Then $G'h_iG'=h_iG'$ and, thus, $n_i=1$. Therefore, the corresponding tensor factor in the decomposition 4.3.2 is simply $H^*((I^\bullet)^{G'})$, where the action of G' on $I^\bullet$ is given by the restriction of the action of $G'\wr\Sigma_n$. The choice of the embedding $G'\hookrightarrow G\hookrightarrow G'\wr\Sigma_n$ implies that $g\in G'$ acts via ${}^{h_i}g=h_i^{-1}gh_i$. Thus, $H^*((I^\bullet)^{G'})$ can be naturally identified with $H^*((I^\bullet)^{{}^{h_i}G'})$ where the action of ${}^{h_i}G'\simeq G'$ is the h_i-conjugate of the original action of G'. Multiplication by h_i gives an isomorphism

$$H^*((I^\bullet)^{{}^{h_i}G'}) \xrightarrow{h_i} H^*((I^\bullet)^{G'}) = H^*(G',k).$$

Since $1+\sigma_{G'}\xi$ is invariant under h_i, we conclude that the factor $1+\sigma x$ corresponding to h_i maps to $1+\sigma_{G'}\xi$.

Case II. $h_i\notin N_\pi(E)$. The assumption implies $n_i>1$. Denote h_i by h and h_{i_j} by h_j for notational convenience. Observe that multiplication by h^{-1} on the right induces a bijection between the set of left coset representatives of $\pi/E=G/G'$ inside the double coset defined by h and the set of left coset representatives of $E/E\cap E^h=G'/G'\cap(G')^h$. Indeed, let $g\in G'$ (for simplicity of notation, we write G' instead of $G'(A)$). Then $gh=h_jg'$ for some $g'\in G'$ and $1\le j\le n_i$. Thus, $g=(h_jh^{-1})hgh^{-1}$. Since h_j and h are in the same right coset $G'h$, we conclude that $hgh^{-1}\in G\cap(G')^h$. A similar calculation shows that $(h_jh^{-1})(h_lh^{-1})^{-1}$ does not belong to $G\cap(G')^h$ whenever $j\neq l$. Thus, we get a bijection as claimed.

Let $E_i=E\cap E^h$ and $G_i=G'\cap(G')^h=G^0\rtimes E_i$. Let further $e_i=|E/E_i|$. Then we have an embedding $G'\hookrightarrow G_i\wr\Sigma_{e_i}$ defined with respect to the set of left coset representatives $h_1h^{-1},\dots h_{n_i}h^{-1}$ of $G'/G_i\simeq E/E_i$. We readily check that the map

$$(hI^\bullet)^{\otimes n_i} \xrightarrow{h^{-1}} (I^\bullet)^{\otimes n_i}$$

is G'-equivariant where G' acts via the restriction of the action of the wreath product $G_i\wr\Sigma_i$ on the left and on the right via the formula $g\circ(\otimes\alpha_j)=\pm\otimes g_{s^{-1}(j)}\alpha_{s^{-1}(j)}$, where s and $g_{s^{-1}(j)}$ are defined as above. Thus, we get a map of complexes

$$((hI^\bullet)^{\otimes n_i})^{G_i\wr\Sigma_i} \xrightarrow{h^{-1}} ((I^\bullet)^{\otimes n_i})^{G'} \tag{4.3.3}$$

and, hence, a well-defined map in cohomology:

$$H^\bullet(((hI^\bullet)^{\otimes n_i})^{G_i\wr\Sigma_i}) \xrightarrow{h^{-1}} H^\bullet(((I^\bullet)^{\otimes n_i})^{G'}). \tag{4.3.4}$$

Observe that the cocycle $(1 + h\sigma x)^{\otimes n_i}$ maps onto $(1 + \sigma x)^{\otimes n_i}$ under the map 4.3.3. On the other hand, since $\sigma_{G'} \downarrow_{G_i} = 0$, and the cohomology class of $(1+\sigma x)$ is invariant under h, we get that $(1+h\sigma x)^{\otimes n_i} - 1^{\otimes n_i}$ vanishes in $H^\bullet(((hI^\bullet)^{\otimes n_i})^{G_i \wr \Sigma_i}$. We finally conclude that the image of the factor $(1 + \sigma x)^{\otimes n_i}$ under the map 4.3.2 is 1.

Combining the two cases with the observation that the number of double cosets considered in case I is precisely $|N_\pi(E)/E|$, we conclude that the map $H^*(((I^\bullet)^{\otimes n})^{G'}) \to H^*(G', k)$ sends the class of the cocycle $(1 + \sigma x)^{\otimes n}$ to $(1 + \sigma_{G'}\xi)^{\otimes |N_\pi(E)/E|}$, which finishes the proof of the claim.

□

We restate the preceding lemma in terms of cohomological spectra.

Corollary 4.4. *Let $G = G^0 \rtimes \pi$ be a finite group scheme, E be an elementary abelian subgroup of π and $G' = G^0 \rtimes E$. Let further $N_\pi(E)$ be the normalizer of E in π. Then the embedding $i : G' \hookrightarrow G$ induces an injective map*

$$\mathit{Spec}\,(H^\bullet(G', k)[\sigma_{G'}^{-1}])/N_\pi(E) \to |G|.$$

Proof. Proposition 4.3 implies that there exists a cohomology class $\rho \in H^\bullet(G, k)$ such that $i^*(\rho) = \sigma_{G'}^n$, and, furthermore, that the map

$$H^\bullet(G, k)[\rho^{-1}] \to (H^\bullet(G', k)[\sigma_{G'}^{-1}])^{N_\pi(E)}$$

is essentially surjective. Note that $\mathrm{Spec}\, H^\bullet(G, k)[\rho^{-1}]$ is naturally embedded into $|G|$. Thus, the map

$$\mathrm{Spec}\,(H^\bullet(G', k)[\sigma_{G'}^{-1}])/N_\pi(E) = \mathrm{Spec}\,((H^\bullet(G', k)[\sigma_{G'}^{-1}])^{N_\pi(E)}) \hookrightarrow$$
$$\mathrm{Spec}\, H^\bullet(G, k)[\rho^{-1}] \hookrightarrow |G|$$

is injective.

□

Lemma 4.5. *Let G be a finite group scheme with group of connected components π. Let $(G^0)^\pi$ be the subgroup scheme of invariants of the connected component G^0 under the action of π. Then the map in cohomology induced by the natural embedding of group schemes $(G^0)^\pi \times \pi \hookrightarrow G^0 \rtimes \pi = G$,*

$$H^\bullet(G, k) \to H^\bullet((G^0)^\pi \times \pi, k), \tag{4.5.1}$$

is essentially surjective.

Proof. Since $H^*((G^0)^\pi \times \pi, k) \simeq H^*((G^0)^\pi, k) \otimes H^*(\pi, k)$, it suffices to show that for any element in $H^\bullet((G^0)^\pi, k) \otimes 1$ or $1 \otimes H^\bullet(\pi, k)$, some power of this element belongs to the image of the map 4.5.1. This is clear for $1 \otimes H^\bullet(\pi, k)$ since the map $H^\bullet(G, k) \to H^\bullet(\pi, k)$ splits. We proceed to show that for any homogeneous $\xi \in H^\bullet((G^0)^\pi, k)$ there exists $\eta \in H^\bullet(G, k)$ such that the restriction of η to $(G^0)^\pi \times \pi$ equals $\xi^n \otimes 1$ for some $n > 0$.

Let π act on $(G^0)^{\times |\pi|}$ by permutations. Let further $k \to I^\bullet$ be an injective resolution of k as a G^0-module. Then $(I^\bullet)^{\otimes |\pi|}$ is an injective resolution of k as a $(G^0)^{\times |\pi|}$-module. By introducing the action of π on this complex by permutations with an appropriate sign,

$$g(\otimes_{h \in \pi} x_h) = \pm \otimes_{h \in \pi} x_{g^{-1}h},$$

we make $(I^\bullet)^{\otimes|\pi|}$ into a π-equivariant resolution. (See [11] for the choice of the sign. The important point for us is that the sign is always + 1 in even degrees.) Let $k \to J^\bullet$ be an injective resolution of k as a π-module. By tensoring $(I^\bullet)^{\otimes|\pi|}$ with $J^\bullet$ we get a resolution of k as a $(G^0)^{\times|\pi|} \rtimes \pi = G^0 \wr \pi$-module. We define an embedding of $G = G^0 \rtimes \pi$ into $(G^0)^{\times|\pi|} \rtimes \pi = G^0 \wr \pi$ by the formula $(g_0, y) \mapsto (\{x^{-1}g_0\}_{x\in\pi}, y)$. Since $(I^\bullet)^{\otimes|\pi|} \otimes J^\bullet$ is an acyclic complex of $G^0 \wr \pi$-modules, we get maps in cohomology

$$(4.5.2)\quad H^\bullet(((I^\bullet)^{\otimes|\pi|} \otimes J^\bullet)^{G^0\wr\pi}) \to H^\bullet(G^0 \wr \pi, k) \to H^\bullet(G, k) \to H^\bullet((G^0)^\pi \times \pi, k).$$

The map in cohomology $H^\bullet(G^0, k) \to H^\bullet((G^0)^\pi, k)$ is essentially surjective since G^0 is an infinitesimal group scheme (cf. [20, 5.2]). Since $H^\bullet((G^0)^\pi, k)$ is fixed by the action of π, we conclude that the restriction of this map to $H^\bullet(G^0, k)^\pi$ is also essentially surjective. Thus, there exists $\zeta \in H^\bullet(G^0, k)^\pi$ such that $\zeta \downarrow_{(G^0)^\pi} = \xi^l$ for some power l. By taking the appropriate power of ξ, we may assume that $\zeta \downarrow_{(G^0)^\pi} = \xi$. Let $\alpha \in I^{G^0}$ be a cocycle representing $\zeta \in H^\bullet(G^0, k)^\pi$ and let $\epsilon : k \to J^\bullet$ be the augmentation map. Then $\alpha^{\otimes|\pi|} \otimes \epsilon$ is a cocycle in

$$(((I^\bullet)^{G^0})^{\otimes|\pi|})^\pi \otimes (J^\bullet)^\pi = ((I^\bullet)^{\otimes|\pi|} \otimes J^\bullet)^{G^0\wr\pi}$$

and defines a cohomology class $\eta \in H^\bullet(G, k)$. It follows directly from the construction that restricting η further to $H^\bullet((G^0)^\pi \times \pi)$ we get $\xi^{|\pi|}t \otimes 1$.

□

The preceding lemmas enable us to prove that Ψ_G is injective for any finite group scheme G.

Theorem 4.6. *For any finite group scheme G,*

$$\Psi_G : P(G) \to Proj|G|$$

is injective.

Proof. Let $G = G^0 \rtimes \pi$, where π is the group of connected components of G. We prove injectivity of Ψ_G in two steps. We first reduce to the case in which π is elementary abelian by using Corollary 4.4, and then treat this case by applying Lemma 4.5.

Let α, β be two abelian p-points of G such that $\ker\{\alpha^*\} = \ker\{\beta^*\}$. Modifying β by a non-zero scalar which does not change its equivalence class we may assume $\alpha^* = \beta^*$. We proceed to show that $\alpha \sim \beta$. Using Proposition 4.2, let E, E' be elementary abelian p-subgroups of π with the property that representatives of the equivalence classes of α, β factor through subgroup schemes

$$k\mathbb{Z}/p \to k(G^0 \rtimes E) \to kG, \quad k\mathbb{Z}/p \to k(G^0 \rtimes E') \to kG$$

respectively. Choose E, E' minimal with this property. Changing α, β within their equivalence class, if necessary, we may assume that they factor as above. We consider the compositions

$$\bar\alpha : k\mathbb{Z}p \to k(G^0 \rtimes E) \to kE \to k\pi, \quad \bar\beta : k\mathbb{Z}p \to k(G^0 \rtimes E') \to kE' \to k\pi.$$

Since the map in cohomology induced by the projection $G \to \pi$ splits, $\bar\alpha^* = \bar\beta^*$.

First, assume that $\bar\alpha^*$ and thus $\bar\beta^*$ are trivial (or, equivalently, $\bar\alpha, \bar\beta$ are not flat). Since some representative of $\alpha \downarrow_{G^0 \rtimes E}$ as a p-point of $G^0 \rtimes E$ factors through a quasi-elementary subgroup scheme of $G^0 \rtimes E$, we may assume that $\alpha \downarrow_{G^0 \rtimes E}$ factors through $(G^0)^E \times E$, where $(G^0)^E$ is the subgroup scheme of G^0 fixed by E. Similarly, β factors through $k((G^0)^{E'} \times E')$. By Proposition 4.1, α is equivalent (as a p-point of $(G^0)^E \times E$) to a sum $\alpha_1 \otimes c_1 + c_2 \otimes \alpha_2$ with α_1 a p-point of $(G^0)^E$ and α_2 a p-point of E. Since $\bar{\alpha}^* = 0$, we conclude that $c_2 = 0$. By the minimality of E, we conclude that E is trivial. Similarly, E' must be trivial.

Next, assume that $\bar{\alpha}^*$ is not trivial. Then, Corollary 3.7 implies that the equivalence classes of $\bar{\alpha}$ and $\bar{\beta}$ in $P(E)$ and $P(E')$ respectively are conjugate by an element of π. Thus, by choosing different representatives, if necessary, and applying conjugation by an element of π (which does not change the equivalence class in $P(G)$) we may assume that $E = E'$.

Thus, in either case we may assume $\alpha, \beta : k\mathbb{Z}/p \to k(G^0 \rtimes E) \to kG$ (i.e., $E = E'$). Denote $G^0 \rtimes E$ by G'. We proceed to show that there is a conjugate of α, α^x, such that $(\alpha^x \downarrow_{G'})^* = (\beta \downarrow_{G'})^* : H^\bullet(G') \to H^\bullet(\mathbb{Z}/p, k)$. This will complete the reduction to the case $\pi = E$. Denote the projection $G' \to E$ by p and recall the elements $\sigma_E \in H^\bullet(E, k)$ and $\sigma_{G'} = p^*(\sigma_E) \in H^\bullet(G', k)$ which were introduced preceding Proposition 4.3. Minimality of E and the bijection $P(E) \simeq \operatorname{Proj}|E|$ established in §3 imply that $(\alpha \downarrow_{G'})^*(\sigma_{G'}) = (\alpha \downarrow_{G'})^*(p^*(\sigma_E)) = (p \circ \alpha \downarrow_{G'})^*(\sigma_E) = (\bar{\alpha} \downarrow_{G'})^*(\sigma_E) \neq 0$. Thus, $\ker(\alpha \downarrow_{G'})^*$ (and, similarly, $\ker(\beta \downarrow_{G'})^*$) belongs to the open subvariety $\operatorname{Spec} H^\bullet(G', k)[\sigma_G^{-1}] \subset |G'|$. Since $\alpha^* = \beta^*$, Corollary 4.4 implies that there exists $x \in N_\pi(E)$ such that

$$((\alpha \downarrow_{G'})^*)^x = (\beta \downarrow_{G'})^*.$$

Since Ψ is compatible with the action by conjugation, we get

$$((\alpha^x) \downarrow_{G'})^* = (\beta \downarrow_{G'})^*.$$

By taking a conjugate of α once again (which does not change the equivalence class in $P(G)$) we may assume that $\pi = E$.

Recall that E was chosen to be a minimal elementary abelian subgroup through which some representatives of classes α and β factor. We therefore may assume that α, β factor through $(G^0)^E \times E$. We consider the following commutative diagram:

$$\text{(4.6.1)} \qquad \begin{array}{ccc} P((G^0)^E \times E) & \longrightarrow & P(G^0 \rtimes E) \\ \Big\downarrow {\scriptstyle \Psi_{(G^0)^E \times E}} & & \Big\downarrow {\scriptstyle \Psi_{G^0 \rtimes E}} \\ \operatorname{Proj}(|(G^0)^E \times E|) & \longrightarrow & \operatorname{Proj}(|G^0 \rtimes E|) \end{array}$$

Recall that we begin with the assumption that $\alpha^* = \beta^*$. Lemma 4.5 implies that $\operatorname{Proj}(|(G^0)^E \times E|)$ embeds into $\operatorname{Proj}(|G^0 \rtimes E|)$, so that the images of the equivalence classes of $\alpha \downarrow_{(G^0)^E \times E}, \beta \downarrow_{(G^0)^E \times E}$ are equal in the lower left of (4.6.1). Since the group algebra of $(G^0)^E \times E$ can be identified with the group algebra of an infinitesimal group scheme $(G^0)^E \times (\mathbb{G}_{a(1)})^{\operatorname{rk} E}$, $\Psi_{(G^0)^E \times E}$ is bijective by Proposition 3.8. Thus, the equivalence classes of $\alpha \downarrow_{(G^0)^E \times E}, \beta \downarrow_{(G^0)^E \times E}$ are equal in the upper left of (4.6.1), so that α, β are equal in the upper right of (4.6.1); thus $\Psi_{G^0 \ltimes E}$ is injective as required.

□

The proof above immediately implies the following generalization of the Corollary 3.7.

Corollary 4.7 (of the proof)**.** *Let G be a finite group scheme, π be the group of connected components of G, and G^0 be the connected component of G. Let further α, β be two abelian p-points of G which factor through some quasi-elementary subgroups of G. Then $\alpha \sim \beta$ if and only if there exists some elementary abelian p-subgroup $E \subset \pi$ such that α, β are π-conjugate to equivalent p-points $\alpha', \beta' : k\mathbb{Z}p \to k((G^0)^E \times E)$ of the subgroup scheme $(G^0)^E \times E \subset G$.*

Proof. Let $\alpha^x : k\mathbb{Z}/p \to kG$ be a conjugate of α (i.e., the composition of α with the automorphism of kG given by conjugation by x for some $x \in \pi$). For any G-module M, the G-module xM is isomorphic to M and yet $(\alpha^x)^*(M) = \alpha^*(xM)$. Thus, $\alpha \sim \alpha^x$ as abelian p-points of G. This shows the "if" part of the statement. The other direction follows directly from the proof of the theorem above. □

As we see in Theorem 4.8 below, to prove surjectivity we need to verify the nilpotence of any homogeneous cohomology class $\zeta \in H^{2j}(G,k)$ whose restriction to each quasi-elementary subgroup scheme is nilpotent. An important result of D. Quillen [17] asserts that a homogeneous cohomology class of a finite group π which restricts to 0 on every elementary abelian p-group is nilpotent in $H^*(\pi, k)$. The analogous theorem was proved in [20, 4.3] for infinitesimal group schemes: if G is an infinitesimal group scheme and if $\zeta_K \in H^n(G_K, K)$ restricts to 0 on every 1-parameter subgroup $\mathbb{G}_{a(r),K} \to G_K$ for every field extension K/k, then $\zeta \in H^*(G, k)$ is nilpotent. This was further extended in [3] to not necessarily connected finite group schemes with unipotent connected component.

Theorem 4.8. *Let G be a finite group scheme satisfying the following condition: for any unital associative G-algebra Λ and any homogeneous cohomology class $\zeta \in H^\bullet(G, \Lambda)$ which is not nilpotent, there exists some field extension K/k and some quasi-elementary subgroup scheme $i_K : \mathcal{E}_K \subset G_K$ such that $i_K^*(\zeta) \in H^\bullet(\mathcal{E}_K, \Lambda_K)$ is not nilpotent. Then*

$$\Psi_G : P(G) \xrightarrow{\sim} Proj\,|G|$$

is a homeomorphism satisfying the property that

$$\Psi_G^{-1}(Proj\,|G|_M) = P(G)_M$$

for every finitely generated G-module M.

Proof. Since Theorem 4.6 establishes injectivity of Ψ_G, to prove that Ψ_G is a homeomorphism, it suffices to prove that Ψ_G is surjective and continuous".

Let $\pi = \pi_0(G)$ be the group of connected components of G, let $\{E < \pi\}$ run through the family of all elementary abelian subgroups of π, and denote by $(G^0)^E$ the subgroup scheme of E-invariants of the connected component G^0 of G.

We consider the commutative square

$$\begin{array}{ccc} \coprod_{\{E<\pi\}} P((G^0)^E \times E) & \longrightarrow & P(G) \\ \Big\downarrow {\scriptstyle \coprod \Psi_{(G^0)^E \times E}} & & \Big\downarrow {\scriptstyle \Psi_G} \\ \coprod_{\{E<\pi\}} \mathrm{Proj}\,|(G^0)^E \times E| & \longrightarrow & \mathrm{Proj}\,|G| \end{array} \tag{4.8.1}$$

Observe that for any field extension K/k and any quasi-elementary abelian subgroup scheme $\mathcal{E}_K \subset G_K$, $\mathcal{E}_K$ factors through $(G^0_K)^{E_K} \times E_K = ((G^0)^E \times E)_K$ for some elementary abelian subgroup $E < \pi$. Thus, our hypothesis implies that the natural map

$$H^\bullet(G,k) \to \bigoplus_{E<\pi} H^\bullet((G^0)^E \times E, k)$$

has a nilpotent kernel. Moreover, since $H^\bullet((G^0)^E \times E, k) \simeq H^\bullet(G, \mathrm{Ind}^G_{(G^0)^E \times E} k)$ is a finite module over $H^\bullet(G,k)$ ([13]), this map is also finite. Hence, the lower horizontal map of (4.8.1) is surjective. Observe that the group algebra $k((G^0)^E \times E)$ can be identified with the group algebra of the infinitesimal group scheme $((G^0)^E \times (\mathbb{G}_{a(1)})^{\mathrm{rk}E}$. Thus, $\Psi_{(G^0)^E \times E}$ is bijective by Proposition 3.8. The surjectivity of $\Psi_G : P(G) \to \mathrm{Proj}\,|G|$ is now immediate from the commutativity of (4.8.1).

To show that $\Psi_G(P(G)_M) \subset \mathrm{Proj}\,|G|_M$ we have to prove that for a p-point $\alpha : k\mathbb{Z}/p \to kG$, $\alpha^*(M)$ is not projective implies that $I_M = \mathrm{Ann}_{H^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \subset \ker(\alpha^*)$. Let $\Lambda = \mathrm{End}_k(M,M)$. Observe that $H^*(G,\Lambda) = \mathrm{Ext}^*_G(M,M)$ and, thus, $I_M = \ker(H^\bullet(G,k) \to H^\bullet(G,\Lambda))$. Consider the following commutative diagram:

(4.8.2)
$$\begin{array}{ccc} H^\bullet(G,k) & \xrightarrow{\alpha^*} & H^\bullet(\mathbb{Z}/p,k) \\ \downarrow & & \downarrow \\ H^\bullet(G,\Lambda) & \xrightarrow{\alpha^*_\Lambda} & H^\bullet(\mathbb{Z}/p,\alpha^*(\Lambda)) \end{array}$$

Note that multiplication by a polynomial generator of $H^\bullet(\mathbb{Z}/p,k) \simeq k[t]$ induces a periodicity isomorphism on $H^\bullet(\mathbb{Z}/p,\alpha^*(\Lambda))$. Since $\alpha^*(M)$, and, hence, $\alpha^*(\Lambda)$, is not free as $k\mathbb{Z}/p$-module, the action of t is non-trivial, and, thus, the map $H^\bullet(\mathbb{Z}/p,k) \to H^\bullet(\mathbb{Z}/p,\alpha^*(\Lambda))$ is injective. This observation readily implies that I_M, which is the kernel of the left vertical map of 4.8.2, is contained in $\ker(\alpha^*)$.

To show $\Psi_G(P(G)_M) = Proj|G|_M$, it is only left to show that $\Psi_G : P(G)_M \to Proj|G|_M$ is surjective. Our hypothesis implies that the kernel of the finite map

$$H^\bullet(G,\Lambda) \to \bigoplus_{E<\pi} H^\bullet((G^0)^E \times E, \Lambda)$$

is nilpotent for any finite-dimensional G-module M in exactly the same way as it does for $M = k$. Thus, an analysis of the commutative square obtained from 4.6.1 by considering supports for M (i.e., by applying $(-)_M$ to each corner of the square) together with another appeal to Proposition 3.8 implies the surjectivity of $\Psi_G : P(G)_M \to Proj|G|_M$.

Since Ψ_G is a bijection which takes closed sets to closed sets, it is a homeomorphism.

□

As mentioned earlier, the detection modulo nilpotents results of [17], [20], and [3] imply that the condition of Theorem 4.8 above is satisfied for both infinitesimal group schemes and finite group schemes with unipotent connected component. Consequently, these results together with the Kunneth Theorem give the following as an immediate corollary of Theorem 4.8.

Corollary 4.9. *If G is a product of an arbitrary infinitesimal group scheme and a finite group scheme with unipotent connected component, then*

$$\Psi_G : P(G) \to Proj|G|$$

is a homeomorphism.

The ultimate version of "detection of cohomology modulo nilpotents" is the following recent theorem of A. Suslin ([18]).

Theorem 4.10. *(A. Suslin) Let G be a finite group scheme, Λ be a unital associative G-algebra, and $\zeta \in H^\bullet(G,\Lambda)$ be a homogeneous cohomology class of even degree. Then ζ is nilpotent if and only if ζ_K restricts to a nilpotent class in the cohomology of every quasi-elementary subgroup scheme of $\mathcal{E}_K$ of G_K for any field extension K/k.*

Theorem 4.10 together with Theorem 4.8 immediately implies the following theorem asserting that Ψ_G is a homeomorphism for any finite group scheme G.

Theorem 4.11. *Let G be a finite group scheme. Then*

$$\Psi_G : P(G) \stackrel{\sim}{\to} Proj|G|$$

is a homeomorphism satisfying the property that

$$\Psi_G^{-1}(Proj|G|_M) = P(G)_M$$

for every finitely generated G-module M.

5. CONSEQUENCES

In this section, we derive a few corollaries of the results of the preceding sections. Several of these are based upon Theorem 4.11 which depends upon the unpublished Theorem 4.10. The reader uncomfortable with this can restrict those statements made for an arbitrary finite group scheme to those finite group schemes discussed in Corollary 4.9.

We begin by observing that Theorem 4.11 gives an equivalent cohomological interpretation of our equivalence relation on abelian p-points. Observe that the fact that (3) implies (1) in Proposition 5.1 asserts that to check that $\alpha \sim \beta$ it suffices to restrict attention to the special family $\{L_\zeta; \zeta \in H^\bullet(G,k)\}$ of finite dimensional G-modules, considered in the proof of Proposition 2.3.

Proposition 5.1. *Let G be a finite group scheme, and consider abelian p-points $\alpha, \beta : k\mathbb{Z}p \to kG$ of G. Then the following are equivalent:*

1. *$\alpha \sim \beta$ (i.e., $\alpha = \beta \in P(G)$).*
2. *$\ker\{\alpha^*\} = \ker\{\beta^*\}$.*
3. *$\alpha^*(L_\zeta)$ is free if and only if $\beta^*(L_\zeta)$ is free for every even dimensional homogeneous cohomology class $\zeta \in H^\bullet(G,k)$.*

Moreover, for any finite dimensional G-module M, the equivalence class of α lies in $P(G)_M$ if and only if $\ker\{\alpha^*\} \supset Ann(Ext^*_G(M,M))$.

Proof. Since $\Psi_G : P(G) \to \mathrm{Proj}\,|G|$ sends the equivalence class of α to $\ker\{\alpha^*\}$, the equivalence of (1) and (2) is the property of bijectivity of Ψ_G given by Theorem 4.11. The equivalence of (2) and (3) is verified in the proof of Proposition 2.3. By definition, $\ker\{\alpha^*\} \in \mathrm{Proj}\,|G|_M$ if and only if $\ker\{\alpha^*\} \supset Ann(Ext^*_G(M,M))$, so that the last statement is also given by Theorem 4.11. □

One evident difference between the context of finite groups and that of infinitesimal group schemes is that an inclusion $i : H \to G$ need not induce an injection $i : |H| \to |G|$ unless G is connected. We investigate this further in the following lemma

Lemma 5.2. *Let $E \subset \pi$ be an elementary abelian subgroup of the group of connected components π of a finite group scheme G, and let $H = (G^0)^E \times E$, where G^0 is the connected component of G. Denote by $i : H \hookrightarrow G$ the embedding of H into G. Let further $P_0(G,H)$ denote the set of the equivalence classes of p-points of G which factor through H but do not factor through $(G^0)^E \times E' \subset H$ for any proper subgroup $E' < E$, and, similarly, $P_0(H)$ denote the set of the equivalence classes of p-points of H which factor through H but do not factor through a subgroup scheme with a strictly smaller group of connected components. Then the map $i_* : P(H) \to P(G)$ induces a homeomorphism*

$$P_0(H)/N_\pi(H) \simeq P_0(G,H)$$

where $P_0(H)/N_\pi(H)$ denotes the coinvariants of the normalizer of H in π acting on $P_0(H)$.

For a finite dimensional G-module M, let $P_0(G,H)_M = P_0(G,H) \cap P(G)_M$ and $P_0(H)_M = P(H)_M \cap P_0(H)$. Then the map above restricts to

$$P_0(H)_M/N_\pi(H) \simeq P_0(G,H)_M.$$

Proof. Since conjugation by an element of π does not change the equivalence class of an abelian p-point of G, the map $i_* : P(H) \to P(G)$ factors through $P(H)/N_\pi(H)$.

Let α, β be abelian p-points of H, which are not equivalent to any p-points which factor through a subgroup scheme with a strictly smaller group of connected components, and such that $i \circ \alpha \sim i \circ \beta$ as p-points of G. Corollary 4.7 implies that there exists an element $x \in \pi$ such that α^x is equivalent to β as a p-point of H. By the assumption on α, x must normalize E and, therefore, $(G^0)^E$. Thus, x belongs to $N_\pi(H)$. We conclude that $i_* : P_0(H)/N_\pi(H) \to P(G)$ is injective. Since $P_0(G,H)$ is clearly the image of $P_0(H)$ under i_*, the bijection $P_0(H)/N_\pi(H) \simeq P_0(G,H)$ follows.

The same argument applied to abelian p-points α, β for which $\alpha^*(M)$, $\beta^*(M)$ are not projective, implies the second asserted bijection. This second bijection also implies that the map induced by i^* takes closed sets to closed sets. Thus, it is a homemorphism. □

As an immediate corollary of the lemma above we get the following decomposition of $P(G)$.

Proposition 5.3. *In the notation of the lemma above, the space $P(G)$ is a disjoint union of subsets of the form $P_0(G,(G^0)^E \times E) \simeq P_0((G^0)^E \times E)/N_\pi((G^0)^E \times E)$, one for each conjugacy class of elementary abelian subrgroups $E < \pi$. Moreover, for any finite dimensional G-module M we have*

$$P(G)_M \;=\; \coprod_{E<\pi} P_0((G^0)^E \times E)_M/N_\pi((G^0)^E \times E).$$

Proof. Let α be an abelian p-point of G, representing an equivalence class in $P(G)_M$ for some finite dimensional G-module M. We may assume that α factors through a quasi-elementary subgroup scheme by Proposition 4.2. Choose the minimal elementary abelian subgroup E of π such that α factors through $(G^0)^E \times E$. Then the equivalence class of α belongs to $P_0(G, (G^0)^E \times E)_M$. Thus, $P(G)_M$ is a union of $P_0(G, (G^0)^E \times E)_M$ where E runs through all elementary abelian subgroups of π. Since conjugate p-points are equivalent, it suffices to take one elementary abelian subgroup from each conjugacy class. Corollary 4.7 implies that the union is disjoint. Finally, taking $M = k$, we get $P(G) = \coprod_{E<\pi} P_0((G^0)^E \times E)/N_\pi((G^0)^E \times E)$. □

We reformulate Proposition 5.3 in terms of colimits so that it resembles the classical statement of the Quillen stratification for finite groups (cf. Example 3.1).

Corollary 5.4. *Let G be a finite group scheme with the group of connected components π and the connected component G^0. Then the natural map*

$$\rho : \varinjlim_{E'<E<\pi} P((G^0)^E \times E') \to P(G)$$

is a homeomorphism, where the colimit is indexed by the category whose objects (E, E') are pairs consisting of an elementary abelian p-subgroup E of π and a subgroup E' of E and whose maps $(E, E') \to (E_1, E_1')$ are compositions of group embeddings $E \supset E_1$ restricting to $E' \subset E_1'$ and isomorphisms $E \simeq E_1$ restricting to $E' \simeq E_1'$ induced by conjugations by elements of π.

Similarly, for any finite dimensional G-module M the natural map

$$\rho : \varinjlim_{E'<E<\pi} P((G^0)^E \times E')_M \to P(G)_M$$

is a homeomorphism.

In the next proposition, we show how one may provide $P(G)$ and $|G|$ with Galois actions with respect to which Ψ_G is equivariant. Even granted the naturality of Ψ, considering p-points might prove to be a better invariant than considering cohomology for finite group schemes G over fields F which are not algebraically closed.

Proposition 5.5. *Let k/F be a field extension and $\sigma : k \to k$ a field automorphism over F. Assume that the finite group scheme G over k is defined over F, so that $G = G_F \times_{Spec\, F} Spec\, k$. Then there is a natural action of σ on $P(G)$ which commutes with the homeomorphism $\Psi_G : P(G) \to Proj\,|G|$, where the action on $|G|$ is that induced by*

$$\sigma \otimes 1 : H^\bullet(G, k) = k \otimes_F H^\bullet(G_F, F) \; \to \; k \otimes_F H^\bullet(G_F, F) = H^\bullet(G, k).$$

Moreover, if M is a finite dimensional kG-module and if the action of kG on M is defined over $F \subset k$, then the action of σ stabilizes $P(G)_M$.

Finally, if K/k is a field extension of algebraically closed fields, then the homeomorphism $\Psi_{G,K} : P(G_K) \to Proj\,|G_K|$ fits naturally in a commutative square with $\Psi_G : P(G) \to Proj\,(|G|)$.

Proof. For a k-vector space V defined over F, $V = k \otimes_F V_F$, we denote by $\sigma \otimes 1 : k \otimes_F V \to k \otimes_F V$ the "σ-conjugate linear" map sending $a \otimes v$ to $\sigma(a) \otimes v$. We define the action of σ on a p-point $\alpha : k[t]/t^p \to kG$ by precomposing with $\sigma^{-1} \otimes 1$ and postcomposing with $\sigma \otimes 1$:

$$\text{(5.5.1)} \qquad \alpha^\sigma : k[t]/t^p = k \otimes_F F[t]/t^p \overset{\sigma^{-1}\otimes 1}{\to} k \otimes_F F[t]/t^p \overset{\alpha}{\to} k \otimes_F FG_F \overset{\sigma\otimes 1}{\to} kG$$

It is immediate from the definition that α^σ is again a p-point of G.

If $\mathrm{Ker}\{\nu : H^\bullet(G,k) \to k[x]\}$ is a point in $\mathrm{Proj}\, H^\bullet(G,k)$, then $\sigma(\mathrm{Ker}\,\nu) = \mathrm{Ker}\,(\nu^\sigma)$, where ν^σ is obtained by conjugating with $\sigma \otimes 1$, i.e.

$$\nu^\sigma : H^\bullet(G,k) = k \otimes_F H^\bullet(G_F,F) \stackrel{\sigma\otimes 1}{\to} k \otimes_F H^\bullet(G_F,F) \stackrel{\nu}{\to} k \otimes_F F[x] \stackrel{\sigma^{-1}\otimes 1}{\to} k[x].$$

Setting $v = \alpha^*$, we have $(\alpha^\sigma)^* = (\alpha^*)^\sigma$ so that

$$(5.5.2)\quad ker\{(\alpha^\sigma)^* : H^\bullet(G,k) \to H^\bullet(G,k)\} = \sigma(ker\{\alpha^* : H^\bullet(G,k) \to H^\bullet(G,k)\}).$$

Equation (5.5.2) together with Theorem 4.8 immediately implies that the action of σ induces an action on equivalence classes (i.e., an action on $P(G)$) and that the homeomorphism Ψ_G is equivariant with respect to this action.

If a finite-dimensional kG-module M is defined over F, $M = k \otimes_F M_0$, then the annihilator ideal $ker\{H^\bullet(G,k) \to Ext^*_G(M,M)\}$ equals $k \otimes_F ker\{H^\bullet(G_F,F) \to Ext^*_{G_F}(M_0,M_0)\}$. Thus, for a finite dimensional kG-module M defined over k, $|G|_M$ and $\mathrm{Proj}\,|G|_M$ are stable under the action of σ, so that the equivariance of Ψ_G with respect to the action of σ together with another appeal to Theorem 4.8 implies that $P(G)_M$ is also stable under the action of σ.

We leave to the reader the straight-forward verification of the last property. □

The following theorem extends Theorem 1.5 to all finite group schemes.

Theorem 5.6. *Let G be a finite group scheme and M be a finite dimensional G-module. Then $M \mapsto P(G)_M$ (as well as $M \mapsto Proj|G|_M$) satisfies the following properties:*

1. *$P(G)_k = P(G)$.*
2. *If $M_1 \to M_2 \to M_3 \to$ is a distinguished triangle in $stmod_G$, then $P(G)_{M_2} \subset P(G)_{M_1} \cup P(G)_{M_3}$.*
3. *$P(G)_{M\oplus N} = P(G)_M \cup P(G)_N$.*
4. *$P(G)_{M\otimes N} = P(G)_M \cap P(G)_N$.*
5. *$P(G)_M = \emptyset$ if and only if M is projective.*
6. *Let $i : H \subset G$ be a closed embedding of finite group schemes. Then $P(H)_M = (i_*)^{-1}(P(G)_M)$*
7. *$dim P(G)_M = cx(M) - 1$.*

Proof. Corollary 3.11 verifies properties (1), (2), (3), and (4). If M is projective, then clearly $\alpha^*(M)$ is projective for every abelian p-point $\alpha : k\mathbb{Z}/p \to kG$ so that $P(G)_M = \emptyset$. Conversely, if $P(G)_M = \emptyset$, then Theorem 4.11 implies that $|G|_M = \{0\}$ so that the radical $\sqrt{J}_M$ of the annihilator ideal $J_M \subset H^\bullet(G,k)$ associated to $H^*(G,k)$ acting on $Ext^*_G(M,M)$ equals the augmentation ideal $I \subset H^\bullet(G,k)$. Since M is finitely generated, $Ext^*_G(M,M)$ is finitely generated as a $H^\bullet(G,k)$-module, so that this implies that $Ext^i_G(M,M) = 0$ for $i \geq N$ for some N sufficiently large. For each simple G-module S, $Ext^*_G(S,M)$ is again a finitely generated module over the algebra $Ext^*_G(M,M)$. Since there are only finitely many simple modules, we conclude, by increasing N if necessary, that $Ext^i_G(S,M) = 0$ for $i \geq N$ and all simple G-modules S. This, in turn, implies that M admits a finite injective resolution. Because G-injectives are also G-projectives, such a finite injective resolution must split. Thus, M is projective.

Since the map of group algebras induced by a closed embedding $i : H \hookrightarrow G$ of finite group schemes is flat, the map $i_* : P(H) \to P(G)$ is well-defined in this case.

To prove (6.) we simply observe that for any abelian p-point $\alpha : k\mathbb{Z}/p \to kH$ and a G-module M, we have an isomorphism of $\mathbb{Z}/p$-modules

$$\alpha^*(M\downarrow_H) \simeq (i \circ \alpha)^*(M).$$

The proof of the equality $\dim |G|_M = cx(M)$ for finite groups (see, for example, [4, II.5.7.2]) applies without changes to any finite group scheme.

□

Proposition 5.7. *Let G be a finite group scheme. Any closed conical subset of $|G|$ can be realized as a support variety $|G|_M$ for some finite-dimensional G-module M.*

Proof. Since closed conical subsets of $|G|$ are in one-to-one correspondence with closed subsets of $\mathrm{Proj}\,|G|$, it suffices to show that any closed subset of $\mathrm{Proj}\,|G|$ has the form $\mathrm{Proj}\,|G|_M$. By Theorem 4.8, we have a homeomorphism $\Psi : P(G) \to \mathrm{Proj}\,|G|$ which takes $P(G)_M$ to $\mathrm{Proj}\,|G|_M$ for all finite-dimensional modules M. Since $P(G)_M$ run through all closed subsets of $P(G)$, $\mathrm{Proj}\,|G|_M$ must give all closed subsets of $\mathrm{Proj}\,|G|$. □

As a final consequence, we give an interpretation in terms of spaces of abelian p-points of the comparison due to J. Carlson, Z. Lin, and D. Nakano of the support varieties of the finite Chevalley group $G(\mathbb{F}_p)$ and the restricted Lie algebra $g = Lie(G)$ associated to a connected reductive algebraic group G [7]. Let $\mathcal{N}_p(g)$ be the p-restricted nullcone of g, i.e. the variety of p-nilpotent elements, and let $\mathcal{U}_p(G)$ be the variety of unipotent elements of G. By a theorem of Friedlander-Parshall ([12]), further extended in [20], there is a natural G-equivariant isogeny $N_p(g) \simeq \mathrm{Spec}\, H^\bullet(g,k)$. We refer the reader to [7] or [16] for the existence of the "good" exponential map: $\exp : \mathcal{N}_p(g) \to \mathcal{U}_p(G)$ under the assumption $p > 2h-2$, where h is the Coxeter number of G. The properties of exp that we require are that it is a G-equivariant isomorphism of varieties such that $exp(x)$, $exp(y)$ commute in G if and only if x, y commute in $\mathcal{N}_p(g)$. We will denote the inverse map by $\ell : \mathcal{U}_p(G) \to \mathcal{N}_p(g)$.

Proposition 5.8. *Let G be a connected reductive algebraic group defined and split over $\mathbb{F}_p$, let $G(\mathbb{F}_p)$ denote the group of $\mathbb{F}_p$-rational points of G, and let $g = Lie(G)$. Assume that $p \geq 2h-1$, where h is the Coxeter number of G. Then ℓ determines a natural embedding*

$$\mathcal{L} : P(G(\mathbb{F}_p)) \to P(G_{(1)})/G(\mathbb{F}_p).$$

Proof. We define a new equivalence relation "$\underset{ab}{\sim}$" on the subset of abelian p-points of $G(\mathbb{F}_p)$ which factor through elementary abelian subgroups to be the transitive closure of the following relation: α is related to β if and only if there exists an elementary abelian subgroup $E \subset G(\mathbb{F}_p)$ such that both α, β factor through kE and are equivalent as p-points of E. We denote the set of equivalence classes of p-points with respect to the equivalence relation $\underset{ab}{\sim}$ by $P(\widetilde{G(\mathbb{F}_p)})$. Observe that $P(\widetilde{G(\mathbb{F}_p)})$ is naturally endowed with an action of $G(\mathbb{F}_p)$ and, moreover, Corollary 3.7 and Lemma 4.2 imply that we have a bijection

$$P(\widetilde{G(\mathbb{F}_p)})/G(\mathbb{F}_p) \xrightarrow{\sim} P(G(\mathbb{F}_p)).$$

We proceed to construct a $G(\mathbb{F}_p)$-equivariant map

$$\widetilde{\mathcal{L}} : P(\widetilde{G(\mathbb{F}_p)}) \to \mathrm{Proj}\,\mathcal{N}_p(g).$$

Let $\alpha : k[u]/u^p \to kG(\mathbb{F}_p)$ be a representative of an equivalence class in $P(\widetilde{G(\mathbb{F}_p)})$. We can write $\alpha(u) = \sum_1^n a_i(g_i - 1)$, where $a_1, \dots, a_n \in k$, $g_1, \dots, g_n$ are pairwise commuting p-unipotent elements of $G(\mathbb{F}_p)$. For any $\sum_i a_i(g_i - 1) \in J(kG(\mathbb{F}_p))$ with $a_i \in k$ and $\{g_i\}$ pairwise commuting (but not necessarily the image of u under some flat map $\alpha : k[u]/u^p \to kG(\mathbb{F}_p)$), define

$$\hat{\mathcal{L}}(\sum_i a_i(g_i - 1)) = a_1\ell(g_1) + \dots a_n\ell(g_n) \in \mathcal{N}_p(g)$$

and define

$$\widetilde{\mathcal{L}}(\alpha) = \overline{\hat{\mathcal{L}}(\alpha(u))}.$$

Here, $J(kG(\mathbb{F}_p))$ denotes the augmentation ideal of $kG(\mathbb{F}_p)$.

We proceed to show that $\widetilde{\mathcal{L}}(\alpha)$ does not depend on the choice of a representative of $[\alpha] \in P(\widetilde{G(\mathbb{F}_p)})$. By the definition of the equivalence relation $\underset{ab}{\sim}$, it suffices to show that for any elementary abelian subgroup $E \subset G(\mathbb{F}_p)$ and any two p-points $\alpha, \beta : k\mathbb{Z}/p \to kE \to kG(\mathbb{F}_p)$ which are equivalent as p-points of E, we have $\widetilde{\mathcal{L}}(\alpha) = \widetilde{\mathcal{L}}(\beta)$. Since $\alpha \sim \beta$ as p-points of E, Proposition 2.2 implies that $(\alpha - \beta)(u) \in J^2(kE)$ (after possibly multiplying α by a suitable non-zero constant). Thus, it suffices to show that $\hat{\mathcal{L}}$ vanishes on $J^2(kE)$. Let $(g-1)(h-1) \in J^2(kE)$ be a generator of the square of the augmentation ideal. Then

$$\hat{\mathcal{L}}((g-1)(h-1)) = \widetilde{\mathcal{L}}((gh-1) - (g-1) - (h-1)) = \ell(gh) - \ell(g) - \ell(h) = 0.$$

The fact that $\widetilde{\mathcal{L}}$ is $G(\mathbb{F}_p)$-equivariant follows immediately from the fact that the exponential map, and thus ℓ, is $G(\mathbb{F}_p)$-equivariant.

Next we show that the map $\widetilde{\mathcal{L}}$ is injective. Let $\alpha : k[u]/u^p \to kE \to kG(\mathbb{F}_p)$, $\beta : k[u]/u^p \to kE' \to kG(\mathbb{F}_p)$ be be two points of $P(\widetilde{G(\mathbb{F}_p)})$ such that $\widetilde{\mathcal{L}}(\alpha) = \widetilde{\mathcal{L}}(\beta)$. Choose E, E' of rank r, r' minimal among elementary abelian p-subgroups of $G(\mathbb{F}_p)$ which factor α, β and let $E_0 = E \cap E'$ whose rank we denote by r_0. Write $E = E_0 \times E_1, E' = E_0 \times E_2$ and choose elements $x_1, \dots, x_{r_0}, \dots, x_r \in g_{\mathbb{F}_p}$ with the property that $\{\exp(x_i)\}$ is a minimal set of generators of E and $\exp(x_1), \dots, \exp(x_{r_0})$ is a minimal set of generators of E_0; choose $y_1 = x_1, \dots, y_{r_0} = x_{r_0}, y_{r_0+1}, \dots, y_{r'} \in g_{\mathbb{F}_p}$ such that $\{\exp(y_1), \dots, \exp(y_{r'})\}$ is a minimal set of generators for E'. Write $\alpha(u) = \sum a_i(\exp x_i - 1)$, $\beta(u) = \sum b_j(\exp y_j - 1)$.

The condition $\widetilde{\mathcal{L}}(\alpha) = \widetilde{\mathcal{L}}(\beta)$ implies that a non-zero multiple of $\sum a_i x_i$ equals $\sum b_j y_j$ in $\mathcal{N}_p(g)$; multiplying by a constant, if necessary, we may assume

$$\sum a_i x_i = \sum b_j y_j.$$

Since $x_i, y_j \in g_{\mathbb{F}_p}$, the existence of linear relation over k implies that there exists a linear relation over $\mathbb{F}_p$, i.e. there exist $a'_i, b'_j \in \mathbb{F}_p$ such that

$$\sum a'_i x_i = \sum b'_j y_j.$$

Exponentiating, we obtain the relation in $G(\mathbb{F}_p)$:

$$\prod_{i=1}^{r} \exp(x_i)^{a'_i} = \prod_{j=1}^{r'} \exp(y_j)^{b'_j}$$

which implies the relation

$$\prod_{j=1}^{r_0} \exp(-y_j)^{b'_j} \prod_{i=1}^{r} \exp(x_i)^{a'_i} = \prod_{j=r_0+1}^{r'} \exp(y_j)^{b'_j}$$

Since the left hand side is an element of E and the right hand side is an element of E_2 and since $E \cap E_2 = \{1\}$, we conclude both sides of the above equality are trivial so that $r_0 = r = r'$. Consequently, the equality $\sum_i a_i x_i = \sum_j b_j y_j$ becomes $\sum_i a_i x_i = \sum_i b_i x_i$ by our choice of $y_j = x_j, j \leq r_0$. We conclude that $a_i = b_i$, so that $\alpha = \beta$.

Since $\widetilde{\mathcal{L}} : P(\widetilde{G(\mathbb{F}_p)}) \to \operatorname{Proj} \mathcal{N}_p(g)$ is $G(\mathbb{F}_p)$-equivariant and injective, it induces an injective map on coinvariants which we denote by $\mathcal{L}$:

$$\mathcal{L} : P(G(\mathbb{F}_p)) \simeq P(\widetilde{G(\mathbb{F}_p)})/G(\mathbb{F}_p) \hookrightarrow \operatorname{Proj} \mathcal{N}_p(g)/G(\mathbb{F}_p) \simeq P(G_{(1)})/G(\mathbb{F}_p),$$

where the last bijection follows from Proposition 3.8 and the homeomorphism $\mathcal{N}_p(g) \simeq V(G_{(1)})$ established in [20].

□

Remark 5.9. The map $\mathcal{L}$ constructed in the proposition is easily seen to be the projectivization of the map $|G(\mathbb{F}_p)| \to \mathcal{N}_p(g)/G(\mathbb{F}_p)$ in [7].

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Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: eric@math.nwu.edu

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540
E-mail address: julia@ias.edu