FUNCTION SPACES AND CONTINUOUS ALGEBRAIC PAIRINGS FOR VARIETIES

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Abstract. Given a quasi-projective complex variety X and a projective variety Y, one may endow the set of morphisms, $Mor(X,Y)$, from X to Y with the natural structure of a topological space. We introduce a convenient technique (namely, the notion of a functor on the category of "smooth curves") for studying these function complexes and for forming continuous pairings of such. Building on this technique, we establish several results, including: (1) the existence of cap and join product pairings in topological cycle theory, (2) the agreement of cup product and intersection product for topological cycle theory, (3) the agreement of the motivic cohomology cup product with morphic cohomology cup product, and (4) the Whitney sum formula for the Chern classes in morphic cohomology of vector bundles.

At first glance, imposing a topology on the set $\mathrm{Mor}(X,Y)$ of morphisms between two complex algebraic varieties seems unnatural. Nevertheless, just such a construction applied to the set of morphisms from X to certain Chow varieties of cycles in projective space leads to the "morphic cohomology" of X as introduced in [FL-1]. In this paper, we show that, in general, the "topology of bounded convergence" (introduced in [FL-2]) on $\mathrm{Mor}(X,Y)$ has a natural algebraic description arising from the enriched structure on $\mathrm{Mor}(X,Y)$ as a contravariant functor on the category of smooth curves. This functorial interpretation leads to a convenient formulation of the technique of demonstrating "uniqueness of specialization" introduced in [F-1] for the construction of continuous algebraic maps. We use this new technique to establish the continuity of various constructions and pairings involving the "function spaces" $\mathcal{M}\mathrm{or}(X,Y)^{an}$, where X and Y are complex (but not necessarily projective) varieties.

More generally, we introduce the notion of a "proper, constructible presentation" of a functor (cf. Definition 2.1), a property which provides a natural topological realization of a contravariant functor on smooth curves. This point of view facilitates (cf. Theorem 2.6) a careful proof of the continuity of the slant product pairing of [FL-1] and the cap product pairing relating Lawson homology and morphic cohomology which plays a central role in [F-3]. Indeed, our techniques provide, not merely a pairing on the level of homology groups, but pairings (in the derived category) of the presheaves of chain complexes used to define Lawson homology and morphic cohomology. Similarly, the join product of cycles in projective spaces

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determines a cup product in morphic cohomology as first recognized in [FL-1]. We provide a definition of this product at the level of presheaves of chain complexes on an arbitrary complex quasi-projective variety (Proposition 3.3). As we make explicit in (4.1.1), there is a natural map of presheaves of chain complexes from those complexes which define motivic cohomology to those which define morphic cohomology. In Theorem 4.4, we show that this natural map commutes with products. On a smooth variety X, we show cup product corresponds to the intersection product of cycles under duality – that is, we refine the intersection product of [F-G] to be a pairing of presheaves of chain complexes on X compatible with our cup product.

In verifying in [F3] that suitably enriched versions of Lawson homology and morphic cohomology satisfy the axioms of Bloch-Ogus [B-O], the first author introduced a cap product whose continuity was not evident. One of the motivations of the present paper is a careful proof of continuity of cap product, set in a more general context. Moreover, the formulation of cup product presented here in terms of a pairing of complexes of sheaves also permitted the verification in [F3] of the stronger result that this "topological cycle theory" satisfies the stronger axioms of H. Gillet [G].

In the final section of this paper, we apply our improved understanding of products to show in Theorem 5.4 that the geometric construction of [FL-1;§10] does indeed determine Chern class maps on $K_0(X)$ for a quasi-projective variety X and that these Chern classes satisfy the expected Whitney sum formula.

Throughout this paper, all varieties considered will be quasi-projective varieties (by which we mean reduced, locally closed subschemes of projective space) over a base field of characteristic 0 (usually the complex field $\mathbb{C}$). We shall frequently consider Chow varieties associated to projective varieties. If $Y \subset \mathbb{P}^N$ is a projective variety provided with a given closed embedding in some projective space $\mathbb{P}^N$, then $C_{r,d}Y$ denotes the Chow variety whose rational points are the effective r-cycles on Y of degree d. We shall consider the Chow monoid $\mathcal{C}_rY \equiv \coprod_{d\geq 0} C_{r,d}Y$ of all effective r-cycles on Y, a monoid whose isomorphism type is independent of the projective embedding $Y \subset \mathbb{P}^N$ (cf. [B]).

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§1 Continuous algebraic maps

One is naturally led to consider continuous algebraic maps to Chow varieties when one is confronted with their construction in terms of elimination theory rather than as a representable functor. Indeed, as we see in Example 1.3, Chow varieties have a natural functorial description in terms of functors on smooth curves. A simple observation which motivates the consideration of such functors is the fact that a continuous algebraic map $X \to Y$ between quasi-projective varieties is equivalent to a natural transformation of associated contravariant functors $\mathrm{Mor}^{\leq 1}(-,X) \to \mathrm{Mor}^{\leq 1}(-,Y)$ (see Proposition 1.1)

The usefulness of this functorial point of view is that the construction of pairings of functors is often straight-forward. In conjunction with the topological realization considered in the next section, our functorial point of view will provide a good formalism for proving the continuity of various pairings.

Eventually, we will be working over the complex numbers, but in this section we work over an arbitrary field K of characteristic 0 and we consider varieties defined over K. Recall that a *continuous algebraic map* $f : X \to Y$ is a closed subvariety $\Gamma_f \subset X \times Y$ with the property that $pr_X : \Gamma_f \to X$ is finite and bijective on geometric points – or, equivalently, that pr_X is a universal homeomorphism (cf. [V1; 3.2.4]). We say that pr_X is a *bicontinuous morphism.* We further recall that a variety X admits a natural normalization $\tilde{X} \to X$ (defined locally by taking integral closures in the total ring of quotients of X), and that this normalization factors as

$$\tilde{X} \longrightarrow X^w \longrightarrow X,$$

where X^w is the *weak normalization* of X (cf. [A-B]). The variety X^w has the property that $X^w \to X$ is bicontinuous and is universal among varieties mapping bicontinuously to X. (The weak normalization X^w coincides with the semi-normalization of X since K has characteristic zero – see [S].) Thus, a continuous algebraic map of algebraic varieties $X \to Y$ is equivalent to a morphism $X^w \to Y$. We say a variety X is *weakly normal* if it is equal to its weak normalization, in which case every continuous algebraic map from X to Y is a morphism of varieties.

We proceed to formalize a technique introduced in [F1] to construct continuous algebraic maps. Let $(Sm/K)_{\leq 1}$ denote the category of smooth affine schemes over $\operatorname{Spec} K$ which are essentially of finite type, connected, and have Krull dimension at most 1. That is, every object of $(Sm/K)_{\leq 1}$ is the scheme associated to a ring A where A has Krull dimension one and is the localization of a finitely generated smooth, integral K-algebra R. In scheme-theoretic language, we observe that any $C \in (Sm/K)_{\leq 1}$ is a filtered limit of smooth varieties of finite type over K (possibly of dimension more that 1) such that the transition maps in the system are open immersions. A typical example of an object of $(Sm/K)_{\leq 1}$ is $\operatorname{Spec} \mathcal{O}_{X;x_1,\dots,x_n}$, where X is a smooth, connected variety, the x_i are the generic points of codimension one subvarieties, and $\mathcal{O}_{X;x_1,\dots,x_n}$ denotes semi-localization at these points. If $C \in (Sm/K)_{\leq 1}$ and X is a quasi-projective variety over K, then we define $\operatorname{Mor}(C, X)$ to be the set of morphisms of schemes over $\operatorname{Spec} K$ from C to X and we write

$$\operatorname{Mor}^{\leq 1}(-, X) : (Sm/K)_{\leq 1} \longrightarrow (Sets)$$

for the functor so defined.

Intuitively, we think of $(Sm/K)_{\leq 1}$ as consisting of all curves and the motivation for its introduction is that a continuous algebraic map on a variety is uniquely determined by its value on all curves. More precisely, we have the following key result.

Proposition 1.1. *For any field K of characteristic 0, a continuous algebraic map $f : X \to Y$ between quasi-projective varieties over K is equivalent to a natural transformation of contravariant functors*

$$\Phi_f : \operatorname{Mor}^{\leq 1}(-, X) \longrightarrow \operatorname{Mor}^{\leq 1}(-, Y) : \ (Sm/K)_{\leq 1} \longrightarrow (Sets).$$

Proof. Assume given such a natural transformation Φ_f. Then Φ_f immediately determines a rational map from X to Y: namely, given a generic point $\eta : \operatorname{Spec} F \to X$, we send η to $\Phi_f(\eta) : \operatorname{Spec} F \to Y$. Let $Y \subset \overline{Y}$ be a projective closure and let $\Gamma_f \subset X \times \overline{Y}$ be the graph of this rational map, so that Γ_f is the closed subvariety whose irreducible components have generic points $(\eta, \Phi_f(\eta)) : \operatorname{Spec} F \to X \times Y \subset X \times \overline{Y}$. It suffices to verify that for any finitely generated field extension L of K and any map $\gamma : \operatorname{Spec} L \to \Gamma_f$, γ is of the form $(p \circ \gamma, \Phi_f(p \circ \gamma)) : \operatorname{Spec} L \to X \times Y$ where $p : X \times Y \to X$ is the projection map.

We argue by induction on the codimension of $\gamma(\operatorname{Spec} L) \in \Gamma_f$. For codimension 0, all maps γ are of the given form by construction. Assume we have verified that all maps $\gamma : \operatorname{Spec} L \to \Gamma_f$ are of the given form if $\gamma(\operatorname{Spec} L)$ has codimension $\leq s$, and consider $\gamma : \operatorname{Spec} L \to \Gamma_f$ with $\gamma(\operatorname{Spec} L) \in \Gamma_f$ of codimension $s+1$. Choose a non-constant map $g : C \to \Gamma_f$ defined over $\operatorname{Spec} L$ from a smooth, connected, affine curve $C \in (Sm/K)_{\leq 1}$ to Γ_f with the property that some L-rational point $c : \operatorname{Spec} L \to C$ maps to γ. By induction, the generic point $\nu : \operatorname{Spec} E \to C$ of C satisfies the condition that $g \circ \nu : \operatorname{Spec} E \to \Gamma_f$ is the map $(p \circ g \circ \nu, \Phi_f(p \circ g \circ \nu))$, which by the naturality of Φ_f is equal to the map $(p \circ g, \Phi_f(p \circ g)) \circ \nu$. Thus, we conclude that $g : C \to \Gamma_f \subset X \times Y$ is of the form $(p \circ g, \Phi_f(p \circ g))$. Naturality of Φ_f now implies that $\gamma = (p \circ g \circ c, \Phi_f(p \circ g) \circ c)$ equals $(\gamma, \Phi_f(\gamma))$. $\square$

Proposition 1.1 motivates the following definition.

Definition 1.2. *Let K be a field of characteristic 0. We define $\mathcal{M}\mathrm{or}^{\leq 1}$ to be the category of contravariant functors $F : (Sm/K)_{\leq 1} \to (Sets)$. For $F, G \in \mathcal{M}\mathrm{or}^{\leq 1}$, we write $\mathrm{Mor}(F,G)$ for the set of natural transformations from F to G. If X is a scheme over K, we also let X denote the functor on $(Sm/K)_{\leq 1}$ sending C to $\mathrm{Hom}_K(C,X)$.*

We provide $\mathrm{Mor}(F,G)$ with the structure of a contravariant functor from $(Sm/K)_{\leq 1}$ to $(Sets)$, written using the calligraphic $\mathcal{M}\mathrm{or}(F,G)$, by defining $\mathcal{M}\mathrm{or}(F,G)(C) = \mathrm{Mor}(C \times F, G)$.

If X and Y are schemes over K, then in light of Proposition 1.1 the functor $\mathcal{M}\mathrm{or}(X,Y)$ may be identified with the functor on $(Sm/K)_{\leq 1}$ which sends C to $\mathrm{Mor}(X \times C, Y)$.

Observe that $\mathcal{M}\mathrm{or}(F,G)$ is an internal Hom-object for the category $\mathcal{M}\mathrm{or}^{\leq 1}$ – for any $H : (Sm/K)_{\leq 1} \to (Sets)$ we have

$$\mathrm{Mor}(H, \mathcal{M}\mathrm{or}(F,G)) \; = \; \mathrm{Mor}(H \times F, G).$$

First of all, a natural transformation $\psi : H \to \mathcal{M}\mathrm{or}(F,G)$ determines $H(C) \to \mathcal{M}\mathrm{or}(F,G)(C) \to \mathrm{Hom}_{(Sets)}(F(C), G(C))$ natural with respect to C. Conversely, a natural transformation $\phi : H \times F \to G$ determines for each C the map $H(C) \to \mathrm{Mor}(C \times F, G)$ associated to the pairing natural with respect to C' determined by ϕ, $H(C) \times \mathrm{Hom}_K(C', C) \to \mathrm{Hom}_{(Sets)}(F(C'), G(C'))$. We readily verify that these constructions are mutually inverse.

Example 1.3. *Let $Y \subset \mathbb{P}^N$ be a quasi-projective variety and consider the functor $\mathcal{C}_r Y : (Sm/K)_{\leq 1} \to (Sets)$ which associates to $C \in (Sm/K)_{\leq 1}$ the monoid of effective cycles in $C \times Y$ which are flat (equivalently, dominant) over C of relative dimension r. The map $(\mathcal{C}_r Y)(C) \to (\mathcal{C}_r Y)(C')$ associated to a morphism $C' \to C$ is given by pullback of cycles (which is well-defined in light of the flatness condition).*

When Y is projective, the functor $\mathcal{C}_r Y$ is represented by the disjoint union of Chow varieties $\coprod_d C_{r,d}(Y)$, where $C_{r,d}(Y)$ parameterizes effective r-cycles of degree d on Y. Each $C_{r,d}Y$ is a projective variety defined over K, and, as shown in [F1], *the Chow monoid*

$$\mathcal{C}_r Y \ \equiv \ \coprod_{d \geq 0} C_{r,d}(Y)$$

is independent of the embedding $Y \subset \mathbb{P}^N$ in the sense that two different embeddings yield monoids which are related by a continuous algebraic map whose graph projects to each via a bicontinuous morphism. We recall that a map from any normal variety X to $\mathcal{C}_r Y$ is equivalent to an effective cycle on $X \times Y$ equidimensional over X of relative dimension r, so that, in particular, the functor $\mathcal{C}_r Y$ is given as $\mathrm{Mor}(-, \mathcal{C}_r Y)$.

If Y is merely quasi-projective with some chosen projective closure $Y \subset \overline{Y}$, the functor $\mathcal{C}_r Y$ is a "quotient" of the representable functor $\mathcal{C}_r \overline{Y}$. Specifically, for any $C \in (Sm/K)_{\leq 1}$, we can realize $(\mathcal{C}_r Y)(C)$ as the quotient of the monoid $\mathrm{Mor}(C, \mathcal{C}_r \overline{Y})$ *by the submonoid* $\mathrm{Mor}(C, \mathcal{C}_r Y_\infty)$, *where $Y_\infty = \overline{Y} \setminus Y$.*

The following proposition verifies the functoriality of the association $Y \mapsto \mathcal{C}_r Y$ for Y projective. This functoriality is a reformulation of the naturality of proper push-forward of cycles.

Proposition 1.4. *Let X, Y be projective varieties. For any $r \geq 0$, there is a natural transformation*

$$\mathcal{M}\mathrm{or}(X, Y) \longrightarrow \mathcal{M}\mathrm{or}(\mathcal{C}_r X, \mathcal{C}_r Y).$$

Proof. We may replace X by its weak normalization, since both $\mathcal{M}\mathrm{or}(X, Y)$ and $\mathcal{C}_r X$ are unaffected by this substitution, so that every continuous algebraic map will be a morphism of varieties.

For $C \in (Sm/K)_{\leq 1}$, consider an element $f : C \times X \to Y$ of $\mathcal{M}\mathrm{or}(X, Y)(C)$. We proceed to define a natural transformation

$$C \times \mathcal{C}_r X \xrightarrow{f_*} \mathcal{C}_r Y : (Sm/K)_{\leq 1} \to (Sets).$$

For any $C' \in (Sm/K)_{\leq 1}$ and any $g = (g_1, g_2) : C' \to C \times \mathcal{C}_r X$ let Z_g denote the effective cycle on $C' \times X$ associated to g_2, so that Z_g equidimensional of relative dimension r over C'. Consider the proper map

$$f * g \ \equiv \ (1_{C'}, f) \circ (1_{C'}, g_1, 1_X) : \ C' \times X \to C' \times C \times X \to C' \times Y$$

and define $f_*(g)$ to be $(f * g)_*(Z_g)$, an effective cycle on $C' \times Y$ equidimensional of relative dimension r over C'.

To verify that f_* is a natural transformation, we consider some $h : C'' \to C'$ in $(Sm/K)_{\leq 1}$. The fact that following diagram consists of Cartesian squares

$$\begin{array}{ccccc}
C'' \times X & \longrightarrow & C'' \times C \times X & \longrightarrow & C'' \times Y \\
{\scriptstyle h \times 1} \big\downarrow & & \big\downarrow {\scriptstyle h \times 1 \times 1} & & \big\downarrow {\scriptstyle h \times 1} \\
C' \times X & \longrightarrow & C' \times C \times X & \longrightarrow & C' \times Y
\end{array}$$

together with the commutativity of push-forward and pull-back implies that h^* applied to $(f * g)_*(Z_g) \in \mathrm{Mor}(C', \mathcal{C}_s Y)$ equals $(f * g')_*(Z_{g'}) \in \mathrm{Mor}(C'', \mathcal{C}_s Y)$, where $g' = g \circ h$ as required by functoriality.

To complete the proof, we must verify the functoriality with respect to C of $f \mapsto f_*$. Consider $k : \tilde{C} \to C \in (Sm/K)_{\leq 1}$ and $f : C \times X \to Y$. To prove functoriality, we must show that

$$(f \circ k)_* = f_* \circ (k, 1) : \tilde{C} \times \mathcal{C}_r X \to \mathcal{C}_r Y.$$

Observe that $f_* : C \times \mathcal{C}_s X \to \mathcal{C}_s Y$ sends a geometric point (c, Z) of $C \times \mathcal{C}_r X$ to $(f_{|c \times X})_*(Z)$, whereas $(f \circ k)_*$ sends a geometric point $(\tilde{c}, Z)$ of $\tilde{C} \times \mathcal{C}_s X$ to $((f \circ k \times 1)_{|\tilde{c} \times X})_*(Z)$. Hence, $(f \circ k)_*$ and $f_* \circ (k, 1)$ agree on geometric points and thus are equal. $\square$

We next present a proof of the well-definedness of the trace map introduced in [FL-1; 7.1] which is more formal and perhaps clearer than the original proof.

Proposition 1.5. *Let Y be a projective variety. For any $C \in (Sm/K)$ and any morphism $f : C \to \mathcal{C}_s(\mathcal{C}_r Y)$, let $Z_f = \Sigma Z_i$ be the associated effective cycle on $C \times \mathcal{C}_r Y$ equidimensional of relative dimension s over C and let $p_i : Z_i \to C$ denote the projection maps of the irreducible components of Z. For each i, let $\tilde{Z}$ be the effective cycle on $Z_i \times Y$ associated to $Z_i \to C \times \mathcal{C}_r Y \to \mathcal{C}_r Y$; thus $\tilde{Z}_i$ is equidimensional of relative dimension r over Z_i. Define $tr(f) = \Sigma(p_i \times 1)_*(\tilde{Z}_i)$, an effective cycle on $C \times Y$ equidimensional of relative dimension $r + s$ over C. Then sending f to $tr(f)$ determines a continuous algebraic map*

$$tr : \mathcal{C}_s(\mathcal{C}_r Y) \ \to \ \mathcal{C}_{r+s} Y.$$

Proof. It suffices to verify the functoriality of the construction $f \mapsto tr(f)$ with respect to maps $g : C' \to C \in (Sm/K)_{\leq 1}$. Observe that $tr(f) \in \mathrm{Mor}(C, \mathcal{C}_{r+s} Y)(C)$ is sent via g to the cycle associated to the pull-back $\Sigma(1 \times p_i)_*(\tilde{Z}_i) \times_C C'$, since $tr(f)$ is flat over C. Similarly, the effective cycle $Z_{f \circ g}$ on $C' \times \mathcal{C}_r Y$ is the cycle associated to the pull-back of Z_f via g. Thus, the required equality

$$tr(f \circ g) \ = \ g^*(tr(f))$$

follows from the commutativity of push-forward (along proper maps) and pull-back (along flat maps). $\square$

The following proposition, in conjunction with the topological realization discussed in the next section, justifies the cap pairing considered in [FL-1; 7.2]. This cap product plays a central role in [F3].

Proposition 1.6. *Let X be a quasi-projective variety and let Y be a projective variety. Then sending a pair (f, Z) with Z an irreducible s-cycle on X to the graph of the composition $Z \to X \to \mathcal{C}_r Y$ determines a "cap product" pairing*

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y) \times \mathcal{C}_s X \ \to \ \mathcal{C}_{r+s}(X \times Y)$$

for any $r, s \geq 0$.

Proof. We may replace X with its weak normalization without loss of generality.

For $C \in (Sm/K)_{\leq 1}$, we define a map

$$\Psi : \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)(C) \times (\mathcal{C}_s X)(C) \longrightarrow \mathcal{C}_{r+s}(X \times Y)(C)$$

by sending $(f : C \times X \longrightarrow \mathcal{C}_r Y, W)$ to the graph of the composite map $W \longrightarrow C \times X \longrightarrow \mathcal{C}_r Y$, where W is a closed, integral subscheme of $C \times X$ that is flat over C. We regard this graph, which is naturally a cycle in $W \times Y$, as being a cycle in $C \times X \times Y$. We extend Ψ linearly, so that Ψ is defined on all cycles. Observe that Ψ sends (f, W) to an element of $\mathcal{C}_{r+s}(X \times Y)(C)$ since the cycle constructed is clearly dominate over C.

To verify functoriality of Ψ with respect to C, we begin by choosing a projective closure $X \subset \overline{X}$. Given $f : C \times X \longrightarrow \mathcal{C}_r Y$, $g : C \longrightarrow \mathcal{C}_s X$ (where g is associated to the W considered above), observe that f determines $f' : C \times X \longrightarrow \mathcal{C}_r(\overline{X} \times Y)$. (One sends the cycle Z on $C \times X \times Y$ determined by f to the push-forward by the diagonal map to a cycle on $C \times X \times \overline{X} \times Y$.) Choose a projective closure $C \times X \subset \overline{C \times X}$ so that f' extends to $\overline{f} : \overline{C \times X} \to \mathcal{C}_r(\overline{X} \times Y)$ and the projection $C \times X \to X$ extends to $\overline{C \times X} \to \overline{X}$. Observe that $g : C \longrightarrow \mathcal{C}_s X$ determines $g' : C \longrightarrow \mathcal{C}_s(C \times X)$. (One sends the cycle W on $C \times X$ flat over C to the push-forward by the diagonal map to a cycle on $C \times C \times X$.) We choose a lifting $\tilde{g} : C \longrightarrow \mathcal{C}_s(\overline{C \times X})$ of g'. Then the pair $(\overline{f}, \tilde{g})$ determines the map

$$\Psi(f, g) : C \longrightarrow \mathcal{C}_s(\overline{C \times X}) \to \mathcal{C}_s(\mathcal{C}_r(\overline{X} \times Y)) \to \mathcal{C}_{r+s}(\overline{X} \times Y) \to \mathcal{C}_{r+s}(X \times Y),$$

where the first map is $\tilde{g}$, the second is induced by $\overline{f}$ using Proposition 1.4, the third is the trace map of Proposition 1.5, and the fourth is the defining projection. One readily verifies that the graph of $\Psi(f, g)$ is precisely $\Psi(f, W)$ by checking this equality at the generic point of C, and, in particular, the map $\Psi(f, g)$ is independent of the choices made.

Assume given $h : C' \to C \in (Sm/K)_{\leq 1}$ as well as (f, g). Provided one chooses $\overline{C' \times X}$ to map to $\overline{C \times X}$ and chooses $(g \circ h)^{\sim} = \tilde{g} \circ h$, one sees immediately that

$$\Psi(f, g) \circ h \;=\; \Psi(f \circ (h \times 1), g \circ h)$$

as required for functoriality. □

In subsequent sections, we shall require the continuity and associativity of composition, which is implied by the next proposition together with the topological realization functor of the next section.

Proposition 1.7. *Let X, Y, W be quasi-projective varieties over K. Composition of morphisms determines a pairing of functors*

$$\mathcal{M}or(X, Y) \times \mathcal{M}or(Y, W) \longrightarrow \mathcal{M}or(X, W)$$

which is associative in the evident sense.

Similarly, if X is a quasi-projective variety, Y and W are projective varieties, then composition together with the trace map of Proposition 1.5 determines a bilinear pairing

$$\mathcal{M}or(X, \mathcal{C}_r Y) \times \mathcal{M}or(Y, \mathcal{C}_s W) \to \mathcal{M}or(X, \mathcal{C}_{r+s} W)$$

which is associative in the evident sense.

Proof. The first pairing is given by sending $f : C \times X \to Y$, $g : C \times Y \to W$ to $g \circ (1_C, f) : C \times X \to W$ for any $C \in (Sm/K)_{\leq 1}$. This is clearly natural in C.

The second pairing is the composition of maps given by applying Proposition 1.4, the first pairing, and the trace map of Proposition 1.5:

$$\begin{aligned}\mathcal{M}or(X, \mathcal{C}_r Y) \times \mathcal{M}or(Y, \mathcal{C}_s W) &\to \mathcal{M}or(X, \mathcal{C}_r Y) \times \mathcal{M}or(\mathcal{C}_r Y, \mathcal{C}_r(\mathcal{C}_s W)) \\ &\to \mathcal{M}or(\mathcal{C}_r x, \mathcal{C}_r(\mathcal{C}_s W)) \to \mathcal{M}or(X, \mathcal{C}_{r+s} W).\end{aligned}$$

□

§2. Topological realization for $K = \mathbb{C}$

Every complex variety admits a realization as a topological space and every morphism of complex varieties induces a continuous map on the associated spaces. The goal of this section is to generalize this simple concept in two ways. Namely, we wish to replace "varieties" with "constructible sets modulo proper equivalence relations" (see Definition 2.1) and also to replace "morphisms" with "natural transformations of the associated functors on $(Sm/K)_{\leq 1}$." The precise statement is Theorem 2.3. This generalized notion of topological realization, together with the results of section 1, allows us to establish the continuity of various maps arising in the study of Lawson homology and morphic cohomology.

Many of the functors on $(Sm/K)_{\leq 1}$ introduced in section 1 admit a kind of presentation in terms of algebro-geometric information. The following definition provides the formal notion which covers all of the cases arising in this paper.

Definition 2.1. *Consider the data: $\mathcal{Y} = \coprod_d Y_d$, a disjoint union of projective varieties over $\operatorname{Spec} K$; $\mathcal{E} = \coprod_d E_d$, where each E_d is a constructible algebraic subset of Y_d; a "proper equivalence relation" $R = \overline{R} \cap (\mathcal{E}^{\times 2})$, where $\overline{R} \subset \mathcal{Y}^{\times 2}$ is a closed equivalence relation such that $R = \overline{R} \cap (\mathcal{E} \times \mathcal{Y})$. Then we say $(\mathcal{Y}, \mathcal{E}, R)$ is a proper, constructible presentation of a functor $F : (Sm/K)_{\leq 1} \to (Sets)$ if F is the functor given by sending $C \in (Sm/K)_{\leq 1}$ to $\operatorname{Mor}(C, \mathcal{E})/\operatorname{Mor}(C, R)$ (where, in general, if E is a constructible subset of variety Y, we define $\operatorname{Mor}(X, E)$ to be the set of those morphisms from X to Y whose set theoretic images land in E).*

As seen in Example 1.3, given a quasi-projective variety Y, the functor $\mathcal{C}_r Y$ admits a proper, constructible presentation. The following proposition implies that $\mathcal{M}\mathrm{or}(X, Y)$ does as well.

Proposition 2.2. *Let X, Y be quasi-projective varieties over $\operatorname{Spec} K$, X^w the weak normalization of X, and assume $X^w \subset \overline{X}^w$, $Y \subset \overline{Y}$ are projective closures. Then $\mathcal{M}\mathrm{or}(X, Y) : (Sm/K)_{\leq 1} \to (Sets)$ admits a proper, constructible presentation $(\tilde{C}(\overline{X}^w \times \overline{Y}), \mathcal{E}_{0,1}(X^w, Y), R)$ defined as follows: $\tilde{C}(\overline{X}^w \times \overline{Y})$ is the Chow variety of effective cycles in $\overline{X}^w \times \overline{Y}$ which have dimension equal to the dimension of X^w (locally); $\mathcal{E}_{0,1}(X^w, Y) \subset \mathcal{C}_d(\overline{X}^w \times \overline{Y})$ is the constructible subset of those cycles whose restriction to $X^w \times \overline{Y}$ are graphs of morphisms from X^w to Y; and R is the equivalence relation associated to the diagonal action of $\tilde{C}(X^w_\infty \times \overline{Y})$, the subset of those cycles supported on $X^w_\infty \times \overline{Y}$, on $(\tilde{C}(\overline{X}^w \times \overline{Y})^{\times 2}$, where $X^w_\infty = \overline{X}^w - X^w$.*

Furthermore, when X and Y are both projective varieties, this presentation of $\mathcal{M}\mathrm{or}(X, Y)$ realizes $\mathcal{M}\mathrm{or}(X, Y)$ as the functor associated to an inductive limit of quasi-projective varieties.

Proof. To simplify notation, we replace X with its weak normalization and omit the superscript w everywhere.

The constructibility of the subset $\mathcal{E}_{0,1}(X, Y) \subset \tilde{C}(\overline{X} \times \overline{Y})$ can be verified by using the incidence correspondence $\mathcal{I}(\overline{X}, \overline{Y}) \subset \tilde{C}(\overline{X} \times \overline{Y}) \times \overline{X} \times \overline{Y}$ consisting of those triples (Z, x, y) with the property that (x, y) lies in the support of the cycle Z. For consider the natural map $\rho : \mathcal{I}(\overline{X}, \overline{Y}) \to \tilde{C}(\overline{X} \times \overline{Y}) \times \overline{X}$. Let $B \subset \mathcal{I}(\overline{X}, \overline{Y})$ be the constructible set of points (Z, x, y) such that $x \in X$ and (Z, x, y) lies in a fiber of ρ consisting of more than one point – i.e., a fiber of dimension more than 0 or of degree more than 1. Then $\mathcal{E}_{0,1}(X, Y)$ is the complement of the projection of B to $\tilde{C}(\overline{X} \times \overline{Y})$.

Observe that the image of the diagonal action of $\tilde{C}(X_\infty \times \overline{Y})$ (which is a proper map between disjoint unions of projective varieties)

$$\tilde{C}(\overline{X} \times \overline{Y}) \times \tilde{C}(X_\infty \times \overline{Y})^{\times 2} \longrightarrow \tilde{C}(\overline{X} \times \overline{Y})^{\times 2}$$

is a closed equivalence relation $\overline{R}$ on $\tilde{C}(\overline{X} \times \overline{Y})$ which satisfies the property

$$R \;\equiv\; \overline{R} \cap \mathcal{E}_{0,1}(X,Y)^{\times 2} \;=\; \overline{R} \cap (\mathcal{E}_{0,1}(X,Y) \times \tilde{C}(\overline{X} \times \overline{Y})).$$

To verify that

$$\mathcal{M}\mathrm{or}(X,Y) = \mathcal{E}_{0,1}(X,Y)/R : (Sm/K)_{\leq 1} \to (Sets),$$

observe that an element of $\mathcal{E}_{0,1}(X,Y)(C)$ is a cycle γ in $C \times \overline{X} \times \overline{Y}$ satisfying the condition that γ is equidimensional over C and that $\gamma \cap (C \times X \times \overline{Y})$ is the graph of a morphism from $C \times X$ to Y. Here, we are using (a) a rational map with domain $C \times X$ is the graph of a morphism if and only if for each geometric point $c \times x \in C \times X$ there is a unique geometric point of the form (c,x,y) in its graph; and (b) the pull-back of γ over C to c has restriction to $\{c\} \times X \times \overline{Y}$ the graph of the map from $X \times \{c\}$ to $\overline{Y}$ given by the image of c in $\mathcal{E}_{0,1}(X,Y) \subset \tilde{C}(\overline{X} \times \overline{Y})$ since γ is flat over C. Thus, there is an evident map $\mathcal{E}_{0,1}(X,Y)(C) \longrightarrow \mathcal{M}\mathrm{or}(X,Y)(C)$ obtained by restriction of cycles to $C \times X \times \overline{Y}$. This map is surjective, since we may lift elements in the target set by taking closures of cycles. (Such closures must remain equidimensional over C since they will dominate C which is one-dimensional and smooth.) Finally, two elements γ and γ' of $\mathcal{E}_r(Y)(X)(C)$ are sent to the same element under this map if and only if their restrictions to $C \times X \times Y$ coincide – that is, if and only if $\gamma + \delta = \gamma' + \delta'$ for some δ, δ' contained in $C \times X_\infty \times \overline{Y}$ and equidimensional over C. In other words, two elements are sent to the same element under this map if only only if their images are the graphs of the same morphism from $C \times X$ to Y.

Finally, if X and Y are both projective varieties, then we take $X = \overline{X}$ and $Y = \overline{Y}$. The constructible subset B defined above is actually closed in this case. Thus, $\mathcal{E}_{0,1}(X,Y)$, which is the complement of the image of B under a proper map, is open in $\tilde{C}(X \times Y)$. The equivalence relation R is clearly trivial in this case, and so $\mathcal{M}\mathrm{or}(X,Y)$ is represented by the ind-variety $\mathcal{E}_{0,1}(X,Y) = \varinjlim_n \mathcal{E}_{0,1}(X,Y)_n$, where $\mathcal{E}_{0,1}(X,Y)_n$ in the quasi-projective variety consisting of those cycles in $\mathcal{E}_{0,1}(X,Y)_n$ of degree at most n. $\square$

We now restrict our attention to complex varieties. For a complex quasi-projective algebraic variety X, we write X^{an} for the set $X(\mathbb{C})$ of $\mathbb{C}$ points of X provided with its topology as an analytic space. If $(\mathcal{E}, \mathcal{Y}, R)$ is a proper, constructible representation of a functor F as in Definition 2.1, we write $\mathcal{E}^{an}$ for the subspace of $\mathcal{Y}^{an}$ consisting of points $\mathcal{E}(\mathbb{C}) \subset \mathcal{Y}(\mathbb{C})$ and we let $(\mathcal{E}/R)^{an}$ denote the space consisting of points $\mathcal{E}(\mathbb{C})/R(\mathbb{C})$ provided with the quotient topology given by the surjective map $\mathcal{E}^{an} \longrightarrow (\mathcal{E}/R)^{an}$. Observe that the set of points of $(\mathcal{E}/R)^{an}$ is simply $F(\mathbb{C})$.

Thus, any functor admitting a proper, constructible presentation has a topological presentation. The following theorem shows that a natural transformation of such functors induces, as one would hope, a continuous map on the associated spaces. This result is particularly useful for establishing the continuity of various pairings, as well as showing the well-definedness of the topology associated to various constructions, which arise in Lawson homology and morphic cohomology. For example, if X a normal, quasi-projective variety and Y projective, then the topology on $\mathrm{Mor}(X, C_r Y)$ as given in [F2] is described by a somewhat different proper, constructible presentation than that given by Proposition 2.2. Theorem 2.3 assures us that these different presentations determine the same topology.

Theorem 2.3. *Let $F, F' : (Sm/\mathbb{C})_{\leq 1} \to (Sets)$ be contravariant functors provided with proper, constructible presentations $(\mathcal{Y}, \mathcal{E}, R)$, $(\mathcal{Y}', \mathcal{E}', R')$. Then a natural transformation $\psi : F \to F'$ induces a continuous map*

$$\psi^{an} : (\mathcal{E}/R)^{an} \longrightarrow (\mathcal{E}'/R')^{an}.$$

Proof. Observe that if $S \subset X$ is a constructible subset of a projective variety X, then S has a canonical expression as a union of irreducible constructible subsets $S = \cup_\beta S_\beta$, where β runs through those (Zariski) points of X which lie in S and which satisfy the condition that they do not lie in the closure of any point $\beta' \neq \beta$ with $\beta' \in S$. Thus, S_β equals $S \cap X_\beta$, where $X_\beta \subset X$ is the closed subvariety with generic point β. Clearly, if R is a proper equivalent relation on $S \subset X$, then $(S/R)^{an} = ((\coprod S_\beta)/R^{\sim})^{an}$ where each $S_\beta \subset X_\beta$ and $R^{\sim}$ is the equivalence relation determined by R. Thus, we may assume that each Y_d and each Y'_d are irreducible and that each $E_d \subset Y_d$ and each $E'_d \subset Y'_d$ are dense.

For each generic point $\eta_\gamma : \operatorname{Spec} k(\gamma) \to \mathcal{E}$ of $\mathcal{E}$, choose some generic point $\tilde{\psi}(\eta_\gamma) : \operatorname{Spec} k(\gamma) \to \mathcal{E}'$ satisfying $\psi \circ p(\eta_\gamma) = q(\tilde{\psi}(\eta_\gamma))$, where $p : \mathcal{E} \to F$ and $q : \mathcal{E}' \to F'$ are the natural quotient maps. Let $\Gamma_\gamma \subset \mathcal{Y} \times \mathcal{Y}'$ denote the irreducible subvariety with generic point

$$(\eta_\gamma, \tilde{\Psi}(\eta_\gamma)) : \operatorname{Spec} k(\gamma) \to \mathcal{E}_\gamma \times \mathcal{E}'_{\tilde{\psi}(\eta_\gamma)} \to \mathcal{Y}_\gamma \times \mathcal{Y}'_{\tilde{\psi}(\eta_\gamma)},$$

and let $\Gamma_\psi \subset \mathcal{Y} \times \mathcal{Y}'$ denote the union

$$\Gamma_\psi \equiv \bigcup_\gamma \Gamma_\gamma.$$

To prove that ψ induces a continuous map ψ^{an}, it suffices to prove that the restriction of $\Gamma_\psi^{an} \subset \mathcal{Y}^{an} \times \mathcal{Y}'^{an}$ to $\mathcal{E}^{an} \times \mathcal{E}'^{an}$ has image in $(\mathcal{E}/R)^{an} \times (\mathcal{Y}/\overline{R'})'^{an}$ which maps bijectively to $(\mathcal{E}/R)^{an}$ and is contained in $(\mathcal{E}/R)^{an} \times (\mathcal{E}/R')^{an}$. (For in this case, the bijective map must be a homeomorphism, since it is a proper map between Hausdorff spaces.) For this, it suffices to prove that for any point $(\chi, \zeta) \in \Gamma_\psi$ such that χ lies in $\mathcal{E}$, we have

$$\psi \circ p(\chi) = \overline{q}(\zeta).$$

(Here, $\mathcal{Y}'/\overline{R}' : (Sm/\mathbb{C})_{\leq 1} \to (Sets)$ is defined as the evident quotient functor of $\mathcal{Y}'$ with projection $\overline{q} : \mathcal{Y}' \to \mathcal{Y}'/\overline{R}'$.) We proceed by induction on the codimension of $\chi \in \mathcal{E}$ (i.e., the maximum over all irreducible components S of $\mathcal{E}$ containing χ of the codimension of $\overline{\chi}$ in $\overline{S}$). By construction, the required equality is valid for all χ of codimension 0 (i.e., for generic points). Assume that the equality is valid for all points of codimension $\leq s$ and let $(x, y) \in \Gamma_\psi$ be such that $x : \operatorname{Spec} k(x) \to \mathcal{E}$

is a point of codimension $s+1$. Let $g_C : C \to \Gamma_\psi$ be a non-constant map from a smooth curve C defined over $k(x)$ with some $k(x)$-rational point $c \in C$ mapping to (x, y). Let $\psi(pr_1 \circ g_C)^\sim$ be any lifting of $\psi(pr_1 \circ g_C)$ to a map from C to $\mathcal{Y}'$. Letting $\gamma : \operatorname{Spec} k(\gamma) \to C$ denote the generic point of C, observe that by hypothesis,

$$\overline{q}\,(\psi(pr_1 \circ g_C)^\sim \circ \gamma) = \overline{q}\,(\psi(pr_1 \circ g_C \circ \gamma)) = \overline{q}(pr_2 \circ g_C \circ \gamma).$$

In other words, the map

$$(\psi(pr_1 \circ g_C)^\sim, pr_2 \circ g_C) : C \longrightarrow \mathcal{Y}' \times \mathcal{Y}'$$

sends the generic point of C into $\overline{R}'$. It thus sends c into $\overline{R}'$ as well, and so

$$\overline{q}\,(\psi(pr_1 \circ g_C)^\sim \circ c) = \overline{q}(pr_2 \circ g_C \circ c),$$

which, by the naturality of ψ, implies that $\psi \circ p(x) = \overline{q}(x)$, as desired. □

It follows from [FL-1; 1.4] that if X and Y are projective varieties, then the set $\operatorname{Mor}(X, Y)$ of morphisms from X to Y has the natural structure of a quasi-projective variety. Taking the analytic topology of this quasi-projective variety gives us a "natural" topology on $\operatorname{Mor}(X, Y)$. For X not necessarily projective, $\operatorname{Mor}(X, Y)$ is no longer a variety but the "analytic" topology on $\operatorname{Mor}(X, Y)$ does have a concrete description as recalled in the following proposition.

Proposition 2.4. *(*[FL-2; A.3]*) Let X be a weakly normal quasi-projective variety, $X \subset \overline{X}$ a projective closure, and Y a projective variety. Then the following topologies on* $\operatorname{Mor}(X, Y)$ *are equivalent:*

a. Identification of $\operatorname{Mor}(X, Y)$ *with* $(\mathcal{E}_{0,1}(X, Y)/R)^{an}$*, where* $(\tilde{C}(\overline{X} \times Y), \mathcal{E}_{0,1}(X, Y), R)$ *is the proper, constructible presentation of Proposition 2.2.*

b. The topology of convergence with bounded degree: a sequence $\{f_i\}$ of morphisms converges if and only if this sequence converges in $\operatorname{Hom}_{cont}(X^{an}, Y^{an})$ *provided with the compact open topology and there exists some upper bound for the degrees of the closures in $\overline{X} \times Y$ of the graphs of f_i*
We let $\mathcal{M}\mathrm{or}(X, Y)^{an}$ *denote the resulting topological space.*

We include the following result which indicates that $\mathcal{M}\mathrm{or}(X, Y)^{an}$ has a "good" topology – i.e., has the homotopy type of C.W. complex. The reader should note that [F2; 1.5] erroneously claims that spaces such as $\mathcal{M}\mathrm{or}(X, Y)^{an}$ admit the structure of C.W. complexes. We give in Proposition 2.5 a slightly weakened (but functionally equivalent) version of this claim, together with a proof.

Proposition 2.5. *Let X be a quasi-projective variety and Y a projective variety. Then* $\mathcal{M}\mathrm{or}(X, Y)^{an}$ *has the homotopy type of a C.W. complex.*

Proof. We may assume X is weakly normal. Choose a projective closure $X \subset \overline{X}$ and use the notations of Proposition 2.2. Additionally, let S_n denote the subset of $\tilde{C}(\overline{X}, Y)$ consisting of cycles of degree n which lie in $\mathcal{E}_{0,1}(X, Y)$. Further, let R_n denote the subset of S_n consisting of cycles with a non-trivial component at infinity – i.e., cycles in the image of the map $\coprod_{k>0} \tilde{C}_k(X_\infty \times Y) \times S_{n-k} \to S_n$ given by addition of cycles. Then $R_n \subset S_n$ is a closed subset (in the Zariski topology) of the constructible subset S_n. Finally, define $\mathcal{E}_n$ to be the constructible subset of $\mathcal{E}_{0,1}(X,Y)$ consisting of cycles whose intersection with $X \times Y$ have closures of degree at most n.

There is an evident push-out square

$$\begin{tikzcd} R_n \times \tilde{C}(X_\infty \times Y) \arrow[r] \arrow[d] & S_n \times \tilde{C}(X_\infty \times Y) \arrow[d] \\ \mathcal{E}_{n-1} \arrow[r] & \mathcal{E}_n \end{tikzcd} \tag{2.5.1}$$

with vertical arrows given by addition of cycles. Note also that the monoid $\tilde{C}(X_\infty \times Y)$ acts on the square (2.5.1). If we mod out by this monoid action, we obtain another push-out square

$$\begin{tikzcd} R_n \arrow[r] \arrow[d] & S_n \arrow[d] \\ X_{n-1} \arrow[r] & X_n, \end{tikzcd} \tag{2.5.2}$$

where $X_n = \mathcal{E}_n / \tilde{C}(X_\infty \times Y)$.

Note that $R_n \subset S_n$ is a closed immersion of constructible subsets of some projective space $\mathbb{P}^N$. By [H] $\mathbb{P}^N$ admits a semi-algebraic triangulation so that S_n and R_n are each unions of open simplices. Now form the barycentric subdivision of this triangulation and define S'_n, R'_n to be the so-called "cores" – namely, S'_n is the union of all closed simplices of the barycentric subdivisions contained entirely in S_n, and R'_n is defined similarly. Observe there is an evident straight-line deformation retract of $S_n \subset R_n$ onto $S'_n \subset R'_n$, and that $S'_n \subset R'_n$ is a cellular extension.

Suppose, by induction on n, we have constructed a homotopy equivalence $X_{n-1} \to Y_{n-1}$, where Y_{n-1} has the structure of a CW complex. Define $f : R_n \to Y_{n-1}$ to be the composition $R_n \twoheadrightarrow R'_n \hookrightarrow R_n \to X_{n-1} \to Y_{n-1}$ and define $\tilde{Y}_n$ so that

$$\begin{tikzcd} R_n \arrow[r] \arrow[d, "f"'] & S_n \arrow[d] \\ Y_{n-1} \arrow[r] & \tilde{Y}_n \end{tikzcd}$$

is a pushout square. Since $R_n \subset S_n$ is an NDR subspace and f is homotopic to the composition $R_n \to X_{n-1} \to Y_{n-1}$, we have by [LW; IV.2.3] that there is a homotopy equivalence $X_n \to \tilde{Y}_n$ causing the triangle

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[d] \\ Y_{n-1} \arrow[r] & \tilde{Y}_n \end{tikzcd}$$

to commute. Finally, define Y_n so that

$$\begin{tikzcd} R'_n \arrow[r] \arrow[d] & S'_n \arrow[d] \\ Y_{n-1} \arrow[r] & Y_n \end{tikzcd}$$

is a pushout square. The deformation retract of $R_n \subset S_n$ onto $R'_n \subset S'_n$ induces a deformation retract of $\tilde{Y}_n$ onto Y_n. Hence there is a homotopy equivalence $X_n \to Y_n$ compatible with the homotopy equivalence $X_{n-1} \to Y_{n-1}$.

Finally, the space $\mathcal{M}\mathrm{or}(X,Y)^{an}$ is the direct limit of the X_n's, which maps via a homotopy equivalence to the direct limit of the Y_n's. Since each map $Y_{n-1} \to Y_n$ is a cellular extension, the proof is complete. □

In light of Theorem 2.3, each of the natural transformations of Propositions 1.4, 1.5, 1.6, and 1.7 (since they are natural transformations of functors admitting proper, constructible presentations) induces a continuous map between the associated topological spaces. We record in the following theorem a specific case of this continuity, since it is used extensively in [F3].

Theorem 2.6. *Let X be a quasi-projective variety, Y a projective variety, and $s, r \geq 0$ integers. Then the pairing of Proposition 1.6 induces a continuous pairing*

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an} \times (\mathcal{C}_s X)^{an} \longrightarrow (\mathcal{C}_{r+s}(X \times Y))^{an}.$$

Proof. The functors $\mathrm{Mor}(X,Y)$, $\mathcal{C}_s X$, and $\mathcal{C}_{r+s}(X \times Y)$ admit proper, constructible presentations by Example 1.3 and Proposition 2.2. The pairing is induced by a natural transformation of functors by Proposition 1.6. Thus, continuity is a consequence of Theorem 2.3. □

We conclude this section with the following explicit description of the set of connected components of $\mathcal{M}\mathrm{or}(X,Y)^{an}$.

Proposition 2.7. *For a quasi-projective varieties X and Y, the set $\pi_0\,\mathcal{M}\mathrm{or}(X,Y)^{an}$ is the quotient of the set $\mathcal{M}\mathrm{or}(X,Y)$ by the equivalence relation generated by declaring two continuous algebraic maps f and g to be equivalent if there is a smooth, connected curve C (of finite type over $\mathbb{C}$) with closed points c, d and a continuous algebraic map $h : X \times C \longrightarrow Y$ such that $f = h|_{X \times \{c\}}$ and $g = h|_{X \times \{d\}}$.*

Proof. In fact, we will describe the set of connected components of $(\mathcal{E}/R)^{an}$, whenever $(\mathcal{Y}, \mathcal{E}, R)$ is a proper, constructible presentation of a contravariant functor $\mathcal{E}/R : (Sm/\mathbb{C})_{\leq 1} \to (Sets)$. This applies to $\mathcal{M}\mathrm{or}(X,Y)$ by Proposition 2.2. We claim two points x, y in $(\mathcal{E}/R)^{an}$ lie in the same component if and only if there is a sequence of smooth, connected curves $C_0, \ldots, C_n$, morphisms $g_i : C_i \to \mathcal{E}$, and points $c_i, c'_i \in C_i$ such that $g_0(c_0) = x$, $g_n(c'_n) = y$, and $(g_i(c'_i), g_{i+1}(c_{i+1})) \in R$ for $0 \leq i < n$.

To establish the claim, first observe that the existence of such a sequence of curves shows that x and y lie in the same component of $(\mathcal{E}/R)^{an}$.

For the converse, observe that we may assume each $E_d \subset Y_d$ is dense. In fact, we may assume each E_d is connected, for whenever we have $E_d = E'_d \coprod E''_d$, we can replace Y_d with $Y'_d \coprod Y''_d$, where Y'_d, Y''_d are the closures in Y_d of E'_d, E''_d. Let A denote the indexing set for the connected Y's and E's. We readily verify in this case that $\pi_0(\mathcal{E}/R)^{an}$ is naturally identified with the set of equivalence classes of A for the equivalence relation generated by pairs $(a, a') \in A^{\times 2}$ with the property that there exists some $t \in E_a$, $t' \in E_{a'}$ with $(t, t') \in R$. It therefore suffices to show for any fixed $\alpha \in A$ that given any two points $x, y \in E_\alpha$, we can connect x and y by a sequence of curves mapping to E_α.

Since E_α is connected, it must contain points of $W \cap Z$ for any two irreducible components Z, W of Y_α (for otherwise we would have $E_\alpha = (E_\alpha \cap Z) \coprod (E_\alpha \cap W)$).

Thus, it suffices to join together any two points on $E_\alpha \cap Z \subset Z$, for any irreducible component Z of Y_α. In other words, we may assume Y_α is irreducible. In this case, E_α contains a dense, irreducible Zariski open subset V_α of Y_α. Let $v \in V_\alpha$ be a chosen closed point. Since Y_α is an irreducible complex variety, there are smooth curves C, D with closed points $c, c' \in C$, $d, d' \in D$, and maps $f : C \longrightarrow Y_\alpha$, $g : D \longrightarrow Y_\alpha$ so that $f(c) = x$, $f(c') = v$, $g(d) = v$, $g(d') = y$. Finally, restrict f, g to $C' = C \cap f^{-1}(E_\alpha)$, $D' = D \cap g^{-1}(E_\alpha)$. Since each of C', D' contain open subsets of C,D (namely, the inverse images of the open subset $V_\alpha \subset Y_\alpha$), both C' and D' are connected, smooth curves mapping to E_α and they connect the points x and y together as desired. $\square$

§3 Construction of pairings

In this section, we build on the foundation of the earlier sections to define a "join pairing" (which is essentially cup product) for morphic cohomology and for its closely related variation, topological cycle cohomology. In fact, the join pairing is defined for the objects in the derived category of presheaves on $Sch/\mathbb{C}$ (the category of quasi-projective complex varieties) which represent these cohomology theories, so that the pairing is natural in a very strong sense. This naturality is needed to establish the main result of the next section (compatibility of the join pairing with the cup product of motivic cohomology – see Corollary 4.5) and also to prove the main result of section 5 (the Whitney sum formula for Chern classes in morphic cohomology – see Theorem 5.4). We also introduce various related pairings and, in particular, show that the join pairing of morphic cohomology and the intersection pairing of Lawson cohomology coincide, for a smooth variety X, under Poincare duality. Here as well, this correspondence is obtained on the level of the representing objects in the derived category of presheaves on X

We begin with a description of the objects used to define morphic cohomology and topological cycle cohomology. For any complex, projective variety Y, the abelian monoid structure on $\mathcal{C}_r Y$ provides the singular complex $\operatorname{Sing} \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}$ associated to the space $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}$ with the structure of a simplicial abelian monoid. We let

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{\sim} \equiv N([\operatorname{Sing} \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}]^+)$$

denote the normalized chain complex associated to the simplicial abelian group $[\operatorname{Sing} \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}]^+$ obtained by level-wise group completion. Following [F3], we define the chain complex

$$\mathcal{M}(X, a) \equiv \operatorname{cone}\left\{\mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^{a-1})^{\sim} \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^{a})^{\sim}\right\}[-2a]$$

and view $\mathcal{M}(-, a)$ as a presheaf on $Sch/\mathbb{C}$. Both the morphic cohomology and the topological cycle cohomology of X (of weight a) are determined using the complex of presheaves $\mathcal{M}(-, a)$ restricted to X (see below). In this definition, we have viewed $\mathbb{P}^{a-1} \subset \mathbb{P}^a$ as the hyperplane obtained as the zero locus of the last coordinate function T_a of $\mathbb{P}^a = \mathbf{P}\mathrm{roj}\, \mathbb{C}[T_0, \dots, T_a]$. On the other hand, the homotopy class of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^{a-1})^{an} \to \mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^{a})^{an}$ is independent of this choice of linear embedding of $\mathbb{P}^{a-1}$ in view of the transitivity of the action of the connected group $PGL_{n+1}(\mathbb{C})$ on the linear hyperplanes of $\mathbb{P}^n$. Thus, the isomorphism class of $\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^a)^\sim \to \mathcal{M}(X, a)[2a]$ in the derived category of presheaves is independent of the choice of linear hyperplane $\mathbb{P}^{a-1} \subset \mathbb{P}^a$.

Observe that $\mathcal{M}(X, a)$ is a chain complex of torsion free abelian groups since for all $k \geq 0$ and all singular k-simplices $\alpha : \Delta^k_{\text{top}} \to \mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^a)^{an}$ if some positive integer multiple of α lies in $\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^{a-1})^{an}$ then α itself lies in $\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^{a-1})^{an}$. Thus, derived tensor products involving $\mathcal{M}(X, a)$ can be represented by ordinary tensor products.

We recall the join pairing

$$\# : \mathcal{C}_r\mathbb{P}^m \times \mathcal{C}_s\mathbb{P}^n \longrightarrow \mathcal{C}_{r+s+1}\mathbb{P}^{m+n+1}$$

defined by sending an irreducible subvariety $Y \subset \mathbb{P}^m$ of dimension r given by homogeneous equations $\{F_i(T_0, \ldots, T_m) : i \in I\}$ and an irreducible subvariety $W \subset \mathbb{P}^n$ of dimension s given by homogeneous equations $\{G_j(S_0, \ldots, S_n) : j \in J\}$ to the irreducible subvariety of $Y\#W \subset \mathbb{P}^{m+n+1}$ of dimension $r + s + 1$ given by the union of these two sets of homogeneous equations viewed as equations in the $m+n+2$ variables $T_0, \ldots, T_m, S_0, \ldots, S_n$. Geometrically, we view $\mathbb{P}^m$, $\mathbb{P}^n$ linearly embedded with disjoint images in $\mathbb{P}^{m+n+1}$ and define $Y\#W$ as the union of lines in $\mathbb{P}^{m+n+1}$ from points on Y to points on W.

Proposition 3.1. *The join map*

$$\mathbb{P}^a \times \mathbb{P}^b \longrightarrow \mathcal{C}_{1,1}\mathbb{P}^{a+b+1}$$

determines a bilinear join pairing

$$\# : C_0(\mathbb{P}^a) \times C_0(\mathbb{P}^b) \longrightarrow C_1(\mathbb{P}^{a+b+1})$$

which induces a pairing in the derived category of presheaves on $Sch/\mathbb{C}$

$$\# : \mathcal{M}(-, a) \overset{\mathbb{L}}{\otimes} \mathcal{M}(-, b) \longrightarrow \mathcal{M}(-, a + b). \tag{3.1.1}$$

Proof. Composition with the bilinear join pairing $\mathcal{C}_0\mathbb{P}^a \times \mathcal{C}_0\mathbb{P}^b \to \mathcal{C}_1\mathbb{P}^{a+b+1}$ determines the bilinear pairing of topological monoids

$$\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^a)^{an} \times \mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^b)^{an} \to \mathcal{M}or(X, \mathcal{C}_1\mathbb{P}^{a+b+1})^{an}.$$

(This pairing sends (f, g) to $f\#_X g : X \to \mathcal{C}_1\mathbb{P}^{m+n+1}$ whose value on $x \in X$ equals the join of $f(x)$ and $g(x)$.) Thus, join determines a pairing of chain complexes natural in X

$$\# : \mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^a)^\sim \otimes \mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^b)^\sim \to \mathcal{M}or(X, \mathcal{C}_1\mathbb{P}^{a+b+1})^\sim. \tag{3.1.2}$$

We recall that the algebraic proof of the Lawson suspension theorem given in [F1] and modified slightly in [F-V] determines a natural transformation for any projective variety Y and any $r \geq 0$

$$\Psi : \mathcal{C}_{r+1}(\Sigma Y) \times \mathbb{A}^1 \longrightarrow \mathcal{C}_{r+1}(\Sigma Y)^{\times 2}$$

sending an effective r-cycle Z on Y and a point $t \in \mathbb{A}^1$ to a pair of effective r-cycles $(\psi_t^+(Z), \psi_t^-(Z))$ such that $\psi_0^+(Z) - \psi_0^-(Z) = Z$ and for $t \neq 0$ both $\psi_t^+(Z)$ and $\psi_t^-(Z)$ lie in the image of the suspension map $\Sigma : \mathcal{C}_r Y \to \mathcal{C}_{r+1}(\Sigma Y)$. As essentially observed in the proof of [FL-1;3.3], this determines a natural transformation

$$\mathrm{Mor}(X, \mathcal{C}_{r+1}(\Sigma Y)) \times \mathbb{A}^1 \longrightarrow \mathrm{Mor}(X, \mathcal{C}_{r+1}(\Sigma Y))^{\times 2}$$

whose induced map

$$\mathrm{Mor}(X, \mathcal{C}_{r+1}(\Sigma Y))^{\sim} \otimes \Delta[1] \longrightarrow \mathrm{Mor}(X, \mathcal{C}_{r+1}(\Sigma Y))^{\sim}$$

is a deformation retraction of the suspension map $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{\sim} \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+1}(\Sigma Y))^{\sim}$.∎ In particular, the suspension maps

$$\Sigma^t : \mathcal{M}\mathrm{or}(X, \mathcal{C}_r \mathbb{P}^n)^{\sim} \xrightarrow{\sim} \mathcal{M}\mathrm{or}(X, \mathcal{C}_r \mathbb{P}^{n+t})^{\sim} \tag{3.1.3}$$

are quasi-isomorphisms, for all t, n, and r. By composing the pairing (3.1.2) with the inverse of

$$\Sigma : \mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^{a+b}) \xrightarrow{\sim} \mathcal{M}\mathrm{or}(X, \mathcal{C}_1 \mathbb{P}^{a+b+1})$$

we obtain a natural (in the derived category) bilinear pairing

$$\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^a)^{\sim} \times \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^b)^{\sim} \to \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^{a+b})^{\sim}. \tag{3.1.4}$$

Recall that $\mathbb{P}^n$ can be viewed as $SP^n(\mathbb{P}^1)$, the n-th symmetric product of $\mathbb{P}^1$. From this point of view, there is a natural map

$$C_{0,d}(\mathbb{P}^n) = SP^d(\mathbb{P}^n) \to C_{0,d\binom{n}{j}}(\mathbb{P}^j)$$

for any $0 < j \leq n$. As shown in [FL-1; 2.10], these maps determine a quasi-isomorphism

$$\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^n)^{\sim} \xrightarrow{\simeq} \bigoplus_{j=0}^{n} \mathcal{M}(X, j)[2j]. \tag{3.1.5}$$

This splitting is natural with respect to X and satisfies

(i.) the composition $\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^j)^{\sim} \to \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^n)^{\sim} \to \mathcal{M}(j)[2j]$ is the natural projection to the cone, and

(ii.) the composition of $\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^i)^{\sim} \to \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^n)^{\sim} \to \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^n)^{\sim} \to \mathcal{M}(j)[2j]$ is trivial for $i < j$ and any linear embedding $\mathbb{P}^i \hookrightarrow \mathbb{P}^n$.

We re-write the bilinear map (3.1.4) as

$$\bigoplus_{i=0}^{a} \mathcal{M}(X, i)[2i] \times \bigoplus_{j=0}^{b} \mathcal{M}(X, j)[2j] \longrightarrow \bigoplus_{k=0}^{a+b} \mathcal{M}(X, k)[2k]. \tag{3.1.6}$$

To prove the proposition it suffices to verify that the composition of the summand inclusion of $\mathcal{M}(X, i)[2i] \times \mathcal{M}(X, j)[2j]$, followed by (3.1.6), followed by the factor projection to $\mathcal{M}(X, a+b)[2a+2b]$ is trivial whenever $i + j < a + b$. This follows from the observation that such a summand inclusion into $\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^a)^{\sim} \times \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^b)^{\sim}$ factors through the natural inclusion of $\mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^i)^{\sim} \times \mathcal{M}or(X, \mathcal{C}_0 \mathbb{P}^j)^{\sim}$.∎

so that the further composition with the join map to $\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^{a+b})^{\sim}$ factors through $\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^{i+j})^{\sim}$. $\square$

Using the suspension quasi-isomorphism (3.1.3), we readily conclude that the pairing of Proposition 3.1 is also induced by the bilinear join pairing

$$\# : \mathcal{C}_r\mathbb{P}^m \times \mathcal{C}_s\mathbb{P}^n \longrightarrow \mathcal{C}_{r+s+1}\mathbb{P}^{m+n+1}$$

whenever $m - r = a$, $n - s = b$.

For any variety X, we let X_{Zar} denote the small Zariski site whose objects are Zariski open subsets of X. If P is a presheaf on $Sch/\mathbb{C}$, we write P_{Zar} for the associated Zariski sheaf on the big Zariski site $(Sch/\mathbb{C})_{\mathrm{Zar}}$. If P is a presheaf on X_{Zar} (or a presheaf on $Sch/\mathbb{C}$ implicitly viewed as a presheaf on X_{Zar} by restriction), then we write P_{Zar} also for the associated Zariski sheaf on X_{Zar}.

Proposition 3.2. *For any quasi-projective variety X, the join pairing induces an internal product pairing in the derived category of presheaves on X_{Zar}*

$$\# : \mathcal{M}(-, a) \overset{\mathbb{L}}{\otimes} \mathcal{M}(-, b) \longrightarrow \mathcal{M}(-, a+b) \tag{3.2.1}$$

which is associative in the appropriate sense.

Similarly, for any quasi-projective varieties X, Y, the join pairing induces an external product pairing in the derived category of presheaves on $(X \times Y)_{\mathrm{Zar}}$

$$\# : pr_X^*\mathcal{M}(-, a) \overset{\mathbb{L}}{\otimes} pr_Y^*\mathcal{M}(-, b) \longrightarrow \mathcal{M}(-, a+b) \tag{3.2.2}$$

which is associative in the appropriate sense.

Proof. The internal product pairing (3.2.1) is obtained from (3.1.1) by simply restricting the presheaves to X_{Zar}.

The bilinear join pairing (3.1.2) admits an external analog

$$\# : \mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^a)^{\sim} \otimes \mathcal{M}or(Y, \mathcal{C}_0\mathbb{P}^b)^{\sim} \to \mathcal{M}or(X \times Y, \mathcal{C}_1\mathbb{P}^{a+b+1})^{\sim} \tag{3.2.3}$$

sending (f, g) to $f\#g : X \times Y \to \mathcal{C}_1\mathbb{P}^{m+n+1}$ whose value on (x, y) equals $f(x)\#g(y)$. As argued in the proof of Proposition 3.1, this leads to a pairing in the derived category of complexes of abelian groups

$$\mathcal{M}(X, a) \otimes \mathcal{M}(Y, b) \longrightarrow \mathcal{M}(X \times Y, a+b)$$

natural with respect to X and Y. Thus, pairing (3.2.2) follows from the observation that there exist canonical maps

$$pr_X^*\mathcal{M}(-, a)_{|X} \to \mathcal{M}(-, a)_{|X\times Y}, \quad pr_Y^*\mathcal{M}(-, b)_{|Y} \to \mathcal{M}(-, b)_{|X\times Y}.$$

The asserted associativity follows easily from the following three facts.

a.) The associativity of the join product – that is, the commutativity of the square

$$\begin{array}{ccc}
\mathcal{C}_0\mathbb{P}^a \times \mathcal{C}_0\mathbb{P}^b \times \mathcal{C}_0\mathbb{P}^c & \longrightarrow & \mathcal{C}_1\mathbb{P}^{a+b+1} \times \mathcal{C}_0\mathbb{P}^c \\
\downarrow & & \downarrow \\
\mathcal{C}_0\mathbb{P}^a \times \mathcal{C}_1\mathbb{P}^{b+c+1} & \longrightarrow & \mathcal{C}_2\mathbb{P}^{a+b+c+2}
\end{array}$$

for all $a, b, c \geq 0$,

b.) the naturality of the suspension isomorphism (3.1.3), and

c.) the naturality of the splitting (3.1.6). $\square$

We recall that "morphic cohomology" of a normal quasi-projective variety X is defined by

$$L^s \mathrm{H}^n(X) \;\equiv\; \pi_{2s-n}(\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^s)^+/\mathcal{M}or(X, \mathcal{C}_0\mathbb{P}^{s-1})^+)$$

which is naturally isomorphic to $\mathrm{H}^n(\mathcal{M}(X,s))$ (cf. [FL-1],[F2]). Here, the superscript + denotes taking naive group completion of the given topological abelian monoid. This definition was modified in [F3], yielding "topological cycle cohomology" defined as

$$\mathcal{H}^n(X,s) \;\equiv\; \mathrm{H}^n_{\mathrm{Zar}}(X, \mathcal{M}(-,s)_{\mathrm{Zar}}).$$

As shown in [F3; 5.7], the canonical map

$$\mathrm{H}^*(\mathcal{M}(X,s)) \longrightarrow \mathcal{H}^*(X,s) \tag{3.3.0}$$

is an isomorphism whenever X is smooth.

Proposition 3.3. *For any quasi-projective variety X, there is a commutative square*

$$\begin{array}{ccc}
\mathrm{H}^m(\mathcal{M}(X,r)) \otimes \mathrm{H}^n(\mathcal{M}(X,s)) & \xrightarrow{\cup} & \mathrm{H}^{m+n}(\mathcal{M}(X,r+s)) \\
\downarrow & & \downarrow \\
\mathcal{H}^m(X,r) \otimes \mathcal{H}^n(X,s) & \xrightarrow{\#} & \mathcal{H}^{m+n}(X,r+s)
\end{array}$$

whose top pairing is the "cup product" pairing of [FL-1]*, whose vertical maps are the canonical maps (3.3.0), and whose bottom pairing is induced by the internal product pairing (3.2.1).*

Proof. Both pairings are induced by the pairing (3.1.2). The cup product in morphic cohomology was obtained from (3.1.2) by passing to homotopy groups, observing that the pairing on homotopy groups commutes with the operations on morphic cohomology induced by the operations introduced in [F-M], and then annihilating those classes in the image of the "h-operation". As verified in the proof of [FL-1; 5.2], this is precisely the effect (on cohomology) of the projection map $\mathcal{M}or(X, \mathbb{P}^a)^\sim \to \mathcal{M}(X,a)[2a]$ used to define the pairing (3.1.2). $\square$

For a projective variety Y, define $\mathcal{C}_r(Y)^\sim$ to be $N\left([\mathrm{Sing}\,\mathcal{C}_r(Y)]^+\right)$ – that is, the normalized chain complex of abelian groups associated to the degree-wise group completed singular simplicial set associated to the topological monoid $\mathcal{C}_r(Y)$. If X is a quasi-projective variety, choose a projective closure $X \subset \overline{X}$ and let $X_\infty = \overline{X} - X$. Define $\mathcal{C}_r(X)^\sim$ to be the chain complex of abelian groups

$$\mathcal{C}_r(X)^\sim \;\equiv\; \mathrm{cone}\left\{\mathcal{C}_r(X_\infty)^\sim \to \mathcal{C}_r(\overline{X})^\sim\right\}.$$

We extend this definition to obtain a presheaf of chain complexes on X_{Zar} as follows. For $U \subset X$ a Zariski open subset, let $U_\infty = \overline{X} - U$. Define $\mathcal{L}(U,r)$ to be

$$\mathcal{L}(U,r) \;\equiv\; \mathrm{cone}\left\{\mathcal{C}_r(U_\infty)^\sim \to \mathcal{C}_r(\overline{X})^\sim\right\}[2r].$$

Since $V_\infty \supset U_\infty$ if $V \subset U$, $\mathcal{L}(-,r)$ is naturally a presheaf on X_{Zar}. As shown in [F3; 3.2], the presheaf $\mathcal{L}(-,r)$ determines Lawson homology:

$$L_r \mathrm{H}_n(X) \;=\; \mathrm{H}^{-n}(\mathcal{L}(X,r)) \xrightarrow{\sim} \mathrm{H}^{-n}_{X_{\mathrm{Zar}}}(X, \mathcal{L}(-,r)).$$

For X of pure dimension d and Y projective, the natural transformation $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y) \to \mathcal{C}_{r+d}(X \times Y)$ sending a continuous algebraic map to its graph defines a continuous map

$$\mathcal{D} : \mathcal{M}or(X, \mathcal{C}_r Y)^{an} \to \mathcal{C}_{r+d}(X \times Y)^{an}$$

called the "duality map". This extends to the map of presheaves on X_{Zar}

$$\mathcal{D} : \mathcal{M}(-,s)[-2s] \longrightarrow \mathcal{L}(-,d-s)[2d-2s]$$

or, equivalently, to the map

$$\mathcal{D} : \mathcal{M}(-,s) \longrightarrow \mathcal{L}(-,d-s)[2d].$$

If X, Y are both smooth, then the main results of [FL-2], [F2] assert that the map

$$\mathcal{D} : \mathcal{M}or(X, \mathcal{C}_r Y)^{\sim} \to \mathcal{C}_{r+d}(X \times Y)^{\sim}$$

is a quasi-isomorphism. In particular, for X smooth of pure dimension d, this duality isomorphism has the form

$$\mathcal{D} : \mathcal{M}(-,s) \xrightarrow{\sim} \mathcal{L}(-,d-s)[2d] \tag{3.4.0}$$

where we have implicitly used the homotopy invariance of Lawson homology (i.e., flat pull-back determines a quasi-isomorphism $\mathcal{C}_j(X)^{\sim} \to \mathcal{C}_{j+1}(X \times \mathbb{A}^1)^{\sim}$).

Proposition 3.4. *Let X, Y be quasi-projective varieties of pure dimension d, e respectively. Then the external product pairing (3.2.2) is compatible via duality isomorphisms with the pairing in homology given by external product of cycles. Namely, the following square commutes in the derived category of presheaves on $X \times Y_{\mathrm{Zar}}$:*

$$\begin{array}{ccc}
pr_X^* \mathcal{M}(-,r) \overset{\mathbb{L}}{\otimes} pr_Y^* \mathcal{M}(-,s) & \xrightarrow{\#} & \mathcal{M}(-,r+s) \\
{\scriptstyle \mathcal{D}\otimes\mathcal{D}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}} \\
pr_X^* \mathcal{L}(-,d-r)[2d] \overset{\mathbb{L}}{\otimes} pr_Y^* \mathcal{L}(-,e-s)[2e] & \xrightarrow{\times} & \mathcal{L}(-,d+e-r-s)[2(d+e)].
\end{array}$$

In particular, taking cohomology of the complexes of global sections on $X \times Y$, we conclude the following commutative square of pairings

$$\begin{array}{ccc}
L^r \mathrm{H}^i(X) \otimes L^s \mathrm{H}^j(Y) & \xrightarrow{\#} & L^{r+s} \mathrm{H}^{i+j}(X \times Y) \\
{\scriptstyle \mathcal{D}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}} \\
L_{d-r} H_{2d-i}(X) \otimes L_{e-s} H_{2e-j}(Y) & \xrightarrow{\times} & L_{d+e-r-s} H_{2d+2e-i-j}(X \times Y).
\end{array}$$

Proof. Let $W \subset \mathbb{P}^r \times \mathbb{P}^s \times \mathbb{P}^{r+s+1}$ denote the "graph" of the join pairing of degree one zero-cycles: a point in W consists of triples (x,y,t) such that $t \in \mathbb{P}^{r+s+1}$ lies on the line joining $x \in \mathbb{P}^r$ with $y \in \mathbb{P}^s$, where $\mathbb{P}^r$, $\mathbb{P}^s$ are embedded in $\mathbb{P}^{r+s+1}$ into the first $r+1$ and last $s+1$ coordinates, respectively. Then the projection $\pi : W \to \mathbb{P}^r \times \mathbb{P}^s$ is the projection of the projectivization of the rank 2 bundle $pr^*_{\mathbb{P}^r}\mathcal{O}(1) \otimes pr^*_{\mathbb{P}^s}\mathcal{O}(1)$ over $\mathbb{P}^r \times \mathbb{P}^s$. Moreover, the join pairing $\mathcal{C}_0\mathbb{P}^r \times \mathcal{C}_0\mathbb{P}^s \to \mathcal{C}_1\mathbb{P}^{r+s+1}$ can be factored as the composition

$$\mathcal{C}_0\mathbb{P}^r \times \mathcal{C}_0\mathbb{P}^s \xrightarrow{\pi^*} \mathcal{C}_1 W \xrightarrow{p_*} \mathcal{C}_1\mathbb{P}^{r+s+1}$$

where $p : W \to \mathbb{P}^{r+s+1}$ is the projection.

We employ the following commutative diagrams natural with respect to maps $U' \to U$ in X_{Zar}, $V' \to V$ in Y_{Zar}:

$$\begin{array}{ccc}
\mathcal{M}or(U, \mathcal{C}_0\mathbb{P}^r) \times \mathcal{M}or(V, \mathcal{C}_0\mathbb{P}^s) & \longrightarrow & \mathcal{C}_d(U \times \mathbb{P}^r) \times \mathcal{C}_e(V \times \mathbb{P}^s) \\
\times \big\downarrow & & \big\downarrow \times \\
\mathcal{M}or(U \times V, \mathcal{C}_0(\mathbb{P}^r \times \mathbb{P}^s)) & \longrightarrow & \mathcal{C}_{d+e}(U \times V \times \mathbb{P}^r \times \mathbb{P}^s) \\
\pi^* \big\downarrow & & \big\downarrow \pi^* \\
\mathcal{M}or(U \times V, \mathcal{C}_1 W) & \longrightarrow & \mathcal{C}_{d+e+1}(U \times V \times W) \\
p_* \big\downarrow & & \big\downarrow p_* \\
\mathcal{M}or(U \times V, \mathcal{C}_1\mathbb{P}^{r+s+1}) & \longrightarrow & \mathcal{C}_{d+e+1}(U \times V \times \mathbb{P}^{r+s+1}) \\
\Sigma \big\uparrow & & \big\uparrow \Sigma \\
\mathcal{M}or(U \times V, \mathcal{C}_0\mathbb{P}^{r+s}) & \longrightarrow & \mathcal{C}_{d+e}(U \times V \times \mathbb{P}^{r+s})
\end{array}$$

where the horizontal maps are given by taking the graph of a continuous algebraic map. The construction of the pairing of (3.2.2) is induced by the left vertical maps as in the proof of Proposition 3.1.

External product on cycles is given by the composition

$$\mathcal{C}_{d-r}(U)^\sim \otimes \mathcal{C}_{e-s}(V)^\sim \to \mathcal{C}_d(U \times \mathbb{P}^r)^\sim \otimes \mathcal{C}_e(V \times \mathbb{P}^s)^\sim \to \mathcal{C}_{d+e}(U \times V \times \mathbb{P}^r \times \mathbb{P}^s)^\sim$$

followed by the projection

$$\mathcal{C}_{d+e}(U \times V \times \mathbb{P}^r \times \mathbb{P}^s)^\sim \to \mathcal{C}_{d+e-r-s}(U \times V)^\sim$$

right inverse to flat pull-back. Thus, to prove the proposition, it suffices to observe the composition of

$$p_* \circ \pi^* : \mathcal{C}_{d+e}(U \times V \times \mathbb{P}^r \times \mathbb{P}^s) \to \mathcal{C}_{d+e+1}(U \times V \times W) \to \mathcal{C}_{d+e+1}(U \times V \times \mathbb{P}^{r+s+1})$$

with flat pull-back $\mathcal{C}_{d+e-r-s}(U \times V) \to \mathcal{C}_{d+e}(U \times V \times \mathbb{P}^r \times \mathbb{P}^s)$ is again flat pull-back. □

As established in [F-G], intersection of cycles on a smooth variety X of pure dimension d determines an intersection product pairing (in the derived category)

$$\bullet : \mathcal{C}_r(X)^\sim \otimes \mathcal{C}_s(X)^\sim \longrightarrow \mathcal{C}_{r+s-d}(X)^\sim. \tag{3.5.0}$$

One aspect of the following theorem is the (implicit) statement that this intersection pairing is sufficiently natural to determine a pairing on the level of presheaves on X_{Zar}. The central point is that it provides a refinement (at the level of presheaves of chain complexes rather than simply cohomology groups) of [FL-2; 4.7],[F2; 4.8] establishing that for a smooth variety the duality map converts cup product in morphic cohomology to intersection product in Lawson homology.

Theorem 3.5. *Let X be a smooth scheme of pure dimension d. Then the internal product pairing X of (3.2.1) and the intersection product pairing $\bullet$ of (3.5.0) are compatible via duality isomorphisms. Namely, the following diagram commutes in the derived category of presheaves on X_{Zar}:*

$$\begin{array}{ccc}
\mathcal{M}(-,r) \overset{\mathbb{L}}{\otimes} \mathcal{M}(-,s) & \xrightarrow{\#} & \mathcal{M}(-,r+s) \\
\mathcal{D}\big\downarrow & & \big\downarrow\mathcal{D} \\
\mathcal{L}(-,d-r)[2d] \overset{\mathbb{L}}{\otimes} \mathcal{L}(-,d-s)[2d] & \xrightarrow{\bullet} & \mathcal{L}(-,d-r-s)[2d]
\end{array}$$

where $\mathcal{D}$ is the duality isomorphism.

Proof. The pairing (3.5.0) is constructed using the external product pairing on cycles and a Gysin map $\Delta^!$ (well defined up to quasi-isomorphism) associated to the regular closed embedding $\Delta : X \to X^{\times 2}$. By Proposition 3.4, it suffices to show that this Gysin map fits in a square commutative in the derived category of presheaves on X_{Zar}:

$$\begin{array}{ccc}
\Delta^*(\mathcal{M}(-,s)) & \xrightarrow{\Delta^*} & \mathcal{M}(-,s) \\
\mathcal{D}\big\downarrow & & \big\downarrow\mathcal{D} \\
\Delta^*(\mathcal{L}(-,2d-s)[2d]) & \xrightarrow{\Delta^!} & \mathcal{L}(-,d-s)[d]
\end{array} \tag{3.5.1}$$

In view of the naturality (in the derived category of presheaves on X_{Zar}) of the projection map $\mathcal{L}(- \times \mathbb{P}^s, j) \to \mathcal{L}(-,j)$, it suffices to choose a representative map of chain complexes for

$$(\Delta_U \times 1)^! : \mathcal{C}_{2d}(U \times U \times \mathbb{P}^s)^{\sim} \to \mathcal{C}_d(U \times \mathbb{P}^s)^{\sim}$$

natural with respect to $U \in X_{\mathrm{Zar}}$ so that the following diagram of presheaves determines a commutative square in the derived category of presheaves on X_{Zar}:

$$\begin{array}{ccc}
\mathcal{M}or((-)^{\times 2}, \mathcal{C}_0(\mathbb{P}^s))^{\sim} & \xrightarrow{\Delta^*} & \mathcal{M}or(-, \mathcal{C}_0(\mathbb{P}^s))^{\sim} \\
\mathcal{D}\big\downarrow & & \big\downarrow\mathcal{D} \\
\mathcal{C}_{2d}((-)^{\times 2} \times \mathbb{P}^s)^{\sim} & \xrightarrow{(\Delta\times 1)^!} & \mathcal{C}_d(- \times \mathbb{P}^s)^{\sim}
\end{array} \tag{3.5.2}$$

For simplicity of notation and consistency with [F-G], we consider the more general case of a regular closed embedding $i_W : W \to Y$ of codimension c. Then

$i^!_W$ is constructed using the technique of deformation to the normal cone, so that a diagram of the following form is considered (cf. [Fu; §5.1]):

$$\begin{array}{ccccc} N_W Y & \longrightarrow & Q_W Y & \xleftarrow{j} & Y \times \mathbb{A}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \{\infty\} & \longrightarrow & \mathbb{P}^1 & \longleftarrow & \mathbb{A}^1 \end{array}$$

where $\pi : N_W Y \to Y$ is the normal bundle of $i_W : W \to Y$. Following [F-G; 3.4] (see also [F-G; 3.3]), we define

$$i^!_W = (\pi^*)^{-1} \circ \tilde{\epsilon}_* : \mathcal{C}_r(Y)^\sim \to \mathcal{C}_r(N_W Y)^\sim \to \mathcal{C}_{r-c}(W)^\sim$$

where $\epsilon : Y = Y \times \{1\} \subset Q_W Y$ and $\tilde{\epsilon}$ is defined as a lifting of $\epsilon_* : \mathcal{C}_r(Y)^\sim \to \mathcal{C}_r(Q_W Y)^\sim$ determined by a choice of null-homotopy for the composition $j^* \circ \epsilon_* : \mathcal{C}_r(Y)^\sim \to \mathcal{C}_r(Q_W Y)^\sim \to \mathcal{C}_r(Y \times \mathbb{A}^1)^\sim$.

We choose our null-homotopy to be parameterized by $\mathbb{P}^1$

$$h : \mathcal{C}_r Y \times \mathbb{P}^1 \to \mathcal{C}_r(Y \times \mathbb{P}^1)/\mathcal{C}_r(Y \times \{\infty\}) = \mathcal{C}_r(Y \times \mathbb{A}^1),$$

and defined by sending $(Z, t) \in \mathcal{C}_r Y \times \mathbb{P}^1$ to $Z \times \{t\} \in \mathcal{C}_r(Y \times \mathbb{P}^1)/\mathcal{C}_r(Y \times \{\infty\})$. This homotopy gives us a homotopy in the category of complexes of presheaves on Y_{Zar}

$$\tilde{h} : \mathcal{C}_r(-)^\sim \otimes \mathcal{C}_0(\mathbb{P}^1)^\sim \longrightarrow \mathcal{C}_r(- \times \mathbb{A}^1)^\sim \tag{3.5.3}$$

which determines

$$\tilde{\epsilon}_* : \mathcal{C}_r(-)^\sim \to \mathcal{C}_r(N_{(-\cap W)}-)^\sim$$

whose composition with the inverse (in the derived category) of flat pull-back $\pi^* : \mathcal{C}_{r-c}(-)^\sim \to \mathcal{C}_r(N_{(-\cap W)}-)^\sim$ gives us our functorial (on Y_{Zar}) representation of $i^!_W$.

Let $Z \subset Y$ be an irreducible r-cycle on Y and consider the following commutative diagram with Cartesian squares

$$\begin{array}{ccccc} \mathbb{P}(C_{W\cap Z} Z \oplus 1) & \longrightarrow & Q_{W\cap Z} Z & \longleftarrow & Z \times \mathbb{A}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}(N_W Y \oplus 1) & \longrightarrow & Q_W Y & \longleftarrow & Y \times \mathbb{A}^1 \end{array}$$

where $C_{W\cap Z} Z$ is the normal cone of $W \cap Z \subset Z$ and $\mathbb{P}(C_{W\cap Z} Z \oplus 1)$ is the associated projective completion of C [Fu; App.B]. If Z meets W properly, then $\mathbb{P}(C_{W\cap Z} Z \oplus 1)$ meets the 0-section of $\mathbb{P}(N_W Y \oplus 1)$ properly and their "classical" intersection $W \bullet Z$ (defined in terms of intersection multiplicities of the components of their intersection) is equal to $i^!_W(Z)$ (cf. [Fu; 7.1]).

To verify the commutativity of (3.5.2), we let $\mathcal{C}_r(Y; W) \subset \mathcal{C}_r Y$ denote the submonoid of those effective r-cycles on Y which meeting W properly (i.e., in codimension c). Then the composition of $\mathcal{C}_r(Y; W) \subset \mathcal{C}_r Y$ with the homotopy h admits a natural lifting

$$H : \mathcal{C}_r(Y; W) \times \mathbb{P}^1 \to \mathcal{C}_r(Q_W Y)$$

given once again by sending (Z, t) to $Z \times \{t\}$ for $t \neq \infty$ and sending (Z, ∞) to $\mathbb{P}(C_{W\cap Z} Z)$ (i.e., this is deformation to the normal cone of cycles meeting W properly).

We now revert to our initial notation in which $W \to Y$ as above becomes $X \times \mathbb{P}^s \to X^{\times 2} \times \mathbb{P}^s$. Observe that the image of the duality map $\mathcal{M}or(X^{\times 2}, \mathcal{C}_0\mathbb{P}^s) \to \mathcal{C}_{2d}(X^{\times 2} \times \mathbb{P}^s)$ lies in $\mathcal{C}_{2d}(X^{\times 2} \times \mathbb{P}^s; \Delta \times \mathbb{P}^s)$ so that H gives us a specific lifting of the homotopy h when restricted to $\mathcal{M}or(X^{\times 2}, \mathcal{C}_0\mathbb{P}^s)$. The naturality of this construction with respect to Zariski open subsets $U \subset X$ implies that H determines a specific lifting

$$\tilde{H} : \mathcal{M}or((-)^{\times 2}, \mathcal{C}_0(\mathbb{P}^s))^{\sim} \times \mathcal{C}_0(\mathbb{P}^1)^{\sim} \to \mathcal{C}_{2d}(Q_{-\cap W}-))^{\sim}$$

of $\tilde{h}$ of (3.5.3). Consequently, we conclude that $\tilde{H}, \tilde{\epsilon}$ give the same map (in the derived category) $\mathcal{M}or((-)^{\times 2}, \mathcal{C}_0(\mathbb{P}^s))^{\sim} \to \mathcal{C}_r(N_{(W\cap -)}-)^{\sim}$. Since the two composition of the square (3.5.2) are obtained from these maps by composing with the quasi-inverse of π^*, and since composition of $\tilde{H}$ with the quasi-inverse of π^* represents intersection with the diagonal, the commutativity of (3.5.2) in the derived category has been proved. □

We recall from [F3] the presheaves $\mathcal{M}_W(-,a)$ on X_{Zar} which determine topological cycle cohomology of X of weight a with supports on the closed subvariety $W \subset X$. Namely, for any Zariski open subset $U \subset X$ we define

$$\mathcal{M}_W(-,a)(U) \equiv \mathrm{cone}\,\{\mathcal{M}(U,a) \to \mathcal{M}(U \cap (X-W), a)\}\,[-1]$$

The next proposition states the evident analog in this context of Proposition 3.2.

Proposition 3.6. *Let $W \subset X$, $Q \subset Y$ be closed embeddings of quasi-projective varieties. Then the pairings of (3.1) and (3.2) determine pairings*

$$\# : \mathcal{M}_W(-,a) \overset{\mathbb{L}}{\otimes} \mathcal{M}_W(-,b) \longrightarrow \mathcal{M}_W(-,a+b) \tag{3.6.1}$$

$$\# : pr_X^*\mathcal{M}_W(-,a) \overset{\mathbb{L}}{\otimes} pr_Y^*\mathcal{M}_Q(-,b) \longrightarrow \mathcal{M}_{W\times Q}(-,a+b) \tag{3.6.2}$$

in the derived category of presheaves on X_{Zar} and $(X \times Y)_{\mathrm{Zar}}$, respectively.

Proof. Using the distinguished triangle

$$\begin{aligned}\mathcal{M}_W(-,&a+b) \longrightarrow \mathcal{M}(-,a+b) \longrightarrow \mathcal{M}(- \cap (X-W), a+b)\\ &\longrightarrow \mathcal{M}_W(-,a+b)[1]\end{aligned}$$

we obtain pairings of the form (3.6.1) and (3.6.2), but do not in this way establish that these pairings are uniquely defined (in the derived category). For this, it suffices to observe that the pairings of (3.1.2) are natural when viewed in the category of chain complexes (not the coarser derived category) and thereby induce pairings on cone complexes

$$\begin{aligned}&\mathrm{cone}\,\{\mathcal{M}or(-,\mathcal{C}_0\mathbb{P}^a)^{\sim} \to \mathcal{M}or(- \cap X - W, \mathcal{C}_0\mathbb{P}^a)^{\sim}\}\\ &\overset{\mathbb{L}}{\otimes} \mathrm{cone}\,\{\mathcal{M}or(-,\mathcal{C}_0\mathbb{P}^b)^{\sim} \to \mathcal{M}or(- \cap (X-W), \mathcal{C}_0\mathbb{P}^b)^{\sim}\}\\ &\longrightarrow \mathrm{cone}\,\{\mathcal{M}or(-,\mathcal{C}_a\mathbb{P}^{a+b+1})^{\sim} \to \mathcal{M}or(- \cap X - W, \mathcal{C}_1\mathbb{P}^{a+b+1})^{\sim}\}.\end{aligned}$$

Repeating the argument in the proof of Proposition 3.1, we conclude as in the proof of Proposition 3.2 that these pairings determine pairings of the form (3.6.1) and (3.6.2) as required. □

We recall from [F3; 5.6] that the duality isomorphism (3.4.0) has an extension to the context of cohomology of supports. Namely, if X is a quasi-projective variety provided with a closed embedding $i : X \subset M$ in a smooth variety M of pure dimension m, then the duality map is a quasi-isomorphism of presheaves on M_{Zar}:

$$\mathcal{D} : \mathcal{M}_X(-,s)[2s] \simeq i_*\mathcal{L}(-,m-s)[2m-2s].$$

We conclude this section with the following proposition asserting that external product of cycles in Lawson homology can be reinterpreted as join product in cohomology with supports. The proof is a merely a repetition of the proof of Proposition 3.4 applied to cone complexes as in the proof of Proposition 3.6.

Proposition 3.7. *Let X, Y be quasi-projective varieties. Choose closed embeddings $X \subset M$, $Y \subset N$ of X, Y in smooth varieties M, N of dimension m, n, respectively. Then the pairing in Lawson homology induced by external product of cycles on X and Y can be obtained from the pairing (3.6.2) as explained in the following commutative diagram of presheaves on $X \times Y$:*

$$\begin{tikzcd}
p_M^*\mathcal{M}_X(-,r) \otimes p_N^*\mathcal{M}_Y(-,s) \arrow[r, "\#"] \arrow[d, "\mathcal{D}\otimes\mathcal{D}"', "\simeq"] & \mathcal{M}_{X\times Y}(-,r+s) \arrow[d, "\simeq"', "\mathcal{D}"] \\
p_M^* i_{X*}\mathcal{L}(-,m-r)[2m] \otimes p_N^* i_{Y*}\mathcal{L}(-,n-s)[2n] \arrow[r, "\times"] & i_{X\times Y*}\mathcal{L}(-,m+n-r-s)[2m+2n]
\end{tikzcd}$$

§4. Compatibility with motivic products

In this section, we describe a morphism from the motivic cohomology of a smooth, complex variety to its morphic cohomology. We then establish that this map is compatible with the cup product in motivic cohomology and the join product in morphic cohomology. Considering hypercohomology with respect to the cdh topology on non-smooth varieties, one could verify this compatibility more generally provided that one modified the definition of topological cycle cohomology to incorporate cdh descent (as is done with motivic cohomology).

For smooth varieties X and Y, let $z_{\mathrm{equi}}(Y,r)(X)$ denote the free abelian group on the collection of closed, integral subschemes of $X \times Y$ that are equidimensional of relative dimension r over X. Then $z_{\mathrm{equi}}(Y,r)(X)$ is a contravariant functor in X. It is also covariant in Y for proper morphisms and contravariant in Y for flat morphisms (cf. [F-V; 2.1]). When Y is a projective variety, we can describe $z_{\mathrm{equi}}(Y,r)(X)$ as the naive group completion of the abelian monoid $\mathrm{Hom}(X, \mathcal{C}_r Y)$, where $\mathcal{C}_r Y$ is the Chow variety parameterizing dimension r effective cycles on Y [F1; 1.4].

To obtain a chain complex from the functor $z_{\mathrm{equi}}(Y,r)(-)$, we introduce the standard cosimplicial variety $\Delta^\bullet$, which is given in degree d by

$$\Delta^d \equiv \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_d]/(x_0 + \cdots + x_d - 1)$$

and which is equipped with the familiar face and degeneracy maps. We then consider the simplicial abelian group

$$z_{\mathrm{equi}}(\mathbb{P}^n, r)(X \times \Delta^\bullet) \equiv d \mapsto z_{\mathrm{equi}}(\mathbb{P}^n, r)(X \times \Delta^d).$$

By an abuse of notation, we also use $z_{\text{equi}}(\mathbb{P}^n, r)(X \times \Delta^\bullet)$ to refer to the associated normalized chain complex of this simplicial abelian group.

For a smooth variety X, we define a chain complex of abelian groups $\mathbb{Z}(X,n)$ by the formula

$$\mathbb{Z}(X,n) \equiv z_{\text{equi}}(\mathbb{A}^n, 0)(X \times \Delta^\bullet)[-2n]. \tag{4.0.1}$$

The following proposition justifies our consideration of the chain complex $\mathbb{Z}(X,n)$ rather than a complex of sheaves as in [F-V].

Proposition 4.1. *If X is a smooth variety, then the motivic cohomology groups of X (as defined in* [F-V; 9.2] *and which are written $\mathrm{H}^q_{\mathcal{M}}(X, \mathbb{Z}(n))$) satisfy*

$$\mathrm{H}^q_{\mathcal{M}}(X, \mathbb{Z}(n)) \cong \mathrm{H}^q(\mathbb{Z}(X,n)).$$

Similarly, the topological cycle cohomology groups of a smooth variety X satisfy

$$\mathcal{H}^q(X,n) \cong \mathrm{H}^q(\mathcal{M}(X,n)).$$

Proof. The first assertion follows from [F-V; 8.1] and second follows from [F3; 5.7] (cf. (4.3.0)). $\square$

Observe that there is a natural sequence

$$z_{\text{equi}}(\mathbb{P}^{n-1}, 0)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{P}^n, 0)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{A}^n, 0)(X \times \Delta^\bullet)$$

of simplicial abelian groups. By [F-V; 5.11,8.1], this sequence induces a distinguished triangle in the derived category of abelian groups (after taking the associated normalized chain complexes) provided that X is smooth. Thus, we have the isomorphism

$$\mathrm{H}^q_{\mathcal{M}}(X, \mathbb{Z}(n)) \cong \mathrm{H}^{q-2n}\left(\text{cone}\left\{z_{\text{equi}}(\mathbb{P}^{n-1}, 0)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{P}^n, 0)(X \times \Delta^\bullet)\right\}\right).$$

For X a smooth variety, we introduce the chain complex

$$\mathcal{M}_{\text{alg}}(X,n) \equiv \text{cone}\left\{z_{\text{equi}}(\mathbb{P}^{n-1}, 0)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{P}^n, 0)(X \times \Delta^\bullet)\right\}[-2n]. \tag{4.1.1}$$

The above results combine to show that for X smooth the chain complexes $\mathbb{Z}(X,n)$ and $\mathcal{M}_{\text{alg}}(X,n)$ are quasi-isomorphic under a natural map

$$\mathcal{M}_{\text{alg}}(X,n) \xrightarrow{\sim} \mathbb{Z}(X,n).$$

We persist in using two notations to refer to essentially the same object since the cup product operation is more directly defined using $\mathbb{Z}(X,n)$, whereas the complex $\mathcal{M}_{\text{alg}}(X,n)$ is more easily compared with the complex defining topological cycle cohomology and admits a naturally defined join product.

In order to construct a map from the motivic cohomology groups of X to the topological cycle cohomology groups of X, we consider the map of simplicial sets

$$\rho : \mathrm{Hom}(X \times \Delta^\bullet, \mathcal{C}_r Y) \longrightarrow \mathrm{Sing}\, \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}$$

defined as follows, for any $r \geq 0$ and projective variety Y. Given an element f of $\mathrm{Hom}(X \times \Delta^n, \mathcal{C}_r Y)$, there is an induced map

$$\tilde{f} : (\Delta^n)^{an} \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}$$

defined by passing to the associated analytic spaces and applying adjointness. The map $\tilde{f}$ is induced by the natural transformation of functors on $(Sm/\mathbb{C})_{\leq 1}$ with proper, constructible representations which sends $g : C \to \Delta^n$ to $f \circ (g \times \mathrm{id}_C)$, and thus $\tilde{f}$ is continuous by Theorem 2.3.

Upon restricting the domain of $\tilde{f}$ to $\Delta^n_{\text{top}} \subset (\Delta^n)^{an}$, where Δ^n_{top} is the subspace of real points having nonnegative coordinates in $(\Delta^n)^{an}$, we obtain the continuous map

$$\rho_n(f) : \Delta^n_{\text{top}} \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_r Y)^{an}.$$

The construction of $\rho_n(f)$ is clearly compatible with the simplicial structures so that we obtain a map of simplicial sets ρ as desired.

In particular, taking Y to be $\mathbb{P}^n$ and setting $r = 0$, we have the map

$$\mathrm{Hom}(X \times \Delta^\bullet, \mathcal{C}_0 \mathbb{P}^n) \longrightarrow \mathrm{Sing}\, \mathcal{M}\mathrm{or}(X, \mathcal{C}_0 \mathbb{P}^n)^{an}.$$

Passing to the category of chain complexes and using the naturality of the construction with respect to the inclusion $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$, we obtain the map

$$\mathcal{M}_{\text{alg}}(X, n) \longrightarrow \mathcal{M}(X, n). \tag{4.1.2}$$

We now proceed to define a join pairing for motivic cohomology. The join pairing will serve as an intermediary for the purposes of comparing the join product in topological cycle cohomology with the cup product in motivic cohomology (whose definition is recalled below). In fact, the definition of the join pairing for motivic cohomology is parallel to the definition of the join pairing for topological cycle cohomology. Namely, let $W \subset \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^{n+m+1}$ be the join correspondence and define the join pairing

$$\# : z_{\text{equi}}(\mathbb{P}^n, r)(X) \otimes z_{\text{equi}}(\mathbb{P}^n, s)(X) \longrightarrow z_{\text{equi}}(\mathbb{P}^{n+m+1}, r+s+1)(X)$$

to the composition of the maps

$$\begin{aligned} z_{\text{equi}}(\mathbb{P}^n, r)(X) \otimes z_{\text{equi}}(\mathbb{P}^m, s)(X) &\xrightarrow{\times} z_{\text{equi}}(\mathbb{P}^n \times \mathbb{P}^m, r+s)(X) \\ &\xrightarrow{\pi_1^*} z_{\text{equi}}(W, r+s+1)(X) \xrightarrow{\pi_{2*}} z_{\text{equi}}(\mathbb{P}^{n+m+1}, r+s+1)(X), \end{aligned}$$

which is natural in X. For any t and k, let

$$\Sigma^k : z_{\text{equi}}(\mathbb{P}^t, q)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{P}^{t+k}, q+k)(X \times \Delta^\bullet)$$

be the map induced by the pairing

$$\# : z_{\text{equi}}(\mathbb{P}^t, q)(-) \otimes z_{\text{equi}}(\mathbb{P}^{k-1}, k-1)(-) \longrightarrow z_{\text{equi}}(\mathbb{P}^{t+k}, q+k)(-)$$

by fixing the element $[\mathbb{P}^{k-1}]$ in $z_{\text{equi}}(\mathbb{P}^{k-1}, k-1)(-)$. It follows from [F-V; 8.3] that the map Σ^k is a quasi-isomorphism. Therefore, in the derived category of abelian groups, we may form the pairing

$$z_{\text{equi}}(\mathbb{P}^m, 0)(X \times \Delta^\bullet) \otimes z_{\text{equi}}(\mathbb{P}^n, 0)(X \times \Delta^\bullet) \longrightarrow z_{\text{equi}}(\mathbb{P}^{m+n}, 0)(X \times \Delta^\bullet) \tag{4.2.0}$$

by composing with the quasi-inverse of Σ^1.

Proposition 4.2. *The pairing (4.2.0) induces a pairing natural in the smooth variety X*

$$\# : \mathcal{M}_{\mathrm{alg}}(X,m) \overset{\mathbb{L}}{\otimes} \mathcal{M}_{\mathrm{alg}}(X,n) \longrightarrow \mathcal{M}_{\mathrm{alg}}(X,m+n).$$

Proof. The construction from the proof of Proposition 3.1 carries over directly into this purely algebraic setting to produce a direct sum decomposition

$$z_{\mathrm{equi}}(\mathbb{P}^t,0)(X\times\Delta^\bullet) \cong \bigoplus_{i=0}^{t} \mathcal{M}_{\mathrm{alg}}(X,i)[2i].$$

As before, it remains to show that the composition of maps

$$\begin{aligned} &z_{\mathrm{equi}}(\mathbb{P}^i,0)(X\times\Delta^\bullet)\otimes z_{\mathrm{equi}}(\mathbb{P}^j,0)(X\times\Delta^\bullet) \\ &\qquad\longrightarrow z_{\mathrm{equi}}(\mathbb{P}^{i+j},0)(X\times\Delta^\bullet) \longrightarrow \mathcal{M}_{\mathrm{alg}}(X,n+m)[2m+2n] \end{aligned}$$

is zero for $i+j<m+n$. This follows as in the proof of Proposition 3.1, since the map

$$z_{\mathrm{equi}}(H,0)(X\times\Delta^\bullet) \longrightarrow \mathcal{M}_{\mathrm{alg}}(X,m+n)[2m+2n]$$

is homotopic to zero for any hyperplane H of $\mathbb{P}^{n+m}$. $\square$

The construction of the join pairing in motivic cohomology leads immediately to the following compatibility with the join pairing in topological cycle cohomology.

Proposition 4.3. *For any smooth variety X, the square*

$$\begin{tikzcd} \mathcal{M}_{\mathrm{alg}}(X,m) \overset{\mathbb{L}}{\otimes} \mathcal{M}_{\mathrm{alg}}(X,n) \arrow[r, "\#"] \arrow[d] & \mathcal{M}_{\mathrm{alg}}(X,m+n) \arrow[d] \\ \mathcal{M}(X,m) \overset{\mathbb{L}}{\otimes} \mathcal{M}(X,n) \arrow[r, "\#"] & \mathcal{M}(X,m+n) \end{tikzcd}$$

commutes in the derived category of abelian groups.

Proof. The proposition follows directly from the observation that the diagram of abelian monoids

$$\begin{tikzcd} \mathrm{Hom}(X\times\Delta^d,\mathcal{C}_0\mathbb{P}^m)\times\mathrm{Hom}(X\times\Delta^d,\mathcal{C}_0\mathbb{P}^n) \arrow[r, "\#"] \arrow[d] & \mathrm{Hom}(X\times\Delta^d,\mathcal{C}_1\mathbb{P}^{m+n+1}) \arrow[d] \\ \mathrm{Hom}(\Delta^d_{\mathrm{top}},\mathcal{M}\mathrm{or}(X,\mathcal{C}_0\mathbb{P}^m))\times\mathrm{Hom}(\Delta^d_{\mathrm{top}},\mathcal{M}\mathrm{or}(X,\mathcal{C}_0\mathbb{P}^m)) \arrow[r, "\#"] & \mathrm{Hom}(\Delta^d_{\mathrm{top}},\mathcal{M}\mathrm{or}(X,\mathcal{C}_1\mathbb{P}^{m+n+1})) \end{tikzcd}$$

commutes. $\square$

Let us recall the definition of the cup product in motivic cohomology. Observe that there is a pairing

$$\times : z_{\mathrm{equi}}(\mathbb{A}^m,0)(X)\otimes z_{\mathrm{equi}}(\mathbb{A}^n,0)(X) \longrightarrow z_{\mathrm{equi}}(\mathbb{A}^{m+n},0)(X),$$

natural in X, defined by sending a pair of generators (V,W) to the cycle associated to $V\times_X W$ [F-V; §8]. By naturality in X, this pairing extends to a map

$$\cup : \mathbb{Z}(X,m)\otimes\mathbb{Z}(X,n) \longrightarrow \mathbb{Z}(X,m+n).$$

As verified in [W; 4.5], the pairing $\cup$ coincides with the cup product on the motivic cohomology groups of X as given in [V2].

In light of Proposition 4.3, the map from the motivic cohomology of X to its topological cycle cohomology will be proven to be compatible with the motivic cup product and the topological join product provided we can establish that the motivic join product coincides with cup product. The key ingredient in establishing the compatibility of join and cup product is the observation that after pulling back along the natural surjection $\mathbb{A}^{i+1} \setminus \{0\} \longrightarrow \mathbb{P}^i$, the join product coincides with cartesian product. This observation motivates the proof of the following theorem.

Theorem 4.4. *For any smooth variety X, the diagram*

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{alg}}(X,n) \overset{\mathbb{L}}{\otimes} \mathcal{M}_{\mathrm{alg}}(X,m) \arrow[r, "\#"] \arrow[d, "\cong"'] & \mathcal{M}_{\mathrm{alg}}(X,n+m) \arrow[d, "\cong"'] \\
\mathbb{Z}(X,n) \overset{\mathbb{L}}{\otimes} \mathbb{Z}(X,m) \arrow[r, "\cup"] & \mathbb{Z}(X,n+m)
\end{tikzcd}$$

commutes in the derived category of abelian groups.

Proof. We will show that the diagram

$$\begin{tikzcd}
z_{\mathrm{equi}}(\mathbb{P}^m,0)(X \times \Delta^\bullet) \otimes z_{\mathrm{equi}}(\mathbb{P}^n,0)(X \times \Delta^\bullet) \arrow[r, "\#"] \arrow[d] & z_{\mathrm{equi}}(\mathbb{P}^{n+m+1},1)(X \times \Delta^\bullet) \arrow[d] \\
z_{\mathrm{equi}}(\mathbb{A}^m,0)(X \times \Delta^\bullet) \otimes z_{\mathrm{equi}}(\mathbb{A}^n,0)(X \times \Delta^\bullet) \arrow[d, "\times"'] & z_{\mathrm{equi}}(\mathbb{A}^{m+n+1},1)(X \times \Delta^\bullet) \arrow[d, "="'] \\
z_{\mathrm{equi}}(\mathbb{A}^{m+n},0)(X \times \Delta^\bullet) \arrow[r, "\pi^*"] & z_{\mathrm{equi}}(\mathbb{A}^{m+n+1},1)(X \times \Delta^\bullet)
\end{tikzcd}$$

commutes up to homotopy, where π is projection on the first $n+m$ coordinates of $\mathbb{A}^{n+m+1}$. This will suffice to prove the theorem since π^* is a quasi-isomorphism by [F-V; 8.3].

Rather than pull back along the surjections $\mathbb{A}^{i+1} \setminus \{0\} \longrightarrow \mathbb{P}^i$ to re-interpret the join product, we use $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \subset \mathbb{A}^{i+1} \setminus \{0\}$ instead (where $\mathbb{G}_{\mathrm{m}} \equiv \mathbb{A}^1 \setminus \{0\}$). Let

$$p_i : \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^i \longrightarrow \mathbb{P}^i$$

be the map sending $(x_0, x_1, \ldots, x_i)$ to $[x_0 : x_1 : \cdots : x_i]$ and

$$q : \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n \times \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m \longrightarrow \mathbb{P}^{n+m+1}$$

the map sending $(x_0, \ldots, x_n, y_0, \ldots, y_m)$ to $[x_0 : \cdots : x_n : y_0 : \cdots : y_m]$. As suggested before, the diagram of abelian groups

$$\begin{tikzcd}
z_{\mathrm{equi}}(\mathbb{P}^n,0)(X) \otimes z_{\mathrm{equi}}(\mathbb{P}^m,0)(X) \arrow[r, "\#"] \arrow[d, "p_n^* \otimes p_m^*"'] & z_{\mathrm{equi}}(\mathbb{P}^{n+m+1},1)(X) \arrow[d, "q^*"'] \\
z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n,1)(X) \otimes z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m,1)(X) \arrow[r, "\times"] & z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n \times \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m,2)(X)
\end{tikzcd} \tag{4.4.1}$$

commutes and is natural in X.

Define

$$\beta_i : \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^i \longrightarrow \mathbb{A}^i$$

by $\beta_i(\lambda, a_1, \ldots, a_i) = (\lambda^{-1}a_1, \ldots, \lambda^{-1}a_i)$. Then observe that the map p_i factors as β_i followed by the standard inclusion of $\mathbb{A}^i$ into $\mathbb{P}^i$ and the map q factors as $\beta_n \times \mathrm{id}_{\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m}$ followed by the standard inclusion of $\mathbb{A}^n \times \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m$ into $\mathbb{P}^{n+m+1}$.

We claim that for any variety Y and integer r, the map

$$\pi^* : z_{\mathrm{equi}}(Y, r)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{G}_{\mathrm{m}}, r+1)(X \times \Delta^\bullet) \tag{4.4.2}$$

is a split injection in the derived category of abelian groups. To see this, observe that by homotopy invariance, the map

$$\pi^* : z_{\mathrm{equi}}(Y, r)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{A}^1, r+1)(X \times \Delta^\bullet)$$

is a weak equivalence [F-V; 8.3]. Further, from [F-V; 5.11] there is a distinguished triangle

$$\begin{aligned} z_{\mathrm{equi}}(Y \times \{0\}, r+1)(X \times \Delta^\bullet) &\longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{A}^1, r+1)(X \times \Delta^\bullet) \longrightarrow \\ z_{\mathrm{equi}}(Y \times \mathbb{G}_{\mathrm{m}}, r+1)(X \times \Delta^\bullet) &\longrightarrow z_{\mathrm{equi}}(Y \times \{0\})(X \times \Delta^\bullet)[1], \end{aligned}$$

and thus it suffices to show that the map

$$z_{\mathrm{equi}}(Y \times \{0\}, r+1)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{A}^1, r+1)(X \times \Delta^\bullet) \tag{4.4.3}$$

is homotopic to the zero map (for then the triangle splits by basic properties of the derived category). It follows from [F-V; 8.3] that

$$z_{\mathrm{equi}}(Y \times \{0\}, r+1)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{P}^1, r+1)(X \times \Delta^\bullet) \xrightarrow{i^*} z_{\mathrm{equi}}(Y \times \mathbb{A}^1, r+1)(X \times \Delta^\bullet)$$

is part of a distinguished triangle, where $i : \mathbb{A}^1 \subset \mathbb{P}^1$ is the open complement of any rational point of $\mathbb{P}^1$. In particular, the composite map

$$z_{\mathrm{equi}}(Y \times \{0\}, r+1)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(Y \times \mathbb{A}^1, r+1)(X \times \Delta^\bullet),$$

which coincides with the map (4.4.3), is zero.

Since (4.4.2) is a split injection, we conclude that

$$\beta_i^* : z_{\mathrm{equi}}(\mathbb{A}^i, 0)(X \times \Delta^\bullet) \longrightarrow z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^i, 1)(X \times \Delta^\bullet)$$

is also a split injection, since β_i differs from the projection map by an automorphism. Thus $\beta_n^* \otimes \beta_m^*$ and $(\beta_n \times \mathrm{inc})^*$ are split injections as well, where $\mathrm{inc} : \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m \longrightarrow \mathbb{A}^{m+1}$ is the evident inclusion. Therefore, to show that the top square in the diagram

$$\begin{tikzcd} z_{\mathrm{equi}}(\mathbb{P}^n, 0) \otimes z_{\mathrm{equi}}(\mathbb{P}^m, 0) \arrow[r, "\#"] \arrow[d] & z_{\mathrm{equi}}(\mathbb{P}^{n+m}, 1) \arrow[d] \\ z_{\mathrm{equi}}(\mathbb{A}^n, 0) \otimes z_{\mathrm{equi}}(\mathbb{A}^m, 0) \arrow[r, "\pi^* \circ \times"] \arrow[d, "\beta_n^* \otimes \beta_m^*"'] & z_{\mathrm{equi}}(\mathbb{A}^n \times \mathbb{A}^{m+1}, 1) \arrow[d, "(\beta_n \times \mathrm{inc})^*"'] \\ z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n, 1) \otimes z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m, 1) \arrow[r, "\times"] & z_{\mathrm{equi}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n \times \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^m, 2) \end{tikzcd}$$

commutes in the derived category, it suffices to establish the commutativity of the outer square. (Here, we have omitted "$(X \times \Delta^\bullet)$" everywhere to simplify the notation.) But the outer square is precisely the commutative diagram (4.4.1). $\square$

The commutative diagram of chain complexes in Theorem 4.4 has the following immediate consequence for the cohomology of these chain complexes.

Corollary 4.5. *For X smooth, there is a natural graded ring homomorphism*

$$\bigoplus_r \mathrm{H}^*_{\mathcal{M}}(X, \mathbb{Z}(r)) \longrightarrow \bigoplus_r \mathcal{H}^*(X, r)$$

where the product on the left is the cup product operation of motivic cohomology and the product on the right is the join product operation of topological cycle cohomology.

§5. Whitney sum formula

The main result of this section will be that the operation of join of cycles is compatible with the direct sum decomposition of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^n)^\sim$ introduced in (3.1.5) (and recalled below). This is a slightly subtle point, whose proof turns out to be rather delicate. The reader should bear in mind that every though we have defined the join pairing

$$\mathcal{M}(X, r) \overset{\mathbb{L}}{\otimes} \mathcal{M}(Y, s) \longrightarrow \mathcal{M}(X, r+s)$$

as being induced under the natural surjection from the pairing

$$\# : \mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^r)^\sim \times \mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^s)^\sim \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_1\mathbb{P}^{r+s+1})^\sim,$$

it does not follow a priori that this latter pairing respects the grading given by the direct sum decomposition

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_k\mathbb{P}^n)^\sim \cong \bigoplus_{i=0}^{n-k} \mathcal{M}(X, i)[2i].$$

However, this is indeed the case, as shown by Theorem 5.3.

We then use Theorem 5.3 to establish that the Chern classes of vector bundles generated by their global sections taking values in morphic cohomology (or topological cycle cohomology), which were first introduced in [FL-1], satisfy the familiar Whitney sum formula. The settles a question left open in [FL-1; §10].

Since we will use it often, we observe here the following consequence of our results from sections 1 and 2. For projective varieties X and Y, it follows from Proposition 1.7 and Theorem 2.3 that an element f of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_rY)$ determines a natural transformation of functors from $Sch/\mathbb{C}$ to topological spaces

$$f_* : \mathcal{M}\mathrm{or}(-, \mathcal{C}_sX)^{an} \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}Y)^{an},$$

for any s. Moreover, two such maps f_1 and f_2 lying in the same component of the space $\mathcal{M}\mathrm{or}(X, \mathcal{C}_rY)^{an}$ define homotopy equivalent maps

$$f_{1*} \sim f_{2*} : \mathcal{M}\mathrm{or}(Z, \mathcal{C}_sX)^{an} \longrightarrow \mathcal{M}\mathrm{or}(Z, \mathcal{C}_{r+s}Y)^{an}$$

for all quasi-projective Z. In fact, choosing a path in $\mathcal{M}\mathrm{or}(X, \mathcal{C}_rY)^{an}$ from f_1 to f_2 determines a homotopy from f_{1*} to f_{2*} which is natural in Z; thus f_{1*} and f_{2*} define the same map

$$f_{1*} = f_{2*} : \mathcal{M}\mathrm{or}(-, \mathcal{C}_sX)^\sim \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}Y)^\sim$$

in the derived category of presheaves on $Sch/\mathbb{C}$.

We shall need to formalize the splitting of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^n)^\sim$ introduced in (3.1.5). Let

$$\rho_{n,i} : \mathbb{P}^n \longrightarrow \mathcal{C}_0\mathbb{P}^i$$

denote the map sending a point $P_1 \cdot \ldots \cdot P_n$ of $\mathrm{SP}^n(\mathbb{P}^1) = \mathbb{P}^n$ to $\sum_{k_1 < \cdots < k_i} P_{k_1} \cdot \ldots \cdot P_{k_i}$. Then for any X,

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^n)^\sim \xrightarrow{\cong} \bigoplus_{i=0}^{n} \mathcal{M}(X, i)[2i],$$

where the map to the i^{th} summand is the composition of the map induced by $\rho_{n,i}$,

$${\rho_{n,i}}_* : \mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^n)^\sim \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^i)^\sim,$$

with the natural split surjection

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^i)^\sim \longrightarrow \mathcal{M}(X, i)[2i].$$

This construction is clearly natural in X.

For any integers r and s and projective varieties X and Y, we define the "external product" map

$$\mathcal{C}_r X \times \mathcal{C}_s Y \xrightarrow{\times} \mathcal{C}_{r+s}(X \times Y)$$

by sending a pair of integral closed subschemes (Z, W) to $Z \times W$ and then extending by linearity. We use the same notation for the induced map

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_r X)^{an} \otimes \mathcal{M}\mathrm{or}(-, \mathcal{C}_s Y)^{an} \xrightarrow{\times} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(X \times Y))^{an}$$

of presheaves of topological spaces on $Sch/\mathbb{C}$.

We use the notation $\rho_{n,i} \times \rho_{m,j}$ to refer to the composition

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathcal{C}_0\mathbb{P}^i \times \mathcal{C}_0\mathbb{P}^j \longrightarrow \mathcal{C}_0(\mathbb{P}^i \times \mathbb{P}^j),$$

where the first map is what one might more accurately write as $\rho_{n_i} \times \rho_{m,j}$ and the second is given by the evident bilinear trace map.

We will need the following simple result.

Lemma 5.1. *The diagram of presheaves on $Sch/\mathbb{C}$*

$$\begin{array}{ccc}
\mathcal{M}\mathrm{or}(-, \mathcal{C}_r\mathbb{P}^n)^\sim \overset{\mathbb{L}}{\otimes} \mathcal{M}\mathrm{or}(-, \mathcal{C}_s\mathbb{P}^m)^\sim & \xrightarrow{\times} & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m))^\sim \\
{\scriptstyle \rho_{n,i} \otimes \rho_{m,j}} \downarrow & & {\scriptstyle \rho_{n,i} \times \rho_{m,j}} \downarrow \\
\mathcal{M}\mathrm{or}(-, \mathcal{C}_r\mathbb{P}^i)^\sim \overset{\mathbb{L}}{\otimes} \mathcal{M}\mathrm{or}(-, \mathcal{C}_s\mathbb{P}^j)^\sim & \xrightarrow{\times} & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j))^\sim
\end{array}$$

commutes.

Proof. Observe that it suffices to check that the diagram of spaces

$$\begin{array}{ccc}
\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^n)^{an} \times \mathcal{M}\mathrm{or}(X, \mathcal{C}_s\mathbb{P}^m)^{an} & \xrightarrow{\times} & \mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m))^{an} \\
{\scriptstyle \rho_{n,i} \times \rho_{m,j}} \downarrow & & {\scriptstyle \rho_{n,i} \times \rho_{m,j}} \downarrow \\
\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^i)^{an} \times \mathcal{M}\mathrm{or}(X, \mathcal{C}_s\mathbb{P}^j)^{an} & \xrightarrow{\times} & \mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j))^{an}
\end{array}$$

commutes, for any X. By Proposition 1.7, it suffices to check that the diagram

$$\begin{array}{ccc} \mathcal{C}_r\mathbb{P}^n \times \mathcal{C}_s\mathbb{P}^m & \xrightarrow{\times} & \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m) \\ {\scriptstyle \rho_{n,i}\times\rho_{m,j}}\big\downarrow & & {\scriptstyle \rho_{n,i}\times\rho_{m,j}}\big\downarrow \\ \mathcal{C}_r\mathbb{P}^i \times \mathcal{C}_s\mathbb{P}^j & \xrightarrow{\times} & \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j) \end{array} \tag{5.1.1}$$

commutes. Observe that the map $\rho_{l,k} : \mathbb{P}^l \longrightarrow \mathcal{C}_0\mathbb{P}^k$ has "graph"

$$\Gamma_{l,k} \subset \mathbb{P}^l \times \mathbb{P}^k$$

which is finite and flat over $\mathbb{P}^l$ (via the map π_1) and proper over $\mathbb{P}^k$ (via the map π_2), so that the map

$$\rho_{l,k_*} : \mathcal{C}_r\mathbb{P}^l \longrightarrow \mathcal{C}_r\mathbb{P}^k$$

is well defined by the formula

$$V \mapsto \pi_{2*}\pi_1^*(V).$$

The commutativity of (5.1.1) follows from the fact that taking external products of cycles commutes with the proper pushforward and flat pullback of cycles (cf. [Fu; 1.10]). □

The following proposition provides the key technique that will be used to prove the main result (Theorem 5.3) of this section.

Proposition 5.2. *For any smooth variety X, there are natural isomorphism*

$$\begin{aligned} &\mathrm{H}_0\left(\mathrm{cone}\left\{\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^{t-1})^\sim \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^t)^\sim\right\}\right) \\ &\quad \cong \pi_0[\mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s}\mathbb{P}^t)^{an} / \mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s}\mathbb{P}^{t-1})^{an}]^+ \\ &\quad \cong A^{t-r}(X), \end{aligned}$$

where $A^n(X)$ is the group of codimension n cycles modulo algebraic equivalence. This isomorphism is induced by sending $f : X \longrightarrow \mathcal{C}_r\mathbb{P}^t$ to the class of the intersection of its graph $\Gamma_f \subset X \times \mathbb{P}^t$ with $X \times \{P\}$ for a general point $P \in \mathbb{P}^t$. Moreover, two morphisms

$$f, g : X \longrightarrow \mathcal{C}_r\mathbb{P}^t$$

define the same map

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_s X)^\sim \longrightarrow \mathrm{cone}\left\{\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^{t-1})^\sim \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^t)^\sim\right\}$$

in the derived category for all s if and only if the classes of f and g coincide in $A^{t-r}(X)$.

Proof. The first isomorphism follows from [FL-2; C4], while the second is a consequence of duality for smooth, projective varieties [FL-2; 5.2].

If f and g determine the same class in $A^{t-r}(X)$, then for suitable choices of elements $h_1, h_2 \in \mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^{t-1})^{an}$, we have that $f + h_1$ and $g + h_1$ lie in the same component of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_r\mathbb{P}^t)^{an}$ (cf. [F2; 7.1]). As indicated previously, Proposition 1.7 tells us that $f + h_1$ and $g + h_2$ define the same morphism

$$(f+h_1)_* = (g+h_2)_* : \mathcal{M}\mathrm{or}(-, \mathcal{C}_sX)^\sim \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^t)^\sim$$

in the derived category. Since $(f-g)_*$ differs from $0 = (f+h_1)_* - (g+h_2)_*$ by a morphism that factors though $\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^{t-1})^\sim$, it follows that we have the desired equality

$$f_* = g_* : \mathcal{M}\mathrm{or}(-, \mathcal{C}_sX)^\sim \longrightarrow \mathrm{cone}\left\{\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^{t-1})^\sim \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}\mathbb{P}^t)^\sim\right\}.$$

Conversely, if $f_* = g_*$, then applying these morphisms to the "identity" map in $\mathcal{M}\mathrm{or}(X, \mathcal{C}_0X)^\sim$, we see immediately that f and g determine the same class in $A^{t-r}(X)$. $\square$

The following theorem asserts that the join product is compatible with the natural direct sum decomposition of $\mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^n)^\sim$.

Theorem 5.3. *For any quasi-projective variety X, the operation of linear join on $\mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^n)^\sim$ is graded in the sense that the diagram*

$$\begin{tikzcd}
\mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^n)^\sim \overset{\mathbb{L}}{\otimes} \mathcal{M}\mathrm{or}(X, \mathcal{C}_0\mathbb{P}^m)^\sim \arrow[r, "\#"] \arrow[d, "\cong"'] & \mathcal{M}\mathrm{or}(X, \mathcal{C}_1\mathbb{P}^{n+m+1})^\sim \arrow[d, "\cong"'] \\
\bigoplus_{i=0}^n \mathcal{M}(X,i)[2i] \otimes \bigoplus_{j=0}^m \mathcal{M}(X,j)[2j] \arrow[r, "\sum \#"] & \bigoplus_{k=0}^m \mathcal{M}(X,k)[2k]
\end{tikzcd}$$

commutes in the derived category.

Proof. It will suffice to establish that the diagram

$$\begin{tikzcd}
\mathcal{M}\mathrm{or}(-, \mathcal{C}_r\mathbb{P}^n)^\sim \overset{\mathbb{L}}{\otimes} \mathcal{M}\mathrm{or}(-, \mathcal{C}_s\mathbb{P}^m)^\sim \arrow[r, "\#"] \arrow[d, "\rho_{n,i} \overset{\mathbb{L}}{\otimes} \rho_{m,j}"'] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{m+n+1})^\sim \arrow[d, "\rho_{m+n+1,k+1}"'] \\
\bigoplus_{i+j=k} \mathcal{M}\mathrm{or}(-, \mathcal{C}_r\mathbb{P}^i)^\sim \overset{\mathbb{L}}{\otimes} \mathcal{M}\mathrm{or}(-, \mathcal{C}_s\mathbb{P}^j)^\sim \arrow[d, "\sum \#"'] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim \arrow[d] \\
\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim \arrow[r] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim / \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k})^\sim
\end{tikzcd} \tag{5.3.1}$$

commutes in the derived category of presheaves. (Actually, we need to know the commutativity of this diagram just for the case $r = s = 0$, but we prove the more general assertion since it is no more difficult.)

Recall that the join operation factors as

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_r\mathbb{P}^i)^\sim \otimes \mathcal{M}\mathrm{or}(-, \mathcal{C}_s\mathbb{P}^j)^\sim \xrightarrow{\times} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j))^\sim \xrightarrow{\#} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{i+j+1})^\sim,$$

where the second map (which we also call "join") is induced by the pairing of Proposition 1.7

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j)) \times \mathcal{M}\mathrm{or}(\mathbb{P}^i \times \mathbb{P}^j, \mathcal{C}_1\mathbb{P}^{i+j+1}) \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}_{r+s+1}\mathbb{P}^{i+j+1})$$

by fixing the element of

$$\mathcal{M}\mathrm{or}(\mathbb{P}^i \times \mathbb{P}^j, \mathcal{C}_1\mathbb{P}^{i+j+1})$$

which sends a pair of points to the line they span in $\mathbb{P}^{i+j+1}$. The commutative diagram of Lemma 5.1 allows us to replace the upper left arrow of (5.3.1) with the map

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_r(\mathbb{P}^n \times \mathbb{P}^m))^\sim \xrightarrow{\rho_{n,i}\times\rho_{m,j}} \bigoplus_{i+j=k} \mathcal{M}\mathrm{or}(-, \mathcal{C}_r(\mathbb{P}^i \times \mathbb{P}^j))^\sim,$$

so that to establish the commutativity of diagram (5.3.1), we need to show the diagram

$$\begin{tikzcd}
\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m))^\sim \arrow[r, "\#"] \arrow[d, "\rho_{n,i}\times\rho_{m,j}"'] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{m+n+1})^\sim \arrow[d, "\rho_{n+m+1,k+1}"'] \\
\bigoplus_{i+j=k} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j))^\sim \arrow[d, "\Sigma\circ\#"'] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim \arrow[d] \\
\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim \arrow[r] & \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim / \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k})^\sim
\end{tikzcd} \tag{5.3.2}$$

commutes in the derived category.

Let us consider first the composition

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m))^\sim \xrightarrow{\#} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{m+n+1})^\sim \xrightarrow{\rho_{m+n+1,k+1}} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim.$$

The associativity condition of Proposition 1.7 implies that this composition is induced by the composition of the maps

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\#} \mathcal{C}_1\mathbb{P}^{n+m+1} \xrightarrow{\mathcal{C}_1(\rho)} \mathcal{C}_1\mathbb{P}^{k+1}.$$

(By $\mathcal{C}_1(\rho)$ we mean the evident map associated to $\rho = \rho_{m+n+1,k+1}$; that is, the map obtained by pairing ρ with the identity on $\mathcal{C}_1\mathbb{P}^{n+m+1}$ in the pairing of Proposition 1.7.) Let us write this composition as ϕ.

Similarly, the associativity and bilinearity conditions of Proposition 1.7 imply that the composition

$$\mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^n \times \mathbb{P}^m))^\sim \longrightarrow \bigoplus_{i+j=k} \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s}(\mathbb{P}^i \times \mathbb{P}^j))^\sim \longrightarrow \mathcal{M}\mathrm{or}(-, \mathcal{C}_{r+s+1}\mathbb{P}^{k+1})^\sim$$

is induced by the sum over all $i + j = k$ of the maps given as the composition of

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\rho_{n,i}\times\rho_{m,j}} \mathcal{C}_0(\mathbb{P}^i \times \mathbb{P}^j) \xrightarrow{\mathcal{C}_0(\#)} \mathcal{C}_1\mathbb{P}^{k+1}.$$

Let us write this map as $\psi_{i,j}$ and write their sum as $\psi = \sum_{i+j=k} \psi_{i,j}$.

To prove the theorem, it suffices to show that two maps

$$\phi = \mathcal{C}_1(\rho) \circ \#, \psi = \sum_{i+j=k} \psi_{i,j} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathcal{C}_1\mathbb{P}^{k+1}$$

induce homotopic natural transformations of functors. By Proposition 5.2, this amounts to showing ϕ and ψ determine the same class in

$$A^k(\mathbb{P}^n \times \mathbb{P}^m) \cong \bigoplus_{p+q=k} \mathbb{Z} \cdot [\mathbb{P}^{n-p} \times \mathbb{P}^{m-q}]$$

upon intersection of their graphs with $\mathbb{P}^n \times \mathbb{P}^m \times \{P\}$ for a general point P. We will show in fact that both maps determine the class

$$\sum_{p+q=k} [\mathbb{P}^{n-p} \times \mathbb{P}^{m-q}]. \tag{5.3.3}$$

To compute the class of ϕ in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$, we begin with the observation that for any t and s, the graph of $\rho_{t,s} : \mathbb{P}^t \longrightarrow \mathcal{C}_0\mathbb{P}^s$, which is an integral subvariety of $\mathbb{P}^t \times \mathbb{P}^s$, forms a projective bundle over $\mathbb{P}^s$ with fibers isomorphic to $\mathbb{P}^{t-s}$. Indeed, the fiber of this graph over the point $P_1 \bullet \cdots \bullet P_s \in \mathrm{SP}^s(\mathbb{P}^1) \cong \mathbb{P}^s$ consists of all points $Q_1 \bullet \cdots \bullet Q_t$ of $\mathrm{SP}^t(\mathbb{P}^1) \cong \mathbb{P}^t$ such that $Q_{j_i} = P_i$, for all i, for some choice $1 \leq j_1 < \cdots < j_s \leq t$, which is precisely the image of $\mathrm{SP}^{t-s}(\mathbb{P}^1) \hookrightarrow \mathrm{SP}^t(\mathbb{P}^1)$ under the closed immersion given by "multiplication" with $P_1 \bullet \cdots \bullet P_s$. Thus, we have

$$\mathrm{graph}(\rho_{t,s}) \cap \big(\mathbb{P}^t \times \{P\}\big) = H \cong \mathbb{P}^{t-s}, \tag{5.3.4}$$

for a general (in fact, every) point P. In particular, the intersection of the graph of $\rho_{n+m+1,k+1}$ with $\mathbb{P}^{n+m+1} \times \{P\}$ for a general point P in $\mathbb{P}^{k+1}$ is a general linear subspace of $\mathbb{P}^{n+m+1}$ of dimension $n+m-k$.

We claim that the intersection of the graph of $\#$, which is the subscheme $W \subset \mathbb{P}^m \times \mathbb{P}^n \times \mathbb{P}^{n+m+1}$ introduced earlier, with $\mathbb{P}^n \times \mathbb{P}^m \times H$ for a general dimension $n+m-k$ linear subspace H pushes forward to $\mathbb{P}^n \times \mathbb{P}^m$ to the class (5.3.3) in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$. This will show that ϕ determines the class (5.3.3) of $A^k(\mathbb{P}^n \times \mathbb{P}^m)$ since we have

$$\begin{aligned} &\mathrm{graph}(\phi) \cap (\mathbb{P}^n \times \mathbb{P}^m \times \{P\}) \\ &\qquad = \mathrm{graph}(\#) \cap \big[\mathbb{P}^n \times \mathbb{P}^m \times \big(\mathrm{graph}(\rho_{n+m+1,k+1}) \cap \mathbb{P}^{n+m+1} \times \{P\}\big)\big] \\ &\qquad = \mathrm{graph}(\#) \cap (\mathbb{P}^n \times \mathbb{P}^m \times H)\,. \end{aligned} \tag{5.3.5}$$

To establish the claim, observe that we need only show that the image under the composition

$$A^*(\mathbb{P}^{n+m+1}) \xrightarrow{\pi_2^*} A^*(W) \xrightarrow{\pi_{1*}} A^*(\mathbb{P}^n \times \mathbb{P}^m)$$

of $[H]$, for H a linear subspace of dimension $n+m-k$, is the class (5.3.3). Recall from the proof of Proposition 4.4 that $W \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$ is the projectivized bundled associated to $\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$, and thus

$$A^*(W) \cong A^*(\mathbb{P}^n \times \mathbb{P}^m)[\zeta]/(\zeta^2 - (\alpha+\beta)\zeta + \alpha \cdot \beta),$$

where $\alpha = [\mathbb{P}^{n-1} \times \mathbb{P}^m]$, $\beta = [\mathbb{P}^n \times \mathbb{P}^{m-1}]$, and ζ is the canonical divisor of the projectivized bundle. The map π_2^* is a ring map and sends $[\mathbb{P}^{n+m}]$ to ζ. The map π_{1*} is a $A^*(\mathbb{P}^n \times \mathbb{P}^m)$-module map which sends ζ to $1 = [\mathbb{P}^n \times \mathbb{P}^m]$ and $[W]$ to 0 (since $W \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$ has relative dimension one). The claimed equality follows for the case $k = 0$ immediately. For general k, observe that

$$\pi_2^*[\mathbb{P}^{n+m-k}] = \zeta^{k+1}.$$

One may easily verify that

$$\zeta^{k+1} = \left(\sum_{p+q=k} \alpha^p \beta^q \right) \zeta + \text{constant term}$$

in $A^k(W)$. Thus, we have, for all k,

$$\pi_{1*}\pi_2^*[H] = \sum_{p+q=k} [\mathbb{P}^{n-p} \times \mathbb{P}^{m-q}]. \tag{5.3.6}$$

As indicated in (5.3.5), it follows that ϕ determines the class (5.3.3) in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$.

Recall that ψ is the sum $\sum_{i+j=k} \psi_{i,j}$. We now compute the class in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$ of the map

$$\psi_{i,j} = \mathcal{C}_0(\#) \circ (\rho_{n,i} \times \rho_{m,j}) \ : \ \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{k+1}.$$

Taking $n = i$, $n = j$, and $k = i+j$ in the equation (5.3.6) shows that the intersection of the graph of $\# : \mathbb{P}^i \times \mathbb{P}^j \longrightarrow \mathbb{P}^k$ with $\mathbb{P}^i \times \mathbb{P}^j \times \{P\}$ for a general point $P \in \mathbb{P}^k$ is the class of a point in $\mathbb{P}^i \times \mathbb{P}^j$. By equation (5.3.4), the intersection of the graph of $\rho_{n,i} \times \rho_{m,j}$ with $\mathbb{P}^n \times \{Q\} \times \mathbb{P}^m \times \{R\}$ for points $Q \in \mathbb{P}^i$ and $R \in \mathbb{P}^j$ is the class $[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j}]$. Since we have

$$\begin{aligned} &\operatorname{graph}(\psi_{i,j}) \cap (\mathbb{P}^n \times \mathbb{P}^m \times \{P\}) \\ &\qquad = \operatorname{graph}(\rho_{n,i} \times \rho_{m,j}) \cap \left(\mathbb{P}^n \times \mathbb{P}^m \times \left[\operatorname{graph}(\#) \cap \left(\mathbb{P}^i \times \mathbb{P}^j \times \{P\}\right)\right]\right) \\ &\qquad = \operatorname{graph}(\rho_{n,i} \times \rho_{m,j}) \cap (\mathbb{P}^n \times \mathbb{P}^m \times \{Q\} \times \{R\}), \end{aligned}$$

it follows that $\psi_{i,j}$ has class $[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j}]$ in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$. Consequently, ψ has class (5.3.3) in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$, since ψ is the sum of the $\psi_{i,j}$.

Since ϕ and ψ determine the same class (5.3.3) in $A^k(\mathbb{P}^n \times \mathbb{P}^m)$, they define the same map in the derived category by Proposition 5.2. The theorem is therefore proven. □

The following was suggested in [FL-1], but was not proven in that paper for lack of a version of Theorem 5.3.

Theorem 5.4. *For any quasi-projective variety X, there are Chern class maps*

$$c_n : K_0(X) \longrightarrow \mathrm{H}^{2n} \mathcal{M}(X, n) \longrightarrow \mathcal{H}^{2n}(X, n)$$

which extend the Chern class maps of [FL-1; 10.3] *for vector bundles generated by global sections. Moreover, these Chern class maps satisfy the Whitney sum formula*

$$c_n(-) \ = \ \bigoplus_{i+j=n} c_i(-) \# c_j(-).$$

Proof. Recall that $\mathrm{H}^* \mathcal{M}(X,*)$ is defined in terms of the weak normalization X^w of X and observe that there is a natural map $K_0(X) \longrightarrow K_0(X^w)$. So we may assume X is weakly normal.

Let $\mathrm{Grass}^e(\mathbb{P}^N)$ be the Grassmannian variety of dimension $e-1$ linear subvarieties of $\mathbb{P}^N$. Then $\mathrm{Grass}^e(\mathbb{P}^N)$ represents the functor sending X to the set of quotient objects $\mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{E}$ (that is, isomorphism classes of surjection), where $\mathcal{E}$ is a rank e vector bundle on X. Moreover, as in Proposition 2.4, the set $\mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))$ comes equipped with a natural topology.

Let

$$\Psi : \mathrm{Grass}^e(\mathbb{P}^N) \longrightarrow \mathcal{C}^e\mathbb{P}^N \equiv \mathcal{C}_{N-e}\mathbb{P}^N$$

be the morphism of varieties which sends a quotient $\pi : \mathbb{C}^{N+1} \twoheadrightarrow \mathbb{C}^e$ to the cycle which is the projectivization of the kernel of π. Then Ψ determines a continuous map of topological spaces

$$\mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))^{an} \longrightarrow \mathcal{M}\mathrm{or}(X, \mathcal{C}^e(\mathbb{P}^N))^{an}$$

by Theorem 2.3.

Observe that for all M, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Grass}^e(\mathbb{P}^N) & \longrightarrow & \mathcal{C}^e\mathbb{P}^N \\ \big\downarrow & & \big\downarrow{\scriptstyle \Sigma^M} \\ \mathrm{Grass}^e(\mathbb{P}^{N+M}) & \longrightarrow & \mathcal{C}^e\mathbb{P}^{N+M}. \end{array}$$

The map on the left in this diagram sends $\mathbb{C}^{N+1} \twoheadrightarrow \mathbb{C}^e$ to the composition $\mathbb{C}^{N+1+M} \twoheadrightarrow \mathbb{C}^{N+1} \twoheadrightarrow \mathbb{C}^e$ (in which the first map is projection onto the first $N+1$ coordinates), and the map on the right is given by suspension. Thus we obtain a map of direct systems

$$\varinjlim_N \mathrm{Grass}^e(\mathbb{P}^N) \longrightarrow \varinjlim_N \mathcal{C}^e\mathbb{P}^N$$

which induces the continuous map of topological spaces

$$\varinjlim_N \mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))^{an} \longrightarrow \varinjlim_N \mathcal{M}\mathrm{or}(X, \mathcal{C}^e\mathbb{P}^N)^{an}.$$

Using the suspension isomorphism (3.1.3), we have canonical isomorphisms

$$\mathcal{M}\mathrm{or}(X, \mathcal{C}^e\mathbb{P}^N)^{\sim} \cong \bigoplus_{i=0}^{e} \mathcal{M}(X,i)[2i]$$

in the derived category of abelian groups. It follows that we obtain a natural map

$$\mathrm{Vect}_e(X) \equiv \varinjlim_N \pi_o \, \mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))^{an} \longrightarrow \bigoplus_{s=0}^{e} \mathrm{H}^{2s} \mathcal{M}(X,s).$$

We may associate to a vector bundle $\mathcal{E}$ on X of rank e which is generated by its global sections an element of $\mathrm{Vect}_e(X)$ by choosing a surjection $\mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{E}$, for some $N >> 0$, and then taking the associated class in $\pi_0 \varinjlim \mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))^{an}$. We claim the resulting class in $\mathrm{Vect}_e(X)$ is independent of the choice made. To see this, suppose $p : \mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{E}$, $q : \mathcal{O}_X^{M+1} \twoheadrightarrow \mathcal{E}$ are two different choices. Then consider the point in $\mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^{N+M+1}))^{an}$ defined by the surjection

$$\mathcal{O}_X^{N+1+M+1} = \mathcal{O}_X^{N+1} \oplus \mathcal{O}_X^{M+1} \overset{(p,q)}{\twoheadrightarrow} \mathcal{E}.$$

There is a path from this point to the point given by the surjection $(p, 0)$ (respectively, $(0, q)$) defined by $(p, \lambda q)$ (respectively, $(\lambda p, q)$) for $\lambda \in [0, 1]$. Similarly, there is a path from the point associated to $(0, q)$ to the point associated to $(q, 0)$. This shows that our two choices coincide at some stage in the direct limit defining $\mathrm{Vect}_e(X)$. In fact, this argument shows that there is a natural, surjective map

$$\mathrm{Iso}\,\mathcal{P}_{gl}(X) \longrightarrow \oplus_e \mathrm{Vect}_e(X), \tag{5.4.1}$$

where $\mathrm{Iso}\,\mathcal{P}_{gl}(X)$ is the set of isomorphism classes of vector bundles on X which are generated by global sections.

Define a pairing

$$\oplus : \mathrm{Grass}^e(\mathbb{P}^N) \times \mathrm{Grass}^{e'}(\mathbb{P}^M) \longrightarrow \mathrm{Grass}^{e+e'}(\mathbb{P}^{N+M+1})$$

by sending the pair $(\mathbb{C}^{N+1} \twoheadrightarrow \mathbb{C}^e, \mathbb{C}^{M+1} \twoheadrightarrow \mathbb{C}^{e'})$ to $\mathbb{C}^{N+1+M+1} = \mathbb{C}^{N+1} \oplus \mathbb{C}^{M+1} \twoheadrightarrow \mathbb{C}^e \oplus \mathbb{C}^{e'}$. It is easy to verify that the diagram

$$\begin{array}{ccc} \mathrm{Grass}^e(\mathbb{P}^N) \times \mathrm{Grass}^{e'}(\mathbb{P}^M) & \xrightarrow{\oplus} & \mathrm{Grass}^{e+e'}(\mathbb{P}^{N+M+1}) \\ \downarrow & & \downarrow \\ \mathcal{C}^e\mathbb{P}^N \times \mathcal{C}^{e'}\mathbb{P}^M & \xrightarrow{\#} & \mathcal{C}^{e+e'}\mathbb{P}^{N+M+1} \end{array} \tag{5.4.2}$$

commutes. If we consider the space of morphisms from X to each variety in diagram (5.4.2), we obtain a commutative diagram of spaces. Further, the maps induced by each arrow in diagram (5.4.2) are compatible with the maps in the direct systems $\varinjlim_N \mathcal{M}\mathrm{or}(X, \mathrm{Grass}^e(\mathbb{P}^N))^{an}$ and $\varinjlim_N \mathcal{M}\mathrm{or}(X, \mathcal{C}^e\mathbb{P}^N)^{an}$ on the level of π_0 (in fact, up to homotopy), and thus the composite map

$$c = (c_n) : \mathrm{Iso}\,\mathcal{P}_{gl}(X) \longrightarrow \tilde{\prod}_e \mathrm{Vect}_e(X) \longrightarrow \bigoplus_{s=0}^{\infty} \mathrm{H}^{2s}\,\mathcal{M}(X, s) \tag{5.4.3}$$

is actually a homomorphism of monoids. Here $\mathrm{Iso}\,\mathcal{P}_{gl}(X)$ is a monoid under direct sum of vector bundles and $\bigoplus_{s=0}^{\infty} \mathrm{H}^{2s}\,\mathcal{M}(X, s)$ is a monoid under the join pairing. The notation $\tilde{\prod}_e \mathrm{Vect}_e(X)$ refers to the restricted direct product, defined as the subset of the product consisting of sequences of elements $\alpha_e \in \mathrm{Vect}_e(X)$ such that α_e coincides with the image of the trivial bundle $\mathcal{O}_X^e$ for almost all e.

Set $\mathcal{H}(X) = \bigoplus_{s=0}^{\infty} \mathrm{H}^{2s}\,\mathcal{M}(X, s)$. Then $\mathcal{H}(X)$ is actually a ring under addition of cycles and the join product. One may easily check that $c_0(\mathcal{E}) = 1 \in \mathrm{H}^0\,\mathcal{M}(X, 0) \cong \mathbb{Z}$. Let $1 + \mathcal{H}(X)^+[[t]]$ denote the subset of the set of formal power series $\mathcal{H}(X)[[t]]$ consisting of those power series whose coefficient of t^n lies in $\mathcal{H}^{2n}(X, n)$ and whose constant term is 1. Then the join operation on $\mathcal{H}(X)$ endows $1 + \mathcal{H}(X)^+[[t]]$ with the structure of multiplicative abelian group. The map

$$c_t : \mathrm{Iso}\,\mathcal{P}_{gl}(X) \longrightarrow 1 + \mathcal{H}^+[[t]]$$

defined by $c_t(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \dots$ is a map of abelian monoids, with target an abelian group. It therefore extends to a map

$$c_t : \operatorname{Iso}\mathcal{P}_{gl}(X)^+ \longrightarrow 1 + \mathcal{H}(X)^+[[t]]$$

from the group completion of $\operatorname{Iso}\mathcal{P}_{gl}(X)$. Observe that the target of c_t is homotopy invariant in X, since there is for any X, Y a natural pairing

$$\mathcal{M}\mathrm{or}(X \times \mathbb{A}^1, Y) \times \mathbb{A}^1 \longrightarrow \mathcal{M}\mathrm{or}(X \times \mathbb{A}^1, Y)$$

relating the identity to the map induced by $i \circ pr : X \times \mathbb{A}^1 \to X \times \{0\} \to X \times \mathbb{A}^1$. Thus, c_t factors though the cokernel

$$\operatorname{Iso}\mathcal{P}_{gl}(X \times \mathbb{A}^1)^+ \overset{\alpha_1 - \alpha_0}{\longrightarrow} \operatorname{Iso}\mathcal{P}_{gl}(X)^+ \tag{5.4.4}$$

where α_i is induced by restriction to $X \times \{i\}$.

We claim the cokernel of (5.4.4) is isomorphic to

$$K_0(X)/(\text{homotopy}) \equiv \operatorname{coker}\left(K_0(X \times \mathbb{A}^1) \overset{\alpha_1 - \alpha_0}{\longrightarrow} K_0(X)\right).$$

Say X is a subvariety of $\mathbb{P}^n$. Then by using the Koszul resolution induced by the canonical surjection $\mathcal{O}_{\mathbb{P}^n}^{n+1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}(1)$, one shows that every class in $K_0(X)$ is a difference of the classes of vector bundle generated by global sections. Further, given a short exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

of vector bundles on X, let

$$0 \longrightarrow \mathcal{E}'[t] \longrightarrow \mathcal{E}[t] \longrightarrow \mathcal{E}''[t] \longrightarrow 0$$

be the pullback of this sequence to $X \times \operatorname{Spec}\mathbb{C}[t] = X \times \mathbb{A}^1$. Now define the vector bundle $\tilde{\mathcal{E}}$ on $X \times \mathbb{A}^1$ so that the square

$$\begin{tikzcd} \tilde{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{E}''[t] \arrow[d, "\cdot t"'] \\ \mathcal{E}[t] \arrow[r] & \mathcal{E}''[t] \end{tikzcd}$$

is cartesian. Then $\tilde{\mathcal{E}}|_{X\times\{0\}} \cong \mathcal{E}' \oplus \mathcal{E}''$ and $\tilde{\mathcal{E}}|_{X\times\{1\}} \cong \mathcal{E}$, and so every short exact sequence may be deformed continuously to a short exact sequence. The claim follows.

We thus obtain the map

$$c_t : K_0(X) \longrightarrow 1 + \mathcal{H}(X)^+[[t]].$$

and we define

$$c_n : K_0(X) \longrightarrow H^{2n}\mathcal{M}(X, n)$$

by taking the coefficient of t^n in c_t. The Whitney sum formula is an obvious consequence of the construction. $\square$

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