SEMI-TOPOLOGICAL K-THEORY USING FUNCTION COMPLEXES

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Abstract. The semi-topological K-theory $K_*^{\text{semi}}(X)$ of a quasi-projective complex algebraic variety X is based on the notion of algebraic vector bundles modulo algebraic equivalence. This theory is given as the homotopy groups of an infinite loop space $\mathcal{K}^{\text{semi}}(X)$ which is equipped with maps $\mathcal{K}^{\text{alg}}(X) \to \mathcal{K}^{\text{semi}}(X)$, $\mathcal{K}^{\text{semi}}(X) \to \mathcal{K}_{\text{top}}(X^{an})$ whose composition is the natural map from the algebraic K-theory of X to the topological K-theory of the underlying analytic space X^{an} of X. We give an explicit description of $K_0^{\text{semi}}(X)$ in terms of $K_0(X)$, a description of $K_q^{\text{semi}}(-)$ in terms of $K_0^{\text{semi}}(-)$ for projective varieties, a Poincaré duality theorem for projective varieties, and a computation of $\mathcal{K}^{\text{semi}}(X)$ whenever X is a product of projective spaces or a smooth complete curve. For X a smooth quasi-projective variety, there are natural Chern class maps from $K_*^{\text{semi}}(X)$ to morphic cohomology compatible with similarly defined Chern class maps from algebraic K-theory to motivic cohomology and compatible with the classical Chern class maps from topological K-theory to the singular cohomology of X^{an}.

We formulate the "semi-topological K-theory" $K_*^{\text{semi}}(X)$ of a quasi-projective complex variety X, which we define as the homotopy groups of a certain infinite loop space $\mathcal{K}^{\text{semi}}(X)$. This theory interpolates between the algebraic K-theory $K_*(X)$ of X and the topological K-theory $K_{\text{top}}^{-*}(X^{\text{an}})$ of the analytic space X^{an} associated to X. Our philosophy is that algebraic K-theory is the study of algebraic vector bundles under rational equivalence, semi-topological K-theory is the study of algebraic vector bundles under algebraic equivalence, and topological K-theory is the study of topological vector bundles under topological equivalence. From this point of view, the commutative diagram for X smooth

$$\begin{array}{ccccc} K_j(X) & \longrightarrow & K_j^{\text{semi}}(X) & \longrightarrow & K_{\text{top}}^{-j}(X) \\ {\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \bigoplus_q \mathrm{H}^{2q-j}(X,\mathbb{Z}(q)) & \longrightarrow & \bigoplus_q L^q\,\mathrm{H}^{2q-j}(X) & \longrightarrow & \bigoplus_q \mathrm{H}^{2q-j}(X^{an},\mathbb{Z}), \end{array} \qquad (A)$$

which we establish in Theorem 6.11, relating these K-theories and their corresponding cohomology theories (motivic, morphic, and singular cohomology) is particularly natural. In fact, diagram (A) arises from a commuting diagram of infinite loop spaces. This partially completes a program outlined in [8].

The semi-topological K-theory space $\mathcal{K}^{\text{semi}}(X)$ of a projective variety X can be viewed as the stabilization of the space of holomorphic maps from X^{an} to

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Grassmann varieties. In particular, we show that the computations of F. Kirwan [20] of homology groups of spaces of maps from a Riemann surface C^{an} to Grassmann varieties has a stable interpretation as the assertion that the natural map $\mathcal{K}^{\mathrm{semi}}(C) \to \mathcal{K}_{\mathrm{top}}(C^{an})$ is a weak equivalence. One might expect that $K_*^{\mathrm{semi}}(X) \to K_{\mathrm{top}}^{-*}(X^{an})$ could also be an isomorphism for "cellular varieties" such as Grassmann varieties. We prove such an isomorphism in the special case in which X is a product of projective spaces.

We establish some fundamental properties for $K_*^{\mathrm{semi}}(X)$ whenever X is a projective complex variety. For example, we establish a version of Poincaré duality

$$K_*^{\mathrm{semi}}(X) \cong G_*^{\mathrm{semi}}(X) \qquad (C)$$

for smooth, projective varieties X, where $G_*^{\mathrm{semi}}(X)$ is defined in analogy with the K-theory $G_*(X) = K'_*(X)$ of coherent sheaves on X (cf. Theorem 4.7). Additionally, in section 5 we establish the "projective bundle formula" for K^{semi} – that is, there is a natural isomorphism

$$K_*^{\mathrm{semi}}(X)^{\times n} \cong K_*^{\mathrm{semi}}(\mathbb{P}(E)) \qquad (D),$$

where X is a projective variety and E is a rank n vector bundle on X. Equation (D) is closely related to the work of [5]. In that paper, the set $\mathrm{Hol}_k(\mathbb{P}^1, BU)$ of algebraic morphisms of degree k from $\mathbb{P}^1$ to BU (defined as a direct limit of certain Grassmann varieties) is endowed with the subspace topology of the space of all continuous maps from $\mathbb{P}^1$ to BU. It is proven in [5] that there is a natural homotopy equivalence

$$\mathrm{Hol}_k(\mathbb{P}^1, BU) \xrightarrow{\sim} BU(k),$$

a result which one can view as an unstable version of (D) in the special case of the trivial rank 2 vector bundle over a point.

Although most of our discussion of semi-topological K-theory applies only to complex varieties, we define $K_0^{\mathrm{semi}}(X)$ for X a quasi-projective variety over an arbitrary field. A natural challenge is the extension of $K_*^{\mathrm{semi}}(-)$ to varieties over an arbitrary field. As one would expect, our understanding of $K_0^{\mathrm{semi}}(-)$ for complex varieties is more complete than that for higher degrees. In particular, we establish in Theorem 1.4 that the rational Chern character is an isomorphism and we exhibit in Proposition 1.8 a portion of the Mayer-Vietoris long exact sequence for open covers.

We anticipate the semi-topological K-theory with rational coefficients of a smooth variety X should be isomorphic in all degrees via a Chern character to the morphic cohomology with rational coefficients of X. We also expect that semi-topological K-theory with finite coefficients should be closely related (perhaps equal) to algebraic K-theory with finite coefficients. One consequence of such an equality in the case of smooth curves would be a confirmation of the Quillen-Lichtenbaum Conjecture (as first formulated in [6]) for curves independent of any consideration of motivic cohomology. We further speculate that, at least for X smooth, $K_*^{\mathrm{semi}}(X)$ and $K_{\mathrm{top}}^*(X)$ become equal after inverting the action of the "Bott element" – i.e., the generator of $K_2^{\mathrm{semi}}(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Z}$, which acts on $K_*^{\mathrm{semi}}(X)$ under the (as of now hypothetical) cup product operation.

These speculations are summarized succinctly in diagram (A) by posing the rather bold conjectures that for a given smooth, quasi-projective complex variety

X the vertical arrows are rational isomorphisms, the left-hand horizontal arrows become isomorphisms upon taking finite coefficients, and the upper right-hand horizontal arrow becomes an isomorphism upon inverting the Bott element. We further conjecture that multiplication by the Bott element in semi-topological K-theory covers (and is thereby equal to, rationally) the "s-map" in morphic cohomology of [11; 4.5].

To give the reader a concrete description of $K_*^{\text{semi}}(X)$, we remark here that $K_0^{\text{semi}}(X)$ is the quotient of $K_0(X)$ by the equivalence relation determined by algebraic equivalence of bundles (cf. Definition 1.1). Further, in section 4 we establish an explicit description of $K_q^{\text{semi}}(X)$ when X is a projective complex variety. A generator of $K_q^{\text{semi}}(X)$ is shown to be an equivalence class consisting of a continuous map $g : S^q \to U^{\text{an}}$, for some complex variety U, together with an algebraic vector bundle E on $U \times X$ – loosely speaking, this amounts to a topological bundle on $S^q \times X$ which is algebraic over each point of S^q.

We should note that our $K_*^{\text{semi}}(X)$ appears to coincide for projective complex varieties with the "holomorphic K-theory" considered in [22], [24]. Additionally, an equivalent version of the right half of diagram (A) is considered in [24; §4].

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§1 $K_0^{\text{semi}}(X)$ and algebraic equivalence

When k is an algebraically closed field, recall that two algebraic cycles γ_0 and γ_1 on a k-variety X are said to be algebraically equivalent if there is a flat family of cycles indexed by a smooth, connected curve C (which one may take to be affine) such that γ_i is the cycle over c_i, for closed points c_0, c_1 of C (cf. [14; §10.3]).

Given the intimate relation between algebraic cycles and algebraic K-theory, it seems natural to introduce the notion of "algebraic equivalence of vector bundles". More precisely, we say that two vector bundles E_0, E_1 on a k-variety X, where k is algebraically closed, are *algebraically equivalent* provided there is a smooth, connected, affine curve C over k, closed points c_0, c_1 on C, and a vector bundle $\tilde{E}$ on $X \times C$ such that E_i is isomorphic to the restriction of $\tilde{E}$ to $X \times \{c_i\} \cong X$. The same argument as that found in [14; 10.3.2] shows this is an equivalence relation. More generally, a class α in $K_0(X)$ is said to be *algebraically equivalent to zero* if it can be written as $\alpha_1 - \alpha_0$, where for some smooth, affine curve C with closed points c_0 and c_1, there is a class $\beta \in K_0(X \times C)$ such that $\alpha_i = \iota_{c_i}^*(\beta)$. Here, $\iota_{c_i} : X \hookrightarrow X \times C$ is the closed immersion at c_i. One easily verifies that the elements algebraically equivalent to zero form a subgroup of $K_0(X)$. In other words, the elements of $K_0(X)$ algebraically equivalent to zero are precisely those that lie in the image of

$$K_0(X \times C) \xrightarrow{\iota_{c_0}^* - \iota_{c_1}^*} K_0(X)$$

for some (C, c_0, c_1).

When k is an arbitrary ground field (not necessarily algebraically closed), the correct notion of algebraic equivalence of cycles on a k-variety X is given in terms of $\mathrm{CH}_*(X)$, the group of cycles modulo rational equivalence. Namely, a cycle α in $\mathrm{CH}_*(X)$ is algebraically equivalent to zero provided there is a smooth, connected variety T with k-rational points t_0 and t_1 and an element $\tilde{\alpha}$ of $\mathrm{CH}_*(X \times T)$ such that $\alpha = \iota_{t_0}^!(\tilde{\alpha}) - \iota_{t_1}^!(\tilde{\alpha})$ [14; 10.3]. This suggests the following definition.

Definition 1.1. *For a quasi-projective variety X over an arbitrary ground field k, an element of $K_0(X)$ is algebraically equivalent to zero provided it lies in the image of*

$$K_0(X \times T) \stackrel{\iota_{t_0}^* - \iota_{t_1}^*}{\longrightarrow} K_0(X),$$

for some smooth, connected k-variety T with k-rational points t_0 and t_1. Define $K_0^{\mathrm{semi}}(X)$ to be the quotient of $K_0(X)$ by the subgroup of elements algebraically equivalent to zero.

For a smooth variety X, the homotopy invariance of K-theory allows one to think of $K_0(X)$ as the group of vector bundles on X modulo "rational equivalence" (i.e., the equivalence relation one gets by taking T to be the affine line in the definition of algebraic equivalence). More generally, we have a sequence of natural maps

$$K_0(X) \longrightarrow K_0(X)/\sim_{\mathrm{rat}} \longrightarrow K_0^{\mathrm{semi}}(X), \tag{1.1.1}$$

where the middle group is defined as

$$K_0(X)/\sim_{\mathrm{rat}} \equiv \mathrm{coker}(K_0(X \times \mathbb{A}^1) \stackrel{\iota_0^* - \iota_1^*}{\longrightarrow} K_0(X)).$$

and ι_ϵ is the evident closed immersion $X \equiv X \times \{\epsilon\} \hookrightarrow X \times \mathbb{A}^1$.

Functoriality of $K_0^{\mathrm{semi}}(X)$ is immediate: if $f : Y \to X$ is any morphism of quasi-projective varieties, then the diagram

$$\begin{array}{ccc} K_0(X \times T) & \xrightarrow{\iota_{t_0}^* - \iota_{t_1}^*} & K_0(X) \\ {\scriptstyle (f\times \mathrm{id})^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ K_0(Y \times T) & \xrightarrow{\iota_{t_0}^* - \iota_{t_1}^*} & K_0(Y) \end{array}$$

commutes for all (T, t_0, t_1). Thus f^* induces a map on quotients $f^* : K_0^{\mathrm{semi}}(X) \to K_0^{\mathrm{semi}}(Y)$. Furthermore, the maps of (1.1.1) clearly represent natural transformations of contravariant functors.

Additionally, $K_0^{\mathrm{semi}}(X)$ is easily seen to form a ring: if $\tilde{E}$ is a bundle on $T \times X$ and $t_0, t_1 \in T$ are rational points on the smooth, connected variety T, then for any bundle F on X, we have that $\tilde{E} \otimes \pi^*(F)$ (where $\pi : T \times X \to X$ is the projection map) gives a bundle on $T \times X$ relating $\tilde{E}|_{X\times\{t_0\}} \otimes F$ to $\tilde{E}|_{X\times\{t_1\}} \otimes F$. This shows the subgroup of elements algebraically equivalent to zero forms an ideal of $K_0(X)$.

Proposition 1.2. *If X is a smooth projective variety over an algebraically closed field k, then the image of the (set-theoretic) map $\mathrm{Pic}^0(X) \to K_0^{\mathrm{semi}}(X)$ is the single element $[\mathcal{O}_X]$.*

Consequently, if C is a smooth, connected complete curve over k, then $K_0^{\mathrm{semi}}(C) \cong \mathbb{Z} \oplus \mathbb{Z}$, and if C is a smooth, connected affine curve, then $K_0^{\mathrm{semi}}(C) \cong \mathbb{Z}$.

Proof. Let X be a smooth projective variety. Then $\mathrm{Pic}^0(X)$ is the collection of closed points on an abelian variety (also written as $\mathrm{Pic}^0(X)$). If L is a line bundle in $\mathrm{Pic}^0(X)$ and if $p, q \in \mathrm{Pic}^0(X)$ denote the closed points of $\mathrm{Pic}^0(X)$ corresponding to L and $\mathcal{O}_X$, then we may find a smooth, irreducible curve C with closed points c and d and a map $C \to \mathrm{Pic}^0(X)$ sending c to p and d to q. If $\mathcal{L} \to \mathrm{Pic}^0(X) \times X$

denotes the universal line bundle, then the pull-back of $\mathcal{L}$ to $C \times X$ provides an algebraic equivalence relating L to $\mathcal{O}_X$. This establishes the first assertion.

If C is a smooth, connected complete curve over k, then $K_0(C) \cong \mathrm{Pic}(C) \oplus \mathbb{Z}$ and $\mathrm{Pic}^0(C)$ is the kernel of the degree map $\mathrm{Pic}(C) \twoheadrightarrow \mathbb{Z}$. Since the degree map is easily seen to respect algebraic equivalence, the first equation follows.

If C is a smooth, connected affine curve with smooth projective closure $\overline{C}$, then $\mathrm{Pic}^0(\overline{C}) \to \mathrm{Pic}(C)$ is surjective and $K_0(C) = \mathrm{Pic}(C) \oplus \mathbb{Z}$. Thus, the second formula follows from the first. □

For a variety X, recall that $\mathrm{CH}_*(X) \equiv \bigoplus_k \mathrm{CH}_k(X)$ refers to the Chow group of algebraic cycles modulo rational equivalence. We write $A_*(X) \equiv \bigoplus A_k(X)$ for the group of algebraic cycles modulo algebraic equivalence, which is naturally a graded quotient group of $\mathrm{CH}_*(X)$.

Proposition 1.3. *Let X be a quasi-projective variety. The composition of the total Chern class map $c_* : K_0(X) \to \mathrm{CH}_*(X)$ and the quotient map $\mathrm{CH}_*(X) \to A_*(X)$ factors through $K_0^{\mathrm{semi}}(X)$. In other words, there is a natural commutative square*

$$\begin{array}{ccc} K_0(X) & \longrightarrow & K_0^{\mathrm{semi}}(X) \\ {\scriptstyle c_*}\downarrow & & {\scriptstyle c_*}\downarrow \\ \mathrm{CH}_*(X) & \longrightarrow & A_*(X) \end{array}$$

with surjective horizontal maps. Similarly, there is a commutative square

$$\begin{array}{ccc} K_0(X) & \longrightarrow & K_0^{\mathrm{semi}}(X) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ \mathrm{CH}_*(X) \otimes \mathbb{Q} & \longrightarrow & A_*(X) \otimes \mathbb{Q}, \end{array}$$

where ch *is the Chern character.*

Proof. To establish the first commutative square, it suffices to show for any smooth variety T with closed points t_0, t_1 that the composition

$$K_0(X \times T) \xrightarrow{\iota_{t_0}^* - \iota_{t_1}^*} K_0(X) \xrightarrow{c_*} \mathrm{CH}_*(X) \longrightarrow A_*(X)$$

is zero. But the map $K_0(X \times T) \longrightarrow \mathrm{CH}_*(X)$ factors through $\mathrm{CH}_*(X \times T) \xrightarrow{\iota_{t_1}^! - \iota_{t_0}^!} \mathrm{CH}_*(X)$, since for a vector bundle E on $X \times T$ we have $\iota_{t_\epsilon}^! c_n(E) = c_n(\iota_{t_\epsilon}^* E)$ in $\mathrm{CH}_*(X)$ (cf. [14; 6.3]).

The existence of the second square follows immediately from the first, since the Chern character of a bundle is defined as a certain rational polynomial in the Chern classes. □

One consequence of the following theorem is the non-triviality of $K_0^{\mathrm{semi}}(X)$ whenever there is a cycle on X no multiple of which is algebraically equivalent to 0. The proof is an application of Grothendieck's Riemann-Roch Theorem [34; Exposé 0].

Theorem 1.4. *If X is a smooth quasi-projective variety over k, then the Chern character map*

$$\mathrm{ch}_* : K_0^{\mathrm{semi}}(X) \otimes \mathbb{Q} \xrightarrow{\cong} A_*(X) \otimes \mathbb{Q}$$

is an isomorphism

Proof. The map $\mathrm{ch}_* : K_0(X) \otimes \mathbb{Q} \to \mathrm{CH}_*(X) \otimes \mathbb{Q}$ is a natural isomorphism on the category of smooth varieties by the Grothendieck-Riemann-Roch Theorem [34; Exposé 0]. Thus there is a commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{(T,t_0,t_1)} K_0(X \times T)_{\mathbb{Q}} & \xrightarrow{\iota_{t_1}^* - \iota_{t_0}^*} & K_0(X)_{\mathbb{Q}} & \longrightarrow & K_0^{\mathrm{semi}}(X)_{\mathbb{Q}} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \oplus \mathrm{ch}_*} & & \Big\downarrow{\scriptstyle \mathrm{ch}_*} & & \Big\downarrow{\scriptstyle \mathrm{ch}_*} & & \\
\bigoplus_{(T,t_0,t_1)} \mathrm{CH}_*(X \times T)_{\mathbb{Q}} & \xrightarrow{\iota_{t_1}^! - \iota_{t_0}^!} & \mathrm{CH}_*(X)_{\mathbb{Q}} & \longrightarrow & A_*(X)_{\mathbb{Q}} & \longrightarrow & 0
\end{array}$$

(where $M_{\mathbb{Q}}$ denotes $M \otimes \mathbb{Q}$) whose rows are exact. (Here, (T, t_0, t_1) ranges over smooth varieties T with rational points t_0 and t_1.) Since the left two vertical arrows are isomorphisms, the right most vertical arrow must be an isomorphism as well. $\square$

Recall that Atiyah and Hirzebruch [1] establish an isomorphism

$$\mathrm{ch}_* : K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{H}^{ev}(X^{\mathrm{an}}, \mathbb{Q}).$$

This is used in the following example.

Example 1.5. *If E is a sufficiently general abelian 3-fold over $\mathbb{C}$, then the kernel of*

$$K_0^{\mathrm{semi}}(E) \otimes \mathbb{Q} \longrightarrow K^0_{\mathrm{top}}(E^{\mathrm{an}}) \otimes \mathbb{Q}$$

is infinitely generated [4].

More generally, for any smooth quasi-projective variety X over $\mathbb{C}$,

$$\ker\left(K_0^{\mathrm{semi}}(X) \otimes \mathbb{Q} \longrightarrow K^0_{\mathrm{top}}(X^{\mathrm{an}}) \otimes \mathbb{Q}\right)$$

is isomorphic to the rational Griffiths group [4] *(i.e., the result of tensoring with $\mathbb{Q}$ the group of algebraic equivalence classes of algebraic cycles homologically equivalent to 0).*

The motivation for defining K_0^{semi} as the quotient of $K_0(X)$ given by algebraic equivalence leads us to also consider $G_0^{\mathrm{semi}}(X)$. This is defined analogously to $K_0^{\mathrm{semi}}(X)$ by starting with $G_0(X)$ (also written $K_0'(X)$), the Grothendieck group of coherent sheaves on X, in place of $K_0(X)$. In fact, we shall require a sort of bivariant group; namely, let $\mathcal{M}(X, Y)$ denote the exact category of coherent $\mathcal{O}_{X \times Y}$-modules with are flat over X, and write $G_0(X, Y)$ for the associated Grothendieck group. Observe that we recover $K_0(X)$ by setting $Y = \operatorname{Spec} k$ and we recover $G_0(Y)$ by setting $X = \operatorname{Spec} k$. There is an evident notion of algebraic equivalence for elements of $G_0(X, Y)$:

Definition 1.6. *For quasi-projective varieties X and Y over a field k, we say an element of $G_0(X, Y)$ is algebraically equivalent to zero if it lies in the image of*

$$\iota_{t_1}^* - \iota_{t_0}^* : G_0(X \times T, Y) \longrightarrow G_0(X, Y),$$

for some smooth variety T with rational points t_0, t_1. We define $G_0^{\mathrm{semi}}(X, Y)$ to be the quotient of $G_0(X, Y)$ by the subgroup of elements algebraically equivalent to zero. We define $G_0^{\mathrm{semi}}(Y)$ to be $G_0^{\mathrm{semi}}(\operatorname{Spec} k, Y)$.

Observe that the group $G_0^{\mathrm{semi}}(Y)$ is the quotient of $G_0(Y)$ by the subgroup

$$\cup_{(T,t_0,t_1)} \operatorname{im}\left(\iota_{t_0}^* - \iota_{t_1}^* : G_0(T, Y) \longrightarrow G_0(Y)\right),$$

where the union ranges over smooth varieties T with rational points t_0, t_1. Since T is smooth, the natural map $G_0(T, Y) \longrightarrow G_0(T \times Y)$ is an isomorphism by Grothendieck's Resolution Theorem for K_0 [2; VIII.4.2]. From this description, contravariant functoriality of $G_0^{\mathrm{semi}}(-)$ for flat morphisms is immediate. If $Y \longrightarrow Y'$ is a proper morphism, then for any smooth variety T, the diagram

$$\begin{array}{ccc} G_0(T, Y) & \longrightarrow & G_0(Y) \\ \downarrow & & \downarrow \\ G_0(T, Y') & \longrightarrow & G_0(Y') \end{array}$$

is seen to commute by applying [29; Proposition 2.11], using the fact that $\{c\} \times Y'$ and $T \times Y$ are Tor-independent over $T \times Y'$. Thus G_0^{semi} is covariantly functorial for proper morphisms.

It is easily established that $G_0^{\mathrm{semi}}(X, Y)$ is a module over the ring $K_0^{\mathrm{semi}}(X)$ under tensor product of $\mathcal{O}_X$-modules.

Proposition 1.7. *For quasi-projective varieties X and Y, there is a natural map $K_0^{\mathrm{semi}}(X \times Y) \to G_0^{\mathrm{semi}}(X, Y)$ which fits in a commutative square*

$$\begin{array}{ccc} K_0(X \times Y) & \longrightarrow & K_0^{\mathrm{semi}}(X \times Y) \\ \downarrow & & \downarrow \\ G_0(X \times Y) & \longrightarrow & G_0^{\mathrm{semi}}(X, Y). \end{array} \tag{1.6.1}$$

If Y is smooth, then the map $K_0^{\mathrm{semi}}(X \times Y) \to G_0^{\mathrm{semi}}(X, Y)$ is an isomorphism.

Proof. It is immediate from the definitions that algebraically equivalent elements of $K_0(X \times Y)$ are sent to algebraically equivalent elements in $G_0(X, Y)$ under the natural map, and thus $K_0^{\mathrm{semi}}(X \times Y) \to G_0^{\mathrm{semi}}(X, Y)$ is uniquely defined so that (1.6.1) commutes.

If Y is smooth, then $K_0(X \times Y) \to G_0(X, Y)$ and $K_0(T \times X \times Y) \to G_0(T \times X, Y)$ are isomorphisms for any smooth T by Grothendieck's Resolution Theorem [2; VIII.4.2]. The result follows immediately. ☐

We conclude this section with the beginnings of a Mayer-Vietoris long exact sequence.

Proposition 1.8. *Let X be a smooth quasi-projective variety over k and let U, V be Zariski open subsets of X with the property that $X = U \cup V$. Then there is a natural exact sequence*

$$K_0^{\text{semi}}(X) \to K_0^{\text{semi}}(U) \oplus K_0^{\text{semi}}(V) \to K_0^{\text{semi}}(U \cap V) \to 0. \tag{1.7.1}$$

Proof. We consider the commutative diagram

$$\begin{array}{ccccccc}
K_0(T \times X) & \longrightarrow & K_0(T \times U) \oplus K_0(T \times V) & \longrightarrow & K_0(T \times (U \cap V)) & \longrightarrow & 0 \\
\iota_{t_0}^* - \iota_{t_1}^* \downarrow & & (\iota_{t_0}^* - \iota_{t_1}^*) \oplus (\iota_{t_0}^* - \iota_{t_1}^*) \downarrow & & \iota_{t_0}^* - \iota_{t_1}^* \downarrow & & \\
K_0(X) & \longrightarrow & K_0(U) \oplus K_0(V) & \longrightarrow & K_0(U \cap V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_0^{\text{semi}}(X) & \longrightarrow & K_0^{\text{semi}}(U) \oplus K_0^{\text{semi}}(V) & \longrightarrow & K_0^{\text{semi}}(U \cap V) & \longrightarrow & 0,
\end{array}$$

for any (T, t_0, t_1). Observe that the top two rows of this diagram are exact by Mayer-Vietoris for algebraic K-theory [29; 3.5]. The lower vertical arrows are surjective, which shows that the bottom row is a chain complex and is surjective on the right. Since $K_0^{\text{semi}}(Y)$ may be defined for any Y as

$$K_0(Y) / \sum_{(T,t_0,t_1)} \operatorname{im}(\iota_{t_0}^* - \iota_{t_1}^* : K_0(T \times Y) \longrightarrow K_0(Y)),$$

exactness at $K_0^{\text{semi}}(U) \oplus K_0^{\text{semi}}(V)$ follows from an easy diagram chase. □

§2 Operads, Grassmannians, and Chow varieties

In this section, we construct the infinite loop space $\mathcal{K}^{\text{semi}}(X)$ whose group of connected components equals $K_0^{\text{semi}}(X)$ for any weakly normal, quasi-projective complex variety X. We also consider parallel constructions of $\mathcal{C}(X)$ and $\mathcal{G}_Y^{\text{semi}}(X)$ of a theory based on Chow varieties and a bivariant theory generalizing $\mathcal{K}^{\text{semi}}(X)$. The technique we employ to construct these infinite loop spaces involves the notion of an E_∞ operad (cf. [25], [27]). Specifically, we shall construct spaces which are "algebras" for an operad we write as $\mathcal{I}^{\text{an}}$. The operad $\mathcal{I}^{\text{an}}$ is closely related to the classical linear isometries operad.

An explanation for the need for the weakly normal hypotheses seems in order at this point. Recall that a variety X is "weakly normal" provided any map $\tilde{X} \longrightarrow X$ which is a universal homeomorphism (that is, a map which after arbitrary base change induces a homeomorphism on underlying topological spaces) is an isomorphism. Over a field of characteristic zero, we may associate to any variety X its weak normalization, written $X^w \to X$, which has the properties that $X^w \to X$ is a universal homeomorphism and X^w is weakly normal. We refer the reader to [33] for details. (Note that the notion of weak normalization and semi-normalization coincide in characteristic 0.)

Weak normality becomes important for us in that our constructions will use the set $\operatorname{Mor}(X, Y)$ of "continuous algebraic maps" between varieties X and Y. Such a map is defined to be a graph – i.e., a reduced, closed subscheme of $X \times Y$ – such that the projection to X is a universal homeomorphism. Thus, if X is weakly normal, the set $\mathrm{Mor}(X, Y)$ coincides with $\mathrm{Hom}(X, Y)$, the set of morphisms of varieties. In characteristic zero, there is a natural isomorphism $\mathrm{Mor}(X, Y) \cong \mathrm{Hom}(X^w, Y)$.

We now begin the construction of the promised infinite loop spaces. Eventually, we will need to work over the complex numbers, but at first we may consider any ground field k. Let $\mathrm{Grass}_m(\mathbb{P}^N)$ denote the projective k-variety of quotient m-planes of k^{N+1} for any $0 \leq m \leq N$. Also let $C_{r,d}(\mathbb{P}^N)$ denote the Chow variety of effective r-cycles of degree d in $\mathbb{P}^N$ for any $r \geq -1$, $d \geq 0$. Here, $\mathcal{C}_{-1,d}(\mathbb{P}^N)$ is by definition $\mathrm{Spec}\, k$ and represents the "empty cycle of degree d". For each fixed r, the scheme $\coprod_{d\in\mathbb{N}} \mathcal{C}_{r,d}(\mathbb{P}^N)$ is an abelian monoid scheme under addition of cycles, with the case $r = -1$ giving the discrete abelian monoid scheme $\mathbb{N}$. Furthermore, we let $I(n)_{N,M}$ denote the affine variety of $M \times nN$ matrices over k of rank nN, which we view as the variety parameterizing n-tuples of linear maps $(f_1 : k^N \to k^M, \dots, f_n : k^N \to k^M)$ such that $\Sigma(f_1, \dots, f_n) : (k^N)^{\oplus n} \equiv k^{nN} \to k^M$ is injective. (To avoid confusion, given maps $g_i : A_i \to B$, $1 \leq i \leq n$, of abelian groups we will use $\Sigma(g_1, \dots, g_n)$ to refer to the map $\bigoplus_i A_i \to B$ sending $(a_1, \dots, a_n)$ to $\Sigma g_i(a_i)$.) There are natural transition maps

$$\mathrm{Grass}_m(\mathbb{P}^{N-1}) \hookrightarrow \mathrm{Grass}_m(\mathbb{P}^N), \quad C_{r,d}(\mathbb{P}^N) \hookrightarrow C_{r,d}(\mathbb{P}^{N+1})$$

given by the projection $k^{N+1} \twoheadrightarrow k^N$ onto the first N standard basis elements and the inclusion $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N+1}$ into the first $N + 1$ homogeneous coordinates, respectively. Also, we have two compatible transition maps

$$I(n)_{N+1,M} \to I(n)_{N,M}, \quad I(n)_{N,M} \hookrightarrow I(n)_{N,M+1},$$

which are defined by

$$(f_1 : k^{N+1} \hookrightarrow k^M, \dots, f_n : k^{N+1} \hookrightarrow k^M) \mapsto (f_1|_{k^N}, \dots, f_n|_{k^N})$$

and by composition with the inclusion $k^M \hookrightarrow k^{M+1}$, respectively.

Definition 2.1. *We define the following functors on k-schemes:*

$$\begin{aligned} K^\infty(X) &\equiv \coprod_{m\geq 0} \varinjlim_{N>0} \mathrm{Mor}(X, \mathrm{Grass}_m(\mathbb{P}^N)) \\ \mathcal{C}^\infty(X) &\equiv \coprod_{r\geq -1, d\geq 0} \varinjlim_{N>0} \mathrm{Mor}(X, C_{r,d}(\mathbb{P}^N)) \\ \mathcal{I}(n)(X) &\equiv \varprojlim_{N>0} \varinjlim_{M>0} \mathrm{Mor}(X, I(n)_{N,M}), \quad n \geq 0. \end{aligned}$$

Here, $\mathcal{I}(0)$ is taken to mean the one point functor.

We view $\mathcal{I}(1)(X)$ (for X weakly normal) as the set of $k[X]$-linear endomorphisms of $k[X]^\infty$ (where $k[X]$ denotes the ring of global functions on X) which are admissible injections. In other words, this is the set of infinite-by-infinite matrices with entries in $k[X]$ such that every column has only finitely many non-zero elements and every submatrix consisting of a finite collection of columns has full rank. Similarly, we view $\mathcal{I}(n)(X)$ as the set of n-tuples $(f_1, \dots, f_n)$ of $k[X]$-linear endomorphisms with the property that

$$\Sigma(f_1, \dots, f_n) : (k[X]^\infty)^{\oplus n} \to k[X]^\infty$$

is an admissible injection (i.e., every matrix formed by choosing finitely many columns from each of the f_i's has full rank).

The utility of the functor $\mathcal{I}(n)$ is that elements of $\mathcal{I}(n)(X)$ allow us to "move" an n-tuple of quotients of $\mathcal{O}_X^\infty$ (respectively, an n-tuple of effective cycles in $X \times \mathbb{P}^\infty$) into general position so that their direct sum (respectively, fiberwise join) is well-defined. This idea is made rigorous in the following proposition.

Proposition 2.2. *For all $n \geq 0$, there exist pairings of contravariant functors on k-schemes*

$$\mathcal{I}(n) \times (K^\infty)^{\times n} \longrightarrow K^\infty,$$

associated to direct sum of locally free coherent sheaves, and

$$\mathcal{I}(n) \times (\mathcal{C}^\infty)^{\times n} \longrightarrow \mathcal{C}^\infty,$$

associated to fibre-wise join of effective cycles. Moreover, each of these pairings is induced by continuous algebraic maps of the systems of varieties which represent these functors.

Proof. When $n = 0$, the pairings are given by the base points of K^∞ and $\mathcal{C}^\infty$: the zero quotient $\mathcal{O}^\infty \twoheadrightarrow 0$ and the empty cycle of degree one $\mathcal{C}_{-1,1}(\mathbb{P}^\infty) \equiv \operatorname{Spec} k$.

Write $\mathrm{Grass}(\mathbb{P}^N)$ for the scheme $\coprod_m \mathrm{Grass}_m(\mathbb{P}^N)$. We have pairings of schemes

$$\mathcal{I}(n)_{N+1,M+1} \times \mathrm{Grass}(\mathbb{P}^N) \times \cdots \times \mathrm{Grass}(\mathbb{P}^N) \longrightarrow \mathrm{Grass}(\mathbb{P}^M), \tag{2.2.1}$$

defined (in terms of the functors they represent) as follows. Send the tuple

$$(f_1 : k[X]^{N+1} \hookrightarrow k[X]^{M+1}, \dots, f_n : k[X]^{N+1} \hookrightarrow k[X]^{M+1},$$
$$p_1 : \mathcal{O}_X^{N+1} \twoheadrightarrow E_1, \dots, p_n : \mathcal{O}_X^{N+1} \twoheadrightarrow E_n),$$

where each E_i is a locally free coherent sheaf, to the quotient object given by the composition of

$$\mathcal{O}_X^{M+1} \xrightarrow{\Sigma(f_1,\dots,f_n)^\wedge} (\mathcal{O}_X^{N+1})^n \to E_1 \oplus \cdots \oplus E_n.$$

Here, $\Sigma(f_1, \dots, f_n)^\wedge$ is the linear dual of $\Sigma(f_1, \dots, f_n)$.

One easily verifies that the pairing (2.2.1) is compatible with the transition maps so that we have commutative squares

$$\begin{array}{ccc} \mathcal{I}(n)_{N+1,M+1} \times \mathrm{Grass}(\mathbb{P}^N)^{\times n} & \longrightarrow & \mathrm{Grass}(\mathbb{P}^M) \\ \downarrow & & \downarrow \\ \mathcal{I}(n)_{N,M+1} \times \mathrm{Grass}(\mathbb{P}^{N-1})^{\times n} & \longrightarrow & \mathrm{Grass}(\mathbb{P}^M) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{I}(n)_{N+1,M} \times \mathrm{Grass}(\mathbb{P}^N)^{\times n} & \longrightarrow & \mathrm{Grass}(\mathbb{P}^{M-1}) \\ \downarrow & & \downarrow \\ \mathcal{I}(n)_{N+1,M+1} \times \mathrm{Grass}(\mathbb{P}^N)^{\times n} & \longrightarrow & \mathrm{Grass}(\mathbb{P}^M), \end{array}$$

which are compatible with each other in the sense that the evident cubical diagram commutes. Taking the direct limit over all M and the inverse limit over all N, we obtain the desired pairing of functors

$$\mathcal{I}(n) \times (K^\infty)^{\times n} \longrightarrow K^\infty.$$

As argued in [13; 1.1] (where characteristic 0 is assumed and a stronger result is proven), a natural transformation of contravariant functors on smooth schemes represented by algebraic varieties induces a continuous algebraic map between these varieties. We therefore establish pairings of varieties (given by a continuous algebraic map)

$$I(n)_{N+1,M+1} \times \mathcal{C}_{r_1,d_1}(\mathbb{P}^N) \times \cdots \times \mathcal{C}_{r_n,d_n}(\mathbb{P}^N) \longrightarrow \mathcal{C}_{r,d}(\mathbb{P}^M), \tag{2.2.2}$$

where $r = r_1 + \cdots + r_n + n - 1$ and $d = d_1 \cdot \cdots \cdot d_n$, by defining them in terms of the functors on the category of smooth k-schemes that they represent as follows. Given a smooth variety X and a tuple

$$\left(f_1 : k[X]^{N+1} \hookrightarrow k[X]^{M+1}, \ldots, f_n : k[X]^{N+1} \hookrightarrow k[X]^{M+1}, \gamma_1, \ldots, \gamma_n\right),$$

where γ_i is an effective cycle on $X \times \mathbb{P}^N$ of dimension r_i and degree d_i, let $f_{i*} : X \times \mathbb{P}^N \hookrightarrow X \times \mathbb{P}^M$ be the closed immersion induced by f_i. Let $f_{i*}(\gamma_i)$ denote the effective cycle in $\mathbb{P}^M$ given by pushforward along f_{i*}. Finally, send this tuple to the effective cycle

$$f_{1*}(\gamma_1) \#_X \cdots \#_X f_{n*}(\gamma_n)$$

in $X \times \mathbb{P}^{nM+n-1}$, where $\#_X$ denotes fiberwise join over X. Observe that the linear subspaces $f_{i*}(\mathbb{P}^N)$ are in general position (in the sense that the linear span of all but one of these spaces is disjoint from the remaining space) and thus this fiberwise join is well-behaved. By convention, the join of any given cycle γ with the empty cycle of degree d (i.e., the unique point of $\mathcal{C}_{-1,d}(\mathbb{P}^\infty)$) is $d\gamma$.

The pairings (2.2.2) are compatible with the transition maps as before, and so by taking inverse and direct limits, we obtain the desired pairing of functors

$$\mathcal{I}(n) \times (\mathcal{C}^\infty)^{\times n} \longrightarrow \mathcal{C}^\infty.$$

□

We now introduce a generalization G_Y^∞ of K^∞ in which the rôle of the Grassmannians is replaced by the Quot-schemes G_Y^r of Grothendieck for a projective variety Y [18].

Definition 2.3. *For Y a projective variety over a field k and $r > 0$, we write G_Y^r for the disjoint union of projective varieties representing the functor*

$$S \mapsto \{\textit{quotient objects } \mathcal{O}_{S\times Y}^r \twoheadrightarrow M \mid M \textit{ is a coherent sheaf on } S \times Y \textit{ flat over } S\},$$

and we define G_Y^∞ to be the functor

$$X \mapsto \{\mathcal{O}_{X\times Y}^\infty \twoheadrightarrow M \mid M \textit{ is a coherent sheaf on } X \times Y \textit{ flat over } X\},$$

where by a quotient of the form $\mathcal{O}_{X\times Y}^\infty \twoheadrightarrow M$, we mean one which has all but finitely many standard basis elements in its kernel. Thus, G_Y^∞ is ind-representable by the ind-variety $\varinjlim_{r>0} G_Y^r$.

Observe that K^∞ coincides with $G_{\operatorname{Spec} k}^\infty$. We will continue to explicitly discuss K^∞ even though it is subsumed by G_Y^∞, because we view K^∞ as the object of primary interest.

Proposition 2.4. *For any projective variety Y over a field k and $n \geq 0$, there is a natural pairing of functors on k-schemes*

$$\mathcal{I}(n) \times (G_Y^\infty)^{\times n} \to G_Y^\infty.$$

Proof. We have pairings

$$\mathcal{I}(n)_{N+1,M+1} \times (G_Y^N)^{\times n} \longrightarrow G_Y^M \tag{2.4.1}$$

defined in precisely the same manner as were the pairings (2.2.1), and these pairing are compatible with the transition maps, just as were the pairings (2.2.1). Taking the direct limit over M and the inverse limit over N, we obtain the desired pairing. □

We now assume $k = \mathbb{C}$ in order to realize sets such as $\mathrm{Mor}(X, \mathrm{Grass}_n(\mathbb{P}^N))$ as topological spaces; these spaces arise naturally using the results of [13]. Let $(Sm/\mathbb{C})_{\leq 1}$ denote the category of smooth, affine varieties essentially of finite type over $\mathbb{C}$ having Krull dimension at most one (so that objects of $(Sm/\mathbb{C})_{\leq 1}$ are localizations of smooth affine varieties over $\mathbb{C}$). For quasi-projective complex varieties X, Y we let $\mathcal{M}or(X,Y)$ denote the functor

$$\mathcal{M}or(X,Y) : (Sm/\mathbb{C})_{\leq 1} \to (sets)$$

from the category $(Sm/\mathbb{C})_{\leq 1}$ to the category of sets, which sends $C \in (Sm/\mathbb{C})_{\leq 1}$ to the set $\mathrm{Mor}(C \times X, Y)$ of continuous algebraic maps from $C \times X$ to Y. When X is weakly normal, $\mathcal{M}or(X,Y)$ is nothing more than the functor $\mathrm{Hom}(- \times X, Y)$ on $(Sm/\mathbb{C})_{\leq 1}$. Furthermore, as shown in [13], the functor $\mathcal{M}or(X,Y)$ admits a "proper, constructible presentation" – that is, this functor is representable by a disjoin union of constructible subsets of projective space modulo a proper equivalence relation. Consequently, by [13; 2.4], the set $\mathrm{Mor}(X,Y)$ admits a natural topology and we write this topological space $\mathcal{M}or(X,Y)^{\mathrm{an}}$. The topology of $\mathcal{M}or(X,Y)^{\mathrm{an}}$ is such that a sequence of continuous algebraic maps f_i converges to f if and only if the sequence of associated continuous maps $f_i^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ converges to f^{an} (in the compact-open topology) and the degrees of the graphs of the f_i's (with respect to a chosen locally closed embedding $X \times Y \hookrightarrow \mathbb{P}^M$) are bounded (cf. [10; A3]). The results of [13] will be especially useful in this paper for establishing the continuity of maps between spaces such as $\mathcal{M}or(X,Y)^{\mathrm{an}}$.

In particular, we have the topological spaces

$$\mathcal{M}or(X, \mathrm{Grass}_m(\mathbb{P}^N))^{\mathrm{an}}, \quad \mathcal{M}or(X, \mathcal{C}_{r,d}(\mathbb{P}^N))^{\mathrm{an}}, \quad \mathcal{M}or(X, G_Y^N)^{\mathrm{an}}, \quad \text{and} \quad I(n)_{N,M}^{\mathrm{an}}.$$

(Recall that G_Y^N is an infinite disjoint union of varieties indexed by Hilbert polynomials.)

Definition 2.5. *For any quasi-projective complex variety X and any projective complex variety Y, we define topological spaces*

$$\begin{aligned}
K^\infty(X)^{\mathrm{an}} &\equiv \coprod_{m \geq 0} \varinjlim_{N>0} \mathcal{M}or(X, \mathrm{Grass}_m(\mathbb{P}^N))^{\mathrm{an}} \\
\mathcal{C}^\infty(X)^{\mathrm{an}} &\equiv \coprod_{r \geq -1, d \geq 0} \varinjlim_{N>0} \mathcal{M}or(X, C_{r,d}(\mathbb{P}^N))^{\mathrm{an}} \\
G_Y^\infty(X)^{\mathrm{an}} &\equiv \varinjlim_{N>0} \mathcal{M}or(X, G_Y^N)^{\mathrm{an}} \\
\mathcal{I}(n)^{\mathrm{an}} &\equiv \varprojlim_{N} \varinjlim_{M} \mathcal{I}(n)_{N,M}^{\mathrm{an}}.
\end{aligned}$$

Proposition 2.6. *Let X be a quasi-projective complex variety. Then, for any $n \geq 0$ and any projective complex variety Y, there are natural continuous pairings*

$$\mathcal{I}(n)^{an} \times (K^\infty(X)^{an})^{\times n} \longrightarrow K^\infty(X)^{an}$$
$$\mathcal{I}(n)^{an} \times (\mathcal{C}^\infty(X)^{an})^{\times n} \longrightarrow \mathcal{C}^\infty(X)^{an}$$
$$\mathcal{I}(n)^{an} \times (G_Y^\infty(X)^{an})^{\times n} \longrightarrow G_Y^\infty(X)^{an}.$$

Proof. Each of these pairing is induced by taking an inverse limit of a direct limit of pairings of functors (i.e., the pairings (2.2.1), (2.2.2), and (2.4.1)). Since all of the functors involved admit proper, constructible presentations, these three pairings induce continuous pairings on the associated topological spaces by [13; 2.3]. □

Returning briefly to an arbitrary base field k, we next observe that there are natural pairings of functors

$$\theta(k; n_1, \dots, n_t) : \ \mathcal{I}(t) \times \mathcal{I}(n_1) \times \cdots \times \mathcal{I}(n_t) \to \mathcal{I}(n_1 + \cdots + n_t)$$

which satisfy the associative, equivariant, and unital conditions required for an operad in the sense of May [25; §1]. Namely, $\theta(k; n_1, \dots, n_t)$ sends

$$(g_1, \cdots, g_t); (f_1^1, \cdots, f_{n_1}^1), \dots, (f_1^t, \cdots, f_{n_t}^t)$$

to the composition

$$\Sigma(g_1, \dots, g_t) \circ \left(\Sigma(f_1^1, \dots, f_{n_1}^1), \dots, \Sigma(f_1^t, \dots, f_{n_t}^t)\right) :$$
$$(k[X]^\infty)^{\times n_1 + \cdots + n_t} = (k[X])^{\times n_1} \times \cdots \times (k[X]^\infty)^{\times n_t} \to (k[X]^\infty)^{\times t} \to (k[X]^\infty).$$

The action of Σ_n on $\mathcal{I}(n)$ is given by permuting the entries in the evident manner, and the unit element of $\mathcal{I}(1)$ is the identity map on k^∞.

To further clarify the rôle of $\mathcal{I}(n)$ in this paper, it might be helpful to recall the "linear isometries operad" of [27]. This is defined by setting $\mathcal{L}(n)$ to be the collection of linear isometric embeddings of $(\mathbb{C}^\infty)^{\oplus n}$ into $\mathbb{C}^\infty$, where $\mathbb{C}^\infty$ is topologized as the inductive limit of its finite dimensional subspaces and is endowed with the evident Hermitian inner product. The topology on $\mathcal{L}(n)$ is the subspace topology of the compactly generated compact-open topology on the collection of all continuous maps from $(\mathbb{C}^\infty)^{\oplus n}$ to $\mathbb{C}^\infty$. That is, $\mathcal{L}(n)$ is a subspace of the space of maps $\mathcal{M}\text{aps}((\mathbb{C}^\infty)^{\oplus n}, \mathbb{C}^\infty) \cong \varprojlim_N \mathcal{M}\text{aps}((\mathbb{C}^N)^{\oplus n}, \mathbb{C}^\infty)$. Since any linear map from a finite dimensional vector space to $\mathbb{C}^\infty$ factors through $\mathbb{C}^M$ for $M >> 0$, we conclude that $\mathcal{L}(n)$ may be identified with a subspace of $\varprojlim_N \varinjlim_M \mathcal{M}\text{aps}((\mathbb{C}^{\mathbb{N}})^{\oplus n}, \mathbb{C}^M)$. Recall that $\mathcal{I}(n)^{\text{an}}$ was defined as $\varprojlim_N \varinjlim_M \mathcal{I}(n)^{\text{an}}_{N,M}$. Since $\mathcal{I}(n)^{\text{an}}_{N,M}$ is clearly a subspace of $\mathcal{M}\text{aps}(\mathbb{C}^{N^{\oplus n}}, \mathbb{C}^M)$, we see that $\mathcal{L}(n)$ is naturally a subspace of $\mathcal{I}(n)^{\text{an}}$. The collection of spaces $\mathcal{L}(n)$ comes with associated pairings $\theta(n_1, \dots, n_t)$, each $\mathcal{L}(n)$ is acted upon by Σ^n, and $\mathcal{L}(1)$ has a unit element. These structures are defined just as for the collection of spaces $\mathcal{I}(n)^{\text{an}}$ and satisfy the same associative, equivariant, and unital conditions as do the spaces $\mathcal{I}(n)^{\text{an}}$. Additionally, for all n, the space $\mathcal{L}(n)$ is contractible and is acted upon freely by Σ_n, making the collection $\mathcal{L}(n)$, $n \geq 0$, into an "E_∞ operad". We now show these latter two properties hold for $\mathcal{I}(n)^{\text{an}}$ as well.

Let $\Delta_k^\bullet$ denote the standard algebraic simplicial object, which in degree n is the variety $\Delta_k^n \equiv \text{Spec}\,(k[x_0, \dots, x_n]/(\sum_i x_i - 1))$. By $\mathcal{I}(n)(\Delta_k^\bullet)$, we mean the simplicial set $d \mapsto \mathcal{I}(n)(\Delta_k^d)$. If no subscript is given, assume $\Delta^\bullet = \Delta_{\mathbb{C}}^\bullet$.

Proposition 2.7. *(cf. [26]) For any $n \geq 0$ and field k, $\mathcal{I}(n)(\Delta_k^\bullet)$ is a contractible simplicial set on which the permutation group Σ_n acts freely. Thus, the sequence $\mathcal{I}(\Delta_k^\bullet) \equiv \{\mathcal{I}(n)(\Delta_k^\bullet); n \in \mathbb{N}\}$ of simplicial sets is a simplicial E_∞ operad.*

Moreover, each space $\mathcal{I}(n)^{an}$ is contractible with a free Σ_n action, so that the sequence of spaces $\mathcal{I}^{\mathrm{an}} \equiv \{\mathcal{I}(n)^{an}; n \in \mathbb{N}\}$ is a (topological) E_∞ operad. Additionally, $\mathcal{I}^{\mathrm{an}}$ contains $\mathcal{L}$ as a sub-operad.

Proof. The freeness of the Σ_n action on $\mathcal{I}(n)(\Delta^\bullet)$ and $\mathcal{I}(n)^{an}$ is evident upon inspection.

Let $\mathrm{ev}, \mathrm{od} \in \mathcal{I}(1)$ be the natural transformations defined by setting

$$\mathrm{ev} : \mathbb{C}[X]^\infty \hookrightarrow \mathbb{C}[X]^\infty, \quad \mathrm{ev}(e_i) = e_{2i}$$
$$\mathrm{od} : \mathbb{C}[X]^\infty \hookrightarrow \mathbb{C}[X]^\infty, \quad \mathrm{od}(e_i) = e_{2i+1}$$

Deformation retractions to a point can be constructed by first deforming any $(f_1, \dots, f_n) \in \mathcal{I}(n)$ to $(\mathrm{ev} \circ f_1, \dots, \mathrm{ev} \circ f_n)$, then deforming this latter to $(\mathrm{od} \circ g_1, \dots, \mathrm{od} \circ g_n)$ for some fixed $(g_1, \dots, g_n) \in \mathcal{I}(n)$. In the case of $\mathcal{I}(n)^{an}$, the first deformation is achieved by sending $(f_1, \dots, f_n)$ to

$$(\mathrm{ev} \circ t \cdot f_1 + (1-t) \cdot f_1, \dots, \mathrm{ev} \circ t \cdot f_n + (1-t) \cdot f_n), \quad 0 \leq t \leq 1;$$

the second deformation is achieved by sending $(\mathrm{ev} \circ f_1, \dots, \mathrm{ev} \circ f_n)$ to

$$(t \cdot \mathrm{ev} \circ f_1 + (1-t) \cdot \mathrm{od} \circ g_1, \dots, t \cdot \mathrm{ev} \circ f_n + (1-t) \cdot \mathrm{od} \circ g_n), \quad 0 \leq t \leq 1.$$

The value of either of these deformations for any $t, 0 \leq t \leq 1$, is easily seen to be an admissible injection.

For $\mathcal{I}(n)(\Delta^\bullet)$, we employ analogous simplicial homotopies.

The final claim of the proposition holds since each $\mathcal{L}(n)$ is a subspace of $\mathcal{I}(n)$ and the operations are clearly compatible with this collection of subspace inclusions. □

The proof of the following proposition is essentially immediate from the definition of the actions of $\mathcal{I}^{\mathrm{an}}$ given in Propositions 2.2 and 2.4 plus the continuity of these actions given in Proposition 2.6. We refer the reader to [25], [27] for details in analogous situations.

Proposition 2.8. *The pairings of Proposition 2.6 satisfy the axioms of* [25; §1], *making the spaces*

$$K^\infty(X)^{an},\ \mathcal{C}^\infty(X)^{an},\ G_Y^\infty(X)^{an}$$

into $\mathcal{I}^{\mathrm{an}}$-algebras for the E_∞ operad $\mathcal{I}^{\mathrm{an}}$.

We shall refer to each of the three spaces of Proposition 2.8 (or any space admitting such an action by $\mathcal{I}^{\mathrm{an}}$) simply as an "$\mathcal{I}^{\mathrm{an}}$-space".

We remind the reader that a homotopy-theoretic group completion $\mu : \mathcal{X} \mapsto \mathcal{X}^\wedge$ of a homotopy associative H-space $\mathcal{X}$ is a map of H-spaces satisfying

(1) $\mu_* : \pi_0\mathcal{X} \to \pi_0\mathcal{X}^\wedge$ is the group completion of the monoid $\pi_0\mathcal{X}$ and
(2) $\mu_* : H_*(\mathcal{X}, A) \to H_*(\mathcal{X}^\wedge, A)$ is the localization map

$$H_*(\mathcal{X}, A) \ \to \ \mathbb{Z}[\pi_0\mathcal{X}^\wedge] \otimes_{\mathbb{Z}[\pi_0\mathcal{X}]} H_*(\mathcal{X}, A),$$

for any commutative ring of coefficients A (cf. [28; §2]).

The action of $\mathcal{I}(2)^{an}$ on an $\mathcal{I}^{\mathrm{an}}$-space $\mathcal{X}$ provides $\mathcal{X}$ with the structure of any H-space. Indeed, the structure of an $\mathcal{I}^{\mathrm{an}}$-space on $\mathcal{X}$ associates to $\mathcal{X}$ an Ω-spectrum whose 0-th space $\Omega^\infty\Sigma^\infty\mathcal{X}$ is equipped with a map from $\mathcal{X}$,

$$\mathcal{X} \longrightarrow \Omega^\infty\Sigma^\infty\mathcal{X},$$

which is a homotopy-theoretic group completion of $\mathcal{X}$ (cf. [28; 6.4], [26, §2]).

Definition 2.9. *Let X be a quasi-projective complex variety and Y a projective complex variety. We define*

$$\begin{aligned} \mathcal{K}^{\mathrm{semi}}(X) &\equiv \Omega^\infty\Sigma^\infty K^\infty(X)^{an} \\ \mathcal{C}(X) &\equiv \Omega^\infty\Sigma^\infty\mathcal{C}^\infty(X)^{an} \\ \mathcal{G}_Y^{\mathrm{semi}}(X) &\equiv \Omega^\infty\Sigma^\infty G_Y^\infty(X)^{an}, \end{aligned}$$

the 0^{th} spaces of the Ω-spectra associated to the $\mathcal{I}^{\mathrm{an}}$-spaces

$$K^\infty(X)^{an},\ \mathcal{C}^\infty(X)^{an},\ G_Y^\infty(X)^{an}.$$

To obtain some understanding of these Ω-spectra, we investigate the monoids of connected components of the corresponding $\mathcal{I}^{\mathrm{an}}$-spaces and their homotopy-theoretic group completions. Notice that the following proposition requires the assumption that X be weakly normal.

Proposition 2.10. *Let X be a weakly normal, quasi-projective complex variety X. Then the set*

$$\pi_0 K^\infty(X)^{\mathrm{an}}$$

consists of equivalence classes of locally free coherent $\mathcal{O}_X$-modules which are generated by their global sections. Two such modules E_1 and E_2 are equivalent if and only if there is a connected, smooth, affine curve C, a locally free coherent $\mathcal{O}_{C\times X}$-module $\tilde{E}$ which is generated by its global sections, and closed points c_1, c_2 of C such that $E_i \cong \tilde{E}_{\{c_i\}\times X}$.

More generally, if Y is a projective complex variety, the set

$$\pi_0 G_Y^\infty(X)^{\mathrm{an}}$$

consists of equivalence classes of coherent $\mathcal{O}_{X\times Y}$-modules which are generated by their global sections and are flat over X. Two such coherent sheaves N_1 and N_2 are equivalent if and only if there exists a smooth, connected, affine curve C, a coherent $\mathcal{O}_{C\times X\times Y}$-module $\tilde{N}$ which is generated by its global sections and which is flat over $X\times C$, and closed points c_1, c_2 of C such that $N_i \cong \tilde{N}|_{\{c_i\}\times X\times Y}$.

Finally, if we assume that X is normal, then the set

$$\pi_0\mathcal{C}^\infty(X)^{\mathrm{an}}$$

consists of effective cycles on $X\times\mathbb{P}^N$ for some $N>0$ which are equidimensional over X modulo the equivalence relation generated by pairs (ζ_1,ζ_2) for which there exists a smooth curve C, points $c_1, c_2 \in C$, and an effective cycle Z on $C\times X\times\mathbb{P}^N$ equidimensional over $X\times C$ whose fibers over c_1, c_2 are ζ_1, ζ_2.

Proof. Since $K^\infty(X)^{\mathrm{an}}$ is a special case of $G_Y^\infty(X)^{\mathrm{an}}$, we do not consider this case separately. The condition that X be weakly normal implies that the sets $\mathrm{Mor}(X, G_Y^\infty)$ and $\mathrm{Mor}(X, \mathcal{C}^\infty)$ coincide with the corresponding sets of morphisms of varieties.

The class of a quotient $\mathcal{O}_{X\times Y}^\infty \twoheadrightarrow M$ depends only on the isomorphism class of M. For suppose $p : \mathcal{O}_{X\times Y}^n \twoheadrightarrow M$ and $q : \mathcal{O}_{X\times Y}^m \twoheadrightarrow M$ are two different choices of surjections, each giving an element of $\mathcal{M}or(X, G_Y^\infty)$. (In general, we regard a quotient $\mathcal{O}_{X\times Y}^l \twoheadrightarrow N$ as giving an element of $\mathcal{M}or(X, G_Y^\infty)$ by passing to the direct limit – i.e., by composing with the canonical surjection $\mathcal{O}_{X\times Y}^\infty \twoheadrightarrow \mathcal{O}_{X\times Y}^l$.) Consider the element of $\mathcal{M}or(X, G_Y^\infty)$ associated to $(p, q) : \mathcal{O}_{X\times Y}^{n+m} \twoheadrightarrow M$. Then $\lambda \mapsto (p, \lambda q)$, for $\lambda \in [0, 1]$, defines a path between the points associated to (p, q) and p. Similar constructions produce paths joining (p, q) to $(0, q)$, $(0, q)$ to (q, q), and (q, q) to q, thereby establishing the claim.

The description of the equivalence relation defining $\pi_0\mathcal{M}or(X, G_Y^\infty)^{\mathrm{an}}$ is a consequence of the description given in [13; 2.7] of π_0 of the space associated to any functor with a proper, constructible presentation.

If we assume X is normal, elements of $\mathrm{Mor}(X, \mathcal{C}^\infty)$ have a natural interpretation as effective cycles on $X \times \mathbb{P}^N$ (for some $N > 0$) equidimensional over X (cf. [10; 1.5]). Since $\mathcal{M}or(X, C_{r,d}(\mathbb{P}^N))(C) = \mathrm{Mor}(C \times X, \mathbb{P}^N)$ for any smooth curve C, the description of $\pi_0 \mathrm{Mor}(X, \mathcal{C}^\infty)$ follows from [13; 2.7]. □

To proceed further, let us make explicit the H-space structure on an $\mathcal{I}^{\mathrm{an}}$-space $\mathcal{X}$. Since $\mathcal{I}(2)^{an}$ is contractible and therefore connected and non-empty, the pairing

$$\mathcal{I}(2)^{an} \times (\mathcal{X})^{\times 2} \ \to \ \mathcal{X}$$

determines an H-space structure once a point in $\mathcal{I}(2)^{an}$ is chosen. Any two choices of points in $\mathcal{I}(2)^{an}$ give homotopy-equivalent H-space actions. Moreover, the connectivity of $\mathcal{I}(3)^{an}$ implies that this H-space structure is both homotopy associative and homotopy commutative. Unless explicit mention is made to the contrary, we choose the "interleaving map" η in $\mathcal{I}(2)^{\mathrm{an}}$, defined as

$$\eta : (\mathbb{C}^\infty)^{\times 2} \to \mathbb{C}^\infty, \quad \eta(e_i, e_j) = e_{2i} + e_{2j+1}. \tag{2.11}$$

As we see in the next proposition, the sets of connected components discussed in Proposition 2.10 have monoid structures with familiar group completions.

Proposition 2.12. *Let X be a weakly normal, quasi-projective complex variety and let Y be a projective complex variety. Then the sets $\pi_0 K^\infty(X)^{\mathrm{an}}$ and $\pi_0 G_Y^\infty(X)^{\mathrm{an}}$, whose elements are given by equivalence classes of coherent sheaves as in Proposition 2.10, are monoids under direct sum. Furthermore, their group completions can be described as follows:*

$$[\pi_0 K^\infty(X)^{\mathrm{an}}]^+ \ \cong \ K_0^{\mathrm{semi}}(X), \quad [\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+ \ \cong \ G_0^{\mathrm{semi}}(X, Y).$$

Proof. The fact that these sets are monoids follows from the fact that $K^\infty(X)^{\mathrm{an}}$ and $G_Y^\infty(X)^{\mathrm{an}}$ are H-spaces. The proof of Proposition 2.10 and definition of the H-space structure induced by the map (2.11) make it clear that the monoid operation is given by direct sum of coherent sheaves.

We write $\mathcal{M}_{gl}(X,Y)$ for the full, additive subcategory of the category of coherent sheaves on $X \times Y$ consisting of those coherent sheaves which are flat over X and generated by their global sections. Observe that $\mathcal{M}_{gl}(X,Y)$ is not naturally an exact category, since it is not closed under extension.

By Proposition 2.10, there is a natural map from the monoid $\pi_0 G_Y^\infty(X)^{\mathrm{an}}$ to the group $G_0^{\mathrm{semi}}(X,Y)$ which necessarily factors through the group completion $\pi_0 G_Y^\infty(X)^{\mathrm{an}} \to [\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+$, yielding the map

$$\theta : [\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+ \longrightarrow G_0^{\mathrm{semi}}(X,Y).$$

Suppose we are given an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of coherent $\mathcal{O}_{X\times Y}$-modules such that M', M, and M'' belong to $\mathcal{M}_{gl}(X,Y)$. By pulling back along the canonical projection $\mathbb{A}^1 \times X \times Y \to X \times Y$, we form the exact sequence

$$0 \to M'[t] \to M[t] \to M''[t] \to 0,$$

where $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. Observe that the objects in this sequence belong to $\mathcal{M}_{gl}(\mathbb{A}^1 \times X, Y)$. Now push out along $M'[t] \xrightarrow{\cdot t} M'[t]$ to form the exact sequence

$$0 \to M'[t] \to \tilde{M} \to M''[t] \to 0.$$

Observe that $\tilde{M}$ also belongs to $\mathcal{M}_{gl}(\mathbb{A}^1 \times X, Y)$, for it is flat over $\mathbb{A}^1 \times X$ (since it is an extension of coherent sheaves that are flat over $\mathbb{A}^1 \times X$) and it is generated by its global sections (since it is a quotient of $M'[t] \oplus M[t]$). Further, the restriction of $\tilde{M}$ to $\{i\} \times X \times Y$ is isomorphic to $M' \oplus M''$ for $i = 1$ and is isomorphic to M for $i = 0$. This shows that $[M] = [M' \oplus M''] = [M'] + [M'']$ in $\pi_0 G_Y^\infty(X)^{\mathrm{an}}$ by Proposition 2.10. It follows that $[\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+$ may be described as the abelian group whose generators are the isomorphism classes of objects of $\mathcal{M}_{gl}(X,Y)$ subject to the relations given by algebraic equivalence and short exact sequences (i.e., sequences of maps of objects of $\mathcal{M}_{gl}(X,Y)$ which are exact as sequences of coherent $\mathcal{O}_{X\times Y}$-modules.)

For any integer j and coherent $\mathcal{O}_{X\times Y}$-module M, we write $M(j)$ for $M \otimes \mathcal{O}(j)$, where $\mathcal{O}(j) \equiv \mathcal{O}(1)^{\otimes j}$ and $\mathcal{O}(1)$ is a chosen very ample line bundle on $X \times Y$. Choose a surjection $\mathcal{O}^{n+1} \twoheadrightarrow \mathcal{O}(1)$ and form the associated Koszul exact sequence (cf. [15; IV.2])

$$0 \to \mathcal{O}(j) \to \mathcal{O}(j+1)^{n+1} \to \cdots \to \mathcal{O}(j+n)^{n+1} \to \mathcal{O}(j+n+1) \to 0 \qquad (2.12.1)$$

of vector bundles on $X \times Y$, for any $j \in \mathbb{Z}$. Further, given a coherent $\mathcal{O}_{X\times Y}$-module M we obtain by taking tensor products the exact sequence

$$0 \to M(j) \to M(j+1)^{n+1} \to \cdots \to M(j+n)^{n+1} \to M(j+n+1) \to 0. \quad (2.12.2)$$

Since for any $M \in \mathcal{M}(X,Y)$ we have that $M(j) \in \mathcal{M}_{gl}(X,Y)$ for $j >> 0$, it follows from the relations imposed by (2.12.2) and descending induction on j that $G_0^{\mathrm{semi}}(X,Y)$ is generated by the classes of objects in $\mathcal{M}_{gl}(X,Y)$. The map θ is therefore onto.

Suppose $\theta([N]) = \theta([N'])$ for some $[N], [N'] \in [\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+$. Then N and N' are related by a finite sequence of equivalences given by algebraic equivalences and short exact sequences of objects in $\mathcal{M}(X,Y)$. For each short exact sequence appearing in this collection of relations, tensoring by $\mathcal{O}(j)$ for $j >> 0$ produces a short exact sequence of objects of $\mathcal{M}_{gl}(X,Y)$. It follows that $[N(j)] = [N'(j)]$ in $[\pi_0 G_Y^\infty(X)]^+$ for all $j >> 0$, since we know that short exact sequences in $\mathcal{M}_{gl}(X,Y)$ give relations.

Observe that for any M belonging to $\mathcal{M}_{gl}(X,Y)$ and any $j \geq 0$, every object and every cokernel and kernel of the maps in the sequence (2.12.2) belong to $\mathcal{M}_{gl}(X,Y)$. This is because the flatness condition passes to kernels and the generation-by-global-sections condition passes to cokernels. Thus, for such an M and j, the sequence (2.12.2) gives the relation

$$[M(j)] = \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} [M(i+j)]$$

in $[\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+$. We conclude that $[N] = [N']$ in $[\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+$ by descending induction on j, and hence θ is injective. □

Remark 2.13. *We could generalize Proposition 2.12 by stating that for an arbitrary quasi-projective complex variety X, we have the formulas*

$$[\pi_0 K^\infty(X)^{\mathrm{an}}]^+ \cong K_0^{\mathrm{semi}}(X^w), \quad [\pi_0 G_Y^\infty(X)^{\mathrm{an}}]^+ \cong G_0^{\mathrm{semi}}(X^w, Y).$$

Indeed, it is clear that $\mathcal{G}^{\mathrm{semi}}(X,Y) = \mathcal{G}^{\mathrm{semi}}(X^w,Y)$ for all X and Y. We do not know whether $K_0^{\mathrm{semi}}(X^w) \cong K_0^{\mathrm{semi}}(X)$ or $G_0^{\mathrm{semi}}(X^w,Y) \cong G^{\mathrm{semi}}(X,Y)$ for all varieties X and Y. Thus, in the remainder of this paper, the reader should be wary that our results concerning $\mathcal{K}^{\mathrm{semi}}(X)$ and $\mathcal{G}^{\mathrm{semi}}(X,Y)$ might not agree with the definitions of section 1 if X is not weakly normal.

The following corollary of Proposition 2.12 relates the infinite loop spaces of this section with the constructions of §1.

Corollary 2.14. *Let X be a weakly normal, quasi-projective complex variety and let Y be a projective complex variety. Then there are natural identifications*

$$\pi_0 \mathcal{K}^{\mathrm{semi}}(X) \; = \; K_0^{\mathrm{semi}}(X), \quad \pi_0 \mathcal{G}_Y^{\mathrm{semi}}(X) \; = \; G_0^{\mathrm{semi}}(X,Y).$$

In light of Corollary 2.14, we are justified in writing $K_q^{\mathrm{semi}}(X)$ and $G_q^{\mathrm{semi}}(X,Y)$ for the q^{th} homotopy groups of $\mathcal{K}^{\mathrm{semi}}(X)$ and $\mathcal{G}_Y^{\mathrm{semi}}(X)$, at least when X is weakly normal. We also write $G_q^{\mathrm{semi}}(Y)$ for $G_q^{\mathrm{semi}}(\mathrm{Spec}\,\mathbb{C}, Y)$.

§3 Interpretations of $\mathcal{G}_Y^{\mathrm{semi}}$, $\mathcal{K}^{\mathrm{semi}}$

It is important for the results of sections 4 and 5 that we have more explicit descriptions of the spaces $\mathcal{G}_Y^{\mathrm{semi}}(X)$ and $\mathcal{K}^{\mathrm{semi}}(X)$ – that is, we need an explicit method for forming the homotopy-theoretic group completions of $G_Y^\infty(X)^{\mathrm{an}}$ and $K^\infty(X)^{\mathrm{an}}$. In this section, we exhibit just such a construction, defined in terms of mapping telescopes. We begin with the following proposition, which shows that to group complete the discrete monoid $\pi_0 G_Y^\infty(X)^{\mathrm{an}}$, one needs only to invert a single element.

Proposition 3.1. *Let X and Y be quasi-projective complex varieties with Y projective. Fix an ample line bundle $\mathcal{O}(1)$ on $X \times Y$ which admits a surjection $\mathcal{O}^{n+1}_{X\times Y} \twoheadrightarrow \mathcal{O}(1)$. Then the abelian monoids*

$$\pi_0 G^\infty_Y(X)^{\mathrm{an}}\,[-[\mathcal{O}(1)]] \quad \textit{and} \quad \pi_0 K^\infty(X)^{\mathrm{an}}\,[-[\mathcal{O}(1)]]\,,$$

formed by adjoining the additive inverse of $[\mathcal{O}(1)]$, are the group completions of $\pi_0 G^\infty_Y(X)^{\mathrm{an}}$ and $\pi_0 K^\infty(X)^{\mathrm{an}}$:

$$\begin{aligned}\pi_0 G^\infty_Y(X)^{\mathrm{an}}\,[-[\mathcal{O}(1)]] &\cong [\pi_0 \mathcal{M}or(X, G^\infty_Y)^{\mathrm{an}}]^+,\\ \pi_0 K^\infty(X)^{\mathrm{an}}\,[-[\mathcal{O}(1)]] &\cong [\pi_0 K^\infty(X)^{\mathrm{an}}]^+.\end{aligned}$$

Proof. The first isomorphism encompasses the second, since $K^\infty = G^\infty_{\operatorname{Spec}\mathbb{C}}$.

Recall that the proof of Proposition 2.12 shows that a sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of maps in $\mathcal{M}_{gl}(X,Y)$ which is exact as a sequence of coherent $\mathcal{O}_{X\times Y}$-modules induces the relation $[M] = [M'] + [M'']$ in $\pi_0 G^\infty_Y(X)$. Let $\mathcal{O}(1)$ be a very ample line bundle on $X \times Y$. Using the Koszul exact sequence (2.12.2) as we did in the proof of Proposition 2.12, we conclude that $[M]$ has an inverse in $\pi_0 G^\infty_Y(X)$ provided $[M(j)]$ does for all $j >> 0$. Additionally, we claim the exact sequence (2.12.1) shows that $[\mathcal{O}(k)]$ has an inverse for all $k \geq 0$ provided $[\mathcal{O}(1)]$ has an inverse. To see this, observe that the case $k = 0$ follows from the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{n+1} \longrightarrow E \longrightarrow 0.$$

For $k \geq 2$, observe that for $i \geq 1$, the kernel of $\mathcal{O}(i)^{n+1} \twoheadrightarrow \mathcal{O}(i+1)$ belongs to $\mathcal{M}_{gl}(X,Y)$ since it is clearly flat over X and is a quotient of $\mathcal{O}(i-1)^{(n+1)(n+2)/2}$, which is generated by its global sections. Thus, $[\mathcal{O}(i+1)]$ has an inverse provided $[\mathcal{O}(i)]$ does, for $i \geq 1$.

Given an $M \in \mathcal{M}_{gl}(X,Y)$, choose a surjection $\mathcal{O}^e \twoheadrightarrow M$ with kernel N. If N is not generated by global sections, twist by $\mathcal{O}(j)$ for $j >> 0$ so that $N(j)$ is generated by global sections. This yields the exact sequence

$$0 \to N(j) \to \mathcal{O}(j)^e \to M(j) \to 0$$

of objects of $\mathcal{M}_{gl}(X,Y)$. It follows that $[M(j)] + [N(j)] = e[\mathcal{O}(j)]$, and thus $[M(j)]$ has an inverse for $j >> 0$ provided $[\mathcal{O}(1)]$ does. Hence, $[M]$ has an inverse provided $[\mathcal{O}(1)]$ does and the result follows. □

Remark 3.2. *The hypotheses on $\mathcal{O}(1)$ in Proposition 3.1 are met when $\mathcal{O}(1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ via a locally closed embedding of $X \times Y$ into projective space $\mathbb{P}^n$. Also, if X is affine and $Y \hookrightarrow \mathbb{P}^n$ is a projective embedding, then the hypothesis are met when $\mathcal{O}(1)$ is taken to be the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ via the composition $X \times Y \to Y \to \mathbb{P}^n$.*

The fact that it suffices to adjoin the additive inverse of $[\mathcal{O}(1)]$ to group complete the (discrete) monoid $\pi_0 G^\infty_Y(X)^{\mathrm{an}}$ motivates the construction described in the following proposition.

Proposition 3.3. *Let X and Y be quasi-projective complex varieties, with Y projective. Suppose $\mathcal{O}(1)$ is an ample line bundle on $X \times Y$ and $p : \mathcal{O}^m_{X\times Y} \twoheadrightarrow \mathcal{O}(1)$ is a surjection. Define a continuous map*

$$\alpha : G^\infty_Y(X)^{\mathrm{an}} \longrightarrow G^\infty_Y(X)^{\mathrm{an}}$$

by sending the quotient $\pi : \mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M$ to the quotient given as the composition of

$$\mathcal{O}^\infty_{X\times Y} \xrightarrow{\sigma} \mathcal{O}^m_{X\times Y} \oplus \mathcal{O}^\infty_{X\times Y} \twoheadrightarrow \mathcal{O}(1) \oplus M,$$

where $\sigma : \mathcal{O}^\infty_{X\times Y} \cong \mathcal{O}^m_{X\times Y} \oplus \mathcal{O}^\infty_{X\times Y}$ is the shift map defined by $e_i \mapsto (e_i, 0)$, $1 \leq i \leq m$ and $e_i \mapsto (0, e_{i-m})$, $m+1 \leq i$, where $e_1, e_2, \dots$ denote the standard basis elements. Then the homotopy colimit of the sequence of maps

$$G^\infty_Y(X)^{\mathrm{an}} \xrightarrow{\alpha} G^\infty_Y(X)^{\mathrm{an}} \xrightarrow{\alpha} \dots, \tag{3.3.1}$$

(that is, the mapping telescope) is a homotopy associative H-space.

In particular, taking $Y = \operatorname{Spec}\mathbb{C}$, the homotopy colimit of

$$K^\infty(X)^{\mathrm{an}} \xrightarrow{\alpha} K^\infty(X)^{\mathrm{an}} \xrightarrow{\alpha} \dots, \tag{3.3.2}$$

is a homotopy associative H-space.

Proof. Recall we obtain equivalent H-space structures on $G^\infty_Y(X)$ for any two choices of element of $\mathcal{I}(2)^{\mathrm{an}}$. For this proof, we consider $G^\infty_Y(X)$ as an H-space whose product map μ is defined by choosing the element $\iota : (\mathbb{C}^\infty)^{\times 2} \to \mathbb{C}^\infty$ of $\mathcal{I}(2)^{\mathrm{an}}$ which is the inverse of the isomorphism which interleaves in groups of m – that is, ι is the inverse of the map sending the sequence $e_1, e_2, \dots$ to

$$(e_1, 0), \dots, (e_m, 0), (0, e_{m+1}), \dots, (0, e_{2m}), (e_{2m+1}, 0), \dots.$$

Let $\mathcal{G}$ denote the homotopy colimit of the sequence (3.3.1) and let us regard the n^{th} member of the sequence (3.3.1), which we write as $G^\infty_Y(X)_n$ when clarification is needed, to be a pointed space with base point given by $\alpha^n(\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow 0)$. (Explicitly, this base point is the quotient $\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow \mathcal{O}^{\oplus nm}_{X\times Y} \twoheadrightarrow \mathcal{O}(1)^{\oplus n}$ given by taking the direct sum of n copies of p.) Let us write g_n for this base point of $G^\infty_Y(X)_n = G^\infty_Y(X)$. The g_n's determine a base point of $\mathcal{G}$, which we write as g.

Observe that $\mathcal{G} \times \mathcal{G}$ is the homotopy colimit of the sequence

$$(G^\infty_Y(X)^{\mathrm{an}})^{\times 2} \xrightarrow{\alpha\times\alpha} (G^\infty_Y(X)^{\mathrm{an}})^{\times 2} \xrightarrow{\alpha\times\alpha} \dots. \tag{3.3.3}$$

Thus, to give a homotopy class of maps

$$\mu^+ : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G},$$

it suffices to map the n^{th} stage of (3.3.3) to the $(2n)^{\mathrm{th}}$ stage of (3.3.1) in such a way that the squares

$$\begin{array}{ccc}
G^\infty_Y(X)_n \times G^\infty_Y(X)_n & \xrightarrow{\alpha\times\alpha} & G^\infty_Y(X)_{n+1} \times G^\infty_Y(X)_{n+1} \\
\downarrow & & \downarrow \\
G^\infty_Y(X)_{2n} & \xrightarrow{\alpha\circ\alpha} & G^\infty_Y(X)_{2n+2}
\end{array}$$

commute up to homotopy. We set each of the vertical maps $G^\infty_Y(X_n) \times G^\infty_Y(X)_n \to G^\infty_Y(X)_{2n}$ equal to the product pairing

$$\mu : \ (G^\infty_Y(X)^{\mathrm{an}})^{\times 2} \to G^\infty_Y(X)^{\mathrm{an}}.$$

One may readily verify that the resulting squares are, in fact, strictly commutating squares of pointed spaces (using the base point conventions we have established).

It remains to verify that μ^+ makes $\mathcal{G}$ into a homotopy associative H-space. To show the base point g of $\mathcal{G}$ behaves as a left identity up to homotopy, observe that the two maps

$$G^\infty_Y(X)_n \to G^\infty_Y(X)_{2n}$$

defined by α^n and multiplication on the left by g_n each send a quotient $\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M$ to a quotient having the form $\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow \mathcal{O}(1)^{\oplus n} \oplus M$ and these quotients differ by a block permutation matrix formed from copies the m-by-m identity matrix I_m. (That is, we can obtain one quotient from the other by precomposing with an infinite-by-infinite permutation matrix with entries in $\mathbb{C}$ formed from blocks of copies of I_m and 0_m.) Write this matrix as B and observe that $\lambda I_\infty + (1-\lambda)B$ is an invertible matrix for any complex number λ except $\frac{1}{2}$. Thus, letting λ denote a path from 0 to 1 in the complex plane which avoids $\frac{1}{2}$, the map

$$(\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M) \mapsto (\mathcal{O}^\infty_{X\times Y} \xrightarrow{\lambda I_\infty + (1-\lambda)B} \mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M)$$

defines a homotopy between α^n and multiplication on the left by g_n. Further, since the homotopy is defined by a block permutation matrix with blocks of size m and the base-point of $G^\infty_Y(X)$ is defined via a direct sum of copies of $\mathcal{O}^m_{X\times Y} \twoheadrightarrow \mathcal{O}(1)$, this homotopy is readily verified to be base-point preserving. Further, the square

$$\begin{array}{ccc} I \times G^\infty_Y(X)_n & \longrightarrow & I \times G^\infty_Y(X)_{n+1} \\ \downarrow & & \downarrow \\ G^\infty_Y(X)_{2n} & \longrightarrow & G^\infty_Y(X)_{2n+2} \end{array}$$

is clearly homotopy commutative (since one endpoint of the homotopy is a strictly commuting square), and thus by choosing homotopies we obtain a base-point preserving homotopy $I \times \mathcal{G} \to \mathcal{G}$ from the identity map to the map given by left multiplication by g. A similar argument shows g is also a right identity on $\mathcal{G}$ up to homotopy.

To show μ^+ is a homotopy associative pairing, observe that $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$ is homotopy equivalent to the telescope of

$$G^\infty_Y(X) \times G^\infty_Y(X) \times G^\infty_Y(X) \to G^\infty_Y(X) \times G^\infty_Y(X) \times G^\infty_Y(X) \to \dots,$$

and that the two possible pairings $\mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are induced by the two possible pairings

$$G^\infty_Y(X)_n \times G^\infty_Y(X)_n \times G^\infty_Y(X)_n \to G^\infty_Y(X)_{3n}.$$

As before, one of these maps can be obtained from the other by composing with a block permutation matrix $B : \mathbb{C}^\infty \cong \mathbb{C}^\infty$ formed from copies of I_m and 0_m, and the map sending $\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M$ to

$$\mathcal{O}^\infty_{X\times Y} \xrightarrow{\lambda I_\infty + (1-\lambda)B} \mathcal{O}^\infty_{X\times Y} \twoheadrightarrow M$$

defines a base point preserving homotopy. Thus, we have homotopy commutative squares

$$\begin{array}{ccc} I \times G_Y^\infty(X)_n \times G_Y^\infty(X)_n \times G_Y^\infty(X)_n & \longrightarrow & I \times G_Y^\infty(X)_{n+1} \times G_Y^\infty(X)_{n+1} \times G_Y^\infty(X)_{n+1} \\ \downarrow & & \downarrow \\ G_Y^\infty(X)_{3n} & \xrightarrow{\alpha^3} & G_Y^\infty(X)_{3n+3} \end{array}$$

and consequently an induced map

$$I \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

proving $\mathcal{G}$ is homotopy associative. $\square$

Theorem 3.4. *Let X and Y be quasi-projective complex varieties with Y projective. Fix an ample line bundle $\mathcal{O}(1)$ on $X \times Y$ and a surjection $p : \mathcal{O}^m_{X\times Y} \twoheadrightarrow \mathcal{O}(1)$. Then there are natural homotopy equivalences of H-spaces:*

$$\text{Telescope}\left(G_Y^\infty(X)^{\text{an}} \xrightarrow{\alpha} G_Y^\infty(X)^{\text{an}} \xrightarrow{\alpha} \dots\right) \xrightarrow{\sim} \mathcal{G}_Y^{\text{semi}}(X),$$
$$\text{Telescope}\left(K^\infty(X)^{\text{an}} \xrightarrow{\alpha} K^\infty(X)^{\text{an}} \xrightarrow{\alpha} \dots\right) \xrightarrow{\sim} \mathcal{K}^{\text{semi}}(X),$$

where α is made explicit in Proposition 3.3

Proof. For notational convenience, we write $\text{Tel}(G_Y^\infty(X)^{\text{an}}, \alpha)$ for

$$\text{Telescope}\left(G_Y^\infty(X)^{\text{an}} \xrightarrow{\alpha} G_Y^\infty(X)^{\text{an}} \xrightarrow{\alpha} \dots\right).$$

Observe that α is homotopic to the map given by left multiplication (under the product map μ given in the proof of 3.3) on $G_Y^\infty(X)$ by the element g_1, defined as the quotient $\mathcal{O}^\infty_{X\times Y} \twoheadrightarrow \mathcal{O}^m_{X\times Y} \twoheadrightarrow \mathcal{O}(1)$. Thus $\text{Tel}(G_Y^\infty(X)^{\text{an}}, \alpha)$ is homotopy equivalent to $\text{Tel}(G_Y^\infty(X)^{\text{an}}, \mu(g_1, -))$, where in forming this latter telescope, we regard the n^{th} member of the sequence as pointed by $\mu(g_1, \mu(\dots, \mu(g_1, g_1)))$.

Recall that $\mathcal{G}_Y^{\text{semi}}(X)$ is provided with a map of H-spaces

$$j : G_Y^\infty(X)^{\text{an}} \to \mathcal{G}_Y^{\text{semi}}(X) \tag{3.4.1}$$

which is a homotopy theoretic group completion – that is, this map has the properties that the induced map on π_0

$$\pi_0 G_Y^\infty(X)^{\text{an}} \longrightarrow \pi_0 \mathcal{G}_Y^{\text{semi}}(X) = [\pi_0 G_Y^\infty(X)^{\text{an}}]^+$$

is group completion and the map on homology with coefficients in A is the localization map

$$\text{H}_*(\mathcal{M}or(X, G_Y^\infty)^{\text{an}}, A) \longrightarrow \mathbb{Z}[(\pi_0 G_Y^\infty(X)^{\text{an}})^+] \otimes_{\mathbb{Z}[\pi_0 G_Y^\infty(X)^{\text{an}}]} \text{H}_*\left(G_Y^\infty(X)^{\text{an}}, A\right).$$

To give a map

$$\psi : \text{Tel}(G_Y^\infty(X)^{\text{an}}, \alpha) \sim \text{Tel}(G_Y^\infty(X)^{\text{an}}, \mu(g_1, -)) \longrightarrow \mathcal{G}_Y^{\text{semi}}(X),$$

it suffices to give maps $h_n : G_Y^\infty(X)^{\mathrm{an}} \longrightarrow \mathcal{G}_Y^{\mathrm{semi}}(X)$ with the property that $h_{n+1} \circ \mu(g_1, -)$ is homotopic to h_n. Since $\mathcal{G}_Y^{\mathrm{semi}}(X)$ is a group-like H-space, the map given by left multiplication by g_1,

$$(\mathcal{G}_Y^{\mathrm{semi}}(X), *) \to (\mathcal{G}_Y^{\mathrm{semi}}(X), g_1),$$

is a homotopy equivalence, and thus there is a pointed homotopy inverse

$$\beta : (\mathcal{G}_Y^{\mathrm{semi}}(X), g_1) \to (\mathcal{G}_Y^{\mathrm{semi}}(X), *).$$

We define

$$h_n = \beta \circ \ldots \circ \beta \circ j : \ G_Y^\infty(X)^{\mathrm{an}} \longrightarrow \mathcal{G}_Y^{\mathrm{semi}}(X),$$

the composition of j of (3.4.1) and n iterates of β. Since $\mathcal{G}_Y^{\mathrm{semi}}(X)$ is homotopy associative, we conclude that $h_{n+1} \circ \alpha$ is homotopic to h_n as required.

Essentially by its very construction, we have

$$\pi_0 \operatorname{Tel}(G_Y^\infty(X)^{\mathrm{an}}, \alpha) \ = \ \pi_0 G_Y^\infty(X)^{\mathrm{an}}[-[\mathcal{O}(1)]],$$

so that Proposition 3.1 implies that ψ induces an isomorphism on groups of connected components. Similarly, the map

$$\mathrm{H}_* \, G_Y^\infty(X)^{\mathrm{an}} \longrightarrow \mathrm{H}_* \operatorname{Tel}(G_Y^\infty(X)^{\mathrm{an}}, \alpha)$$

is localization with respect to the action of $[\mathcal{O}(1)] \in \pi_0 G_Y^\infty(X)^{\mathrm{an}}$, and hence is isomorphic to the map $H_* G_Y^\infty(X)^{\mathrm{an}} \to H_* \mathcal{G}_Y^{\mathrm{semi}}(X)$.

We conclude that ψ is a homology equivalence of group-like H-spaces and is therefore a weak homotopy equivalence. Namely, the connected components containing the identity elements of $\operatorname{Tel}(G_Y^\infty(X)^{\mathrm{an}}, \alpha)$ and $\mathcal{G}_Y^{\mathrm{semi}}(X)$ are connected H-spaces and therefore simple [31] and a homology equivalence of simple spaces is a weak homotopy equivalence. Moreover, translation between connected components within either $\operatorname{Tel}(G_Y^\infty(X)^{\mathrm{an}}, \alpha)$ or $\mathcal{G}_Y^{\mathrm{semi}}(X)$ is a homotopy equivalence.

The second weak equivalence is a special case of the first. □

We make Theorem 3.4 more explicit by providing a simpler description of $\mathcal{K}^{\mathrm{semi}}(X)$, which involves only (limits of) finite Grassmannians.

Corollary 3.5. *Let X be a quasi-projective complex variety, $\mathcal{O}(1)$ an ample line bundle, and $\mathcal{O}_X^m \twoheadrightarrow \mathcal{O}(1)$ a surjective map. Then $\mathcal{K}^{\mathrm{semi}}(X)$ is homotopy equivalent to*

$$\mathcal{M}or(X, \mathbb{Z})^{\mathrm{an}} \times \operatorname{Telescope}(\mathcal{M}or(X, \operatorname{Grass}_0(\mathbb{P}^0))^{\mathrm{an}} \xrightarrow{\theta} \mathcal{M}or(X, \operatorname{Grass}_1(\mathbb{P}^{m+1}))^{\mathrm{an}} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \mathcal{M}or(X, \operatorname{Grass}_j(\mathbb{P}^{j(m+1)}))^{\mathrm{an}} \xrightarrow{\theta} \cdots),$$

where the transition map θ sends $\mathcal{O}_X^{e(m+1)+1} \twoheadrightarrow \mathcal{E}$ to the composition of

$$\mathcal{O}_X^{(e+1)(m+1)+1} \ \cong \ \mathcal{O}_X^m \oplus \mathcal{O}_X^{e(m+1)+1} \oplus \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X^m \oplus \mathcal{O}_X^{e(m+1)+1} \twoheadrightarrow \mathcal{O}(1) \oplus \mathcal{E}.$$

In particular, for $i > 0$, we have

$$K_i^{\mathrm{semi}}(X) \ \cong \ \varinjlim_{j \in \mathbb{N}} \pi_i \mathcal{M}or(X, \operatorname{Grass}_j(\mathbb{P}^{j(m+1)}))^{\mathrm{an}},$$

where the maps of the limit are induced by θ.

Proof. We may assume X is connected, so that $\mathcal{M}or(X,\mathbb{Z})^{\mathrm{an}} = \mathbb{Z}$.

Write $\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty)), \alpha)$ for the mapping telescope of

$$\mathcal{M}or(X, \mathrm{Grass}_0(\mathbb{P}^\infty)) \xrightarrow{\alpha} \mathcal{M}or(X, \mathrm{Grass}_1(\mathbb{P}^\infty)) \xrightarrow{\alpha} \cdots .$$

For each $n \geq 0$, the collection of maps

$$\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}} \xrightarrow{\alpha^{\circ n}} \mathcal{M}or(X, \mathrm{Grass}_{n+i}(\mathbb{P}^\infty))^{\mathrm{an}}$$

determines a map on telescopes

$$\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}}, \alpha) \xrightarrow{\alpha^{\circ n}} \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha).$$

We thus obtain a map of H-spaces

$$\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}}, \alpha) \times \mathbb{N} \to \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha),$$

defined to be the map $\alpha^{\circ n}$ on $\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty)), \alpha) \times \{n\}$. The induced map on homotopy theoretic group completions

$$\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}}, \alpha) \times \mathbb{Z} \to \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha)$$

is evidently a homology equivalence of group-like H-spaces and thus a homotopy equivalence.

Observe now that we have

$$\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}}, \alpha) \;\cong\; \mathrm{Tel}(\varinjlim_N \mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^N))^{\mathrm{an}}, \alpha).$$

Further, if we also write α for the map

$$\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^N))^{\mathrm{an}} \to \mathcal{M}or(X, \mathrm{Grass}_{i+1}(\mathbb{P}^{N+m}))^{\mathrm{an}}$$

which sends $\mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{E}$ to $\mathcal{O}_X^{m+N+1} \cong \mathcal{O}_X^m \oplus \mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{O}(1) \oplus \mathcal{E}$, then α commutes with the transition maps in the direct system. Since these transition maps are in fact cofibrations, we may replace the direct limit with a homotopy direct limit to obtain a homotopy equivalence

$$\mathrm{Tel}(\mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^\infty))^{\mathrm{an}}, \alpha) \sim \mathrm{hocolim}\, \mathcal{M}or(X, \mathrm{Grass}_i(\mathbb{P}^N))^{\mathrm{an}},$$

where the directed system on the right is indexed by $\mathbb{N} \times \mathbb{N}$. The telescope appearing in the statement of this corollary is formed from a "diagonal" cofinal subsystem of this direct system, and thus the result follows. $\square$

§4 Compacta mapping to function complexes

In this section, we develop a technique for describing the homotopy groups of $\mathcal{K}^{\text{semi}}(X)$ and $\mathcal{G}^{\text{semi}}_Y(X)$ when X and Y are both projective complex varieties. The key advantage the projective hypothesis adds is that the spaces $K^\infty(X)^{\text{an}}$ and $G^\infty_Y(X)^{\text{an}}$, where X is projective, have the structures of inductive limits of algebraic varieties. The technique we develop is then used to prove "Poincaré duality" relating $\mathcal{K}^{\text{semi}}$ and $\mathcal{G}^{\text{semi}}$.

Lemma 4.1. *For X and Y projective complex varieties, the set* $\text{Mor}(X, Y)$ *may be identified with the complex points of an infinite disjoint union of quasi-projective varieties (indexed by the degrees of each connected component of $X \times Y$ with respect to some projective embedding) which we write as* $\underline{\text{Mor}}(X, Y)$*. Moreover, for any complex variety S, we have a natural isomorphism*

$$\text{Mor}(S, \underline{\text{Mor}}(X, Y)) \;\cong\; \text{Mor}(S \times X, Y),$$

and thus the space $\mathcal{M}or(X, Y)^{\text{an}}$ is the analytic realization of $\underline{\text{Mor}}(X, Y)$*.*

Proof. We may assume X and Y are connected. The proof of [13; 2.2] shows that the collection of effective cycles on $X \times Y$ of degree d which have dimension 0 and degree 1 over every point of X form a locally closed subvariety of the Chow variety $C_{*,d}(X \times Y)$. We define $\underline{\text{Mor}}(X, Y)$ to be the disjoint union over all d of these subvarieties. The proof of [13; 2.2] also shows that $\underline{\text{Mor}}(X, Y)$ represents the functor $C \mapsto \text{Mor}(X \times C, Y)$ on the category $(Sm/\mathbb{C})_{\leq 1}$. It now follows from [13; 1.1] that there is a natural isomorphism

$$\text{Mor}(S, \underline{\text{Mor}}(X, Y)) \;\cong\; \text{Mor}(S \times X, Y),$$

for all complex varieties S (using the fact that $\mathcal{M}or(-, -)$ is an internal Hom-object for the category of functors $(Sm/\mathbb{C})_{\leq 1} \to (sets)$). $\square$

Remark. *If X and Y are projective complex varieties (possibly not weakly normal), then there is an alternative to the space $\mathcal{M}or(X, Y)^{\text{an}}$ considered in this paper. Namely, the set* $\text{Hom}(X, Y)$ *of morphism of varieties has the structure of a disjoint union of schemes, given by taking an appropriate locally closed subscheme of the Hilbert scheme of $X \times Y$. Let us write this scheme as* $\underline{\text{Hom}}(X, Y)$*. Then the topological space associated to* $\underline{\text{Hom}}(X, K^\infty) \equiv \varinjlim_N \underline{\text{Hom}}(X, \coprod_n \text{Grass}_n(\mathbb{P}^N))$ *is a more likely candidate for defining the infinite loop space $\mathcal{K}^{\text{semi}}(X)$ when X is projective but not weakly normal. (The two candidates coincide when X is weakly normal, and it is possible they coincide up to homotopy in general.) However, this construction does not generalize well to non-projective varieties, since* $\text{Hom}(X, Y)$ *no longer has the structure of a disjoint union of varieties if X is not projective. This explains our use of $K^\infty(X)$ in this paper.*

For topological spaces T and S, we write $\text{Maps}(T, S)$ for the set of continuous maps from T to S. The key technical result for this section is the following simple assertion.

Proposition 4.2. *Assume that T is finite C. W. complex and that Y is a quasi-projective complex variety. Let Aff^T denote the category whose objects are continuous maps $T \to U^{\text{an}}$, where U is an affine complex variety, and for which a* *morphism from $\alpha : T \to U^{\mathrm{an}}$ to $\beta : T \to V^{\mathrm{an}}$ is a morphism of varieties $f : V \to U$ such that $f^{\mathrm{an}} \circ \beta = \alpha$. Then we have natural isomorphisms*

$$\mathrm{Maps}(T, Y^{an}) \;\cong\; \varinjlim_{(T \to U^{\mathrm{an}}) \in Aff^T} \mathrm{Mor}(U, Y) \;\cong\; \varinjlim_{A \subset \mathcal{C}(T)} \mathrm{Mor}(\mathrm{Spec}\, A, Y),$$

where A ranges over the collection of finitely generated $\mathbb{C}$-subalgebras of $\mathcal{C}(T)$, the algebra of continuous complex-valued functions on T.

Proof. Given a continuous map $T \to U^{\mathrm{an}}$ and a continuous algebraic map $U \to Y$, we clearly obtain a continuous map from T to Y^{an} by composition. This defines a map

$$\Psi : \varinjlim_{(T \to U^{\mathrm{an}}) \in Aff^T} \mathrm{Mor}(U, Y) \longrightarrow \mathrm{Maps}(T, Y^{an}).$$

We recall Jouanolou's device [19], which constructs an affine variety J together with a morphism $J \to Y$ which is a torsor for some vector bundle of rank n over Y. In particular, there is a finite Zariski open cover $Y = \cup_i U_i$ of Y such that the morphism $J \times_Y U_i \to U_i$ is isomorphic to $\mathbb{A}^n \times U_i \to U_i$ for each i. Thus, the associated continuous map $J^{an} \to Y^{an}$ is a local fibration and hence a fibration by Hurewicz's Theorem (cf. [35; p. 33]). Since $J^{an} \to Y^{an}$ is a homotopy equivalence, any continuous map $f : T \to Y^{an}$ lifts to a map $T \to J^{an}$, so that f factors as $T \to J^{\mathrm{an}} \to Y^{\mathrm{an}}$. Thus, Ψ is onto.

Suppose we are given continuous maps $\alpha : T \to U^{\mathrm{an}}$, $\beta : T \to V^{\mathrm{an}}$ and continuous algebraic maps $f : U \to Y$, $g : V \to Y$ such that $f^{\mathrm{an}} \circ \alpha = g^{\mathrm{an}} \circ \beta$. Then there is an induced continuous map

$$(\alpha, \beta) : T \longrightarrow U^{\mathrm{an}} \times_{Y^{\mathrm{an}}} V^{\mathrm{an}} \;\cong\; (U \times_Y V)^{\mathrm{an}}.$$

Use Jouanolou's device once again to construct a vector bundle torsor $W \to U \times_Y V$, with W affine. As before, the continuous map (α, β) factors through $W^{\mathrm{an}} \to (U \times_Y V)^{\mathrm{an}}$. But then the transition maps associated to the morphisms $W \to U$ and $W \to V$ in Aff^T send (α, f) and (β, g) to the same pair $(T \to W^{\mathrm{an}}, W \to Y)$. Thus, Ψ is injective.

The second isomorphism of the proposition results from the fact that the direct indexing system $\{A \subset \mathcal{C}(T)\}$ forms a cofinal subsystem of Aff^T by sending $A \subset \mathcal{C}(T)$ to the continuous map $T \to (\mathrm{Spec}\, A)^{\mathrm{an}}$. To see this, observe that given $T \to U^{\mathrm{an}}$ with $U = \mathrm{Spec}\, B$ affine, one obtains a natural ring map $B \to \mathcal{C}(T)$. Now take A to be the image of B in $\mathcal{C}(T)$, so that the continuous map $T \to U^{\mathrm{an}}$ factors as $T \to (\mathrm{Spec}\, A)^{\mathrm{an}} \to (\mathrm{Spec}\, B)^{\mathrm{an}}$. □

Corollary 4.3. *Assume that X and Y are projective complex varieties. Then there are natural isomorphisms*

$$\begin{aligned} \mathrm{Maps}(T, \mathcal{M}or(X, Y)^{\mathrm{an}}) \;&\cong\; \varinjlim_{(T \to U^{\mathrm{an}}) \in Aff^T} \mathrm{Mor}(U \times X, Y) \\ &\cong\; \varinjlim_{A \subset \mathcal{C}(T)} \mathrm{Mor}(\mathrm{Spec}\, A \times X, Y). \end{aligned}$$

Proof. By Lemma 4.1, the space $\mathcal{M}or(X, Y)^{\mathrm{an}}$ is the disjoint union $\coprod_{\underline{d}} \underline{\mathrm{Mor}}^{\underline{d}}(X, Y)^{\mathrm{an}}$, where $\underline{\mathrm{Mor}}^{\underline{d}}(X, Y)$ is the quasi-projective variety parameterizing continuous algebraic maps of multi-degree $\underline{d} = (d_1, \ldots, d_n)$ (where we have chosen some closed embedding of $X \times Y$ in projective space and the connected components of $X \times Y$ are indexed from 1 to n). Observe that we may assume T is connected. Thus, using Proposition 4.2 we obtain the isomorphisms

$$\begin{aligned}
\mathrm{Maps}(T, \mathcal{M}or(X,Y)^{\mathrm{an}}) &\cong \coprod_{\underline{d}} \mathrm{Maps}(T, \underline{\mathrm{Mor}}^{\underline{d}}(X,Y)^{\mathrm{an}}) \\
&\cong \coprod_{\underline{d}} \varinjlim_{T \to U^{\mathrm{an}}} \mathrm{Mor}(U, \underline{\mathrm{Mor}}^{\underline{d}}(X,Y)) \\
&\cong \varinjlim_{T \to U^{\mathrm{an}}} \coprod_{\underline{d}} \mathrm{Mor}(U, \underline{\mathrm{Mor}}^{\underline{d}}(X,Y)) \\
&\cong \varinjlim_{T \to U^{\mathrm{an}}} \mathrm{Mor}(U, \underline{\mathrm{Mor}}(X,Y)) \\
&\cong \varinjlim_{T \to U^{\mathrm{an}}} \mathrm{Mor}(U \times X, Y).
\end{aligned}$$

The other isomorphism follows as in the proof of Proposition 4.2. $\square$

We also require the following analogue of Corollary 4.3 in which we consider homotopy classes of continuous maps from a C. W. complex T to the space $\mathcal{M}or(X,Y)^{\mathrm{an}}$. In general, for topological spaces T and S, we write $[T,S]$ for the collection of homotopy classes of continuous maps from T to S. Let $hAff^T$ denote the indexing category in which an object is a homotopy class of continuous maps from T to U^{an}, where U is an affine variety. The morphisms of $hAff^T$ are defined as in Aff^T (so that $hAff^T$ is a quotient category of Aff^T).

Proposition 4.4. *Assume T is a finite C. W. complex and let X and Y be projective complex varieties. Then we have a natural isomorphism*

$$[T, \mathcal{M}or(X,Y)^{\mathrm{an}}] \cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, Y)^{\mathrm{an}}.$$

Proof. Using Corollary 4.3, we see there is a natural surjection

$$p : \mathrm{Maps}(T, \mathcal{M}or(X,Y)^{\mathrm{an}}) \twoheadrightarrow \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, Y)^{\mathrm{an}}.$$

Suppose $h \in \mathrm{Maps}(T \times I, \mathcal{M}or(X,Y)^{\mathrm{an}})$ is represented by the pair $(\alpha : T \times I \to V^{\mathrm{an}}, g : V \times X \to Y)$ (under the isomorphism of Corollary 4.3), and let h_i be the restriction of h to $T \times \{i\}$, for $i = 0, 1$. Then h_i is represented by the pair $(\alpha_i : T \times \{i\} \to V^{\mathrm{an}}, g : V \times X \to Y)$. Clearly, p sends h_0 and h_1 to the same element since α_0 is homotopic to α_1. We thus obtain a surjection

$$q : [T, \mathcal{M}or(X,Y)^{\mathrm{an}}] \twoheadrightarrow \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, Y)^{\mathrm{an}}.$$

Now suppose q sends two continuous maps $\beta_0, \beta_1 : T \longrightarrow \mathcal{M}or(X,Y)^{\mathrm{an}}$ to the same element. Since the indexing category Aff^T is directed, we may assume β_i is represented by the pair $(\alpha : T \to V, g_i : V \times X \to Y)$, for $i = 0, 1$ – that is, the first components may be taken to be the same. Furthermore, since $hAff^T$ is also directed, the fact that $q(\beta_1) = q(\beta_2)$ may be taken to mean g_1 and g_2 lie in the same component of the space $\mathcal{M}or(U \times X, Y)^{\mathrm{an}}$. But then by [13; 2.7], there exists a smooth, connected affine curve C with closed points c_0, c_1 and a continuous algebraic map $\gamma : C \times U \times X \to Y$ such that $g_i = \gamma_{\{c_i\} \times U \times X}$. Choose a continuous path $\delta : I \to C^{\mathrm{an}}$ joining c_0 to c_1, and consider the map $h : T \times I \to \mathcal{M}or(X, Y)^{\mathrm{an}}$ associated to the pair $(\delta \times \alpha : I \times T \to C \times U, \gamma : C \times U \times X \to Y)$. Clearly, h defines a homotopy from β_0 to β_1, and so q is injective. $\square$

The following result is essentially a special case of Proposition 4.4 obtained by taking $Y = K^\infty$. However, some care is needed to deal with the direct limits involved in the construction of K^∞.

Proposition 4.5. *For any projective complex varieties X and Y and any finite C. W. complex T, we have natural isomorphisms*

$$[T, \mathcal{G}_Y^{\mathrm{semi}}(X)] \cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} G_0^{\mathrm{semi}}(X \times U, Y).$$

and

$$[T, \mathcal{K}^{\mathrm{semi}}(X)] \cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} K_0^{\mathrm{semi}}(X \times U).$$

Proof. The second isomorphism is the special case of the first obtained by setting $Y = \operatorname{Spec} \mathbb{C}$.

Fix $p : \mathcal{O}_{X \times Y}^m \twoheadrightarrow \mathcal{O}(1)$ as in Proposition 3.3 and recall the construction of the map α. From the basic properties of the mapping telescope and Theorem 3.4, we have

$$[T, \mathcal{G}_Y^{\mathrm{semi}}(X)] \;\cong\; \varinjlim_{\mathbb{N}} [T, G_Y^\infty(X)^{\mathrm{an}}]$$

where the transition maps are induced by the map α. Recall that

$$G_Y^\infty(X)^{\mathrm{an}} = \varinjlim_r \mathcal{M}or(X, G_Y^r)^{\mathrm{an}},$$

where G_Y^r is a disjoint union of algebraic varieties. We therefore have the natural isomorphism

$$\operatorname{Maps}(T, G_Y^\infty(X)^{\mathrm{an}}) \;\cong\; \varinjlim_r \operatorname{Maps}(T, \mathcal{M}or(X, G_Y^r)^{\mathrm{an}}),$$

by [32; 9.3]. Hence, by Proposition 4.4, we have

$$\begin{aligned}
[T, G_Y^\infty(X)^{\mathrm{an}}] &\cong \varinjlim_r [T, \mathcal{M}or(X, G_Y^r)^{\mathrm{an}}] \\
&\cong \varinjlim_r \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, G_Y^r)^{\mathrm{an}} \\
&\cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, G_Y^\infty)^{\mathrm{an}}.
\end{aligned}$$

Since U is affine, we may form the group completion of $\pi_0 \mathcal{M}or(U \times X, G_Y^\infty)^{\mathrm{an}}$ by using the transition maps α arising from the pullback of $\mathcal{O}_{X \times Y}^m \twoheadrightarrow \mathcal{O}(1)$ to

$U \times X \times Y$ (see Remark 3.2). By Propositions 2.12 and 3.1, we have a chain of natural isomorphisms

$$\begin{aligned}[T, \mathcal{G}_Y^{\mathrm{semi}}(X)] &\cong \varinjlim_{\mathbb{N}} [T, G_Y^{\infty}(X)^{\mathrm{an}}] \\ &\cong \varinjlim_{\mathbb{N}} \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \pi_0 \mathcal{M}or(U \times X, G_Y^{\infty})^{\mathrm{an}} \\ &\cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} \varinjlim_{\mathbb{N}} \pi_0 \mathcal{M}or(U \times X, G_Y^{\infty})^{\mathrm{an}} \\ &\cong \varinjlim_{[T \to U^{\mathrm{an}}] \in hAff^T} G_0^{\mathrm{semi}}(U \times X, Y).\end{aligned}$$

□

As an easy application of Proposition 4.5, we obtain an interesting interpretation of groups $\mathcal{K}_q^{\mathrm{semi}}$ for $q > 0$ in terms of K_0^{semi}. This should be viewed as an analogue of the formula of algebraic K-theory (which is valid for X smooth)

$$K_q(X) \cong K_0(X \times \partial\Delta^{q+1})/K_0(X),$$

where $\partial\Delta^{q+1}$ is defined to be $\mathrm{Spec}\,(k[x_0, \dots, x_{q+1}]/(\sum_i x_i - 1, \prod_i x_i))$ (i.e., the "algebraic q-sphere") and by $K_0(X \times \partial\Delta^{q+1})/K_0(X)$ we mean the cokernel of the split injection $K_0(X) \hookrightarrow K_0(X \times \partial\Delta^{q+1})$ induced by the projection map. Given quasi-projective complex varieties X and U, let $K_0^{\mathrm{semi}}(X \times U)/K_0^{\mathrm{semi}}(X)$ denote the cokernel of the natural split injection $K_0^{\mathrm{semi}}(X) \hookrightarrow K_0^{\mathrm{semi}}(X \times U)$ induced by the projection map.

Corollary 4.6. *Let S^q denote the topological q-sphere. For a projective complex variety X, we have natural isomorphisms*

$$K_q^{\mathrm{semi}}(X) \cong \varinjlim_{[S^q \longrightarrow U^{\mathrm{an}}] \in hAff^{S^q}} K_0^{\mathrm{semi}}(X \times U)/K_0^{\mathrm{semi}}(X).$$

Proof. Since $\mathcal{K}^{\mathrm{semi}}(X)$ is a homotopy commutative group-like H-space, there is a short exact sequence

$$0 \longrightarrow \pi_q \mathcal{K}^{\mathrm{semi}}(X) \longrightarrow [S^q, \mathcal{K}^{\mathrm{semi}}(X)] \longrightarrow \pi_0 \mathcal{K}^{\mathrm{semi}}(X) \longrightarrow 0 \qquad (4.6.1)$$

of abelian groups, where the last map in this sequence is induced by restricting maps to the base point of S^q. By Proposition 4.5, we have

$$[S^q, \mathcal{K}^{\mathrm{semi}}(X)] \cong \varinjlim_{[S^q \to U] \in hAff^{S^q}} K_0^{\mathrm{semi}}(X \times U)$$

and the last map of (4.6.1) is induced by the map $K_0^{\mathrm{semi}}(X \times U) \to K_0^{\mathrm{semi}}(X)$ given by restricting to the image of the base point of S^q in U. This map is clearly split by the map $\mathcal{K}^{\mathrm{semi}}(X) \hookrightarrow \mathcal{K}^{\mathrm{semi}}(X \times U)$ induced by projection, from which the result follows. □

For projective varieties X and Y, there is a natural map

$$\mathcal{K}^{\mathrm{semi}}(Y \times X) \longrightarrow \mathcal{G}_Y^{\mathrm{semi}}(X)$$

of infinite loop spaces. Indeed, this map is induced via homotopy-theoretic group completion by the map of $\mathcal{I}^{\mathrm{an}}$-spaces $K^{\infty}(X \times Y)^{\mathrm{an}} \longrightarrow G_Y^{\infty}(X)^{\mathrm{an}}$. The existence of this latter map is seen by observing that there is an evident natural transformation of functors on $(Sm/\mathbb{C})_{\leq 1}$. Namely, given $C \in (Sm/\mathbb{C})_{\leq 1}$, send a quotient $\mathcal{O}_{C \times X \times Y} \twoheadrightarrow E$ (where E is locally free on $C \times X \times Y$) to the same quotient (but where E is regarded as being merely flat over $C \times X$).

Theorem 4.7. *Let X and Y be projective complex varieties with Y smooth. For any finite C. W. complex T, the natural map*

$$[T, \mathcal{K}^{\rm semi}(X \times Y)] \longrightarrow [T, \mathcal{G}_Y^{\rm semi}(X)]$$

is an isomorphism of abelian groups. Consequently, we have natural isomorphisms

$$K_q^{\rm semi}(X \times Y) \xrightarrow{\cong} G_q^{\rm semi}(X, Y),$$

for all q.

Proof. By Proposition 4.5, there are natural isomorphisms

$$[T, \mathcal{K}^{\rm semi}(X \times Y)] \ \cong \ \varinjlim_{[T \to U^{\rm an}] \in hAff^T} K_0^{\rm semi}(U \times X \times Y) \qquad (4.7.1)$$

and

$$[T, \mathcal{G}_Y^{\rm semi}(X)] \ \cong \ \varinjlim_{[T \to U^{\rm an}] \in hAff^T} G_0^{\rm semi}(U \times X, Y). \qquad (4.7.2)$$

Further, the map $[T, \mathcal{K}^{\rm semi}(X \times Y)] \to [T, \mathcal{G}_Y^{\rm semi}(X)]$ is compatible with the natural maps $K_0^{\rm semi}(U \times X \times Y) \to G_0^{\rm semi}(U \times X, Y)$ of the the direct limits in (4.7.1) and (4.7.2). Since for each U this map is an isomorphism by Proposition 1.7, the desired result follows. □

§5 Projective bundle formula

In this section we establish the projective bundle formula for semi-topological K-theory. In particular, this gives a computation of $K_*^{\rm semi}(P)$ whenever P is a product of projective spaces.

We begin by introducing the beginnings of a multiplicative structure on $\mathcal{K}^{\rm semi}(X)$ as follows. As in (3.5.1) we consider maps $\alpha : K^\infty(X) \to K^\infty(X)$ associated to taking the direct sum with an ample line bundle. Since we shall have need to vary this line bundle, we introduce the following notation. Let $\mathcal{O}_X^m \twoheadrightarrow L$ be a quotient map to an ample line bundle L and define

$$\alpha_L : K^\infty(X) \to K^\infty(X)$$

by sending $\mathcal{O}_X^\infty \twoheadrightarrow E$ to $\mathcal{O}_X^\infty \cong \mathcal{O}_X^m \oplus \mathcal{O}_X^\infty \twoheadrightarrow L \oplus E$, where the isomorphism is the usual shift of coordinates. (Note that α_L depends not only upon L by also the choice of quotient map $\mathcal{O}_X^m \twoheadrightarrow L$.) Choose a possibly different ample line bundle L' with a chosen quotient map $\mathcal{O}_X^n \twoheadrightarrow L'$ and define

$$\beta_{L'} : K^\infty(X) \to K^\infty(X)$$

by sending $\mathcal{O}_X^\infty \twoheadrightarrow E$ to $\mathcal{O}_X^\infty \cong \mathcal{O}_X^\infty \otimes \mathcal{O}_X^n \twoheadrightarrow E \otimes L'$. Here, the isomorphism is the inverse of the isomorphism sending $e_i \otimes f_j$ to (e_{in+j}), where $e_0, e_1, ...$ is a basis of $\mathcal{O}_X^\infty$ and $f_0, ..., f_{n-1}$ is a basis of $\mathcal{O}_X^n$.

Given two quotients $\mathcal{O}_X^m \twoheadrightarrow L$ and $\mathcal{O}_X^n \twoheadrightarrow L'$, let $\mathcal{O}_X^{nm} \cong \mathcal{O}_X^m \otimes \mathcal{O}_X^n \twoheadrightarrow L \otimes L'$ be the evident induced quotient, where the isomorphism here is the inverse of the map sending $e_i \otimes f_j$ to e_{in+j}. With this choice, the square

$$\begin{array}{ccc} K^\infty(X) & \xrightarrow{\alpha_L} & K^\infty(X) \\ {\scriptstyle \beta_{L'}} \downarrow & & {\scriptstyle \beta_{L'}} \downarrow \\ K^\infty(X) & \xrightarrow{\alpha_{L \otimes L'}} & K^\infty(X) \end{array}$$

commutes. Iterating this procedure, we get a system of maps of spaces indexed by $\mathbb{N} \times \mathbb{N}$:

$$\begin{array}{ccccc}
K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_L} & K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_L} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_{L\otimes L'}} & K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_{L\otimes L'}} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & K^\infty(X)^{\mathrm{an}} & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
\vdots & & \vdots & &
\end{array} \tag{5.1.1}$$

We write the two-dimensional array (5.1.1) as $\{K^\infty(X)^{\mathrm{an}}; \alpha, \beta\}$.

Proposition 5.1. *Let X be a projective complex variety and consider the two dimensional array $\{K^\infty(X)^{\mathrm{an}}; \alpha, \beta\}$.*

(1) *The mapping telescope of any row of this array is naturally homotopy equivalent to $\mathcal{K}^{\mathrm{semi}}(X)$.*
(2) *The map between the mapping telescopes of any two consecutive rows of this array gives a homotopy class of maps*

$$\rho : \mathcal{K}^{\mathrm{semi}}(X) \longrightarrow \mathcal{K}^{\mathrm{semi}}(X),$$

such that for any finite C. W. complex T, the induced automorphism of

$$[T, \mathcal{K}^{\mathrm{semi}}(X)] \cong \varinjlim_{[T\to U^{\mathrm{an}}]\in hAff} K_0^{\mathrm{semi}}(U \times X)$$

is multiplication by $[pr_X^* L']$ on $K_0(U \times X)$ for each fixed $[T \to U^{\mathrm{an}}]$.

Proof. The telescope of any row of $\{K^\infty(X)^{\mathrm{an}}; \alpha, \beta\}$ is homotopy equivalent to $\mathcal{K}^{\mathrm{semi}}(X)$ by Theorem 3.4 and the proof of Corollary 3.5.

To verify (2), fix a finite C. W. complex T and an integer j and consider the natural isomorphisms

$$\begin{aligned}
[T, \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha_{L\otimes L'^{\otimes j}})] &\cong \varinjlim_{\mathbb{N}} ([T, K^\infty(X)^{\mathrm{an}}], \alpha_{L\otimes L'^{\otimes j}}) \\
&\cong \varinjlim_{\mathbb{N}} \varinjlim_{[T\to U^{\mathrm{an}}]\in hAff^T} \pi_0 \mathcal{M}or(U, K^\infty(X))^{\mathrm{an}} \\
&\cong \varinjlim_{[T\to U^{\mathrm{an}}]\in hAff^T} K_0^{\mathrm{semi}}(U \times X).
\end{aligned}$$

The first of these isomorphisms is a well known property of mapping telescopes, the second is given by Proposition 4.4, and the last by Proposition 3.1. The map β from row j to row $j+1$ induces for each fixed $n \in \mathbb{N}$ and $[T \to U^{\mathrm{an}}]$ the map $\pi_0\mathcal{M}or(U, K^\infty(X))^{\mathrm{an}} \to \pi_0\mathcal{M}or(U, K^\infty(X))^{\mathrm{an}}$ given by tensoring locally free $\mathcal{O}_{U\times X}$-modules by the line bundle $pr_X^* L'$. Therefore, the induced map from $[T, \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha_{L\otimes L'^{\otimes j}})]$ to $[T, \mathrm{Tel}(K^\infty(X)^{\mathrm{an}}, \alpha_{L\otimes L'^{\otimes j+1}})]$ is the self map on $\varinjlim_{[T\to U^{\mathrm{an}}]\in hAff^T} K_0^{\mathrm{semi}}(U\times X)$ given by sending $([T\to U^{\mathrm{an}}],[E])$ to $([T\to U^{\mathrm{an}}],[E\otimes pr_X^* L'])$, which is an isomorphism. □

We proceed to apply this in the special case in which X is of the form $X=\mathbb{P}(E)$, where Y is a projective complex variety and E a vector bundle of rank n on Y. Here, $\mathbb{P}(E)$ is defined by applying $\mathrm{Proj}(-)$ to the graded $\mathcal{O}_Y$-algebra $\mathrm{Sym}\,E$, the symmetric algebra on E. Let $\mathcal{O}_X^m \twoheadrightarrow L$ be any surjection with L a very ample line bundle on $\mathbb{P}(E)$. Let L' denote $\mathcal{O}_{\mathbb{P}(E)}(1)$, the canonical line bundle of $X=\mathbb{P}(E)$, and recall that there is a canonical surjection $\mathcal{O}_X^n \to L'$.

Theorem 5.2. *Let Y be a projective complex variety, E a rank n vector bundle on Y, and $\pi:\mathbb{P}(E)\to Y$ the associated projective bundle over Y. Define $\rho:\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E))\to\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E))$ to be the weak equivalence associated to the map between mapping telescopes of the first two rows of $\{K^\infty(\mathbb{P}(E))^{\mathrm{an}};\alpha,\beta\}$.*

Then the maps $\rho^{\circ i}\circ\pi^*:\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E))\to\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E))$ for $i=0,\dots,n-1$ induce a natural homotopy class of weak equivalences

$$\sum_i \rho^{\circ i}\circ\pi^*:\mathcal{K}^{\mathrm{semi}}(Y)^{\times n}\xrightarrow{\sim}\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E)).$$

Proof. It suffices to fix a finite C. W. complex and show that the induced map

$$[T,\mathcal{K}^{\mathrm{semi}}(Y)^{\times n}]\equiv[T,\mathcal{K}^{\mathrm{semi}}(Y)]^{\times n}\longrightarrow[T,\mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E))]$$

is an isomorphism. Theorem 5.2 (2) implies that the the map $\rho^{\circ i}\circ\pi^*$ induces the map

$$\varinjlim_{[T\to U^{an}]\in hAff^T} K_0^{\mathrm{semi}}(U\times Y)\longrightarrow \varinjlim_{[T\to U^{an}]\in hAff^T} K_0^{\mathrm{semi}}(U\times\mathbb{P}(E))$$

which for each fixed $[T\to U^{\mathrm{an}}]$ gives the map $K_0^{\mathrm{semi}}(U\times Y)\longrightarrow K_0^{\mathrm{semi}}(U\times\mathbb{P}(E))$ defined by $[E]\mapsto[\pi^*E\otimes pr^*_{\mathcal{O}_{P(E)}}\mathcal{O}(i)]$. It therefore suffices to show that

$$K_0^{\mathrm{semi}}(U\times Y)^{\times n}\longrightarrow K_0^{\mathrm{semi}}(U\times\mathbb{P}(E))$$

given by $\Big([E_0],\dots,[E_{n-1}])\mapsto\sum_i[\pi^*E_i\otimes pr^*_{\mathcal{O}_{P(E)}}\mathcal{O}(i)]\Big)$ is an isomorphism for all quasi-projective complex varieties U. For any smooth variety T with closed points t_0, t_1 the diagram

$$\begin{array}{ccc}
K_0(T\times U\times X)^{\times n} & \longrightarrow & K_0(T\times U\times\mathbb{P}(E))\\
\downarrow{\scriptstyle \iota_{t_0}^*-\iota_{t_1}^*} & & \downarrow{\scriptstyle \iota_{t_0}^*-\iota_{t_1}^*}\\
K_0(U\times X)^{\times n} & \longrightarrow & K_0(U\times\mathbb{P}(E))\\
\downarrow & & \downarrow\\
K_0^{\mathrm{semi}}(U\times X)^{\times n} & \longrightarrow & K_0^{\mathrm{semi}}(U\times\mathbb{P}(E))
\end{array}$$

is easily seen to commute. The top two horizontal arrows are isomorphisms from the projective bundle formula for algebraic K_0 [34; VI.1.1], and thus the bottom horizontal arrow is an isomorphism as well. □

Applying induction on the number k of factors, we immediately obtain the following explicit calculation.

Corollary 5.3. *For any projective complex variety X, there is a natural homotopy equivalence*

$$\mathcal{K}^{\mathrm{semi}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times X) \;\cong\; \mathcal{K}^{\mathrm{semi}}(X)^{\times n},$$

where $n = \prod_i (n_i + 1)$.

In particular, for any $j \geq 0$

$$K^{\mathrm{semi}}_{2j}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) = \mathbb{Z}^n, \quad K^{\mathrm{semi}}_{2j+1}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) = 0.$$

With the aid of Corollary 3.5, we can restate Corollary 5.3 in terms of spaces of morphisms between projective varieties.

Corollary 5.4. *Let X be a projective complex variety. If $\mathcal{O}_X(1)$ is an ample line bundle on X and $\mathcal{O}_X^m \twoheadrightarrow \mathcal{O}_X(1)$ is a chosen surjection, then there is a natural isomorphism*

$$\varinjlim_{j \in \mathbb{N}} \pi_i \mathcal{M}or(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times X, \mathrm{Grass}_j(\mathbb{P}^{j(m+2)}))^{\mathrm{an}} \cong$$
$$\left(\varinjlim_{j \in \mathbb{N}} \pi_i \mathcal{M}or(X, \mathrm{Grass}_j(\mathbb{P}^{j(m+2)}))^{\mathrm{an}} \right)^{\times n},$$

where $n = \Sigma_i (n_i + 1)$ and the transition maps maps are defined as in Corollary 3.5.

§6 Total Segre Class

In this section, we place $\mathcal{K}^{\mathrm{semi}}$ in a commutative diagram (namely, (6.11.1)) of infinite loop spaces involving K-theories and cohomology theories. The existence of such a diagram was suggested in [8]. An equivalent version of the right-hand portion of diagram (6.11.1) is considered in [24].

In the proposition below, we introduce the Segre map $s_{N,n}$. We remind the reader that the total Segre class $s_*(E)$ of a vector bundle E satisfies

$$s_*(E) \;=\; c_*(-E) \;=\; (c_*(E))^{-1}$$

where c_* denotes the total Chern class.

Proposition 6.1. *For any $N > k \geq 0$ define the morphism of projective complex varieties*

$$s_{N,k} : \mathrm{Grass}_k(\mathbb{P}^N) \to C_{k-1,1}(\mathbb{P}^N)$$

to be the map which is defined (in terms of the functors they represent) by sending a quotient $\mathcal{O}_X^{N+1} \twoheadrightarrow E$, for some variety X, to the cycle $\mathbb{P}(E^\wedge)$ (where $E^\wedge$ is the $\mathcal{O}_X$-linear dual of E) on $X \times \mathbb{P}^N$ which is equidimensional of relative dimension $k-1$ over X. Then the maps $s_{N,k}$ determine a natural transformation of functors on smooth $\mathbb{C}$-schemes

$$s_\infty : K^\infty \to \mathcal{C}^\infty$$

by taking the direct limit indexed by N and the disjoin union indexed by k. This natural transformation commutes with the action of $\mathcal{I}^{\mathrm{an}}$ discussed in §2, in the sense that for each $n \geq 0$ we have commutative squares of functors

$$\begin{array}{ccc} I(n) \times (K^\infty)^{\times n} & \longrightarrow & K^\infty \\ {\scriptstyle 1 \times s_\infty^{\times n}} \downarrow & & \downarrow {\scriptstyle s_\infty} \\ I(n) \times (\mathcal{C}^\infty)^{\times n} & \longrightarrow & \mathcal{C}^\infty. \end{array} \qquad (6.1.1)$$

Proof. As defined, the $s_{N,k}$ stabilize with respect to N for each k and thereby determine the functor $s_\infty : K^\infty \to \mathcal{C}^\infty$.

The verification that s_∞ commutes with the action of $\mathcal{I}^{\mathrm{an}}$ is a straight-forward application of the observation that the projective bundle $\mathbb{P}((E \oplus E')^\wedge)$ associated to the (external) direct sum $\mathcal{O}_X^{N+1} \oplus \mathcal{O}_X^{N+1} \to E \oplus E'$ is the (external) fibre-wise join $\mathbb{P}(E^\wedge) \#_X \mathbb{P}(E'^\wedge) \subset \mathbb{P}^{2N+1} \times X$. □

Let X be a quasi-projective complex variety. For any $r \geq -1$, we introduce the notation $\mathcal{Z}_r(X, \mathbb{P}^\infty)$ to denote the naive group completion of the topological abelian monoid $\coprod_{d \geq 0} \mathcal{M}or(X, C_{r,d}(\mathbb{P}^\infty))^{an}$. We recall that the natural map

$$\coprod_{d \geq 0} \mathcal{M}or(X, C_{r,d}(\mathbb{P}^\infty))^{an} \longrightarrow \mathcal{Z}_r(X, \mathbb{P}^\infty)$$

is a homotopy-theoretic group completion (cf. [9]) whenever X is weakly normal.

Observe that the additive map (of schemes)

$$\deg : \mathcal{C}^\infty = \coprod_{r \geq -1} \coprod_{d \geq 0} \varinjlim_N C_{r,d}(\mathbb{P}^N) \longrightarrow \mathbb{N}$$

sending $C_{r,d}(\mathbb{P}^N)$ to d extends to a continuous map

$$\deg : \mathcal{Z}_r(X, \mathbb{P}^\infty) \longrightarrow \mathbb{Z}$$

whenever X is connected. In the corollary below, we consider the subspace

$$\mathcal{Z}_r(X, \mathbb{P}^\infty)_1 \;\equiv\; \deg^{-1}(1) \;\subset\; \mathcal{Z}_r(X, \mathbb{P}^\infty).$$

Corollary 6.2. *For a quasi-projective complex variety X, the map s_∞ of Proposition 6.1 determines maps of $\mathcal{I}^{\mathrm{an}}$-spaces*

$$K^\infty(X)^{an} \to \mathcal{C}^\infty(X)^{an} \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty).$$

Moreover, if X is connected, then the composition of these maps factors as a composition of $\mathcal{I}^{\mathrm{an}}$-spaces

$$K^\infty(X)^{an} \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty)_1 \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty).$$

We call the map $K^\infty(X)^{\mathrm{an}} \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty)_1$ the "Segre map" and write it as s.

Proof. The fact that s_∞ is a map of $\mathcal{I}^{\mathrm{an}}$-spaces follows from the observation made in Proposition 2.6 that the natural transformations of functors given in Proposition 6.1 and fitting in the commutative squares (6.1.1) induce continuous maps on associated analytic spaces. The factorization follows from the observation that for any quotient $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{O}_X^{N+1} \twoheadrightarrow E$ the cycle $\mathbb{P}(E^\wedge) \subset \mathbb{P}^N \times X$ is fiberwise linear. □

We next formulate two lemmas which will enable us to "identify" the space $\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$ with its H-space structure given by the $\mathcal{I}(2)^{an}$-pairing

$$\mathcal{I}(2)^{\mathrm{an}} \times \left(\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \right)^{\times 2} \to \coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1.$$

We recall from [10] the notation

$$\mathcal{Z}^j(X) \;\equiv\; \mathcal{Z}_0(X,\mathbb{P}^j)/\mathcal{Z}_0(X,\mathbb{P}^{j-1}), \quad j \geq 1,$$

for the quotient topological abelian group associated to the map $\mathcal{Z}_0(X,\mathbb{P}^{j-1}) \hookrightarrow \mathcal{Z}_0(X,\mathbb{P}^j)$ induced by inclusion $\mathbb{P}^{j-1} \hookrightarrow \mathbb{P}^j$ into the first j homogeneous coordinates. For $j = 0$, we may identify $\coprod_d \mathcal{M}or(X, C_{0,d}(\mathbb{P}^0))^{\mathrm{an}}$ with the free abelian monoid on $\pi_0(X)$ and $\mathcal{Z}^0(X)$ denotes the group completion of this monoid.

In the following lemma, we give an alternative description of the spaces $\mathcal{Z}_r(X,\mathbb{P}^\infty)$.

Lemma 6.3. *Let X be a quasi-projective complex variety. For each $r \geq 0$, there is a natural homotopy equivalence given by algebraic suspension*

$$\Sigma^r : \; \mathcal{Z}_0(X,\mathbb{P}^\infty) \xrightarrow{\sim} \mathcal{Z}_r(X,\mathbb{P}^\infty). \tag{6.3.1}$$

Moreover, there is a natural homotopy equivalence

$$\mathcal{Z}_0(X,\mathbb{P}^\infty) \xrightarrow{\sim} \varinjlim_N \prod_{j=0}^{N} \mathcal{Z}^j(X). \tag{6.3.2}$$

If X is connected, the map (6.3.1) restricts to give a natural homotopy equivalence

$$\Sigma^r : \; \mathcal{Z}_0(X,\mathbb{P}^\infty)_1 \xrightarrow{\sim} \mathcal{Z}_r(X,\mathbb{P}^\infty)_1$$

and the map (6.3.2) restricts to a homotopy equivalence

$$\mathcal{Z}_0(X,\mathbb{P}^\infty)_1 \xrightarrow{\sim} \{1\} \times \varinjlim_N \prod_{j=1}^{N} \mathcal{Z}^j(X).$$

Finally, for any X the space $\mathcal{Z}_{-1}(X,\mathbb{P}^\infty)$ is the discrete free abelian group on $\pi_0(X)$, so that if X is connected then $\mathcal{Z}_{-1}(X,\mathbb{P}^\infty)_1$ consists of a single point.

Proof. The fact that algebraic suspension is a weak equivalence is verified in [10; 3.3]; the fact that the spaces involved have the homotopy type of C. W. complexes is proved in [9]. The splitting (6.3.2) is verified in [10; 2.10]. The weak equivalences of (6.3.1) are easily seen to be compatible with the degree maps, which gives the third formula. For X connected, the degree map $\deg : \mathcal{Z}_0(X,\mathbb{P}^\infty) \to \mathbb{Z}$ is easily seen to correspond to the map $\varinjlim_N \prod_{j=0}^N \mathcal{Z}^j(X) \to \mathbb{Z}$ given by projection to $\mathcal{Z}^0(X) \cong \mathbb{Z}$ under the weak equivalence of (6.3.2), and thus the fourth formula holds. The last assertion is evident. □

The following lemma identifies $\pi_0\left(\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1\right)$.

Lemma 6.4. *Let X be a connected quasi-projective complex variety. Then as an abelian monoid with additive structure given by the action of $\mathcal{I}(2)^{an}$,*

$$\pi_0\left(\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1\right) \subset \mathbb{N}\times\pi_0(\mathcal{Z}_0(X,\mathbb{P}^\infty)_1) \cong \mathbb{N}\times\bigoplus_{j\geq 1}\pi_0(\mathcal{Z}^j(X))$$

consists of pairs (m,ζ) such that $\zeta = 0$ if $m = 0$, where addition is defined by $(m,[\zeta])+(m',[\zeta']) = (m+m',[\zeta\#_X\zeta'])$. Here, $\zeta\#_X\zeta'$ is the cup product pairing of [10; §6] *induced by fiberwise join.*

Furthermore, if X is also smooth, then

$$pr_2(\pi_0(\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1)\cong\pi_0(\mathcal{Z}_0(X,\mathbb{P}^\infty)_1)\cong\bigoplus_{j\geq 1}\pi_0(\mathcal{Z}^j(X))\cong\{1\}\times A^{\geq 1}(X)$$

is an abelian group whose additive structure is given by intersection of cycles.

Proof. The first statement follows from the identification of $\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$ given in Lemma 6.3 plus the observation that the H-space structure given by the $\mathcal{I}(2)^{an}$ pairing determines fiberwise join of cycles. Here, we are describing the pairing after applying Σ^{-r+1}, a chosen homotopy inverse of Σ^{r-1}, to $\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$, and thus we have implicitly used the fact that the fiberwise join pairing commutes up to homotopy with fiberwise suspension.

For X smooth of dimension D, the duality theorem of [9] implies the existence of a homotopy equivalence

$$\mathcal{Z}^j(X)\xrightarrow{\sim}\mathcal{Z}_{D-j}(X),\quad j\leq D,$$

where $\mathcal{Z}_{D-j}(X)$ denotes the naive group completion of $\coprod_{d\geq 0}C_{D-j,d}(X)^{an}$ whose group of connected components is $A^j(X)\equiv A_{D-j}(X)$ by [7]. For $j>D$, [7] once again implies

$$\mathcal{Z}^j(X)\cong\mathcal{Z}_0(X\times\mathbb{A}^{D-j})\equiv\mathcal{Z}_0(X\times\mathbb{P}^{D-j})/\mathcal{Z}_0(X\times\mathbb{P}^{D-j-1})$$

whose group of connected components is trivial. □

The preceding lemmas permit us to determine the homotopy theoretic group completion of $\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$. We require X to be smooth in order to guarantee that the direct sum $\bigoplus_{j\geq 1}\pi_0(\mathcal{Z}^j(X))$ is finite.

Proposition 6.5. *If X is a smooth, connected quasi-projective complex variety, then the homotopy theoretic group completion*

$$\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1\to\Omega^\infty\Sigma^\infty\left(\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1\right)$$

can be identified with the map

$$\theta:\coprod_{r\geq 0}\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1\to\mathbb{Z}\times\mathcal{Z}_0(X,\mathbb{P}^\infty)_1$$

which restricts to the homotopy equivalence

$$\theta|_{\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1} = \Sigma^{-r+1} : \ \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \xrightarrow{\sim} \{r\} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1$$

for each $r \geq 1$ (and which restricts to the evident constant map when $r = 0$).

Consequently, there is a natural isomorphism

$$\pi_j \left(\Omega^\infty \Sigma^\infty (\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1) \right) \ \cong \ \bigoplus_{q \geq 0} L^q \, \mathrm{H}^{2q-j}(X)$$

for any $j \geq 0$, where $L^q \, \mathrm{H}^{2q-j}(X)$ denotes the morphic cohomology of X as introduced in [10] *and formulated in* [9]*.*

Proof. The description of $\pi_0 \left(\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \right)$ given in Lemma 6.4 shows that to group complete this monoid we merely need to invert the element $(1,[X]) \in \mathbb{N} \times A_D(X)$, where $D = \dim X$. Let $\epsilon \in \mathcal{Z}_0(X,\mathbb{P}^\infty)$ correspond to the effective cycle $X \times \{[1:0:\dots]\} \subset X \times \mathbb{P}^\infty$, and observe that the class of ϵ in $\pi_0 \left(\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \right)$ corresponds to $(1,[X])$ under the isomorphism of Lemma 6.4. The H-space structure of $\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$ determined by the interleaving map (2.11) defines the map

$$\alpha_r : \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \longrightarrow \mathcal{Z}_r(X,\mathbb{P}^\infty)_1$$

given by addition by ϵ. In fact, we readily verify that α_r is homotopic to the suspension map Σ. Taking the disjoint union of the α_r, we obtain the map

$$\alpha : \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \longrightarrow \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1.$$

The same argument as that given in the proof of Theorem 3.4 establishes that the mapping telescope

$$\mathrm{Tel}(\coprod_r \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1, \alpha)$$

gives the homotopy-theoretic group completion of $\coprod_r \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$.

Observe that the square

$$\begin{array}{ccc} \coprod_r \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 & \xrightarrow{\alpha} & \coprod_r \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1 \\ \theta\downarrow & & \theta\downarrow \\ \mathbb{Z} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1 & \xrightarrow{\beta} & \mathbb{Z} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1 \end{array}$$

commutes up to homotopy, where β is the shift map, sending $\{i\} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1$ identically to $\{i+1\} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1$. Thus, there is an induced map on telescopes

$$\mathrm{Tel}(\coprod_r \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1, \alpha) \longrightarrow \mathrm{Tel}(\mathbb{Z} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1, \beta) \cong \mathbb{Z} \times \mathcal{Z}_0(X,\mathbb{P}^\infty)_1$$

which is a homology equivalence of homotopy commutative group-like H-spaces, and hence a homotopy equivalence.

Since $L^q \, \mathrm{H}^{2q-j}(X)$ is defined to be $\pi_j(\mathcal{Z}^q(X))$, the second statement follows immediately from the splitting $\mathcal{Z}_0(X,\mathbb{P}^\infty) \cong \varinjlim_N \prod_{q=0}^N \mathcal{Z}^q(X)$. □

The following is a generalization (in the case of smooth varieties) to $K_*^{\mathrm{semi}}(-)$ of a Whitney sum formula for $K_0^{\mathrm{semi}}(-)$ established in [13;5.4]

Proposition 6.6. *For a smooth quasi-projective complex variety X, write*

$$s : \mathcal{K}^{\mathrm{semi}}(X) \longrightarrow \Omega^\infty\Sigma^\infty(\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1) \cong \mathbb{Z}\times\left(\{1\}\times \varinjlim_N \prod_{j=1}^N \mathcal{Z}^j(X)\right)$$

for the map induced by the map of Corollary 6.2. Write

$$\sum_q s_q : K_j^{\mathrm{semi}}(X) \longrightarrow \bigoplus_q L^q\,\mathrm{H}^{2q-j}(X)$$

for the induced map on homotopy groups, using the decomposition of the target given in Proposition 6.5. Then the Whitney sum formula

$$s_n(\alpha+\beta) = \sum_{i+j=n} s_i(\alpha)\# s_j(\beta),$$

where # is the pairing of [13; 3.2], *holds for all $\alpha,\beta \in K_j^{\mathrm{semi}}(X)$.*

Proof. It follows from [13; 5.3] that the H-space structure on

$$\Omega^\infty\Sigma^\infty(\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1) \cong \mathbb{Z}\times\left(\{1\}\times \varinjlim_N \prod_{j=1}^N \mathcal{Z}^j(X)\right),$$

which is given by linear join of cycles, is a graded operation. That is, using the fact that this space is equivalent to the direct sum of topological abelian groups $\mathbb{Z}\oplus\bigoplus_{j=1}^\infty \mathcal{Z}^j(X)$, the H-space product map restricts to the bilinear pairing

$$\# : \mathcal{Z}^j(X)\times\mathcal{Z}^k(X) \to \mathcal{Z}^{j+k}(X)$$

given in [13; 3.2]. (Here, we identify $\mathcal{Z}^0(X)$ with $\mathbb{Z}$.) Since s is a map of H-spaces, the result follows immediately. □

We now repeat the above construction replacing the space of continuous algebraic maps by the space of continuous maps of associated analytic spaces. Observe that for any r, $C_r(\mathbb{P}^\infty)^{\mathrm{an}} \equiv \varinjlim_N \coprod_d C_{r,d}(\mathbb{P}^N)^{\mathrm{an}}$ is a topological abelian monoid under addition of cycles. We let $\mathcal{Z}_r(\mathbb{P}^N)^{\mathrm{an}}$ stand for the homotopy-theoretic group completion of this abelian monoid – i.e., we set $\mathcal{Z}_r(\mathbb{P}^N)^{\mathrm{an}} \equiv \Omega B C_r(\mathbb{P}^\infty)^{\mathrm{an}}$. Additionally, for a complex variety X, we let $\mathcal{Z}_r(X^{\mathrm{an}},\mathbb{P}^\infty)$ denote the mapping space $\mathcal{M}\mathrm{aps}(X^{\mathrm{an}},\mathcal{Z}_r(\mathbb{P}^\infty)^{\mathrm{an}})$, where in general $\mathcal{M}$aps refers to the internal Hom-object for the category of compactly generated spaces – i.e., $\mathcal{M}\mathrm{aps}(S,T)$ is the compactly generated space associated to the space of continuous maps from S to T endowed with the compact-open topology (cf. [32]). Equivalently, we could define $\mathcal{Z}_r(X^{\mathrm{an}},\mathbb{P}^\infty)$ as $\Omega B\left(\varinjlim_N \mathcal{M}\mathrm{aps}(X^{\mathrm{an}}, C_r(\mathbb{P}^N)^{\mathrm{an}})\right)$.

As above, we have a continuous degree map $\deg : \mathcal{Z}_r(X^{an},\mathbb{P}^\infty)\to\mathbb{Z}$ provided that X is connected, and we set

$$\mathcal{Z}_r(X^{an},\mathbb{P}^\infty)_1 \;\equiv\; \deg^{-1}(1) \;\subset\; \mathcal{Z}_r(X^{an},\mathbb{P}^\infty).$$

The evident analogue of the pairing of Proposition 2.6 gives a continuous pairing

$$\mathcal{I}(n)^{\mathrm{an}} \times \left(\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X^{\mathrm{an}}, \mathbb{P}^{\infty})_1 \right)^{\times n} \longrightarrow \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X^{\mathrm{an}}, \mathbb{P}^{\infty})_1,$$

making $\coprod_r \mathcal{Z}_r(X^{\mathrm{an}}, \mathbb{P}^{\infty})_1$ into an $\mathcal{I}^{\mathrm{an}}$-space.

The following proposition is a restatement of the main result of [3]. Our discussion of this result differs from that in [3] in part because we have emphasized the somewhat surprising fact that the degree 1 part of the $\mathcal{I}^{\mathrm{an}}$-space $\coprod_r \mathcal{Z}_{r-1}(X^{an}, \mathbb{P}^{\infty})_1$ nearly coincides with its homotopy-theoretic group completion, whereas [3] emphasizes the observation that $(K^{\infty})^{an} \to (\mathcal{C}^{\infty})^{an}$ induces a map of infinite loop spaces. Another difference is that we do not consider spaces of the form $\varinjlim_{N,r} C_{r,d}(\mathbb{P}^N)$ for each $d \geq 0$.

The reader should observe that the construction of [23] leading to that of [3] sends a linear subspace V of codimension n in $\mathbb{C}^{N+1}$ to the associated codimension n cycle $\mathrm{Proj}(\mathrm{Sym}\, V^*) \subset \mathrm{Proj}(\mathrm{Sym}\, \mathbb{C}^{N+1}) = \mathbb{P}^N$. On the other hand, our map s sends such a linear subspace V to the $n-1$-dimensional subspace $\mathrm{Proj}(\mathrm{Sym}(\mathbb{C}^{N+1}/V)^*) \subset \mathbb{P}^N$. Thus, while [23] realizes the total Chern class of a vector bundle $\mathcal{E} \subset \mathcal{O}_X^{N+1}$, we are realizing the total Chern class of $\mathcal{O}_X^{N+1}/\mathcal{E}$ which is the total Segre class of $\mathcal{E}$.

Let $\underline{bu}(T)$ denote the connective topological K-theory spectrum of a space T. We define $\underline{K(\mathbb{Z}, 2*)}(X^{\mathrm{an}})$ to be the Ω-spectrum defined by the $\mathcal{I}^{\mathrm{an}}$-space $\coprod_r \mathcal{Z}_r(X^{\mathrm{an}}, \mathbb{P}^{\infty})_1$. (The choice of notation is justified by the proposition.)

Proposition 6.7. *(cf.* [3]*) For any connected quasi-projective complex variety X, the map s_{∞} of Proposition 6.1 determines a map of $\mathcal{I}^{an}$-spaces*

$$s : \mathcal{M}\mathrm{aps}(X^{\mathrm{an}}, (K^{\infty})^{\mathrm{an}}) \longrightarrow \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X^{\mathrm{an}}, \mathbb{P}^{\infty})_1,$$

and consequently a map of Ω-spectra

$$s : \underline{bu}(X^{\mathrm{an}}) \longrightarrow \underline{K(\mathbb{Z}, 2*)}(X^{\mathrm{an}}). \tag{6.7.1}$$

Moreover, the 0^{th} space of $\underline{K(\mathbb{Z}, 2*)}(X^{\mathrm{an}})$ is homotopy equivalent to

$$\mathcal{M}\mathrm{aps}(X^{\mathrm{an}}, \prod_{q \geq 0} K(\mathbb{Z}, 2q)),$$

and the map on homotopy groups associated to (6.7.1) is the total Segre class map

$$s : K_{\mathrm{top}}^{-j}(X^{\mathrm{an}}) \longrightarrow \bigoplus_{q \geq 0} \mathrm{H}^{2q-j}(X^{\mathrm{an}}, \mathbb{Z}), \quad j \geq 0.$$

Proof. The existence of the map s follows from Proposition 6.1 exactly as does the corresponding statement in Corollary 6.2. The identification of the Ω-spectrum associated to $\mathcal{M}\mathrm{aps}(X^{\mathrm{an}}, (K^{\infty})^{\mathrm{an}})$ with $\underline{bu}(X^{\mathrm{an}})$ is given in [27; I.1] (since we may equivalently form this Ω-spectrum using the linear isometries operad).

To identify the 0^{th} space of $\underline{K(\mathbb{Z},2*)}(X^{\text{an}})$, we observe that an argument parallel to the proof of Proposition 6.5 shows that the homotopy theoretic group completion of $\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X^{an},\mathbb{P}^\infty)_1$ is given by

$$\mathbb{Z}\times \mathcal{Z}_0(X^{an},\mathbb{P}^\infty)_1.$$

We next observe that $\mathcal{Z}_0(\mathbb{P}^N)^{an} \simeq \prod_{q=0}^N K(\mathbb{Z},2q)$ by the Lawson suspension theorem [21], so that

$$\mathcal{Z}_0(X^{an},\mathbb{P}^\infty)_1 \equiv \mathcal{M}\text{aps}(X^{\text{an}},\mathcal{Z}_0(\mathbb{P}^\infty))_1 \cong \mathcal{M}\text{aps}(X^{\text{an}},\prod_{q=1}^{\infty} K(\mathbb{Z},2q)).$$

The verification that the resulting map on homotopy groups gives the total Segre class is essentially done in [23]. □

For varieties X and Y, we write $\text{Hom}(X,Y)$ for the set of morphisms of varieties, and we let $\text{Hom}(X\times\Delta^\bullet,Y)$ denote the simplicial set $n\mapsto \text{Hom}(X\times\Delta^n,Y)$.

Proposition 6.8 (cf. [17, 3.3]**).** *For any quasi-projective variety X over a field k, the Ω-spectrum associated to the $|\mathcal{I}(\Delta^\bullet)|$-space $|\text{Hom}(X\times\Delta^\bullet,K^\infty)|$ is equivalent to the K-theory spectrum $\underline{\mathcal{K}}(X\times\Delta^\bullet)$, defined as the geometric realization of the simplicial spectrum $n\mapsto\mathcal{K}(X\times\Delta^n)$.*

Proof. This is basically a special case of Theorem 3.3 of [17], but in that paper Segal's notion of a Γ-space [30] is used to exhibit an infinite loop space structure on $|\text{Hom}(X\times\Delta^\bullet,K^\infty)|$ in lieu of the $|\mathcal{I}(\Delta^\bullet)|$-space technique employed here. We show how the results of [28] establish that these two potentially different infinite loop space structures are in fact equivalent.

In [17], the Γ-space $\left|\text{Hom}(X\times\Delta^\bullet,(K^\infty)^{(n)})\right|$, $n\geq 0$, is introduced, where $(K^\infty)^{(n)}$ is the ind-scheme parameterizing n-tuples of quotients of k^∞ which are in general position. Let us write $Y^{(n)}$ for the space $|\text{Hom}(X\times\Delta^\bullet,(K^\infty)^{(n)}|$, observing that $Y\equiv Y^{(1)}$ is precisely the $|\mathcal{I}(\Delta^\bullet)|$-space considered in this paper. In the language of [28], the collection $Y^{(n)}$, $n\geq 0$, is an $\mathcal{F}$-space, where $\mathcal{F}$ is a (in fact, the canonical) "category of operators" (cf. [28; 1.2]). (Note that the category $\mathcal{F}$ is written as Γ^{op} in [17].)

Let $\hat{\mathcal{I}}$ denote the category of operators associated to the operad $|\mathcal{I}(\Delta^\bullet)|$ (cf. [28; 4.1]) and observe there is a natural map $\hat{\mathcal{I}}\to\mathcal{F}$ of categories of operators. Thus, we may view the collection $Y^{(n)}$, $n\geq 0$, as an $\hat{\mathcal{I}}$-space as well. Since Y is an $|\mathcal{I}(\Delta^\bullet)|$-space (in our terminology), the collection $Y^{\times n}$, $n\geq 0$, is an $\hat{\mathcal{I}}$-space by [28; 4.2]. Moreover, the natural inclusion maps

$$Y^{(n)} \xrightarrow{\sim} Y^{\times n}$$

are homotopy equivalences of spaces, for all n, by [17; 2.2]. These equivalences are clearly compatible with the actions of $\hat{\mathcal{I}}$, and thus give an equivalence of $\hat{\mathcal{I}}$-spaces and hence an equivalence of the associated infinite loop spaces.

The infinite loop space associated to the $\hat{\mathcal{I}}$-space $Y^{(n)}$, $n\geq 0$, is equivalent to the infinite loop space constructed in [17] by [28; 2.5], while the infinite loop space associated to the $\hat{\mathcal{I}}$-space $Y^{\times n}$, $n\geq 0$, is equivalent to the infinite loop space considered in this paper by [28; 6.4]. □

Homotopy invariance of algebraic K-theory for smooth k-schemes implies that the natural map

$$K_q(X) \;\to\; \pi_q \underline{\mathcal{K}}(X \times \Delta^\bullet)$$

is an isomorphism for all q whenever X is smooth.

We let

$$Z_r(X \times \Delta^\bullet) \;\equiv\; n \mapsto \left(\varinjlim_N \coprod_d \operatorname{Mor}(X \times \Delta^n, C_{r,d}(\mathbb{P}^N)) \right)^+$$

denote the simplicial abelian group defined as the (level-wise) group completion of the simplicial monoid $n \mapsto \varinjlim_N \coprod_d \operatorname{Mor}(X \times \Delta^n, C_{r,d}(\mathbb{P}^N))$. Once again, we have a well defined degree map $\deg : \; Z_r(X \times \Delta^\bullet) \to \mathbb{Z}$ whenever X is connected, and we write $Z_r(X \times \Delta^\bullet)_1$ for the pre-image of $\{1\}$.

In view of Proposition 6.8, we can repeat the constructions of Corollary 6.2 and Proposition 6.7 in the context of algebraic K-theory and motivic cohomology.

Proposition 6.9. *For any connected quasi-projective variety X over a field k, the map s_∞ of Proposition 6.1 determines a map of $|\mathcal{I}(\Delta^\bullet)|$-spaces*

$$s : |\operatorname{Hom}(X \times \Delta^\bullet, K^\infty)| \to \coprod_{r \geq 0} |Z_r(X \times \Delta^\bullet)_1|.$$

Furthermore, if k has characteristic 0 and X is smooth, then the homotopy groups of $\Omega^\infty \Sigma^\infty \coprod_{r \geq 0} |Z_{r-1}(X \times \Delta^\bullet)_1|$ are the motivic cohomology groups of X.

Proof. Observe that for arbitrary varieties X and Y, there is a natural map $\operatorname{Hom}(X, Y) \to \operatorname{Mor}(X, Y)$, and thus a natural map $\operatorname{Hom}(X \times \Delta^\bullet, K^\infty) \to \operatorname{Mor}(X \times \Delta^\bullet, K^\infty)$ of simplicial sets. Define a map $\operatorname{Mor}(X \times \Delta^\bullet, K^\infty) \to \coprod_r Z_{r-1}(X \times \Delta^\bullet, \mathbb{P}^\infty)_1$ in the same manner as the corresponding maps of Corollary 6.2 and Proposition 6.7. The map s is defined as the evident composition.

In the terminology of [12], the second statement is the assertion that, for X smooth of characteristic 0, the naive motivic cohomology of X equals the motivic cohomology of X [12; 8.1]. $\square$

Remark 6.10. *In light of Proposition 6.7, we feel justified in referring to the maps s of Corollary 6.2 and Proposition 6.8, as well as the maps on homotopy groups they induce, as "the total Segre class maps". Especially in the context of Proposition 6.8, however, it seems likely that the axiomatic approach of* [16] *leads to a definition of Chern class maps (and hence Segre class maps) from algebraic K-theory to motivic cohomology. Once such an axiomatic approach is completed, one should then compare the resulting Segre classes with those associated to the map s of Proposition 6.8.*

We now proceed to summarize the formal relationships between algebraic, semi-topological, and topological K-theory and their corresponding cohomology theories in a "double square". The existence of such a diagram was sketched in [8], even before the definition of $\mathcal{K}^{\text{semi}}(X)$ was formalized.

To formulate the statement of the theorem, it is convenient to establish equivalent simplicial versions of each of the theories we have discussed so far. Namely, let $\operatorname{Sing.} \mathcal{Z}_r(X, \mathbb{P}^\infty)_1$ denote the level-wise group completion of the simplicial abelian monoid $\mathrm{Sing.}\,\mathcal{M}or(X,\mathcal{C}_r(\mathbb{P}^\infty))^{\mathrm{an}}$. Similarly, define $\mathrm{Sing.}\,\mathcal{Z}_r(X^{\mathrm{an}},\mathbb{P}^\infty)$ to be the level-wise group completion of $\mathrm{Sing.}\,\mathcal{M}\mathrm{aps}(X^{\mathrm{an}},\mathcal{C}_r(\mathbb{P}^\infty)^{\mathrm{an}})$. (Note that we have abused notation slightly: $\mathrm{Sing.}\,T$ denotes the simplicial set associated to a space T, but $\mathrm{Sing.}\,\mathcal{Z}_r(X,\mathbb{P}^\infty)$ as defined here is not the simplicial set associated to the space $\mathcal{Z}_r(X,\mathbb{P}^\infty)$. However, the two are naturally homotopy equivalent.) There are natural homotopy equivalences

$$|\,\mathrm{Sing.}\,\mathcal{Z}_r(X,\mathbb{P}^\infty)_1| \xrightarrow{\sim} \mathcal{Z}_r(X,\mathbb{P}^\infty)_1$$

and

$$|\,\mathrm{Sing.}\,\mathcal{Z}_r(X^{\mathrm{an}},\mathbb{P}^\infty)_1| \xrightarrow{\sim} \mathcal{Z}_r(X^{\mathrm{an}},\mathbb{P}^\infty)_1.$$

Observe that for each n there is a natural map of contractible simplicial sets

$$\mathcal{I}(n)(\Delta^\bullet) \longrightarrow \mathrm{Sing.}\,\mathcal{I}(n)^{\mathrm{an}},$$

and also a natural equivalence of contractible spaces

$$|\,\mathrm{Sing.}\,\mathcal{I}(n)^{\mathrm{an}}| \longrightarrow \mathcal{I}(n)^{\mathrm{an}}.$$

These maps are compatible with the operad structures, giving us a map of E_∞ simplicial operads $\mathcal{I}(\Delta^\bullet) \longrightarrow \mathrm{Sing.}\,\mathcal{I}^{\mathrm{an}}$ (where $\mathrm{Sing.}\,\mathcal{I}^{\mathrm{an}}$ is the simplicial operad consisting of simplicial sets $\mathrm{Sing.}\,\mathcal{I}(n)^{\mathrm{an}}$, $n \geq 0$) and a map of E_∞ operads $|\,\mathrm{Sing.}\,\mathcal{I}^{\mathrm{an}}| \longrightarrow \mathcal{I}^{\mathrm{an}}$. In particular, any $\mathcal{I}^{\mathrm{an}}$-space may be equivalently regarded as an $|\mathcal{I}(\Delta^\bullet)|$-space (i.e., the associated Ω-spectra will be equivalent). Moreover, each of the simplicial sets $\mathrm{Sing.}\,K^\infty(X)^{\mathrm{an}}$, $\coprod_r \mathrm{Sing.}\,\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1$, $\mathrm{Sing.}\,\mathcal{M}\mathrm{aps}(X^{\mathrm{an}},(K^\infty)^{\mathrm{an}})$, and $\coprod_r \mathrm{Sing.}\,\mathcal{Z}_{r-1}(X^{\mathrm{an}},\mathbb{P}^\infty)_1$ is an algebra over the simplicial operad $\mathcal{I}(\Delta^\bullet)$. Upon taking geometric realizations, we obtain $|\mathcal{I}(\Delta^\bullet)|$-spaces $|\,\mathrm{Sing.}\,K^\infty(X)^{\mathrm{an}}|$, etc., which are homotopy equivalent to the $\mathcal{I}^{\mathrm{an}}$-spaces $K^\infty(X)^{\mathrm{an}}$, etc. under maps compatible with the action of the operad $|\mathcal{I}(\Delta^\bullet)|$. Thus, the Ω-spectrum associated to $|\,\mathrm{Sing.}\,K^\infty(X)^{\mathrm{an}}|$ is equivalent to the Ω-spectrum associated to $K^\infty(X)^{\mathrm{an}}$, and similarly for the other three $\mathcal{I}^{\mathrm{an}}$-spaces.

Theorem 6.11. *For any connected quasi-projective complex variety X, there is a commutative diagram of $|\mathcal{I}(\Delta^\bullet)|$-spaces*

$$\begin{array}{ccc}
|\mathrm{Mor}(X\times\Delta^\bullet, K^\infty)| & \xrightarrow{s} & \coprod_{r\geq 0} |Z_{r-1}(X\times\Delta^\bullet)_1| \\
\downarrow & & \downarrow \\
|\mathrm{Sing.}\,K^\infty(X)^{\mathrm{an}}| & \xrightarrow{s} & \coprod_{r\geq 0} |\mathrm{Sing.}\,\mathcal{Z}_{r-1}(X,\mathbb{P}^\infty)_1| \\
\downarrow & & \downarrow \\
|\mathrm{Sing.}\,\mathcal{M}\mathrm{aps}(X^{\mathrm{an}},(K^\infty)^{\mathrm{an}})| & \xrightarrow{s} & \coprod_{r\geq 0} |\mathrm{Sing.}\,\mathcal{Z}_{r-1}(X^{an},\mathbb{P}^\infty)_1|
\end{array} \qquad (6.11.1).$$

Applying $\Omega^\infty\Sigma^\infty$ to (6.11.1), we obtain a commutative diagram of infinite loop spaces $\Omega^\infty\Sigma^\infty(6.11.1)$.

If X is smooth and connected, then upon applying $\pi_j(-)$ for some $j \geq 0$ to $\Omega^\infty\Sigma^\infty(6.11.1)$ we obtain the following commutative diagram of abelian groups

$$\begin{array}{ccccc}
K_j(X) & \longrightarrow & K_j^{\mathrm{semi}}(X) & \longrightarrow & K_{\mathrm{top}}^{-j}(X) \\
{\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
\bigoplus_q \mathrm{H}^{2q-j}(X,\mathbb{Z}(q)) & \longrightarrow & \bigoplus_q L^q\,\mathrm{H}^{2q-j}(X) & \longrightarrow & \bigoplus_q \mathrm{H}^{2q-j}(X^{an},\mathbb{Z})
\end{array}$$

(6.11.2)

whose vertical arrows are the total Segre class maps.

Proof. The map from the top pair to the middle pair of (6.11.1) is induced by the natural map of simplicial sets

$$(n \mapsto \mathrm{Mor}(X \times \Delta^n, Y)) \to \big(n \mapsto \mathrm{Maps}(\Delta^n_{\mathrm{top}}, \mathcal{M}or(X,Y)^{an})\big)$$

which exists for any Y. The map from the middle pair to the bottom pair is induced by the map of simplicial sets associated to the natural inclusions of spaces

$$\mathcal{M}or(X,Y)^{an} \to \mathcal{M}\mathrm{aps}(X^{an}, Y^{an}),$$

whose continuity is given by [13; 2.4]. Naturality of these constructions implies the commutativity of (6.11.1).

The observation that $\pi_j(\Omega^\infty\Sigma^\infty(6.11.1))$ has the form (6.11.2) follows from the discussion preceding the theorem and Propositions 6.5, 6.7, 6.8, and 6.9. □

§7 Varieties X satisfying $\mathcal{K}^{\mathrm{semi}}(X) \simeq \mathcal{K}_{\mathrm{top}}(X^{an})$

In this section, we investigate the natural map

$$\mathcal{K}^{\mathrm{semi}}(X) \longrightarrow \mathcal{K}_{\mathrm{top}}(X^{an}) \equiv \underline{bu}(X^{\mathrm{an}}) \tag{7.0}$$

which we define to be the map of homotopy-theoretic group completions determined by the map of $\mathcal{I}^{\mathrm{an}}$-spaces

$$K^\infty(X)^{\mathrm{an}} \longrightarrow \mathcal{M}\mathrm{aps}(X^{\mathrm{an}}, (K^\infty)^{\mathrm{an}}) \equiv M^\infty(X)$$

adjoint to the map of (6.11.1).

The goal of this section is to establish a few situations in which the map (7.0) is a weak equivalence. An easy example occurs when $X = \mathrm{Spec}\,\mathbb{C}$, in which case this map is a weak equivalence by Proposition 6.7. Our first non-trivial example is a consequence of Corollary 5.3.

Proposition 7.1. *Let $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces. Then the natural map*

$$\mathcal{K}^{\mathrm{semi}}(P) \longrightarrow \mathcal{K}_{\mathrm{top}}(P^{an})$$

is a weak equivalence.

Proof. Let X be any projective complex variety. Given quotients $\mathcal{O}^m \twoheadrightarrow L$ and $\mathcal{O}^n \twoheadrightarrow L'$, the natural map $K^\infty(X)^{an} \to M^\infty(X)$ determines a map of two dimensional arrays from $\{K^\infty(X)^{an}; \alpha, \beta\}$ to $\{M^\infty(X); \alpha, \beta\}$, where the latter is the array

$$\begin{array}{ccccc}
M^\infty(X) & \xrightarrow{\alpha_L} & M^\infty(X) & \xrightarrow{\alpha_L} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
M^\infty(X) & \xrightarrow{\alpha_{L\otimes L'}} & M^\infty(X) & \xrightarrow{\alpha_{L\otimes L'}} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
M^\infty(X) & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & M^\infty(X) & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & \cdots \\
{\scriptstyle \beta_{L'}}\downarrow & & {\scriptstyle \beta_{L'}}\downarrow & & \\
\vdots & & \vdots & &
\end{array} \tag{7.1.1}$$

Here, the maps α and β in (7.1.1) are defined analogously to the maps of (5.1.1) by regarding $\mathcal{O}^m \twoheadrightarrow L$ and $\mathcal{O}^n \twoheadrightarrow L'$ as quotients of topological vector bundles.

Given a rank n vector bundle Y on a projective variety E, we can mimic the construction of the map $\rho : \mathcal{K}^{\mathrm{semi}}(\mathbb{P}^n) \to \mathcal{K}^{\mathrm{semi}}(\mathbb{P}^n)$ using digram (7.1.1) in place of (5.1.1) to obtain a natural weak equivalence

$$\rho_{\mathrm{top}} : \mathcal{K}_{\mathrm{top}}(\mathbb{P}(E)^{\mathrm{an}}) \longrightarrow \mathcal{K}_{\mathrm{top}}(\mathbb{P}(E)^{\mathrm{an}})$$

whose effect on homotopy groups is multiplication by the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$ (regarded as a topological vector bundle) in $\mathcal{K}^0_{\mathrm{top}}(\mathbb{P}(E)^{\mathrm{an}})$. Since there is a natural map from diagram (5.1.1) to diagram (7.1.1), the diagram

$$\begin{array}{ccc} \mathcal{K}^{\mathrm{semi}}(Y) & \xrightarrow{\rho^{\circ i}\circ \pi^*} & \mathcal{K}^{\mathrm{semi}}(\mathbb{P}(E)) \\ \downarrow & & \downarrow \\ \mathcal{K}_{\mathrm{top}}(Y^{\mathrm{an}}) & \xrightarrow{\rho^{\circ i}_{\mathrm{top}}\circ \pi^*} & \mathcal{K}_{\mathrm{top}}(\mathbb{P}(E)^{\mathrm{an}}) \end{array}$$

commutes for all $i \geq 0$. Performing this construction for the sequence of varieties and bundles $(\operatorname{Spec}\mathbb{C}, \mathcal{O}^{n_1+1}_{\operatorname{Spec}\mathbb{C}}), (\mathbb{P}^{n_1}, \mathcal{O}^{n_2+1}_{\mathbb{P}^{n_1}}), \ldots$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{K}^{\mathrm{semi}}(\operatorname{Spec}\mathbb{C})^{\times n_1+\cdots+n_k} & \longrightarrow & \mathcal{K}^{\mathrm{semi}}(P) \\ \downarrow & & \downarrow \\ \mathcal{K}_{\mathrm{top}}(pt)^{\times n_1+\cdots+n_k} & \longrightarrow & \mathcal{K}_{\mathrm{top}}(P^{\mathrm{an}}) \end{array}$$

whose horizontal and left-hand vertical maps are weak equivalences. The result follows. □

We now proceed to show that a theorem of F. Kirwan (Theorem 7.2 below) admits a stabilization which asserts that the natural map $\mathcal{K}^{\mathrm{semi}}(C) \to \mathcal{K}_{\mathrm{top}}(C)$ is a weak equivalence for any smooth, complex projective curve.

Let C be a smooth complex projective complex curve (i.e., a Riemann surface) and let g be its genus. Given a continuous map $f : C \to \mathrm{Grass}_n(\mathbb{P}^N)$ corresponding to a quotient $\mathcal{O}^{N+1}_C \twoheadrightarrow E$ of topological vector bundles, we say that f has degree d if $c_1(E) = d \in H^2(C,\mathbb{Z}) = \mathbb{Z}$. We recall the notation $A_d(n,m)$ of [20], defined as the collection of quotients $\mathcal{O}^m_C \twoheadrightarrow E$ such that E is a vector bundle on C having rank n and degree d and which satisfies the condition that $\mathrm{H}^1(C,E) = 0$. By [20], $A_d(n,m)$ has the structure of an open subvariety of the variety $\underline{\mathrm{Mor}}_d(C, \mathrm{Grass}_n(\mathbb{P}^{m-1}))$,

$$A_d(n,m) \subset \underline{\mathrm{Mor}}_d(C, \mathrm{Grass}_n(\mathbb{P}^{m-1})) \equiv \underline{\mathrm{Mor}}_d(n,m),$$

where $\underline{\mathrm{Mor}}(C, \mathrm{Grass}_n(\mathbb{P}^{m-1})) = \coprod_{d\geq 0} \underline{\mathrm{Mor}}_d(C, \mathrm{Grass}_n(\mathbb{P}^{m-1})$ is discussed in Lemma 4.1 and where the subscript d indicates degree in the above sense. Observe that the stabilization of $\underline{\mathrm{Mor}}_d(C, \mathrm{Grass}_n(\mathbb{P}^{m-1}))$ with respect to m restricts to a stabilization of $A_d(n,m)$, thereby defining an ind-variety

$$A_d(n) \subset \underline{\mathrm{Mor}}_d(C, \mathrm{Grass}_n(\mathbb{P}^{\infty}) \equiv \underline{\mathrm{Mor}}_d(n).$$

For each $d, n \geq 0$, there is a natural map

$$A_d(n)^{\mathrm{an}} \to \underline{\mathrm{Mor}}_d(n)^{an} \to \varinjlim_m \mathcal{M}\mathrm{aps}_d(C, \mathrm{Grass}_n(\mathbb{P}^m)) \equiv \mathcal{M}\mathrm{aps}_d(n)$$

where the subscript d on the right-hand side refers to the subspace of maps having degree d.

We recall the following result of [20].

Theorem 7.2. [20; 1.1] *As above, let C denote a smooth complex projective curve. Let k be any positive integer with $k \geq n - 2g$. Then the composition*

$$A_d(n)^{\text{an}} \to \underline{\text{Mor}}_d(n)^{an} \to \mathcal{M}\text{aps}_d(n)$$

induces an isomorphism in cohomology up to dimension k provided

$$d \geq 2n(2g+k+1) + n\max(k+1+n(2g+k+1), \frac{1}{4}n^2 g).$$

In parallel with the notation $K^\infty(C)$ and $M^\infty(C)$, we introduce the notation $A^\infty(C)$:

$$A^\infty(C) \equiv \coprod_{d,n} A_d(n); \quad K^\infty(C) \equiv \coprod_{d,n} \underline{\text{Mor}}_d(n); \quad M^\infty(C) \equiv \coprod_{d,n} \mathcal{M}\text{aps}_d(n).$$

We observe that the maps of the two-dimensional array $\{K^\infty; \alpha, \beta\}$ of (5.1.1) restrict to determine the sub-array of spaces

$$\begin{array}{ccccc}
A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_L} & A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_L} & \cdots \\
\beta_{L'}\downarrow & & \beta_{L'}\downarrow & & \\
A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_{L\otimes L'}} & A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_{L\otimes L'}} & \cdots \\
\beta_{L'}\downarrow & & \beta_{L'}\downarrow & & \\
A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & A^\infty(X)^{\text{an}} & \xrightarrow{\alpha_{L\otimes L'\otimes L'}} & \cdots \\
\beta_{L'}\downarrow & & \beta_{L'}\downarrow & & \\
\vdots & & \vdots & &
\end{array} \tag{7.2.1}$$

We write the two-dimensional array (7.2.1) as $\{A^\infty(X)^{\text{an}}; \alpha, \beta\}$.

Proposition 7.3. *The natural map of homotopy colimits*

$$\text{hocolim}\{A^\infty(X)^{an}; \alpha, \beta\} \to \text{hocolim}\{K^\infty(X)^{an}; \alpha, \beta\}$$

is a weak homotopy equivalence.

Proof. We use the fact that we may realize these homotopy colimits by first taking the homotopy colimit of each column, then taking the homotopy colimit of the resulting row of spaces. Yet, the map clearly gives a weak homotopy equivalence on each column (i.e., the infinite mapping telescope for $\beta_{L'}$), since upon restriction to any $\underline{\text{Mor}}_d(n)$, one can factor a sufficiently high iteration of $\beta_{L'}$ through A^∞. □

Proposition 7.4. *The natural composition of homotopy colimits*

$$\text{hocolim}\{A^\infty(C)^{an}; \alpha, \beta\} \to \text{hocolim}\{K^\infty(C)^{an}; \alpha, \beta\} \to \text{hocolim}\{M^\infty(C); \alpha, \beta\}$$

is a homology equivalence.

Proof. We use the fact that we may realize these homotopy colimits by first taking mapping telescopes of the columns of (7.1.2) and (7.1.1) and then taking mapping telescopes along the resulting rows. For each fixed column, to show

$$\mathrm{Tel}(A^\infty(C)^{\mathrm{an}}, \beta) \longrightarrow \mathrm{Tel}(M^\infty(C), \beta)$$

is a homology equivalence, it suffices to show the map

$$\varinjlim \left(\mathrm{H}_q(A^\infty(C) \xrightarrow{\beta} \cdots\right) \longrightarrow \varinjlim \left(\mathrm{H}_q(M^\infty(C) \xrightarrow{\beta} \cdots\right) \qquad (7.4.1)$$

is an isomorphism for all q. Observe that the map from (7.2.1) to (7.1.1) respects degree and rank, so that to establish that (7.4.1) is an isomorphism, it suffices to show the map

$$\varinjlim \left(\mathrm{H}_q(A_d(n)^{\mathrm{an}}) \xrightarrow{\beta} \mathrm{H}_q(A_{d+e}(n)^{\mathrm{an}}) \xrightarrow{\beta} \cdots\right) \longrightarrow$$
$$\varinjlim \left(\mathrm{H}_q(\mathcal{M}\mathrm{aps}_d(n)) \xrightarrow{\beta} \mathrm{H}_q(\mathcal{M}\mathrm{aps}_{d+e}(n)) \xrightarrow{\beta} \cdots\right) \qquad (7.4.2)$$

is an isomorphism for all fixed d and n. Here, $e > 0$ is the degree of the line bundle L'. Since at some sufficiently large stage of the direct limit the degree is large enough to satisfy the hypothesis of this Theorem 7.2, it follows that (7.4.2) is an isomorphism for all q, n and d.

Thus the map $\mathrm{hocolim}\{A^\infty(C)^{\mathrm{an}}; \alpha, \beta\} \to \mathrm{hocolim}\{M^\infty(C)^{\mathrm{an}}; \alpha, \beta\}$ may be realized as a telescope of maps which are homology equivalences. Since homology commutes with taking mapping telescopes, the result follows. □

Theorem 7.5. *For a smooth, projective complex curve C the natural map*

$$\mathcal{K}^{\mathrm{semi}}(C) \longrightarrow \mathcal{K}_{\mathrm{top}}(C^{an})$$

is a weak equivalence. Thus, if C has genus g, then for any $j \geq 0$

$$K_{2j}^{\mathrm{semi}}(C) \cong \mathbb{Z}^{2g}, \quad K_{2j+1}^{\mathrm{semi}}(C) \cong \mathbb{Z}^2.$$

Proof. By Proposition 5.1, the homotopy colimit of any row of $\{K^\infty(C)^{\mathrm{an}}; \alpha, \beta\}$ – that is, $\mathcal{K}^{\mathrm{semi}}(C)$ – is weakly equivalent to the homotopy colimit of the whole diagram under the natural map.

We easily modify the proof of Proposition 5.1 so that it applies to $M^\infty(C)$ in place of $K^\infty(C)$. Namely, under the identification

$$[T, \mathrm{Tel}(M^\infty(C), \alpha_{L\otimes L'^{\otimes j}})] \;=\; K^0_{\mathrm{top}}(T \times C^{an}),$$

the map between consecutive rows of (7.2.1) induces the self-map on $K^0_{\mathrm{top}}(T \times C^{an})$ which sends a topological vector bundle E on $T \times C^{an}$ to $E \otimes pr_C^* L'$. Thus, we conclude that the homotopy colimit of any row of $\{M^\infty(C); \alpha, \beta\}$ – that is, $\mathcal{K}_{\mathrm{top}}(C^{an})$ – is weakly equivalent to the homotopy colimit of the whole diagram under the natural map.

Thus, Propositions 7.4 and 7.5 imply that $\mathcal{K}^{\mathrm{semi}}(C) \longrightarrow \mathcal{K}_{\mathrm{top}}(C^{an})$ is a homology equivalence. Since this is a map of group-like H-spaces, we conclude that the map is a weak homotopy equivalence.

The calculation of $K_*^{\mathrm{semi}}(C) \cong K_{\mathrm{top}}^{-*}(C)$ is standard. □

The following is an immediate corollary of Theorems 5.2 and 7.5.

Corollary 7.6. *Let C be a smooth, projective complex curve of genus g and E a rank n vector bundle on C. Then for any $j \geq 0$,*

$$K_{2j}^{\mathrm{semi}}(\mathbb{P}(E)) \cong \mathbb{Z}^{2ng}, \quad K_{2j+1}^{\mathrm{semi}}(\mathbb{P}(E)) \cong \mathbb{Z}^{2n}.$$

References

[1] M. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Differential Geometry, Proceedings of a conference held at the University of Arizona in 1960, vol. 3, Proc. Sympos. Pure Math., Amer. Math. Society, 1961, pp. 7–38.

[2] H. Bass, *Algebraic K-theory*, Benjamin, 1968.

[3] C. Boyer, H.B. Lawson, P. Lima-Filho, B. Mann, and M.-L. Micheslsohn, *Algebraic cycles on equivariant cohomology theories*, Invent. Math. **113** (1993), 373 – 388.

[4] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, I.H.E.S. Pub. Math **58** (1984), 19-38.

[5] Ralph L. Cohen, Ernesto Lupercio, and Graeme Segal, *Holomorphic spheres in loop groups and Bott periodicity*, Preprint.

[6] E. Friedlander, *Etale K-theory II. Connections with algebraic K-theory*, Ann. Sci Ecole Norm. Sup **15** (1982), 231–256.

[7] E. Friedlander, *Algebraic cocycles, Chow varieties, and Lawson homology*, Compositio Math. **77** (1991), 55-93.

[8] E. Friedlander, *A double cube diagram relating cycles, K-theory, and homology; Oberwolfach 1996*, K-theory preprints, available at http://www.math.uiuc.edu/K-theory **138**.

[9] E. Friedlander, *Algebraic cocycles on normal, quasi-projective varieties*, Compositio Mathematica **110** (1998), 127-162.

[10] E. Friedlander and H.B. Lawson, *A theory of algebraic cocycles*, Annals of Math. **136** (1992), 361-428.

[11] E. Friedlander and H.B. Lawson, *Duality relating spaces of algebraic cocycles and cycles*, Topology **36** (1997), 533-565.

[12] E. Friedlander and V. Voevodsky, *Bivariant cycle cohomology*, Cycles, Transfers, and Motivic Homology Theories (V. Voevodsky, A. Suslin and E. Friedlander, eds.), Annals of Math Studies.

[13] E. Friedlander and M. Walker, *Function spaces and continuous algebraic pairings for varieties*, Preprint, available at http://www.math.uiuc.edu/K-theory.

[14] W. Fulton, *Intersection Theory*, 1984.

[15] W. Fulton and S. Lang, *Riemann-Roch algebra*, Grundlerhren der Mathematischen Wissenschaften 277, Springer-Verlag, 1985.

[16] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. in Math **40** (1981), 203-289.

[17] D. Grayson and M. Walker, *Geometric models for algebraic K-theory*, Preprint, available at http://www.math.uiuc.edu/K-theory.

[18] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert*, Séminaire Bourbaki, vol. 221, 1961.

[19] J.P. Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique*, Algebraic K-theory I: Higher algebraic K-theories, Lec. Notes Math 341, 1973, pp. 293-316.

[20] F. Kirwan, *On spaces of maps from Riemann surfaces to Grassmannians and applitcations to the cohomology of moduli of vector bundles*, Arkiv för matematik **24 2** (1986), 221–275.

[21] H.B. Lawson, *Algebraic cycles and homotopy theory*, Annals of Math **129** (1989), 253–291.

[22] H. B. Lawson, P. Lima-Filho and M.-L. Michelsohn, *Algebriac cycles and equivariant cohomology theories*, Proc. London Math. Soc. (3) **73 3**, 679–720.

[23] H.B. Lawson and M.-L. Michelsohn, *Algebraic cycles, Bott periodicity, and the Chern characteristic map*, The mathematical heritage of Hermann Weyl, Proc. Sympos. Pure Math., vol. 48, Amer. Math. Soc., 1988., pp. 241–263.

[24] P. Lima-Filho, *Algebraic families of E_∞ spectra*, Preprint.

[25] J. P. May, *The geometry of infinite loop spaces*, Lec. Notes Math. 271, Springer-Verlag, 1972.

[26] J. P. May, *E_∞ spaces, group completions, and permutative categories*, New developments in topology, London Math. Soc. Lec. Notes Series, pp. 61–94.

[27] J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, Lec. Notes Math. 577, Springer-Verlag, 1977.

[28] J. P . May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology **17** (1978), 205–224.
[29] D. Quillen, *Higher algebraic K-theory, I*, Algebraic K-theory I: Higher algebraic K-theories, Lec. Notes Math 341, 1973, pp. 85-147.
[30] G. Segal, *Categories and cohomology theories*, Topology **13**, 293–312.
[31] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[32] N. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.
[33] R. Swan, *On seminormality*, J. of Agebra **67** (1980), 210-229.
[34] P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des Intersections et Théorème de Riemann-Roch*, Lect. Notes Math. 225.
[35] G. Whitehead, *Elements of homotopy theory, Graduate Texts in Math 61*, Springer-Verlag, 1978.

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