COMPARING K-THEORIES FOR COMPLEX VARIETIES

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Abstract. The semi-topological K-theory of a complex variety was defined in [FW-2] with the expectation that it would prove to be a theory lying "part way" between the algebraic K-theory of the variety and the topological K-theory of the associated analytic space, and thus would share properties with each of these other theories. In this paper, we realize these expectations by proving among other results that (1) the algebraic K-theory with finite coefficients and the semi-topological K-theory with finite coefficients coincide on all projective complex varieties, (2) semi-topological K-theory and topological K-theory agree on certain types of generalized flag varieties, and (3) (assuming a result asserted by Cohen and Lima-Filho) the semi-topological K-theory of any smooth projective variety becomes isomorphic to the topological K-theory of the underlying analytic space once the Bott element is inverted. To illustrate the utility of our results, we observe that a new proof of the Quillen-Lichtenbaum conjecture for smooth, complete curves is obtained as a corollary.

In the recent paper [FW-2], the authors introduced "semi-topological K-theory" $K_*^{\text{semi}}(X)$ for a complex quasi-projective algebraic variety X, showed that the natural map from algebraic to topological K-theory $K_*(X) \to K_{\text{top}}^{-*}(X^{an})$ factors through this new theory, and showed that $K_*^{\text{semi}}(X)$ is related to Friedlander-Lawson morphic cohomology $L^*H^*(X)$ as algebraic K-theory is related to motivic cohomology and topological K-theory is related to integral singular cohomology. A few computations (projective smooth curves, projective spaces) were provided, but the general behavior of this theory remained inaccessible.

In this paper, we introduce a new theory $K_*(\Delta^{\bullet}_{top} \times X)$ which is less geometric in character but more accessible to computation. The relevance of $K_*(\Delta^{\bullet}_{top} \times X)$ to our goal of understanding $K_*^{\text{semi}}(X)$ is that these theories agree whenever X is projective (and possibly in general). As we shall see, $K_*(\Delta^{\bullet}_{top} \times X)$ admits further computations and satisfies Mayer-Vietoris for smooth varieties. More remarkably, we prove for any $n > 0$ the existence of natural isomorphisms for all quasi-projective varieties X

$$K_*(X;\mathbb{Z}/n) \simeq K_*(\Delta^{\bullet}_{top} \times X;\mathbb{Z}/n). \tag{0.1}$$

Starting from a somewhat different point of view, R. Cohen and P. Lima-Filho have considered "holomorphic K-theory" $K_{hol}^{-*}(X)$ of a projective algebraic variety X [CL]. This theory appears to be equivalent to $K_*^{\text{semi}}(X)$ when the latter is restricted to weakly normal, projective varieties. In [CL], it is asserted that the map

$$K_*^{\text{semi}}(X)[1/\beta] \otimes \mathbb{Q} \simeq K_{\text{top}}^{-*}(X^{an}) \otimes \mathbb{Q} \tag{0.2}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX is a rational isomorphism for X a smooth, projective variety, where $\beta \in K_2^{\mathrm{semi}}(\operatorname{Spec}\mathbb{C})$ denotes the Bott element. Building on this result, we prove that (0.2) is in fact an isomorphism integrally; i.e., without having to tensor with $\mathbb{Q}$. We conclude that our $K_*^{\mathrm{semi}}(X)$ fits tightly between algebraic and topological K-theory.

In the first section of this paper, we define the spectrum-valued theory $\mathcal{K}(\Delta^\bullet_{top} \times X)$ whose homotopy groups are the groups $K_*(\Delta^\bullet_{top} \times X)$ mentioned above. We establish maps involving this new theory, algebraic K-theory, and semi-topological K-theory, and we establish the existence of the natural weak equivalence

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \xrightarrow{\sim} \mathcal{K}^{\mathrm{semi}}(X) \tag{0.3}$$

when X is projective and weakly normal.

In section two we employ (0.3) and a result of Panin [P] to conclude that the natural map

$$\mathcal{K}^{\mathrm{semi}}(X) \to \mathcal{K}_{\mathrm{top}}(X^{\mathrm{an}})$$

is a weak equivalence for a class of varieties which includes quotients of certain classical algebraic groups by parabolic subgroups and bundles of such over smooth, complete curves. This result can be viewed as a generalization of a stable version of results of Kirwan [K] comparing the algebraic and topological mapping spaces from Riemann surfaces to Grassmann varieties. In degree 0, this has been proven by Cohen and Lima-Filho [CL].

In section three we establish the weak equivalence (0.1) which in conjunction with (0.2) implies the weak equivalence

$$\mathcal{K}(X;\mathbb{Z}/n) \xrightarrow{\sim} \mathcal{K}^{\mathrm{semi}}(X;\mathbb{Z}/n), \tag{0.4}$$

for $n > 0$ and X a weakly normal projective variety. An interesting consequence of (0.4) in conjunction with [FW-2; 7.5] is a new proof of the Quillen-Lichtenbaum conjecture for smooth, complete curves.

In section four, we use the weak equivalence (0.4), the previously mentioned result of [CL], and a result of R. Thomason [T; 4.11] to establish (0.2): "Bott-inverted" semi-topological K-theory of any smooth, projective variety X coincides with the topological K-theory of X^{an}. This settles affirmatively a conjecture made in [FW-2].

Section five focuses primarily on the theory $\mathcal{K}(\Delta^\bullet_{top} \times -)$. In it, we establish that the Mayer-Vietoris property for open covers is satisfied by this theory, giving long exact sequences which may prove useful for studying $\mathcal{K}(\Delta^\bullet_{top} \times -)$. Using the techniques developed in this section, we also show that the natural map

$$K_q^{\mathrm{semi}}(C) \to K_{\mathrm{top}}^{-q}(C^{\mathrm{an}})$$

is an isomorphism for $q \geq 0$ whenever C is a (possibly singular) complete curve.

All varieties considered in this paper are quasi-projective over the complex field $\mathbb{C}$.

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§1. $\mathcal{K}(\Delta^\bullet_{top} \times X)$

In this first section, we introduce a naturally defined Ω-spectrum $\mathcal{K}(\Delta^\bullet_{top} \times X)$ for any quasi-projective variety X and verify that this Ω-spectrum is weakly equivalent to the Ω-spectrum $\mathcal{K}^{\mathrm{semi}}(X)$ of [FW-2] whenever X is projective and weakly normal. Although we view $\mathcal{K}^{\mathrm{semi}}(X)$ as the primary object of interest, $\mathcal{K}(\Delta^\bullet_{top} \times X)$ proves to be a more convenient object of study for it appears to be better behaved when applied to varieties which are not projective.

We begin by briefly recalling the construction of $\mathcal{K}^{\mathrm{semi}}(X)$. For any quasi-projective varieties X and Y, define $\mathrm{Hom}(X,Y)$ to be the collection of morphisms of varieties from X to Y. Similarly, we define $\mathrm{Mor}(X,Y)$ to be the collection of all *continuous algebraic maps* from X to Y – that is, $\mathrm{Mor}(X,Y)$ is the set $\mathrm{Hom}(X^w,Y)$, where $X^w \to X$ denotes the weak normalization of X. In particular, we consider the set of continuous algebraic maps $\mathrm{Mor}(X, \mathrm{Grass}_m(\mathbb{P}^N))$. As shown in [FW-1], this set has a natural topology; we let $\mathcal{M}or(X, \mathrm{Grass}_m(\mathbb{P}^N))^{\mathrm{an}}$ denote the resulting topological space. If X is projective, this topology has the simple description as the subspace topology of the set of all continuous maps from X^{an} to $\mathrm{Grass}_m(\mathbb{P}^N)^{an}$ equipped with the compact-open topology.

We consider the space

$$\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}} \equiv \coprod_{m \geq 0} \varinjlim_N \mathcal{M}or(X, \mathrm{Grass}_m(\mathbb{P}^N))^{\mathrm{an}}.$$

External Whitney sum

$$\mathrm{Grass}_m(\mathbb{P}^N) \times \mathrm{Grass}_{m'}(\mathbb{P}^{N'}) \to \mathrm{Grass}_{m+m'}(\mathbb{P}^{N+N'})$$

determines a product on this space which is enhanced to admit the action of a certain E_∞-operad $\mathcal{I}^{an}$, consisting of spaces $\mathcal{I}(n)$, $n \geq 0$. The space $\mathcal{I}(n)$ is the collection of n-tuples of linear maps $\mathbb{C}^\infty \to \mathbb{C}^\infty$ which induce an injection $(\mathbb{C}^\infty)^n \hookrightarrow \mathbb{C}^\infty$ (cf. [FW-1] for further details). As with any space admitting an action by an E_∞-operad, we have an associated Ω-spectrum

$$\Omega^\infty \mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}$$

whose 0^{th} term is the homotopy-theoretic group completion of $\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}$,

$$\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}} \to [\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}]^+.$$

This homotopy-theoretic group completion is the semi-topological K-theory space of X,

$$\mathcal{K}^{\mathrm{semi}}(X) \equiv [\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}]^+.$$

Note in particular that $\mathcal{K}^{\mathrm{semi}}(X)$ and $\mathcal{K}^{\mathrm{semi}}(X^w)$ coincide by definition, where $X^w \to X$ is the weak normalization of X. The space $\mathcal{K}^{\mathrm{semi}}(X)$ becomes somewhat more comprehensible once one observes that it is homotopy equivalent to the infinite mapping telescope of a self-map

$$\alpha : \mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}} \to \mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}},$$

which is defined by taken external sum with a chosen very ample line bundle $\mathcal{O}(1)$ (cf. [FW-2; 3.4]).

We proceed to introduce $\mathcal{K}(\Delta^d_{top} \times X)$ following a suggestion of V. Voevodsky. Given a compact Hausdorff space T, we write Var^T for the category whose objects are continuous maps $T \to U^{\mathrm{an}}$, for $U \in Sch/\mathbb{C}$ (i.e., for U a quasi-projective variety). A morphism in Var^T from $T \to U^{\mathrm{an}}$ to $T \to V^{\mathrm{an}}$ is a morphism of complex varieties $V \to U$ causing the evident triangle to commute. Given a contravariant functor F from $Sch/\mathbb{C}$ to the category of sets, abelian groups, spaces, spectra, etc. and given a compact topological space T, define

$$F(T) \equiv \varinjlim_{(T \to U^{\mathrm{an}}) \in Var^T} F(U).$$

More generally, given such an F and T and a variety X, we define $F(T \times X)$ by using such a direct limit, regarding $F(- \times T)$ as a functor from $Sch/\mathbb{C}$ to sets, abelian groups, etc.

In particular, let $\mathcal{K} : (Sch/\mathbb{C})^{op} \to (spectra)$ be a functor giving (connective) algebraic K-theory (for example, chosen as in [FS]). Then given a quasi-projective variety X, we define $\mathcal{K}(\Delta^d_{top} \times X)$ as

$$\mathcal{K}(\Delta^d_{top} \times X) \equiv \varinjlim_{(\Delta^d_{top} \to U^{\mathrm{an}}) \in Var^{\Delta^d_{top}}} \mathcal{K}(U \times X),$$

where Δ^d_{top} stands for the standard d-dimensional topological simplex. Since Var^T is a directed category for any T, we have

$$\pi_q \mathcal{K}(\Delta^d_{top} \times X) \cong K_q(\Delta^d_{top} \times X).$$

Definition 1.1. *Let $\mathcal{K}(-)$ be an Ω-spectrum valued contravariant functor on $(Sch/\mathbb{C})$ giving algebraic K-theory. Define*

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \equiv |d \mapsto \mathcal{K}(\Delta^d_{top} \times X)|,$$

the geometric realization of the indicated simplicial Ω-spectrum. By $K_q(\Delta^\bullet_{top} \times X)$ we mean $\pi_q \mathcal{K}(\Delta^\bullet_{top} \times X)$.

We begin our analysis of $\mathcal{K}(\Delta^\bullet_{top} \times X)$ by verifying the following useful property. For F as in the following lemma, we write $F(\Delta^\bullet_{top})$ for the geometric realization of the simplicial space $d \mapsto F(\Delta^d_{top})$ and $F(\Delta^\bullet_{top} \times \Delta^\bullet)$ for the geometric realization of the bisimplicial space $d, e \mapsto F(\Delta^d_{top} \times \Delta^e)$. Here, $\Delta^\bullet$ denotes the standard cosimplicial variety which in degree d is $\Delta^d \equiv \operatorname{Spec} \mathbb{C}[x_0, \dots, x_d]/(\sum_i x_i - 1)$.

Lemma 1.2. *Let $F : (Sch/\mathbb{C}) \to (spaces)$ be a contravariant functor. Then the natural map induced by projection*

$$F(\Delta^\bullet_{top}) \to F(\Delta^\bullet_{top} \times \Delta^\bullet)$$

induces an isomorphism in homology.

Proof. We first verify that $H_r \circ F(\Delta^\bullet_{top} \times -)$ is homotopy invariant for any $r \geq 0$, arguing as in [Sw1; 4.1]. Let $\psi_i : \Delta^{n+1} \to \Delta^n \times \Delta^1$ be the linear map sending the j^{th} vertex v_j of Δ^{n+1} to $v_j \times 0$ if $j \leq i$ and to $v_{j-1} \times 1$ otherwise. Then ψ_i induces a continuous map $\psi_i^* : F(\Delta^n_{top} \times \Delta^1 \times X) \to F(\Delta^{n+1}_{top} \times X)$ for any quasi-projective variety X, since a continuous map $g : \Delta^n_{top} \to U^{an}$ determines $g \circ \psi^{an}_j : \Delta^{n+1}_{top} \to \Delta^n_{top} \times \Delta^1_{top} \to (U \times \Delta^1)^{an}$. Thus, the maps s_n for $n \geq 0$,

$$s_n = \sum_{i=0}^{n} (-1)^i H_r \circ \psi_i^* : H_r(F(\Delta^n_{top} \times \Delta^1 \times X)) \to H_r(F(\Delta^{n+1}_{top} \times X)),$$

determine a chain homotopy relating the two natural maps

$$i_0, i_1 : H_r(F(\Delta^\bullet_{top} \times \Delta^1 \times X)) \to H_r(F(\Delta^\bullet_{top} \times X)).$$

Homotopy invariance now follows as in [Sw1; 4.1] by applying the above argument with X of the form $Y \times \Delta^1$ and composing with the map $\Delta^1 \times \Delta^1 \to \Delta^1$ induced by multiplication of functions.

We conclude for any $r \geq 0$ that the complex $\{n \mapsto H_r(F(\Delta^\bullet_{top} \times \Delta^n))\}$ is quasi-isomorphic to the constant complex $H_r(F(\Delta^\bullet_{top}))$, thereby implying the isomorphism

$$H_r(F(\Delta^\bullet_{top})) \cong H_r(F(\Delta^\bullet_{top} \times \Delta^\bullet)).$$

□

There is an "algebraic" version of the E_∞-operad $\mathcal{I}^{an}$, defined by letting $\mathcal{I}(j)(\Delta^\bullet)$ denote the simplicial set

$$d \mapsto \varprojlim_N \varinjlim_M \operatorname{Hom}(\Delta^d, \mathcal{I}(j)_{N,M}).$$

Here, $\Delta^\bullet$ denotes the standard cosimplicial variety and $\mathcal{I}(j)_{N,M}$ is the algebraic variety parameterizing all injective maps $\mathbb{C}^{jN} \hookrightarrow \mathbb{C}^M$. Taking geometric realizations of the simplicial sets $\mathcal{I}(j)(\Delta^\bullet)$, $j \geq 0$, we form an E_∞-operad, which we write as $|\mathcal{I}(\Delta^\bullet)|$. Just as for $\mathcal{I}^{\mathrm{an}}$ and $\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))$, the operad $|\mathcal{I}(\Delta^\bullet)|$ acts on the space $|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))|$ via natural pairings

$$\mathcal{I}(\Delta^\bullet)(j) \times \operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))^{\times j} \to \operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))$$

of simplicial sets.

Proposition 1.3. *For any quasi-projective variety X, there is a natural weak equivalence of Ω-spectra*

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \to |\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))|^+,$$

where $|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))|^+$ is the 0^{th} space of the Ω spectrum associated to the $|\mathcal{I}(\Delta^\bullet)|$-space $|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))|$.

Proof. Lemma 1.2 implies that

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \to \mathcal{K}(\Delta^\bullet_{top} \times X \times \Delta^\bullet)$$

is a homology equivalence and thus a homotopy equivalence. By [GW; 3.3] and [FW-2; 6.8], the Ω-spectrum $\mathcal{K}(U \times X \times \Delta^\bullet)$ is naturally equivalent (as an Ω-spectrum) to the Ω-spectrum associated to the $|\mathcal{I}(\Delta^\bullet)|$-space $|\operatorname{Hom}(U \times X \times \Delta^\bullet, \operatorname{Grass}(\mathbb{P}^\infty))|$ for any U. Taking limits, we have that $\mathcal{K}(\Delta^\bullet_{top} \times X \times \Delta^\bullet)$ is naturally weakly equivalent to $|\operatorname{Hom}(\Delta^\bullet_{top} \times X \times \Delta^\bullet, \operatorname{Grass}(\mathbb{P}^\infty))|^+$. Finally, the proof of Lemma 1.2 establishes the homotopy invariance of $|\operatorname{Hom}(\Delta^\bullet_{top} \times -, \operatorname{Grass}(\mathbb{P}^\infty))|^+$, so that

$$|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))|^+ \to |\operatorname{Hom}(\Delta^\bullet_{top} \times X \times \Delta^\bullet, \operatorname{Grass}(\mathbb{P}^\infty))|^+$$

is also a weak equivalence. □

The following theorem compares $\mathcal{K}(\Delta^\bullet_{top} \times X)$ and $\mathcal{K}^{\mathrm{semi}}(X)$. Although we would like to prove that the natural map exhibited below is always a weak equivalence, we have been unable so far to do so. The essential difficulty of a such a general comparison is the somewhat awkward nature of the topology of $\mathcal{M}or(X, \operatorname{Grass}_n(\mathbb{P}^N))^{\mathrm{an}}$ whenever X is not projective.

Theorem 1.4. *For any quasi-projective variety X, there is a natural map of Ω-spectra*

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \to \mathcal{K}^{\mathrm{semi}}(X)$$

induced by the map of $|\mathcal{I}(\Delta^\bullet)|$ spaces

$$|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))| \to |Sing.\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}|.$$

This map of Ω-spectra is a weak equivalence whenever X is projective and weakly normal.

Proof. The first assertion follows from Proposition 1.3, using the fact that an element of $\operatorname{Hom}(\Delta^d_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))$, which is represented by a pair $\Delta^d_{top} \to U^{\mathrm{an}}, U \times X \to \operatorname{Grass}(\mathbb{P}^\infty)$, naturally defines a continuous map given by the composition of $\Delta^d_{top} \to U^{\mathrm{an}} \to \mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))$ (cf. the "internal hom" description of $\mathcal{M}or$ in [FW-1; §1]).

Assume X is projective and weakly normal. By Proposition 1.3, it suffices to exhibit an isomorphism of $|\mathcal{I}(\Delta^\bullet)|$-spaces

$$|\operatorname{Hom}(\Delta^\bullet_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty))| \to |Sing.\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}|.$$

Since X is projective, $\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}$ is the analytic space associated to an inductive limit of quasi-projective varieties by [FW-2; 4.1]. Thus, every continuous map $\phi : \Delta^d_{top} \to \mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}$ factors as

$$g^{an} \circ \phi' : \Delta^d_{top} \to U^{an} \to \mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}$$

for some (weakly normal) variety U and some $g \in \operatorname{Mor}(U \times X, \operatorname{Grass}(\mathbb{P}^\infty))$ (cf. [FW-2; 4.3]). Since X is also assumed to be weakly normal, the ind-variety defining $\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))$ represents the functor $V \mapsto \operatorname{Hom}(V \times X, \operatorname{Grass}(\mathbb{P}^\infty))$, and thus we have

$$\operatorname{Hom}(\Delta^d_{top} \times X, \operatorname{Grass}(\mathbb{P}^\infty)) \;=\; Sing_d\mathcal{M}or(X, \operatorname{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}.$$

□

The utility of Theorem 1.4 is that it allows us to replace the geometrically interesting, but difficult to manipulate space $\mathcal{K}^{\mathrm{semi}}(X)$ with the space $\mathcal{K}(\Delta^\bullet_{top} \times X)$ for projective (and weakly normal) X. Although seemingly more awkward, the space

$\mathcal{K}(\Delta^\bullet_{top} \times X)$ proves much easier to compare with other theories. This theme will be exploited in the remainder of the paper.

As shown in [FW-2; 6.11], there is a natural map of $|\mathcal{I}(\Delta^\bullet)|$-spaces

$$|\operatorname{Hom}(X \times \Delta^\bullet, \mathrm{Grass}(\mathbb{P}^\infty))| \to |Sing.\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}|$$

which induces a natural map of Ω-spectra

$$\mathcal{K}(X \times \Delta^\bullet) \to \mathcal{K}^{\mathrm{semi}}(X) \to \mathcal{K}_{\mathrm{top}}(X^{an}).$$

(Observe that $\mathcal{K}(X \times \Delta^\bullet)$ receives a natural map from the algebraic K-theory spectrum of X,

$$\mathcal{K}(X) \to \mathcal{K}(X \times \Delta^\bullet),$$

which is a weak-equivalence when X is smooth.) We next verify that this map factors through the natural transformation of Theorem 1.4.

Proposition 1.5. *The natural transformation $\mathcal{K}(- \times \Delta^\bullet) \to \mathcal{K}^{\mathrm{semi}}(-)$ of functors from $(Sch/\mathbb{C})$ to Ω-spectra factors through the natural transformation $\mathcal{K}(\Delta^\bullet_{top} \times -) \to \mathcal{K}^{\mathrm{semi}}(-)$ of Theorem 1.4.*

Proof. An element of $\operatorname{Hom}(X \times \Delta^d, \mathrm{Grass}(\mathbb{P}^\infty))$ naturally defines a continuous map

$$\Delta^d_{top} \subset (\Delta^d)^{\mathrm{an}} \to \mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))$$

using [FW-1; §1]. Thus, for each $d \geq 0$ the natural map of sets $\operatorname{Hom}(X \times \Delta^d, \mathrm{Grass}(\mathbb{P}^\infty)) \to Sing_d(\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}})$ determining $\mathcal{K}(X \times \Delta^\bullet) \to \mathcal{K}^{\mathrm{semi}}(X)$ factors as

$$\begin{aligned}\operatorname{Hom}(X \times \Delta^d, \mathrm{Grass}(\mathbb{P}^\infty)) \to & \varinjlim_{(\Delta^d_{top} \to U^{an}) \in Var^{\Delta^d_{top}}} \operatorname{Hom}(U \times X, \mathrm{Grass}(\mathbb{P}^\infty)) \\ & \to Sing_d(\mathcal{M}or(X, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}}).\end{aligned}$$

□

§2. Generalized flag varieties

For a topological space T, we let $\mathcal{K}_{\mathrm{top}}(T)$ denote the connective K-theory of T, so that as a space $\mathcal{K}_{\mathrm{top}}(T)$ is -1-connected. In fact, we define $\mathcal{K}_{\mathrm{top}}(T)$ as the homotopy-theoretic group completion of the $\mathcal{I}$-space

$$\mathcal{M}\mathrm{aps}(T, \mathrm{Grass}(\mathbb{P}^\infty)^{\mathrm{an}}) = \coprod_{m \geq 0} \varinjlim_N \mathcal{M}\mathrm{aps}(T, \mathrm{Grass}_m(\mathbb{P}^N)^{\mathrm{an}})$$

where $\mathcal{M}$aps signifies the mapping space endowed with the compact open topology. Equivalently, we could define $\mathcal{K}_{\mathrm{top}}(T)$ by first taking the associated simplicial set

$$d \mapsto \mathrm{Maps}(\Delta^d_{top} \times T, \mathrm{Grass}(\mathbb{P}^\infty)^{an})$$

and then forming its homotopy-theoretic group completion as an $\mathcal{I}(\Delta^\bullet_{top})$-space. Note that the the q^{th} homotopy group of $\mathcal{K}_{\mathrm{top}}(T)$ is written $K^{-q}_{\mathrm{top}}(T)$.

If we take T to be the analytic realization $Y^{\rm an}$ of a complex variety, then using Theorem 1.4 and the natural continuous map

$$\mathcal{M}or(Y, \mathrm{Grass}(\mathbb{P}^\infty)) \to \mathcal{M}\mathrm{aps}(Y^{\rm an}, \mathrm{Grass}(\mathbb{P}^\infty)^{\rm an}),$$

we see that there are evident natural maps of simplicial sets

$$\begin{aligned}\mathrm{Hom}(\Delta^\bullet_{top} \times Y, \mathrm{Grass}(\mathbb{P}^\infty)) &\to \mathrm{Maps}(\Delta^\bullet_{top}, \mathcal{M}or(Y, \mathrm{Grass}(\mathbb{P}^\infty))^{\rm an}) \\ &\to \mathrm{Maps}(\Delta^\bullet_{top} \times Y^{\rm an}, \mathrm{Grass}(\mathbb{P}^\infty)^{an}),\end{aligned}$$

which give maps of $\mathcal{I}(\Delta^\bullet_{top})$-spaces upon taking geometric realizations. Taking homotopy-theoretic group completions, we obtain the natural maps

$$\mathcal{K}(\Delta^\bullet_{top} \times Y) \to \mathcal{K}^{\rm semi}(Y) \to \mathcal{K}_{\rm top}(Y^{\rm an}). \tag{2.0}$$

In [FW-2; 7.6], we show that the map

$$\mathcal{K}^{\rm semi}(Y) \to \mathcal{K}_{\rm top}(Y^{\rm an})$$

is a weak equivalences when Y is projective space or a projective bundle over a smooth, complete curve. In this section we apply Theorem 1.4 to extend this result to the case where Y is any quotient of a certain type of a classical group scheme by a parabolic subgroup scheme (or a bundle of such over a smooth, complete curve). This result can be seen as an extension to such Y of the stabilization of Kirwan's analysis of $\mathcal{M}or(C, \mathrm{Grass}(\mathbb{P}^\infty))^{\rm an}$, where C is a projective smooth curve (cf. [K], [FW-2; 7.5]).

We require the following result of I. Panin.

Theorem 2.1. *(Panin* [P; 5.2, 5.4]*) Let G denote one of the following group schemes over $\mathbb{Z}$: $GL_{n,\mathbb{Z}}, SL_{n,\mathbb{Z}}, Spin_{n,\mathbb{Z}}$ or $Sp_{2n,\mathbb{Z}}$. Choose a split Borel subgroup scheme B of G, and consider a parabolic subgroup scheme $P \subset G$ containing B. Then for any scheme S and any G-torsor $f : \mathcal{G} \to S$ with associated generalized flag fiber bundle $g : \mathcal{G}/P \to S$, the product structure in K-theory induces an isomorphism*

$$K_0(\mathcal{G}/P) \otimes_{K_0(S)} K_*(S) \xrightarrow{\cong} K_*(\mathcal{G}/P).$$

Moreover, $K_0(\mathcal{G}/P)$ is a free $K_0(S)$-module with basis consisting of vector bundles $E_P(\lambda)$ associated with dominant weights λ of P in a certain subset $\Lambda(P) \subset X(P)^+$.

Panin's theorem easily gives us further examples of projective varieties X for which the natural map $\mathcal{K}^{\rm semi}(X) \to \mathcal{K}_{\rm top}(X^{an})$ is a weak equivalence. This is a strengthened version of a theorem of Cohen and Lima-Filho [CL] who prove that $\pi_0\mathcal{K}^{semi}(X) \to \pi_0\mathcal{K}_{top}(X^{an})$ is an isomorphism for X a flag variety.

Theorem 2.2. *Let X be the generalized flag variety $G_{\mathbb{C}}/P_{\mathbb{C}}$, where G is one of the classical group schemes over $\mathbb{Z}$ considered in Theorem 2.1 and $P \subset G$ is a parabolic subgroup. Then the natural map*

$$\mathcal{K}^{\rm semi}(X) \to \mathcal{K}_{\rm top}(X^{an})$$

is a weak equivalence.

More generally, if Y is a projective variety such that $\mathcal{K}^{\text{semi}}(Y) \to \mathcal{K}_{\text{top}}(Y^{an})$ is a weak equivalence (e.g., a smooth projective curve or a generalized flag variety as above) and if X_Y is a generalized flag bundle over Y associated to a G-torsor $\mathcal{G} \to Y$ for G as above, then

$$\mathcal{K}^{\text{semi}}(X_Y) \to \mathcal{K}_{\text{top}}(X_Y^{an})$$

is a weak equivalence.

Proof. Observe that if $Y = \operatorname{Spec} \mathbb{C}$, then the natural map $\mathcal{K}^{\text{semi}}(Y) \to \mathcal{K}_{\text{top}}(Y^{an})$ is a weak equivalence, so that the first assertion is a special case of the second.

For any quasi-projective variety U, Theorem 2.1 applied to the generalized flag bundle $U \times X_Y \to U \times Y$ associated to the G-torsor $U \times \mathcal{G} \to U \times Y$ shows that there is a weak equivalence

$$\mathcal{K}(U \times Y)^{\times k} \xrightarrow{\sim} \mathcal{K}(U \times X_Y), \tag{2.2.1}$$

where $k = \operatorname{rank}_{K_0(Y)} K_0(X_Y)$. Further, this weak equivalence is defined using the vector bundles $E_P(\lambda)$ on $G_{\mathbb{C}}/P_{\mathbb{C}}$ of Theorem 2.1, and thus is natural in the variable U. Taking limits over $Var^{\Delta^d_{top}}$, we obtain a natural weak equivalence

$$\mathcal{K}(\Delta^d_{top} \times Y)^{\times k} \xrightarrow{\sim} \mathcal{K}(\Delta^d_{top} \times X_Y)$$

and thus a weak equivalence

$$\mathcal{K}(\Delta^\bullet_{top} \times Y)^{\times k} \xrightarrow{\sim} \mathcal{K}(\Delta^\bullet_{top} \times X_Y) \xrightarrow{\sim} \mathcal{K}^{\text{semi}}(X_Y).$$

Similarly, and more classically (cf. [Pi; 3], [AH; 3.6]), the same vector bundles $E_P(\lambda)$ viewed as topological vector bundles over $(G_{\mathbb{C}}/P_{\mathbb{C}})^{an}$ determine a natural weak equivalence

$$\mathcal{K}_{\text{top}}(Y^{\text{an}})^{\times k} \xrightarrow{\sim} \mathcal{K}_{\text{top}}(X_Y^{\text{an}})$$

fitting in the commutative square

$$\begin{array}{ccc} \mathcal{K}^{\text{semi}}(Y)^{\times k} & \xrightarrow{\sim} & \mathcal{K}^{\text{semi}}(X_Y) \\ \downarrow & & \downarrow \\ \mathcal{K}_{\text{top}}(Y^{\text{an}})^{\times k} & \xrightarrow{\sim} & \mathcal{K}_{\text{top}}(X_Y^{\text{an}}) \end{array} .$$

□

The following corollary is a translation of Theorem 2.2 in terms of mapping spaces, thereby relating this theorem to earlier work of Kirwan and others. We recall from [FW-2; §3] the continuous map

$$\alpha_L : \mathcal{M}or(Y, \text{Grass}(\mathbb{P}^\infty))^{\text{an}} \to \mathcal{M}or(Y, \text{Grass}(\mathbb{P}^\infty))^{\text{an}}$$

associated to a quotient $\mathcal{O}_X^m \twoheadrightarrow L$, where L is an ample line bundle on Y. The map α_L sends a point $\mathcal{O}_Y^\infty \twoheadrightarrow E$ to the quotient determined by the composition of

$$\mathcal{O}_Y^\infty \cong \mathcal{O}_Y^\infty \oplus \mathcal{O}_Y^m \twoheadrightarrow E \oplus L,$$

where the displayed isomorphism is defined by "interleaving" as in [FW-2; 3.3]. Replacing all algebraic vector bundles with topological ones, we have a similarly defined continuous map

$$\alpha_L^{top} : \mathcal{M}\text{aps}(Y^{\text{an}}, \text{Grass}(\mathbb{P}^\infty)^{\text{an}}) \to \mathcal{M}\text{aps}(Y^{\text{an}}, \text{Grass}(\mathbb{P}^\infty)^{\text{an}}).$$

Corollary 2.3. *Let Y be as in Theorem 2.2 and let α_L, α_L^{top} be defined as above for some quotient $\mathcal{O}_Y^m \twoheadrightarrow L$ with L an ample line bundle on Y. Then the diagram of spaces*

$$\begin{array}{ccc}
\mathcal{M}or(Y, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}} & \xrightarrow{\subset} & \mathcal{M}\mathrm{aps}(Y^{\mathrm{an}}, \mathrm{Grass}(\mathbb{P}^\infty)^{an}) \\
{\scriptstyle \alpha_L}\downarrow & & {\scriptstyle \alpha_L^{top}}\downarrow \\
\mathcal{M}or(Y, \mathrm{Grass}(\mathbb{P}^\infty))^{\mathrm{an}} & \xrightarrow{\subset} & \mathcal{M}\mathrm{aps}(Y^{\mathrm{an}}, \mathrm{Grass}(\mathbb{P}^\infty)^{an}) \\
{\scriptstyle \alpha_L}\downarrow & & {\scriptstyle \alpha_L^{top}}\downarrow \\
\vdots & & \vdots
\end{array}$$

induces a weak-equivalence on the mapping telescopes of the columns.

Proof. The mapping telescope of the left column is shown to be weakly equivalent to the homotopy-theoretic group completion of $\mathcal{M}or(Y, \mathrm{Grass}(\mathbb{P}^\infty))$ in [FW-2; 3.4]. The same proof applies to show the mapping telescope of the right column is weakly equivalent to the homotopy-theoretic group completion of $\mathcal{M}\mathrm{aps}(Y^{\mathrm{an}}, \mathrm{Grass}(\mathbb{P}^\infty)^{an})$. □

§3. K-THEORIES WITH FINITE COEFFICIENTS

In this section, we show $\mathcal{K}^{\mathrm{semi}}(-;\mathbb{Z}/n)$ and $\mathcal{K}^{\mathrm{alg}}(-;\mathbb{Z}/n)$ agree on all projective varieties for any $n > 0$. More generally, we verify that $\pi_*(\mathcal{K}(\Delta^\bullet_{top} \times -;\mathbb{Z}/n))$ and $K_*(-;\mathbb{Z}/n)$ agree on all quasi-projective varieties. A major ingredient in our proof of these results is a version of rigidity (Proposition 3.2) adapted from [SV].

We begin with the definition of a "pseudo pretheory", a slight generalization of Voevodsky's pretheories given in [FS]. This generalization is required in view of the fact that $K_*(-)$ is not a pretheory, but only a pseudo pretheory.

Definition 3.1. [FS] *A pseudo pretheory F on $(Sm/\mathbb{C})$ is an abelian-group valued contravariant functor equipped with transfer maps $Tr_D : F(X) \to F(S)$ for any (relative) smooth affine curve X/S and any effective Cartier divisor $D \subset X$ finite and surjective over S. These transfer maps are required to satisfy*

1. *Tr_D is compatible with pullbacks,*
2. *$Tr_D + Tr_{D'} = Tr_{D.D'}$ whenever the restriction of the line bundle I_D to D' is trivial, and*
3. *$Tr_D = i^*$, if D is the divisor associated to an S-point $i : S \hookrightarrow X$ of X.*

A pseudo pretheory F is said to be homotopy invariant if $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism for all varieties X.

Given a presheaf F on $Sm/\mathbb{C}$, we write $F(\Delta^\bullet)$ (respectively, $F(- \times \Delta^\bullet)$) for the normalized chain complex associated to the simplicial abelian group $d \mapsto F(\Delta^d)$ (resp., to the simplicial presheaf $d \mapsto F(- \times \Delta^d)$). Also, we write $F^\sim_{\text{ét}}$ for the sheafification of F on the big site $(Sm/\mathbb{C})_{\text{ét}}$ and $F(- \times \Delta^\bullet)^\sim_{\text{ét}}$ for complex of sheaves formed by taking the degree-wise sheafification of $F(- \times \Delta^\bullet)$.

The following proposition is essentially [SV; 7.6] stated in the context of pseudo pretheories.

Proposition 3.2. *Suppose $F : Sch/\mathbb{C} \to Ab$ is a presheaf whose restriction to $Sm/\mathbb{C}$ is a homotopy invariant pseudo pretheory. Then for any positive integer n the natural maps*

$$F(\Delta^\bullet) \to F(- \times \Delta^\bullet)^\sim_{\text{ét}} \leftarrow F^\sim_{\text{ét}}$$

induce isomorphisms

$$\mathrm{Ext}^*_{Ab}(F(\Delta^\bullet), \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{Ext}^*_{\text{ét}}(F(- \times \Delta^\bullet)^\sim_{\text{ét}}, \mathbb{Z}/n) \xleftarrow{\sim} \mathrm{Ext}^*_{\text{ét}}(F^\sim_{\text{ét}}, \mathbb{Z}/n),$$

for all $*$.

Proof. Note that since the étale topology for $\mathrm{Spec}\,\mathbb{C}$ is trivial, we have

$$\mathrm{Ext}^*_{\text{ét}}(F(\Delta^\bullet), \mathbb{Z}/n) \cong \mathrm{Ext}^*_{Ab}(F(\Delta^\bullet), \mathbb{Z}/n),$$

where $F(\Delta^\bullet)$ is regarded as a chain complex of constant sheaves. Thus, it suffices to show that the application of the functor $\mathrm{Ext}^*_{\text{ét}}(-, \mathbb{Z}/n)$, from chain complexes of étale sheaves on $Sch/\mathbb{C}$ to graded abelian groups, to the maps

$$F(\Delta^\bullet) \to F(- \times \Delta^\bullet)^\sim_{\text{ét}} \leftarrow F^\sim_{\text{ét}}$$

induces isomorphisms

$$\mathrm{Ext}^*_{\text{ét}}(F(\Delta^\bullet), \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{Ext}^*_{\text{ét}}(F(- \times \Delta^\bullet)^\sim_{\text{ét}}, \mathbb{Z}/n) \xleftarrow{\sim} \mathrm{Ext}^*_{\text{ét}}(F^\sim_{\text{ét}}, \mathbb{Z}/n).$$

(Here, we regard $F^\sim_{\text{ét}}$ as a chain complex concentrated in degree 0.)

The proof is parallel to the proof of [SV; 7.6], once we verify that certain properties of a "presheaf with transfer" of [SV; 4.1] are satisfied by a pseudo pretheory. We give references for these properties and sketch the proof.

By [FS; 10.1], associated to any smooth affine curve $X \to S$ with S smooth and semi-local, we have a natural pairing

$$H_0^{sing}(X/S) \otimes G(X) \to G(S)$$

for any homotopy invariant pseudo pretheory G. This natural pairing is all that is needed for the proofs of [SV; 4.3] and [SV; 4.4] to hold for pseudo pretheories. Therefore, if in addition $nG = 0$, we have that

$$G(X^h_x) = G(\mathrm{Spec}\,\mathbb{C}), \tag{3.2.1}$$

where X^h_x is the Henselization of any smooth variety X at a closed point x. Now we claim that the analogue of [SV; 4.5] holds as well; namely, given a homotopy invariant pseudo pretheory G, we have a natural isomorphism

$$\mathrm{Ext}^*_{\text{ét}}(G^\sim_{\text{ét}}, \mathbb{Z}/n) \cong \mathrm{Ext}^*_{Ab}(G(\mathrm{Spec}\,\mathbb{C}), \mathbb{Z}/n). \tag{3.2.2}$$

The proof of (3.2.2) parallels the proof of [SV; 4.5]; specifically, observe that if we take L to be the cokernel of the injective map of presheaves $G(\mathrm{Spec}\,\mathbb{C}) \to G$, and then take M to be either the kernel or cokernel of the map of presheaves $L \xrightarrow{\cdot n} L$, then M is a pseudo pretheory and thus (3.2.1) implies $M^\sim_h$ vanishes. Here, $M^\sim_h$ denotes sheafification in the so-called "h-topology" (cf. [SV; §10]), and we use the fact that every complex variety is smooth "locally in the h-topology" (That is, using resolutions of singularities [H1], every variety admits a covering in the h-topology by smooth varieties.) It follows that $L_h^\sim$ is uniquely n-divisible and so we have

$$\mathrm{Ext}_h^*(G_h^\sim, \mathbb{Z}/n) \cong \mathrm{Ext}_h^*(G(\mathrm{Spec}\,\mathbb{C}), \mathbb{Z}/n) \cong \mathrm{Ext}_{Ab}(G(\mathrm{Spec}\,\mathbb{C}), \mathbb{Z}/n).$$

Finally, by [SV; 10.10], there is a natural isomorphism

$$\mathrm{Ext}_{et}^*(G_{\text{ét}}^\sim, \mathbb{Z}/n) \cong \mathrm{Ext}_h^*(G_h^\sim, \mathbb{Z}/n)$$

and (3.2.2) follows.

Now the fact that

$$F(\Delta^\bullet) \longrightarrow F(- \times \Delta^\bullet)_{\text{ét}}^\sim$$

becomes an isomorphism after applying $\mathrm{Ext}_{\text{ét}}^*(-, \mathbb{Z}/n)$ follows by a spectral sequence argument just as in the proof of [SV; 7.6]. The key point is that since the homology presheaves of $F(- \times \Delta^\bullet)$ are homotopy invariant pseudo pretheories, equation (3.2.2) applies with G taken to be $H_q F(- \times \Delta^\bullet)$ for any q.

The fact that $F_{\text{ét}}^\sim \to F(- \times \Delta^\bullet)_{\text{ét}}^\sim$ induces an isomorphism on $\mathrm{Ext}_{\text{ét}}^*(-, \mathbb{Z}/n)$ is a consequence of [SV; 7.1, 10.10]. □

Using the technique of [SV; §9], we conclude a "topological analogue" of Proposition 3.2.

Proposition 3.3. *Let $F : Sch/\mathbb{C} \to Ab$ be a presheaf of abelian groups which restricts to a homotopy invariant pseudo pretheory on $Sm/\mathbb{C}$. Then the natural map*

$$\mathrm{Ext}_{Ab}^*(F(\Delta^\bullet), \mathbb{Z}/n) \to \mathrm{Ext}_{Ab}^*(F(\Delta_{top}^\bullet), \mathbb{Z}/n)$$

is an isomorphism.

Proof. This is essentially the argument found in [SV; §9].

Let CW_{lh} denote the site consisting of triangulable spaces equipped with the Grothendieck topology of local homeomorphisms. There is a natural morphism of sites

$$j : CW_{lh} \to (Sch/\mathbb{C})_{\text{ét}}$$

since every quasi-projective complex variety admits an open triangulation by [H2]. Given a presheaf (respectively, sheaf) F on $Sch/\mathbb{C}$, write $j^p F$ (respectively, $j^* F$) for the pullback of F to CW_{lh}.

By Proposition 3.2, the natural maps

$$F(\Delta^\bullet) \to F(- \times \Delta^\bullet)_{\text{ét}}^\sim \leftarrow F_{\text{ét}}^\sim$$

of complexes of étale sheaves induce isomorphisms upon applying $\mathrm{Ext}_{\text{ét}}^*(-, \mathbb{Z}/n)$. If we apply the functor j^* to these two arrows, the resulting maps induce isomorphisms on $\mathrm{Ext}_{lh}^*(-, \mathbb{Z}/n)$ by the "classical comparison theorem". That is, using [SGA4; XVI.4.1] we have that the étale sheaf $\mathbb{R}^i j_* \mathbb{Z}/n$ is given by the étale sheafification of the presheaf

$$U \mapsto H_{lh}^i(U^{\mathrm{an}}, \mathbb{Z}/n) \cong H_{\text{ét}}^i(U, \mathbb{Z}/n),$$

and is thus isomorphic to $\mathbb{Z}/n$ if $i = 0$ and 0 for $i > 0$. Consequently, we have $\mathrm{Ext}_{lh}^*(j^* G, \mathbb{Z}/n) \cong \mathrm{Ext}_{\text{ét}}^*(G, \mathbb{R} j_* \mathbb{Z}/n) \cong \mathrm{Ext}_{\text{ét}}^*(G, \mathbb{Z}/n)$ for any étale sheaf G.

Regarding $F(\Delta^\bullet)$ as a complex of constant étale sheaves, we observe the existence of natural maps $j^*(F(\Delta^\bullet)) \to (j^pF)(\Delta^\bullet_{top})$, $j^*(F(-\times\Delta^\bullet)^\sim_{\text{ét}}) \to (j^pF)(-\times\Delta^\bullet_{top})^\sim_{lh}$, and $j^*(F^\sim_{\text{ét}}) \to (j^pF)^\sim_{lh}$. (Note that $(j^pF)(\Delta^n_{top})$ is by definition the abelian group $\varinjlim_{\Delta^n_{top}\to U^{\text{an}}} F(U)$, regarded as a constant sheaf.) The map $j^*(F^\sim_{\text{ét}}) \to (j^pF)^\sim_{lh}$ is in fact an isomorphism by the very definition of j^*.

We obtain the following diagram of complexes of sheaves:

$$\begin{array}{ccccc} j^*F(\Delta^\bullet) & \longrightarrow & j^*(F(-\times\Delta^\bullet)^\sim_{\text{ét}}) & \longleftarrow & j^*(F^\sim_{\text{ét}}) \\ \downarrow & & \downarrow & & \cong\downarrow \\ (j^pF)(\Delta^\bullet_{top}) & \longrightarrow & (j^pF)(-\times\Delta^\bullet_{top})^\sim_{lh} & \longleftarrow & (j^pF)^\sim_{lh}. \end{array}$$

All four horizontal arrows induce isomorphisms after applying $\mathrm{Ext}^*_{lh}(-,\mathbb{Z}/n)$ – the top map in each square induces an isomorphism by the preceding argument and bottom map in each square does so by [SV; 8.1]. Since the right-most vertical map is an isomorphism, we conclude that the map of complexes of constant sheaves

$$F(\Delta^\bullet) \equiv j^*F(\Delta^\bullet) \to (j^pF)(\Delta^\bullet_{top}) \equiv F(\Delta^\bullet_{top})$$

induces an isomorphism after applying $\mathrm{Ext}^*_{lh}(-,\mathbb{Z}/n)$. This immediately gives the proposition, since these complexes consist of constant sheaves and $\mathrm{Ext}^*_{lh}(-,\mathbb{Z}/n) \cong \mathrm{Ext}^*_{Ab}(-,\mathbb{Z}/n)$ for such sheaves. □

Proposition 3.4. *For any $q \geq 0, n > 0$, and any variety Y the presheaf $K_q(-\times Y;\mathbb{Z}/n)$ is a homotopy invariant pseudo pretheory.*

Proof. The proof of [FS; 11.3] suffices to show $K_q(-\times Y;\mathbb{Z}/n)$ is a pseudo pretheory, where the transfer map

$$Tr_D : K_q(X\times Y;\mathbb{Z}/n) \to K_q(S\times Y;\mathbb{Z}/n)$$

associated to a relative Cartier divisor $i : D \subset X$ of a smooth, affine curve $\pi : X \to S$ is defined by taking homotopy groups with coefficients of the composition

$$\mathcal{K}(X\times Y) \xrightarrow{i^*} \mathcal{K}(D\times Y) \xrightarrow{p_*} \mathcal{K}(S\times Y).$$

Here $p : D \to S$ is the restriction of π to D and is necessarily flat (as seen by the local criterion of flatness). The homotopy invariance of $K_q(-\times Y;\mathbb{Z}/n)$ for $n > 0$ is given by [W1; 3.4]. □

We will write $K_q(-\times X;\mathbb{Z}/n)(\Delta^\bullet_{top})$ to refer to the normalized chain complex of abelian groups associated to the simplicial abelian group

$$d \mapsto K_q(\Delta^d_{top}\times X;\mathbb{Z}/n).$$

Theorem 3.5. *For any variety X, any $q \geq 0$, and any $n > 0$, the chain complex $K_q(-\times X;\mathbb{Z}/n)(\Delta^\bullet_{top})$ is quasi-isomorphic to the constant chain complex $K_q(X;\mathbb{Z}/n)$.*

Proof. Observe that, in general, a map between chain complexes of $\mathbb{Z}/n$-modules $A_\bullet \to B_\bullet$ which induces an isomorphism

$$\mathrm{Ext}^q_{Ab}(A_\bullet,\mathbb{Z}/n) \to \mathrm{Ext}^q_{Ab}(B_\bullet,\mathbb{Z}/n)$$

for all q is necessarily a quasi-isomorphism. For let $C_\bullet$ be the mapping cone of $A_\bullet \to B_\bullet$. Then $C_\bullet$ is a complex of $\mathbb{Z}/n$-modules so that $\mathrm{Ext}^q_{Ab}(C_\bullet, \mathbb{Z}/n) = 0$ for all q. By considering the long exact sequence associated to

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and the fact that $C_\bullet$ consists of $\mathbb{Z}/n$-modules, it follows that $\mathrm{Ext}^q_{Ab}(C_\bullet, \mathbb{Q}/\mathbb{Z}) = 0$ for all q. Since $\mathbb{Q}/\mathbb{Z}$ is injective, we have that $\mathrm{Hom}(\mathrm{H}^q(C_\bullet), \mathbb{Q}/\mathbb{Z}) = 0$ and hence $\mathrm{H}^q(C_\bullet) = 0$ for all q.

It follows from this observation and Propositions 3.3 and 3.4 that the natural map

$$K_q(- \times X; \mathbb{Z}/n)(\Delta^\bullet) \to K_q(- \times X; \mathbb{Z}/n)(\Delta^\bullet_{top})$$

is a quasi-isomorphism. The result now follows from the homotopy invariance of $K_q(- \times X; \mathbb{Z}/n)$. □

We use the following well-known spectral sequence for the homotopy groups of a simplicial space.

Lemma 3.6. [BF; B.5] *Consider a simplicial spectrum $\{d \mapsto \mathcal{S}(d)\}$ with the property that $\mathcal{S}(d)$ is $N - d$ connected for some integer N. Then for any $n > 0$, there is a natural convergent first quadrant spectral sequences of homological type of the form*

$$E^2_{s,t} = \pi_s\{d \mapsto \pi_t(\mathcal{S}(d); \mathbb{Z}/n)\} \Longrightarrow \pi_{s+t}(|d \mapsto \mathcal{S}(d)|; \mathbb{Z}/n).$$

Theorem 3.7. *For any quasi-projective variety X, and any $q \geq 0, n > 0$, there is a natural isomorphism*

$$K_q(X; \mathbb{Z}/n) \cong \mathcal{K}_q(\Delta^\bullet_{top} \times X; \mathbb{Z}/n).$$

Proof. Observe that there exists a natural map of simplicial spectra

$$\{d \mapsto \mathcal{K}(X \times \Delta^d)\} \to \{d \mapsto \mathcal{K}(\Delta^d_{top} \times X)\}$$

(induced by the canonical map $\mathcal{K}(X \times \Delta^d) \to \mathcal{K}(\Delta^d_{top} \times X)$), and hence a natural map of simplicial spectra

$$\mathcal{K}(X; \mathbb{Z}/n) \sim \{d \mapsto \mathcal{K}(X \times \Delta^d; \mathbb{Z}/n)\} \to \{d \mapsto \mathcal{K}(\Delta^d_{top} \times X; \mathbb{Z}/n)\}$$

obtained by smashing with the mod-n Moore spectrum. Thus the theorem follows by comparing the map of spectral sequences given by Lemma 3.6 associated to this map of simplicial spectra and applying Theorem 3.5. □

Corollary 3.8. *For X a projective variety and $n > 0$, the natural map induces an isomorphism*

$$K_q(X; \mathbb{Z}/n) \cong K^{\mathrm{semi}}_q(X; \mathbb{Z}/n)$$

for all $q \geq 0$.

Proof. If X is weakly normal, the result follows immediately from Theorems 1.4 and 3.7. More generally, if $X^w \to X$ is the weak normalization of X, then

$$K^{\mathrm{semi}}_q(X; \mathbb{Z}/n) \to K^{\mathrm{semi}}_q(X^w; \mathbb{Z}/n)$$

is an isomorphism by definition and

$$K_q(X;\mathbb{Z}/n) \to K_q(X^w;\mathbb{Z}/n)$$

is an isomorphism by [W2;1.6,3.2], and so the result follows by naturality. □

We obtain the following result, which in particular verifies the Quillen-Lichtenbaum conjecture for projective smooth complex curves. (The Quillen-Lichtenbaum conjecture in this case was previously established by [Su], [PW] using methods different than ours.)

Corollary 3.9. *For X a projective, smooth curve, the maps*

$$K_*(X;\mathbb{Z}/n) \to K_*^{\mathrm{semi}}(X;\mathbb{Z}/n) \to K_{\mathrm{top}}^{-*}(X^{an};\mathbb{Z}/n)$$

are isomorphisms.

Proof. This follows immediately from Corollary 3.8 and [FW-2; 7.5]. □

§4 Inverting the Bott element.

In this section we prove that the natural map

$$\bigoplus_{q\in\mathbb{Z}} K_q^{\mathrm{semi}}(X)[\frac{1}{\beta}] \to \bigoplus_{q\in\mathbb{Z}} K_{\mathrm{top}}^{-q}(X^{\mathrm{an}}) \tag{4.0}$$

is an isomorphism for any smooth, projective variety X, under the assumption that it induces an isomorphism after tensoring with $\mathbb{Q}$. Here, $\beta \in K_2^{\mathrm{semi}}(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Z}$ is the so-called "Bott element" – i.e., an explicit generator of $K_2^{\mathrm{semi}}(\mathrm{Spec}\,\mathbb{C})$ – which acts upon the graded ring $\bigoplus_{q\geq 0} K_q^{\mathrm{semi}}(X)$ via a cup product. The notation $\bigoplus_{q\in\mathbb{Z}} K_q^{\mathrm{semi}}(X)[\frac{1}{\beta}]$ thus refers to the $\mathbb{Z}$-graded ring resulting from the inversion of the action of β.

The map (4.0) is asserted to induce an isomorphism rationally by Cohen and Lima-Filho in the recent preprint [CL; 35] by comparing it with the corresponding map from rational morphic cohomology to rational singular cohomology under the Chern character isomorphism. Here, we show that one may extend this result to the integral case by applying our Theorem 3.7 and the well-known result of Thomason [T; 4.11] which identifies Bott-inverted algebraic K-theory with finite coefficients with topological K-theory with finite coefficients.

Implicit in our consideration of $K_q^{\mathrm{semi}}(X)[\frac{1}{\beta}]$ is the observation that $K_*^{semi}(X)$ admits a natural ring structure with the property that the natural map $K_*^{semi}(X) \to K_{\mathrm{top}}^{-*}(X^{an})$ is a ring homomorphism. This was asserted without proof in [CL], but a rigorous development of ring structures for $K_*^{semi}(X)$ and $K_{\mathrm{top}}^{-*}(X^{an})$ and a proof that the natural map is a ring homomorphism may be found in [FW-4; App. A].

Theorem 4.1. *For any quasi-projective variety X and any integer $n > 1$, the natural maps*

$$\mathcal{K}(X) \to \mathcal{K}(\Delta^\bullet_{top} \times X) \to \mathcal{K}_{\mathrm{top}}(X^{an})$$

induce ring homomorphisms

$$K_*(X) \to K_*(\Delta^\bullet_{top} \times X) \to K_{\mathrm{top}}^{-*}(X^{an})$$

$$K_*(X;\mathbb{Z}/n) \to K_*(\Delta^\bullet_{top} \times X;\mathbb{Z}/n) \to K^{-*}_{\mathrm{top}}(X^{an};\mathbb{Z}/n).$$

Define $\beta \in K_2^{semi}(\operatorname{Spec}\mathbb{C}) \simeq K^{-2}_{top}(pt) = K^0_{top}(S^2)$ *to be the classical topological Bott element. Then the algebraic Bott element* $\beta \in K_2(X;\mathbb{Z}/n)$ *maps to a class in* $K_2(\Delta^\bullet_{top} \times \operatorname{Spec}\mathbb{C};\mathbb{Z}/n) \simeq K^{-2}_{top}(pt;\mathbb{Z}/n)$ *which is the reduction mod-n of the topological Bott element in* $K^{-2}_{top}(pt)$.

Proof. The functor $\mathcal{K}(-)$ takes values in ring spectra (with product induced by the tensor product of projective modules), so that $\mathcal{K}(\Delta^d_{top} \times X)$ is a ring spectrum for each $d \geq 0$. Thus, the canonical map $\mathcal{K}(X) = \mathcal{K}(\Delta^0_{top} \times X) \to \mathcal{K}(\Delta^\bullet_{top} \times X)$ is a map of ring spectra. (See [FW-4; App. A] for a more rigorous treatment of the multiplicative properties of these spectra.) Moreover, the homotopy invariance of $K^{-*}_{\mathrm{top}}(-)$ implies that the canonical map $\mathcal{K}_{\mathrm{top}}(X^{an}) \to \mathcal{K}_{\mathrm{top}}(\Delta^\bullet_{top} \times X^{an})$ is a weak equivalence of ring spectra. (Here, $\mathcal{K}_{\mathrm{top}}(\Delta^\bullet_{top} \times X^{\mathrm{an}})$ is the geometric realization of the simplicial spectrum $d \mapsto \mathcal{K}_{\mathrm{top}}(\Delta^d_{top} \times X^{\mathrm{an}})$, and in particular is not defined using a direct limit as in 1.1.) Finally, the natural transformation

$$\mathcal{K}(-) \to \mathcal{K}_{\mathrm{top}}((-)^{an})$$

is induced by the functor sending an algebraic vector bundle to its underlying topological vector bundle. Since this commutes with tensor products, $\mathcal{K}(-) \to \mathcal{K}_{top}((-)^{an})$ respects ring structures, so that

$$\mathcal{K}(\Delta^\bullet_{top} \times X) \to \mathcal{K}_{\mathrm{top}}(\Delta^\bullet_{top} \times X^{an})$$

is also a map of ring spectra.

Recall that "the" algebraic Bott element in $K_2(\mathbb{C};\mathbb{Z}/n)$ actually depends on a choice of primitive n-th root of unity ζ_n; such a choice is a generator of the n-torsion subgroup of $K_1(\mathbb{C}) \simeq \mathbb{C}^*$. Thus, such a choice of generator determines a generator of the n-torsion subgroup of $\pi_1(B\mathbb{C}^*)$ which is naturally isomorphic to $\pi_2(B\mathbb{C}^*;\mathbb{Z}/n) \simeq \pi_2(K(\mathbb{Z},2);\mathbb{Z}/n)$. Since the last group is naturally isomorphic to $K^{-2}_{top}(pt;\mathbb{Z}/n)$, we conclude that the reduction of the generator of the topological Bott element is the algebraic Bott element.

These maps of ring spectra induce the indicated ring homomorphisms. Observe that because $\sqrt{-1} \in \mathbb{C}$, there is no problem with the multiplicative structure of $K_*(-;\mathbb{Z}/n)$ even when $n = 2$ (cf. [AT; 10.7]). □

We now show how the rational equivalence of [CL; 35] can be refined to a homotopy equivalence.

Theorem 4.2. *If X is a projective variety such that the natural map of ring spectra*

$$\mathcal{K}^{\mathrm{semi}}(X) \to \mathcal{K}_{\mathrm{top}}(X^{an})$$

induces a rational equivalence after inverting the Bott element:

$$\bigoplus_{q\in\mathbb{Z}} K_q^{\mathrm{semi}}(X)[\frac{1}{\beta}] \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{q\in\mathbb{Z}} K^{-q}_{\mathrm{top}}(X^{\mathrm{an}}) \otimes \mathbb{Q} \tag{4.2.1}$$

(for example, assuming the asserted result [CL; 3.5]*, this is valid if X is smooth and projective), then (4.2.1) is actually an integral isomorphism. In other words, in this situation*

$$\bigoplus_{q\in\mathbb{Z}} K_q^{\mathrm{semi}}(X)[\frac{1}{\beta}] \to \bigoplus_{q\in\mathbb{Z}} K^{-q}_{\mathrm{top}}(X^{\mathrm{an}}) \tag{4.2.2}$$

is an isomorphism of graded rings.

Proof. Since the map (4.2.2) is a rational isomorphism by hypothesis, it suffices to show

$$\bigoplus_{q\in\mathbb{Z}} K_q^{\text{semi}}(X;\mathbb{Z}/n)[\frac{1}{\beta}] \to \bigoplus_{q\in\mathbb{Z}} K_{\text{top}}^{-q}(X^{\text{an}};\mathbb{Z}/n) \tag{4.2.3}$$

is an isomorphism for all $n > 0$. By [T; 4.11], the map

$$\bigoplus_{q\in\mathbb{Z}} K_q(X;\mathbb{Z}/n)[\frac{1}{\beta}] \to \bigoplus_{q\in\mathbb{Z}} K_{\text{top}}^{-q}(X^{\text{an}};\mathbb{Z}/n)$$

is an isomorphism for all $n > 0$. Further, the natural map

$$\bigoplus_{q\geq 0} K_q(X;\mathbb{Z}/n) \to \bigoplus_{q\geq 0} K_q^{\text{semi}}(X;\mathbb{Z}/n)$$

is an isomorphism of rings by Corollary 3.8 and is sends the algebraic Bott element on the left to the mod-n reduction of the topological Bott element on the right. The result follows immediately. □

§5. K-regularity and Mayer-Vietoris for smooth varieties

In this section, we establish the Mayer-Vietoris property for Zariski covers of smooth varieties for the theory $\mathcal{K}(\Delta^{\bullet}_{top} \times -)$. Ideally, one would like to know such a descent property for $\mathcal{K}^{\text{semi}}(-)$ itself, but since these two theories are known to coincide only for projective varieties, such a goal remains elusive.

This intuition behind our proof of Zariski descent is that the simplicial space $d \mapsto \mathcal{K}(\Delta^d_{top} \times -)$ satisfies descent for each fixed degree d and thus its geometric realization ought to as well. This simple argument is complicated, however, by the fact that $\mathcal{K}(\Delta^d_{top} \times -)$ is defined in terms of maps $\Delta^d_{top} \to U^{\text{an}}$ to possibly singular varieties U. Thus, while Zariski descent for K-theory is known even for singular varieties thanks to [TT], there exists a "problem at π_0" which must be addressed. This problem is resolved with the aid of the following theorem. (A special case of this result would follow from [R1; 2.4] – however, the proof of that result is flawed.)

In the following theorem and the throughout the remainder of this section, we will be referring to the K-groups K_q for $q < 0$. We remind the reader that such groups are formed from Bass's construction [B] and are sometimes written K_q^B, although we suppress the superscript B here.

Theorem 5.1. *(Compare with* [R1; 2.4]*.) If* $V = \operatorname{Spec} R$ *is any regular affine variety then the groups*

$$K_q(\Delta^n_{top} \times V) \equiv \varinjlim_{(\Delta^n_{top}\to U^{\text{an}})\in Var^{\Delta^n_{top}}} K_q(U \times V)$$

vanish for all $n \geq 0$ *and* $q < 0$.

Proof. We first show that for any compact CW complex D, we have a natural isomorphism

$$K_q(D \times V) \cong K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R),$$

where $\mathcal{C}(D)$ is the ring of continuous complex-valued functions on D. Given any quasi-projective complex variety U, we may employ "Jouanolou's device" to find a vector bundle torsor $J \to U$ with J affine [J]. The associated map on analytic spaces is a fibration and a homotopy equivalence, and thus every continuous map $D \to U^{\mathrm{an}}$ factors through $J \to U$. This shows that the collection of maps $D \to U^{\mathrm{an}}$ with U affine forms a cofinal indexing category of Var^D. Moreover, taking just those maps of the form $D \to U^{\mathrm{an}}$, where $U = \operatorname{Spec} A$ for some finitely generated subalgebra A of $\mathcal{C}(D)$, also gives a cofinal system, and hence

$$K_q(\Delta^n_{top} \times V) \cong \varinjlim_{A \subset \mathcal{C}(D)} K_q(A \otimes_{\mathbb{C}} R) \cong K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R),$$

since K-theory commutes with filtered direct limits of rings.

Now let D be Δ^n_{top}. Elements of $K_q(D \times V)$ are represented by pairs (α, γ), where $\alpha : D \to Y^{\mathrm{an}}$ is a continuous map, Y a quasi-projective variety, and γ belongs to $K_q(Y \times V)$. Two such elements (α, γ) and $(\alpha' : D \to Z^{\mathrm{an}}, \gamma' \in K_0(Z \times V))$ are equivalent if there is a map of varieties $f : Z \to Y$ such that $f^{\mathrm{an}} \circ \alpha' = \alpha$ and $f^*(\gamma) = \gamma'$. Choose an element of $K_q(D \times V)$ represented by (α, γ). We need to show this element is equivalent to the trivial element. Observe that if Y is smooth, (α, γ) is trivial since $K_q(Y \times V) = 0$. The remainder of the proof amounts to reducing to the case Y is smooth.

If Y is not smooth, by resolution of singularities [H1] there exists a closed subscheme $Z \subset Y$ of smaller dimension and a proper map $p : \tilde{Y} \to Y$ such that $\tilde{Y}$ is smooth and the induced map $\tilde{Y} - \tilde{Z} \to Y - Z$ is an isomorphism. (Here, $\tilde{Z}$ is $p^{-1}(Z)$.) For the next stage of the proof, we show it suffices to consider the case where $\alpha : D \to Y^{\mathrm{an}}$ is the inclusion of a closed simplex of Y for some semi-algebraic triangulation of Y and that $D \cap Z^{\mathrm{an}} \subset D$ is the inclusion of a proper face of the simplex D.

Suppose E is any star-like simplicial complex (i.e., it is the star-neighborhood of one of its vertices) and assume $\{F_i\}$ is the set of maximal simplices of E. Write $E = A \cup B$ where A and B are formed by partitioning the F_i's in any non-trivial manner. Then the square

$$\begin{array}{ccc} \mathcal{C}(E) & \longrightarrow & \mathcal{C}(A) \\ \downarrow & & \downarrow \\ \mathcal{C}(B) & \longrightarrow & \mathcal{C}(A \cap B) \end{array} \tag{5.1.1}$$

is clearly a Milnor square of rings, and furthermore $\mathcal{C}(A) \to \mathcal{C}(A \cap B)$ is a split surjection since $A \cap B \hookrightarrow A$ is inclusion of contractible polyhedron. Tensoring (5.1.1) by R we obtain another split Milnor square and consequently [B; XII.8.3] a long exact sequence (recall $q \le 0$)

$$K_{q+1}(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \oplus K_{q+1}(\mathcal{C}(B) \otimes_{\mathbb{C}} R) \twoheadrightarrow K_{q+1}(\mathcal{C}(A \cap B) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(E) \otimes_{\mathbb{C}} R)$$
$$\to K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \oplus K_q(\mathcal{C}(B) \otimes_{\mathbb{C}} R) \to \cdots,$$

from which the injectivity of

$$K_q(\mathcal{C}(E) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \oplus K_q(\mathcal{C}(B) \otimes_{\mathbb{C}} R)$$

follows. By induction on the number of simplices (and using the fact that A and B are each star-like) we get that

$$K_q(\mathcal{C}(E) \otimes_{\mathbb{C}} R) \to \bigoplus_i K_q(\mathcal{C}(F_i) \otimes_{\mathbb{C}} R) \tag{5.1.2}$$

is an injection.

Now choose a semi-algebraic triangulation of Y such that Z is a subcomplex [H2]. Subdivide D by taking repeated barycentric subdivisions so that the image of each n-dimensional simplex of D lies in a star neighborhood of some vertex of Y. Let $\{T_i\}$ be the collection of n-simplices of this subdivision of D. We claim

$$K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to \bigoplus_i K_q(\mathcal{C}(T_i) \otimes_{\mathbb{C}} R) \tag{5.1.3}$$

is an injection. By induction, it suffices to consider the case where a single barycentric subdivision has been performed. But then D is star-like and thus the injectivity of (5.1.2) applies to show (5.1.3) is an injection as well.

Since (5.1.3) is an injection, we may replace D by each T_i in a sufficiently fine barycentric subdivision, and thus it suffices to assume the image of $\alpha : D \to Y^{\mathrm{an}}$ lies in a star neighborhood E of some vertex of Y. But then (α, γ) lifts to the element $(i : E \hookrightarrow Y^{\mathrm{an}}, \gamma)$ of $K_q(\mathcal{C}(E) \otimes R)$, and so we may as well assume α is the inclusion of a star-neighborhood of Y. Using the injectivity of (5.1.2) again, we may as well assume $\alpha : D \hookrightarrow Y^{\mathrm{an}}$ is the inclusion of a single simplex of Y. Finally, $Z^{\mathrm{an}} \cap D$ is a union of some of the proper faces of D. Now replace the chosen semi-algebraic triangulation of Y with its barycentric subdivision. Using the injectivity of (5.1.2) yet again, we may replace $\alpha : D \hookrightarrow Y^{\mathrm{an}}$ with the inclusion of one of the n-simplices $D' \hookrightarrow Y^{\mathrm{an}}$ for this new triangulation. But now we claim that $D' \cap Z^{\mathrm{an}}$ consists of a single lower-dimensional simplex. For the vertices of D' (which are the barycenters of certain faces of D) are totally ordered by inclusion of the corresponding faces of D, and a vertex $v \in D'$, which is the barycenter of a face $F \subset D$, belongs to Z if and only if $F \subset Z$. Thus if $v \in D'$ is a vertex belonging to Z, then $w \in Z$ for all $w \leq v$, from which it follows that $D' \cap Z$ is a single face of D'.

We have thus reduced the proof to consideration of the case where D is an n-simplex of Y and $A \equiv D \cap Z^{\mathrm{an}}$ is a proper face. Let $\tilde{D} = p^{-1}(D)$ and $\tilde{A} = p^{-1}(A)$. These compact sets form a square

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\subset} & \tilde{D} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{\subset} & D \end{array}$$

which is bicartesian and whose horizontal maps are NDR subspace inclusions. Thus the square of rings

$$\begin{array}{ccc} \mathcal{C}(D) & \longrightarrow & \mathcal{C}(A) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}(\tilde{D}) & \longrightarrow & \mathcal{C}(\tilde{A}) \end{array}$$

is a Milnor square. Since $-\otimes_{\mathbb{C}} R$ is an exact functor, tensoring the above square with R results in another Milnor square, and consequently we have the long exact sequence

$$K_{q+1}(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \oplus K_{q+1}(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \to K_{q+1}(\mathcal{C}(\tilde{A}) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \oplus K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \to \cdots . \quad (5.1.4)$$

We claim the map

$$K_{q+1}(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \to K_{q+1}(\mathcal{C}(\tilde{A}) \otimes_{\mathbb{C}} R)$$

is surjective. To see this, observe that $\tilde{A}$ is the deformation retract of some compact neighborhood N in $\tilde{D}$. In particular $\mathcal{C}(N) \to \mathcal{C}(\tilde{A})$ is a split surjection of rings and thus $K_{q+1}(\mathcal{C}(N) \otimes_{\mathbb{C}} R) \to K_{q+1}(\mathcal{C}(\tilde{A}) \otimes_{\mathbb{C}} R)$ is a surjection. Further, a sufficiently small compact epsilon neighborhood M of A in D has the property that $\tilde{A} \subset p^{-1}(M) \subset N$. Since $M \subset D$ splits and $\tilde{D} - \tilde{A}$ is homeomorphic to $D - A$, it follows that $p^{-1}(M) \subset \tilde{D}$ splits as well. Thus $\mathcal{C}(\tilde{D}) \to \mathcal{C}(p^{-1}(M))$ is a split surjection of rings and thus $K_{q+1}(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \to K_{q+1}(\mathcal{C}(p^{-1}(M) \otimes_{\mathbb{C}} R)$ is also surjective. It follows that the composition

$$K_{q+1}(\mathcal{C}(\tilde{D})\otimes_{\mathbb{C}}R) \to K_{q+1}(\mathcal{C}(N)\otimes_{\mathbb{C}}R) \to K_{q+1}(\mathcal{C}(p^{-1}(M))\otimes_{\mathbb{C}}R) \to K_{q+1}(\mathcal{C}(\tilde{A})\otimes_{\mathbb{C}}R)$$

must be a surjection, as claimed.

Using the long exact sequence (5.1.4), it now follows that

$$K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \oplus K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R) \quad (5.1.5)$$

is an injection. Under the map $K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R)$, the element represented by (α, γ) maps to the element represented by $(\tilde{D} \to \tilde{Y}^{\mathrm{an}}, \tilde{\gamma})$. But this is the trivial element since $\tilde{Y}$ is smooth. Under the map $K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R)$, (α, γ) is sent to $(A \to Z^{\mathrm{an}}, \gamma')$, where Z has smaller dimension than Y. By induction on the dimension of Y, the proof is complete. (Observe that we may assume Y is reduced, so that Y is smooth when $\dim Y = 0$.) □

Corollary 5.2. *The theory $\mathcal{K}(\Delta^{\bullet}_{top} \times -)$ satisfies Zariski decent on the category of smooth, affine complex varieties.*

Proof. It is known from [TT; 8.1] that the spectrum valued functor $\mathcal{K}^B(U \times -)$ satisfies Zariski descent for any variety U. Here $\mathcal{K}^B$ is "Bass's K-theory", a spectrum valued theory which has nonzero negative homotopy groups in general and which admits a natural map $\mathcal{K} \to \mathcal{K}^B$ inducing an isomorphism on non-negative homotopy groups. (The negative homotopy groups of $\mathcal{K}^B$ are the usual negative K-groups $K^B_q \equiv K_q$.) Thus, given any cover by two open subsets $V = V_1 \cap V_2$ of a smooth quasi-projective variety V, there is a natural long exact sequence

$$\cdots \to K_q(U\times V) \to K_q(U\times V_1)\oplus K_q(U\times V_2) \to K_q(U\times W) \to K_{q-1}(U\times V_1 \cap V_2) \to \cdots$$

extending possibly into the negative indices. If T is any compact Hausdorff space, then by taking direct limits indexed by Var^T, we obtain the long exact sequence

$$\cdots \to K_q(T_{top}\times V) \to K_q(T_{top}\times V_1)\oplus K_q(T_{top}\times V_2) \to K_q(T_{top}\times V_1 \cap V_2) \to K_{q-1}(T_{top}\times V) \to \cdots .$$

By Theorem 5.1, the negative homotopy groups in this sequence all vanish for $T = \Delta^n_{top}$. The square

$$\begin{array}{ccc} \mathcal{K}(\Delta^n_{top} \times V) & \longrightarrow & \mathcal{K}(\Delta^n_{top} \times V_1) \\ \downarrow & & \downarrow \\ \mathcal{K}(\Delta^n_{top} \times V_2) & \longrightarrow & \mathcal{K}(\Delta^n_{top} \times V_1 \cap V_2) \end{array}$$

is thus homotopy cartesian for each n and

$$K_0(\Delta^n_{top} \times V_1) \oplus K_0(\Delta^n_{top} \times V_2) \to K_0(\Delta^n_{top} \times V_1 \cap V_2)$$

is surjective. Consequently, a standard argument shows that the square

$$\begin{array}{ccc} \mathcal{K}(\Delta^\bullet_{top} \times V) & \longrightarrow & \mathcal{K}(\Delta^\bullet_{top} \times V_1) \\ \downarrow & & \downarrow \\ \mathcal{K}(\Delta^\bullet_{top} \times V_2) & \longrightarrow & \mathcal{K}(\Delta^\bullet_{top} \times V_1 \cap V_2) \end{array}$$

is homotopy cartesian. By [BG; 4], this suffices to establish Zariski descent. □

To generalize Theorem 5.1 and Corollary 5.2 to all quasi-projective complex varieties, we employ "Jouanolou's device". That is, for any quasi-projective complex variety X, there is an affine variety J and a map $J \to X$ which is a torsor of a vector bundle over X [J]. To prove Theorem 5.1 for an arbitrary X, we will, roughly speaking, use the fact that Theorem 5.1 is known for J and that $K(\Delta^\bullet_{top} \times -)$ sends $J \to X$ to a weak equivalence. To make the second part of this argument work, we need to generalize (and fix) a result of Rosenberg [R2; 3.1] establishing the so-called K-regularity of the ring $\mathcal{C}(T)$, where T is a compact Hausdorff space.

Theorem 5.3. *(Compare with* [R2; 3.1].*) Let X be any smooth, quasi-projective complex variety and $J \to X$ a vector bundle torsor. The natural map*

$$K_q(\Delta^n_{top} \times X) \to K_q(\Delta^n_{top} \times J)$$

is a weak equivalence for all $n \geq 0$, $q \in \mathbb{Z}$.

Proof. Let $D = \Delta^n_{top}$. The fact that $\mathcal{K}^B(Y \times -)$ satisfies Zariski descent for each $D \to Y^{an}$ (cf. [TT;8.1]) allows us to reduce to the case when $X = \operatorname{Spec} R$ is affine and $J \to X$ is isomorphic to $X \times \mathbb{A}^k \to X$. Recall that for any scheme V (resp., ring A), the group $N^k K_q(V)$ (resp., $N^k K_q(A)$) is defined as the kernel of the split surjection $K_q(V \times \mathbb{A}^k) \twoheadrightarrow K_q(V)$ (resp. $K_q(A[y_1, \dots, y_n]) \twoheadrightarrow K_q(A)$). Observe that we have a natural isomorphism

$$N^k K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \cong N^k K_q(D \times \operatorname{Spec} R),$$

using the same argument as given in the first paragraph of the proof of Theorem 5.1. Therefore, it suffices to show

$$N^k K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) = 0,$$

for all $k > 0$, for any smooth, finitely generated $\mathbb{C}$-algebra R.

We next observe that rings of the form $\mathcal{C}_0(T) \otimes_{\mathbb{C}} R$, where T is a locally compact Hausdorff space and $\mathcal{C}_0$ denotes functions which vanish at infinity, are "H-unital" [SW]. From this, it follows that the square one obtains by tensoring (5.1.1) with R induces a long exact sequences for the functors K_* and $N^k K_*$ for all indices $* \in \mathbb{Z}$ [SW].

The remainder of the proof is parallel to the proof of Theorem 5.1. Specifically, note that an element of $N^k K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R)$ is represented by a pair $(\alpha : D \to Y^{\text{an}}, \gamma \in N^k K_q(Y \times X))$ and that such a representative is trivial if Y is smooth since $N^k K_q(Y \times X) = 0$ in that case. If Y is not smooth, choose a resolution of singularities $p : \tilde{Y} \to Y$, $Z \subset Y$.

If E is any star-like simplicial complex with maximal dimensional simplices $\{T_i\}$, then the proof of the injectivity of (5.1.2) also establishes that

$$N^k K_q(\mathcal{C}(E) \otimes_{\mathbb{C}} R) \to \bigoplus_i N^k K_q(\mathcal{C}(T_i) \otimes_{\mathbb{C}} R)$$

is injective for all k, q. This allows us to assume $\alpha : D \to Y^{\text{an}}$ is the inclusion of a simplex and that $A \equiv D \cap Z^{\text{an}}$ is a proper face. Let $\tilde{D} = p^{-1}(D)$, $\tilde{A} = p^{-1}(A)$ as before. The proof of the injectivity of (5.1.5) carries through to establish that

$$N^k K_q(\mathcal{C}(D) \otimes_{\mathbb{C}} R) \to N^k K_q(\mathcal{C}(\tilde{D}) \otimes_{\mathbb{C}} R) \oplus N^k K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R)$$

is injective for all k, q. Since $\tilde{Y}$ is smooth, we are reduced to showing the element of $N^k K_q(\mathcal{C}(A) \otimes_{\mathbb{C}} R)$ represented by $(A \to Z, \gamma')$ vanishes. By induction on the dimension of Y, we are finished. □

Corollary 5.4. *For any smooth, quasi-projective variety X, the groups $K_q(\Delta^k_{top} \times X)$ vanish for any $k \geq 0$ and $q < 0$.*

Proof. This following immediately from Theorem 5.3 and Theorem 5.1, using the fact that any quasi-projective variety X admits an affine vector bundle torsor $J \to X$ [J]. □

Corollary 5.5. *The spectrum-valued theory $\mathcal{K}(\Delta^\bullet_{top} \times -)$ satisfies Zariski descent on the category of smooth, quasi-projective complex varieties.*

Proof. The proof is parallel to the proof of Corollary 5.2, using Corollary 5.4 instead of Theorem 5.1. □

We now calculate the $\mathcal{K}^{\text{semi}}$-groups of a possibly singular complete curve. The calculation of $\mathcal{K}^{\text{semi}}(C)$ when C is a smooth complete curve was done in [FW-2; 7.5] and serves as the starting point for the calculation here.

Theorem 5.6. *Let C be a complete, possibly singular, complex curve. Then the natural map*

$$K_q^{\text{semi}}(C) \to K^{-q}_{\text{top}}(C^{\text{an}})$$

is an isomorphism for all $q \geq 0$.

Proof. By its very definition, K^{semi} does not distinguish between a variety and its weak normalization. Thus, we may assume C is weakly normal.

Let S be the finite number of singular points of C, let $\tilde{C} \to C$ be the normalization of C, and let $\tilde{S}$ be $S \times_C \tilde{C}$. Since C is weakly normal, we claim the scheme $\tilde{S}$ is reduced – i.e., is a finite collection of points as well. To see this, let $\operatorname{Spec} R$ be an affine neighborhood of S in C and write $\operatorname{Spec} \tilde{R}$ for its inverse image in $\tilde{C}$, so that $\tilde{R}$ is the normalization of R. Then S is the closed subset of $\operatorname{Spec} R$ defined by the conductor ideal $J \equiv ann_R(\tilde{R}/R)$. In fact, J is also an ideal of $\tilde{R}$ and cuts out $\tilde{S} \subset \operatorname{Spec} \tilde{R}$. An elementary argument using the fact that

$$x \in \tilde{R}, \quad \text{and} \quad x^2, x^3 \in R \Rightarrow x \in R$$

now shows that J is a radical ideal of $\tilde{R}$. (The displayed property of R is by definition the semi-normality property, which is equivalent to weak-normality in characteristic zero – see [Sw2].)

For any variety U, we have by [W2; 4.9] a homotopy cartesian square of spectra

$$\begin{array}{ccc} \mathcal{K}H(U \times C) & \longrightarrow & \mathcal{K}H(U \times \tilde{C}) \\ \downarrow & & \downarrow \\ \mathcal{K}H(U \times S) & \longrightarrow & \mathcal{K}H(U \times \tilde{S}), \end{array}$$

where $\mathcal{K}H$ denotes Weibel's "homotopy K-theory". Taking limits over $Var^{\Delta^d_{top}}$, we obtain the homotopy cartesian square of spectra

$$\begin{array}{ccc} \mathcal{K}H(\Delta^d_{top} \times C) & \longrightarrow & \mathcal{K}H(\Delta^d_{top} \times \tilde{C}) \\ \downarrow & & \downarrow \\ \mathcal{K}H(\Delta^d_{top} \times S) & \longrightarrow & \mathcal{K}H(\Delta^d_{top} \times \tilde{S}). \end{array}$$

Theorem 5.3 and Corollary 5.4 suffice to show that $\mathcal{K}H(\Delta^d_{top} \times \tilde{C})$, $\mathcal{K}H(\Delta^d_{top} \times \tilde{S})$, and $\mathcal{K}H(\Delta^d_{top} \times S)$ each coincide with the usual K-theory spaces $K(\Delta^d_{top} \times \tilde{C})$, $K(\Delta^d_{top} \times \tilde{S})$, and $K(\Delta^d_{top} \times S)$ under the natural map (cf. [W2; 1.3]). It follows that $KH_q(\Delta^d_{top} \times C) = 0$ for $q \leq -2$.

Let $C = C_1 \cup C_2$ be an open covering by affine curves such that C_1 contains the singular locus S and $C_2 \cap S = \emptyset$. Let $\tilde{C}_1$ denote the evident pullback. Then there is a natural exact sequence

$$K_0(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \to K_0(\Delta^d_{top} \times \tilde{C}_1 \times \mathbb{A}^n) \oplus K_0(\Delta^d_{top} \times S \times \mathbb{A}^n) \to$$
$$K_0(\Delta^d_{top} \times \tilde{S} \times \mathbb{A}^n) \to K_{-1}(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \to 0,$$

for any n, using the fact that the rings of regular functions on C_1, S, $\tilde{S}$, and $\tilde{C}_1$ form a Milnor square. Since S and $\tilde{S}$ are merely disjoint unions of copies of $\operatorname{Spec} \mathbb{C}$, we obtain from Theorem 5.3 and the fact that Δ^d_{top} is contractible the isomorphisms $K_0(\Delta^d_{top} \times S \times \mathbb{A}^n) \cong \mathrm{H}^0(S, \mathbb{Z})$ and $K_0(\Delta^d_{top} \times \tilde{S} \times \mathbb{A}^n) \cong \mathrm{H}^0(\tilde{S}, \mathbb{Z})$. Further, the map

$$K_0(\Delta^d_{top} \times \tilde{C}_1 \times \mathbb{A}^n) \to K_0(\Delta^d_{top} \times \tilde{S} \times \mathbb{A}^n)$$

in the above exact sequence factors as

$$K_0(\Delta^d_{top} \times \tilde{C}_1 \times \mathbb{A}^n) \stackrel{\text{rank}}{\to} \mathrm{H}^0(\tilde{C}_1, \mathbb{Z}) \to \mathrm{H}^0(\tilde{S}, \mathbb{Z}) = K_0(\Delta^d_{top} \times \tilde{S} \times \mathbb{A}^n).$$

From this we conclude

$$K_{-1}(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \cong \operatorname{coker}\left(\mathrm{H}^0(\tilde{C}_1, \mathbb{Z}) \oplus \mathrm{H}^0(S, \mathbb{Z}) \to \mathrm{H}^0(\tilde{S}, \mathbb{Z})\right).$$

Since C_2 is smooth, we know $K_q(\Delta^d_{top} \times C_2 \times \mathbb{A}^n) = 0$ for all n, d and $q \leq -1$ by Theorem 5.1. Now consider the following portion of the long exact Mayer-Vietoris sequence:

$$\cdots \to K_0(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \oplus K_0(\Delta^d_{top} \times C_2 \times \mathbb{A}^n) \to K_0(\Delta^d_{top} \times (C_1 \cap C_2) \times \mathbb{A}^n)$$
$$\to K_{-1}(\Delta^d_{top} \times C \times \mathbb{A}^n) \to K_{-1}(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \to 0.$$

We claim that the map

$$K_0(\Delta^d_{top} \times C_1 \times \mathbb{A}^n) \oplus K_0(\Delta^d_{top} \times C_2 \times \mathbb{A}^n) \to K_0(\Delta^d_{top} \times (C_1 \cap C_2) \times \mathbb{A}^n) \quad (5.6.1)$$

is onto, which will show that $K_{-1}(\Delta^d_{top} \times C \times \mathbb{A}^n)$ is naturally isomorphic to

$$\operatorname{coker}\left(\mathrm{H}^0(\tilde{C}_1, \mathbb{Z}) \oplus \mathrm{H}^0(S, Z) \to \mathrm{H}^0(\tilde{S}, Z)\right).$$

To see this, observe that the covering $C = C_1 \cup C_2$ induces the open covering $\tilde{C} = \tilde{C}_1 \cup C_2$ and $\tilde{C}_1 \cap C_2 = C_1 \cap C_2$. Further, the map (5.6.1) factors through

$$K_0(\Delta^d_{top} \times \tilde{C}_1 \times \mathbb{A}^n) \oplus K_0(\Delta^d_{top} \times C_2 \times \mathbb{A}^n) \to K_0(\Delta^d_{top} \times (C_1 \cap C_2) \times \mathbb{A}^n),$$

which is onto since $K_{-1}(\Delta^d_{top} \times \tilde{C} \times \mathbb{A}^n) = 0$ by Theorem 5.1.

We have therefore shown that the natural map

$$K_{-1}(C) \to K_{-1}(\Delta^d_{top} \times C \times \mathbb{A}^n)$$

is an isomorphism, and in particular the simplicial abelian group $e \mapsto K_{-1}(\Delta^d_{top} \times C \times \Delta^e)$ is constant. Moreover, from above we have that $K_q(\Delta^d_{top} \times C \times \mathbb{A}^n) = 0$ for all d, n provided $q < -1$. Thus, for a fixed d, the spectral sequence (which is constructed using [W2; 1.3] and Jouanolou's device)

$$\pi_p \left| e \mapsto K_q(\Delta^d_{top} \times C \times \Delta^e) \right| \Longrightarrow KH_{p+q}(\Delta^d_{top} \times C)$$

shows that the homotopy cofiber (that is, the delooping of the homotopy fiber) of the natural map of spectra

$$\mathcal{K}(\Delta^d_{top} \times C) \to \mathcal{K}H(\Delta^d_{top} \times C)$$

has a single non-vanishing homotopy group, which is located in degree -1 and is isomorphic to $K_{-1}(C)$. And so the simplicial spectrum

$$d \mapsto \mathcal{K}H(\Delta^d_{top} \times C)$$

agrees with $\mathcal{K}(\Delta^\bullet_{top} \times C) \sim \mathcal{K}^{\mathrm{semi}}(C)$ in nonnegative degrees.

Using these facts, we get that the homotopy cartesian square of spectra

$$\begin{array}{ccc} \left|d \mapsto \mathcal{K}H(\Delta^d_{top} \times C)\right| & \longrightarrow & \left|d \mapsto \mathcal{K}H(\Delta^d_{top} \times \tilde{C})\right| \\ \downarrow & & \downarrow \\ \left|d \mapsto \mathcal{K}H(\Delta^d_{top} \times S)\right| & \longrightarrow & \left|d \mapsto \mathcal{K}H(\Delta^d_{top} \times \tilde{S})\right|. \end{array}$$

induces the long exact sequence

$$\cdots \to K^{\mathrm{semi}}_1(\tilde{S}) \to K^{\mathrm{semi}}_0(C) \to K^{\mathrm{semi}}_0(\tilde{C}) \oplus K^{\mathrm{semi}}_0(S) \to K^{\mathrm{semi}}_0(\tilde{S}) \to K^{\mathrm{alg}}_{-1}(C) \to 0.$$

The result now follows from the five lemma, since we know that the natural map $K^{\mathrm{semi}}_q(-) \to \mathcal{K}^{-q}_{\mathrm{top}}(-)$ is an isomorphism on S, $\tilde{S}$, and $\tilde{C}$ for $q \geq 0$. $\square$

Remark 5.7. *The proof of Theorem 5.6 suggests that it might be reasonable to define a variation of $\mathcal{K}^{\text{semi}}$ (or perhaps, even to redefine $\mathcal{K}^{\text{semi}}$) for a possibly singular projective variety X by using the formula*

$$\left| d \mapsto \mathcal{K}H(\Delta^d_{top} \times X) \right|.$$

(This construction is easily seen to coincide with $\left| d \mapsto \mathcal{K}^B(\Delta^d_{top} \times X) \right|$.) When X is smooth, one recovers the original definition of $\mathcal{K}^{\text{semi}}(X)$. When X is a singular curve, this new formula leads to one additional group, namely the algebraic K-group $K^{\text{alg}}_{-1}(C)$. Simple examples show that $K^{\text{alg}}_{-1}(C)$ need not be isomorphic to $K^1_{\text{top}}(C^{\text{an}})$ for a singular complete curve C.

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