1 Semi-topological K-theory

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1.1 Introduction

The *semi-topological K-theory* of a complex variety X, written $K_*^{sst}(X)$, interpolates between the algebraic K-theory, $K_*^{alg}(X)$, of X and the topological K-theory, $K_{top}^*(X^{an})$, of the analytic space X^{an} associated to X. (The superscript "sst" stands for "singular semi-topological".) In a similar vein, the *real semi-topological K-theory*, written $K\mathbb{R}_*^{sst}(Y)$, of a real variety Y interpolates between the algebraic K-theory of Y and the Atiyah Real K-theory of the associated space with involution $Y_{\mathbb{R}}(\mathbb{C})$. We intend this survey to provide both motivation and coherence to the field of semi-topological K-theory. We explain the many foundational results contained in the series of papers by the authors [31, 27, 32], as well as in the recent paper by the authors and Christian Haesemeyer [21]. We shall also mention various conjectures that involve challenging problems concerning both algebraic cycles and algebraic K-theory.

Our expectation is that the functor $K_*^{sst}(-)$ is better suited for the study of complex algebraic varieties than either algebraic K-theory or topological K-theory. For example, applied to the point $X = \operatorname{Spec}\mathbb{C}$, $K_i^{alg}(-)$ yields uncountable abelian groups for $i > 0$, whereas $K_i^{sst}(\operatorname{Spec}\mathbb{C})$ is 0 for i odd and $\mathbb{Z}$ for i even (i.e., it coincides with the topological K-theory of a point). On the other hand, topological K-theory is a functor on homotopy types and ignores finer algebro-geometric structure of varieties, whereas semi-topological and algebraic K-theory agree on finite coefficients

$$K_*^{alg}(-,\mathbb{Z}/n) \cong K_*^{sst}(-,\mathbb{Z}/n) \tag{1.1}$$

and the rational semi-topological K-groups $K_*^{sst}(X,\mathbb{Q})$ contain information about the cycles on X and, conjecturally, the rational Hodge filtration on singular cohomology $H^*(X^{an},\mathbb{Q})$.

To give the reader some sense of the definition of semi-topological K-theory, we mention that $K_0^{sst}(X)$ is the Grothendieck group of algebraic vector bundles modulo algebraic equivalence: two bundles on X are *algebraically equivalent* if each is given as the specialization to a closed point on a connected curve C of a common vector bundle on $C \times X$. In particular,

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the ring $K_0^{sst}(X)$ (with product given by tensor product of vector bundles) is rationally isomorphic to the ring $A^*(X)$ of algebraic cycles modulo algebraic equivalence (with the product given by intersection of cycles) under the Chern character map

$$ch : K_0^{sst}(X, \mathbb{Q}) \xrightarrow{\cong} A^*(X, \mathbb{Q}).$$

This should be compared with the similar relationship between $K_0^{alg}(X)$ and the Chow ring $CH^*(X)$ of algebraic cycles modulo rational equivalence. Taking into consideration also the associated topological theories, we obtain the following heuristic diagram, describing six cohomology theories of interest.

Table 1.1. Six cohomology theories together with their "base values" for a smooth variety X

K-theory	Cohomology (i.e., cycle theory)
Algebraic K-theory, $K_*^{alg}(-)$ $K_0^{alg}(X)$ = algebraic vector bundles modulo rational equivalence	*Motivic Cohomology*, $H^*_{\mathcal{M}}(-, \mathbb{Z}(*))$ $H^{2*}_{\mathcal{M}}(X, \mathbb{Z}(*)) = CH^*(X)$ = cycles modulo rational equivalence
Semi-topological K-theory, $K_*^{sst}(-)$ $K_0^{sst}(X)$ = algebraic vector bundles modulo algebraic equivalence	*Morphic Cohomology*, $L^*H^*(-)$ $L^*H^{2*}(X) = A^*(X)$ = cycles modulo algebraic equivalence
Topological K-theory, $K^*_{top}(-)$ $K^0_{top}(X^{an})$ = topological vector bundles modulo topological equivalence	*Singular Cohomology*, $H^*_{sing}(-)$ $H^{2*}_{sing}(X^{an})$ = integral, rectifiable cycles modulo topological equivalence

As we discuss below, the authors have constructed a precise counter-part of this heuristic diagram by establishing a homotopy commutative diagram of spectra — see (1.9) in Section 1.3 below. For example, there are Chern character maps joining the theories in the left column and the theories (with rational coefficients) in the right column, and if X is smooth, these Chern character maps are rational isomorphisms in all degrees. In particular, such isomorphisms extend the rational isomorphism $K_0^{sst}(-)_{\mathbb{Q}} \cong A^*(-)_{\mathbb{Q}}$ mentioned above.

For certain special varieties X (e.g., projective smooth toric varieties), the natural map

$$K_*^{sst}(X) \to K_{top}^{-*}(X^{an})$$

is an isomorphism whenever $* \geq 0$ (see [21]). Such an isomorphism can be interpreted (as we now interpret Lawson's original theorem) as asserting that some construction involving algebraic morphisms is a "small" homotopy-theoretic model for an analogous construction involving continuous maps between analytic spaces. In general, however, $K_*^{sst}(X)$ differs considerably from topological K-theory; for example, $K_0^{sst}(X)$ need not be finitely generated even for a smooth projective variety X. Nonetheless, there are natural transformations

$$\mathcal{K}^{alg}(-) \to \mathcal{K}^{sst}(-) \to \mathcal{K}_{top}(-)$$

from $(Sch/\mathbb{C})$ to $\underline{Spectra}$ with many good properties, perhaps the most striking of which is the isomorphism for finite coefficients mentioned above (1.1). Understanding multiplication in $K_*^{sst}(X)$ by the Bott element

$$\beta \in K_2^{sst}(\mathrm{Spec}\,\mathbb{C}) \cong K_{top}^{-2}(pt)$$

is also interesting: for X is smooth, the natural map $K_*^{sst}(X) \to K_{top}^{-*}(X^{an})$ induces an isomorphism

$$K_*^{sst}(X)[\beta^{-1}] \cong K_{top}^{-*}(X^{an})[\beta^{-1}] = KU^{-*}(X^{an})$$

upon inverting the Bott element. On the other hand, the kernel of $K_0^{sst}(X) \to K_{top}^0(X^{an})$ is rationally isomorphic to the Griffiths group (of algebraic cycles on X homologically equivalent to 0 modulo those algebraically equivalent to 0). Moreover, a filtration on $K_{top}^*(X) \otimes \mathbb{Q}$ associated to $\mathcal{K}^{sst}(-) \to \mathcal{K}_{top}(-)$ and multiplication by $\beta \in K_2^{sst}(pt)$ is conjecturally equivalent to the rational Hodge filtration.

To formulate K-theories, we require some process of "homotopy theoretic group completion" as first became evident in Quillen's formulation of algebraic K-theory of rings using the Quillen plus construction. This can be contrasted with the simpler constructions of cohomology theories: these derive from structures (e.g., the monoid of effective cycles) which are commutative, whereas the direct sum of vector bundles is only commutative up to coherent isomorphism. For this reason, the constructions we present involve use of the machinery of operads and the utilization of certain other homotopy-theoretic techniques.

As the reader will see, the analytic topology on a real or complex variety is used in the construction of semi-topological K-theory. Thus far, there is no reasonable definition of semi-topological K-theory for varieties over other base fields, although one might anticipate that p-adic fields and real closed fields might lend themselves to such a theory.

We conclude this introduction with a few brief comments to guide the reader toward more details about the topics we discuss. First, the original paper of H. B. Lawson [39] initiated the study of the analytic spaces of Chow varieties, varieties that parametrize effective algebraic cycles on a complex projective variety. Lawson's remarkable theorem enables one to compute the homotopy groups of the topological abelian group of algebraic cycles of a given dimension on a projective space $\mathbb{P}^n$ (i.e., his theorem enables one to compute what is now known as the Lawson homology of $\mathbb{P}^n$). In [14] the first author pointed out that the group of connected components of the topological abelian group of r-cycles on a complex projective variety X is naturally isomorphic to the group of algebraic r-cycles on X modulo algebraic equivalence. A key insight was provided by Lawson and M.-L. Michelson [41] who proved that the universal total Chern class can be interpreted as a map induced by the inclusion of linear cycles on projective spaces into all algebraic cycles. Important formal properties of Lawson homology were developed by the first author in collaboration with O. Gabber [20] and by P. Lima-Filho [46]. B. Mazur and the first author investigated filtrations on homology associated with Lawson homology in [24], and the first author studied complementary filtrations on cycles in [15, 19]. H. B. Lawson and the first author introduced the concept of a cocycle leading to morphic cohomology theory [23] and established a duality relationship between morphic cohomology and Lawson homology in [22]. Consideration of quasi-projective varieties using similar methods has led to awkward questions of point-set topology, so that plausible definitions are difficult to handle when the varieties are not smooth (cf. [28]). When contemplating the formulation of semi-topological K-theory for quasi-projective varieties, the authors introduce singular semi-topological complexes, which appear to give a good formulation of morphic cohomology for any quasi-projective algebraic variety [32].

As Lawson homology and morphic cohomology developed, it became natural to seek a companion K-theory. In [23], H. B. Lawson and the first author showed how to obtain characteristic classes in morphic cohomology for algebraic vector bundles. "Holomorphic K-theory" was briefly introduced by Lawson, Lima-Filho, and Michelsohn in [40]. Following an outline of the first author [16], the authors established the foundations and general properties of semi-topological K-theory in a series of papers [31, 27, 32] and extended this theory to real quasi-projective varieties in [30]. Many of the results sketched in this survey were first formulated and proved in these papers. A surprisingly difficult result proved by the authors is the assertion that there is a natural rational isomorphism (given by the Chern character) relating semi-topological K-theory and morphic cohomology of smooth varieties [32, 4.7].

Most recently, the authors together with C. Haesemeyer established a spectral sequence relating morphic cohomology and semi-topological K-theory compatible with the motivic and Atiyah-Hirzebruch spectral sequences [21]. Moreover, this paper uses the notion of integral weight filtrations on Borel-Moore homology (due to Deligne [11] and Gillet-Soulé [33]), which, in conjunction with the spectral sequence, enable them to establish that $K_*^{sst}(X) \to K_{top}^{-*}(X^{an})$ is an isomorphism for many of the special varieties for which one might hope this to be true.

The subject now is ready for the computation of $K_*^{sst}(-)$ for more complicated varieties and for applications of this theory to the study of geometry. Since many of the most difficult and long-standing conjectures about complex algebraic varieties are related to such computations, one suspects that general results will be difficult to achieve. We anticipate that the focus on algebraic equivalence given by morphic cohomology and semi-topological K-theory might lead to insights into vector bundles and algebraic cycles on real and complex varieties.

1.2 Definition of Semi-topological K-theory

Originally [31] semi-topological K-theory was defined only for projective, weakly normal complex algebraic varieties and these original constructions involved consideration of topological spaces of algebraic morphisms from such a variety X to the family of Grassmann varieties $\mathrm{Grass}_n(\mathbb{C}^N)$. The assumption that X is projective implies that the set of algebraic morphisms $\mathrm{Hom}(X, \mathrm{Grass}_n(\mathbb{C}^N))$ coincides with the set of closed points of an ind-variety, and thus we may topologize $\mathrm{Hom}(X, \mathrm{Grass}_n(\mathbb{C}^N))$ by giving it the associated analytic topology. If, in addition, X is weakly normal, then $\mathrm{Hom}(X, \mathrm{Grass}_n(\mathbb{C}^N))$ maps injectively to $\mathrm{Maps}(X^{an}, \mathrm{Grass}_n(\mathbb{C}^N)^{an})$, the set of all continuous maps, and we may also endow $\mathrm{Hom}(X, \mathrm{Grass}_n(\mathbb{C}^N))$ with the subspace topology of the space $\mathcal{M}aps(X^{an}, \mathrm{Grass}_n(\mathbb{C}^N)^{an})$, the set $\mathrm{Maps}(X^{an}, \mathrm{Grass}_n(\mathbb{C}^N)^{an})$ endowed with the usual compact-open topology. In fact, these topologies coincide, and $\mathcal{M}or(X, \mathrm{Grass}_n(\mathbb{C}^N))$ denotes this topological space. The collection of spaces $\mathcal{M}or(X, \mathrm{Grass}_n(\mathbb{C}^N))$ for varying n and N leads to the construction of a spectrum $\mathcal{K}^{semi}(X)$ whose homotopy groups are the semi-topological K-groups of X.

The authors [31] subsequently extended the theory so constructed to all quasi-projective complex varieties U by providing the set of algebraic morphisms $\mathrm{Hom}(U^w, \mathrm{Grass}_n(\mathbb{C}^N))$ (where $U^w \to U$ is the weak normalization of U) with a natural topology (again using $\mathcal{M}or(U, \mathrm{Grass}_n(\mathbb{C}^N))^{an}$ to denote the resulting space). We were, however, unable to verify many of the desired formal properties of this construction $K_*^{semi}(-)$ when applied to non-projective varieties.

Inspired by a suggestion of V. Voevodsky, the authors reformulated semi-topological K-theory in [27]. The resulting functor from quasi-projective complex varieties to spectra, $\mathcal{K}^{sst}(-)$, when applied to a weakly normal projective variety X gives a spectrum weakly homotopy equivalent to the spectrum $\mathcal{K}^{semi}(X)$. We have shown that the functor $U \mapsto K_*^{sst}(U)$ satisfies many desirably properties, and thus now view the groups K_*^{sst} as *the* semi-topological K-groups of a variety.

In this section, we begin with the definition of $\mathcal{K}^{semi}(X)$ (restricted to weakly normal, projective complex varieties). Although supplanted by the more general construction $\mathcal{K}^{sst}$ discussed below, the motivation underlying the construction of $\mathcal{K}^{semi}$ is more geometric and transparent.

We shall see that the definition is formulated so that there are natural homotopy classes of maps of spectra

$$\mathcal{K}^{alg}(X) \to \mathcal{K}^{semi}(X) \to \mathcal{K}_{top}(X^{an}). \tag{1.2}$$

(Here, $\mathcal{K}_{top}(X^{an})$ denotes the (-1)-connected cover of $ku(X^{an})$, the mapping spectrum from X^{an} to **bu**.) These maps are induced by the natural maps of simplicial sets given in degree d by

$$\begin{aligned}\mathrm{Hom}(\Delta^d \times X, \mathrm{Grass}_n(\mathbb{C}^N)) &\to \mathrm{Maps}(\Delta^d_{top}, \mathcal{M}or(X, \mathrm{Grass}_n(\mathbb{C}^N))^{an}) \\ &\to \mathrm{Maps}(\Delta^d_{top}, \mathcal{M}aps(X^{an}, \mathrm{Grass}_n(\mathbb{C}^N))^{an}).\end{aligned}$$

In formulating $\mathcal{K}^{semi}(-)$ (and later $\mathcal{K}^{sst}(-)$), we are motivated by the property that if one applies the connected component functor $\pi_0(-)$ to the maps of (1.2), then one obtains the natural maps

$$K_0^{alg}(X) \to K_0^{alg}(X)/\text{algebraic equivalence} \to K^0_{top}(X^{an})$$

from the Grothendieck group of algebraic vector bundles to the Grothendieck group of algebraic vector bundles modulo algebraic equivalence to the Grothendieck group of topological vector bundles.

The reader will find that in order to define the spectra appearing in (1.2), we use operads to stabilize and group complete the associated mapping spaces. This use of operads makes the following discussion somewhat technical.

1.2.1 Semi-topological *K*-theory of Projective Varieties: $\mathcal{K}^{semi}$

Let X be a projective, weakly normal complex variety and define $\mathrm{Grass}(\mathbb{C}^N) = \coprod_n \mathrm{Grass}_n(\mathbb{C}^N)$ where $\mathrm{Grass}_n(\mathbb{C}^N)$ is the projective variety parameterizing rank n quotient complex vector spaces of $\mathbb{C}^N$. Since X and $\mathrm{Grass}(\mathbb{C}^N)$ are projective varieties, the set $\mathrm{Hom}(X, \mathrm{Grass}(\mathbb{C}^N))$ is the set of closed points of an infinite disjoint union (indexed by degree) of quasi-projective complex varieties. We write this ind-variety as $\mathcal{M}or(X, \mathrm{Grass}(\mathbb{C}^N))$ and we let $\mathcal{M}or(X, \mathrm{Grass}(\mathbb{C}^N))^{an}$ denote the associated topological space endowed with the analytic topology. Since X is weakly normal, one can verify that $\mathcal{M}or(X, \mathrm{Grass}(\mathbb{C}^N))^{an}$ is naturally a subspace of $\mathcal{M}aps(X^{an}, \mathrm{Grass}(\mathbb{C}^N)^{an})$, the space of all continuous maps endowed with the compact-open topology. Considering the system of ind-varieties $\mathcal{M}or(X, \mathrm{Grass}(\mathbb{C}^N)$ for $N \geq 0$, where the map $\mathrm{Grass}(\mathbb{C}^N) \to \mathrm{Grass}(\mathbb{C}^{N+1})$ is given by composing with the projection map $\mathbb{C}^{N+1} \to \mathbb{C}^N$ onto the first N coordinates, gives the ind-variety $\mathcal{M}or(X, \mathrm{Grass})$ and the associated space

$$\mathcal{M}or(X, \mathrm{Grass})^{an} = \varinjlim_N \mathcal{M}or(X, \mathrm{Grass}(\mathbb{C}^N))^{an},$$

which we may identify with a subspace of $\mathcal{M}aps(X^{an}, \mathrm{Grass}^{an})$.

The following proposition indicates that the space $\mathcal{M}or(X, \mathrm{Grass})^{an}$ possesses interesting K-theoretic information. We shall say that two algebraic bundles $V_1 \to X$, $V_2 \to X$ are *algebraically equivalent* if there exists a connected smooth curve C and an algebraic vector bundle $\mathcal{V} \to C \times X$ such that each of $V_1 \to X$, $V_2 \to X$ is given by restriction of $\mathcal{V}$ to some $\mathbb{C}$-point of C. We define *algebraic equivalence* on either the monoid of isomorphism classes of algebraic vector bundles or the associate Grothendieck group $K_0^{alg}(X)$ to be equivalence relation generated by algebraic equivalence of bundles.

Proposition 1.2.1. *(cf. [31, 2.10, 2.12]) There is a natural isomorphism of abelian monoids*

$$\pi_0\left(\mathcal{M}or(X, \mathrm{Grass})^{an}\right) \cong \frac{\{\textit{algebraic vector bundles on } X \textit{ generated by global sections}\}}{\textit{algebraic equivalence}},$$

(where the abelian monoid law for the left-hand side is described below). Moreover, the group completion of the above map can be identified with the following natural isomorphism of abelian groups

$$K_0^{semi}(X) = \pi_0\left(\mathcal{M}or(X, \mathrm{Grass})^{an}\right)^+ \cong \frac{K_0(X)}{\textit{algebraic equivalence}}.$$

The proof of Proposition 1.2.1 is straight-forward, perhaps disguising several interesting and important features. First, the condition that two points in $\mathcal{M}or(X, \mathrm{Grass})^{an}$ lie in the same topological component is equivalent to the condition that they lie in the same Zariski component of the ind-variety $\mathcal{M}or(X, \mathrm{Grass})$. Second, upon group completion, one obtains all (virtual) vector bundles so that one may drop the condition that the vector bundles be generated by their global sections.

To define the higher semi-topological K-groups, we introduce the structure of an H-space on $\mathcal{M}or(X, \mathrm{Grass})$. Direct sum of bundles determines algebraic pairings

$$\mathcal{M}or(X, \mathrm{Grass}_n(\mathbb{C}^N)) \times \mathcal{M}or(X, \mathrm{Grass}_m(\mathbb{C}^M)) \to \mathcal{M}or(X, \mathrm{Grass}_{n+m}(\mathbb{C}^{N+M})),$$

for all M, N. Once one chooses a linear injection $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \hookrightarrow \mathbb{C}^\infty$, these pairing may be stabilized in a suitable fashion by letting $M, N \mapsto \infty$ to endow $\mathcal{M}or(X, \mathrm{Grass})$ with an operation. Under this operation, the associated space $\mathcal{M}or(X, \mathrm{Grass})^{an}$ is a homotopy-commutative H-space. Using one of several techniques of infinite loop spaces, one shows that this H-space admits a *homotopy-theoretic group completion*

$$\mathcal{M}or(X, \mathrm{Grass})^{an} \to (\mathcal{M}or(X, \mathrm{Grass})^{an})^{h+};$$

by definition, this is a map of H-spaces which induces group completion on π_0 and whose map on (integral, singular) homology can be identified with the map

$$H_*(\mathcal{M}or(X, \mathrm{Grass})^{an}) \to \mathbb{Z}[\pi_0^+] \otimes_{\mathbb{Z}[\pi_0]} H_*(\mathcal{M}or(X, \mathrm{Grass})^{an}).$$

Definition 1.2.2. Let X be a weakly normal, projective complex variety. We define

$$\mathcal{M}or(X, \mathrm{Grass})^{an} \to \mathcal{K}^{semi}(X)$$

to be a homotopy-theoretic group completion of the homotopy commutative H-space $\mathcal{M}or(X, \mathrm{Grass})^{an}$. We call $\mathcal{K}^{semi}(X)$ the *semi-topological K-theory space* of X, and we define the *semi-topological K-groups* of X by the formula

$$K_n^{semi}(X) = \pi_n(\mathcal{K}^{semi}(X)), \quad n \geq 0.$$

In particular, $K_0^{semi}(X)$ is naturally isomorphic to $K_0^{alg}(X)/(\text{algebraic equivalence})$. For any finitely generated abelian group A, we define the semi-topological K-groups of X with coefficients in A by the formula

$$K_n^{sst}(X, A) = \pi_n(\mathcal{K}^{semi}(X; A)), \quad n \geq 0.$$

Remark 1.2.3. An equivalent construction of $\mathcal{K}^{semi}(X)$ is given by Lawson, Lima-Filho, and Michelsohn in [40]. See also [49]. In these papers, the term "holomorphic K-theory" is used instead of "semi-topological K-theory".

The construction of the group-like H-space $\mathcal{K}^{semi}(X)$ from the H-space $\mathcal{M}or(X, \mathrm{Grass})^{an}$ can be enriched to yield an Ω-spectrum in several ways. For example, one can let $\mathcal{I} = \mathcal{I}(n), n \geq 0$, denote the E_∞-operad with $\mathcal{I}(n)$ defined to be the contractible space of all linear injections from $(\mathbb{C}^\infty)^{\oplus n}$ into $\mathbb{C}^\infty$. (Thus $\mathcal{I}$ is closely related to the linear isometries operad.) Then $\mathcal{I}$ "acts" on $\mathcal{M}or(X, \mathrm{Grass})^{an}$ via a family of pairings

$$\mathcal{I}(n) \times (\mathcal{M}or(X, \mathrm{Grass})^{an})^n \to \mathcal{M}or(X, \mathrm{Grass})^{an}, \quad n \geq 0.$$

Intuitively, this action can be describe as follows: given a point in $\mathcal{I}(n)$ and n quotients of the form $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{E}_i$, the point of $\mathcal{I}(n)$ allows one to move the n quotient objects into general position so that one may take their internal direct sum. In particular, the case $n = 2$ together with a specific choice of a point in $\mathcal{I}(2)$ defines the H-space operation for $\mathcal{M}or(X, \mathrm{Grass})^{an}$ given above. Such an action of the operad $\mathcal{I}$ determines an Ω-spectrum $\Omega^\infty \mathcal{M}or(X, \mathrm{Grass})^{an}$ using the machinery of May [50, §14], and the 0-th space of this spectrum provides a model for the homotopy-theoretic group completion $\mathcal{K}^{semi}(X)$.

It is useful to know that the algebraic K-theory space defined for a variety Y over an arbitrary field F admits a parallel construction. Recall that the standard algebraic k-simplex Δ^k over $\operatorname{Spec} F$ is the affine variety $\operatorname{Spec} F[x_0, \ldots, x_k]/(\Sigma_i x_i - 1)$ and that these standard simplices determine a cosimplicial variety $\Delta^\bullet_F$. The simplicial set $d \mapsto \operatorname{Hom}(X \times_F \Delta^d_F, \mathrm{Grass}(F^\infty))$ admits the structure of a homotopy-commutative H-space (if "space" is interpreted to mean "simplicial set"). In fact, $\operatorname{Hom}(X \times_F \Delta^\bullet_F, \mathrm{Grass}(F^\infty))$ is an $\mathcal{I}(\Delta^\bullet_F)$-space where $\mathcal{I}(\Delta^\bullet_F)$ is a suitable simplicial analogue of the E_∞-operad $\mathcal{I}$ introduced above. The following result is due to the second author and D. Grayson.

Theorem 1.2.4. *(cf. [31, 6.8] [35, 3.3]) Given a smooth, algebraic variety X over a field F, the homotopy-theoretic group completion of the homotopy-commutative H-space*

$$\mathrm{Hom}(X \times_F \Delta^\bullet_F, \mathrm{Grass}(F^\infty))$$

is weakly homotopy equivalent to $\mathcal{K}^{alg}(X)$, the algebraic K-theory space of X. In fact, the spectrum associated to the $\mathcal{I}(\Delta^\bullet)$-space $\mathrm{Hom}(X \times_F \Delta^\bullet_F, \mathrm{Grass}(F^\infty))$ is weakly equivalent to the algebraic K-theory spectrum of X.

Theorem 1.2.4 leads easily to the existence of a natural map

$$\mathcal{K}^{alg}(X) \to \mathcal{K}^{semi}(X)$$

of spectra (more precisely, $\mathcal{K}^{alg}(X) \to Sing.(\mathcal{K}^{semi}(X))$ representing the algebraic K-theory and semi-topological K-theory of smooth, projective complex varieties. Furthermore, if we replace $\mathcal{M}or(X, \mathrm{Grass})^{an}$ by $\mathcal{M}aps(X^{an}, \mathrm{Grass}^{an})$, then we can proceed with the same construction as above to form a space (in fact, an Ω-spectrum) $\mathcal{K}_{top}(X^{an})$ that receives a map from $\mathcal{K}^{semi}(X)$. It follows from [51, I.1] that we have

$$K^{-n}_{top}(X^{an}) := \pi_n \mathcal{K}_{top}(X^{an}) \cong ku^{-n}(X^{an}), \quad n \geq 0,$$

where ku^* denotes the connective topological K-theory of a space (i.e., the generalized cohomology theory represented by the connective spectrum **bu**). In other words, the spectrum $\mathcal{K}_{top}(X^{an})$ is the (-1)-connected cover of the mapping spectrum from X^{an} to **bu**. Moreover, the subspace inclusion $\mathcal{M}or(X, \mathrm{Grass})^{an} \subset \mathcal{M}aps(X^{an}, \mathrm{Grass}^{an})$ induces a natural map $\mathcal{K}^{semi}(X) \to \mathcal{K}_{top}(X^{an})$ such that the composition

$$\mathcal{K}^{alg}(X) \to \mathcal{K}^{semi}(X) \to \mathcal{K}_{top}(X) \tag{1.3}$$

induces the usual map from algebraic to topological K-theory.

If $\mathrm{Spec}\, \mathbb{C}$ is a point, we clearly have $\mathcal{K}^{semi}(\mathrm{Spec}\, \mathbb{C}) = \mathcal{K}_{top}(pt)$. A more interesting computation is the following integral analogue of the Quillen-Lichtenbaum conjecture for smooth projective complex curves.

Theorem 1.2.5. *(cf. [31, 7.5]) If C is a smooth, projective complex curve, then the natural map*

$$\mathcal{K}^{semi}(C) \to \mathcal{K}_{top}(C^{an})$$

is a weak homotopy equivalence, inducing isomorphisms

$$K^{semi}_n(C) \cong K^{-n}_{top}(C^{an}), \quad n \geq 0.$$

The proof of Theorem 1.2.5 uses a result of F. Kirwan [38, 1.1] on the moduli space of vector bundles on curves. Specifically, Kirwan shows that the composition of

$$A_d(n)^{an} \to \mathcal{M}or(C, \mathrm{Grass}_n(\mathbb{C}^\infty))^{an}_d \to \mathcal{M}aps(C^{an}, \mathrm{Grass}_n(\mathbb{C}^\infty)^{an})_d$$

induces an isomorphism in cohomology up to dimension k provided

$$d \geq 2n(2g+k+1) + n\max(k+1+n(2g+k+1), \frac{1}{4}n^2 g). \tag{1.4}$$

Here, g is the genus of C, the subscripts d refer to taking the open and closed subspaces consisting of degree d maps, and $A_d(n)$ refers to the subvariety of $\mathcal{M}or_d(C, \mathrm{Grass}_n(\mathbb{C}^\infty))$ parameterizing quotients $\mathcal{O}_X^\infty \twoheadrightarrow V$ satisfying the extra condition that $H^1(C, V) = 0$. Theorem 1.2.5 is deduced from this result of Kirwan by showing that the homotopy-theoretic group completions of each of the spaces in the chain

$$\coprod_{d,n} A_d(n)^{an} \to \coprod_{d,n} \mathcal{M}or_d(C, \mathrm{Grass}_n(\mathbb{C}^\infty))^{an} \to \coprod_{d,n} \mathcal{M}aps(C^{an}, \mathrm{Grass}_n(\mathbb{C}^\infty)^{an})_d$$

can be obtained by taking suitable limits (technically, mapping telescopes) of self-maps of each of the spaces. The point is that the first map here becomes an equivalence upon taking such limits since the condition defining $A_d(n)$ as a subvariety of $\mathcal{M}or_d(C, \mathrm{Grass}_n(\mathbb{C}^\infty))^{an}$ becomes trivial, and the second map becomes an equivalence since the inequality (1.4) is met in all degrees in the limit.

For higher dimensional varieties X, the map $\mathcal{K}^{semi}(X) \to \mathcal{K}_{top}(X^{an})$ is rarely a weak homotopy equivalence. For example, if S is a smooth, projective complex surface, the map $\mathcal{K}^{semi}(S) \to \mathcal{K}_{top}(S^{an})$ usually fails to induce an isomorphism at π_0 (although it does induce an isomorphism on all higher homotopy groups.) In fact, $K_0^{semi}(S, \mathbb{Q}) \cong K^0_{top}(S^{an}, \mathbb{Q})$ if and only if $H^2(S^{an}, \mathbb{Q})$ consists only of algebraic cohomology classes [21, 6.17].

1.2.2 Semi-topological K-theory of Quasi-projective Varieties: $\mathcal{K}^{sst}$

The extensions of the definition of semi-topological K-theory from projective complex varieties to quasi-projective complex varieties has a somewhat confusing history. Initially, the authors (see especially [31]) carried out this extension in seemingly the most natural way possible: one imposes a suitable topology on the set $\mathrm{Hom}(X^w, \mathrm{Grass})$ to form a space $\mathcal{M}or(X, \mathrm{Grass})^{an}$ and then repeats the constructions of the previous section to yield a group-like H-space (in fact, a spectrum) $\mathcal{K}^{semi}(X)$. We will not go into the details of the topology on $\mathcal{M}or(X, \mathrm{Grass})^{an}$ — we refer the interested reader to [28] for a careful description.

It gradually became apparent that annoying point-set topology considerations prevents one from establishing the desired formal properties of the theory $\mathcal{K}^{semi}(-)$ for arbitrary quasi-projective varieties. On the other hand, the authors have developed a closely related and conjecturally equivalent theory that allows for such properties to be proven. This newer theory, $\mathcal{K}^{sst}$, is now viewed by the authors as *the* semi-topological K-theory.

To motivate the definition of $\mathcal{K}^{sst}(-)$, we return to the case of weakly normal, projective complex varieties and consider what happens if we replace spaces with singular simplicial sets in the construction of $\mathcal{K}^{semi}(-)$. That is, for such a variety X we replace the space $\mathcal{M}or(X, \mathrm{Grass})^{an}$ with the simplicial set $d \mapsto \mathrm{Maps}(\Delta^d_{top}, \mathcal{M}or(X, \mathrm{Grass})^{an})$ and we replace the E_∞ topological operad $\mathcal{I}$ with the associated simplicial one, $\mathcal{I}(\Delta^\bullet_{top})$, defined by $\mathcal{I}(\Delta^\bullet_{top})(n) = \big(d \mapsto \mathrm{Maps}(\Delta^d_{top}, \mathcal{I}(n))\big)$. An important observation is that since Δ^d_{top} is compact and since $\mathcal{M}or(X, \mathrm{Grass})^{an}$ is an inductive limit of analytic spaces associated to quasi-projective varieties, we have the natural isomorphism

$$\mathrm{Maps}(\Delta^d_{top}, \mathcal{M}or(X, \mathrm{Grass})^{an}) \cong \varinjlim_{\Delta^d_{top} \to U^{an}} \mathrm{Hom}(U \times X, \mathrm{Grass}),$$

where the limit ranges over the filtered category whose objects are continuous maps $\Delta^d_{top} \to U^{an}$, with U a quasi-projective complex variety, and in which a morphism is given by a morphism of varieties $U \to V$ causing the evident triangle to commute. In other words, if we define

$$\mathrm{Hom}(\Delta^d_{top} \times X, \mathrm{Grass}) = \varinjlim_{\Delta^d_{top} \to U^{an}} \mathrm{Hom}(U \times X, \mathrm{Grass})$$

then we have $\mathrm{Maps}(\Delta^d_{top}, \mathcal{M}or(X, \mathrm{Grass})) \cong \mathrm{Hom}(\Delta^d_{top} \times X, \mathrm{Grass})$.

For readers inclined to categorical constructions it might be helpful to observe that $\mathrm{Hom}(\Delta^d_{top} \times X, \mathrm{Grass})$ is the result of applying to the topological space Δ^d_{top} the Kan extension of the presheaf $\mathrm{Hom}(- \times X, \mathrm{Grass})$ on $Sch/\mathbb{C}$ along the functor $Sch/\mathbb{C} \to Top$ given by $U \mapsto U^{an}$.

Just as in the construction of $\mathcal{K}^{semi}$, it's easy to show that we have the action of the simplicial E_∞ operad $\mathcal{I}(\Delta^\bullet_{top})$ on the simplicial set $\mathrm{Hom}(\Delta^\bullet_{top} \times X, \mathrm{Grass})$, and hence we obtain an associated Ω-spectrum

$$\Omega^\infty |\mathrm{Hom}(\Delta^\bullet_{top} \times X, \mathrm{Grass})|.$$

Finally, this Ω-spectrum is readily seen to be equivalent to the spectrum $\mathcal{K}^{semi}(X)$ constructed above (assuming X is projective and weakly normal).

The idea in defining $\mathcal{K}^{sst}$, then, is to just take the simplicial set $\mathrm{Hom}(X \times \Delta^\bullet_{top}, \mathrm{Grass})$ itself for the starting point of the construction. For observe that the definition of this simplicial set does not depend on X being either projective or weakly normal, and so we may use it for arbitrary varieties. Theorem 1.2.4 suggests another alternative — one could simply take the algebraic K-theory functor taking values in spectra, $\mathcal{K}^{alg}$, and "semi-topologize" it by applying it degree-wise to $\Delta^\bullet_{top} \times X$ via the Kan extension formula. The following proposition shows that the two constructions result in equivalent theories.

Proposition 1.2.6. *(cf. [27, 1.3]) For any quasi-projective complex variety X, there are natural weak homotopy equivalences of spectra*

$$|d \mapsto \mathcal{K}^{alg}(\Delta^d_{top} \times X)| \to \Omega^\infty |\operatorname{Hom}(\Delta^\bullet_{top} \times \Delta^\bullet \times X, \mathrm{Grass})| \leftarrow \Omega^\infty |\operatorname{Hom}(\Delta^\bullet_{top} \times X, \mathrm{Grass})|$$

where $\mathcal{K}^{alg}(-) : (Sch/\mathbb{C}) \to \underline{Spectra}$ is a fixed choice of functorial model of the algebraic K-theory spectrum of quasi-projective complex varieties and $\mathcal{K}^{alg}(\Delta^d_{top} \times X)$ is the value of the Kan extension of $\mathcal{K}^{alg}(- \times X)$ applied to Δ^d_{top}.

The choice of $|d \mapsto \mathcal{K}^{alg}(\Delta^d_{top} \times X)|$ as the primary definition of semi-topological K-theory is justified by the "Recognition Principle", which appears below as Theorem 1.2.12. As we shall see, this definition is but one of an interesting collection of "singular semi-topological constructions".

Definition 1.2.7. For any quasi-projective complex variety X, the *(singular) semi-topological K-theory spectrum* of X is the Ω-spectrum

$$\mathcal{K}^{sst}(X) = \mathcal{K}(\Delta^\bullet_{top} \times X) = |d \mapsto \mathcal{K}^{alg}(\Delta^d_{top} \times X)|.$$

The semi-topological K-groups of X with coefficients in the abelian group A are given by

$$K^{sst}_n(X, A) = \pi_n \mathcal{K}^{sst}(X, A), \quad n \geq 0.$$

We find that we may easily construct maps as in (1.3) for any quasi-projective complex variety.

Proposition 1.2.8. *(cf. [27, 1.4]) There are natural maps of spectra (in the stable homotopy category)*

$$\mathcal{K}^{alg}(X) \to \mathcal{K}^{sst}(X) \to \mathcal{K}_{top}(X^{an}). \tag{1.5}$$

Furthermore, if X is projective and weakly normal, there is a weak equivalence of spectra

$$\mathcal{K}^{sst}(X) \simeq \mathcal{K}^{semi}(X).$$

The map $\mathcal{K}^{alg}(X) \to \mathcal{K}^{sst}(X)$ is the canonical map $\mathcal{K}^{alg}(\Delta^0_{top} \times X) \to |d \mapsto \mathcal{K}(\Delta^d \times X)|$. The map $\mathcal{K}^{sst}(X) \to \mathcal{K}_{top}(X^{an})$ is the map in the stable homotopy category (using Proposition 1.2.6) associated to the map

$$\Omega^\infty |\operatorname{Hom}(\Delta^\bullet_{top} \times X, \mathrm{Grass})| \to \mathcal{K}_{top}(X^{an})$$

that is given by the adjoint of the map $|\operatorname{Sing}_\bullet(-)| \to id$ together with the natural inclusion $\operatorname{Hom}(-, \mathrm{Grass}) \subset \operatorname{Maps}((-)^{an}, \mathrm{Grass}^{an})$.

Upon taking homotopy groups with coefficients in an abelian group A, we thus have the chain of maps

$$K^{alg}_*(X, A) \to K^{sst}_*(X, A) \to K^{-*}_{top}(X^{an}, A) \tag{1.6}$$

whose composition is the usual map from algebraic to topological K-theory with coefficients in A.

The following property of $\mathcal{K}^{sst}(-)$ is one indication that Definition 1.2.7 is a suitable generalization of $\mathcal{K}^{semi}(-)$ to all quasi-projective complex varieties.

Proposition 1.2.9. *[30, 2.5] For any quasi-projective complex variety X, there is a natural isomorphism*

$$\mathcal{K}_0^{sst}(X) \cong \frac{K_0^{alg}(X)}{\textit{algebraic equivalence}}.$$

1.2.3 The Recognition Principle

The formulation of $\mathcal{K}^{sst}(-)$ in Definition 1.2.7 is a special case of the following *singular topological construction.*

Definition 1.2.10. Let $\mathcal{F}$ be a contravariant functor from $Sch/\mathbb{C}$ to a suitable category $\mathcal{C}$ (such as chain complexes of abelian groups, spaces, and spectra). For any compact Hausdorff space T and variety $X \in Sch/\mathbb{C}$, define

$$\mathcal{F}(T \times X) = \varinjlim_{T \to U^{an}} \mathcal{F}(U \times X).$$

Define $\mathcal{F}^{sst}$ to be the functor from $Sch/\mathbb{C}$ to $\mathcal{C}$ by

$$\mathcal{F}^{sst}(X) = Tot(d \mapsto \mathcal{F}(\Delta^d_{top} \times X))$$

where Tot refers to a suitable notion of "total object" (such as total complex of a bicomplex or geometric realization of a bisimplicial space or spectrum). We call $\mathcal{F}^{sst}$ the *singular semi-topological functor associated to $\mathcal{F}$.*

The usefulness of this singular semi-topological construction arises in large part from the validity of the following *Recognition Principle*. This theorem should be compared with an analogous theorem of V. Voevodsky [58, 5.9].

Theorem 1.2.11. *[32, 2.7] Suppose $\mathcal{F} \to \mathcal{G}$ is a natural transformation of contravariant functors from $Sch/\mathbb{C}$ to chain complexes of abelian groups, group-like H-spaces, or spectra. Suppose this map is a weak equivalence locally in the h-topology (for example, suppose it is a weak equivalence on all smooth varieties). Then the associated map*

$$\mathcal{F}(\Delta^\bullet_{top}) \to \mathcal{G}(\Delta^\bullet_{top})$$

is a weak homotopy equivalence.

As a sample application (many more will be discussed below) of the Recognition Principle, we have following theorem relating algebraic and semi-topological K-theory. The authors had originally established the validity of Theorem 1.2.12 using a much more involved argument (see [27, 3.8]), an argument which pointed the way toward the formulation and applications of Theorem 1.2.11.

Theorem 1.2.12. *(cf. [27, 3.7]) For a quasi-projective complex variety X and positive integer n, we have an isomorphism*

$$K_q^{alg}(X;\mathbb{Z}/n) \xrightarrow{\cong} K_q^{sst}(X;\mathbb{Z}/n), \quad q \geq 0.$$

Sketch of Proof. Via an evident spectral sequence argument, it suffices to prove $K_q^{alg}(X;\mathbb{Z}/n) \to \left(d \mapsto K_q^{alg}(\Delta^d_{top} \times X;\mathbb{Z}/n)\right)$ is a homotopy equivalence of simplicial abelian groups (the source being constant). By the Recognition Principle, it suffices to prove the map of presheaves $K_q^{alg}(X;\mathbb{Z}/n) \to K_q^{alg}(- \times X;\mathbb{Z}/n)$ is locally an isomorphism in the h topology (where again the source is constant). This holds already for the étale topology by Suslin rigidity [55]. □

1.2.4 Semi-topological *K*-theory for Real Varieties

In this section, we summarize results of [30] which show that the semi-topological K-theory for real varieties satisfies analogues of the pleasing properties of $\mathcal{K}^{sst}(-)$ for complex varieties. We take these properties as confirmation that the definition of $\mathcal{K}\mathbb{R}^{sst}(-)$ given here is the "correct" analogue of $\mathcal{K}^{sst}(-)$, but are frustrated by the fact that this extension does not suggest a generalization to other fields.

The reader should observe that the definition of $\mathcal{K}\mathbb{R}^{sst}(-)$ below is so formulated that if Y is a quasi-projective complex variety then (see Proposition 1.2.18)

$$\mathcal{K}\mathbb{R}^{sst}(Y|_{\mathbb{R}}) = \mathcal{K}^{sst}(Y),$$

where $Y|_{\mathbb{R}}$ denotes the complex variety Y regarded as a real variety via restriction of scalars. Thus, any result concerning $\mathcal{K}\mathbb{R}^{sst}(-)$ that applies to all quasi-projective real varieties incorporates the analogous statement for $\mathcal{K}^{sst}(-)$ applied to quasi-projective complex varieties.

As in the complex case, the motivation for the definition of real semi-topological K-theory is most easily seen in the projective case first, and in this case we first define $\mathcal{K}\mathbb{R}^{semi}$, an equivalent but more geometric version of $\mathcal{K}\mathbb{R}^{sst}$ defined below.

Definition 1.2.13. Let Y be a projective real variety. We define

$$\mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass})^{an} = \varinjlim_N \mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass}(\mathbb{R}^N))(\mathbb{R}),$$

where $\mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass}(\mathbb{R}^N))(\mathbb{R})$ denotes the space of real points of the real ind-variety $\mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass}(\mathbb{R}^N))$ parameterizing morphisms over $\mathbb{R}$ from Y to $\mathrm{Grass}(\mathbb{R}^N)$. As in the complex setting, $\mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass})^{an}$ admits the structure of a homotopy-commutative H-space and we let

$$\mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass})^{an} \to \mathcal{K}\mathbb{R}^{semi}(Y)$$

denote the homotopy-theoretic group completion. We call $\mathcal{K}\mathbb{R}^{semi}(Y)$ the *real semi-topological K-theory space.*

As in the complex case, for Y a projective, weakly normal real variety, we have that

$$\mathrm{Maps}(\Delta^d_{top}, \mathcal{M}or_{\mathbb{R}}(Y, \mathrm{Grass})) = \varinjlim_{\Delta^d_{top} \to U(\mathbb{R})} \mathrm{Hom}(U \times_{\mathbb{R}} Y, \mathrm{Grass}),$$

where the limit ranges over pairs $(U, \Delta^d_{top} \to U(\mathbb{R}))$ consisting of a real variety U and a continuous maps from Δ^d_{top} to the its space of real points $U(\mathbb{R})$. As before, this leads naturally to the following definition:

Definition 1.2.14. For a quasi-projective real variety Y, the *real (singular) semi-topological K-theory space* of Y is defined by

$$\mathcal{K}\mathbb{R}^{sst}(Y) \equiv |d \mapsto \mathcal{K}^{alg}(\Delta^d_{top} \times_{\mathbb{R}} Y)|$$

where

$$\mathcal{K}^{alg}(\Delta^d_{top} \times_{\mathbb{R}} Y) = \varinjlim_{\Delta^d_{top} \to U(\mathbb{R})} \mathcal{K}^{alg}(U \times_{\mathbb{R}} Y).$$

The *real (singular) semi-topological K-groups* of Y are defined by

$$K\mathbb{R}^{sst}_n(Y) = \pi_n \mathcal{K}\mathbb{R}^{sst}(Y).$$

In other words, the theory $\mathcal{K}\mathbb{R}^{sst}$ is induced from algebraic K-theory using Kan extension along the functor $Sch/\mathbb{R} \to Top$ sending U to $U(\mathbb{R})$.

Proposition 1.2.15. *(cf. [30, 2.5] If Y is a weakly normal, projective real variety, then there is a natural weak equivalence of spectra*

$$\mathcal{K}\mathbb{R}^{semi}(Y) \simeq \mathcal{K}\mathbb{R}^{sst}(Y).$$

The explicit description of $K\mathbb{R}^{sst}_0(Y)$ is perhaps a bit unexpected. If $V_1 \to X, V_2 \to Y$ are algebraic vector bundles on the real variety Y, then we say that V_1, V_2 are *real algebraically equivalent* if there exists a smooth, connected real curve C, an algebraic vector bundle $\mathcal{V} \to Y \times C$, and real points $c_1, c_2 \in C(\mathbb{R})$ *lying in the same real analytic component of* $C(\mathbb{R})$ such that $V_i \to Y$ is the fibre of $\mathcal{V} \to Y \times C$ over $Y \times \{c_i\}$. We refer to the equivalence relation on $K^{alg}_0(Y)$ generated by real algebraic equivalence as *real algebraic equivalence.*

The condition that two bundles be joined via real algebraic equivalence is significantly stronger than what might be termed "algebraic equivalence for real varieties" (i.e., requiring only that c_1, c_2 belong to the same algebraic component of C). Nevertheless, the next proposition and the subsequent theorem indicated that this stronger condition is the appropriate one.

Proposition 1.2.16. *(cf. [30, 1.6]) For any quasi-projective real variety Y,*

$$K\mathbb{R}^{sst}_0(Y) \cong \frac{K^{alg}_0(Y)}{\textit{real algebraic equivalence}}.$$

If Y is a real variety, we write $Y_{\mathbb{R}}(\mathbb{C})$ for the topological space $Y(\mathbb{C})^{an}$ equipped with the involution $y \mapsto \overline{y}$ induced by complex conjugation — in Atiyah's terminology [4] $Y_{\mathbb{R}}(\mathbb{C})$ is a *Real space*. As with any Real space, we may associated to $Y_{\mathbb{R}}(\mathbb{C})$ its *Atiyah's Real K-theory* space $\mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C}))$. We remind the reader that $\mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C}))$ is constructed using the category of *Real vector bundles*. Such a bundle is a complex topological vector bundle $V \to Y_{\mathbb{R}}(\mathbb{C})$ equipped with an involution $\tau : V \to V$ covering the involution of $Y_{\mathbb{R}}(\mathbb{C})$ such that for each $y \in Y_{\mathbb{R}}(\mathbb{C})$, the map $\mathbb{C}^r \cong V_y \stackrel{\tau}{\longrightarrow} V_{\overline{y}} \cong \mathbb{C}_r$ is given by complex conjugation.

Proposition 1.2.17. *(cf. [30, 2.5, 4.3] For a quasi-projective real variety Y, there is a natural (up to weak equivalence) triple of spectra*

$$\mathcal{K}^{alg}(Y) \to \mathcal{K}\mathbb{R}^{sst}(Y) \to \mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C})). \tag{1.7}$$

We may of course view any quasi-projective complex variety Y as a quasi-projective real variety $Y|_{\mathbb{R}}$ via restriction of scalars, and, conversely, any real variety U admits a base change $U_{\mathbb{C}} = U \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ to a complex variety. If Y is a quasi-projective complex variety, then $(Y|_{\mathbb{R}})_{\mathbb{C}} = Y \coprod Y$ and the non-trivial element of the Galois group $Gal(\mathbb{C}/\mathbb{R})$ acts on $(Y|_{\mathbb{R}})_{\mathbb{C}}$ by interchanging the copies of Y. It follows that $\operatorname{Hom}(U \times_{\mathbb{R}} Y|_{\mathbb{R}}, \operatorname{Grass}) = \operatorname{Hom}(U_{\mathbb{C}} \times_{\mathbb{C}} Y, \operatorname{Grass})$ for any real variety U, from which the following result may be deduced.

Proposition 1.2.18. *[30, 2.4, 4.3] If Y is a complex, quasi-projective variety and $X = Y|_{\mathbb{R}}$, then*

$$\mathcal{K}\mathbb{R}^{sst}(X) = \mathcal{K}^{sst}(Y) \quad \textit{and} \quad \mathcal{K}\mathbb{R}_{top}(X_{\mathbb{R}}(\mathbb{C})) = \mathcal{K}_{top}(Y)$$

and, moreover, in this case the maps of (1.7) coincide with the maps of (1.5).

The following theorem, generalizing Theorems 1.2.5 and 1.2.12, provides further evidence of the "correctness" of our definition of $\mathcal{K}\mathbb{R}^{sst}(-)$.

Theorem 1.2.19. *(cf. [30, 3.9, 6.9]) Let Y be a quasi-projective real variety. Then*

$$K_*^{alg}(Y, \mathbb{Z}/n) \cong K\mathbb{R}_*^{sst}(Y, \mathbb{Z}/n)$$

for any positive integer n.

Furthermore, if C is a smooth real curve, then

$$K\mathbb{R}_q^{sst}(C) \cong K\mathbb{R}_{top}^{-q}(C_{\mathbb{R}}(\mathbb{C})), \quad q \geq 0.$$

For example, if C is a smooth, projective real curve of genus g such that $C(\mathbb{R}) \neq \emptyset$, then we have

$$K\mathbb{R}_0^{semi}(C) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^{c-1},$$

where c is the number of connected components of the space $C(\mathbb{R})^{an}$ (cf. [30, 1.7]). This example shows that real algebraic equivalence differs from algebraic equivalence for real varieties, since modding out $K_0(C)$ by the latter equivalence relation yields the group $\mathbb{Z} \oplus \mathbb{Z}$ (cf. [30, 1.8]).

We interpret the next theorem as asserting that the triple (1.7) for the real variety Y is a retract of the triple (1.5) for its base change to $\mathbb{C}$, $Y_{\mathbb{C}}$, once one inverts the prime 2. In establishing this theorem, the authors first constructed a good transfer map $\pi_* : \mathcal{K}^{sst}(Y_{\mathbb{C}}) \to \mathcal{K}^{sst}(Y)$ (see [30, §5]).

Theorem 1.2.20. *(cf. [30, 5.4, 5.6]) Let Y be a quasi-projective real variety and let*

$$\pi : Y_{\mathbb{C}} = Y \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C} \to Y$$

denote the natural map of $\mathbb{R}$-varieties. Then we have a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathcal{K}(Y) & \longrightarrow & \mathcal{K}\mathbb{R}^{sst}(Y) & \longrightarrow & \mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C})) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ \mathcal{K}(Y_{\mathbb{C}}) & \longrightarrow & \mathcal{K}^{sst}(Y_{\mathbb{C}}) & \longrightarrow & \mathcal{K}_{top}(Y(\mathbb{C})^{an}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathcal{K}(Y) & \longrightarrow & \mathcal{K}\mathbb{R}^{sst}(Y) & \longrightarrow & \mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C})) \end{array} \tag{1.8}$$

with the property that the vertical compositions are weakly equivalent to multiplication by 2 with respect to the H-space structures.

The reader seeking to extend the construction of $\mathcal{K}^{sst}(-)$ and $\mathcal{K}\mathbb{R}^{sst}(-)$ to varieties over some other ground field F would likely have to address the following two questions:

– What is the correct notion of "F-algebraic equivalence" if F is not algebraically closed and not equal to $\mathbb{R}$? Specifically, what condition on a pair of F-points of a variety is analogous to the condition that two points $c_1, c_2 \in C(\mathbb{R})$ lie in the same real analytic component of $C(\mathbb{R})^{an}$?
– What should play the role of $\mathcal{K}\mathbb{R}_{top}(-)$ or $\mathcal{K}_{top}$ if F is not equal to $\mathbb{R}$ or $\mathbb{C}$?

1.3 Algebraic, Semi-topological, and Topological Theories

In this section we state the major results relating semi-topological K-theory to algebraic K-theory, topological K-theory, and morphic cohomology and we provide some indications of proofs.

The connections between semi-topological K-theory and these other cohomology theories are well summarized by the existence of and properties enjoyed by the commutative diagram

$$\begin{array}{ccccc} K_*^{alg}(X;A) & \longrightarrow & K_*^{sst}(X;A) & \longrightarrow & K_{top}^{-*}(X^{an};A) \\ \vdots & & \vdots & & \vdots \\ \bigoplus_q H_{\mathcal{M}}^{2q-*}(X,A(q)) & \longrightarrow & \bigoplus_q L^q H^{2q-*}(X;A) & \longrightarrow & \bigoplus_q H_{sing}^{2q-*}(X^{an};A), \end{array} \tag{1.9}$$

where X is a smooth, quasi-projective complex variety and A is an arbitrary abelian group. In this diagram, the top row is the chain of maps (1.6) defined in the previous section, and the maps in the bottom row are defined in a similar manner below. The vertical arrows are dashed to indicate that one must interpret them non-literally in one of three ways: (1) One may interpret them as homomorphisms from K-theories to cycle theories (heading downward in the diagram) whose targets land, to put it a bit imprecisely, in the groups of units of the ring cohomology theories along the bottom row — that is, one may take these arrows to be total Chern class maps; (2) one may interpret them as natural transformations of ring theories from K-theories to cycles theories (heading downward in the diagram) provided one takes $A = \mathbb{Q}$ — that is, one may take these arrows to be Chern character maps; or (3) one may interpret these dashed lines as indicating the existence of three compatible spectral sequences whose E_2-terms are cycles theories and whose abutments are K-theories. We discuss the first two interpretations of these vertical arrows in this section and leave the spectral sequence interpretation for the next.

1.3.1 Motivic, Morphic, and Singular Cohomology

Before describing the many nice properties enjoyed by diagram (1.9), we first remind the reader of the definition of morphic cohomology and define the maps along the bottom row of this diagram.

Once again, the definition of morphic cohomology is more intuitive in the case of projective varieties, and, in fact, we first describe Lawson homology, the homology theory dual to morphic cohomology for smooth varieties, in this case. The definition of Lawson homology can be motivated by the Dold-Thom Theorem [12] that gives the isomorphism

$$\pi_n\left((\amalg_d S^d(Y))^+\right) \cong H_n^{sing}(Y)$$

where Y is a compact space, $S^d(-)$ denotes taking the d-th symmetric power of a space, and $(-)^+$ denotes forming the topological abelian group associated to a topological abelian monoid via (naive) group completion. Observe that if we take Y to be the analytic realization of a projective variety X, then $S^d(X^{an})$ is the space of effective 0-cycles of degree d on X and this space coincides with the analytic realization of the Chow variety $C_{0,d}(X)$. Thus, in this context, the Dold-Thom theorem becomes

$$\pi_n \mathcal{Z}_0(X)^{an} \cong H_n^{sing}(X^{an}), \quad \text{where } \mathcal{Z}_0(X)^{an} = (\coprod_d \mathcal{C}_{0,d}(X)^{an})^+.$$

Observe that $\mathcal{Z}_0(X)^{an}$ coincides the space of all 0-cycles on X. A fascinating theorem of F. Almgren [2] generalizes the Dold-Thom Theorem by asserting that for sufficiently well-behaved spaces Y (i.e., for Lipschitz neighborhood retracts) and for any $r \geq 0$ the topological abelian group $\mathcal{Z}_r^{curr}(Y)$ of "integral r-cycles" (i.e., closed rectifiable currents on Y) has the property that

$$\pi_n \mathcal{Z}_r^{curr}(Y) = H_{n+r}^{sing}(Y).$$

This result motivated Blaine Lawson to investigate spaces of *algebraic* r-cycles on a complex projective variety X as a "small model" for the space of integral $2r$-cycles on X^{an}. Namely, the collection of effective r-cycles for any $r \geq 0$ of a fixed degree d on a projective variety X is given as the set of closed points of the Chow variety $\mathcal{C}_{r,d}(X)$. Letting $\mathcal{C}_r(X) = \coprod_d \mathcal{C}_{r,d}(X)$, we see that $\mathcal{C}_r(X)^{an}$ is a topological abelian monoid under addition of cycles. We define

$$\mathcal{Z}_r(X)^{an} = (\mathcal{C}_r(X)^{an})^+,$$

the associated topological abelian group given by (naive) group completion. (Up to homotopy, one may equivalently use a homotopy-theoretic group completion, defined by the bar construction, in place of naive group completion [47].) Then $\mathcal{Z}_r(X)^{an}$ is the topological space of all r-cycles on X, and the Lawson homology groups are defined to be the homotopy groups of this space:

Definition 1.3.1. For a projective, complex variety X, we define the *Lawson homology groups* of X to be

$$L_r H_n(X) = \pi_{n-2r} \mathcal{Z}_r(X)^{an}$$

where

$$\mathcal{Z}_r(X)^{an} = (\mathcal{C}_r(X)^{an})^+ \text{ and } \mathcal{C}_r(X) = \coprod_d \mathcal{C}_{r,d}(X).$$

In fact, this definition generalizes is a straightforward fashion to quasi-projective varieties. Namely, for U quasi-projective, one chooses a compactification $U \subset X$ (i.e., an open, dense embedding with X projective) and defines

$$L_r H_n(U) = \pi_{n-2r} \mathcal{Z}_r(X)^{an} / \mathcal{Z}_r(X-U)^{an}$$

where $\mathcal{Z}_r(X)^{an}/\mathcal{Z}_r(X-U)^{an}$ denotes the quotient topological abelian group. By the Dold-Thom Theorem [12], we have

$$L_0 H_n(U) = H_n^{BM}(U^{an}), \text{ for all } n,$$

where H^{BM} denotes Borel-Moore homology, and it is easy to prove (cf. [20]) that

$$L_r H_{2r}(X) = \pi_0 \mathcal{Z}_r(X)^{an} \cong A_r(X)$$

where $A_r(X)$ denotes the group of cycles of dimension r on X modulo algebraic equivalence.

The definition of the morphic cohomology groups is also quite natural. The key motivational observation is that the quotient topological group

$$\mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an}$$

is a model for the Eilenberg-MacLane space $K(\mathbb{Z}, 2q)$ by the Dold-Thom theorem, and thus represents the functor $H^{2q}_{sing}(-, \mathbb{Z})$. That is, the homotopy groups of $\mathcal{M}aps(Y, \mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an})$ give the singular homology groups of a space Y. Replacing $\mathcal{M}aps(-,-)$ by $\mathcal{M}or(-,-)^{an}$ as in the definition of $\mathcal{K}^{semi}$, we arrive at the definition of morphic cohomology for projective varieties.

Definition 1.3.2. For a smooth, projective complex variety X, we define the *morphic cohomology groups of* X to be

$$L^qH^n(X) = \pi_{2q-n}\mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an}$$

where we define

$$\mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an} = [\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^q))^{an}]^+/[\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^{q-1}))^{an}]^+.$$

As before, the definition extends naturally to all quasi-projective varieties, but we omit the details.

The connection between morphic cohomology and singular cohomology can be seen from the definition of the former: since $\mathcal{M}or(-,-)^{an}$ is a subspace of $\mathcal{M}aps((-)^{an}, (-)^{an})$, we obtain a natural map

$$\mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an} \to \mathcal{M}aps(X^{an}, \mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an})$$

which induces the map

$$L^qH^n(X) \to H^n_{sing}(X^{an}).$$

The connection between morphic cohomology and motivic cohomology is suggested by the following fact:

Proposition 1.3.3. *(cf. [26, 4.4, 8.1], [56, 2.1]) For a smooth, quasi-projective complex variety* X*, we have*

$$\pi_n \operatorname{Hom}(X \times \Delta^\bullet, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})) \cong H^{2q-n}_{\mathcal{M}}(X, \mathbb{Z}(q))$$

where $\operatorname{Hom}(X \times \Delta^\bullet, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))$ *denotes the quotient simplicial abelian group*

$$\operatorname{Hom}(X \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^q))^+/\operatorname{Hom}(X \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^{q-1}))^+.$$

(Here, the superscript + signifies taking degree-wise group completion of a simplicial abelian monoid.)

For a projective variety X, it's not hard to establish the isomorphism

$$L^q H^n(X) \cong \pi_{2q-n} \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^q))^+ / \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^{q-1}))^+$$

in much the same way that the equivalence $\mathcal{K}^{semi} \simeq \mathcal{K}^{sst}$ is proven for such varieties. Indeed, we may thus use this isomorphism to *define* morphic cohomology for non-projective varieties. Although less "geometric", the construction given in the following definition of morphic cohomology is more amenable.

Definition 1.3.4 (Revised Definition of Morphic Cohomology). For a smooth, quasi-projective complex variety X, the *morphic cohomology groups* of X are defined to be

$$L^q H^n(X) = \pi_{2q-n} \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))$$

where

$$\operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})) = \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^q))^+ / \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^{q-1}))^+.$$

(The superscripts $+$ denote taking degree-wise group completion of a simplicial abelian monoid.)

In other words, we simply define morphic cohomology to be the "semi-topologized" theory associated to motivic cohomology.

Definition 1.3.5. The maps along the bottom row of (1.9) are given by applying π_* to the sequence of natural maps

$$\begin{aligned}\operatorname{Hom}(X \times \Delta^{\bullet}, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})) &\to \operatorname{Hom}(X \times \Delta^{\bullet}_{top}, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})) \\ &\to \operatorname{Maps}(X^{an} \times \Delta^{\bullet}_{top}, \mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an}).\end{aligned}$$

In summary, we have the following heuristic overview: Motivic, morphic, and singular cohomology are defined as the homotopy groups of, respectively, the "algebraic space" of algebraic morphisms, the topological space of algebraic morphisms, and the topological space of topological morphisms from a given variety to the object $\mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})$. Moreover, the maps joining these three theories are given by the canonical maps from the algebraic space of algebraic morphisms to the topological space of algebraic morphisms to the topological space of topological morphisms.

1.3.2 The Chern Class Maps

The relation between semi-topological K-theory and morphic cohomology is given, as one would expect, by the total Chern class map and the closely related Chern character. The former has the advantage that it is defined integrally, whereas the later has the advantage that it determines a natural transformation of ring-valued cohomology theories. Each map induces a rational isomorphism from the rational semi-topological K-groups to the rational morphic cohomology groups of a smooth, quasi-projective complex variety. These isomorphisms generalize the isomorphisms on π_0 groups

$$c : K_0^{sst}(X)_{\mathbb{Q}} \xrightarrow{\cong} A^0(X)_{\mathbb{Q}} \times \left(\{1\} \times \bigoplus_{q \geq 1} A^q(X)_{\mathbb{Q}} \right)^{\times}$$

and

$$ch : K_0^{sst}(X)_{\mathbb{Q}} \xrightarrow{\cong} A^*(X)_{\mathbb{Q}} = L^* H^{2*}(X; \mathbb{Q}).$$

The first of these isomorphisms, the total Chern class map

$$c(\alpha) = (\mathrm{rank}(\alpha), 1 + c_1(\alpha) + \cdots),$$

is an isomorphism of abelian groups, where the group law for the second component of the target is given by cup product (i.e., intersection of cycles). The second of these isomorphisms, the Chern character, is an isomorphism of rings and is defined via the usual universal polynomials (with $\mathbb{Q}$ coefficients) in the individual Chern classes $c_i, i \geq 1$. Each of these isomorphisms may be deduced easily from the corresponding and well-known isomorphisms relating $K_0^{alg}(X)$ and $CH^*(X)$ by simply modding out by algebraic equivalence.

Lawson and Michelsohn recognized that sending an arbitrary subspace $W \subset \mathbb{C}^{N+1}$ to the linear cycle $\mathbb{P}(W^*) \subset \mathbb{P}((\mathbb{C}^{N+1})^*) \cong \mathbb{P}^N$ stablizes (by letting N approach infinite) to give the universal total Chern class map [41]. (Here, $\mathbb{P}(W^*) = \mathrm{Proj}(S^*(W^*))$ is the projective variety parameterizing one dimensional subspaces of W^*, the linear dual of W.) Indeed, in [9], Boyer, Lawson, Lima-Filho, Mann, and Michelsohn show that this model of the total Chern class is a map of infinite loop spaces, thereby answering a question of G. Segal. This result is extended in Theorem 1.3.7 below. We find it more convenient when stabilizing with respect to N and when considering the pairing determined by external direct sum of vector spaces to send a *quotient* space of the form $\mathbb{C}^{N+1} \twoheadrightarrow V$ to the linear cycle $\mathbb{P}(V) \subset \mathbb{P}^N$. This becomes a model for the total Segre class. The total Segre and Chern class maps differ only slightly: We define $Seg(\alpha) = (\mathrm{rank}(\alpha), 1 - s_1(\alpha) + s_2(\alpha) - \cdots)$ where s_q are the Segre class maps, defined by the formula

$$1 + s_1(x) + s_2(x) + \cdots = (1 + c_1(x) + c_2(x) + \cdots)^{-1}.$$

It follows from [32, 1.4] that there is a natural isomorphism of the form

$$\mathrm{Hom}(X \times \Delta^\bullet_{top}, \mathcal{C}_r(\mathbb{P}^N))^+ \cong \bigoplus_{q=0}^{N-r} \mathrm{Hom}(X \times \Delta^\bullet_{top}, \mathcal{C}_0(\mathbb{P}^q)) / \mathrm{Hom}(X \times \Delta^\bullet_{top}, \mathcal{C}_0(\mathbb{P}^{q-1}))$$

from which one deduces the isomorphism

$$\pi_n\left(\mathrm{Hom}(X \times \Delta^\bullet_{top}, \mathcal{C}_r(\mathbb{P}^\infty))^+\right) \cong \bigoplus_{q \geq 0} L^q H^{2q-n}(X),$$

for any $r \geq 0$ and any smooth, quasi-projective complex variety X. Thus, morphic cohomology is "represented" by the ind-variety $\mathcal{C}_r(\mathbb{P}^\infty)$ for any $r \geq 0$ just as semi-topological K-theory is represented by $\mathrm{Grass}(\mathbb{C}^\infty)$. Now, a point in the latter ind-variety is given by a quotient $\mathbb{C}^\infty \twoheadrightarrow V$ (that factors through $\mathbb{C}^N$ for $N \gg 0$), which in turn determines an effective cycle $\mathbb{P}(V) \subset \mathbb{P}^\infty$ of degree 1 and dimension $\dim(V) - 1$ by taking associated projective spaces. Thus we have a map

$$\mathrm{Grass}(\mathbb{C}^\infty) \to \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)^+_1, \tag{1.10}$$

where $\mathcal{C}_{r-1}(\mathbb{P}^\infty)^+_1$ denotes the subset of the abelian group $\mathcal{C}_{r-1}(\mathbb{P}^\infty)^+$ consisting of (not necessarily effective) cycles of degree 1. (As a technical point, when $r = 0$ one sets $\mathcal{C}_{-1}(\mathbb{P}^\infty)^+ = \mathbb{Z}$, the free abelian group generated by the "empty cycle" which has degree 1 by convention.) In essence, the map (1.10) induces the total Segre class map by taking homotopy-theoretic group completions, although some further details are needed to make this precise.

The ind-variety $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)$ admits a natural product given by linear join of cycles. Namely, we first specify an linear isomorphism $\mathbb{P}^\infty \coprod \mathbb{P}^\infty \cong \mathbb{P}^\infty$ by choosing an isomorphism $\mathbb{C}^\infty \cong \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ of vector spaces. Then given a pair of effective cycles α and β in $\mathbb{P}^\infty$, we may embed them as cycles in general position in $\mathbb{P}^\infty$ by use of the isomorphism $\mathbb{P}^\infty \coprod \mathbb{P}^\infty \cong \mathbb{P}^\infty$ (regarding α as a cycle on the first copy of $\mathbb{P}^\infty$ and β as a cycle on the second), so that their linear join (i.e., the cycle formed by the union of all lines intersecting both α and β) is well-behaved. This pairing extends to all cycles by linearity and restricts to a pairing

$$\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)^+_1 \times \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)^+_1 \to \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)^+_1$$

since the linear join of cycles having degrees d and e has degree de. For a complex variety X, this product endows

$$\left(\mathrm{Hom}(X \times \Delta^\bullet_{top}, \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty))^{an}\right)^+_1$$

with the structure of a homotopy-commutative H-space, whose associated homotopy-theoretic group completion is written $\mathcal{H}^{sst}_{mult}(X)$. It is apparent from its definition that the space $\mathcal{H}^{sst}_{mult}(X)$ should be be closely related to the morphic cohomology of X and, since cup product in morphic cohomology can be defined by linear join, that the H-space structure of this space should be related to cup product. The precise connection is given by the isomorphism of groups

$$\pi_n \mathcal{H}^{sst}_{mult}(X) \cong L^0H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q\geq 1} L^qH^{2q-n}(X)\right)^{\times},$$

where $\left(\{1\} \times \bigoplus_{q\geq 1} L^qH^{2q-n}(X)\right)^{\times}$ is a subgroup of the multiplicative group of units of the ring $L^*H^*(X)$.

A key observation is that the map (1.10) is additive in that it takes direct sum to linear join — that is, given any projective variety X, the induced map

$$\mathrm{Hom}(X \times \Delta^{\bullet}_{top}, \mathrm{Grass}(\mathbb{C}^{\infty})) \to \left(\mathrm{Hom}(X \times \Delta^{\bullet}_{top}, \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^{\infty}))^{an}\right)^{+}_{1},$$

is a map of H-spaces. In fact, it can easily be enriched to be a map of $\mathcal{I}$-spaces, where $\mathcal{I}$ is the the E_∞ operad discussed above. Upon taking homotopy-theoretic group completion of this map, we obtain the *total Segre class* map

$$Seg^{sst} : \mathcal{K}^{sst}(X) \to \mathcal{H}^{sst}_{mult}(X),$$

which is a map of group-like H-spaces (in fact, of spectra). Upon taking homotopy groups, we get

$$Seg^{sst} : K^{sst}_n(X) \to L^0H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q\geq 1} L^qH^{2q-n}(X)\right)^{\times}.$$

Remark 1.3.6. Lima-Filho [49, 4.1] has constructed a similar map of spectra, resulting in a total Chern class map.

The above construction of the (semi-topological) total Segre class involves, in a suitable sense, only constructions on the ind-varieties $\mathrm{Grass}(\mathbb{C}^{\infty})$ and $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^{\infty})$ representing semi-topological K-theory and morphic cohomology. Given that $\mathrm{Grass}(\mathbb{C}^{\infty})$ and $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^{\infty})$ (resp., the corresponding analytic spaces) can also be used to define algebraic K-theory and motivic cohomology (resp., topological K-theory and singular cohomology), it is unsurprising that one also obtains total Segre class maps

$$Seg^{alg} : K^{alg}_n(X) \to H^{-n}_{\mathcal{M}}(X, \mathbb{Z}(0)) \times \left(\{1\} \times \bigoplus_{q\geq 1} H^{2q-n}_{\mathcal{M}}(X, \mathbb{Z}(q))\right)^{\times}.$$

and

$$Seg^{top} : K^{-n}_{top} \to H^{-n}_{sing}(X, \mathbb{Z}) \times \left(\{1\} \times \bigoplus_{q\geq 1} H^{2q-n}_{sing}(X, \mathbb{Z})\right)^{\times}$$

in the algebraic and topological setting via highly analogous constructions. We obtain the following theorem.

Theorem 1.3.7. *(cf. [30, 8.6]) The total Segre class maps*

$$Seg^{alg} : K_n^{alg}(X) \to L^0H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q \geq 1} L^q H^{2q-n}(X) \right)^{\times},$$

$$Seg^{sst} : K_n^{sst}(X) \to H_{\mathcal{M}}^{-n}(X, \mathbb{Z}(0)) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q)) \right)^{\times}, \quad \textit{and}$$

$$Seg^{top} : K_{top}^{-n} \to H_{sing}^{-n}(X, \mathbb{Z}) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{sing}^{2q-n}(X, \mathbb{Z}) \right)^{\times}$$

are induced by natural transformations of H-spaces (in fact, of spectra). Moreover, they form the vertical arrows of the commutative diagram (1.9), provided one interprets the entries along the bottom row as groups in a suitable fashion.

The topological version of this theorem was first proven by Boyer et al in [9], settling in the affirmative a conjecture of Segal that the total Chern class map is a natural transformation of generalized cohomology theories. In addition, Lima-Filho [49, §4] has established an equivalent version of the right half of diagram (1.9) in which the verticle arrows are the total Chern class maps.

By applying the familiar universal polynomials (which have coefficients in $\mathbb{Q}$) that define the Chern character from the individual Chern classes, we obtain the Chern character maps

$$\begin{aligned} ch^{alg} &: K_*^{alg}(X) \to H_{\mathcal{M}}^*(X; \mathbb{Q}(*)), \\ ch^{sst} &: K_*^{sst}(X) \to L^*H^*(X; \mathbb{Q}), \quad \text{and} \\ ch^{top} &: K_{top}^*(X^{an}) \to H_{sing}^*(X; \mathbb{Q}) \end{aligned}$$

Theorem 1.3.8. *(cf. [32, 4.7] For a smooth, quasi-projective complex variety X, the Chern character maps are ring maps and they induce rational isomorphisms:*

$$\begin{aligned} ch^{alg} &: K_*^{alg}(X)_{\mathbb{Q}} \xrightarrow{\cong} H_{\mathcal{M}}^*(X; \mathbb{Q}(*)), \\ ch^{sst} &: K_*^{sst}(X)_{\mathbb{Q}} \xrightarrow{\cong} L^*H^*(X; \mathbb{Q}), \quad \textit{and} \\ ch^{top} &: K_{top}^*(X^{an})_{\mathbb{Q}} \xrightarrow{\cong} H_{sing}^*(X; \mathbb{Q}) \end{aligned}$$

Sketch of Proof. The result is well-known in the algebraic [7, 42] and topological [5] settings. For the semi-topological context, the proof is easy to describe at a heuristic level (although the rigorous details turn out to be more complicated than one might guess): One first shows, without much difficulty, that it suffices to prove that the semi-topological total Segre class map induces an isomorphism on rational homotopy groups. Since this is a map of H-spaces and since the result is known in the algebraic context for all smooth varieties, the Recognition Principle (Theorem 1.2.11) implies the desired result. (One difficulty in making this argument rigorous is proving that the usual Chern character isomorphism in algebraic K-theory coincides with the map given by universal polynomials from the map Seg^{alg}.) □

Remark 1.3.9. Cohen and Lima-Filho [10] have claimed a proof of the second isomorphism of Theorem 1.3.8, but their proof is invalid.

1.3.3 Finite Coefficients and the Bott Element

In this section, we describe two important properties of the horizontal maps in the diagram (1.9) — that is, we describe results about the comparison of algebraic and semi-topological theories and about the comparison of semi-topological and topological theories.

The first property is given by the following result, the first half of which has already been stated above as Theorem 1.2.12.

Theorem 1.3.10. *(cf. [27, 3.7], [54]) The left-hand horizontal maps of (1.9) are isomorphisms if X is smooth and $A = \mathbb{Z}/n$ for $n > 0$. That is, for $n > 0$ we have isomorphisms*

$$K_m^{alg}(X;\mathbb{Z}/n) \xrightarrow{\cong} K_m^{sst}(X;\mathbb{Z}/n)$$

and

$$H^p_{\mathcal{M}}(X,\mathbb{Z}/n(q)) \xrightarrow{\cong} L^qH^p(X,\mathbb{Z}/n)$$

for all integers p,q,m and all quasi-projective complex varieties X.

Remark 1.3.11. In fact, this theorem is valid even for X singular. For the second isomorphism, one must define the morphic cohomology so that *cdh* descent holds.

As mentioned in the sketch of proof of Theorem 1.2.12, the first isomorphism follows from the Recognition Principle and rigidity for algebraic K-theory with finite coefficients. The second isomorphism was proven originally by Suslin-Voevodsky [54] but the Recognition Principle can also be used to give another proof (but one which mimics much of Suslin-Voevodsky's original proof): The map in question is induced by the natural transformation of functors from $Sch/\mathbb{C}$ to chain complexes

$$\begin{aligned} &z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^q)/z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \\ &\qquad \longrightarrow z_0^{equi}(- \times X \times \Delta^\bullet, \mathbb{P}^q)/z_0^{equi}(- \times X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \end{aligned}$$

(with the first one being constant). By rigidity [54], this map is locally a quasi-isomorphism for the étale topology, and hence the induced map

$$z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^q)/z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \\ \longrightarrow z_0^{equi}(\Delta^\bullet_{top} \times X \times \Delta^\bullet, \mathbb{P}^q)/z_0^{equi}(\Delta^\bullet_{top} \times X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n$$

is a quasi-isomorphism by the Recognition Principle. These complexes define the motivic cohomology and morphic cohomology of smooth varieties, respectively.

Since using finite coefficients makes the maps from algebraic theories to semi-topological theories into equivalences, one might ask what modification of the maps from semi-topological theories to topological theories converts them to equivalences. The answer is that one needs only invert the Bott element (for K-theory) and the s element (for cycle theories). In other words, the *integral* analogue of Thomason's theorem [57, 4.11] relating Bott inverted algebraic K-theory and topological K-theory with finite coefficients holds in the context of semi-topological K-theory.

Note that since we have the isomorphism $K_*^{sst}(\operatorname{Spec}\mathbb{C}) \cong K_{top}^{-*}(pt)$ we have in particular that $K_2^{sst}(\operatorname{Spec}\mathbb{C}) \cong \mathbb{Z}$. Let β be the generator of $K_2^{sst}(\operatorname{Spec}\mathbb{C})$ associated to the canonical map $S^2 = \mathbb{P}^1(\mathbb{C}) \to$ Grass induced by the surjection $\mathbb{C}^\infty \twoheadrightarrow \mathbb{C}^2$ (defined by projection onto the first 2 coordinate) and call β the "Bott element". Obviously, under the map from semi-topological K-theory to topological K-theory, β maps to the usual Bott element in topology. Moreover, it's evident that that under the composition

$$K_2^{sst}(\operatorname{Spec}\mathbb{C}) \to K_2^{sst}(\operatorname{Spec}\mathbb{C}; \mathbb{Z}/n) \cong K_2^{alg}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}/n) \cong \mu_n(\mathbb{C}), \quad n > 0,$$

the element β maps to a generator of $\mu_n(\mathbb{C})$ (i.e., a primitive n-th root of unity), so that β maps to the Bott element in algebraic K-theory with finite coefficients.

Since $K_*^{sst}(X)$ is a (graded) module over the (graded) ring $K_*^{sst}(\operatorname{Spec}\mathbb{C}) \cong \mathbb{Z}[\beta]$, we may formally invert the action of β on $K_*^{sst}(X)$. Doing the same to $K^*_{top}(X^{an})$ results in the 2-periodic (non-connective) K-theory ring $KU^*(X^{an})$. Clearly, $K_*^{sst}(X)\left[\beta^{-1}\right]$ maps to $KU^*(X^{an})$, and the theorem is simply that this map is an isomorphism in all degrees:

Theorem 1.3.12. *(cf. [32], [60]) For a smooth, quasi-projective complex variety X, the right-hand horizontal maps of (1.9) become isomorphisms upon inverting the Bott element:*

$$K_*^{sst}(X)\left[\beta^{-1}\right] \xrightarrow{\cong} K_{top}^{-*}(X^{an})\left[\beta^{-1}\right] = KU^{-*}(X^{an})$$

We know of three separate proofs of this theorem, two of which use the analogous result for morphic cohomology. This result involves inverting the so-called "s operation" in morphic cohomology, defined originally the first author and B. Mazur [24] in the context of Lawson homology. The original definition involved a map defined on the level of cycle spaces; the definition given here is equivalent, under duality, to the induced map on homotopy groups.

Definition 1.3.13. For a quasi-projective complex variety X, the *s operation*

$$s : L^t H^n(X) \to L^{t+1} H^n(X)$$

is defined as multiplication by the s element $s \in L^1 H^0(\operatorname{Spec} \mathbb{C})$, which is given by $s = c_{2,1}(\beta)$ where $c_{2,1} : K_2^{sst}(\operatorname{Spec} \mathbb{C}) \to L^1 H^0(\operatorname{Spec} \mathbb{C})$ is the Chern class map.

The element s is a generator of $L^1 H^0(\operatorname{Spec} \mathbb{C}) \cong \mathbb{Z}$ and it clearly maps to a unit of the graded ring $H^*_{sing}(X^{an})$, for all X, but is never a unit in the bigraded ring $L^* H^*(X)$. Let $L^t H^n(X)[s^{-1}]$ denote the degree (t, n) piece of the result of inverting s in the bi-graded ring $L^* H^*(X)$.

Proposition 1.3.14. *For a smooth, quasi-projective complex variety X, the canonical map*

$$L^t H^n(X) \left[s^{-1}\right] \to H^n_{sing}(X^{an})$$

is an isomorphism for all t, n.

The proposition follows directly from the facts that morphic cohomology is isomorphic to Larson homology, that under this isomorphism multiplication by s corresponds to cup product by s (which is a map of the form $L_t H_n(X) \to L_{t-1} H_n(X)$), and that $L_t H_n(X) \cong H_n^{BM}(X^{an})$ for $t \leq 0$.

One proof of Theorem 1.3.12 (cf. [32, 5.8]) is given by establishing the desired isomorphism in the case of $\mathbb{Z}/n$ coefficients and $\mathbb{Q}$ coefficients separately. For $\mathbb{Z}/n$ coefficients, by using Theorem 1.3.10 it suffices to establish the analogous result comparing Bott inverted algebraic K-theory with $\mathbb{Z}/n$ coefficients to topological K-theory with $\mathbb{Z}/n$ coefficients — that this map is an isomorphism is (a special case of) Thomason's theorem [57, 4.11]. For $\mathbb{Q}$ coefficients, the result follows directly from the rational isomorphisms of Theorem 1.3.8, using Proposition 1.3.14 and the fact that $ch^{sst}(\beta) = s$.

A second proof (one which is not in yet in the literature at the time of this writing) is quite similar to the proof of the $\mathbb{Q}$ coefficients case above, except that one uses the integral "Atiyah-Hirzebruch-like" spectral sequence relating morphic cohomology and semi-topological K-theory that has been established by the authors and Christian Haesemeyer [21, 2.10]. This spectral sequence is described in the next section. Here again the point is that inverting the Bott element corresponds under this spectral sequence to inverting the s element in morphic cohomology, and thus Proposition 1.3.14 applies.

The third proof of Theorem 1.3.12 does not use morphic cohomology in any fashion, but it applies only to smooth, projective varieties. This proof is given by the second author in [60].

Note that the whereas the first proof uses Thomason's theorem, the latter two proofs do not. In light of Theorem 1.2.12, these proofs therefore represent, in particular, new proofs of Thomason's theorem for the special case of smooth complex varieties.

We close this section by presenting diagram (1.9) again, this time with arrows suitable decorated to indicate their properties:

$$\begin{array}{ccccc} K_*^{alg}(X) & \xrightarrow{\mathbb{Z}/n\text{-equiv.}} & K_*^{sst}(X) & \xrightarrow{\frac{1}{\beta}\text{-equiv.}} & K_{top}^{-*}(X^{an}) \\ \downarrow{\scriptstyle \mathbb{Q}\text{-equiv.}} & & \downarrow{\scriptstyle \mathbb{Q}\text{-equiv.}} & & \downarrow{\scriptstyle \mathbb{Q}\text{-equiv.}} \\ \bigoplus_q H_{\mathcal{M}}^{2q-*}(X, \mathbb{Z}(q)) & \xrightarrow{\mathbb{Z}/n\text{-equiv.}} & \bigoplus_q L^q H^{2q-*}(X) & \xrightarrow{\frac{1}{s}\text{-equiv.}} & \bigoplus_q H_{sing}^{2q-*}(X^{an}), \end{array} \tag{1.11}$$

1.3.4 Real Analogues

The real analogues of the results of Section 1.3.3 are developed by the authors in [30]. In particular, the real morphic cohomology of the variety X defined over $\mathbb{R}$ is formulated in terms of morphisms defined over $\mathbb{R}$ from X to Chow varieties, and semi-topological real Chern and Segre classes are defined. Moreover, the real analogue of Theorem 1.2.12 is proved. In [32], the semi-topological real total Segre class is shown to be a rational isomorphism for smooth, quasi-projective varieties defined over $\mathbb{R}$. Indeed, once one inverts the prime 2, the semi-topological total Segre class is a retract of the semi-topological total Segre class of the complexified variety $X_{\mathbb{C}}$, thanks to an argument using transfers.

1.4 Spectral Sequences and Computations

In this section we describe the construction of the "semi-topological Atiyah-Hirzebruch spectral sequence" relating morphic cohomology to semi-topological K-theory. We also provide computations of semi-topological K-groups for certain special varieties. These computations essentially all boil down to proving that for certain varieties, the map from semi-topological to topological K-theory is an isomorphism, at least in a certain range. These two topics are related, since the primary technique exploited in this section for such computations is the fact that the map from Lawson homology to Borel-Moore singular homology is an isomorphism in certain degrees for a special class of varieties. Such isomorphisms, in the case of smooth varieties, imply isomorphisms from morphic cohomology to singular cohomology, which, by using the spectral sequence, imply isomorphisms relating semi-topological to topological K-theory. Nearly all of the results in this section are found in the recent paper [21] of the two authors and C. Haesemeyer.

1.4.1 The Spectral Sequence

The "classical" Atiyah-Hirzebruch spectral sequence relates the singular cohomology groups of a finite dimensional CW complex Y with its topological K-groups, and is given by

$$E_2^{p,q}(top) = H^{p-q}_{sing}(Y, \mathbb{Z}) \Longrightarrow ku^{p+q}(Y).$$

Recall that ku^* denotes the generalized cohomology theory associated to the (-1)-connected spectrum $\mathbf{bu}$. (In non-positive degrees, ku^* coincides with K^*_{top}.) One method of constructing this spectral sequence is to observe that the homotopy groups of the spectrum $\mathbf{bu}$ are $\pi_{2n}\mathbf{bu} = \mathbb{Z}$, $\pi_{2n+1}\mathbf{bu} = 0$, for $n \geq 0$. Thus, the Postnikov tower of the spectrum $\mathbf{bu}$ is the tower of spectra

$$\cdots \to \mathbf{bu}[4] \to \mathbf{bu}[2] \to \mathbf{bu}$$

and there are fibration sequences

$$\mathbf{bu}[2q+2] \to \mathbf{bu}[2q] \to \mathbf{K}(\mathbb{Z}, 2q), \quad q \geq 0,$$

where $\mathbf{K}(\mathbb{Z}, 2q)$ denotes the Eilenberg-Maclane spectrum whose only non-vanishing homotopy group is $\mathbb{Z}$ in degree $2q$. By applying $\mathcal{M}aps(Y, -)$, one obtains the tower of spectra

$$\cdots \to \mathcal{M}aps(Y, \mathbf{bu}[4]) \to \mathcal{M}aps(Y, \mathbf{bu}[2]) \to \mathcal{M}aps(Y, \mathbf{bu}) \tag{1.12}$$

and fibration sequences of spectra

$$\mathcal{M}aps(Y, \mathbf{bu}[2q+2]) \to \mathcal{M}aps(Y, \mathbf{bu}[2q]) \to \mathcal{M}aps(Y, \mathbf{K}(\mathbb{Z}, 2q)), \; q \geq 0.$$

These data determine a collection of long exact sequences that form an exact couple, and the isomorphisms

$$\pi_n\mathcal{M}aps(Y, \mathbf{K}(\mathbb{Z}, 2q)) \cong H^{2q-n}_{sing}(Y, \mathbb{Z}) \quad \text{and} \quad \pi_n\mathcal{M}aps(Y, \mathbf{bu}) = ku^{-n}(Y^{an}), \quad n \in \mathbb{Z},$$

show that the associated spectral sequence has the form indicated above.

One of the more significant developments in algebraic K-theory in recent years is the construction of a purely algebraic analogue of the Atiyah-Hirzebruch spectral sequence, one which relates the motivic cohomology groups of a smooth variety to its algebraic K-groups. The construction of this spectral sequence is given (in various forms) in the papers [8, 25, 43, 34, 53]. To construct the spectral sequence for arbitrary smooth varieties (as is done in [25, 43, 34, 53]), the essential point is to reproduce the tower (1.12) at the algebraic level. Namely, for a smooth, quasi-projective variety over an arbitrary ground field F, one constructs a natural tower of spectra

$$\cdots \to \mathcal{K}^{(q+1)}(X) \to \mathcal{K}^{(q)}(X) \to \cdots \to \mathcal{K}^{(1)}(X) \to \mathcal{K}^{(0)}(X) = \mathcal{K}^{alg}(X) \tag{1.13}$$

together with natural fibration sequences of the form

$$\mathcal{K}^{(q+1)}(X) \to \mathcal{K}^{(q)}(X) \to \mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q)), \tag{1.14}$$

where $\mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q))$ is a suitable spectrum (in fact, a spectrum associated to a chain complex of abelian groups) whose homotopy groups give the motivic cohomology groups of X:

$$\pi_n \mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q)) = H^{2q-n}_{\mathcal{M}}(X, \mathbb{Z}(q)).$$

Such a tower and collection of fibration sequences leads immediately to the *motivic spectral sequence*:

$$E_2^{p,q}(alg) = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(q)) \Longrightarrow K^{alg}_{-p-q}(X).$$

The *semi-topological spectral sequence* is defined by simply "semi-topologizing" the motivic version. That is, once one observes that the spectra appearing in (1.13) and (1.14) are defined for all $X \in Sch/\mathbb{C}$ (not just smooth varieties) and that they represent functors from $Sch/\mathbb{C}$ to spectra (see [25] and [21, §2]), then one may form the tower

$$\begin{aligned} \cdots \to \mathcal{K}^{(q+1)}(X \times \Delta^\bullet_{top}) \to \mathcal{K}^{(q)}(X \times \Delta^\bullet_{top}) \to \cdots \\ \cdots \to \mathcal{K}^{(1)}(X \times \Delta^\bullet_{top}) \to \mathcal{K}^{(0)}(X \times \Delta^\bullet_{top}) = \mathcal{K}^{alg}(X \times \Delta^\bullet_{top}) \end{aligned} \tag{1.15}$$

and the collection of fibration sequences

$$\mathcal{K}^{(q+1)}(X \times \Delta^\bullet_{top}) \to \mathcal{K}^{(q)}(X \times \Delta^\bullet_{top}) \to \mathcal{H}_{\mathcal{M}}(X \times \Delta^\bullet_{top}, \mathbb{Z}(q)) \tag{1.16}$$

in the usual manner. Since we have $\mathcal{K}^{sst}(X) = \mathcal{K}^{alg}(X \times \Delta^\bullet_{top})$ and we also have (essentially by definition — at least, using the definition given in this paper)

$$\pi_n \mathcal{H}_{\mathcal{M}}(X \times \Delta^\bullet_{top}, \mathbb{Z}(q)) = L^q H^{2q-n}(X),$$

the collection of long exact sequences associated to (1.16) determines the semi-topological spectral sequence

$$E_2^{p,q}(sst) = L^q H^{p-q}(X) \Longrightarrow K^{sst}_{-p-q}(X). \tag{1.17}$$

Moreover, just as there is a natural map $\mathcal{K}^{sst}(X) \to \mathcal{K}_{top}(X^{an})$, one can define natural maps $\mathcal{K}^{(q)}(X \times \Delta^\bullet_{top}) \to \mathcal{M}aps(X^{an}, \mathbf{bu}[2q])$ for all $q \geq 0$ [21, 3.4], and thus one obtains a map from the semi-topological version of the Atiyah-Hirzebruch spectral sequence to the classical one. There is an obvious map from the motivic spectral sequence to the semi-topological spectral sequence, and thus we have the following theorem.

Theorem 1.4.1. *[21, 3.6] For a smooth, quasi-projective complex variety X, we have natural maps of convergent spectral sequences of "Atiyah-Hirzebruch type"*

$$\begin{array}{c} E_2^{p,q}(alg) = H^{p-q}_{\mathcal{M}}(X,\mathbb{Z}(q)) \Longrightarrow K^{alg}_{-p-q}(X) \\ \downarrow \\ E_2^{p,q}(sst) = L^qH^{p-q}(X) \Longrightarrow K^{sst}_{-p-q}(X) \\ \downarrow \\ E_2^{p,q}(top) = H^{p-q}_{sing}(Y,\mathbb{Z}) \Longrightarrow ku^{p+q}(Y) \end{array}$$

given by the usual maps from motivic to morphic to singular cohomology and from algebraic to semi-topological to topological K-theory.

1.4.2 Generalized Cycle Map and Weights

The concept of a *weight filtration* on the singular cohomology of a complex variety was introduce by Deligne [11] (for rational coefficients). This notion was extended to arbitrary coefficients in a paper of Gillet-Soulé [33]. For our purposes, the analogous notion of a weight filtration for Borel-Moore singular cohomology, H^{BM}_*, turns out to be of more use.

For any U, the weight filtration on $H^{BM}_n(U^{an})$ has the form

$$\cdots \subset W_tH^{BM}_n(U^{an}) \subset W_{t+1}H^{BM}_n(U^{an}) \subset \cdots \subset H^{BM}_n(U^{an})$$

and is "concentrated" in the range $-n \leq t \leq d-n$, where $d = \dim(U)$, in the sense that $W_tH^{BM}_n(U^{an}) = 0$ for $t < -n$ and $W_tH^{BM}_n(U^{an}) = H^{BM}_n(U^{an})$ for $t \geq d-n$. (That the filtration is concentrated in this range is not so obvious from the definition below, but see [33, §2].) We have found it useful to consider a slight variation on the groups $W_tH^{BM}_n(U^{an})$, which are written $\tilde{W}_tH^{BM}_n(U)$. The groups $\tilde{W}_tH^{BM}_n(U)$ do not form a filtration on $H^{BM}_n(U^{an})$, but rather map surjectively to $W_tH^{BM}_n(U^{an})$ (with torsion kernel). The groups $\tilde{W}_tH^{BM}_n(U)$, however, enjoy better formal properties than do the groups comprising the weight filtration.

The essential idea underlying the definition of the weight filtration is that the n-th homology group of a smooth, projective complex variety X is of pure weight $-n$, by which we mean

$$W_tH^{BM}_n(X^{an}) = \begin{cases} 0, & \text{if } t < -n \text{ and} \\ H^{BM}_n(X^{an}), & \text{if } t \geq -n, \end{cases}$$

and, more generally, elements in $H^{BM}_*(U^{an})$ have weight t if they "come from" elements of weight t in $H^{BM}_*(X^{an})$ for X smooth and projective under a suitable construction.

In detail, given a quasi-projective complex variety U, one chooses a projective compactification $\overline{U}$ (so that $U \subset \overline{U}$ is open and dense) and lets $Y = \overline{U} - U$ be the reduced closed complement. One then constructs a pair of "smooth hyperenvelopes" $\overline{U}_\bullet \to \overline{U}$ and $Y_\bullet \to Y$ together with a map $Y_\bullet \to \overline{U}_\bullet$ of such extending the map $Y \hookrightarrow \overline{U}$. In general, a "smooth hyperenvelope" $X_\bullet \to X$ is an augmented simplicial variety such that each X_n is smooth and the induced map $X_n \to (\mathrm{cosk}_{n-1}(X_\bullet))_n$ is a proper map that is surjective on F-points for any field F. Loosely speaking, such a smooth hyper-envelope over X is formed by first choosing a resolution of singularities $X_0 \to X$ of X (more specifically, a projective map with X_0 smooth that induces a surjection on F-points for any field F), then by choosing a resolution of singularities $X_1 \to X_0 \times_X X_0 = (\mathrm{cosk}_0(X_\bullet))_1$, then by choosing a resolution of singularities $X_2 \to (\mathrm{cosk}_2(X_\bullet))_1$, and so on.

Let $\mathbb{Z}\,\mathrm{Sing}_\bullet(-)$ denote the functor taking a space to the chain complex that computes its singular homology (i.e., $\mathbb{Z}\,\mathrm{Sing}_\bullet(-)$ is the chain complex associated to the simplicial abelian group $d \mapsto \mathbb{Z}\,\mathrm{Maps}(\Delta^d_{top}, -)$). Then the total complex associated to the map of bicomplexes (i.e., the tri-complex)

$$\mathbb{Z}\,\mathrm{Sing}_\bullet(Y_\bullet^{an}) \to \mathbb{Z}\,\mathrm{Sing}_\bullet(\overline{U}_\bullet^{an})$$

gives the Borel-Moore homology of U^{an}. That is, letting $U_i = \overline{U}_i \coprod Y_{i-1}$ (with $Y_{-1} = \emptyset$), we have

$$H_n^{BM}(U^{an}) \cong h_n(Tot(\cdots \to \mathbb{Z}\,\mathrm{Sing}_\bullet(U_1) \to \mathbb{Z}\,\mathrm{Sing}_\bullet(U_0))).$$

Observe that the definition of $W_t H_n$ for a smooth, projective variety X given above amounts to setting

$$W_t H_n(X) = h_n(tr_{\geq -t}\mathbb{Z}\,\mathrm{Sing}_\bullet(X^{an})),$$

where $tr_{\geq -t}$ denotes the good truncation of chain complexes at homological degree $-t$. In heuristic terms, the weight filtration on $H_*^{BM}(U^{an})$ is defined from the "left-derived functor" of $tr_{\geq -t}$, if we interpret the smooth, projective varieties U_i as forming a resolution of U.

This idea is formalized in the following definition, which also includes a definition of related functors $\tilde{W}_t H_n^{BM}$.

Definition 1.4.2. Given a quasi-projective complex variety U, define

$$\tilde{W}_t H_n^{BM}(U) = h_n(\cdots \to tr_{\geq -t}\mathbb{Z}\,\mathrm{Sing}_\bullet(U_1) \to tr_{\geq -t}\mathbb{Z}\,\mathrm{Sing}_\bullet(U_0)),$$

where $tr_{\geq -t}$ denotes the good truncation of a chain complex at homological degree $-t$ and the U_i's are constructed as above.

Define $W_t H_n^{BM}(U^{an})$ to be the image of $\tilde{W}_t H_n^{BM}(U)$ in $H_n^{BM}(U^{an})$ under the canonical map:

$$W_t H_n^{BM}(U^{an}) = \mathrm{image}(\tilde{W}_t H_n^{BM}(U^{an}) \to H_n^{BM}(U^{an})).$$

For an alternative formulation of the weight filtration, observe that associated to the bicomplex $\cdots \to \mathbb{Z}\,\mathrm{Sing}_\bullet(U_1) \to \mathbb{Z}\,\mathrm{Sing}_\bullet(U_0)$, we have the spectral sequence

$$E^2_{p,q} = h_p(\cdots \to H_q(U_1^{an}) \to H_q(U_0^{an})) \Longrightarrow H^{BM}_{p+q}(U^{an}). \tag{1.18}$$

The weight filtration on $H^{BM}_*(U^{an})$ may equivalently be defined to be the filtration induced by this spectral sequence [33, §3]:

$$W_t H^{BM}_n(U^{an}) = \mathrm{image}\left(h_n\left(\mathbb{Z}\,\mathrm{Sing}_\bullet(U_{n+t}) \to \cdots \to \mathbb{Z}\,\mathrm{Sing}_\bullet(U_0)\right) \to H^{BM}_n(U^{an})\right).$$

In other words, the groups $W_t H^{BM}_n(U^{an})$ are the D_∞ terms of the spectral sequence (1.18). What's more, the groups $\tilde{W}_t H^{BM}_n(U)$ are equal to the D_2 terms of this spectral sequence. This implies that for a situation in which the spectral sequence (1.18) degenerates at the E_2 terms, the map $\tilde{W}_t H^{BM}_n(U) \to W_t H^{BM}_n(U^{an})$ is an isomorphism for all t and n. In particular, since (1.18) degenerates rationally by Deligne's result [11], we have $\tilde{W}_t H^{BM}_n(U)_\mathbb{Q} \cong W_t H^{BM}_n(U^{an})_\mathbb{Q}$.

For a simple example, suppose U happens to be smooth and admits a smooth compactification X such that $Y = X - U$ is again smooth. Then the spectral sequence (1.18) degenerates (integrally) and it really just amounts to a single long exact sequence

$$\cdots \to H^{sing}_n(Y^{an}) \to H^{sing}_n(X^{an}) \to H^{BM}_n(U^{an}) \to H^{sing}_{n-1}(Y^{an}) \to \cdots.$$

It follows that $\tilde{W}_t H^{BM}_n(U) = W_t H^{BM}_n(U^{an})$ and

$$W_t H^{BM}_n(U^{an}) = \begin{cases} 0, & \text{if } t < -n, \\ \mathrm{image}(H^{sing}_n(X^{an}) \to H^{BM}_n(U^{an})), & \text{if } t = -n, \text{ and} \\ H^{BM}_n(U^{an}), & \text{if } t > -n. \end{cases}$$

Thus, in this situation the information encoded by the weight filtration on $H^{BM}_*(U^{an})$ concerns which classes in $H^{BM}_*(U^{an})$ can be lifted to the homology of a smooth compactification $H_*(X^{an})$.

It is a non-trivial theorem, due to Deligne [11] for $\mathbb{Q}$-coefficients and Gillet-Soulé [33, §2] in general, that the weight $W_* H^{BM}_*$ filtration is independent of the choices made in its construction. Using their techniques, the authors and Christian Haesemeyer have also proven that $\tilde{W}_t H^{BM}$ is independent of the choices made [21, 5.9].

1.4.3 Computations

The *generalized cycle map* (see [24] and [48]) is the map from the Lawson homology of a complex variety X to its Borel-Moore homology

$$L_t H_n(X) \to H^{sing}_n(X^{an}).$$

For smooth varieties, the generalized cycle map corresponds under duality to the map from morphic cohomology to singular cohomology

$$L^{d-t}H^{2d-n}(X) \to H^{2d-n}_{sing}(X^{an})$$

described in Section 1.3.1. For projective (but possibly singular) varieties, the generalized cycle map is defined by applying π_{n-2t} to the diagram of spaces

$$\mathcal{Z}_t(X) \xrightarrow{\ s\ } \Omega^{2t}\mathcal{Z}_t(X \times \mathbb{A}^t) \xleftarrow{\ \sim\ } \Omega^{2t}\mathcal{Z}_0(X),$$

where the first map is the "s map" defined by Friedlander and Mazur [24] (see also Definition 1.3.13) and the second map is the homotopy equivalence induced by flat pullback of cycles along the projection $X \times \mathbb{A}^t \to X$. This definition is extended to quasi-projective varieties in [20].

Observe that the singular chain complex associated to the space $\Omega^{2t}\mathcal{Z}_0(X)$ is quasi-isomorphic to $tr_{\geq -2t}\mathbb{Z}\,\mathrm{Sing}_\bullet(X^{an})[2t]$, the complex used to define $\tilde{W}_{2t}H^{BM}_*$. This observation leads to a proof that the generalized cycle class map from L_tH_n lands in the weight $-2t$ part of Borel-Moore homology. This fact, as well as other properties relating the weight filtration on Borel-Moore homology and the generalized cycle map, is formalized by the following result.

Proposition 1.4.3. *(cf. [21, 5.11, 5.12])*

1. *For any quasi-projective variety U, the generalized cycle map factors as*

$$L_tH_n(U) \to \tilde{W}_{-2t}H^{BM}_n(U) \twoheadrightarrow W_{-2t}H^{BM}_n(U) \subset H^{BM}_n(U),$$

and each of these maps is covariantly functorial for proper morphisms and contravariantly functorial for open immersions. The map $L_tH_n(U) \to \tilde{W}_{-2t}H^{BM}_n(U)$ is called the refined cycle map.

2. *Each of the theories $L_tH_*(-)$, $\tilde{W}_tH^{BM}_*(-)$, $H^{BM}_*((-)^{an})$ has a long exact localization sequence associated to an open immersion $U \subset X$ with closed complement $Y = X - U$, and the maps*

$$L_tH_*(-) \to \tilde{W}_tH^{BM}_*(-) \to H^{BM}_*((-)^{an})$$

are compatible with these long exact sequences.

Remark 1.4.4. The weight filtration itself, $W_tH^{BM}_*(-)$, is *not* always compatible with localization sequences, and the construction $\tilde{W}_tH^{BM}_*$ was introduced to rectify this defect.

The first part of Proposition 1.4.3 clearly provides an obstruction for the generalized cycle map to be an isomorphism in certain degrees for certain kinds of varieties.

Definition 1.4.5. Define $\mathcal{C}$ to be the collection of smooth, quasi-projective complex varieties U such that the refined cycle map $L_tH_n(U) \to \tilde{W}_tH_n(U)$ is an isomorphism for all t and n.

Theorem 1.4.6. *(cf. [21, 6.3]) Assume X is a quasi-projective complex variety of dimension d that belongs to the class $\mathcal{C}$ and let A be any abelian group.*

1. *The generalized cycle map*

$$L_tH_n(X,A) \to H_n^{BM}(X^{an},A)$$

is an isomorphism for $n \geq d+t$ and a monomorphism for $n = d+t-1$. If X is smooth and projective, this map is an isomorphism for $n \geq 2t$.

2. *If X is smooth, the canonical map*

$$K_q^{sst}(X,A) \to K_{top}^{-q}(X^{an},A)$$

is an isomorphism for $q \geq d-1$ and a monomorphism for $q = d-2$. If X is smooth and projective, this map is an isomorphism for $q \geq 0$.

The proof of the first part of Theorem 1.4.6 is achieved via a careful analysis of the spectral sequence (1.18), and the proof of the second part follows from a careful analysis of the semi-topological spectral sequence and it's comparison with the classical Atiyah-Hirzebruch spectral sequence (Theorem 1.4.1).

The conclusion of the second part of Theorem 1.4.6 (in the not-necessarily-projective case) is what we term the *Semi-topological Quillen-Lichtenbaum Conjecture*, discussed in more detail in Section 1.5 below.

The validity of the following assertions for the class $\mathcal{C}$ is the primary reason the groups $\tilde{W}_*H_*^{BM}(-)$ were introduced. The corresponding statement for the class of varieties for which $L_tH_n(-) \to W_{-2t}H_n^{BM}((-)^{an})$ is an isomorphism in all degrees is false.

Proposition 1.4.7. *(cf. [21, 6.9]) The class $\mathcal{C}$ is closed under the following constructions:*

1. *Closure under localization: Let $Y \subset X$ be a closed immersion with Zariski open complement U. If two of X, Z, and U belong to $\mathcal{C}$, so does the third.*
2. *Closure for bundles: For a vector bundle $E \to X$, the variety X belongs to $\mathcal{C}$ if and only if $\mathbb{P}(E)$ does. In this case, E belongs to $\mathcal{C}$ as well.*
3. *Closure under blow-ups: Let $Z \subset X$ be a regular closed immersion and such that Z belongs to $\mathcal{C}$. Then X is in $\mathcal{C}$ if and only if the blow-up X_Z of X along Z is in $\mathcal{C}$.*

Recall that the class of *linear varieties* is the smallest collection $\mathcal{L}$ of complex varieties such that (1) $\mathbb{A}^n$ belongs to $\mathcal{L}$, for all $n \geq 0$, and (2) if X is a quasi-projective complex variety, $Z \subset X$ is a closed subscheme, $U = X - Z$ is the open complement, and Z and either X or U belongs to $\mathcal{L}$, then so does the remaining member of the triple (X, Z, U). Examples of linear varieties include toric and cellular varieties.

Theorem 1.4.8. *(cf. [21, §6]) The following complex varieties belong to $\mathcal{C}$:*

1. *A quasi-projective curve.*
2. *A smooth, quasi-projective surface having a smooth compactification with all of H^2_{sing} algebraic.*
3. *A smooth projective rational three-fold.*
4. *A smooth quasi-projective linear variety (e.g., a smooth quasi-projective toric variety).*
5. *A toric fibration (e.g., an affine or projective bundle) over one of the above varieties.*

In particular, if X is smooth and one of the above types of varieties, then for any abelian group A the natural map

$$K_n^{sst}(X, A) \to K_{top}^{-n}(X^{an}, A)$$

is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$. If X is in addition projective, this map is an isomorphism for all $n \geq 0$.

As mentioned in the introduction, when X is a smooth, projective complex variety belonging to $\mathcal{C}$, Theorem 1.4.8 implies that the subspace inclusion

$$\mathcal{M}or(X, \mathrm{Grass}) \subset \mathcal{M}aps(X^{an}, \mathrm{Grass}^{an})$$

becomes a homotopy equivalence upon taking homotopy-theoretic group completions. In fact, both homotopy-theoretic group completions can be described precisely by taking mapping telescopes of self-maps (essentially defined as "addition by a fixed ample line bundle") of the spaces above [31, 3.5]. This result therefore gives examples when the stabilized space of all continuous maps between two complex (ind-)varieties can represented up-to-homotopy equivalence by the stabilized space of all algebraic morphisms between them.

1.5 Conjectures

In this section, we discuss various conjectures relating semi-topological K-theory to topological K-theory and relating morphic cohomology to singular cohomology.

1.5.1 Integral Versions of the "Classical" Conjectures

One important feature of semi-topological K-theory and morphic cohomology is that they allow for the formulation of plausible analogues for arbitrary coefficients of the classical conjectures in algebraic K-theory and motivic cohomology for finite coefficients. For example, the Quillen-Lichtenbaum and Beilinson-Lichtenbaum Conjectures, each of which concerns theories with $\mathbb{Z}/n$-coefficients, admit *integral* analogues in the semi-topological world. Moreover, in light of Theorem 1.3.10, these semi-topological conjectures imply their classical counter-parts (for complex varieties).

Perhaps the most fundamental of these conjectures, formulated originally by A. Suslin, concerns a conjectural description of morphic cohomology in terms of singular cohomology. To understand *Suslin's Conjecture*, as we have termed it, recall that if we define $\mathbb{Z}^{sst}$ to be the complex of abelian sheaves

$$\mathbb{Z}^{sst}(t) = \mathrm{Hom}(- \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^t))^+ / \mathrm{Hom}(- \times \Delta^{\bullet}_{top}, \mathcal{C}_0(\mathbb{P}^{t-1}))^+[-2t],$$

then the morphic cohomology groups of a smooth variety X are given by

$$L^tH^n(X) = h^{n-2t}\Gamma(X, \mathbb{Z}^{sst}(t)).$$

In fact, Zariski descent for morphic cohomology [18] implies that

$$L^tH^n(X) \cong \mathbb{H}^{n-2t}_{Zar}(X, \mathbb{Z}^{sst}(t))$$

for X smooth, where $\mathbb{H}_{Zar}$ denotes taking the hypercohomology in the Zariski topology. The comparison of $\mathbb{Z}^{sst}(t)$ with singular cohomology uses the morphism of sites $\epsilon : CW_{open} \to (Sch/\mathbb{C}_{Zar})$, where CW denotes the category of topological spaces homeomorphic to finite dimensional CW-complexes, associated to the functor $U \mapsto U^{an}$ taking a complex variety to its associated analytic space. If $\mathbb{Z}$ denotes the sheaf associated to the constant presheaf $T \mapsto \mathbb{Z}$ defined on CW, then we have

$$H^n_{sing}(X^{an}, \mathbb{Z}) \cong H^n_{sheaf}(X^{an}, \mathbb{Z}) \cong \mathbb{H}^n_{Zar}(X^{an}, \mathbb{R}\epsilon_*\mathbb{Z}),$$

for any $X \in Sch/\mathbb{C}$. It's not hard to see that the map from morphic cohomology to singular cohomology is induced by a map of chain complexes of sheaves

$$\mathbb{Z}^{sst}(q) \to \mathbb{R}\epsilon_*\mathbb{Z}.$$

More generally, for any abelian group A, if we define $A^{sst}(q) = \mathbb{Z}^{sst}(q) \otimes A$, then there is a natural map

$$A^{sst}(q) \to \mathbb{R}\epsilon_*A$$

of complexes of sheaves that induces the map from morphic cohomology with A-coefficients to singular cohomology with A-coefficients.

To formulate Suslin's Conjecture, we need the following result:

Theorem 1.5.1. *[21, 7.3] For any abelian group A, the map $A^{sst}(q) \to \mathbb{R}\epsilon_*A$ factors (in the derived category of sheaves) as*

$$A^{sst}(q) \to tr^{\leq q}\mathbb{R}\epsilon_*A \to \mathbb{R}\epsilon_*A,$$

where $tr^{\leq q}$ represents the "good truncation" at degree q of a cochain complex.

The proof of Theorem 1.5.1 may be of independent interest, and so we sketch it here. (This is proved formally in [21], building on ideas from [13].) It suffices to prove $L^tH^n(-, A)$ vanishes locally on a smooth variety whenever $n > t$. Using duality relating morphic cohomology to Lawson homology [22, 17], we see that it suffices to prove that $L_tH_m(-, A)$ vanishes at the generic point of X for $m < t + \dim(X)$. Localization for Lawson homology and the rational injectivity of the Hurewicz map for a topological abelian group shows that it suffices to verify that the canonical map

$$\varinjlim_{Y\subset X, \mathrm{codim}(Y)\geq 1} H_m^{sing}(Z_t(Y), A) \to H_m^{sing}(Z_t(X), A)$$

is an isomorphism for $n < d - t - 1$ and a surjection for $n = d - t - 1$. For a given X and a given $t \geq 0$, the proof of this statement can be reduced to proving the analogous statements for

$$\varinjlim_{Y\subset X, \mathrm{codim}(Y)\geq 1} H_m^{sing}(C_{t,e}(Y), A) \to H_m^{sing}(C_{t,e}(X), A), \quad e > 0.$$

Finally, these statements concerning the (singular) algebraic varieties $C_{t,e}(X)$ are then proved using the Lefschetz theorem as proved by Andreotti and Frankel [3].

Conjecture 1.5.2 (Suslin's Conjecture). For any abelian group A, the map of complexes

$$A^{sst}(q) \to tr^{\leq q}\mathbb{R}\epsilon_* A$$

is a quasi-isomorphism on the category of smooth, quasi-projective complex varieties.

Equivalently (see [21, 7.9]), for all smooth, quasi-projective complex varieties X, the map

$$L^tH^n(X, A) \to H^n_{sing}(X, A)$$

is an isomorphism for $n \leq t$ and a monomorphism for $n = t + 1$.

Suslin's Conjecture is clearly analogous to the Beilinson-Lichtenbaum Conjecture, which can be stated as follows.

Conjecture 1.5.3 (Beilinson-Lichtenbaum Conjecture). (See [6] and [45].) Let F be an arbitrary field, let $\pi : (Sch/F)_{\acute{e}t} \to (Sch/F)_{Zar}$ be the evident morphism of sites, and let m be a positive integer not divisible by the characteristic of F. Then the canonical map of complexes of sheaves on $(Sm/F)_{Zar}$

$$\mathbb{Z}/m(q) \to tr^{\leq q}\mathbb{R}\pi_*\mu_m^{\otimes q}$$

is a quasi-isomorphism.

Equivalently, for all smooth, quasi-projective F-varieties X, the map

$$H^n_{\mathcal{M}}(X, \mathbb{Z}/n(q)) \to H^n_{\acute{e}t}(X, \mu_m^{\otimes q})$$

is an isomorphism for $n \leq t$ and a monomorphism for $n = t + 1$.

In light of Theorem 1.3.10 and the fact that étale and singular cohomology with finite coefficients of complex varieties coincide, the following result is evident.

Proposition 1.5.4. *Suslin's Conjecture implies the Beilinson-Lichtenbaum Conjecture for complex varieties.*

In a parallel fashion, the Quillen-Lichtenbaum Conjecture, which asserts an isomorphism between algebraic and topological K-theory with finite coefficients in a certain range, admits an integral, semi-topological analogue:

Conjecture 1.5.5 (Semi-topological Quillen-Lichtenbaum Conjecture). For a smooth, quasi-projective complex variety X and abelian group A, the canonical map

$$K_n^{sst}(X, A) \to K_{top}^{-n}(X, A)$$

is an isomorphism for $n \geq \dim(X)-1$ and a monomorphism for $n = \dim(X)-2$.

Using the isomorphism of Theorem 1.3.10, we see in the case $A = \mathbb{Z}/m$ that this conjecture is equivalent to the assertion that

$$K_n^{alg}(X, \mathbb{Z}/m) \to K_{top}^{-n}(X, \mathbb{Z}/m)$$

is an isomorphism for $n \geq \dim(X)-1$ and a monomorphism for $n = \dim(X)-2$. This special case is the "classical" *Quillen-Lichtenbaum Conjecture* (see [52] and [44]) for complex varieties.

Evidence for the semi-topological Quillen-Lichtenbaum Conjecture is supplied by Theorem 1.3.12, which may be interpreted as saying the map $K_*^{sst}(X) \to K_{top}^{-*}(X^{an})$ is an isomorphism "stably". In addition, we have the following result establishing split surjectivity of this map in a range.

Theorem 1.5.6. *For a smooth, quasi-projective complex variety X, the map*

$$K_n^{sst}(X) \to K_{top}^{-n}(X^{an})$$

is a split surjection for $n \geq 2\dim(X)$.

When X is projective, this is proven by the second author in [60] using the theory of "semi-topological K-homology". Thanks to the recently established motivic spectral sequence (1.17), a proof for the gereral case is obtained by mimicking the argument of [29, 1.4].

Using the semi-topological Atiyah-Hirzebruch spectral sequence (1.17), one may readily deduce that Suslin's Conjecture implies the semi-topological Quillen-Lichtenbaum Conjecture.

Theorem 1.5.7. *[21, 6.1] For a smooth, quasi-projective complex variety X and an abelian group A, if*

$$L^q H^n(X, A) \to H^n(X^{an}, A)$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$, then the map

$$K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$$

is an isomorphism for $i \geq \dim(X)-1$ and a monomorphism for $i = \dim(X)-2$.

In other words, Suslin's Conjecture implies the semi-topological Quillen-Lichtenbaum Conjecture.

The results described in Section 1.4.3 lead to the following theorem:

Theorem 1.5.8. *(cf. [21, 7.14]) Suslin's Conjecture and the and semi-topological Quillen-Lichtenbaum Conjectures hold for the following complex varieties:*

1. *smooth quasi-projective curves,*
2. *smooth quasi-projective surfaces,*
3. *smooth projective rational three-folds,*
4. *smooth quasi-projective linear varieties (for example, smooth quasi-projective toric and cellular varieties), and*
5. *smooth toric fibrations (e.g., affine and projective bundles) over one of the above varieties.*

Consequently, the "classical" Quillen-Lichtenbaum Conjecture and the Beilinson-Lichtenbaum Conjecture hold for these varieties.

We remind the reader that Voevodsky has recently proven the Beilinson-Lichtenbaum and "classical" Quillen-Lichtenbaum Conjectures [59].

1.5.2 *K*-theoretic Analogue of the Hodge Conjecture

The Hodge conjecture [37] concerns which rational singular cohomology classes of a smooth, complex variety arise from cycles — more precisely, it asserts that for a smooth, projective complex variety X, every class in $H^{p,p}_{sing}(X^{an}, \mathbb{Q})$ lies in the image of the rational cycle class map $A^p(X)_{\mathbb{Q}} \to H^{2p}_{sing}(X^{an}, \mathbb{Q})$, where

$$H^{p,p}(X^{an}, \mathbb{Q}) = H^{2p}_{sing}(X^{an}, \mathbb{Q}) \cap H^{p,p}(X^{an}, \mathbb{C})$$

and $H^{p,p}(X^{an}, \mathbb{C})$ refers to the Hodge decomposition of a complex Kahler manifold. It is easy to show the image of the rational cycle class map is contained in $H^{p,p}_{sing}(X^{an}, \mathbb{Q})$, and thus the Hodge Conjecture becomes (in the language of morphic cohomology) that the rational cycle class map

$$L^p H^{2p}(X, \mathbb{Q}) \to H^{p,p}(X^{an}, \mathbb{Q})$$

is a surjection.

The Generalized Hodge Conjecture, as corrected by Grothendieck [36], asserts that the *Hodge filtration* and the *coniveau filtration* on the rational homology of a smooth, projective complex variety coincide. The rational Hodge filtration is given as

$$H^m_{sing}(X^{an},\mathbb{Q}) = F^0_h H^m_{sing}(X^{an},\mathbb{Q}) \supset F^1_h H^m_{sing}(X^{an},\mathbb{Q}) \supset F^2_h H^m_{sing}(X^{an},\mathbb{Q}) \supset \cdots,$$

where $F^j_h H^m_{sing}(X^{an},\mathbb{Q})$ is defined by Grothendieck [36] to be the maximal sub-mixed Hodge structure of $H^m_{sing}(X^{an},\mathbb{Q})$ contained in $H^m_{sing}(X^{an},\mathbb{Q}) \cap \bigoplus_{p\geq j} H^{p,m-p}_{sing}(X^{an},\mathbb{C})$. The coniveau filtration, $N^*H^m_{sing}(X^{an},\mathbb{Q})$, is given by

$$N^j H^m_{sing}(X^{an},\mathbb{Q}) = \bigcup_{\substack{Y\subset X \\ \operatorname{codim}(Y)=j}} \ker\left(H^m_{sing}(X^{an},\mathbb{Q}) \to H^m_{sing}((X-Y)^{an},\mathbb{Q})\right)$$

The containment $H^j H^m_{sing}(X^{an},\mathbb{Q}) \subset F^j_h H^m_{sing}(X^{an},\mathbb{Q})$ always holds, and so the Generalized Hodge Conjecture amounts to the assertion that the opposite containment also holds — i.e., every class in $F^j_h H^m_{sing}(X^{an},\mathbb{Q})$ vanishes on the complement of a closed subscheme of codimension j.

For any smooth, quasi-projective complex variety X, the *topological filtration* of $H^m_{sing}(X^{an})$ is given by considering the images of the powers of the s map:

$$T^j H^m_{sing}(X^{an},\mathbb{Z}) = \operatorname{image}\left(L^{m-j}H^m(X) \to H^m_{sing}(X^{an},\mathbb{Z})\right).$$

(We set $T^j = T^0$ if $j<0$.) Recall that Suslin's Conjecture predicts that the map $L^m H^m(X) \to H^m_{sing}(X^{an},\mathbb{Z})$ is an isomorphism, so that, conjecturally, $T^j H^m_{sing}(X^{an},\mathbb{Z})$ may be identified with the image of $s^j : L^{m-j}H^m_{sing}(X) \to L^m H^m(X)$.

The following result of the first author and B. Mazur (originally stated in the context of Lawson homology) relates the three filtrations above. Note that we have modified the indexing conventions here from those of the original.

Proposition 1.5.9. *(cf. [24]) For a smooth, projective complex variety X, we have*

$$T^j H^m_{sing}(X^{an},\mathbb{Q}) \subset N^j H^m_{sing}(X^{an},\mathbb{Q}) \subset F^j_h H^m_{sing}(X^{an},\mathbb{Q}),$$

for all j and m.

The following conjecture thus represents a (possibly stronger) version of the Generalized Hodge Conjecture:

Conjecture 1.5.10 (Friedlander-Mazur Conjecture). (cf. [24]) For a smooth, projective complex variety, we have

$$T^j H^m_{sing}(X^{an},\mathbb{Q}) = F^j_h H^m_{sing}(X^{an},\mathbb{Q}),$$

for all m and j.

Proposition 1.5.9 shows that the Friedlander-Mazur Conjecture implies the Generalized Hodge Conjecture. In the case of abelian varieties, S. Abdulali has established the following converse.

Theorem 1.5.11. *(cf. [1]) For abelian varieties for which the Generalized Hodge Conjecture is known the Friedlander-Mazur Conjecture also holds.*

Semi-topological K-theory provides another perspective on the Hodge Conjecture, one which could prove to be of some use. Since the topological filtration on singular cohomology is defined by in terms of the s map, it is natural to define the *topological filtration* on topological K-theory in terms of the Bott map (i.e., multiplication by the Bott element) in semi-topological K-theory. That is, an element of $K^0_{top}(X^{an})$ lies in the j-th filtered piece of the topological filtration if it comes from semi-topogical K-theory after applying the j-th power of the Bott map:

$$T^j K^0_{top}(X^{an}) = \text{image}(K^{sst}_{2d-2j}(X) \xrightarrow{\beta^j} K^{sst}_{2d}(X) \to K^{-2d}_{top}(X^{an}) \xleftarrow[\cong]{\beta^d} K^0_{top}(X^{an})). \tag{1.19}$$

(As before, $T^j = T^0$ for $j < 0$.) The form of this definition appears more sensible once one recalls that the map

$$K^{sst}_{2d}(X) \to K^{-2d}_{top}(X^{an})$$

is known to be a split surjection (see Theorem 1.5.6). We thus have a filtration of the form

$$K^0_{top}(X^{an}) = T^0 K^0_{top}(X^{an}) \supset T^1 K^0_{top}(X^{an}) \supset \cdots T^d K^0_{top}(X^{an}) \supset T^{d+1} K^0_{top}(X^{an}) = 0,$$

for a smooth, quasi-projective complex variety X. In fact, for $j \leq \frac{d+1}{2}$, the Quillen-Lichtenbaum Conjecture predicts that each map in (1.19) is an isomorphism, so that conjecturally we have $T^0 K^0_{top}(X^{an}) = T^1 K^0_{top}(X^{an}) = \cdots = T^{\frac{d+1}{2}} K^0_{top}(X^{an})$.

Rationally, under the Chern character isomorphism, the Bott element in K-theory corresponds to the s element in cohomology, and thus the topological filtrations for K-theory and cohomology defined above are closely related. The precise statement is the following.

Theorem 1.5.12. *(cf. [32, 5.9]) For any smooth, quasi-projective complex variety X of dimension d, the Chern character restricts to an isomorphism*

$$ch^{top} : T^j K^0_{top}(X^{an})_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{q \geq 0} T^{q+j-d} H^{2q}_{sing}(X^{an}, \mathbb{Q}),$$

for all j. In other words, the weight q piece of $T^j K^0_{top}(X^{an})_{\mathbb{Q}}$ is mapped isomorphically via the Chern character to $T^{q+j-d} H^{2q}_{sing}(X^{an}, \mathbb{Q})$.

Using the Chern character isomorphism, one can transport the rational Hodge Filtration in singular cohomology to a filtration of $K^0_{top}(X)_{\mathbb{Q}}$. In this manner, the Friedlander-Mazur Conjecture, which implies the Generalized Hodge Conjecture, can be stated in purely K-theoretic terms. It would be interesting to find an intrinsic description of the "Hodge filtration" of $K^0_{top}(X)_{\mathbb{Q}}$.

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