SEMI-TOPOLOGICAL K-THEORY OF REAL VARIETIES

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Abstract. The semi-topological K-theory of real varieties, $\mathcal{K}\mathbb{R}^{\text{semi}}(-)$, is an oriented multiplicative (generalized) cohomology theory which extends the authors' earlier theory, $\mathcal{K}^{\text{semi}}(-)$, for complex algebraic varieties. Motivation comes from consideration of algebraic equivalence of vector bundles (sharpened to real semi-topological equivalence), consideration of $\mathbb{Z}/2$-equivariant mapping spaces of morphisms of algebraic varieties to Grassmannian varieties, and consideration of the algebraic K-theory of real varieties.

The authors verify that the semi-topological K-theory of a real variety X interpolates between the algebraic K-theory of X and Atiyah's Real K-theory of the associated Real space of complex points, $X_{\mathbb{R}}(\mathbb{C})$. The resulting natural maps of spectra

$$\mathcal{K}^{\text{alg}}(X) \to \mathcal{K}\mathbb{R}^{\text{semi}}(X) \to \mathcal{K}\mathbb{R}_{\text{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

satisfy numerous good properties: the first map is a mod-n equivalence for any projective real variety and any $n > 0$; the second map is an equivalence for smooth projective curves and flag varieties; the triple fits in a commutative diagram of spectra mapping via total Segre classes to a triple of cohomology theories. The authors also establish results for the semi-topological K-theory of real varieties, such as Nisnevich excision and a type of localization result, which were previously unknown even for complex varieties.

Introduction

In the papers [FW2], [FW3], we introduced and studied *semi-topological $\mathcal{K}$-theory*, which is a spectrum valued theory $\mathcal{K}^{\text{semi}}(-)$ defined on the category of quasi-projective complex varieties. The definition of $\mathcal{K}^{\text{semi}}$ was originally suggested in [F3], and more recently an equivalent theory defined for smooth, projective complex varieties, called *holomorphic $\mathcal{K}$-theory*, has been studied by R. Cohen and P. Lima-Filho in [CL2]. The semi-topological $\mathcal{K}$-theory of a complex variety X fits in between the algebraic K-theory of X and the topological K-theory of the associated topological space of complex points $X(\mathbb{C})$:

$$\mathcal{K}^{\text{alg}}(X) \to \mathcal{K}^{\text{semi}}(X) \to \mathcal{K}_{\text{top}}(X(\mathbb{C})).$$

The theory $\mathcal{K}^{\text{semi}}(X)$ is a good interpolation between $\mathcal{K}^{\text{alg}}(X)$ and $\mathcal{K}^{\text{semi}}(X)$: the map $\mathcal{K}^{\text{alg}}(X) \to \mathcal{K}^{\text{semi}}(X)$ induces a isomorphism on homotopy groups with finite coefficients, while the map $\mathcal{K}^{\text{semi}}(X) \to \mathcal{K}_{\text{top}}(X(\mathbb{C}))$ apparently induces an isomorphism on homotopy groups once the action of the so-called Bott element in $\mathcal{K}_2^{\text{semi}}(\operatorname{Spec}\mathbb{C})$ is inverted. Moreover, we view $\mathcal{K}^{\text{semi}}(X)$ as having intrinsic interest,

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for it can be viewed as the stabilization of function complexes of algebraic morphisms of X into Grassmannians, a topic of study in papers such as [Ki], [CLS]. Thus, computations of $\mathcal{K}^{\text{semi}}(X)$ arising from either topological or algebraic K-theory can provide information about such morphisms, whereas computations of invariants of certain moduli spaces (such as those in [Ki]) can provide information about algebraic K-theory.

In this paper, we extend the definition of $\mathcal{K}^{\text{semi}}$ to real varieties, establish numerous foundational properties of our theory, and compute various examples. We also show how this new theory, $\mathcal{K}\mathbb{R}^{\text{semi}}$, is related to various other constructions in algebraic geometry and topology. As was the case for $\mathcal{K}^{\text{semi}}(X)$, with a complex variety X, the initial motivation for the definition of $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$, with X a real variety, is our intention of constructing a theory based on algebraic vector bundles and algebraic equivalence. A subtlety arises in that we must use the real analytic topology in our definition of the equivalence relation giving $K\mathbb{R}_0^{\text{semi}}(X)$ as a quotient of $K_0^{\text{alg}}(X)$, for a real variety X. This equivalence relation, *real semi-topological equivalence*, provides an invariant finer than ordinary algebraic equivalence (as suggested by [Fu; 10.3]) and thus $K\mathbb{R}_0^{\text{semi}}(X)$ is an invariant more closely approximating $K_0^{alg}(X)$ (cf. Proposition 1.6). As for most constructions of higher K-groups, to define $K\mathbb{R}_q^{\text{semi}}(X)$ for $q > 0$ we require some machinery to provide a suitable homotopy-theoretic group completion; we find the "machine" using E_∞-operads convenient for most our purposes (see [M1]), although the equivalent "machine" stemming from Segal's notion of a Γ-space (see [Se]) is used instead in Section 5. Thanks to a stabilization theorem proved in Section 7, we find that this homotopy-theoretic group completion can be viewed as a colimit of more familiar spaces of algebraic morphisms.

In analogy with $\mathcal{K}^{\text{semi}}$, the theory $\mathcal{K}\mathbb{R}^{\text{semi}}$ fits in between the algebraic K-theory, $\mathcal{K}^{\text{alg}}$, of real varieties and the so-called Atiyah's Real K-theory, $\mathcal{K}\mathbb{R}_{\text{top}}$, of *Real spaces* – i.e., spaces equipped with continuous involutions (cf. [At]). Namely, if X is a quasi-projective real variety, then writing $X_{\mathbb{R}}(\mathbb{C})$ for the analytic space of complex points equipped with the involution given by complex conjugation, we have natural maps of spectra

$$\mathcal{K}^{\text{alg}}(X) \to \mathcal{K}\mathbb{R}^{\text{semi}}(X) \to \mathcal{K}\mathbb{R}_{\text{top}}(X_{\mathbb{R}}(\mathbb{C})).$$

We mention three theorems which should give the reader some impression of how $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ relates to these other K-theories and also to cohomology theories arising from cycles. To prove these theorems, we follow [FW3] in introducing a variation of $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$, written $\mathcal{K}^{alg}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$, which is weakly equivalent to $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ whenever X is projective (and weakly normal), but which is better behaved for arbitrary quasi-projective real varieties.

Theorem 0.1. *(cf. Corollary 3.10) For a projective real variety X, the natural map*

$$K_q^{\text{alg}}(X; \mathbb{Z}/n) \to K\mathbb{R}_q^{\text{semi}}(X; \mathbb{Z}/n)$$

is an isomorphism for all $q \geq 0$, $n > 0$.

Theorem 0.2. *(cf. Propositions 6.1 and 6.2) Suppose X is one of the following projective real varieties: (1) a smooth, projective real curve, or (2) G/P, where G*

is one of the linear algebraic groups $GL_{n,\mathbb{R}}$, $SL_{n,\mathbb{R}}$, $Spin_{n,\mathbb{R}}$, or $Sp_{2n,\mathbb{R}}$ and P is a parabolic subgroup containing a split Borel subgroup. Then the natural map

$$K\mathbb{R}_q^{\mathrm{semi}}(X) \to K\mathbb{R}_{\mathrm{top}}^{-q}(X)$$

is an isomorphism for all $q \geq 0$.

Theorem 0.3. *(cf. Theorem 8.8) Let X be a smooth, projective real variety. Then there is a natural commutative diagram*

$$\begin{array}{ccccc} K_i^{\mathrm{alg}}(X) & \longrightarrow & K_i^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & K\mathbb{R}^{-i}_{top}(X_{\mathbb{R}}(\mathbb{C})) \\ {\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow \\ \bigoplus_{q\geq 0} H^{2q-i}_{\mathcal{M}}(X,\mathbb{Z}(q)) & \longrightarrow & \bigoplus_{q\geq 0} L^q H\mathbb{R}^{2q-i}(X) & \longrightarrow & \bigoplus_{q\geq 0} H^{q,q-i}_{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}),\underline{\mathbb{Z}}). \end{array}$$

for all $i \geq 0$, where $H^*_{\mathcal{M}}(-,\mathbb{Z}(*))$ denotes the motivic cohomology of a real variety, $H^{*,*}_{\mathbb{Z}/2}(-,\underline{\mathbb{Z}})$ denotes the twisted equivariant cohomology of a Real space, $L^*H\mathbb{R}^*(-)$ is the real analogue of morphic cohomology, and the vertical maps are the so-called "total Segre class maps". Moreover, this diagram is obtained from a commuting diagram of spectra by applying $\pi_i(-)$.

As shown in Proposition 1.5, $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(-)$ and $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ really do extend the constructions of $\mathcal{K}^{\mathrm{semi}}(-)$ and $\mathcal{K}^{\mathrm{semi}}(\Delta^\bullet_{top} \times_{\mathbb{C}} -)$ of [FW2], [FW3], in that whenever X is a quasi-projective complex variety, there are natural weak equivalences

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \xrightarrow{\sim} \mathcal{K}^{\mathrm{semi}}(X) \quad \text{and} \quad \mathcal{K}\mathbb{R}^{\mathrm{semi}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \xrightarrow{\sim} \mathcal{K}^{\mathrm{semi}}(\Delta^\bullet_{top} \times_{\mathbb{C}} X).$$

In particular, theorems proven about $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(-)$ for real varieties give theorems about $\mathcal{K}^{\mathrm{semi}}$ for complex varieties. Some of the results in this paper involve extending known results for $\mathcal{K}^{\mathrm{semi}}$ to $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ for complex varieties. This includes the three theorems mentioned above. However, we also establish a few results for $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ and $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ which were unknown before even for the $\mathcal{K}^{\mathrm{semi}}$- and $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{C}} -)$-theories of complex varieties. For example, we establish that $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ satisfies Nisnevich descent and then use this to prove a localization type result:

Theorem 0.4. *(cf. Theorem 3.5) The theory $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ satisfies Nisnevich descent on the category of smooth, quasi-projective real varieties. That is, let X be a smooth, quasi-projective real variety, $p : X' \to X$ be an étale map, $i : Z \hookrightarrow X$ be a closed immersion which factors as $Z \xrightarrow{j} X' \xrightarrow{p} X$. Then setting $U = X - i(Z)$ and $U' = X' - j(Z)$, there is a natural long exact sequence*

$$\cdots \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X') \oplus K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U)$$
$$\to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U') \to K_{q-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \cdots .$$

Theorem 0.5. *(cf. Corollary 6.7) Given a regular closed immersion $i : Z \hookrightarrow X$ of smooth, quasi-projective real varieties, there is a natural long exact sequence*

$$\cdots \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \\ \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X - Z) \to K_{q-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \to \cdots .$$

In Section 9 we demonstrate that $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ satisfies the axioms of a *multiplicative oriented cohomology theory* in the sense of Panin-Smirnov [PS]. One significance of such axioms is that they allow for easy verification of Riemann-Roch type results involving a suitably multiplicative map between such theories. Ideally, we would like to apply this machinery to the Chern character map from $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ to real morphic cohomology with rational coefficients (defined in Section 8). But for now, a proof of the required multiplicative property remains elusive, and so such a Riemann-Roch result is still conjectural. (However, in [CL1; 4.11] the multiplicativity of a Chern character is asserted for complex varieties.)

An interesting aspect of our extension of $\mathcal{K}^{\mathrm{semi}}$ to real varieties is the potential for an equivariant point of view in the study of algebraic K-theory and semi-topological K-theory. Namely, we envision associating to a real variety X a so-called $\mathbb{Z}/2$-spectrum (intuitively, a spectrum equipped with a suitably defined notion of an action by the group $\mathbb{Z}/2$), written $\mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(X)$. This construct should have the property that $K^{\mathrm{semi}}(X_{\mathbb{C}})$ is recovered by forgetting the $\mathbb{Z}/2$-action, whereas $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ is obtained by taking $\mathbb{Z}/2$-fixed points. Unfortunately, some of the foundational results for equivariant spectra are missing from the literature (although they are apparently known to the experts). Consequently, we relegate the discussion of $\mathbb{Z}/2$-spectra to an appendix, which should be viewed as the optimal way to develop the material in this paper, once the requisite foundational results become available.

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§1 Formulation of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$

In this section, we introduce a spectrum $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ associated to a quasi-projective real variety X, which is analogous to the spectrum $\mathcal{K}^{\mathrm{semi}}(Y)$ constructed in [FW2] for complex varieties Y. The spectrum-valued functor $X \mapsto \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ defines a cohomology theory on the category of quasi-projective real varieties which, in a suitable sense, lies part-way between the algebraic K-theory of the real variety X, $\mathcal{K}^{\mathrm{alg}}(X)$, and Atiyah's Real K-theory (cf. [At]) of the $\mathbb{Z}/2$-space $X_{\mathbb{R}}(\mathbb{C})$, $\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$. Here, we write $X_{\mathbb{R}}(\mathbb{C})$ for the analytic space of complex points of a real variety equipped with the $\mathbb{Z}/2$-action given by complex conjugation. The sense in which $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ lies between $\mathcal{K}^{\mathrm{alg}}(X)$ and $\mathcal{K}\mathbb{R}_{\mathrm{top}}(X)$ will be made precise throughout the remainder of this paper.

We use the notation of [FW1; 2.4], where for quasi-projective k-varieties X and G, for some ground field k, the set $\mathrm{Mor}_k(X, G)$ consists of all *continuous algebraic morphisms* of k-varieties. Assuming k has characteristic 0, then when X is *weakly normal* (equivalently, *semi-normal* – see [Swa]), $\mathrm{Mor}_k(X, G)$ coincides with the set of all morphisms of k-varieties. More generally, when $\mathrm{char}\, k = 0$, one may form the so-called *weak normalization* $X^w \to X$, and then $\mathrm{Mor}_k(X, G)$ coincides with $\mathrm{Hom}_k(X^w, G)$, the collection of morphisms of k-varieties from X^w to G.

As in [FW1; §2], when $k = \mathbb{C}$ we impose a topology on $\mathrm{Mor}_{\mathbb{C}}(X, G)$, resulting in a topological space which we will write as $\mathcal{M}or_{\mathbb{C}}(X, G)$ here (but which was written as $\mathcal{M}or(X, G)^{an}$ in [FW1]). We briefly recall the description of this topology so as to extend it to cover the case $k = \mathbb{R}$. First, one chooses projective closures $X \subset \overline{X}$ and $G \subset \overline{G}$ and then defines Y to be the Chow variety of cycles on $\overline{X} \times \overline{G}$ of appropriate degree. Then $E \subset Y$ is defined as the constructible subset consisting of those cycles which give graphs of morphisms from X to G upon restriction to the open subscheme $X \times \overline{G}$. Next, $R \subset E \times E \subset Y \times Y$ is defined as the constructible subset of pairs of cycles which induce the same graph upon intersection with $X \times \overline{G}$. Finally, the topological space $\mathcal{M}or_{\mathbb{C}}(X, G)$ is defined to be the quotient of the analytic space $E(\mathbb{C})$ by the equivalence relation $R(\mathbb{C})$. It is shown that this is a *proper equivalence relation*, in the sense that each of the two projection maps $R(\mathbb{C}) \to E(\mathbb{C})$ is a proper map of topological space, or equivalently the quotient map $E(\mathbb{C}) \twoheadrightarrow E(\mathbb{C})/R(\mathbb{C})$ is proper.

We observe that these constructions apply equally well with $\mathbb{R}$ replacing $\mathbb{C}$, since $E(\mathbb{R})$ and $R(\mathbb{R})$ have natural topologies and $R(\mathbb{R})$ is again a proper equivalence relation.

Definition 1.1. *Let X and G be quasi-projective real varieties. Define $\mathcal{M}or_{\mathbb{R}}(X, G)$ to be the set $\mathrm{Mor}_{\mathbb{R}}(X, G)$ endowed with the topology given as the quotient topology of $E(\mathbb{R})$ by the proper equivalence relation $R(\mathbb{R})$.*

Observe that there is a natural involution on the set $\mathrm{Mor}_{\mathbb{C}}(X_{\mathbb{C}}, G_{\mathbb{C}})$ induced by complex conjugation. Explicitly, assuming X is weakly normal (or if not, replacing it by its weak normalization), this involution is the involution on $\mathrm{Hom}_{\mathbb{C}}(X_{\mathbb{C}}, G_{\mathbb{C}})$ which sends a morphism f to $\sigma_G \circ f \circ \sigma_X$, where σ_Y is the real map on the complexification $Y_{\mathbb{C}}$ of a real variety Y induced by complex conjugation. (One may easily check that $\sigma_G \circ f \circ \sigma_X$ is indeed a morphism of complex varieties.)

Lemma 1.2. *For X and G quasi-projective real varieties, the space $\mathcal{M}or_{\mathbb{R}}(X, G)$ is homeomorphic to the fixed point subspace of $\mathcal{M}or_{\mathbb{C}}(X_{\mathbb{C}}, G_{\mathbb{C}})$ under the action of complex conjugation.*

Proof. If Y is the Chow variety of real cycles (of some degree) on $\overline{X} \times \overline{G}$, then $Y_{\mathbb{C}}$ is the Chow variety of complex cycles on $\overline{X}_{\mathbb{C}} \times \overline{G}_{\mathbb{C}}$ [F2; 1.1]. Moreover, complex conjugation defines a continuous involution on the space $Y(\mathbb{C})$ and the fixed point subspace of this involution is $Y(\mathbb{R})$. This involution restricts to an involution on the subspace $E(\mathbb{C})$ whose fixed points give the subspace $E(\mathbb{R})$. Similarly, complex conjugation defines a continuous involution on $R(\mathbb{C})$ with fixed point subspace $R(\mathbb{R})$. Since the equivalence relations $R(\mathbb{R})$ and $R(\mathbb{C})$ on $E(\mathbb{R})$ and $E(\mathbb{C})$ are proper, it follows that there is an induced involution on $\mathcal{M}or_{\mathbb{C}}(X_{\mathbb{C}}, G_{\mathbb{C}})$ which is continuous. This involution is easily seen to coincide with the involution introduced above and its fixed point subspace is $\mathcal{M}or_{\mathbb{R}}(X, G)$. □

As an aide to the reader, we provide a simpler description of the topology on $\mathcal{M}or_{\mathbb{R}}(X, G)$ in the special case when X is a projective real variety. This turns out to be the primary case of interest for this paper. Let $\mathrm{Hom}_{\mathbb{R}}(U, V)$ denote the collection of morphisms of $\mathbb{R}$-schemes from U to V. If X and G are both projective, then the functor on real schemes $\mathrm{Hom}_{\mathbb{R}}(- \times X, G)$ is representable by an infinite disjoint union (indexed by degree) of quasi-projective real varieties, written $\underline{\mathrm{Hom}}_{\mathbb{R}}(X,G)$. Thus if additionally we assume X is weakly normal, then $\mathcal{M}or_{\mathbb{R}}(X,G)$ is merely the associated analytic space of real points $\underline{\mathrm{Hom}}(X,G)(\mathbb{R})$ of the ind-variety $\underline{\mathrm{Hom}}_{\mathbb{R}}(X,G)$. If X and G are projective but X is not weakly normal, then $\mathcal{M}or_{\mathbb{R}}(X,G)$ is homeomorphic to the space $\mathcal{M}or_{\mathbb{R}}(X^w,G) = \underline{\mathrm{Hom}}(X^w,G)(\mathbb{R})$, where $X^w \to X$ is a weak normalization.

When X is projective (and weakly normal) and G is merely quasi-projective, then a similar result holds. Namely, observe if we choose a projective closure $\overline{G}$ of G, then the evaluation map

$$X \times \underline{\mathrm{Hom}}_{\mathbb{R}}(X,\overline{G}) \to \overline{G}$$

given by $(x,f) \mapsto f(x)$ is a morphism of ind-varieties – more formally, this morphism corresponds to the natural transformation of functors on real schemes

$$\mathrm{Hom}_{\mathbb{R}}(-,X) \times \mathrm{Hom}_{\mathbb{R}}(- \times X,\overline{G}) \to \mathrm{Hom}_{\mathbb{R}}(-,\overline{G})$$

given by sending $(f : Y \to X, g : Y \times X \to \overline{G})$ to $g \circ (\mathrm{id}, f)$. Then $B \subset X \times \underline{\mathrm{Hom}}_{\mathbb{R}}(X,\overline{G})$, defined as the set of (x,f) such that $f(x) \notin G$, is easily seen to be a closed subscheme. The complement of the image of B under the proper map $X \times \underline{\mathrm{Hom}}_{\mathbb{R}}(X,\overline{G}) \to \underline{\mathrm{Hom}}_{\mathbb{R}}(X,\overline{G})$ defines an open subscheme $\underline{\mathrm{Hom}}_{\mathbb{R}}(X,G)$ of $\underline{\mathrm{Hom}}_{\mathbb{R}}(X,\overline{G})$ which represents the functor $\mathrm{Hom}_{\mathbb{R}}(- \times X, G)$. In this case, we have $\mathcal{M}or_{\mathbb{R}}(X,G) \cong \underline{\mathrm{Hom}}(X,G)(\mathbb{R})$, and a similar statement holds for X not weakly normal.

Observe that if X happens to be a complex variety (i.e., if the structure map $X \to \operatorname{Spec}\mathbb{R}$ is provided with a factorization $X \to \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$), then $X_{\mathbb{C}} \cong X \coprod X$ and complex conjugation acts on $\mathcal{M}or_{\mathbb{C}}(X_{\mathbb{C}},G_{\mathbb{C}}) = \mathcal{M}or_{\mathbb{C}}(X,G_{\mathbb{C}}) \times \mathcal{M}or_{\mathbb{C}}(X,G_{\mathbb{C}})$ by interchanging the two factors. Consequently, by Lemma 1.2 the space $\mathcal{M}or_{\mathbb{R}}(X,G)$ is naturally homeomorphic to $\mathcal{M}or_{\mathbb{C}}(X,G_{\mathbb{C}})$ in this case.

We now give the construction of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ for a quasi-projective real variety X. The construction parallels the construction of $\mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}})$ in [FW2]. Write $\mathrm{Grass}_{\mathbb{R}}$ for the real ind-variety $\coprod_n \varinjlim_N \mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})$, where $\mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})$ parameterizes all rank n subspaces of $\mathbb{R}^{N+1}$. Taking transposes, we may equivalently regard $\mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})$ as parameterizing rank n quotients of $\mathbb{R}^{N+1}$, and we use this dual perspective to regard $\mathrm{Hom}_{\mathbb{R}}(X, \mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}}))$ as parameterizing quotients $\mathcal{O}_X^{N+1} \twoheadrightarrow E$, with E locally free of rank n. The space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is defined as the inductive limit of spaces $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}}))$, and points of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ are represented by quotients $\mathcal{O}_X^{\infty} \twoheadrightarrow E$ with E locally free which factor through the canonical map $\mathcal{O}_X^{\infty} \twoheadrightarrow \mathcal{O}_X^N$ for $N \gg 0$.

Set $\mathcal{I}(n)$ to be the space of all $\mathbb{R}$-linear injective maps $(\mathbb{R}^{\infty})^n \to \mathbb{R}^{\infty}$. The topology on $\mathcal{I}(n)$ is given as a subspace of the compactly generated compact-open topology for the set of all continuous maps from $(\mathbb{R}^{\infty})^n$ to $\mathbb{R}^{\infty}$, where $\mathbb{R}^{\infty}$ is topologized as a direct limit of its finite dimensional subspaces. An element $\alpha = (\alpha_1, \ldots, \alpha_n) : (\mathbb{R}^{\infty})^{\times n} \to \mathbb{R}^{\infty}$ of $\mathcal{I}(n)$ induces a natural map

$$\alpha^* : \mathcal{O}_X^{\infty} \twoheadrightarrow (\prod_{i=0}^{\infty} \mathcal{O}_X)^{\times n}$$

for any real variety X, by taking transposes of the matrices defining each α_i and extending scalars to $\mathcal{O}_X$. Given a point of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})^{\times n}$ – that is, quotients

$\mathcal{O}_X^\infty \twoheadrightarrow E_i$, $i = 1, \dots, n$, each of which factors through the canonical quotient $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{O}_X^N$ for $N \gg 0$ – one may readily verify that the composition

$$\mathcal{O}_X^\infty \to (\mathcal{O}_X^\infty)^{\times n} \to \bigoplus_i E_i$$

is well-defined and belongs to $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$. In fact, we have a well-defined continuous pairings

$$\mathcal{I}(n) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})^{\times n} \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$$

for $n = 0, 1, \dots$. As in [FW2; 2.8], these pairings are readily verified to satisfy the axioms of an action of an operad on a space in the sense of [M1; §1]. As shown in [M1; §14], given the action of an operad $\mathcal{I}$ on a space Y, there is a functorial construction of a spectrum whose zeroth space is the homotopy-theoretic group completion of Y (see below for a precise definition of this latter term).

Definition 1.3. *For any quasi-projective real variety X, define $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ to be the spectrum associated to the $\mathcal{I}$-space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$, as described in* [M1; §14]. *In particular, if $X^w \to X$ is the weak normalization of X, then*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \equiv \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X^w).$$

The homotopy groups of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ are written

$$K\mathbb{R}_n^{\mathrm{semi}}(X) \equiv \pi_n \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X).$$

In particular, there is an induced natural map of H-spaces $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$, which is a homotopy-theoretic group completion. Namely, this map induces isomorphisms

$$(\pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}))^+ \xrightarrow{\cong} K\mathbb{R}_0^{\mathrm{semi}}(X)$$

and

$$H_*(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), A) \otimes_{\mathbb{Z}[\pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})]} \mathbb{Z}\left[(\pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}))^+\right]$$
$$\xrightarrow{\cong} H_*(\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X), A),$$

where A is any commutative coefficient ring and the superscript "+" denotes the group completion of an abelian monoid. This gives some sense of the space $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$. In Section 7 of this paper, we give a more explicit description of this space in terms of a mapping telescope.

The natural map

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X \times \mathbb{A}^1)$$

is a weak homotopy equivalence of spectra, since by the techniques of [FW1], for any real quasi-projective varieties X and G, there is a continuous map

$$\mathbb{A}^1(\mathbb{R}) \times \mathcal{M}or_{\mathbb{R}}(X \times \mathbb{A}^1, G) \to \mathcal{M}or_{\mathbb{R}}(X \times \mathbb{A}^1, G)$$

sending (t,f) to $f(-,-\cdot t)$. This map shows that $\mathcal{M}or_{\mathbb{R}}(X,G)$ is a deformation retract of $\mathcal{M}or_{\mathbb{R}}(X\times\mathbb{A}^1,G)$. Thus the theory $\mathcal{K}\mathbb{R}^{\text{semi}}(-)$ is homotopy invariant.

We now turn to the issue of relating $\mathcal{K}\mathbb{R}^{\text{semi}}$ to algebraic K-theory of real varieties. To begin the construction, we define an algebraic version of the operad $\mathcal{I}$. Namely, define $\mathcal{I}(n)(\Delta^\bullet)$ to be the simplicial set

$$d\mapsto \varprojlim_N \varinjlim_M \operatorname{Hom}(\Delta^d_{\mathbb{R}},\mathcal{I}(j)_{N,M}).$$

Here, $\Delta^\bullet_{\mathbb{R}}$ is the standard cosimplicial object in the category of real varieties and $\mathcal{I}(j)_{N,M}$ denotes the real variety parameterizing injective $\mathbb{R}$-linear maps $\mathbb{R}^{jN}\hookrightarrow\mathbb{R}^M$. Upon taking geometric realizations, we obtain the collection of space $|\mathcal{I}(j)(\Delta^\bullet_{\mathbb{R}})|$, $j=0,1,\dots$. The same structure used to define $\mathcal{I}$ gives this collection of spaces the structure of an E_∞-operad written $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$. Indeed, there is a natural map of E_∞-operads

$$|\mathcal{I}(\Delta^\bullet)|\to\mathcal{I}$$

and thus we may equivalently regard the spectrum $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ as being defined by the action of $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$ on the space $\mathcal{M}or_{\mathbb{R}}(X,\text{Grass}_{\mathbb{R}})$.

For any real variety, consider the simplicial set

$$d\mapsto \operatorname{Hom}_{\mathbb{R}}(\Delta^d_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}}).$$

There are evident pairings of simplicial sets

$$\mathcal{I}(j)(\Delta^d_{\mathbb{R}})\times\operatorname{Hom}_{\mathbb{R}}(\Delta^d_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})^{\times j}\to\operatorname{Hom}_{\mathbb{R}}(\Delta^d_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})$$

defined just as above for the action of $\mathcal{I}$ on $\mathcal{M}or_{\mathbb{R}}(X,\text{Grass}_{\mathbb{R}})$. Indeed, taking geometric realizations, we obtain an $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-space

$$\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})\equiv|d\mapsto\operatorname{Hom}_{\mathbb{R}}(\Delta^d_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})|.$$

Using the main result of [GW] and mimicking the argument of [FW2; 6.8], which permits a change of contexts from Segal's Γ-spaces to spaces with operad actions, it follows that the spectrum associated to $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-space $|\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})|$ gives a model for the algebraic K-theory space $|d\mapsto\mathcal{K}^{\text{alg}}(\Delta^d_R\times X)|$ for any quasi-projective real variety X. In particular, when X is smooth, we recover the algebraic K-theory of X in this manner.

For X and G quasi-projective real varieties, a continuous algebraic morphism $\Delta^n_{\mathbb{R}}\times X\to G$ induces a continuous map $\Delta^n(\mathbb{R})\to\mathcal{M}or_{\mathbb{R}}(X,G)$ (this follows from the "internal Hom" description of $\mathcal{M}or$ found in [FW1; §1]). Moreover, the standard topological simplex Δ^n_{top} is naturally a subspace of $\Delta^n(\mathbb{R})$. There is thus a natural map $\operatorname{Mor}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}}\times X,\text{Grass}_{\mathbb{R}})\to\operatorname{Maps}(\Delta^\bullet_{top},\mathcal{M}or_{\mathbb{R}}(X,\text{Grass}_{\mathbb{R}}))$ of simplicial sets, where Maps denotes the set of continuous maps between topological spaces. Moreover, the geometric realization of this map is easily seen to be a map of $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-spaces and there is a natural map of $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-spaces

$$|\operatorname{Maps}(\Delta^\bullet_{top},\mathcal{M}or_{\mathbb{R}}(X,\text{Grass}_{\mathbb{R}}))|\to\mathcal{M}or_{\mathbb{R}}(X,\text{Grass}_{\mathbb{R}}).$$

We therefore obtain the following analogue of [FW2; 6.11].

Proposition 1.4. *For a quasi-projective real variety X, there is a natural map of spectra*

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet \times X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$$

induced by the composition of the maps of $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-spaces

$$\begin{aligned} &|\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}} \times X, \mathrm{Grass}_{\mathbb{R}})| \to |\operatorname{Mor}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}} \times X, \mathrm{Grass}_{\mathbb{R}})| \\ &\qquad \to |\operatorname{Maps}(\Delta^\bullet_{top}, \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}))| \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}). \end{aligned}$$

The following proposition verifies that our construction of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ gives us the infinite loop space $\mathcal{K}^{\mathrm{semi}}(X)$ of [FW2] whenever X has the structure of a complex variety – i.e., if the structure map $X \to \operatorname{Spec}\mathbb{R}$ is given a factorization $X \to \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$.

Proposition 1.5. *If X is a quasi-projective complex variety, then there is a natural homotopy equivalence*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \sim \mathcal{K}^{\mathrm{semi}}(X).$$

Consequently, for any quasi-projective real variety X, there exists a natural map

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X_{\mathbb{C}}) \sim \mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}}).$$

Proof. As shown after Definition 1.1, when X is a complex quasi-projective variety, there is a natural homeomorphism

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \cong \mathcal{M}or_{\mathbb{C}}(X, \mathrm{Grass}_{\mathbb{C}}),$$

and this homeomorphism is clearly compatibly with the $\mathcal{I}$-space structures used to define each of $\mathcal{K}^{\mathrm{semi}}(X)$ and $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$. (Observe that $\mathcal{K}^{\mathrm{semi}}(X)$ was actually defined in [FW2] using a complex version of the operad $\mathcal{I}$. Since there is a natural map from $\mathcal{I}$ to this complex version of itself, up to homotopy equivalence, we can equivalently define $\mathcal{K}^{\mathrm{semi}}(X)$ using the operad $\mathcal{I}$.) The first result follows immediately, and the second result follows from the first using the contravariant functoriality of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$. □

We next provide an explicit description of $K\mathbb{R}_0^{\mathrm{semi}}(-)$ for projective real varieties it terms of vector bundles modulo an equivalence relation. The reader should be warned that *real topological equivalence* as defined in the following Proposition 1.6 differs from the algebraic equivalence over the field $k = \mathbb{R}$ considered in [FW2; 1.1]. The difference is that here the analytic topology is used; i.e., the equivalence relation is generated by pairs of real algebraic vector bundles which are related by a family parametrized by an analytically connected portion of a real smooth curve. By contrast, the equivalence relation suggested in [FW1] would require merely an algebraically connected parameterizing variety (i.e., a variety connected in the Zariski topology). In defining $K_0^{\mathrm{semi}}(X)$ for a complex variety X, this distinction disappears since algebraic and analytic connectedness are the same.

Proposition 1.6. *For any weakly normal, projective real variety X, the group $K\mathbb{R}_0^{\mathrm{semi}}(X)$ is the quotient of $K_0^{\mathrm{alg}}(X)$ defined by real semi-topological equivalence, the equivalence relation generated by the following equivalence: given a smooth, connected real curve C and real points t_0, t_1 which lie in the same (analytic) connected* *component of the space $C(\mathbb{R})$ and a vector bundle E on $X \times_{\mathbb{R}} C$, the classes of $E|_{t_0}$ and $E|_{t_1}$ are equivalent.*

Proof. We show that $\pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ can be described as the collection of isomorphism classes of vector bundles generated by their global sections modulo real semi-topological equivalence of such bundles (that is, the equivalence relation defined as in the statement of this proposition in which all bundles considered are generated by their global sections). Once this assertion is proven, the proofs of [FW2; 2.10] and [FW2; 2.12] carry over into this context to show that $K\mathbb{R}_0^{\mathrm{semi}}(X)$ is the quotient of $K_0^{\mathrm{alg}}(X)$ by real semi-topological equivalence, thereby proving the proposition.

As in the proof of [FW2; 2.10], $\pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is a quotient of the set, $\mathrm{Isom}\, \mathcal{P}_{gl}(X)$, of isomorphism classes of algebraic vector bundles on X generated by their global sections because the connected component of a point $\mathcal{O}_X^{\infty} \twoheadrightarrow E$ of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ depends only on the isomorphism type of E. A bundle on $X \times_{\mathbb{R}} C$ together with a choice of global sections which generate it, for C a smooth real curve, defines a morphism of varieties $X \times_{\mathbb{R}} C \to \mathrm{Grass}_{\mathbb{R}}$ and consequently a continuous map $C(\mathbb{R}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$. It follows that the equivalence relation defining the surjection $\mathrm{Isom}\, \mathcal{P}_{gl}(X) \to \pi_0 \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ contains real semi-topological equivalence.

For the opposite containment, recall that since we have assumed X is projective and weakly normal, the space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is given as the real points of $\underline{\mathrm{Hom}}_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$, an infinite disjoint union of quasi-projective varieties. Thus, for any two points P_0, P_1 of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ in the same topological component (say representing the bundles E_0 and E_1), there exists a real quasi-projective variety T mapping to the ind-variety underlying $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ (via a morphism of real ind-varieties) and a pair of points $t_0, t_1 \in T(\mathbb{R})$ which lie in the same topological component and which map to P_0, P_1. Such a map determines a vector bundle on $X \times_{\mathbb{R}} T$ which restricts to the bundle E_i on t_i. It therefore suffices to show the following: given a quasi-projective real variety T and real points t_0, t_1 as above, there exists a finite collection of smooth real curves $C_1, \dots, C_n$, points $a_i, b_i \in C_i(\mathbb{R})$ lying in the same topological component for each i, and morphisms of real varieties $C_i \to T$ such that a_0 maps to t_0, b_i and a_{i+1} map to the same point of T for $i = 1, \dots n-1$, and b_n maps to the point t_1.

As an intermediate step, we claim that such a chain exists if we allow each C_i to be an arbitrary smooth variety T_i. By [H1], there exists a proper map $p : T' \to T$ with T' smooth and a proper closed subscheme $Z \subset T$ such that $T' - Z'$ maps isomorphically to $T - Z$, where $Z' \equiv Z \times_T T'$. Choose a semi-algebraic triangulation of $T(\mathbb{R})$ such that $Z(\mathbb{R})$ forms a subcomplex of this triangulation and the points t_0, t_1 are vertices of this triangulation (cf. [H2]). The points t_0 and t_1 may be joined by a path $\sigma : [0,1] \to T(\mathbb{R})$ which follows the edges of this triangulation. In particular, the path σ can be subdivided into finitely many subpaths $\sigma_i : [0,1] \to T(R)$ so that either $\sigma_i([0,1]) \subset Z(\mathbb{R})$ or $\sigma((0,1)) \subset T(\mathbb{R}) - Z(\mathbb{R})$ (where $(0,1)$ denotes the open unit interval). We claim that for all i, the points $\sigma_i(0), \sigma_i(1)$ can be joined by a path which either factors through $Z(\mathbb{R}) \hookrightarrow T(\mathbb{R})$ or through $T'(\mathbb{R}) \to T(\mathbb{R})$. Since T' is smooth and Z is a proper closed subvariety, this suffices to prove our claim by using Noetherian induction on T. Fix an i. If $\sigma_i([0,1]) \subset Z(\mathbb{R})$, there is nothing to show. If $\sigma_i((0,1)) \subset T(\mathbb{R}) - Z(\mathbb{R})$, then observe that since $T'(\mathbb{R}) - Z'(\mathbb{R}) = T(\mathbb{R}) - Z(\mathbb{R})$, we may lift $\sigma_i|_{(0,1)} : (0,1) \to T(\mathbb{R})$

to $\sigma'_i : (0,1) \to T'(\mathbb{R})$. Further, since the map $p : T'(\mathbb{R}) \to T(\mathbb{R})$ is proper, the subset $p^{-1}(\sigma_i([0,1])) \subset T'(\mathbb{R})$ is compact. It follows that the closure of $\sigma'_i((0,1))$ in $T'(\mathbb{R})$ is a compact, connected subset of $T'(\mathbb{R})$ which necessarily maps surjectively onto $\sigma_i([0,1])$ under p. We may thus find lifts of $\sigma_i(0)$ and $\sigma_i(1)$ which lie in the same path component of $T'(\mathbb{R})$. Our claim is proven.

Finally, by [I; Prop. 5], for any smooth, real variety T' with points $t_0, t_1 \in T'(\mathbb{R})$ which lie in the same path component, there exists a smooth real curve C, points $c_0, c_1 \in C(\mathbb{R})$ lying in the same path component, and a morphism $C \to T'$ of real varieties which sends c_i to t_i. Thus, we can replace each T_i in the chain constructed above with a smooth real curve. □

We close this section with a calculation of $K\mathbb{R}_0^{\text{semi}}(X)$ when X is a smooth, projective real curve. As observed in Remark 1.8, this example illustrates in particular that real semi-topological equivalence of vector bundles differs from algebraic equivalence

Proposition 1.7. *(cf.* [PW]*) Let X be a smooth, projective real curve of genus g. Assume $X(\mathbb{R}) \neq \emptyset$ and let c denote the number of connected components of the space $X(\mathbb{R})$. Then we have*

$$K\mathbb{R}_0^{\text{semi}}(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2^{\oplus(c-1)}.$$

Proof. Recall that $K_0^{\text{alg}}(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus Pic^0(X)$, where $Pic^0(X)$ is the group of isomorphism classes of line bundles of degree zero, and the projection to $\mathbb{Z} \oplus \mathbb{Z}$ sends a bundle E to $(rank(E), degree(E))$. Since the rank and degree maps define continuous maps $\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}) \to \mathbb{Z}$, it follows that

$$K\mathbb{R}_0^{\text{semi}}(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{Pic^0(X)}{\sim},$$

where the equivalence relation on $Pic^0(X)$ is given by semi-topological equivalence of line bundles.

A line bundle L of degree 0 on $X \times_{\mathbb{R}} U$ determines a morphism $U \to \underline{\text{Pic}}^0(X)$, where $\underline{\text{Pic}}^0(X)$ is the projective real variety parameterizing degree zero line bundles on X [G]. Clearly two points of $U(\mathbb{R})$ lying in the same topological component determine two real points in $\underline{\text{Pic}}^0(X)$ lying in the same topological component of $\underline{\text{Pic}}^0(X)(\mathbb{R})$, and it follows immediately that $Pic^0(X)/\sim$ is isomorphic to $\pi_0 \, \underline{\text{Pic}}^0(X)(\mathbb{R})$.

Finally, as shown in [PW; §1], a classical argument due to Weichold [We] shows that $\underline{\text{Pic}}^0(X)(\mathbb{R})$ is the topological subgroup of $\underline{\text{Pic}}^0(X_{\mathbb{C}})(\mathbb{C}) \cong (\mathbb{R}/\mathbb{Z})^{\times 2g}$ fixed by complex conjugation and moreover we have

$$\underline{\text{Pic}}^0(X)(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^g \times \mathbb{Z}/2^{\times(c-1)}$$

as topological groups. The result follows. □

Remark 1.8. *The example of Proposition 1.7 demonstrates that semi-topological equivalence, used to define $K\mathbb{R}_0^{\text{semi}}$, differs from algebraic equivalence, used in* [FW2; 1.1] *to define K_0^{semi} over an arbitrary ground field. For observe that (using the notation of the proof of 1.7) $\underline{\text{Pic}}^0(X)$ is a non-singular, (Zariski) connected algebraic variety. Thus, since every degree 0 line bundles on X is given as the restriction of the universal line bundle on $X \times_{\mathbb{R}} \underline{\text{Pic}}^0(X)$ to a real point of $\underline{\text{Pic}}^0(X)$, it follows that $K_0^{\text{alg}}(X)$ modulo algebraic equivalence is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}$.*

§2 Definitions and basic properties of $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$

Our construction of $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$ for a real quasi-projective variety X is a generalization of the construction for complex varieties presented in [FW3]. Indeed, as observed in Proposition 2.4, this construction generalizes that of [FW3]. Proposition 2.5 demonstrates that the natural map

$$\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \to \mathcal{K}\mathbb{R}^{\text{semi}}(X)$$

is a weak equivalence of spectra whenever X is weakly normal and projective. (Conceivably, this map is a weak equivalence for all quasi-projective real varieties X.) The advantage of $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$ over $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is that general properties of algebraic K-theory can often be transported to $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$. In particular, we establish the projective bundle theorem (Proposition 2.8), Nisnevich excision for smooth varieties (Theorem 3.5 of the next section), and localization (Theorem 6.6 of Section 6) for $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$.

For any compact CW complex T, let $Var^T(\mathbb{R})$ denote the category in which an object is a continuous map $T \to U(\mathbb{R})$ with U an affine, real variety. A morphism from $T \to U(\mathbb{R})$ to $T \to V(\mathbb{R})$ in Var^T is a morphism $V \to U$ of real varieties causing the evident triangle of spaces to commute. As argued in [FW2; 4.2] in the complex context, the category $Var^T(\mathbb{R})$ is directed and there is a natural bijection of sets

$$\text{Maps}(T, Y(\mathbb{R})) \simeq \varinjlim_{(T \to U(\mathbb{R})) \in Var^T(\mathbb{R})} \text{Hom}_{\mathbb{R}}(U, Y)$$

whenever Y is an inductive limit of quasi-projective real varieties.

As before, Δ^d_{top} denotes the standard d simplex, a subspace of $\mathbb{R}^{d+1}$. For any contravariant functor F from quasi-projective real varieties to sets, spaces, etc. and for any real quasi-projective variety X, we define a simplicial set by the formula

$$F(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \equiv d \mapsto \varinjlim_{(\Delta^d_{top} \to U(\mathbb{R})) \in Var^{\Delta^d_{top}}(\mathbb{R})} F(U \times_{\mathbb{R}} X).$$

In particular, we can take F to be the functor $\text{Hom}_{\mathbb{R}}(-, \text{Grass}_{\mathbb{R}})$ from quasi-projective real varieties to sets to form the simplicial set

$$\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}}) \equiv d \mapsto \varinjlim_{\Delta^d_{top} \to U(\mathbb{R})} \text{Hom}_{\mathbb{R}}(U \times_{\mathbb{R}} X, \text{Grass}_R).$$

Just as for $|\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{\mathbb{R}} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})|$, the space $|\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})|$ obtained by geometric realization has the structure of on $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-space and moreover the natural map

$$|\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{\mathbb{R}} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})| \to |\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})|$$

is a map of $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-spaces.

Definition 2.1. *For a quasi-projective real variety X, $\mathcal{K}^{\text{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$ denotes the spectrum associated to the $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-space $|\text{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})|$.*

As recalled in Appendix A of this paper, there is a strict functor sending a quasi-projective real variety X to a prespectrum $\mathcal{K}^{\text{alg}}(X)$ representing the algebraic K-theory of X (cf. [FS; App. B]). The word "strict" is used here merely for emphasis –

we just mean a functor in the ordinary sense. Note that this means given $f : T \to Y$ and $g : Y \to X$, it follows that the maps $(gf)^*, f^*g^* : \mathcal{K}^{\mathrm{alg}}(X) \to \mathcal{K}^{\mathrm{alg}}(T)$ coincide exactly on the spaces comprising these spectra, not merely up to homotopy. We can thus view $\mathcal{K}^{\mathrm{alg}}(-\times X)$ as a functor from quasi-projective real varieties to prespectra. In particular, we can extend the definition of the functor $\mathcal{K}^{\mathrm{alg}}(- \times X)$ to compact spaces via the formula

$$T \mapsto \mathcal{K}^{\mathrm{alg}}(T \times_{\mathbb{R}} U) \equiv \varinjlim_{T \to U(\mathbb{R})} \mathcal{K}^{\mathrm{alg}}(U \times_{\mathbb{R}} X),$$

and we can form a simplicial spectrum $d \mapsto \mathcal{K}^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X)$.

As the notation would suggest, we have the following result.

Lemma 2.2. *For any quasi-projective real variety X, there is a natural weak equivalence between $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ and $|d \mapsto \mathcal{K}^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X)|$.*

Proof. The proof is identical to the proof of [FW3; 1.3]. □

In light of the lemma, we are justified in writing $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ to refer to either of these equivalent spectra. We write $K^{\mathrm{alg}}_q(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ for the qth homotopy group of this spectrum.

Since we shall need it, we describe an extension of $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ which incorporates supports in a closed subscheme of X. As described in Appendix A, there is a model for the algebraic K-theory of X with supports in a closed subscheme Z, written $\mathcal{K}^{\mathrm{alg}}_Z(X)$, which is also strictly functorial. This induces the functor $U \mapsto \mathcal{K}^{\mathrm{alg}}_{U\times Z}(U \times X)$ on the category of real varieties. Taking $Z = X$, we have $\mathcal{K}^{\mathrm{alg}}_X(X) = \mathcal{K}^{\mathrm{alg}}(X)$. As explained in the appendix, $\mathcal{K}^{\mathrm{alg}}_Z(X)$ admits functorial deloopings, which we write here as

$$\Omega^{-1}\mathcal{K}^{\mathrm{alg}}_Z(X), \Omega^{-2}\mathcal{K}^{\mathrm{alg}}_Z(X), \dots,$$

such that $\Omega^{-j}\mathcal{K}^{\mathrm{alg}}_Z(X)$ is $(j-1)$-connected and such that there are natural weak equivalences

$$\mathcal{K}^{\mathrm{alg}}_Z(X) \xrightarrow{\sim} \Omega(\Omega^{-1}\mathcal{K}^{\mathrm{alg}}_Z(X)), \qquad \Omega^{-j}\mathcal{K}^{\mathrm{alg}}_Z(X) \xrightarrow{\sim} \Omega(\Omega^{-j-1}\mathcal{K}^{\mathrm{alg}}_Z(X)), \quad j \geq 1.$$

Definition 2.3. *For any quasi-projective real variety X and closed subscheme Z, the space*

$$\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X).$$

denotes the geometric realization of the simplicial space

$$d \mapsto \varinjlim_{(\Delta^d_{top} \to U(\mathbb{R})) \in Var^{\Delta^d_{top}}(\mathbb{R})} K^{\mathrm{alg}}_{U\times_{\mathbb{R}} Z}(U \times_{\mathbb{R}} X).$$

The qth homotopy group of $\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ is written $K^{\mathrm{alg}}_{Z,q}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$. The space $\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ in enriched to a spectrum by taking

$$\Omega^{-j}\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \equiv |d \mapsto \varinjlim_{(\Delta^d_{top} \to U(\mathbb{R})) \in Var^{\Delta^d_{top}}(\mathbb{R})} \Omega^{-j}K^{\mathrm{alg}}_{U\times_{\mathbb{R}} Z}(U \times_{\mathbb{R}} X)|$$

for its jth delooping and the structure maps induced in the evident manner.

Recall in [FW3] that we introduced the analogous construction for a complex quasi-projective variety Y:

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{C}} Y) \equiv |d \mapsto \varinjlim_{(\Delta^d_{top} \to V(\mathbb{C}) \in Var^{\Delta^d_{top}}(\mathbb{C})} \mathcal{K}(V \times X)|$$

where $Var^T(\mathbb{C})$ denotes the category analogous to $Var^T(\mathbb{R})$ with objects given by continuous map $T \to U(\mathbb{C})$, with U an affine, *complex* variety and morphisms induced by suitable morphisms of complex varieties. In view of the following proposition, the similarity of notation should offer no confusion.

Proposition 2.4. *Let Y be a quasi-projective complex variety and let $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{C}} Y)$ denote the construction of* [FW3] *mentioned above. Then there is a natural isomorphism*

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) \simeq \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{C}} Y).$$

Proof. The proposition follows easily from the following general result for any functor F on the category of schemes of finite type over $\mathbb{C}$ (with values in sets, groups, spaces, etc.): The natural map

$$\varinjlim_{T \to V(\mathbb{R})} F(V_{\mathbb{C}}) \to \varinjlim_{T \to U(\mathbb{C})} F(U)$$

is an isomorphism.

To prove the assertion, we first prove that the natural map is onto. An element of the target is represented by a pair $(\alpha : T \to U(\mathbb{C}), \gamma \in F(U))$. Let U' denote the Weil transform of U, which is by definition a real variety representing the functor $\mathrm{Hom}_{\mathbb{C}}(- \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}, U)$. Then there is a natural map $\pi : U'_{\mathbb{C}} \to U$ such that the composition $U'(\mathbb{R}) \to U'_{\mathbb{C}}(\mathbb{C}) \to U(\mathbb{C})$ is a homeomorphism of topological spaces. Thus the map $\alpha : T \to U(\mathbb{C})$ can be lifted to $\alpha' : T \to U'(\mathbb{R})$, and we see that the pair (α, γ) is equivalent to $(\alpha', \pi^*\gamma)$. Thus, this map is onto.

For injectivity, suppose given $(\alpha : T \to V(\mathbb{R}), \gamma \in F(V_{\mathbb{C}}))$ and $(\beta : T \to W(\mathbb{R}), \delta \in F(W_{\mathbb{C}}))$ which become equivalent in the target of the map of this claim. We need to show they are equivalent in the source. Since the indexing set $Var^T(\mathbb{C})$ is directed, it follows there is a complex variety U, a continuous map $h : T \to U(\mathbb{C})$, and a pair of maps (of complex varieties) $f : U \to V_{\mathbb{C}}$ and $g : U \to W_{\mathbb{C}}$ so that the following conditions hold: (1) the triangle formed from the maps $T \to U(\mathbb{C})$, $U(\mathbb{C}) \to V_{\mathbb{C}}(\mathbb{C})$, and $T \to V(\mathbb{R}) \to V_{\mathbb{C}}(\mathbb{C})$ commutes and similarly for the triangle involving W in place of V and (2) $f^*\gamma = g^*\delta$.

Take Weil transforms of the maps $f : U \to V_{\mathbb{C}}$ and $g : U \to W_{\mathbb{C}}$ to get maps $f' : U' \to (V_{\mathbb{C}})'$ and $g' : U' \to (W_{\mathbb{C}})'$. There are natural maps of real varieties $V \to (V_{\mathbb{C}})'$ and $W \to (W_{\mathbb{C}})'$. Define $\alpha_1 : T \to (V_{\mathbb{C}})'(\mathbb{R})$ to be the composition $T \to V(\mathbb{R}) \to (V_{\mathbb{C}})'(\mathbb{R})$, and define β_1 similarly. The given map $h : T \to U(\mathbb{C})$ determines a map $h_1 : T \to U'(\mathbb{R})$, and a diagram chase verifies that the the triangle formed from $T \to U'(\mathbb{R})$, $U'(\mathbb{R}) \to (V_{\mathbb{C}})'(\mathbb{R})$, and $T \to (V_{\mathbb{C}})'(\mathbb{R})$ commutes and similarly for W replacing V. Thus we have natural morphisms in $Var^T(\mathbb{R})$ from $T \to (V_{\mathbb{C}})'(\mathbb{R})$ to $T \to U'(\mathbb{R})$ and from $T \to (W_{\mathbb{C}})'(\mathbb{R})$ to $T \to U'(\mathbb{R})$.

In general, given any complex variety X, the complex variety $X'_{\mathbb{C}}$ (i.e., the complexification of the Weil transform of X) is isomorphic to $X \times_{\mathbb{C}} X$. In particular, the complexification of the map $V \to (V_{\mathbb{C}})'$ is isomorphic to the diagonal map $V_{\mathbb{C}} \to V_{\mathbb{C}} \times_{\mathbb{C}} V_{\mathbb{C}}$, as is thus a split injection, and similarly for W replacing V. Thus, the element (α, γ) is equivalent to the element (α_1, γ_1), where γ_1 is taken to be any lifting of $\gamma \in F(V_{\mathbb{C}})$ under the split surjection $F((V_{\mathbb{C}})'_{\mathbb{C}}) \to F(V_{\mathbb{C}})$. Say we choose γ_1 by choosing the splitting of $V_{\mathbb{C}} \to (V_{\mathbb{C}})'_{\mathbb{C}}$ given by projection onto the first factor. Similarly define $\delta_1 \in F((W_{\mathbb{C}})'_{\mathbb{C}})$ and note that (β, δ) is equivalent to (β_1, δ_1).

Finally, a diagram chase shows that the pullback of γ_1 and δ_1 under the maps $U'_{\mathbb{C}} \to (V_{\mathbb{C}})'_{\mathbb{C}}$ and $U'_{\mathbb{C}} \to (W_{\mathbb{C}})'_{\mathbb{C}}$ (which are the complexifications of the maps f' and g' introduced above) coincide. This suffices to show (α_1, γ_1) and (β_1, δ_1) are equivalent, and thus (α, γ) and (β, δ) are equivalent. □

Exactly as in the complex context considered in [FW3], we can easily relate $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$ and $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ whenever X is projective.

Proposition 2.5. *Let X be a quasi-projective real variety. Then there are natural maps of spectra*

$$\mathcal{K}^{\mathrm{alg}}(X) \to \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$$

whose composition is the map of Proposition 1.4. The map $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ is a weak equivalence of spectra whenever X is both weakly normal and projective. Moreover, for any quasi-projective real variety X, the natural map $K_0^{\mathrm{alg}}(X) \to K_0^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$ is the surjection given by modding out by real semi-topological equivalence.

Proof. The desired maps are induced by the maps of $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-spaces

$$|\operatorname{Hom}_{\mathbb{R}}(\Delta^{\bullet} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}})| \to |\operatorname{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}})| \to |Sing.\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})|.$$

Here, the first map is induced by the evident continuous map $\Delta^n_{top} \to \Delta^n(\mathbb{R})$, and the second map is induced by sending a pair $(\Delta^d_{top} \to U(\mathbb{R}), U \times_{\mathbb{R}} X \to \mathrm{Grass}_{\mathbb{R}})$ to the composition $\Delta^d_{top} \to U(\mathbb{R}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$. To prove the second assertion, it suffices to verify that the underlying map

$$\operatorname{Hom}_{\mathbb{R}}(\Delta^d_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}) \to Sing_d \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$$

is a bijection for each d when X is projective and weakly normal. The same argument as [FW2; 4.1] shows that if X and G are projective varieties with X weakly normal, then the space $\mathcal{M}or_{\mathbb{R}}(X, G_{\mathbb{R}})$ is the space of real points of the infinite disjoint union of real quasi-projective varieties which represents the functor (on real schemes) $U \mapsto \operatorname{Hom}_{\mathbb{R}}(U \times_{\mathbb{R}} X, G_{\mathbb{R}})$. Thus, the required bijection follows from the natural isomorphism

$$\operatorname{Maps}(T, Y(\mathbb{R})) \cong \varinjlim_{(T \to U(\mathbb{R})) \in Var^T(\mathbb{R})} \operatorname{Hom}_{\mathbb{R}}(U, Y),$$

whose proof is identical to that of [FW2; 4.2].

The final assertion follows just as in the proof of 1.6, starting with the observation that the map

$$K_0^{\mathrm{alg}}(\Delta^1_{top} \times_{\mathbb{R}} X) \to K_0^{\mathrm{alg}}(X),$$

given as the difference of the two face maps, has for its image all elements of the form $[E_{u_0}] - [E_{u_1}]$, where E is a bundle on $U \times_{\mathbb{R}} X$ and U is a quasi-projective real variety with $u_0, u_1 \in U(\mathbb{R})$ in the same topological component. □

We easily verify homotopy invariance of $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times -)$ as stated in the next proposition.

Proposition 2.6. *Let $\mathbb{A}^1$ denote the real affine line,* $\operatorname{Spec}\mathbb{R}[t]$. *For any real quasi-projective variety X, pull-back via the projection map*

$$\pi^* : \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} \mathbb{A}^1)$$

is a weak equivalence of spectra.

Proof. The proof of [FW3; 1.2] establishes that the asserted pull-back via the projection map induces an isomorphism in homology,

$$H_* \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \simeq H_* \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} \mathbb{A}^1).$$

Since these K-theory spaces are group-like H-spaces and therefore simple, such a homology equivalence is necessarily a weak homotopy equivalence. □

We close this section by establishing the projective bundle theorem for $\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$, a result which follows easily from the projective bundle theorem for algebraic K-theory. The analogous bundle theorem for complex varieties is proven in [FW3].

In Appendix A, a natural multiplication pairing is defined for any two pairs of closed subscheme inclusions $Z \subset X$, $W \subset Y$,

$$\mathcal{K}^{\text{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\text{alg}}_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) \to \mathcal{K}^{\text{alg}}_{Z \times W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y),$$

which satisfies the expected properties. (This pairing is actually only defined in the homotopy category of spaces, since the inverses of certain homotopy equivalences are needed for it's definition.) In the special case $Z = X = W = Y$, this induces a natural pairing

$$\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$$

by composing with pullback along the diagonal $X \hookrightarrow X \times X$. In particular, an element of $K_0^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ induces by multiplication a natural homotopy class of endomorphisms of the space $\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$. This multiplication is used in the following result.

Proposition 2.7. *Let X be a quasi-projective real variety and let $E \to X$ be a rank n vector bundle. Let $\gamma \in K_0^{\text{alg}}(\mathbb{P}(E))$ denote $[\mathcal{O}_{\mathbb{P}(E)}(1)] - [\mathcal{O}]$ as well as its image in $K_0^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E))$. Then the map*

$$\Sigma_{i=0}^{n-1} \gamma^i \cdot (-) : \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)^n \to \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E))$$

is a weak homotopy equivalence.

Proof. For any quasi-projective real variety U, the projective bundle theorem for the the pull-back of E to $U \times X$ establishes a weak homotopy equivalence

$$\Sigma_{i=0}^{n-1} \gamma^i \cdot (-) : \mathcal{K}(U \times_{\mathbb{R}} X)^n \to \mathcal{K}(U \times_{\mathbb{R}} \mathbb{P}(E))$$

for each $d \geq 0$, which is natural in U. Taking direct limits over $Var^{\Delta^n_{top}}(\mathbb{R})$, we obtain a natural weak equivalence

$$\Sigma_{i=0}^{n-1} \gamma^i \cdot (-) : \mathcal{K}^{\text{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X)^n \to \mathcal{K}^{\text{alg}}(\Delta^d_{top} \times_{\mathbb{R}} \mathbb{P}(E)),$$

for each $d \geq 0$. Since these maps fit into a map of simplicial spaces, we conclude (e.g., by applying [BF]) that the resulting map of geometric realizations of simplicial spaces

$$\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)^n \to \mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E))$$

is also a weak homotopy equivalence. □

§3 Nisnevich descent and finite coefficients

In this section, we establish two significant properties of the theory $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$ – namely, Nisnevich descent and the equivalence with algebraic K-theory when coefficients are taken in $\mathbb{Z}/n$, $n > 0$. These two seemingly unrelated results follow from similar technical considerations and thus are included together in this section.

Since we will use the concept multiple times, we introduce the notion of an *excisive* sequence of presheaves on the category of CW complexes. Namely, suppose $F = (F_q)_{q \in \mathbb{Z}}$ is sequence presheaves of abelian groups defined on the category of compact CW-pairs (that is, pairs of spaces (T, A) such that T admits the structure of a finite CW complex such that A is a subcomplex) equipped with long exact sequences

$$\cdots \to F^{q+1}(A) \xrightarrow{\partial_{(T,A)}} F^q(T, A) \to F^q(T) \to F^q(A) \to \cdots,$$

which are natural for maps of pairs. Here, we write $F^q(T)$ for $F^q(T, \emptyset)$. Moreover, assume that excision holds in the sense that if a map of pairs $(S, B) \to (T, A)$ restricts to a homeomorphism $S - B \cong T - A$, then the induced map $F^q(S, B) \to F^q(T, A)$ is an isomorphism for all q. Call such a sequence of functors an *excisive theory.* (Essentially, we are describing a cohomology theory except that homotopy invariance is not assumed to hold.) A map of excisive theories $F \to G$ is a collection of maps of presheaves $F_q \to G_q$, $q \in \mathbb{Z}$, which commute with the boundary maps.

The examples of excisive theories which we shall consider arise as in the following lemma.

Lemma 3.1. *Fix a quasi-projective real variety X with closed subvariety Z and fix an abelian group C. For any CW pair (T, A), define $F^q(T, A)$ to be the $-q$th homotopy group with coefficients in C of the homotopy fiber of the map of spectra*

$$\mathcal{K}^B_Z(T \times_{\mathbb{R}} X) \to \mathcal{K}^B_Z(A \times_{\mathbb{R}} X),$$

where $\mathcal{K}^B$ denotes Bass' K-theory as defined by Thomason in [TT]. *Then the collection F^q, $q \in \mathbb{Z}$, forms an excisive theory. A morphism $f : X \to Y$ together with a closed subscheme $W \subset Y$ such that $f^{-1}(W) \subset Z$ determines a map of excisive theories in the evident manner.*

Proof. If X is affine, say $X = \operatorname{Spec} R$, and $Z = X$, then the argument in the proof of [FW2; 5.1] shows that we can identify $\mathcal{K}^B(T \times_{\mathbb{R}} X)$ with $\mathcal{K}^B(\mathcal{C}_{\mathbb{R}}(T) \otimes_{\mathbb{R}} R)$, where $\mathcal{C}_{\mathbb{R}}(T)$ is the ring of real-valued continuous functions on a compact CW complex T. It follows from the results of [SW] that there is a fibration sequence of spectra

$$\mathcal{K}^B(\mathcal{C}_0(T/A) \otimes_{\mathbb{R}} R) \to \mathcal{K}^B(\mathcal{C}_{\mathbb{R}}(T) \otimes_{\mathbb{R}} R) \to \mathcal{K}^B(\mathcal{C}_{\mathbb{R}}(A) \otimes_{\mathbb{R}} R),$$

where $\mathcal{C}_0(T/A)$ denotes the ideal in $\mathcal{C}_{\mathbb{R}}(T/A)$ of functions vanishing at the point given by A. In general, the spectrum $\mathcal{K}^B(I)$ for a non-unital $\mathbb{R}$-algebra I is defined as the homotopy fiber of the split surjection of spectra

$$\mathcal{K}^B(I_+) \to \mathcal{K}^B(\mathbb{R}),$$

where I_+ is the unital $\mathbb{R}$-algebra formed by adjoining a multiplicative identity. In this case, $(\mathcal{C}_0(T/A) \otimes_{\mathbb{R}} R)_+$ is isomorphic to the $\mathbb{R}$-algebra $\mathcal{C}_{\mathbb{R}}(T/A) \otimes_{\mathbb{R}} R$. Excision then follows immediately, since given a relative homeomorphism $(S, B) \to$

(T, A) of compact CW pairs, we have a natural isomorphism of non-unital rings $\mathcal{C}_0(S/B) \cong \mathcal{C}_0(T/A)$. Thus, the induced map on the homotopy fibers of the rows of the commutative square

$$\begin{array}{ccc} \mathcal{K}^B_Z(S \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^B_Z(A \times_{\mathbb{R}} X) \\ \downarrow & & \downarrow \\ \mathcal{K}^B_Z(B \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^B_Z(A \times_{\mathbb{R}} X) \end{array} \tag{3.1.1}$$

is a weak equivalence when X is affine and $Z = X$. Since $\mathcal{K}^B_Z(T \times_{\mathbb{R}} X)$ is weakly equivalent to the homotopy fiber of

$$\mathcal{K}^B(T \times_{\mathbb{R}} X) \to \mathcal{K}^B(T \times_{\mathbb{R}} (X - Z)),$$

it follows that the map on homotopy fibers of the rows of (3.1.1) is a weak equivalence when X is affine and Z is any closed subscheme such that $X - Z$ is also affine. Finally, in general, we may cover X by a finite number of open affine subsets, $X = U_1 \cup \cdots \cup U_k$, such that $U_i - (U_i \cap Z)$ is also affine for all i. Using Mayer-Vietoris – i.e., the fact that the square

$$\begin{array}{ccc} \mathcal{K}^B_Z(T \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^B_{Z\cap U}(T \times_{\mathbb{R}} U) \\ \downarrow & & \downarrow \\ \mathcal{K}^B_{Z\cap V}(T \times_{\mathbb{R}} V) & \longrightarrow & \mathcal{K}^B_{Z\cap U\cap V}(T \times_{\mathbb{R}} U \cap V) \end{array}$$

is a homotopy Cartesian squares of spectra for any open cover $X = U \cup V$ [TT; 7.4] – and an easy induction argument, we see that the map on the homotopy fibers of the rows of (3.1.1) is a weak equivalence in all cases.

The naturality of the constructions used here establishes the final claim. □

The following technical result will be used in the proofs of the major results which follow.

Lemma 3.2. *Let F^q, $q \in \mathbb{Z}$, be an excisive theory. Given a contractible finite CW-complex D and a closed covering by CW subcomplexes, $D = D_1 \cup \cdots \cup D_k$, such that for any subset I of $\{1, \ldots, k\}$ the subcomplex $\cap_{i\in I} D_i$ is either contractible or empty (for example, if the D_i's are the maximal simplices in a triangulation of D), the map*

$$F^q(D) \to F^q(D_1) \oplus \cdots \oplus F^q(D_k)$$

is split injective. Moreover, given a map $F \to G$ of excisive theories, the splittings can be chosen to cause the evident square to commute.

Proof. We proceed by induction on k with the case $k = 1$ being obvious. Consider the square

$$\begin{array}{ccc} D_1 \cap D_j & \xrightarrow{\beta} & D_j \\ \downarrow & & \downarrow \\ D_j & \xrightarrow{\alpha} & D, \end{array}$$

for any $1 < j \leq k$. The map α admits a retraction since it is a cofibration of contractible spaces, which in turn induces a compatible retraction of β. Thus, the long-exact sequences associated to the rows of this diagram split to give a diagram with compatibly split exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^q(D_j, D_1 \cap D_j) & \longrightarrow & F^q(D_j) & \longrightarrow & F^q(D_1 \cap D_j) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^q(D, D_1) & \longrightarrow & F^q(D) & \longrightarrow & F^q(D_1) & \longrightarrow & 0. \end{array}$$

Taking direct sum over all $2 \leq j \leq k$, we obtain the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F^q(D, D_1) & \longrightarrow & \bigoplus_{j=2}^{k} F^q(D_j, D_1 \cap D_j) \\ \downarrow & & \downarrow \\ F^q(D) & \longrightarrow & \bigoplus_{j=2}^{k} F^q(D_j) \\ \downarrow & & \downarrow \\ F^q(D_1) & \longrightarrow & \bigoplus_{j=2}^{k} F^q(D_1 \cap D_j) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{3.2.1}$$

which has compatibly split exact columns. Now we claim that the quotient CW complex D/D_1 is covered by CW subcomplexes $D_2/(D_1 \cap D_2), \ldots, D_k/(D_1 \cap D_k)$ and satisfies the same hypotheses as the original covering of D by the D_i's. (Note that if $D_1 \cap D_j = \emptyset$, then we interpret $D_j/(D_1 \cap D_j)$ as being D_j and $(D_1 \cap D_j)/(D_1 \cap D_j)$ as being the empty set.) To see this, first observe that each $D_j/(D_1 \cap D_j)$ is indeed a CW subcomplex of the CW complex D/D_1 by [LW; 5.7]. Now observe that for $I \subset \{2, \ldots, k\}$, we have that

$$\cap_{i \in I} D_i/(D_1 \cap D_i) \cong (\cap_{i \in I} D_i)/(D_1 \cap (\cap_{i \in I} D_i)),$$

where the space on the right is the cofiber of the cofibration of contractible spaces

$$(D_1 \cap (\cap_{i \in I} D_i)) \to (\cap_{i \in I} D_i),$$

and is thus contractible. By induction, the map

$$F^q(D/D_1) \to \bigoplus_{j=2}^{k} F^q(D_j/(D_1 \cap D_j))$$

admits a natural splitting. Using the diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F^q(D,D_1) & \longrightarrow & \bigoplus_{j=2}^{k} F^q(D_j, D_1\cap D_j) \\
\downarrow & & \downarrow \\
F^q(D/D_1) & \longrightarrow & \bigoplus_{j=2}^{k} F^q(D_j/(D_1\cap D_j)) \\
\downarrow & & \downarrow \\
F^q(D_1/D_1) & \longrightarrow & \bigoplus_{j=2}^{k} F^q((D_1\cap D_j)/(D_1\cap D_j)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

we conclude the top arrow of (3.2.1) admits a natural splitting. A simple diagram chase completes the proof. □

The key ingredient for proving Nisnevich descent is the following result, which identifies the homotopy fiber of the map on $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$ associated to the restriction to an open subscheme of a smooth variety.

Theorem 3.3. *For a smooth, quasi-projective real variety X, an open subscheme $U \subset X$, and a closed subscheme $W \subset X$, the natural map*

$$K^{\mathrm{alg}}_{W,0}(\Delta^d_{top} \times X) \to K^{\mathrm{alg}}_{U\cap W,0}(\Delta^d_{top} \times U)$$

is a surjection for all $d \geq 0$. Consequently, if $Z = X - U$, the sequence of spectra

$$\mathcal{K}^{\mathrm{alg}}_{Z\cap W}(\Delta^d_{top} \times X) \to \mathcal{K}^{\mathrm{alg}}_{W}(\Delta^d_{top} \times X) \to \mathcal{K}^{\mathrm{alg}}_{U\cap W}(\Delta^d_{top} \times U)$$

is a homotopy fibration sequence of spectra.

Proof. Let $D = \Delta^d_{top}$ for $d \geq 0$. Recall from Appendix A that the spaces $\mathcal{K}^{\mathrm{alg}}_Z(D \times X)$, etc., admit functorial deloopings which we write here as

$$\Omega^{-1}\mathcal{K}^{\mathrm{alg}}_Z(D \times X), \Omega^{-2}\mathcal{K}^{\mathrm{alg}}_Z(D \times X), \dots.$$

We claim the first assertion shows that

$$\Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{Z\cap W}(D \times X) \to \Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{W}(D \times X) \to \Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{U\cap W}(D \times U)$$

is a fibration sequence for all j, thus proving the second assertion of the theorem. For note that there is a natural map

$$\Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{Z\cap W}(D \times X) \to \text{homotopy fiber}(\Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{W}(D \times X) \to \Omega^{-j}\mathcal{K}^{\mathrm{alg}}_{W\cap U}(D \times U))$$

which is an isomorphism on homotopy groups π_n for $n \geq j$ by [TT; 5.1]. The space $\Omega^{-j}\mathcal{K}^{\text{alg}}_{Z\cap W}(D \times X)$ is $(j-1)$-connected by construction and the homotopy fiber is $(j-1)$-connected provided the first assertion holds.

To establish the first claim, regard the map

$$K^{\text{alg}}_{W,0}(- \times X) \to K^{\text{alg}}_{W\cap U,0}(- \times U)$$

as a natural transformation of contravariant abelian-group-valued functors on the category of compact CW complexes. By Lemma 3.1, this map is part of the collection of maps defining a map of excisive theories

$$\Big((T,A) \mapsto \tilde{K}^{B}_{W,q}(T/A \times X)\Big) \to \Big((T,A) \mapsto \tilde{K}^{B}_{W\cap U,q}(T/A \times U)\Big).$$

For T any compact CW complex, an element of $K^{\text{alg}}_{W\cap U,0}(T \times U)$ is described by a pair $\Big(\alpha : T \to M(\mathbb{R}), \gamma \in K^{\text{alg}}_{M\times(W\cap U),0}(M \times U)\Big)$ and elements of $K^{\text{alg}}_{W,0}(T \times X)$ are similarly described. Pick an element $(f : D \to M(\mathbb{R}), \gamma \in K^{\text{alg}}_{M\times(U\cap W),0}(M \times U))$ representing an element of $K^{\text{alg}}_{W\cap U,0}(D \times U)$. We need to show it lifts to an element of $K^{\text{alg}}_{W,0}(D \times X)$. Observe that if M is smooth, such a lifting exists since

$$K^{\text{alg}}_{M\times W,0}(M \times X) \to K^{\text{alg}}_{M\times(U\cap W),0}(M \times U)$$

is surjective in this case because we have $K^{\text{alg}}_{M\times W,0}(M \times X) \cong K_0'(M \times W)$ and $K^{\text{alg}}_{M\times(U\cap W),0}(M \times U) \cong K_0'(M \times (U \cap W))$. Otherwise, using [H1] we can find a smooth variety $\tilde{M}$, a proper map $\pi : \tilde{M} \to M$, a closed proper subscheme $W \subset M$ such that if $\tilde{W} = \pi^{-1}(W)$, then the restriction of π maps $\tilde{M} - \tilde{W}$ isomorphically onto $M - W$. Moreover, there is a semi-algebraic triangulation of the space $M(\mathbb{R})$ such that $W(\mathbb{R})$ is a sub-complex of this triangulation [H2]. By replacing this triangulation with its barycentric subdivision, we have that the (closed) star-neighborhood of any point is contractible and the intersection of any simplex of $M(\mathbb{R})$ with the subcomplex $W(\mathbb{R})$ consists of a single face.

For a sufficiently fine barycentric subdivision of D with maximal simplices $\{D_i\}$, each D_i is mapped by f into a star-like neighborhood of $M(\mathbb{R})$. Since the horizontal maps of

$$\begin{array}{ccc} K^{\text{alg}}_{W,0}(D \times X) & \longrightarrow & \bigoplus_i K^{\text{alg}}_{W,0}(D_i \times X) \\ \downarrow & & \downarrow \\ K^{\text{alg}}_{W\cap U,0}(D \times U) & \longrightarrow & \bigoplus_i K^{\text{alg}}_{W\cap U,0}(D_i \times U) \end{array}$$

admit compatible splittings by Lemma 3.2, it suffices to check that each element of the form

$$(f|_{D_i} : D_i \to M(\mathbb{R}), \gamma \in K^{\text{alg}}_{W\cap U,0}(D_i \times U))$$

lifts to an element of $K^{\text{alg}}_{W,0}(D_i \times X)$. In other words, we may assume α maps D itself into a star-like neighborhood S of $M(\mathbb{R})$. But then (f, γ) lifts to an element of $K^{\text{alg}}_{W\cap U,0}(S \times U)$ and it clearly suffices to show this element lifts to an element of $K^{\text{alg}}_{W,0}(S \times X)$. Using Lemma 3.2 again, we see that it suffices to restrict attention to a single simplex of S. In other words, we may assume that D is a simplex of $M(\mathbb{R})$ and f is the inclusion map.

Recall that $A = D \cap W(\mathbb{R})$ is a single face of D. Let $\tilde{D} = \pi^{-1}(D)$ and $\tilde{A} = \pi^{-1}(A)$ where $\pi : \tilde{M} \to M$ is the proper map introduced above. Then the Cartesian square of compact CW complexes

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\subset} & \tilde{D} \\ \downarrow & & \downarrow \pi \\ A & \xrightarrow{\subset} & D \end{array}$$

has the properties that the horizontal maps are cofibrations, the vertical maps are proper, and π maps $\tilde{D} - \tilde{A}$ homeomorphically onto $D - A$. As argued in the proof of [FW3; 5.1], we can therefore find sufficiently small compact neighborhoods $A \subset B \subset D$ and $\tilde{A} \subset \tilde{B} \subset \tilde{D}$ so that $\tilde{A} \hookrightarrow \tilde{B}$ splits, $\pi^{-1}(B) \subset \tilde{B}$, and $\pi^{-1}(B) \hookrightarrow \tilde{D}$ splits. It follows that the inclusion $\tilde{A} \hookrightarrow \tilde{D}$ must split as well.

As in the proof of 3.2, a splitting of $\tilde{A} \hookrightarrow \tilde{D}$ defines a compatible splitting of the horizontal maps of the diagram

$$\begin{array}{ccc} K^{\text{alg}}_{W,0}(D \times X) & \longrightarrow & K^{\text{alg}}_{W,0}(\tilde{D} \times X) \oplus K^{\text{alg}}_{W,0}(A \times X) \\ \downarrow & & \downarrow \\ K^{\text{alg}}_{W\cap U,0}(D \times U) & \longrightarrow & K^{\text{alg}}_{W\cap U,0}(\tilde{D} \times U) \oplus K^{\text{alg}}_{W\cap U,0}(A \times U). \end{array}$$

Thus, it suffices to show the images of (α, δ) in $K^{\text{alg}}_{W\cap U,0}(\tilde{D} \times U)$ and $K^{\text{alg}}_{W\cap U,0}(A \times U)$ can be lifted. For the former image, this follows immediately since the map $\tilde{D} \to M(\mathbb{R})$ factors through the space of real points of the smooth variety $\tilde{M}$. By Noetherian induction, the image in $K^{\text{alg}}_{W\cap U,0}(\tilde{D} \times U)$ lifts as well. □

Corollary 3.4. *For X a smooth, quasi-projective real variety and Z, W closed sub-varieties, the sequence*

$$\mathcal{K}^{\text{alg}}_{Z\cap W}(\Delta^\bullet_{top} \times X) \to \mathcal{K}^{\text{alg}}_{W}(\Delta^\bullet_{top} \times X) \to \mathcal{K}^{\text{alg}}_{W-Z}(\Delta^\bullet_{top} \times (X - Z))$$

is a homotopy fibration sequence of spectra.

Proof. There is a compatible family of sequences of spaces

$$\Omega^{-j}\mathcal{K}^{\text{alg}}_{Z\cap W}(\Delta^\bullet_{top} \times X) \to \Omega^{-j}\mathcal{K}^{\text{alg}}_{W}(\Delta^\bullet_{top} \times X) \to \Omega^{-j}\mathcal{K}^{\text{alg}}_{W-Z}(\Delta^\bullet_{top} \times (X - Z)) \quad (3.4.1)$$

for all $j \geq 1$. For each such j, the underlying sequence of simplicial spaces is a degree-wise homotopy fibration sequence satisfying the hypotheses of [BF; B.4], and therefore (3.4.1) is a homotopy fibration sequence of spaces. The result follows. □

The following result, *Nisnevich excision*, follows easily from Theorem 3.3.

Theorem 3.5. *Let X be a smooth, quasi-projective real variety, $p : X' \to X$ be an étale map, $i : Z \hookrightarrow X$ be a closed subvariety. Assume $i : Z \hookrightarrow X$ factors through $p : X' \to X$ and write Z also for the induced closed subscheme of X'. Then the natural map*

$$p^* : \mathcal{K}^{\text{alg}}_{Z}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\text{alg}}_{Z}(\Delta^\bullet_{top} \times_{\mathbb{R}} X')$$

is a weak equivalence. In particular, if $U = X - Z$ and $U' = X' - Z$, there is a natural long exact sequence

$$\cdots \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X') \oplus K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U) \\ \to K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U') \to K_{q-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \cdots .$$

Proof. Let $Z' = Z \times_X X'$. Since $Z' \to Z$ is an étale map which admits a section, Z' decomposes as a disjoint union of closed subvarieties of X' of the form $Z' = Z \coprod Z''$. For each fixed d, consider the sequence

$$\mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X) \to \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X') \to \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} (X' - Z'')).$$

The maps $X' - Z'' \to X'$ and $X' - Z'' \to X$ are isomorphisms "infinitely near Z" in the sense of [TT; 2.6.2.1], and thus each of the maps

$$\mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X) \to \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} (X' - Z'')), \quad \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X') \to \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} (X' - Z''))$$

is a weak equivalence. Thus

$$\mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X) \to \mathcal{K}_Z^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} X')$$

is a weak equivalence for all d, and the first result follows by taking geometric realizations.

The long exact sequence follows from Theorem 3.3, since $\mathcal{K}_Z^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ and $\mathcal{K}_Z^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X')$ are the homotopy fibers of the rows of the diagram

$$\begin{array}{ccc} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U) \\ \downarrow & & \downarrow \\ \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X') & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U'). \end{array}$$

□

Remark 3.6. *Note that given an open cover $X = U \cup V$, if we take $X' = V$ and $Z = X - U$, then the hypotheses of Theorem 3.5 are satisfied and we obtain the familiar Mayer-Vietoris property – i.e., Zariski descent – for $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ on the category of smooth, quasi-projective real varieties.*

We next observe that Zariski descent allows for a generalization of Proposition 2.7 in the smooth case.

Corollary 3.7. *Let X be a smooth, quasi-projective real variety and let $J \to X$ be a vector bundle or a torsor for a vector bundle. Then the natural map*

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} J)$$

is a weak equivalence.

Proof. The hypothesis ensures that X admits a covering by open affine subschemes $U_1, \ldots, U_m$ so that the restriction of $J \to X$ to any subscheme of the form $V =$

$U_{i_1} \cap \cdots \cap U_{i_k}$ is isomorphic to the canonical projection $\mathbb{A}^d \times V \to V$. The result now follows immediately from Proposition 2.7 and Theorem 3.5 by induction on m. □

We now turn to the proof that the natural map

$$\mathcal{K}^{\mathrm{alg}}(X) \to \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$$

induces an isomorphism on homotopy groups with coefficients in $\mathbb{Z}/n$, $n > 0$, for any quasi-projective real variety X. The analogous statement for complex varieties – namely, that the natural map

$$\mathcal{K}^{\mathrm{alg}}(X) \to \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{C}} X)$$

induces an isomorphism on homotopy groups with finite coefficients for any quasi-projective complex variety – holds by [FW3; 3.7].

As with the corresponding complex result, the key ingredient is a "rigidity" result, as formulated in [SV; 4.4]. Recall from [FW3] that the presheaf $K_q^{\mathrm{alg}}(- \times_{\mathbb{R}} X; \mathbb{Z}/n)$ is a *pseudo-pretheory* in the sense of [FS]. As argued in [FW3; 3.2], the properties of a pseudo-pretheory established in [FS] suffice for the proof of [SV; 4.4] to work, thereby giving the following result.

Lemma 3.8. *(cf.* [SV; 4.4]*) If F is a homotopy invariant pseudo-pretheory defined on $Sch/\mathbb{R}$ such that $nF = 0$ for some $n > 0$, then for any smooth real variety Y and rational point $y \in Y(\mathbb{R})$, the natural map*

$$F(\operatorname{Spec} \mathcal{O}^h_{Y,y}) \to F(\operatorname{Spec} \mathbb{R})$$

is an isomorphism, where $\mathcal{O}^h_{Y,y}$ denotes the Henselization of the local ring at y.

Proof. The pairing established for any pseudo-pretheory in [FS; 10.1] is all that is needed for the proof of [SV; 4.4] to carry through for the presheaf F, which shows that if $Y = \mathbb{A}^n_{\mathbb{R}}$ and y is the origin, then the statement of this lemma holds. More generally, given Y smooth (which we may assume to be affine) with rational point y, there is a map $Y \to \mathbb{A}^d_{\mathbb{R}}$ sending y to the origin and which is étale in some neighborhood of y. Such a map induces an isomorphism $\mathcal{O}^h_{Y,y} \cong \mathcal{O}^h_{\mathbb{A}^d,0}$ and the result follows by the naturality of F. □

Theorem 3.9. *For any quasi-projective real variety Y and integers $q \geq 0$, $n > 0$, the natural map*

$$K_q^{\mathrm{alg}}(Y; \mathbb{Z}/n) \to K_q^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n)$$

is an isomorphism.

Proof. Regard the map

$$\mathcal{K}(Y; \mathbb{Z}/n) \to \left(d \mapsto \mathcal{K}^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n)\right)$$

as a map of simplicial spaces, in which the source is constant in the simplicial direction. By consideration of the map of associated spectral sequences [BF; B.5] from

$$\pi_p|d \mapsto K_q^{\mathrm{alg}}(Y; \mathbb{Z}/n)| \Longrightarrow K_{p+q}^{\mathrm{alg}}(Y; \mathbb{Z}/n)$$

to

$$\pi_p|d \mapsto K_q^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n)| \Longrightarrow K^{\mathrm{alg}}_{p+q}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n),$$

we see that it suffices to prove the natural map

$$K_q^{\mathrm{alg}}(Y; \mathbb{Z}/n) \to K_q^{\mathrm{alg}}(\Delta^d_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n) \tag{3.9.1}$$

is an isomorphism for all $q \geq 0$ and all $d \geq 0$.

We regard $K_q^B(Y; \mathbb{Z}/n) \to K_q^B(\Delta^d_{top} \times_{\mathbb{R}} Y; \mathbb{Z}/n)$ as the induced map for the excisive theory $F^* = K^B_{-*}(- \times_{\mathbb{R}} Y; \mathbb{Z}/n)$ associated to the map $\Delta^d_{top} \to pt$. Thus, it suffices to show $F^q(\Delta^d_{top}, pt) = 0$ for all $q \in \mathbb{Z}$. In fact, we show that if T is any contractible simplicial complex containing a point P, we have $F^q(T, P) = 0$.

Let (T, P) be any such pair. Observe that the long exact sequence for F splits to give the short exact sequence

$$0 \to F^q(T, P) \to F^q(T) \to F^q(P) \to 0,$$

for each q. It follows that an element α of $F^q(T, P)$ may be represented by a pair $(g : T \to U(\mathbb{R}), \gamma \in K^B_{-q}(U \times_{\mathbb{R}} Y; \mathbb{Z}/n))$ such that the restriction of γ to $g(P)$ gives the zero element of $K^B_{-q}(Y; \mathbb{Z}/n)$. Observe that since T is connected, we can assume U is algebraically connected. More generally, let $\alpha = (g : T \to U(\mathbb{R}), \gamma \in K^{\mathrm{alg}}_{-q}(U \times_{\mathbb{R}} Y; \mathbb{Z}/n))$ be any element of $F^q(T)$ such that U is algebraically connected and γ vanishes upon restriction to at least one $\mathbb{R}$-point u of U. We show $\alpha = 0$ under just this hypothesis.

As a preliminary step, we prove that any such α has the property that the associated element $\gamma \in K^B_{-q}(U \times_{\mathbb{R}} Y; \mathbb{Z}/n)$ vanishes upon restriction to any point in $U(\mathbb{R})$ lying in the same topological connected component of $U(\mathbb{R})$ as u. In particular, this shows that the image of α in $F^q(Q)$ vanishes for any point Q of T. To see this, let v any other point in the same topological connected component of u. Then as argued in the proof of Proposition 1.6, there exists a chain of maps from smooth real curves $h_i : C_i \to U$ joining u to v in a piecewise fashion. Let γ_i denote the pullback of γ to $K^B_{-q}(C_i \times_{\mathbb{R}} Y; \mathbb{Z}/n)$. By rigidity (cf. Lemma 3.8), for any real point c on C_i, the map

$$K^B_{-q}(\operatorname{Spec} \mathcal{O}^h_{C_i,c} \times_{\mathbb{R}} Y; \mathbb{Z}/n) \to K^B_{-q}(Y; \mathbb{Z}/n)$$

given by restriction to the closed fiber is an isomorphism. In particular, this shows that the vanishing of γ_i is both an open an closed condition on the set of real points of C_i, since if γ_i restricts to the constant element δ at a point c, then γ_i and δ agree in some Nisnevich neighborhood of c. Thus, if γ_i restricts to zero at one real point of C, then it does so for every real point in the same Zariski connected component. From this, it follows that since γ vanishes at u, it must vanish at v too.

We claim that if U is smooth, then $\alpha = 0$. Since γ vanishes locally in the Nisnevich topology on U, by quasi-compactness we can find a finite collection of étale maps $\{V_i \to U\}$ such that the image of $\coprod_i V_i(\mathbb{R}) \to U(\mathbb{R})$ contains $f(T)$ and such that the pullback of γ to $K^B_{-q}(V_i \times_{\mathbb{R}} Y; \mathbb{Z}/n)$ vanishes for all i. Since $\coprod_i V_i(\mathbb{R}) \to U(\mathbb{R})$ is a local homeomorphism, taking a sufficiently fine subdivision of the triangulation of T with maximal simplices $\{T_j\}$, each map $g|_{T_j} : T_j \to U(\mathbb{R})$

factors through $V_i(\mathbb{R}) \to U(\mathbb{R})$ for some i. It follows that the image of α in $F^q(T_i)$ is zero; but the map

$$F^q(T) \to \bigoplus_i F^q(T_i)$$

is injective by Lemma 3.2, and our claim is established.

More generally, if U is singular, then we proceed by Noetherian induction on U. As in the proof of Theorem 3.3, using [H1], [H2], there exists a proper map $\pi : U' \to U$ with U' smooth and a closed proper subscheme $Z \subset U$ such that setting $Z' = Z \times_U U'$ the induced map $U' - Z' \to U - Z$ is an isomorphism. Moreover, there is a semi-algebraic triangulation of $U(\mathbb{R})$ such that $Z(\mathbb{R})$ is a subcomplex, the intersection of any simplex with $Z(\mathbb{R})$ is a single face, and the star-neighborhood of any vertex is contractible. Note that if $g : T \to U(\mathbb{R})$ lands in $Z(\mathbb{R})$, we are done by Noetherian induction. In particular, we may assume $U'(\mathbb{R})$ is not empty.

Replacing T by any of its maximal simplices $\{T_j\}$ for a sufficiently fine triangulation and using the injectivity of

$$F^q(T) \to \bigoplus_j F^q(T_j)$$

again, we may assume g maps T into a star neighborhood S of $U(\mathbb{R})$. But then α lifts to the element of $F^q(S)$ represented by $(S \hookrightarrow U(\mathbb{R}), \gamma)$. It suffices to show this element vanishes. Replacing S by its maximal simplices and using Lemma 3.2 again, we see that it suffices to assume T is a simplex of $U(\mathbb{R})$.

Let $A = T \cap Z(\mathbb{R})$, which is a face of T, and write T' for $\pi^{-1}(T)$ and A' for $\pi^{-1}(A)$. Triangulate T' such that each simplex T'_j intersects A' in a single face written A'_j (or not at all). Then we claim that the map

$$F_q(T) \to F_q(A) \oplus \bigoplus_j F_q(T'_j)$$

is an injection. To see this, note that A'_j is mapped into A under $T' \to T$. Thus we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^q(T,A) & \longrightarrow & F^q(T) & \longrightarrow & F^q(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^q(T'_j, A'_j) & \longrightarrow & F^q(T'_j) & \longrightarrow & F^q(A'_j) & \longrightarrow & 0, \end{array}$$

for each j, whose rows are exact since $A \to T$ and $A'_j \to T'_j$ admit retractions. Taking direct sums, we obtain the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^q(T,A) & \longrightarrow & F^q(T) & \longrightarrow & F^q(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_j F^q(T'_j, A'_j) & \longrightarrow & \bigoplus_j F^q(T'_j) & \longrightarrow & \bigoplus_j F^q(A'_j) & \longrightarrow & 0. \end{array}$$

A simple diagram chase shows that it suffices to prove

$$F^q(T,A) \to \bigoplus_j F^q(T'_j, A'_j)$$

is injective. Notice that, using excision, we can also fit this map into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^q(T,A) & \longrightarrow & F^q(T/A) & \longrightarrow & F^q(A/A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_j F^q(T'_j, A'_j) & \longrightarrow & \bigoplus_j F^q(T'_j/A'_j) & \longrightarrow & \bigoplus_j F^q(A'_j/A'_j) & \longrightarrow & 0, \end{array}$$

with exact rows. (As before, if $A'_j = \emptyset$, we interpret T'_j/A'_j as being T'_j and A'_j/A'_j as empty.) Observe that the collection $\{T'_j/A'_j\}$ cover $T'/A' \cong T/A$ and the subset formed by removing redundancies satisfy the hypotheses of Lemma 3.2. To see this, note that using [LW; 5.7] we have that T'/A' is a CW complex whose cells are the images of cells of T'; in particular, its maximal closed cells are given by $\{T'_j/T'_j\}$. Furthermore, since each A'_j is a face of T'_j, it follows that for any subset J of our indexing set, the spaces $\cap_{j\in J} A'_j$, $\cap_{j\in J} T'_j$, and consequently $\cap_{j\in J} T'_j/A'_j \cong (\cap_{j\in J} T'_j)/(\cap_{j\in J} A'_j)$ are contractible. Thus, by Lemma 3.2, the middle arrow and consequently the left-hand vertical arrow of this diagram is an injection.

Note that α is sent to the element $\alpha_A \in F_q(A)$ represented by the pair $(A \to Z(\mathbb{R}), \gamma|_Z)$, and that $\gamma|_Z$ vanishes at every real point of Z in the image of $A \to Z(\mathbb{R})$. So by Noetherian induction, $\alpha_A = 0$. The image α'_j of α in $F_q(T'_j)$ is represented by $(T'_j \to U'(\mathbb{R}), \gamma')$, where γ' is the pullback of γ to $K^B_{-q}(U' \times_{\mathbb{R}} Y; \mathbb{Z}/n)$. Clearly, γ' vanishes at every real point of U' in the image of $T'_j \to U'(\mathbb{R})$, and since U' is smooth, we know from above that $\alpha'_j = 0$ for all j. □

Corollary 3.10. *For any projective real variety X, the natural map $\mathcal{K}^{\mathrm{alg}}(X) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ induces an isomorphism*

$$K_q^{\mathrm{alg}}(X; \mathbb{Z}/n) \to K\mathbb{R}_q^{\mathrm{semi}}(X; \mathbb{Z}/n)$$

for all $q \geq 0$, $n > 0$.

Proof. If X is weakly normal, this follows immediately from the theorem and Proposition 2.5. More generally, let $X^w \to X$ be the weak normalization of X and observe that we the natural isomorphisms $K\mathbb{R}_q^{\mathrm{semi}}(X; \mathbb{Z}/n) \cong K\mathbb{R}_q^{\mathrm{semi}}(X^w; \mathbb{Z}/n)$, given by definition, and $K_q^{\mathrm{alg}}(X; \mathbb{Z}/n) \cong K_q^{\mathrm{alg}}(X^w; \mathbb{Z}/n)$, given by [W; 1.6,3.2]. The result follows by naturality. □

Remark 3.11. *The proof of 3.9 can be extended slightly to show that $T \mapsto \mathcal{K}^B(T \times_{\mathbb{R}} Y; \mathbb{Z}/n)$ is not just excisive, but also homotopy invariant, as a functor from finite CW complexes to spectra. In other words, the collection of abelian group valued functors $T \mapsto K_q^B(T \times_{\mathbb{R}}; \mathbb{Z}/n)$ forms a generalized cohomology theory on the category of compact CW complexes. The value of this theory at a point is clearly given by the homotopy groups of the spectrum $\mathcal{K}^B(Y; \mathbb{Z}/n)$. In particular, taking $Y = \operatorname{Spec} \mathbb{R}$ and using Example 4.5 (of the next section), we obtain the natural isomorphism*

$$K_q^B(\mathcal{C}_{\mathbb{R}}(T); \mathbb{Z}/n) \cong KO^{-q}(T; \mathbb{Z}/n)$$

for $q \in \mathbb{Z}$ and $n > 0$. Here $\mathcal{C}_{\mathbb{R}}(T)$ denotes the ring of continuous real valued functions on T and KO denotes connective real topological K-theory. Similarly, taking $Y = \operatorname{Spec} \mathbb{C}$ and using Proposition 2.4, we obtain the natural isomorphism

$$K_q^B(\mathcal{C}_{\mathbb{C}}(T); \mathbb{Z}/n) \cong K_{\mathrm{top}}^{-q}(T; \mathbb{Z}/n)$$

for $q \in \mathbb{Z}$ and $n > 0$. We have thus recovered a result of Fischer [Fi] *for compact CW complexes.*

For an original application of 3.9, observe that by taking Y to be $\operatorname{Spec}\mathbb{R}[t, t^{-1}]$ *and* $\operatorname{Spec}\mathbb{C}[t, t^{-1}]$*, we obtain the natural isomorphisms*

$$K_q^B(\mathcal{C}_{\mathbb{R}}(T)[t, t^{-1}]; \mathbb{Z}/n) \cong KO^{-q}(T; \mathbb{Z}/n) \oplus KO^{-q+1}(T; \mathbb{Z}/n)$$

and

$$K_q^B(\mathcal{C}_{\mathbb{C}}(T)[t, t^{-1}]; \mathbb{Z}/n) \cong K_{\text{top}}^{-q}(T; \mathbb{Z}/n) \oplus K_{\text{top}}^{-q+1}(T; \mathbb{Z}/n),$$

for $q \in \mathbb{Z}$, $n > 0$.

§4 Comparison with $K\mathbb{R}_{\text{top}}^{-*}(-)$

Recall that for a complex quasi-projective variety Y, $\mathcal{K}^{semi}(Y)$ interpolates between the algebraic K-theory of Y and the topological K-theory of the associated analytic space Y^{an} (cf. [FW2], [FW3]). In this section, we verify the natural generalization of this observation to real varieties X: $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ interpolates between the algebraic K-theory of X and the Atiyah Real K-theory of $X_{\mathbb{R}}(\mathbb{C})$, the analytic space $X(\mathbb{C})$ equipped with the involution provided by complex conjugation.

We begin with the following definition of Atiyah's Real K-theory of a space equipped with a continuous involution (i.e., a *Real space*). Note that for compactly generated spaces X and Y, we write $\mathcal{M}\text{aps}(X, Y)$ for the topological space of all continuous maps from X to Y, endowed with the compactly generated topology associated to the compact-open topology. Further, if X and Y admit continuous actions by a group G (typically, $G = \mathbb{Z}/2$), then $\mathcal{M}\text{aps}^G(X, Y)$ denotes the subspace of $\mathcal{M}\text{aps}(X, Y)$ consisting of G-equivariant maps.

Definition 4.1. *For X a quasi-projective variety over $\mathbb{R}$, observe that*

$$\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \text{Grass}_{\mathbb{R}}(\mathbb{C})),$$

the space of all $\mathbb{Z}/2$-equivariant continuous maps

$$X_{\mathbb{R}}(\mathbb{C}) \to \text{Grass}_{\mathbb{R}}(\mathbb{C}) \equiv \coprod_n \text{Grass}_n(\mathbb{P}^\infty)_{\mathbb{R}}(\mathbb{C}),$$

has the structure of an $|\mathcal{I}(\Delta^\bullet_{\mathbb{R}})|$-space. We define (connective) Atiyah's Real K-theory of $X_{\mathbb{R}}(\mathbb{C})$, $\mathcal{K}\mathbb{R}_{\text{top}}(X_{\mathbb{R}}(\mathbb{C}))$, to be the associated spectrum as defined by [M1; §14]. *The qth homotopy group of $\mathcal{K}\mathbb{R}_{\text{top}}(X_{\mathbb{R}}(\mathbb{C}))$ is written $K\mathbb{R}_{\text{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C}))$ and is called the qth Atiyah Real K-group of the Real space $X_{\mathbb{R}}(\mathbb{C})$.*

We recall that if G is a discrete group acting on a space T, then the homotopy fixed point space T^{hG} of this action is the space

$$\mathcal{M}\text{aps}^G(EG, T).$$

Alternatively, this space can be identified with the space of sections of the Borel construction $EG \times_G T \to BG$. This homotopy fixed point space has the advantage that its homotopy groups are sometimes computable via a (not always convergent) spectral sequence of the form

$$E_2^{p,q} = H^p(BG, \pi_{-q}(T)) \Rightarrow \pi_{-p-q}(T).$$

The following result is a special case of a theorem recently proven by M. Karoubi.

Proposition 4.2. *(*[Ka]*) For any real quasi-projective variety X, the natural map*

$$\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \to \mathcal{K}_{\mathrm{top}}(X(\mathbb{C}))^{h\mathbb{Z}/2}$$

is a weak homotopy equivalence, where $\mathcal{K}_{\mathrm{top}}(X(\mathbb{C}))^{h\mathbb{Z}/2}$ denotes the homotopy fixed point space of the conjugation action on $\mathcal{K}_{\mathrm{top}}(X(\mathbb{C}))$. Furthermore, there is a strongly convergent spectral sequence

$$E_2^{p,q} = H^p(B\mathbb{Z}/2, K^q_{\mathrm{top}}(X(\mathbb{C})) \Rightarrow K\mathbb{R}^{p+q}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})). \tag{4.2.1}$$

As indicated at the beginning of this section, we now verify that Atiyah's Real K-theory plays the role of a suitable target for the semi-topological K-theory of real varieties.

Proposition 4.3. *Let X be a quasi-projective variety over $\mathbb{R}$. Then there exists a natural map of $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-spaces*

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C}))$$

which induces a map of spectra

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})).$$

Moreover, if X is provided with the structure of a complex variety, then this map is equivalent to the map considered in [FW2], [FW3]

$$\mathcal{K}^{semi}(X) \to \mathcal{K}_{\mathrm{top}}(X(\mathbb{C})).$$

Proof. The indicated map of $|\mathcal{I}(\Delta^{\bullet}_{\mathbb{R}})|$-spaces is merely the natural inclusion: a map over $\mathbb{R}$ induces a $\mathbb{Z}/2$-equivariant map of spaces of complex points and the topology on $\mathcal{M}or_{\mathbb{C}}(X, \mathrm{Grass}_{\mathbb{C}})$ is always at least as fine as the compact open topology.

The second assertion follows from Proposition 1.4 and the observation that for any complex variety X

$$\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C})) = \mathcal{M}\mathrm{aps}(X(\mathbb{C}), \mathrm{Grass}(\mathbb{C}))$$

since $X_{\mathbb{R}}(\mathbb{C}) = X(\mathbb{C})^{\times 2}$ (where the involution interchanges the two factors) whenever $X \to \operatorname{Spec}\mathbb{R}$ factors through $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$. $\square$

By applying the naturality of Proposition 4.3 to the map $X \to X_{\mathbb{C}}$, we immediately conclude that for any quasi-projective variety X over $\mathbb{R}$, the diagram

$$\begin{array}{ccccccc} \mathcal{K}^{\mathrm{alg}}(X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}^{\mathrm{alg}}(X_{\mathbb{C}}) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{C}} X_{\mathbb{C}}) & \longrightarrow & \mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}}) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \end{array} \tag{4.4}$$

is a homotopy commutative diagram of spectra (induced by a commutative diagram of $|\mathcal{I}(\Delta^{\bullet})|$-spaces).

Example 4.5. *In Section 6, we will establish various situations in which the map* $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$ *is a weak equivalence. For now, observe that there is the "trivial" example of such an equivalence – namely, the case* $X = \operatorname{Spec}\mathbb{R}$*. Note that in this case* $X_{\mathbb{R}}(\mathbb{C})$ *is a single point (with the trivial action) and* $\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$ *is homotopy equivalent to* $BO \times \mathbb{Z}$*. Thus, we conclude that*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(\operatorname{Spec}\mathbb{R}) \simeq \mathcal{K}\mathbb{R}_{\mathrm{top}}(\operatorname{Spec}\mathbb{R}(\mathbb{C})) \simeq BO \times \mathbb{Z}.$$

Note that in conjunction with Theorem 3.9, this example recovers Suslin's computation of the algebraic K*-theory of* $\mathbb{R}$ *with finite coefficients* [Su; 4.1]*. (Of course, we have used here the main technique – namely, rigidity – invented by Suslin for his original calculation.)*

For future reference, we record the following elementary observation.

Proposition 4.6. *Let* X *be a quasi-projective variety over* $\mathbb{R}$*. Then the natural maps of* $|\mathcal{I}(\Delta^\bullet)|$*-spaces*

$$\mathcal{M}\mathrm{aps}(X(\mathbb{C}), \mathrm{Grass}(\mathbb{C})) \leftarrow \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X(\mathbb{C}), \mathrm{Grass}(\mathbb{C})) \to \mathcal{M}\mathrm{aps}(X(\mathbb{R}), \mathrm{Grass}(\mathbb{R}))$$

determine natural maps of spectra

$$\mathcal{K}_{\mathrm{top}}(X(\mathbb{C})) \leftarrow \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \to \mathcal{K}O(X(\mathbb{R}))$$

where $\mathcal{K}O(-)$ *denotes real orthogonal connective topological* K*-theory.*

The following observation is suggestive of a way to consider varieties over fields more general that $\mathbb{R}$ and $\mathbb{C}$. This observation is made for the reader familiar with the construction of étale K-theory by W. Dwyer and E. Friedlander [DF].

Proposition 4.7. *Let* X *be a quasi-projective variety over* $\mathbb{R}$*. Then for any prime* ℓ *the profinite* ℓ*-completion of* $\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$ *is weakly equivalent to the étale* K*-theory space* $\mathcal{K}_{\text{ét}}(X)$ *for the prime* ℓ*.*

Proof. We recall that $\mathcal{K}_{\text{ét}}(X)$ can be realized as the homotopy fixed point space of $\mathcal{K}_{\text{ét}}(X_{\mathbb{C}})$ with respect to the Galois action of $\mathbb{Z}/2$. The proposition follows from the observation that $\mathcal{K}_{\text{ét}}(X_{\mathbb{C}})$ is the ℓ-adic completion of $\mathcal{K}_{\mathrm{top}}(X(\mathbb{C}))$. Namely, comparing Karoubi descent spectral sequences (4.2.1) taken mod-ℓ, we conclude that the natural map

$$\mathcal{K}\mathbb{R}_{top}(X_{\mathbb{R}}(\mathbb{C})) \simeq (\mathcal{K}_{\mathrm{top}}(X(\mathbb{C}))^{h\mathbb{Z}/2} \to \mathcal{K}_{\text{ét}}(X_{\mathbb{C}})^{h\mathbb{Z}/2} \simeq \mathcal{K}_{\text{ét}}(X)$$

induces an isomorphism in mod-ℓ homotopy groups and thus induces a weak equivalence

$$(\mathcal{K}\mathbb{R}_{top}(X_{\mathbb{R}}(\mathbb{C})))^{\wedge} \xrightarrow{\sim} \mathcal{K}_{\text{ét}}(X).$$

□

§5 Transfer maps

In this section, we define transfer maps associated to a finite, étale map $X' \to X$ of quasi-projective real varieties for each of the theories $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} -)$, $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$, $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$, and $\mathcal{K}\mathbb{R}_{\mathrm{top}}$, and we show these transfer maps are compatible with the maps between these theories. The case of primary interest is when $X' = X_{\mathbb{C}}$, regarded as a real variety, which will allow us among other things to compare the $\mathcal{K}\mathbb{R}^{\text{semi}}$-theory of a real variety with the $\mathcal{K}^{\text{semi}}$-theory of its complexification.

It turns out that in order to define transfer maps as maps of infinite loop spaces, it is convenient to introduce a different, but homotopy equivalent, model for the infinite loop spaces $\mathcal{K}^{\text{alg}}(\Delta^\bullet \times X)$, $\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$, $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$, and $\mathcal{K}\mathbb{R}_{\text{top}}(X_{\mathbb{R}}(\mathbb{C}))$. These new models use Segal's notion of a Γ-space in place of operads.

To begin, we define $\text{Grass}_{\mathbb{R}}(\mathbb{P}^N)^{(n)} \subset \text{Grass}_{\mathbb{R}}(\mathbb{P}^N)^{\times n}$ to be the quasi-projective subvariety parameterizing n-tuples of subspaces $V_i \subset \mathbb{R}^{N+1}$ in general position – i.e., which satisfy $V_i \cap \sum_{j \neq i} V_j = 0$ for all i. This is shown to be an open subvariety of the projective variety $\text{Grass}_{\mathbb{R}}(\mathbb{P}^N)^{\times n}$ in [GW; §2]. Define $\text{Grass}_{\mathbb{R}}^{(n)}$ to be the ind-variety $\varinjlim_N \text{Grass}_{\mathbb{R}}(\mathbb{P}^N)^{(n)}$. Then $\text{Grass}_{\mathbb{R}}(\mathbb{P}^N)^{(n)}$ (respectively, $\text{Grass}_{\mathbb{R}}^{(n)}$) represents the functor on quasi-projective real varieties sending X to the collection of n-tuples $(p_i : \mathcal{O}_X^{N+1} \twoheadrightarrow E_i)_{i=1,\dots,n}$ (respectively, n-tuples $(p_i : \mathcal{O}_X^{\infty} \twoheadrightarrow E_i)_{i=1,\dots,n}$ such that each quotient factors through the canonical projection $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{O}_X^M$ for $M \gg 0$) which are in general position in the sense that the induced map

$$(p_1, \dots, p_n)^t : \mathcal{O}_X^{N+1} \to \bigoplus_i E_i$$

is a surjection (respectively, the same map with $N+1$ replaced by ∞ is a surjection).

Recall that Γ^{op} is the category of finite, pointed sets of the form $\underline{n} = \{*, 1, \dots, n\}$, with $*$ the base point, whose morphisms are base point preserving functions. A Γ-object in a category $\mathcal{C}$ is a functor $\Gamma^{op} \to \mathcal{C}$. The map $\underline{n} \mapsto \text{Grass}_{\mathbb{R}}^{(n)}$ defines a Γ-ind-variety as follows. Given $f : \underline{n} \to \underline{m}$, let $f_* : \text{Grass}_{\mathbb{R}}^{(n)} \to \text{Grass}_{\mathbb{R}}^{(m)}$ send $(V_i \subset \mathbb{R}^\infty)_{i=1,\dots n}$ to $(W_j \subset \mathbb{R}^\infty)_{j=1,\dots m}$, where $W_j = \bigoplus_{i \in f^{-1}(j)} V_i$. Consequently, for any variety X, the collection of sets $\text{Hom}_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}^{(n)})$, $n = 0, 1, \dots,$ form a Γ-set.

For X a quasi-projective real variety, we associate to each $\underline{n}$ the topological space $\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}^{(n)})$ as defined in Definition 1.1. (More accurately, the space $\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}(\mathbb{P}_{\mathbb{R}}^N)^{(n)})$ is defined in 1.1 for each N and then we take the direct limit as N goes to infinity in the category of compactly generated topological spaces.) When X is weakly normal and projective, this space is the set of n-tuples of quotient $(\mathcal{O}_X^\infty \twoheadrightarrow E_i)_{i=1,\dots,n}$ with the topology induced by realizing this set as the real points of an ind-variety. For any X, observe that $\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}^{(-)})$ is a functor from Γ^{op} to spaces – i.e., it is a Γ-space.

We also consider the following three variations on the Γ-space $\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}^{(-)})$:

(1) the Γ-space
$$|\text{Hom}_{\mathbb{R}}(\Delta^\bullet \times X, \text{Grass}_{\mathbb{R}}^{(-)})|$$
obtained by taking geometric realizations of the simplicial sets $d \mapsto \text{Hom}_{\mathbb{R}}(\Delta^d \times X, \text{Grass}_{\mathbb{R}}^{(n)})$, for $n = 0, 1, \dots$;

(2) the Γ-space
$$|\text{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}}^{(-)})|$$
defined analogously as in the previous construction; and

(3) the Γ-space
$$\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \text{Grass}_{\mathbb{R}}^{(-)}(\mathbb{C})).$$

For this last example, we view $X_{\mathbb{R}}(\mathbb{C})$ and $\mathrm{Grass}_{\mathbb{R}}^{(n)}(\mathbb{C})$ as Real spaces in the sense of Atiyah and $\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}$ denotes the space of equivariant continuous maps endowed with the compact-open topology.

Moreover, there are natural maps of Γ-spaces

$$(5.1)\quad |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet \times X, \mathrm{Grass}_{\mathbb{R}}^{(-)})| \to |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}^{(-)})| \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}^{(-)}) \to \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(Y_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}^{(-)}(\mathbb{C})),$$

and the map

$$|\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}^{(-)})| \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}^{(-)})$$

is a weak-equivalence of Γ-spaces (i.e., the map induces a weak-equivalence of topological spaces for each fixed $\underline{n}$) whenever X is projective and weakly normal.

Recall that a Γ-space $G(-)$ is *special* if the canonical map $G(n) \to G(1)^{\times n}$ is a homotopy equivalence for all n.

Lemma 5.2. *For X a quasi-projective real variety, the Γ-spaces $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}^{(-)})$, $\operatorname{Hom}_{\mathbb{R}}(X \times \Delta^\bullet, \mathrm{Grass}_{\mathbb{R}}^{(-)})$, $\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}^{(-)})$, and $\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}^{(-)}(\mathbb{C}))$ are all special.*

Proof. The proof of [GW; 2.2] suffices to establish each of these claims. □

Naturally associated to any Γ-space $G(-)$ is a spectrum $\mathbb{S}(G)$, formed in the following manner. The zeroth space of the spectrum is the space $G(1)$. Observe that there is a functor $\Delta^{op} \to \Gamma^{op}$, where Δ is the category of finite, non-empty, totally ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, whose morphisms are non-decreasing functions. This functor sends $[n]$ to $\underline{n}$ and a order preserving function $f : [n] \to [m]$ to $f^* : \underline{m} \to \underline{n}$ where $f^*(i)$ is the smallest j in the set $f^{-1}(i)$ or $*$ if $j = 0$ or no such j exists. The first space in the spectrum $\mathbb{S}(G)$ is the geometric realization of the simplicial space $d \mapsto G(d)$ defined by restriction along $\Delta^{op} \to \Gamma^{op}$. Further, there is a functor for each d

$$(\Gamma^{op})^{\times d} \to \Gamma^{op}$$

sending $(\underline{n}_1, \ldots, \underline{n}_d)$ to $\underline{n_1 \cdot \cdots \cdot n_d}$, which allows any Γ-space to be viewed as a d-fold Γ-space and consequently a d-fold multi-simplicial space. The geometric realization of this associated d-fold multi-simplicial space gives $\mathbb{S}(G)_d$, the dth space of the spectrum.

For any Γ space G, there is in particular a natural map

$$G(1) \to \Omega |d \mapsto G(d)| \equiv \Omega \mathbb{S}(G)_1.$$

When G is a special Γ space, which holds for all cases we are interested in, this map gives a homotopy-theoretic group completion of the H-space $G(1)$, and moreover all the maps $\mathbb{S}(G)_d \to \Omega \mathbb{S}(G)_{d+1}$ are weak equivalences for $d \geq 1$ [Se]. Here, $G(1)$ is endowed with an H-space structure by choosing a homotopy inverse of the equivalence $G(2) \to G^{\times 2}$ and composing with the multiplication map $\mu : G(2) \to G(1)$ induced by the morphism $\underline{2} \to \underline{1}$ sending both 1 and 2 to 1.

Proposition 5.3. *For X projective and Y quasi-projective real varieties, there are natural weak equivalences of spectra*

$$\mathbb{S}(\mathrm{Hom}_{\mathbb{R}}(\Delta^{\bullet} \times_{\mathbb{R}} Y, \mathrm{Grass}_{\mathbb{R}}^{(-)})) \sim \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{R}} Y)$$
$$\mathbb{S}(\mathrm{Hom}_{\mathbb{R}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} Y, \mathrm{Grass}_{\mathbb{R}}^{(-)})) \sim \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} Y)$$
$$\mathbb{S}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}^{(-)})) \sim \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$$
$$\mathbb{S}(\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(Y_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}^{(-)}(\mathbb{C}))) \sim \mathcal{K}\mathbb{R}_{\mathrm{top}}(Y_{\mathbb{R}}(\mathbb{C})).$$

Moreover, these equivalences are compatible with the maps of (5.1) and Proposition 5.3.

Proof. The proof of each of these assertions is strictly parallel to the proof of [FW2; 6.8]. □

In short, everywhere where we have used operads in the study of $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{R}} -)$, $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$, $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$, and $\mathcal{K}\mathbb{R}_{\mathrm{top}}$, we may equivalently use the Γ-space constructions presented here instead.

As mentioned, the purpose of introducing these Γ-space models is to define transfer maps. Let $\pi : X' \to X$ be a finite, flat map of real quasi-projective varieties. This hypothesis ensures that the push-forward $\pi_* E$ of a vector bundle E on Y is a vector bundle on X, and heuristically the transfer map is merely given by pushforward of vector bundles. In addition, we assume there is a surjection $\phi : \mathcal{O}_X^k \twoheadrightarrow \pi_* \mathcal{O}_{X'}$ of $\mathcal{O}_X$-modules, which will become part of the construction. (Note that such a quotient determines a closed immersion $X' \hookrightarrow X \times \mathbb{A}^k$ through which the map π factors. The surjection ϕ should therefore be seen as playing a role analogous to the embedding $X' \hookrightarrow X \times int(I^k)$ used to define transfer maps for generalized cohomology theories in algebraic topology. See [Ad; §5.1].)

The pair (π, ϕ) allows us to associate to any quotient $\mathcal{O}_{X'}^{\infty} \twoheadrightarrow E$ the quotient given by the composition of

$$\mathcal{O}_X^k \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\infty} \xrightarrow{\phi \otimes \mathrm{id}} \pi_*(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\infty} \xrightarrow{\cong} \pi_*(\mathcal{O}_{X'}^{\infty}) \to \pi_* E, \tag{5.4}$$

where the middle isomorphism is the evident one. Finally, define θ to be the isomorphism $\theta : \mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{R}^{\infty} \xrightarrow{\cong} \mathbb{R}^{\infty}$ sending $e_i \otimes f_j$ to $f_{k(j-1)+i}$, where $e_1, \ldots, e_k$ and $f_1, f_2, \ldots$ are the standard bases of $\mathbb{R}^k$ and $\mathbb{R}^{\infty}$. (Really, any choice of isomorphism will work in place of θ – this particular isomorphism allows the proof of Theorem 5.8 below to proceed most smoothly. Since the definition of θ is somewhat arbitrary, we will include θ are part of the data defining the transfer map below.) The map θ induces an isomorphism

$$\mathcal{O}_X^k \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\infty} \cong \mathcal{O}_X^{\infty}$$

by extension of scalars and we let θ denote this isomorphism too. Precomposing the quotient object (5.4) with θ, we obtain the quotient object

$$\mathcal{O}_X^{\infty} \twoheadrightarrow \pi_* E.$$

Thus to any quotient $\mathcal{O}_{X'}^{\infty} \twoheadrightarrow E$, we associate a quotient of the form $\mathcal{O}_X^{\infty} \twoheadrightarrow \pi_* E$, and this map will give our transfer map. Note that this construction depends on the choices of π, ϕ, and θ.

We now extend these ideas to the topological setting. Suppose that the map $\pi : X' \to X$ is not merely finite and flat, but is also étale . Then π induces a finite covering space map of Real spaces $\pi : X'_{\mathbb{R}}(\mathbb{C}) \to X_{\mathbb{R}}(\mathbb{C})$ (that is, it is an $\mathbb{Z}/2$-equivariant covering space map). Moreover, the surjection $\phi : \mathcal{O}_X^k \twoheadrightarrow \pi_*\mathcal{O}_{X'}$ induces a surjection of Real topological vector bundles $X_{\mathbb{R}}(\mathbb{C}) \times \mathbb{C}^k \twoheadrightarrow X'_{\mathbb{R}}(\mathbb{C}) \times \mathbb{C}$. (For any Real space T, by $T \times \mathbb{C}^m$ we mean the Real vector bundle with involution given by $\overline{(t,v)} = (\bar{t}, \overline{v})$, where $\overline{v}$ denotes complex conjugation performed on each component of v.) Consequently, parallel to the preceding construction, the data (π, ϕ, θ) determines a map sending a quotient Real bundle $X'_{\mathbb{R}}(\mathbb{C}) \times \mathbb{C}^\infty \twoheadrightarrow E$ to a quotient Real bundle $X_{\mathbb{R}}(\mathbb{C}) \times \mathbb{C}^\infty \twoheadrightarrow \pi_*E$. Here, π_*E is the Real bundle on X whose stalk at x is $\bigoplus_{\pi(y)=x} E_y$, endowed with the evident topology and Real structure.

Associated to a finite, étale morphism $\pi : X' \to X$, the constructions we have described above associate to the triple (π, ϕ, θ) the maps

$$\mathrm{Hom}_{\mathbb{R}}(X', \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}),$$

$$\mathcal{M}or_{\mathbb{R}}(X', \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}),$$

and

$$\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X'_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C})) \to \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C})).$$

Each of these functions is natural in X in the following sense: a morphism $Y \to X$ determines by pullback a finite, étale map $\pi_Y : Y' \to Y$ and a surjection $\phi_Y : \mathcal{O}_Y^k \twoheadrightarrow \pi_{Y*}\mathcal{O}_{Y'}$. The transfer maps associated to (π_Y, ϕ_Y, θ) and to (π, ϕ, θ) fit into an evident commuting square in each of the three cases.

Using this naturality, one concludes that the transfer maps for $\mathcal{M}or$ is continuous, since it extends to the natural transformations of representable functors

$$\mathcal{M}or_{\mathbb{R}}(- \times_{\mathbb{R}} X', \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(- \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}).$$

Similarly, the transfer map for $\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}$ is easily seen to be continuous.

Naturality also gives a natural transformation of set-valued functors

$$\mathrm{Hom}_{\mathbb{R}}(- \times_{\mathbb{R}} X', \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{R}}(- \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}),$$

from which we define transfer maps

$$\mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X', \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}})$$

and

$$\mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X', \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}).$$

Finally, we define transfer maps

$$\mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X', \mathrm{Grass}^{(-)}) \to \mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X, \mathrm{Grass}^{(-)})$$
$$\mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X', \mathrm{Grass}^{(-)}) \to \mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}^{(-)})$$
$$\mathcal{M}or_{\mathbb{R}}(X', \mathrm{Grass}^{(-)}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}^{(-)})$$
$$\mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X'_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}^{(-)}_{\mathbb{R}}(\mathbb{C})) \to \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}^{(-)}_{\mathbb{R}}(\mathbb{C}))$$

by applying the previously constructed transfer maps component-wise (recalling that $\mathrm{Grass}^{(n)}_{\mathbb{R}}$ is a subvariety of $\mathrm{Grass}^{\times n}_{\mathbb{R}}$). Although these maps depend on the choices of π, ϕ, and θ, we will write each of them simply as π_*. The following proposition summarizes our result concerning the transfer map so far.

Proposition 5.5. *Given a finite, étale map $\pi : X' \to X$ and a surjection $\phi : \mathcal{O}_X^k \twoheadrightarrow \mathcal{O}_{X'}$, there are associated "transfer maps" of Γ-spaces*

$$\pi_* : |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X', \operatorname{Grass}^{(-)})| \to |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X, \operatorname{Grass}^{(-)})|$$
$$\pi_* : |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X', \operatorname{Grass}^{(-)})| \to |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \operatorname{Grass}^{(-)})|$$
$$\pi_* : \mathcal{M}or_{\mathbb{R}}(X', \operatorname{Grass}^{(-)}) \to \mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}^{(-)})$$
$$\pi_* : \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X'_{\mathbb{R}}(\mathbb{C}), \operatorname{Grass}^{(-)}_{\mathbb{R}}(\mathbb{C})) \to \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \operatorname{Grass}^{(-)}_{\mathbb{R}}(\mathbb{C})),$$

which commute with the natural transformations of these theories introduced in (5.1). Moreover, these transfer maps are natural in the sense that if $Y \to X$ is a morphism of varieties, then π_* and π_{Y*} (where π_{Y*} is defined by the induced maps $\pi_Y : Y' = Y \times_X X' \to Y$ and $\phi_Y : \mathcal{O}_Y^k \twoheadrightarrow \pi_*\mathcal{O}_{Y'}$) fit into the evident commuting squares of Γ-spaces.

Proof. The only fact remaining to be established is that π_* is actually a map of Γ-spaces. This is verified by a routine computation. □

Any map of Γ-spaces induces a map of associated spectra, and we will write π_* for each of the four associated transfer maps of spectra. Thus we have the following corollary.

Corollary 5.6. *Given a finite, étale map of quasi-projective varieties $\pi : X' \to X$ and a surjection $\phi : \mathcal{O}_X^k \twoheadrightarrow \pi_*\mathcal{O}_{X'}$, the transfer maps of Γ-spaces of Proposition 5.5 determine transfer maps of spectra π_* such that the following diagram commutes up to weak homotopy*

$$\begin{array}{ccccccc}
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X') & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X') & \longrightarrow & \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X') & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X'_{\mathbb{R}}(\mathbb{C})) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})).
\end{array}$$

Moreover, this diagram is natural up to weak homotopy in the same sense the maps of Proposition 5.5 are natural.

Remark 5.7. *We have abused notation a bit by using merely π_* to denote the transfer maps induced by the triple (π, ϕ, θ). We presume that, up to weak homotopy, the transfer map only depends on the morphism π, although we have not attempted to prove this result since we have no need for it in this paper. Additionally, there are well-known transfer maps for algebraic K-theory and topological K-theory associated to a finite, flat map and finite covering space map, respectively. We presume also that the transfer maps we have defined here are equivalent, up to weak homotopy, to these classical transfer maps. Again, since we will not need this result in this paper, we have not attempted to prove it.*

In the remainder of this section, we consider the special case $X' = X_{\mathbb{C}} \equiv X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ and we take $\pi : X' \to X$ to be the canonical map and $\phi : \mathcal{O}_X^2 \twoheadrightarrow \pi_*\mathcal{O}_{X'} \equiv \mathcal{O}_X \otimes_{\mathbb{R}} \mathbb{C}$ to be the surjection (in fact, isomorphism) induced by a chosen isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ of real vector spaces.

Recall that there are weak homotopy equivalences $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X_{\mathbb{C}}) \sim \mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}})$, $\mathcal{K}\mathbb{R}_{\mathrm{top}}((X_{\mathbb{C}})_{\mathbb{R}}(\mathbb{C})) \sim \mathcal{K}_{\mathrm{top}}(X_{\mathbb{C}}(\mathbb{C}))$, etc. Thus the transfer maps π_* actually produce maps

$$\pi_* : \mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}}) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \quad \text{and} \quad \pi_* : \mathcal{K}_{\mathrm{top}}(X_{\mathbb{C}}(\mathbb{C})) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

relating $\mathcal{K}^{\mathrm{semi}}$ with $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ and $\mathcal{K}_{\mathrm{top}}$ with $\mathcal{K}\mathbb{R}_{\mathrm{top}}$. The following theorem says, effectively, that we the homotopy groups of $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$, etc. are easily determined "away from 2" by the homotopy groups of $\mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}})$.

Theorem 5.8. *For $\pi : X_{\mathbb{C}} \to X$ and ϕ as above, the compositions*

$$\begin{array}{ccccc} \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{R}} X) & \xrightarrow{\pi^*} & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{C}} X_{\mathbb{C}}) & \xrightarrow{\pi_*} & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{R}} X) \\ \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) & \xrightarrow{\pi^*} & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{C}} X_{\mathbb{C}}) & \xrightarrow{\pi_*} & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \\ \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) & \xrightarrow{\pi^*} & \mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}}) & \xrightarrow{\pi_*} & \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \\ \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) & \xrightarrow{\pi^*} & \mathcal{K}_{\mathrm{top}}(X_{\mathbb{C}}(\mathbb{C})) & \xrightarrow{\pi_*} & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \end{array}$$

are each weakly homotopic to multiplication by 2 with respect to the H-space structures of each of these infinite loop spaces. That is, the results of precomposing $\pi_* \circ \pi^*$ and multiplication by 2 with any pointed map $T \to \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet} \times_{\mathbb{R}} X)$, $T \to \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$, etc., where T is a compact pointed CW-complex, are homotopic.

Proof. The proofs of all four assertions are essentially the same; we give the proof of the claim involving $\mathcal{K}\mathbb{R}^{\mathrm{semi}}$ here.

The composition of π^* with the transfer map π_* induces an endomorphisms of the Γ-space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}^{(-)}_{\mathbb{R}})$, and consequently we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) & \xrightarrow{\pi_* \circ \pi^*} & \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \\ \downarrow & & \downarrow \\ \Omega|d \mapsto \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}^{(d)}_{\mathbb{R}})| & \xrightarrow{\pi_* \circ \pi^*} & \Omega|d \mapsto \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}^{(d)}_{\mathbb{R}})|. \end{array} \tag{5.8.1}$$

The vertical map of (5.8.1) is a homotopy-theoretic group completion as defined in [CCMT, §1]. Here, the H-space structure on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is determined by choosing a homotopy inverse to the canonical map

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}^{(2)}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})^{\times 2}.$$

The endomorphism $\pi_* \circ \pi^*$ on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is readily checked to coincide with multiplication by 2 with respect to the H-space structure induced by the following choice of a homotopy inverse: Send $(\mathcal{O}^{\infty}_X \twoheadrightarrow E_1, \mathcal{O}^{\infty}_X \twoheadrightarrow E_2)$ to the quotient object associated to the composition

$$\mathcal{O}^{\infty}_X \cong \mathcal{O}^{\infty}_X \oplus \mathcal{O}^{\infty}_X \twoheadrightarrow E_1 \oplus E_2,$$

where the isomorphism is the "interleaving map" defined by

$$e_i \mapsto \begin{cases} (e_{(i+1)/2}, 0), \text{if } i \text{ is odd}, \\ (0, e_{i/2}), \text{if } i \text{ is even}. \end{cases}$$

Since $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is homotopy commutative, multiplication by 2 is an H-map, and thus the square (5.8.1) is a homotopy commutative square of H-spaces.

(The bottom arrow is clearly a map of H-spaces since it is in fact an infinite loop space map.) Since multiplication by 2 on

$$\Omega|d \mapsto \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}^{(d)})|$$

would also cause this square to commute up to weak homotopy, by [CCMT; 1.2] it follows that $\pi_* \circ \pi^*$ is weakly homotopic to multiplication by 2. □

Recall that if A is a finitely generated abelian group and X is a pointed space, one defines $\pi_q(X; A)$, the qth homotopy group of X with coefficients in A, as the homotopy classes of maps from the Moore space of type (A, q) to X. For an arbitrary abelian group A, $\pi_q(X; A)$ is the direct limit of $\pi_q(X; B)$, where B ranges over all finitely generated subgroups of A.

Corollary 5.9. *For any abelian group A, quasi-projective real variety X, and integer q, the maps*

$$\begin{aligned} K_q^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X; A) &\xrightarrow{\pi_* \circ \pi^*} K_q^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X; A) \\ K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X; A) &\xrightarrow{\pi_* \circ \pi^*} K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X; A) \\ K\mathbb{R}_q^{\mathrm{semi}}(X; A) &\xrightarrow{\pi_* \circ \pi^*} K\mathbb{R}_q^{\mathrm{semi}}(X; A) \\ K\mathbb{R}_{\mathrm{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C}); A) &\xrightarrow{\pi_* \circ \pi^*} K\mathbb{R}_{\mathrm{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C}); A) \end{aligned}$$

are given by multiplication by 2*. In particular, if A is a $\mathbb{Z}[\frac{1}{2}]$-module, then $K_q^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X; A)$, $K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X; A)$, $K\mathbb{R}_q^{\mathrm{semi}}(X; A)$, and $K\mathbb{R}_{\mathrm{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C}); A)$ are naturally summands of the groups $K_q^{\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{C}} X_{\mathbb{C}}; A)$, $K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{C}} X_{\mathbb{C}}; A)$, $K_q^{\mathrm{semi}}(X_{\mathbb{C}}; A)$, and $K_{\mathrm{top}}^{-q}(X_{\mathbb{C}}(\mathbb{C}); A)$, respectively.*

Proof. If A is a finitely generated abelian group, the Moore space $M(A, q)$ is a finite CW complex, and thus the result follows immediately from Theorem 5.8. More generally, the result follows by taking direct limits over all finitely generated subgroups of A. □

§6 Examples and Localization

In this section, we establish that the natural maps give isomorphisms

$$K\mathbb{R}_q^{\mathrm{semi}}(X) \cong K\mathbb{R}_{\mathrm{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C})), \quad q \geq 0,$$

and

$$K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \cong K\mathbb{R}_{\mathrm{top}}^{-q}(X_{\mathbb{R}}(\mathbb{C})), \quad q \geq 0,$$

for certain classes of real varieties X. For example, we obtain such weak equivalences when X is a projective, smooth curve, or a certain type of generalized flag variety. As seen in the next section, this computation can be viewed as a computation of the "stabilized homotopy groups" of spaces of morphisms. Thus, the examples in this section represent (stable) real analogues of generalizations of results found in [Ki], [CLS].

To extend our examples of such weak equivalences involving $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ to the case where X is affine, it is useful to prove a localization result for $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}}$

$-$) analogous to the known localization property of algebraic K-theory (but which here only applies to smooth varieties). This result is also proven in this section, although we expect its applications to extend beyond the use to which it is employed here.

The following proposition can be viewed as a generalization of the standard projective space formula and is the real analogue of [FW3; 2.1].

Proposition 6.1. *Suppose G is one of the linear algebraic groups $GL_{n,\mathbb{R}}$, $SL_{n,\mathbb{R}}$, $Spin_{n,\mathbb{R}}$, or $Sp_{2n,\mathbb{R}}$ and P is a parabolic subgroup containing a split Borel subgroup. Then the natural map*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(G/P) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}((G/P)_{\mathbb{R}}(\mathbb{C}))$$

is a weak equivalence.

Proof. Let $X = S \times_{\mathbb{R}} G/P$, for any S. As argued in [FW3; 2.2] using [P; 5.2, 5.4], for a choice of dominant weights $\lambda_1, \dots, \lambda_k$, the vector bundles $E_P(\lambda_1), \dots, E_P(\lambda_k)$ determine a basis $\mathcal{B}$ of $K_0^{\mathrm{alg}}(G/P)$ as a free $K_0^{\mathrm{alg}}(\mathbb{R})$-module. More generally, the pullback of $\mathcal{B}$ to X gives a basis of $K_0^{\mathrm{alg}}(X)$ as a free $K_0^{\mathrm{alg}}(S)$-module. Moreover, multiplication by these basis elements defines a natural weak equivalence

$$\mathcal{K}^{\mathrm{alg}}(S)^{\times k} \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}(S \times_{\mathbb{R}} G/P).$$

Replacing S by Δ^n_{top} by taking direct limits, we obtain the natural weak equivalence

$$\mathcal{K}^{\mathrm{alg}}(\Delta^n_{top} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{R})^{\times k} \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}(\Delta^n_{top} \times_{\mathbb{R}} G/P),$$

and consequently a weak equivalence

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{R})^{\times k} \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} G/P).$$

The image of $\mathcal{B} \subset K_0^{\mathrm{alg}}(G/P)$ in $K^0_{\mathrm{top}}((G/P)(\mathbb{C}))$ defines an analogous weak equivalence

$$(BU \times \mathbb{Z})^{\times k} \equiv \mathcal{K}_{\mathrm{top}}(pt)^{\times k} \xrightarrow{\sim} \mathcal{K}_{\mathrm{top}}((G/P)(\mathbb{C})),$$

by [Pi; 3] and [AH; 3.6]. Using [Sey; 4.3], it follows that the image of $\mathcal{B}$ in $K\mathbb{R}^0_{\mathrm{top}}((G/P)_{\mathbb{R}}(\mathbb{C}))$ defines a weak equivalence

$$(BO \times \mathbb{Z})^{\times k} \equiv \mathcal{K}\mathbb{R}_{\mathrm{top}}(pt)^{\times k} \xrightarrow{\sim} \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})).$$

Finally, since the map $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(-)$ is compatible with products by Theorem A.5 of the appendix, the result now follows from Example 4.5. □

In [FW2], we established a equivalence between $\mathcal{K}^{\mathrm{semi}}$ and $\mathcal{K}_{\mathrm{top}}$ of a smooth, projective complex curve. This result, together with a result of M. Karoubi and C. Weibel [KW] on the algebraic K-theory with finite coefficients of smooth real curves, leads directly to the following real analogue of [FW2; 7.5].

Proposition 6.2. *Let C be a smooth, projective real curve. Then the natural map*

$$K\mathbb{R}_q^{\mathrm{semi}}(C) \to K\mathbb{R}_{\mathrm{top}}^{-q}(C_{\mathbb{R}}(\mathbb{C}))$$

is an isomorphism for $q \geq 0$.

Proof. The composition

$$\mathcal{K}^{\mathrm{alg}}(C) \to \mathcal{K}\mathbb{R}^{\mathrm{semi}}(C) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(C_{\mathbb{R}}(\mathbb{C}))$$

is shown to induce an isomorphism upon applying $\pi_q(-;\mathbb{Z}/2)$, $q \geq 0$, in [KW]. It follows from Theorem 3.9 that

$$K\mathbb{R}_q^{\mathrm{semi}}(C;\mathbb{Z}/2) \cong K\mathbb{R}_{\mathrm{top}}^{-q}(C_{\mathbb{R}}(\mathbb{C});\mathbb{Z}/2)$$

for $q \geq 0$. Finally, in the diagram

$$\begin{array}{ccccc} K\mathbb{R}_q^{\mathrm{semi}}(C;\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\pi^*} & K_q^{\mathrm{semi}}(C_{\mathbb{C}};\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\pi_*} & K\mathbb{R}_q^{\mathrm{semi}}(C;\mathbb{Z}[\frac{1}{2}]) \\ \downarrow & & \downarrow & & \downarrow \\ K\mathbb{R}_{\mathrm{top}}^{-q}(C_{\mathbb{R}}(\mathbb{C});\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\pi^*} & K_{\mathrm{top}}^{-q}(C(\mathbb{C});\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\pi_*} & K\mathbb{R}_{\mathrm{top}}^{-q}(C_{\mathbb{R}}(\mathbb{C});\mathbb{Z}[\frac{1}{2}]) \end{array}$$

the composition of each horizontal row is an isomorphism by Corollary 5.9 and the middle vertical map is an isomorphism by [FW3; 7.5]. A diagram chase completes the proof. □

Remark 6.3. *If X is a smooth real toric variety, then an argument due to the second author and C. Weibel can be adapted to conclude that the natural map*

$$\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

is a weak equivalence. The essential point of this argument is that a map between two theories on real varieties which are homotopy invariant, which satisfy Zariski descent, and which agree on projective spaces, must agree on all smooth toric varieties. In particular, if X is a smooth, projective real toric variety, then the natural map

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

is a weak equivalence.

The Quillen-Lichtenbaum conjecture (cf. [F1]) for a smooth real variety X of dimension d asserts that for $n > 0$ the natural map $K_q^{\mathrm{alg}}(X;\mathbb{Z}/n) \to K_{\text{ét}}^{-q}(X;\mathbb{Z}/n)$ is an isomorphism for $q \geq d-1$. In light of Proposition 4.7 and Corollary 3.10, such an isomorphism holds, at least in the range $q \geq d$, provided the map $K\mathbb{R}_q^{\mathrm{semi}}(X) \to K\mathbb{R}_{\mathrm{top}}^q(X)$ is an isomorphism for $q \geq d-1$. The following result has been proven in [KW], using different means.

Corollary 6.4. *The Quillen-Lichtenbaum conjecture holds for a smooth, complete real curve C.*

Proof. The map $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(C) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(C_{\mathbb{R}}(\mathbb{C}))$ is a weak equivalence by Proposition 6.2. The result follows by considering the long exact sequence in homotopy groups and by using Proposition 4.7 and Corollary 3.10. □

To extend the above examples to affine varieties (and for further applications), we establish a localization result for $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} -)$. The statement and proof of this result were inspired by the recent preprint of I. Panin and A. Smirnov [PS]. The basic technique is the use of the deformation to the normal bundle, an idea due to R. MacPherson (cf. [BFM]). We follow the treatment of this concept found in the work of F. Morel [Mo].

Let $i : Z \hookrightarrow X$ be a regular closed immersion and let $W_Z(X)$ be the smooth variety given by the construction of [Mo; 3.2.2] (which is written $D(i)$ there). The variety $W_Z(X)$ is defined as the complement of X_Z in $(X \times \mathbb{A}^1)_Z$, where X_Z is the blow-up of X along Z and $(X \times \mathbb{A}^1)_Z$ is the blow-up of $X \times \mathbb{A}^1$ along $Z \times \{0\} \hookrightarrow X \times \mathbb{A}^1$. We briefly describe the key properties of $W_Z(X)$ (see [Mo; 3.2.6]). There exists a map $W_Z(X) \to X \times \mathbb{A}^1$ and a regular closed immersion $Z \times \mathbb{A}^1 \hookrightarrow W_X(Z)$ such that the squares appearing in the commutative diagrams

$$\begin{array}{ccccc} Z & \longrightarrow & X & \xrightarrow{=} & X \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle i_1}\downarrow \\ Z \times \mathbb{A}^1 & \longrightarrow & W_Z(X) & \longrightarrow & X \times \mathbb{A}^1 \end{array}$$

and

$$\begin{array}{ccccc} Z & \longrightarrow & N_Z(X) & \longrightarrow & X \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle i_0}\downarrow \\ Z \times \mathbb{A}^1 & \longrightarrow & W_Z(X) & \longrightarrow & X \times \mathbb{A}^1 \end{array}$$

are Cartesian. Here, $N_Z(X)$ denotes the normal bundle for the immersion $Z \hookrightarrow X$, $Z \to N_Z(X)$ is the zero section embedding, and $N_Z(X) \to X$ is the composition $N_Z(X) \to Z \hookrightarrow X$. Finally, if $X = U \cup V$ is an open cover, then $W_{Z\cap U}(U)$ and $W_{Z\cap V}(V)$ form an open cover of $W_Z(X)$ with intersection $W_{Z\cap U\cap V}(U \cap V)$ (cf. proof of [Mo; 3.2.8]).

Theorem 6.5. *For any closed immersion of smooth quasi-projective varieties $i : Z \hookrightarrow X$, we have that*

$$\mathcal{K}_Z^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) \xleftarrow{\alpha^*} \mathcal{K}_{Z\times\mathbb{A}^1}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} W_Z(X)) \xrightarrow{\beta^*} \mathcal{K}_Z^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} N_Z(X))$$

are weak equivalences.

Proof. Our proof mimics the proof of [Mo; 3.2.8]. Let $F_A(B)$ stand for $\mathcal{K}_A^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} B)$, for any closed immersion $A \hookrightarrow B$.

Suppose we can find an open covering of X, $X = \cup_i W_i$, such that the statement of the theorem holds for each closed immersion $Z \cap W \hookrightarrow W$, where W is any open subscheme of the form $W_{j_1} \cap \cdots \cap W_{j_k}$. Then by induction, using Mayer-Vietoris of $F_{-}(-)$ (see Remark 3.6) and the fact that $W_{Z\cap U}(U)$ and $W_{Z\cap V}(V)$ form an open cover of $W_Z(X)$ for any open cover $X = U \cup V$, it follows that the statement of the theorem holds for $Z \hookrightarrow X$ itself. In other words, it suffices to prove the theorem locally on X.

As is well known (cf. [EGAIV; 17.12.2 d]), for each $x \in X$, there is an open neighborhood $x \in V \subset X$ and an étale map $f : V \to \mathbb{A}^{n+d}$ such that the closed immersion $V \cap Z \hookrightarrow V$ is the pullback along f of the closed immersion $\mathbb{A}^d \hookrightarrow \mathbb{A}^{n+d}$

given by inclusion into the first d coordinates. Thus, by appealing to the previous paragraph, we may as well assume X itself admits an étale map

$$f : X \to \mathbb{A}^{n+d}$$

such that $Z \hookrightarrow X$ is the pullback along f of the closed immersion $i : \mathbb{A}^d \hookrightarrow \mathbb{A}^{n+d}$.

In this situation, by [Mo; 3.2.9], there exists étale maps $U \to X$ and $U \to Z \times \mathbb{A}^d$ and a closed immersion $Z \hookrightarrow U$ such that the squares

$$\begin{array}{ccc} Z & \longrightarrow & U \\ {\scriptstyle =}\downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

and

$$\begin{array}{ccc} Z & \longrightarrow & U \\ {\scriptstyle =}\downarrow & & \downarrow \\ Z & \xrightarrow{0} & Z \times \mathbb{A}^d \end{array}$$

are Cartesian. By [Mo; 3.2.7 (3)], these diagrams induce étale maps $W_Z(U) \to W_Z(X)$ and $W_Z(U) \to W_Z(Z \times \mathbb{A}^d)$ such that the diagrams

$$\begin{array}{ccc} Z \times \mathbb{A}^1 & \longrightarrow & W_Z(U) \\ {\scriptstyle =}\downarrow & & \downarrow \\ Z \times \mathbb{A}^1 & \longrightarrow & W_Z(X) \end{array}$$

and

$$\begin{array}{ccc} Z \times \mathbb{A}^1 & \longrightarrow & W_Z(U) \\ {\scriptstyle =}\downarrow & & \downarrow \\ Z \times \mathbb{A}^1 & \longrightarrow & W_Z(Z \times \mathbb{A}^d) \end{array}$$

are also Cartesian. Now consider the diagram

$$\begin{array}{ccccc} F_Z(X) & \longleftarrow & F_{Z\times\mathbb{A}^1}(W_Z(X)) & \longrightarrow & F_Z(N_Z(X)) \\ \downarrow & & \downarrow & & \downarrow \\ F_Z(U) & \longleftarrow & F_{Z\times\mathbb{A}^1}(W_Z(U)) & \longrightarrow & F_Z(N_Z(U)) \\ \uparrow & & \uparrow & & \uparrow \\ F_Z(Z \times \mathbb{A}^d)) & \longleftarrow & F_{Z\times\mathbb{A}^1}(W_Z(Z \times \mathbb{A}^d)) & \longrightarrow & F_Z(N_Z(Z \times \mathbb{A}^d)). \end{array} \tag{6.5.1}$$

The vertical maps in the left and middle columns of (6.5.1) are weak equivalences by Nisnevich excision (Theorem 3.5), using the preceding diagrams. The vertical arrows in the right column of (6.5.1) are weak equivalences, since each of $N_Z(U) \to N_Z(X)$ and $N_Z(U) \to N_Z(Z \times \mathbb{A}^d)$ is readily verified to be a Nisnevich neighborhoods of Z (i.e., they also satisfy the hypothesis of Theorem 3.5). As shown in the proof of [Mo; 3.2.11], the immersion $Z \times \mathbb{A}^1 \hookrightarrow W_Z(Z \times \mathbb{A}^d)$ is naturally isomorphic to the map $Z \times \mathbb{A}^1 \hookrightarrow Z \times \mathbb{A}^d \times \mathbb{A}^1$ given by inclusion at 0. It follows from homotopy invariance that the maps in the bottom row of (6.5.1) are weak equivalences. A diagram chase completes the proof. □

Theorem 6.6. *Given a regular closed immersion of smooth, quasi-projective real varieties $i : Z \hookrightarrow X$, there is a weak equivalence in the homotopy category of spectra*

$$\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z).$$

This equivalence is natural in the sense that given a Cartesian square of smooth, quasi-projective real varieties

$$\begin{array}{ccc} W & \xrightarrow{j} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

whose rows are regular closed immersions, the diagram in the homotopy category

$$\begin{array}{ccc} \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \xrightarrow{\sim} & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{K}^{\mathrm{alg}}_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) & \xrightarrow{\sim} & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} W) \end{array}$$

commutes.

Proof. By Theorem 6.5, the maps

$$\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \xleftarrow{\alpha^*} \mathcal{K}^{\mathrm{alg}}_{Z\times\mathbb{A}^1}(\Delta^\bullet_{top} \times_{\mathbb{R}} W_Z(X)) \xrightarrow{\beta^*} \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} N_Z(X))$$

are weak equivalences. Thus there is a weak equivalence in the homotopy category

$$\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} N_Z(X)) \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X),$$

and this map is readily verified to be natural in the same sense as in the statement of this theorem.

We now claim that there is a natural weak equivalence

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \xrightarrow{\sim} \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} N_Z(X)).$$

Indeed, let $p : E \to Z$ be any vector bundle of rank n with zero section s_0, and form the projectivized bundle $q : \mathbb{P}(E \oplus 1) \to Z$, which comes with the evident open immersion $j : E \hookrightarrow \mathbb{P}(E \oplus 1)$ and complementary closed immersion at infinity $\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus 1)$. Following the proof given by [PS], let

$$th(E) \in K^{\mathrm{alg}}_{Z,0}(\mathbb{P}(E \oplus 1))$$

be the class associated to the chain complex

$$\cdots \to 0 \to \mathcal{O}_{\mathbb{P}(E\oplus 1)}(-n) \otimes \Lambda^n(E) \to \cdots \to \mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1) \otimes \Lambda^1(E) \to \mathcal{O}_{\mathbb{P}(E\oplus 1)} \to 0.$$

(Here Z is regarded as a closed subscheme of $\mathbb{P}(E \oplus 1)$ via the composition $j \circ s_0$.) Write $t(E)$, the *Thom class*, for the restriction of $th(E)$ to $K^{\mathrm{alg}}_{Z,0}(E)$ via j. We also write $th(E)$ and $t(E)$ for the images of these classes in $K^{\mathrm{alg}}_{Z,0}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1))$ and $K^{\mathrm{alg}}_{Z,0}(\Delta^\bullet_{top} \times_{\mathbb{R}} E)$, respectively. Then in the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) & \xrightarrow{th(E)\cup q^*} & \mathcal{K}^{\mathrm{alg}}_{\mathbb{P}(1)}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1)) \\
{\scriptstyle =}\downarrow & & {\scriptstyle j^*}\downarrow \\
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) & \xrightarrow{t(E)\cup p^*} & \mathcal{K}^{\mathrm{alg}}_{Z}(\Delta^\bullet_{top} \times_{\mathbb{R}} E),
\end{array}$$

the right-hand vertical arrow is a weak equivalence by Nisnevich excision (Theorem 3.5). The bottom arrow will therefore be the desired weak equivalence provided the top arrow is a weak equivalence. To show this, observe that there is a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \xrightarrow{th(E)\cup q^*} & \mathcal{K}^{\mathrm{alg}}_{Z}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1)) \\
\downarrow & & \downarrow \\
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)^{\oplus n+1} & \xrightarrow{(th(E)\cup q^*, 1, \zeta_{E\oplus 1}, \ldots, \zeta^{n-1}_{E\oplus 1})} & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1)) \\
\downarrow & & \downarrow \\
\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)^{\oplus n} & \xrightarrow{\theta} & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1) - X) \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array} \tag{6.6.1}$$

where in general we write $\zeta_{\mathbb{P}(F)}$ for the class $1-[\mathcal{O}_{\mathbb{P}(F)}(-1)] \in K^{\mathrm{alg}}_0(\mathbb{P}(F))$ associated to a vector bundle F over X. Here, the map θ is the composition of the weak equivalence (cf. Proposition 2.7)

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)^{\oplus n} \xrightarrow{(1, \zeta_E, \ldots, \zeta^{n-1}_E)} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E))$$

with the pullback along the projection map $\mathbb{P}(E \oplus 1) - X \to \mathbb{P}(E)$. Since this projection map is a vector bundle, it follows θ is a weak equivalence, and thus it suffices to show the middle arrow of (6.6.1) is a weak equivalence. By Zariski descent (Theorem 3.5), it suffices to check this locally on X, so that we may assume $E = \mathcal{O}^n_X$, the trivial bundle of rank n. In this case, $th(E)$ coincides with $\zeta^n_{\mathbb{P}(E\oplus 1)}$ in $K^{\mathrm{alg}}_0(\Delta^\bullet_{top} \times_{\mathbb{R}} \mathbb{P}(E \oplus 1))$, and thus the middle arrow of (6.6.1) is a weak equivalence by Proposition 2.7.

The naturality of the map $t(E) \cup q^*$ follows from the naturality of the Thom class $t(E)$. $\square$

Combining Theorems 3.3 and 6.6, we obtain the following statement of *localization* for $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$.

Corollary 6.7. *Given a regular closed immersion $i : Z \hookrightarrow X$ of smooth, quasi-projective real varieties, there is a "Gysin map"*

$$i_* : \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \to \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$$

which fits into a fibration sequence in the homotopy category of spectra

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \xrightarrow{i_*} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X - Z).$$

This sequence is natural in the sense that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X - Z) \\ {\scriptstyle g^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle (f|_{Y-W})^*}\downarrow \\ \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} W) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X - Z) \end{array}$$

associated to a map of pairs as in Theorem 6.6.

Proposition 6.8. *Let $Z \hookrightarrow X$ be a closed immersion of smooth, projective real varieties and let $U = X - Z$. If the natural maps*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(Z) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(Z_{\mathbb{R}}(\mathbb{C})), \quad \mathcal{K}\mathbb{R}^{\mathrm{semi}}(U) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C}))$$

are weak equivalences, then so is

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})).$$

Proof. As shown in Appendix A, there is a natural model $\mathcal{K}\mathbb{R}^{Z_{\mathbb{R}}(\mathbb{C})}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$ for the homotopy fiber of

$$\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C})).$$

Moreover, by Theorem A.5 of Appendix A, there is a natural map

$$\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}\mathbb{R}^{Z_{\mathbb{R}}(\mathbb{C})}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

fitting into a homotopy commutative diagram

$$\begin{array}{ccccc} \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}\mathbb{R}^{Z_{\mathbb{R}}(\mathbb{C})}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C})), \end{array} \tag{6.8.1}$$

all of whose arrows are compatible with cup product pairings.

From standard topological results, the theory with supports

$$(X, Z) \mapsto \mathcal{K}\mathbb{R}^{Z_{\mathbb{R}}(\mathbb{C})}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

satisfies the needed properties (i.e., excision, homotopy invariance, and existence of a bundle formula) for the proofs of 6.6 and 6.7 to carry over. In particular, there is a natural weak equivalence

$$\mathcal{K}\mathbb{R}^{Z_{\mathbb{R}}(\mathbb{C})}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \xrightarrow{\sim} \mathcal{K}\mathbb{R}_{\mathrm{top}}(Z_{\mathbb{R}}(\mathbb{C})), \tag{6.8.2}$$

and consequently a natural fibration sequence

$$\mathcal{K}\mathbb{R}_{\mathrm{top}}(Z_{\mathbb{R}}(\mathbb{C})) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C})).$$

Moreover, since the equivalence (6.8.2) and that of Theorem 6.6 are constructed using the same deformation argument, they are compatible with the maps in diagram (6.8.1) and we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} Z) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X) & \longrightarrow & \mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}\mathbb{R}_{\mathrm{top}}(Z_{\mathbb{R}}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C})). \end{array}$$

The result follows immediately. $\square$

Corollary 6.9. *If X is a smooth (possibly affine) real curve, then the natural map*

$$\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X) \to \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$$

is a weak equivalence.

Proof. By considering a smooth projective closure of X, this result follows immediately from Proposition 6.8, Proposition 6.2, and Example 4.5. $\square$

Combining Corollary 6.9 with Proposition 4.7 and Corollary 3.10 gives the following result, which has also been established in [KW].

Corollary 6.10. *The Quillen-Lichtenbaum conjecture holds for an arbitrary smooth real curve.*

§7 Stabilization of mapping spaces of morphisms

The process of forming the homotopy-theoretic group completion of an $\mathcal{I}$-space (or a Γ-space) is slightly mysterious, rendering the infinite loop space $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ rather less explicit than one would hope. In this section, we show how to form the homotopy-theoretic group completions of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ and $\mathrm{Hom}_{\mathbb{R}}(X \times \Delta^{\bullet}_{top}, \mathrm{Grass}_{\mathbb{R}})$ in terms of explicit mapping telescopes. In particular, this description exhibits a connection between $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ and the spaces of holomorphic maps studied in [Ki] and [CLS]. Additionally, such a construction has the advantage that it allows for a nice description of the homotopy groups $K\mathbb{R}^{\mathrm{semi}}_{*}(X)$ in terms of ordinary group completions of the monoids of homotopy classes of maps to $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ (see Corollary 7.4).

As a preliminary step, recall that $K\mathbb{R}^{\mathrm{semi}}_{0}(X)$ is the group completion of the abelian monoid $\pi_0\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$, and a similar statement holds for $K^{\mathrm{alg}}_{0}(\Delta^{\bullet}_{top} \times_{\mathbb{R}} X)$. In fact, this group completion is attained by inverting a single element of $\pi_0\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$, as shown in the following lemma.

Lemma 7.1. *Let X be a quasi-projective real variety and $p : \mathcal{O}_X^{\infty} \twoheadrightarrow L$ any quotient (factoring through $\mathcal{O}_X^N$ for $N \gg 0$) such that L is an ample line bundle on X. If X is projective, the induced map*

$$\pi_0\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})\,[-[p]] \to K\mathbb{R}^{\mathrm{semi}}_{0}(X)$$

is an isomorphism. If X is an arbitrary quasi-projective variety, then the induced map

$$\pi_0|\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \operatorname{Grass}_{\mathbb{R}})|\,[-[p]] \to K_0^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$$

is an isomorphism. (Here, $[-[p]]$ refers to formally inverting the class of p in the indicated abelian monoid).

Proof. The proof of the first isomorphism is identical to [FW2; 3.1], using the fact that $\pi_0\mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}})$ is the set of isomorphism classes of bundles on X^w (where $X^w \to X$ is the weak normalization of X) generated by global section modulo real semi-topological equivalence of such bundles, which is shown in the proof of Proposition 1.6. The second isomorphism follows in the same manner once one knows that $\pi_0|\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \operatorname{Grass}_{\mathbb{R}})|$ is also the set of isomorphism classes of bundles generated by their global section modulo real semi-topological equivalence. This is proven just as is the last statement of Proposition 2.5, using the latter half of the proof of Proposition 1.6. □

For a quasi-projective real variety X, fix a very ample line bundle L together with a surjection

$$\mathcal{O}_X^\infty \xrightarrow{(l_1, l_2, \dots)} L$$

(with $l_i = 0$ for $i \gg 0$). Define

$$\alpha_L : \mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}})$$

to be the map sending

$$\mathcal{O}_X^\infty \xrightarrow{(e_1, e_2, \dots)} E$$

to the quotient

$$\mathcal{O}_X^\infty \xrightarrow{(l_1, e_1, l_2, e_2, \dots)} L \oplus E.$$

(The definition of α_L actually depends on the choice of surjection $\mathcal{O}_X^\infty \twoheadrightarrow L$.) Now form the mapping telescope of the infinite sequence of maps

$$\mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}}) \xrightarrow{\alpha_L} \mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}}) \xrightarrow{\alpha_L} \cdots,$$

and write this as $\operatorname{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \operatorname{Grass}_{\mathbb{R}}), \alpha_L)$. Here, we regard the nth space in this telescope, G_n for short, as being pointed by the image of the base point under the nth iterate of α_L. We remind the reader that, in general, the mapping telescope of a sequence of maps of pointed spaces

$$(G_0, g_0) \xrightarrow{f_0} (G_1, g_1) \xrightarrow{f_1} \cdots,$$

written $\operatorname{Tel}(G_i, f_i)$, is defined as

$$\frac{\coprod_{n \geq 0} G_n \wedge [n, n+1]_+}{(x, n+1) \sim (f_n(x), n+1)},$$

where $x \in G_n$, $n \geq 0$ and $[n, n+1] \subset \mathbb{R}$ is the evident interval. Observe that $\operatorname{Tel}(G_n, f_n)$ is weakly equivalent to the ordinary direct limit $\varinjlim_n G_n$ but is better behaved in that it has the homotopy type of a CW complex provided each G_n does. Theorem 7.3 below shows that the evident map

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), \alpha_L)$$

is a homotopy-theoretic group completion, when X projective.

The construction of the group completion of $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ presented here is similar to that found in [FW2], except that there the space of complex morphisms $\mathcal{M}or_{\mathbb{C}}(Y, \mathrm{Grass}_{\mathbb{C}})$, for Y a complex variety, is considered. (Additionally, the definition of α_L in [FW2] differs slightly from that used here.) It turns out to be a delicate matter to adapt the proof found in [FW2], which shows that the mapping telescope of $(\mathcal{M}or_{\mathbb{C}}(Y, \mathrm{Grass}_{\mathbb{C}}), \alpha_L)$ gives a homotopy-theoretic group completion, to the situation here. The difficulty lies in the fact that we must use the real operad $\mathcal{I}$ here, whereas its complex version is used in [FW2], and this complex version provides more freedom to construct the needed homotopies. Thus, in lieu of adapting the proof in [FW2] directly, we instead provide an easier proof in Theorem 7.3 below, one which unfortunately does not apply to the study of G-theory considered in [FW2]. The following lemma provides the key to the proof of Theorem 7.3.

Lemma 7.2. *Let (G, μ) be a strictly associative H-space with strict identity e. Fix $g \in G$ and let G_n denote the space G pointed by the base point g^n (where $g^n = \mu(g, g^{n-1})$, $g^0 = e$). Assume that the maps*

$$\mu(g, -), \mu(-, g) : G_n \to G_{n+1}$$

are homotopic via a base point preserving homotopy. Then the mapping telescope $\mathrm{Tel}(G_n, \mu(g, -))$ of the sequence

$$G = G_0 \xrightarrow{\mu(g,-)} G_1 \xrightarrow{\mu(g,-)} G_2 \xrightarrow{\mu(g,-)} \cdots$$

has the structure of a homotopy associative H-space such that the map

$$G \to \mathrm{Tel}(G_n, \mu(g, -))$$

is a map of H-spaces. Furthermore, the induced map on π_0 is the map given by inverting the class of g in the abelian monoid $\pi_0(G)$ and the induced map on homology groups with coefficients in any commutative ring A,

$$H_*(G; A) \to H_*(\mathrm{Tel}(G_i, \mu(g, -)); A),$$

is localization with respect to the action of $[g] \in \pi_0(G)$. Finally, if H is homotopy commutative in such a fashion that for all $n \geq 0$ the two maps

$$\mu, \mu \circ \tau : G_n \times G_n \to G_{2n},$$

are homotopic via a pointed homotopy (where τ interchanges the two factors), then $\mathrm{Tel}(G_n, \mu(g, -))$ is homotopy commutative.

Proof. We use the following general fact about mapping telescopes: Given an infinite "ladder" of pointed spaces

$$\begin{array}{ccccc} X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & \\ Y_0 & \xrightarrow{\beta_0} & Y_1 & \xrightarrow{\beta_1} & \cdots \end{array}$$

such that each square is homotopy commutative via a homotopy

$$h_n : X_n \wedge I_+ \to Y_{n+1}, \quad n \geq 0,$$

such that $h_n(-,0) = \beta_n \circ f_n$ and $h_n(-,1) = f_{n+1} \circ \alpha_n$, then there is a map on telescopes

$$\mathrm{Tel}(X_i, \alpha_i) \to \mathrm{Tel}(Y_i, \beta_i)$$

defined by sending $(x,t) \in X_n \times [n, n+1]$ to

$$\begin{cases} (f_n(x), 2t-n)) \in Y_n \times [n, n+1], & \text{if } t \in [n, n+\frac{1}{2}] \\ (h_n(x, 2t-2n-1), n+1) \in Y_{n+1} \times \{n+1\}, & \text{if } t \in [n+\frac{1}{2}, n+1]. \end{cases}$$

Moreover, the map

$$\mathrm{Tel}(X_i, \alpha_i) \to \mathrm{Tel}(Y_i, \beta_i)$$

is independent up to homotopy of the choice of the h_n's.

Write $\chi : G_n \to G_{n+1}$ for the pointed map $\mu(g,-)$, and consider the infinite ladder

$$\begin{array}{ccccc} G_0 \times G_0 & \xrightarrow{\chi\times\chi} & G_1 \times G_1 & \xrightarrow{\chi\times\chi} & \cdots \\ \mu\downarrow & & \mu\downarrow & & \\ G_0 & \xrightarrow{\chi^2} & G_2 & \xrightarrow{\chi^2} & \cdots . \end{array} \tag{7.2.1}$$

A choice of homotopy joining

$$\mu(g,-), \mu(-,g) : G_n \to G_{n+1}$$

determines a homotopy $G_n \times G_n \wedge I_+ \to G_{2n+2}$ exhibiting that the nth square of (7.2.1) commutes up to homotopy, and thus determines a map

$$\mathrm{Tel}(G_i \times G_i, \chi \times \chi) \to \mathrm{Tel}(G_{2i}, \chi^2).$$

There is a natural homotopy equivalence

$$\mathrm{Tel}(G_i \times G_i, \chi \times \chi) \xrightarrow{\sim} \mathrm{Tel}(G_i, \chi)^{\times 2},$$

induced by the pair of maps of sequences of spaces

$$(G_i \times G_i, \chi \times \chi) \to (G_i, \chi)$$

given by the two projection maps. Similarly, there is a natural homotopy equivalence

$$\mathrm{Tel}(G_n, \chi) \xrightarrow{\sim} \mathrm{Tel}(G_{2n}, \chi^2)$$

induced by the map of sequences determined by $\mu(g^n, -) : G_n \to G_{2n}$, $n \geq 0$. By choosing homotopy inverses, this gives the pairing

$$\mu_\infty : \mathrm{Tel}(G_n, \chi) \times \mathrm{Tel}(G_n, \chi) \to \mathrm{Tel}(G_n, \chi)$$

which we claim makes $\mathrm{Tel}(G_n, \chi)$ into a homotopy associative H-space.

We must show, first of all, that the base point is a two-sided identity up to homotopy. Consider the two diagrams of the form

$$\begin{array}{ccccc} \mathrm{Tel}(G_n,\chi) & \xrightarrow{=} & \mathrm{Tel}(G_n,\chi) & \xrightarrow{=} & \mathrm{Tel}(G_n,\chi) \\ \downarrow & & \downarrow & & \sim\downarrow \\ \mathrm{Tel}(G_n,\chi)\times\mathrm{Tel}(G_n,\chi) & \xleftarrow{\sim} & \mathrm{Tel}(G_i\times G_i,\chi\times\chi) & \xrightarrow{\sim} & \mathrm{Tel}(G_{2n},\chi^2), \end{array} \tag{7.2.2}$$

where the bottom horizontal maps and the right-hand vertical map are given as above and the left-hand and middle vertical maps are induced by either $\{g^n\}\times id : G_n \to G_n \times G_n$ or $id \times \{g^n\} : G_n \to G_n \times G_n$. One may readily verify that, in either case, the left-hand square commutes strictly. If the maps $\{g^n\}\times id : G_n \to G_n \times G_n$, $n \geq 0$, are used to define the two vertical maps, then the right-hand square of (7.2.2) is also strictly commutative. If the maps $id \times \{g^n\} : G_n \to G_n \times G_n$ are used to define these vertical maps, then it suffices to show the two maps

$$\mu(-, g^n), \mu(g^n, -) : G_n \to G_{2n}$$

induce homotopic maps on telescopes in order to establish the homotopy commutativity of the right-hand square. To do this, choose pointed homotopies

$$h_n : G_n \wedge I_+ \to G_{2n}, \quad n \geq 0,$$

joining $\mu(g^n, -)$ to $\mu(-, g^n)$, which exist by hypothesis. Consider the sequence $(G_n \wedge I_+, \chi \wedge id)$ and the infinite ladder formed by $h_n : G_n \wedge I_+ \to G_{2n}$, $n \geq 0$. Observe that the squares in this ladder are homotopy commutative, since $G_n \xrightarrow{0} G_n \wedge I_+$ is a homotopy equivalence and the ladder formed by the composition $G_n \xrightarrow{0} G_n \wedge I_+ \xrightarrow{h_n} G_{2n}$ commutes strictly. Thus we obtain an induced map on telescopes

$$\mathrm{Tel}(G_n \wedge I_+, \chi \wedge id) \to \mathrm{Tel}(G_{2n}, \chi^2).$$

Moreover, the two maps

$$\mathrm{Tel}(G_n, \chi) \xrightarrow{i} \mathrm{Tel}(G_n \wedge I_+, \chi \wedge id)$$

given by inclusion at $i = 0$ or $i = 1$ are homotopy inverses to the map induced by projection, and are thus homotopic. Further, the compositions of $\mathrm{Tel}(G_n, \chi) \xrightarrow{i} \mathrm{Tel}(G_n \wedge I_+, \chi \wedge id)$ with $\mathrm{Tel}(G_n \wedge I_+, \chi \wedge id) \to \mathrm{Tel}(G_{2n})$, for $i = 0, 1$, give the maps induced by $\mu(g^n, -)$ and $\mu(-, g^n)$.

Thus, (7.2.2) is homotopy commutative in both cases, and now a simple diagram chase shows that multiplication on the left or the right by the base point of $\mathrm{Tel}(G_n, \chi)$ is homotopic to the identity map.

To show homotopy associativity, consider the diagram

$$\begin{array}{ccccc}
\mathrm{Tel}(G_n^{\times 3}, \chi^{\times 3}) & \xrightarrow{\sim} & \mathrm{Tel}(G_n^{\times 2}, \chi^{\times 2}) \times \mathrm{Tel}(G_n, \chi) & \xrightarrow{\sim} & \mathrm{Tel}(G_n, \chi)^{\times 3} \\
{\scriptstyle \mu}\downarrow & & {\scriptstyle \mu \times id}\downarrow & & \\
\mathrm{Tel}(G_{3n}, \chi^3) & \xleftarrow[\mu]{\sim} & \mathrm{Tel}(G_{2n}, \chi^2) \times \mathrm{Tel}(G_n, \chi) & & \\
{\scriptstyle \mu(g^{2n},-)}\uparrow{\scriptstyle \sim} & & {\scriptstyle \mu(g^n,-)\times id}\uparrow{\scriptstyle \sim} & & \\
\mathrm{Tel}(G_n, \chi) & \xrightarrow{\{g^n\} \times id} & \mathrm{Tel}(G_n, \chi)^{\times 2} & & \\
{\scriptstyle =}\downarrow & & {\scriptstyle \mu}\downarrow & & \\
\mathrm{Tel}(G_n, \chi) & \xrightarrow[\mu(g^n,-)]{\sim} & \mathrm{Tel}(G_{2n}, \chi^2). & &
\end{array} \tag{7.2.3}$$

The diagram is easily seen to be homotopy commutative since μ is strictly associative. The map $\mathrm{Tel}(G_n, \chi)^{\times 3} \to \mathrm{Tel}(G_n, \chi)$ given by $\mu_\infty(\mu_\infty(-,-),-)$ is, up to homotopy, the map given by choosing homotopy inverses for the homotopy equivalences in (7.2.3) and then forming the composite map which starts in the upper right corner, proceeds down the middle column, and ends at $\mathrm{Tel}(G_n, \chi)$ via the inverse of $\mu(g^n,-)$. This map is homotopic to the map formed by going across the entire top of (7.2.3), from right to left, and then down the left-hand side. A similar diagram shows that $\mu_\infty(-, \mu_\infty(-,-))$ is, up to homotopy, the composition of the map going across the entire top of (7.2.3), the upper left vertical map of (7.2.3), and the homotopy inverse of $\mu(-, g^{2n}) : \mathrm{Tel}(G_n, \chi) \to \mathrm{Tel}(G_{3n}, \chi^3)$. Consequently, to show homotopy associativity, it suffices to show $\mu(g^{2n}, -)$ and $\mu(-, g^{2n})$ induce homotopic maps from $\mathrm{Tel}(G_n, \chi)$ to $\mathrm{Tel}(G_{3n}, \chi^3)$, which is proven just as above for the maps $\mu(g^n, -)$ and $\mu(-, g^n)$. Thus $\mathrm{Tel}(G_n, \chi)$ is homotopy associative.

It is evident that the natural map $G \to \mathrm{Tel}(G_n, \chi)$ is a map of H-spaces. The claims concerning the induced maps on π_0 and homology follow from the observations that $\pi_0(-)$ and $H_*(-; A)$ commute with direct limits and $\mathrm{Tel}(G_n, \chi)$ is weakly equivalent to the direct limit $\varinjlim_n G_n$.

Finally, if there exist a pointed homotopy

$$C_n : G_n \times G_n \wedge I_+ \to G_{2n}$$

from μ to $\mu \circ \tau$, for each $n \geq 0$, then the collection of C_n's form a ladder of spaces in the evident matter. Each of the squares in this ladder is homotopy commutative, since $G_n \times G_n \xrightarrow{0} G_n \times G_n \wedge I_+$ is a homotopy equivalence and the induced square

$$\begin{array}{ccc}
G_n \times G_n & \xrightarrow{\chi \times \chi} & G_{n+1} \times G_{n+1} \\
{\scriptstyle \mu}\downarrow & & {\scriptstyle \mu}\downarrow \\
G_{2n} & \xrightarrow{\chi^2} & G_{2n+2}
\end{array}$$

commutes up to homotopy. Consequently, there is an induced map

$$\mathrm{Tel}(G_n \times G_n \wedge I_+, \chi \times \chi \wedge id) \to \mathrm{Tel}(G_{2n}, \chi^2)$$

which establishes the homotopy commutativity of $\mathrm{Tel}(G_n, \chi)$. $\square$

Theorem 7.3. *Let X be a quasi-projective real variety and fix a surjection $p : \mathcal{O}_X^\infty \twoheadrightarrow L$, with L a very ample line bundle. With α_L defined as above, if X is projective, the natural map*

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), \alpha_L)$$

is a homotopy-theoretic group completion and thus there is a weak equivalence of spaces

$$\mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), \alpha_L) \xrightarrow{\sim} \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$$

which fits into the evident commutative triangle. For any quasi-projective X, there is a similarly defined endomorphism

$$\alpha_L : |\operatorname{Hom}_{\mathbb{R}}(X \times_{\mathbb{R}} \Delta^\bullet_{top}, \mathrm{Grass}_{\mathbb{R}})| \to |\operatorname{Hom}_{\mathbb{R}}(X \times_{\mathbb{R}} \Delta^\bullet_{top}, \mathrm{Grass}_{\mathbb{R}})|$$

and the associated mapping telescope is weakly equivalent to $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$.

Proof. We prove the assertion involving $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$; the proof of the assertion involving $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}})$ differs only superficially. If X is not weakly normal, replacing it by its weak normalization does not affect any of the relevant spaces, and thus we may assume X is weakly normal.

Recall that the H-space structure on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ arises by choosing any element of $\mathcal{I}(2)$ and using the pairing

$$\mathcal{I}(2) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})^{\times 2} \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}),$$

and furthermore any two such choices yield equivalent homotopy commutative, homotopy associative H-spaces. In fact, we use for this proof an H-space structure for $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$, which does not actually arise in this manner, but which is nevertheless equivalent to those that do. Note that it suffices to assume X is connected (as an algebraic variety), for otherwise the map

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), \alpha_L)$$

decomposes into a Cartesian product indexed by the components of X, and it suffices to check the result for each component. Define

$$\mu : \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$$

by sending a pair $(\mathcal{O}_X^\infty \twoheadrightarrow E, \mathcal{O}_X^\infty \twoheadrightarrow F)$ to the quotients determined by the composition

$$\mathcal{O}_X^\infty \cong \mathcal{O}_X^\infty \oplus \mathcal{O}_X^\infty \twoheadrightarrow E \oplus F,$$

where the isomorphism is the inverse of the map sending $((e_1, e_2, \dots), (f_1, f_2, \dots))$ to

$$(e_1, \dots, e_r, f_1, \dots, f_s, e_{r+1}, \dots, e_{2r}, f_{s+1}, \dots),$$

where $r = rank(E)$ and $s = rank(F)$. One readily verifies that μ is strictly associative and has as a strict identity $\mathcal{O}_X^\infty \twoheadrightarrow 0$.

We claim μ endows $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ with an H-space structure which is H-equivalent to the H-space determined by a point in $\mathcal{I}(2)$. To see this, observe that $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ decomposes into a disjoint union indexed by the rank of the quotients

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) = \coprod_{r\geq 0} \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=r},$$

and thus the pairing

$$\mathcal{I}(2) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \qquad (7.3.1)$$

can be written as a disjoint union of the pairings

$$\mathcal{I}(2) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=r} \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=s} \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=r+s}.$$

For any fixed r, s, the pairing μ on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=r} \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{rk=s}$ is determined from the pairing (7.3.1) by a point $\mu_{r,s} \in \mathcal{I}(2)$ – namely, the point given by the isomorphism

$$\mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$$

sending $((e_1, e_2, \dots), (f_1, f_2, \dots))$ to $(e_1, \dots, e_r, f_1, \dots, f_s, e_{r+1}, \dots)$. Fix any point $\eta \in \mathcal{I}(2)$ and choose a path in the connected space $\mathcal{I}(2)$ from $\mu_{r,s}$ to η for each $r, s \geq 0$. This determines a pointed homotopy

$$I \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \times \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$$

from the pairing given by η to the pairing μ, and proves that these pairings determine equivalent H-spaces.

Note that the map

$$\alpha_L : \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$$

coincides with $\mu(p, -)$. Thus, to prove the theorem, it suffices to show

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_n, \mu(p, -))$$

is a homotopy-theoretic group completion, where $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_n$ is the space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ pointed by p^n. Clearly, the telescope $\mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_n, \mu(p, -))$ is homotopy equivalent to the telescope $\mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{2n}, \mu(p^2, -))$. Thus, setting $g = p^2$ and $G_n = \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})_{2n}$, it suffices to show

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(G_n, \mu(g, -))$$

is a homotopy-theoretic group completion, which will be accomplished using Lemma 7.2.

We claim $\mu(g, -), \mu(-, g) : G_n \to G_{n+1}$ are homotopic via a base point preserving homotopy. Note that G_n decomposes into a disjoint union index by rank, $G_n = \coprod_{r\geq 0} G_{n,rk=r}$, and it suffices to establish homotopies joining the two maps

$$\mu(g, -), \mu(-, g) : G_{n,rk=r} \to G_{n+1,rk=r+1},$$

for each $r \geq 0$. For any $x \in G_{n,rk=r}$, the quotient $\mu(g, x)$ can be obtained from the quotient $\mu(x, g)$ by precomposition with the automorphism of $\mathcal{O}_X^\infty$ given by an infinite direct sum of copies of the $(r+2)$-by-$(r+2)$ permutation matrix P associated to the permutation

$$1 \mapsto r+1, 2 \mapsto r+2, 3 \mapsto 1, 4 \mapsto 2, \dots, r+2 \mapsto r.$$

This is an even permutation and thus there is a path from P to the identity matrix in the space $\mathrm{GL}_{r+2}(\mathbb{R})$, which determines a pointed homotopy of the desired type.

Now we show $\mu, \mu \circ \tau : G_n \times G_n \to G_{2n}$ are homotopic via a base point preserving homotopy. As before, it suffices to construct an appropriate homotopy

$$G_{n,rk=r} \times G_{n,rk=s} \times I \to G_{2n,rk=r+s}$$

for each $r, s \geq 0$. For all x, y, the quotient $\mu(x,y)$ is obtained from the quotient $\mu(y,x)$ by precomposition with an automorphism of $\mathcal{O}_X^\infty$ defined by the direct sum of infinitely many copies of an $(r+s)$-by-$(r+s)$ permutation matrix $P_{r,s}$. If r and s are both even numbers, then, as one can readily verify, the matrix $P_{r,s}$ is even, and there is thus a path in $\mathrm{GL}_{r+s}(\mathbb{R})$ from $P_{r,s}$ to the identity matrix. This path determines a homotopy joining μ and $\mu \circ \tau$, when r and s are both even. If r and s are not both even, notice that $\mu(x,y)$ is obviously obtained from $\mu(y,x)$ via precomposition with the direct sum of infinitely many copies of $P_{r,s}^{\oplus 2}$ and $P_{r,s}^{\oplus 2}$ is even. Thus by choosing a path from $P_{r,s}^{\oplus 2}$ to the identity, we construct a homotopy in this case as well. Finally, since the base point of $G_n \times G_n$ lies in $G_{n,rk=2n} \times G_{n,rk=2n}$ (i.e., the base points all have even rank), the homotopy is readily seen to be base point preserving.

Thus, using Lemma 7.2, the map

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \mathrm{Tel}(G_n, \mu(g, -)) \tag{7.3.2}$$

is a map of homotopy associative, homotopy commutative H-spaces, the induced map on π_0 is given by inverting $[g]$, and the induced map on homology with coefficients in a commutative ring A is given by localizing with respect to the action of $[g]$. By Lemma 7.1, this means that the target of (7.3.2) is in particular a group-like H-space, and thus by definition (see [CCMT; §1]) the map (7.3.2) is a homotopy-theoretic group completion. The fact that this characterizes $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ up to weak equivalence, i.e., that there is a weak equivalence

$$\mathrm{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}), \alpha_L) \sim \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X),$$

follows from [CCMT; 1.2]. □

Corollary 7.4. *Let X be a real projective variety and T a finite CW complex. Let $[-,-]$ denote the set of homotopy classes of (unpointed) maps. Then the natural map*

$$[T, \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})] \to [T, \mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)]$$

identifies the target with the group completion of the abelian monoid of the source. Consequently, the group $K\mathbb{R}_q^{\mathrm{semi}}(X)$ may be identified as the kernel of the split surjection

$$[S^q, \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})]^+ \twoheadrightarrow K\mathbb{R}_0^{\mathrm{semi}}(X).$$

Proof. From Theorem 7.3 we know that $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is weakly equivalent to the mapping telescope of

$$\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}) \xrightarrow{\alpha_L} \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}}) \xrightarrow{\alpha_L} \dots.$$

Thus, the natural map $[T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})] \to [T, \mathcal{K}\mathbb{R}^{\text{semi}}(X)]$ identifies the target with the direct limit

$$[T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})] \to [T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})] \to \cdots, \tag{7.4.1}$$

in which each map is multiplication by the constant map $c : T \to \{p\}$ in the abelian monoid $[T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})]$. It follows that the direct limit of (7.4.1) is isomorphic to the abelian monoid obtain by inverting the class of the constant map c in the abelian monoid $[T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})]$. As this direct limit is known to be an abelian group (since $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is a group-like H-space), it follows that

$$[T, \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{\mathbb{R}})]^{+} \cong [T, \mathcal{K}\mathbb{R}^{\text{semi}}(X)].$$

Since $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is a group-like H-space, it is H-equivalent to the product of H-spaces $K\mathbb{R}_0^{\text{semi}}(X) \times \mathcal{K}\mathbb{R}^{\text{semi}}(X)_0$, where $\mathcal{K}\mathbb{R}^{\text{semi}}(X)_0$ is the connected component of the identity [CCMT; 1.1]. Using this decomposition and [Wh; III.1.11], it follows that the cofibration sequence

$$S^0 \hookrightarrow S^q_+ \twoheadrightarrow S^q$$

induces a short exact sequence of abelian groups

$$0 \to \pi_q \mathcal{K}\mathbb{R}^{\text{semi}}(X) \to [S^q, \mathcal{K}\mathbb{R}^{\text{semi}}(X)] \to K\mathbb{R}_0^{\text{semi}}(X) \to 0,$$

from which the second claim follows. □

The following corollary describes the space $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ as an explicit direct limit of spaces of maps of real varieties.

Corollary 7.5. *Let X be a quasi-projective real variety and assume $p : \mathcal{O}_X^m \twoheadrightarrow L$ is a surjection with L a very ample line bundle. Then the space $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is weakly equivalent to the space*

$$\mathcal{M}or_{\mathbb{R}}(X, \mathbb{Z}) \times \text{Tel}(\mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_j(\mathbb{P}_{\mathbb{R}}^{j(m+1)})), \theta_j)$$

where the transition maps

$$\theta_j : \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_j(\mathbb{P}_{\mathbb{R}}^{j(m+1)})) \to \mathcal{M}or_{\mathbb{R}}(X, \text{Grass}_{j+1}(\mathbb{P}_{\mathbb{R}}^{(j+1)(m+1)}))$$

in the telescope are induced by the map sending a quotient $q : \mathcal{O}_X^{j(m+1)+1} \twoheadrightarrow E$ to the quotient determined by the composition of

$$\mathcal{O}_X^{(j+1)(m+1)+1} \equiv \mathcal{O}_X^m \oplus \mathcal{O}_X^{j(m+1)+1} \oplus \mathcal{O}_X \xrightarrow{\begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}} L \oplus E.$$

Proof. The proof is identical to the proof of [FW2; 3.5]. □

The significance of Corollary 7.5 is that it relates our theory $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ to the space of algebraic maps of X to various (finite-dimensional) Grassmann varieties. For example, when X is projective and weakly normal, $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ is explicitly a direct limit of the spaces of algebraic maps from X to the Grassmann varieties $\text{Grass}_j(\mathbb{P}^{jn+j}_{\mathbb{R}})$, where these spaces are topologized as subspaces of the collection of all continuous, equivariant maps from $X_{\mathbb{R}}(\mathbb{C})$ to $\text{Grass}_j(\mathbb{P}^{2j+j})_{\mathbb{R}}(\mathbb{C})$. The complex version of these spaces appears in the work of Kirwan [Ki] and Cohen-Lupercio-Segal [CLS], among others, especially when X is taken to be a complete complex curve (i.e., a Riemann surface). Thus, just as $\mathcal{K}^{\text{semi}}(X)$ for a complex projective variety X is a "stabilization" of the spaces of holomorphic maps appearing in [Ki] and [CLS], the space $\mathcal{K}\mathbb{R}^{\text{semi}}(X)$ for a real projective variety X is related to the study of spaces of "equivariant holomorphic" maps between the spaces of complex points of real varieties. We suggest that these latter "unstable" spaces are worthy of study in their own right.

§8 Real morphic cohomology and characteristic classes

We introduce real morphic cohomology, the analogue in the real context of morphic cohomology considered in [FL1]. This is a cohomology theory presumably dual (for smooth varieties) to real Lawson homology as constructed by P. dos Santos in [Sa]. We show that there are natural Chern classes from $K^{\text{alg}}_*(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ to this real morphic cohomology for any real quasi-projective variety X. These Chern classes should be viewed as a simultaneous generalization of the total Chern class map of [FW2] and [BLLMM]. Indeed, we generalize to the real context the "double square" of infinite loop spaces presented in [FW2] relating K-theories to cohomology theories of complex varieties. As we shall see, the double indexing in $\mathbb{Z}/2$-equivariant singular cohomology receiving Atiyah's Real K-theory is compatible with the double indexing in motivic cohomology receiving algebraic K-theory as well as the double indexing in real morphic cohomology receiving $K^{\text{alg}}_*(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$.

Let $\text{Chow}_r(\mathbb{P}^N_{\mathbb{R}})$ be the infinite disjoint union of quasi-projective varieties

$$\text{Chow}_r(\mathbb{P}^N_{\mathbb{R}}) \equiv \coprod_{d \geq 0} C_{r,d}(\mathbb{P}^N),$$

where $C_{r,d}$ parameterizes all effective cycles of dimension r and degree d on $\mathbb{P}^N_{\mathbb{R}}$. We write $\text{Chow}_{r,\mathbb{R}}$ for the ind-variety $\varinjlim_N \text{Chow}_r(\mathbb{P}^N_{\mathbb{R}})$, where the transition maps are induced by the closed immersions $\mathbb{P}^N_{\mathbb{R}} \hookrightarrow \mathbb{P}^{N+1}_{\mathbb{R}}$ given by inclusion into the first $N+1$ homogeneous coordinates. Notice that $\text{Chow}_r(\mathbb{P}^N_{\mathbb{R}})$ and $\text{Chow}_{r,\mathbb{R}}$ are each abelian monoids under addition of cycles in the category of ind-varieties.

In general, if M is a topological abelian monoid, its "naive group completion" is the quotient space $M^+ = M \times M/(x,y) \sim (x+m, y+m)$, which has the induced structure of a topological abelian group.

Definition 8.1. *Let X be a real quasi-projective variety. For each $r \geq 0$, define*

$$\mathcal{M}or_{\mathbb{R}}(X, \text{Chow}_r(\mathbb{P}^N_{\mathbb{R}})) \to \mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}})$$

to be the naive group completion of the topological abelian monoid $\mathcal{M}or_{\mathbb{R}}(X, \text{Chow}_r(\mathbb{P}^N_{\mathbb{R}}))$. *Define*

$$\mathcal{M}or_{\mathbb{R}}(X, \text{Chow}_{r,\mathbb{R}}) \to \mathcal{Z}_r(X, \mathbb{P}^\infty_{\mathbb{R}})$$

to be the direct limit of the maps

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^N_{\mathbb{R}})) \to \mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}})$$

as N goes to infinity. For any $d \geq 0$, we let $\mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}})_d$ denote the pre-image under the degree map $\mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}}) \to \mathbb{Z}$ of $d \in \mathbb{Z}$, and $\mathcal{Z}_r(X, \mathbb{P}^\infty_{\mathbb{R}})_d$ is defined similarly. Furthermore, we write $\mathcal{Z}^q(X)_{\mathbb{R}}$ for the naive group completion of the topological abelian quotient monoid

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^q_{\mathbb{R}}))/\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^{q-1}_{\mathbb{R}})).$$

Finally, the Real morphic cohomology groups of X are defined by the formula

$$L^qH\mathbb{R}^p(X) \equiv \pi_{2q-p}\mathcal{Z}^q(X)_{\mathbb{R}}.eqno(8.1.1)$$

Proposition 8.2. *With hypotheses and notation as in Definition 8.1, for any $r \geq 0$ the natural monoid homomorphisms*

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^N_{\mathbb{R}})) \to \mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}})$$

and

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_{r,\mathbb{R}}) \to \mathcal{Z}_r(X, \mathbb{P}^\infty_{\mathbb{R}})$$

are homotopy-theoretic group completions. Moreover, for any $q \geq 0$, the sequence of maps

$$\mathcal{Z}_0(X, \mathbb{P}^{q-1}_{\mathbb{R}}) \to \mathcal{Z}_0(X, \mathbb{P}^q_{\mathbb{R}}) \to \mathcal{Z}^q(X)_{\mathbb{R}}$$

is a fibration sequence up to homotopy.

Proof. The proof is parallel to the proof of [F4; 1.7]. Indeed, the key technical arguments of [FL2; App. C] concerning the topology of constructible subsets of $\mathbb{P}^m_{\mathbb{C}}$ are equally valid for constructible subsets of $\mathbb{P}^m_{\mathbb{R}}$, and in particular the topological monoid $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^N))$ is "tractable". This is enough to conclude that there is a natural weak homotopy equivalence

$$\mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}}) \xrightarrow{\sim} \Omega B\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^N_{\mathbb{R}}))$$

by using the argument given in the proof of [FL2; C.4]. (Recall that given a topological abelian monoid M, its homotopy-theoretic groups completion can be given as ΩBM.) Since the conditions defining a homotopy-theoretic group completion (cf. [CCMT; §1]) commute with direct limits, we conclude

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_{r,\mathbb{R}}) \to \mathcal{Z}_r(X, \mathbb{P}^\infty_{\mathbb{R}})$$

is also a homotopy-theoretic group completion.

The proof of the final assertion follows from the fact that that

$$\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^{q-1}_{\mathbb{R}})) \xrightarrow{\subseteq} \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^q_{\mathbb{R}}))$$

is a cofibration of tractable monoids and $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^q_{\mathbb{R}}))$ is tractable as an $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Chow}_r(\mathbb{P}^{q-1}_{\mathbb{R}}))$-space, using the argument found in the proof of [FG; 1.6]. □

Since it turns out to be convenient, we introduce the notion of the *empty cycle* in $\mathbb{P}^N_{\mathbb{R}}$, which by definition has degree 1 and dimension -1. Thus, $C_{-1,d}(\mathbb{P}^N_{\mathbb{R}})$ is a one element set consisting of the formal sum of d copies of the empty cycle. Further, $\mathcal{Z}_{-1}(X, \mathbb{P}^N_{\mathbb{R}})$ is isomorphic to the abelian group $\mathbb{Z}$ containing all integer multiples of the empty cycle. Note that $\mathcal{Z}_{-1}(X, \mathbb{P}^N_{\mathbb{R}})_1$ corresponds to the singleton $\{1\} \subset \mathbb{Z}$.

Proposition 8.3. *Let X be a quasi-projective real variety. Then, in analogy with the complex context, the following hold:*

(a.) (Lawson suspension) Algebraic suspension of cycles induces weak equivalences

$$\Sigma : \mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{Z}_{r+1}(X, \mathbb{P}^{N+1}_{\mathbb{R}}), \quad \Sigma : \mathcal{Z}_r(X, \mathbb{P}^{\infty}_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{Z}_{r+1}(X, \mathbb{P}^{\infty}_{\mathbb{R}}).$$

(b.) (Splitting) There is a natural weak equivalence

$$\mathcal{Z}_0(X, \mathbb{P}^{\infty}_{\mathbb{R}}) \xrightarrow{\sim} \varinjlim_N \prod_{q=0}^{N} \mathcal{Z}^q(X)_{\mathbb{R}}.$$

Proof. The suspension map

$$\Sigma : \mathcal{Z}_r(X, \mathbb{P}^N_{\mathbb{R}}) \to \mathcal{Z}_{r+1}(X, \mathbb{P}^{N+1}_{\mathbb{R}})$$

is given by embedding $\mathbb{P}^N_{\mathbb{R}}$ in $\mathbb{P}^{N+1}_{\mathbb{R}}$ by using the *last* $N+1$ homogeneous coordinates and then taking linear join of cycles with the one-point cycle corresponding to the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^{N+1}_{\mathbb{R}}$. In particular, the suspension map is readily verified to be compatible with the maps

$$\mathcal{Z}_r(X, \mathbb{P}^M_{\mathbb{R}}) \to \mathcal{Z}_r(X, \mathbb{P}^{M+1}_{\mathbb{R}}), \quad \mathcal{Z}_{r+1}(X, \mathbb{P}^M_{\mathbb{R}}) \to \mathcal{Z}_{r+1}(X, \mathbb{P}^{M+1}_{\mathbb{R}}),$$

in the direct limits which define $\mathcal{Z}_r(X, \mathbb{P}^{\infty}_{\mathbb{R}})$ and $\mathcal{Z}_{r+1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})$. The suspension map

$$\Sigma : \mathcal{Z}_r(X, \mathbb{P}^{\infty}_{\mathbb{R}}) \to \mathcal{Z}_{r+1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})$$

is defined by taking the direct limits of these "finite" suspension maps, and so in particular it suffices to establish the first suspension equivalence.

To show this, we merely observe that the homotopies constructed in the proof of [FL1; 3.3] are in fact "algebraic" – i.e., are induced by continuous algebraic maps of the form $T \times \mathbb{A}^1 \to S$. Thus, the argument of [op. cit.] carries over to our context, word for word, replacing the topological realization functor $U \mapsto U(\mathbb{C})$ with $U \mapsto U(\mathbb{R})$ everywhere.

Similarly, part (b) follows directly from the proof of [FL1; 2.10], since the splittings constructed are algebraic. □

As in [FW2], we find it convenient to introduce the total Segre class rather than the total Chern class. Recall that the total Segre class $s(E)$ of a vector bundle E equals the total Chern class of the virtual bundle $-E$. In particular, the total Chern class of E coincides with $1/s(E)$.

For any $N > n > 0$, we let

$$s_{N,n} : \mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}}) \to C_{n-1,1}(\mathbb{P}^N_{\mathbb{R}})$$

denote the map of [FW2; 6.1] which sends the universal locally free rank n quotient $\mathcal{O}^{\oplus N+1}_{\mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})} \twoheadrightarrow U^{n,N}$ to the effective, relative dimension $n-1$ cocycle $\mathbb{P}((U^{n,N})^{\#})$ on $\mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})$. Here, $(U^{n,N})^{\#}$ is the $\mathcal{O}_{\mathrm{Grass}_n(\mathbb{P}^N_{\mathbb{R}})}$-linear dual of $U^{n,N}$. Taking limits as N goes to infinity, we obtain the map

$$s = \coprod_r s_r : \mathrm{Grass}_{\mathbb{R}} \to \mathrm{Chow}_{\mathbb{R},1} \equiv \coprod_{r \geq 0} \varinjlim_{N \to \infty} C_{r-1,1}(\mathbb{P}^N_{\mathbb{R}}),$$

which induces the map (also written as s)

$$s : \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1,$$

obtained by composition.

The action of the operad $\mathcal{I}$ on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ suggests the following definition of an action of $\mathcal{I}$ on $\coprod_{r\geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})$: Given effective cycles $\gamma_i \in C_{r_i,d_i}(\mathbb{P}^{\infty}_{\mathbb{R}})$, for $i = 1, \ldots, n$ and a linear injection $\alpha : (\mathbb{R}^{\infty})^{\times n} \hookrightarrow \mathbb{R}^{\infty}$, associate to α the evident closed immersion

$$\alpha : \mathbb{P}^{\infty}_{\mathbb{R}} \amalg \cdots \amalg \mathbb{P}^{\infty}_{\mathbb{R}} \hookrightarrow \mathbb{P}^{\infty}_{\mathbb{R}}$$

and to the each cycle γ_i, the pushforward $\alpha_*(\gamma_i)$. Now form a new cycle by taking the *linear join* of $\alpha_*(\gamma_1), \ldots, \alpha_*(\gamma_n)$ – that is, take the union of all n-dimensional linear subvarieties of $\mathbb{P}^{\infty}_{\mathbb{R}}$ containing one point from each of $\alpha_*(\gamma_1), \ldots, \alpha_*(\gamma_n)$. Extending by linearity, we define natural pairings

$$\mathcal{I}(n) \times \left(\coprod_{r \geq 0} \mathcal{Z}_{r-1}(\mathbb{P}^{\infty}_{\mathbb{R}}) \right)^{\times n} \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(\mathbb{P}^{\infty}_{\mathbb{R}}),$$

where by convention the linear join of the empty cycle with any other cycle δ is δ itself. These in turn extend by functoriality to give natural pairings

$$\mathcal{I}(n) \times \left(\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}}) \right)^{\times n} \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}}),$$

which are readily checked to satisfy the axioms of an operad action. Moreover, these pairings clearly behave in a multiplicative fashion on degree, and thus restrict to endow

$$\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1$$

with the structure of an $\mathcal{I}$-space such that total Segre class map

$$s : \mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) \to \coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1$$

is a morphism of $\mathcal{I}$-spaces.

Proposition 8.4. *Let X be a smooth, connected, quasi-projective real variety. Then the homotopy-theoretic group completion of the $\mathcal{I}$-space*

$$\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1$$

is equivalent to the map

$$\coprod_{r \geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1 \to \mathbb{Z} \times \mathcal{Z}_0(X, \mathbb{P}^{\infty}_{\mathbb{R}})_1$$

which is given on the rth component, $r > 0$, by choosing a homotopy inverse of the map

$$\{r\} \times \mathcal{Z}_0(X, \mathbb{P}^\infty_\mathbb{R})_1 \xrightarrow{\Sigma^{\circ r-1}} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty_\mathbb{R})_1,$$

and which is given by the evident constant map for $r = 0$. Thus the Segre map s introduced above induces a map of infinite loop spaces of associated group completions,

$$s : \mathcal{K}\mathbb{R}^{\text{semi}}(X) \to \mathbb{Z} \times \mathcal{Z}_0(X, \mathbb{P}^\infty_\mathbb{R})_1,$$

which we call the total Segre class.

Proof. This is proven exactly as is [FW2; 6.5]. □

Note that in Proposition 8.4, the H-space structure on the factor $\mathbb{Z}$ is the usual one given by addition. The composition of s with projection to $\mathbb{Z}$ is, of course, the rank map. Thus, we have implicitly defined the 0th Segre class of an element of $\mathcal{K}\mathbb{R}^{\text{semi}}_0(X)$ to be its rank.

We now focus on the space

$$\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_r(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C})))$$

where $\mathcal{Z}_r(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C}))$ denotes the naive group completion of the topological abelian monoid with $\mathbb{Z}/2$-action obtained by taking the space of complex points of the ind-variety $\coprod_d C_{r,d}(\mathbb{P}^\infty_\mathbb{R})$. (We adhere to the conventions introduced above for the case $r = -1$.) Further, we write $\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_r(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C})))_1$ for the subspace of degree one cocycles – i.e., the fiber over 1 of the degree map

$$deg : \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_r(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C}))) \to \mathbb{Z}.$$

We have the following analogue of Proposition 8.4.

Proposition 8.5. *The operad $\mathcal{I}$ acts naturally on the space*

$$\coprod_{r \geq 0} \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_{r-1}(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C})))_1$$

and the homotopy-theoretic group completion with respect to this action is weakly equivalent to

$$\coprod_{r \geq 0} \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_{r-1}(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C})))_1 \to \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathbb{Z}) \times \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \mathcal{Z}_0(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C})))_1.$$

Proof. The proof of [FW2; 6.5] applies. □

We next recall a fundamental result due to P. dos Santos [Sa; 3.2], which identifies the $\mathbb{Z}/2$-equivariant homotopy type of $\mathcal{Z}_0(\mathbb{P}^\infty_\mathbb{R}(\mathbb{C}))$ with the $\mathbb{Z}/2$-spectrum

$$\varinjlim_{N \to \infty} \prod_{q=0}^{N} K(\underline{\mathbb{Z}}, \mathbb{R}^{q,q}).$$

Here, $K(\underline{\mathbb{Z}}, \mathbb{R}^{p,q})$ denotes the $\mathbb{Z}/2$-equivariant Eilenberg-MacLane spectrum arising from the constant Mackey functor $\underline{\mathbb{Z}}$ associated to $\mathbb{Z}$ and the $\mathbb{Z}/2$-space $\mathbb{R}^{p,q}$, defined as $\mathbb{R}^{p+q}$ equipped with the $\mathbb{Z}/2$ action given as multiplication by -1 on the last q coordinates. Thus we may identify $\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{R}}(\mathbb{C})))$ with

$$\varinjlim_{N\to\infty} \prod_{q=0}^{N} \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), K(\underline{\mathbb{Z}}, \mathbb{R}^{q,q})).$$

The homotopy groups of $\mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), K(\underline{\mathbb{Z}}, \mathbb{R}^{q,q}))$ give the equivariant cohomology groups of the $\mathbb{Z}/2$-space $X_{\mathbb{R}}(\mathbb{C})$ with twisted coefficients. We write these groups more simply as

$$H^{q-p,q}_{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \underline{\mathbb{Z}}) \equiv \pi_p \, \mathcal{M}\text{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), K(\underline{\mathbb{Z}}, \mathbb{R}^{q,q})). \tag{8.6}$$

As in [FW2; §6], we also define the purely algebraic cycle theory

$$Z_{r-1}(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^\infty_{\mathbb{R}})_1$$

as the geometric realization of the fiber over 1 of the map of simplicial abelian groups

$$\left(d \mapsto \mathcal{M}or_{\mathbb{R}}(\Delta^d \times_{\mathbb{R}} X, \mathcal{Z}_{r-1}\mathbb{P}^\infty_{\mathbb{R}})^+\right) \to \mathbb{Z},$$

where the superscript + denotes taking group completion of the indicated abelian monoid. By [FW2; 6.9], the space

$$\coprod_{r\geq 0} Z_{r-1}(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^\infty_{\mathbb{R}})_1$$

has a natural structure as an $|\mathcal{I}(\Delta^\bullet)|$-space, there is a map

$$s : |\operatorname{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X, \text{Grass}_{\mathbb{R}})| \to Z_{r-1}(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^\infty_{\mathbb{R}})_1,$$

of $|\mathcal{I}(\Delta^\bullet)|$-spaces defined analogously to the maps s above, and the homotopy-theoretic group completion of

$$\coprod_{r\geq 0} Z_{r-1}(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^\infty_{\mathbb{R}})_1$$

decomposes up to weak homotopy as

$$\bigoplus_{q=0}^{\infty} \mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^q_{\mathbb{R}})^+ / \mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^{q-1}_{\mathbb{R}})^+.$$

Moreover, the collection of quotient chain complexes of abelian groups $\mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^q_{\mathbb{R}})^+ / \mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^{q-1}_{\mathbb{R}})^+$, $q \geq 0$, gives the motivic cohomology groups of X, when X is smooth (as defined, for example, in [FV]), by the formula

$$H^{2q-i}_{\mathcal{M}}(X, \mathbb{Z}(q)) = \pi_i \mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^q_{\mathbb{R}})^+ / \mathcal{Z}_0(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^{q-1}_{\mathbb{R}})^+. \tag{8.7}$$

Theorem 8.8. *Let X be a quasi-projective real variety. Then there is a natural commutative double square of $|\mathcal{I}(\Delta^\bullet)|$-spaces*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}}) & \xrightarrow{s} & \coprod_{r\geq 0} Z_{r-1}(\Delta^\bullet \times_{\mathbb{R}} X, \mathbb{P}^\infty_{\mathbb{R}})_1 \\
\downarrow & & \downarrow \\
\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}}) & \xrightarrow{s} & \coprod_{r\geq 0} \mathcal{Z}_{r-1}(X, \mathbb{P}^\infty_{\mathbb{R}})_1 \\
\downarrow & & \downarrow \\
\mathcal{M}\mathrm{aps}^{Z/2}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C})) & \xrightarrow{s} & \coprod_{r\geq 0} \mathcal{M}\mathrm{aps}^{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \mathcal{Z}_{r-1}(\mathbb{P}^\infty_{\mathbb{R}}(\mathbb{C})))_1.
\end{array}$$

If X is smooth, by taking homotopy groups of the induced diagram of homotopy-theoretic group completions, we obtain for each $i \geq 0$ a commutative double square

$$\begin{array}{ccccc}
K_i^{\mathrm{alg}}(X) & \longrightarrow & K_i^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) & \longrightarrow & K\mathbb{R}^{-i}_{top}(X_{\mathbb{R}}(\mathbb{C})) \\
{\scriptstyle s_i}\downarrow & & {\scriptstyle s_i}\downarrow & & {\scriptstyle s_i}\downarrow \\
\bigoplus_{q\geq 0} H^{2q-i}_{\mathcal{M}}(X, \mathbb{Z}(q)) & \longrightarrow & \bigoplus_{q\geq 0} L^q H\mathbb{R}^{2q-i}(X) & \longrightarrow & \bigoplus_{q\geq 0} H^{q-i,q}_{\mathbb{Z}/2}(X_{\mathbb{R}}(\mathbb{C}), \underline{\mathbb{Z}}),
\end{array} \tag{8.8.1}$$

where the group laws for the bottom row are induced by join of cycles. Here, the groups in the lower left corner of (8.8.1) are the motivic cohomology groups as defined in (8.7), those in the lower middle are defined in (8.1.1), and those the the lower right corner are defined in (8.5.1).

Proof. The commutativity of the top diagram is essentially given by construction. The second diagram is obtained from the first by applying π_j to the associated diagram of infinite loop spaces obtained from homotopy-theoretic group completion. The fact that the homotopy-theoretic group completion of $\mathrm{Hom}(\Delta^\bullet \times_{\mathbb{R}} X, \mathrm{Grass}_{\mathbb{R}})$ is $\mathcal{K}^{\mathrm{alg}}(X)$ for X smooth is given by Proposition 1.4. The groups appearing in the bottom-left corner of (8.8.1) are correct by [FW2; 6.9]. Those appearing in the bottom middle of (8.8.1) are shown to be correct by using Proposition 8.3(b), which shows that there is a weak equivalence

$$\mathcal{Z}_0(X, \mathbb{P}^\infty_{\mathbb{R}})_1 \sim \{1\} \times \varinjlim_N \prod_{j=1}^N \mathcal{Z}^j(X)_{\mathbb{R}},$$

together with Proposition 8.4. The bottom-right conner is correct by Proposition 8.5 and the calculation of dos Santos [Sa; 3.2] mentioned above. □

Recall that for a smooth variety X, the group law of $\bigoplus_{q\geq 0} L^q H\mathbb{R}^{2q-i}(X)$ is given in terms of the group-like H-space

$$\mathbb{Z} \times \mathcal{Z}_0(X, \mathbb{P}^\infty_{\mathbb{R}})_1.$$

The H-space structure on $\mathbb{Z}$ is the obvious one given by addition and the H-space structure for $\mathcal{Z}_0(X, \mathbb{P}^\infty_{\mathbb{R}})_1$ is described as follows: Given two cycles γ_1, γ_2 in $X \times \mathbb{P}^\infty_{\mathbb{R}}$

which have degree 1 and are equidimensional over X, their product is given by first moving them into disjoint linear subspaces in a predetermined manner and then taking their linear join.

As seen in the proof of Theorem 8.7, there is an equivalence

$$\mathcal{Z}_0(X, \mathbb{P}^\infty_\mathbb{R})_1 \sim \left(\varinjlim_N \prod_{j=0}^N \mathcal{Z}^j(X)_\mathbb{R} \right)_1 = \left(\{1\} \times \varinjlim_N \prod_{j=1}^N \mathcal{Z}^j(X)_\mathbb{R} \right). \tag{8.8}$$

From this decomposition we see that an arguably more logical pairing for

$$\bigoplus_{q \geq 0} L^q H\mathbb{R}^{2q-i}(X)$$

could be given by the join pairing

$$\# : \mathcal{Z}^p(X) \times \mathcal{Z}^q(X) \to \mathcal{Z}^{p+q}(X), \tag{8.9}$$

defined for all $p, q \geq 1$. Here, (8.9) is defined by choosing a homotopy inverse for the suspension map in the diagram

$$\begin{array}{ccc} \mathcal{M}or_\mathbb{R}(X, \mathcal{Z}_0(\mathbb{P}^p_\mathbb{R})) \times \mathcal{M}or_\mathbb{R}(X, \mathcal{Z}_0(\mathbb{P}^q_\mathbb{R})) & \xrightarrow{\#} & \mathcal{M}or_\mathbb{R}(X, \mathcal{Z}_1(\mathbb{P}^{p+q+1}_\mathbb{R})) \\ & & \text{suspension} \Big\uparrow \wr \\ & & \mathcal{M}or_\mathbb{R}(X, \mathcal{Z}_0(\mathbb{P}^{p+q}_\mathbb{R})), \end{array}$$

where $\#$ is defined by embedding $\mathbb{P}^p_\mathbb{R}$ and $\mathbb{P}^q_\mathbb{R}$ in $\mathbb{P}^{p+q+1}_\mathbb{R}$ via the first $p+1$ and last $q+1$ homogeneous coordinates, respectively, and then taking linear join of cycles.

The following proposition asserts that these two potentially different product rules for $\bigoplus_{q \geq 0} L^q H\mathbb{R}^{2q-i}(X)$ are indeed equivalent, and that an analogous result holds for both $\bigoplus_{q \geq 0} H^{2q-i}_\mathcal{M}(X, \mathbb{Z}(q))$ and $\bigoplus_{q \geq 0} H^{q-i,q}_{\mathbb{Z}/2}(X_\mathbb{R}(\mathbb{C}), \underline{\mathbb{Z}})$. One way to view this result is that it asserts that the join pairing that gives $\mathbb{Z} \times \mathcal{Z}_0(X, \mathbb{P}^\infty_\mathbb{R})_1$ the structure of an H-space is a *graded* pairing with respect to the decomposition (8.8), at least up to homotopy. Alternatively, one can view this result as asserting that the *Whitney sum* formula holds for the pairing (8.9).

Proposition 8.10. *The middle vertical maps in (8.8.1) are homomorphisms of groups where the group law for the target is given by the pairings*

$$L^p H\mathbb{R}^{2p-i}(X) \times L^q H\mathbb{R}^{2q-i}(X) \to L^{p+q} H\mathbb{R}^{2p+2q-i}(X)$$

induced by the pairings (8.9). Thus, writing $s_{q,i}$ for the composition of s_i with projection to $L^q H\mathbb{R}^{2q-i}(X)$, given $x, y \in K^{\text{alg}}_i(\Delta^\bullet_{top} \times_\mathbb{R} X)$, the familiar Whitney sum formula holds:

$$s_{q,i}(x+y) = \sum_{k+l=q} s_{k,i}(x) \# s_{l,i}(y),$$

for all i, q. The other two vertical maps also satisfy the Whitney sum formulas with respect to the analogous pairings.

Proof. This is proven just as is [FW2; 6.6] by using the real analogue of [FW1; 5.3]. The Whitney sum formula holds since the proof of [FW1; 5.3] carries over, word for word, to the real context, replacing $\mathcal{M}or_\mathbb{C}(-,-)$ with $\mathcal{M}or_\mathbb{R}(-,-)$ everywhere. □

Remark 8.11. *Recall the pairing on higher homotopy groups (i.e., on π_i for $i > 0$) induced by an H-space structure necessarily coincides with the usual product rule for homotopy groups. It follows immediately that in the case $i > 0$, the formula of Proposition 8.10 has to be simply*

$$s_{q,i}(x+y) = s_{q,i}(x) + s_{q,i}(y).$$

This seeming contradiction is explain by the fact that for any $p, q \geq 1$ the join pairing (8.9) factors through $\mathcal{Z}^p(X) \wedge \mathcal{Z}^q(X)$ and thus induces the trivial pairing upon applying $\pi_i(-)$, $i \geq 1$. However, the pairing induced by (8.9)

$$\# : \mathcal{Z}^p(X) \wedge \mathcal{Z}^q(X) \to \mathcal{Z}^{p+q}(X), \quad p, q \geq 0,$$

is of interest even when considering higher homotopy groups, in that by taking smash product of maps in the usual fashion, one defines multiplicative pairings

$$L^q H\mathbb{R}^p(X) \otimes L^s H\mathbb{R}^r(X) \to L^{q+s} H\mathbb{R}^{p+r}(X).$$

These pairings give the "cup product" for real morphic cohomology.

§9 The Panin-Smirnov Axioms

In a recent preprint [PS], I. Panin and A. Smirnov discuss "oriented multiplicative cohomology theories" on smooth algebraic varieties over a field. Their general discussion shows how such a structure naturally associates to a projective map of smooth varieties a push-forward map on the theory. Furthermore, they analyze how such push-forward maps compare with respect to a natural transformation of such theories. The latter is exactly the Riemann-Roch problem, as formulated by A. Grothendieck.

In this section, we verify that the theory $K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ satisfies the axioms of Panin and Smirnov. We then observe that the push-forward maps associated for this theory arise naturally from the discussion of localization in Section 6. Finally, we point out that one would have a Riemann-Roch theorem relating this theory to real morphic cohomology provided that one could verify that the Chern character is multiplicative. (The multiplicativity of a Chern character map in the complex setting is asserted in [CL1; 4.11].)

Recall that we introduced in Definition 2.3 the spectrum $\mathcal{K}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$, representing the theory $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ with "supports" in a closed subvariety $Z \subset X$. We write $Sm^2_{\mathbb{R}}$ for the category of pairs (X, U) where X is a smooth, quasi-projective real variety and $U \subset X$ is a Zariski open subset.

Theorem 9.1. *Consider the graded-abelian-group-valued functor*

$$(X, X-Z) \mapsto K^{\mathrm{alg}}_{Z,*}(\Delta^\bullet_{top} \times X),$$

defined on $Sm^2_{\mathbb{R}}$ and equipped with the external cup product operation introduced in Theorem A.1. Then this theory satisfies the Panin-Smirnov axioms of a strictly commutative multiplicative cohomology theory. Specifically, letting (X, U) and (Y, V) denote any two smooth pairs with $Z = X - U, W = Y - V$, the following hold:

(a) There is a natural long-exact sequence

$$\cdots \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times X) \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times U) \xrightarrow{\partial} K_{Z,*-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X)$$
$$\to K_{*-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X) \to \cdots .$$

(b) Nisnevich excision holds – i.e., if there is an étale map $\pi : X' \to X$ such that the inclusion $Z \hookrightarrow X$ factors through π, then the natural map

$$K_{Z,*}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X) \to K_{Z,*}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X')$$

is an isomorphism for all $*$.

(c) Homotopy invariance holds – i.e., the natural map

$$K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times X) \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times X \times \mathbb{A}^1)$$

is an isomorphism for all $*$.

(d) The cup product operation of Theorem A.1 is associative – i.e., we have $a \cup (b \cup c) = (a \cup b) \cup c$ for all appropriate a, b, and c.

(e) The element 1 in $K_0^{\mathrm{alg}}(\Delta^\bullet_{top} \times \operatorname{Spec} \mathbb{R}) \cong \mathbb{Z}$ is a two-sided identity for cup product.

(f) For all $a \in K_p^{\mathrm{alg}}(\Delta^\bullet_{top} \times U)$, $b \in K_{W,q}^{\mathrm{alg}}(\Delta^\bullet_{top} \times Y)$, $c \in K_{Z,p}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X)$ and $d \in K_q^{\mathrm{alg}}(\Delta^\bullet_{top} \times V)$, the formulas

$$\partial(a \cup b) = \partial(a) \cup b$$
$$\partial(c \cup d) = (-1)^p c \cup \partial d$$

hold in $K_{Z \times W, p+q-1}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X \times Y)$.

(g) The formula $a \cup b = (-1)^{pq} b \cup a$ holds for all $a \in K_{Z,p}^{\mathrm{alg}}(\Delta^\bullet_{top} \times X)$ and $b \in K_{W,q}^{\mathrm{alg}}(\Delta^\bullet_{top} \times Y)$.

Proof. Property (a) follows from Corollary 6.7; property (b) is given by Proposition 3.5; and property (c) is given by Proposition 2.6.

To prove property (d), we use the explicit functorial model $A_Z(X)$ for the spectrum $\mathcal{K}_Z^{alg}(X)$ given in Appendix A. Then, given three pairs $(X,U),(Y,V),(D,E)$ with closed complements Z,W,C, the two possible pairings

$$A_Z(X)_i \wedge A_W(Y)_j \wedge A_C(D)_k \to A_{Z \times_{\mathbb{R}} W \times_{\mathbb{R}} C}(X \times_{\mathbb{R}} Y \times_{\mathbb{R}} D)_{i+j+k}$$

are naturally homotopic. Naturality of the homotopy ensures that we obtain a similar associativity claim by inserting the simplicial direction $\Delta^\bullet_{top}$. The desired result now follows by using suitable choices for the space-level pairings as given in the proof of Theorem A.1.

The identity element of $K_0^{\mathrm{alg}}(\Delta^\bullet_{top} \times \operatorname{Spec} \mathbb{R})$ is induced by pull-back from an element ι of $K_0^{\mathrm{alg}}(\operatorname{Spec} \mathbb{R}) = \pi_1 A(\operatorname{Spec} \mathbb{R})_1$ represented by a map $S^1 \to A(\operatorname{Spec} \mathbb{R})_1$. This element induces a map

$$S^1 \wedge A_Z(X)_i \to A_Z(X)_{i+1}$$

naturally homotopic to the structure map of the prespectrum $\mathcal{K}_Z^{\mathrm{alg}}(X)$ for all i. Thus, property (e) follows immediately.

Property (g) is a standard property of the multiplication for prespectra. Property (f) follows from a standard argument using the fact that there is a commutative diagram (in the homotopy category of CW complexes)

$$\begin{array}{ccc}
A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_i \wedge A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_j & \longrightarrow & A_{Z \times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{i+j} \\
\downarrow & & \downarrow \\
A(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_i \wedge A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_j & \longrightarrow & A_{X \times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{i+j} \\
\downarrow & & \downarrow \\
A(\Delta^\bullet_{top} \times_{\mathbb{R}} U)_i \wedge A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_j & \longrightarrow & A_{U \times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} U \times_{\mathbb{R}} Y)_{i+j}
\end{array}$$

whose rows are homotopy fibration sequences (and such that the null-homotopies of the compositions of the rows are compatible). □

From the point of view taken by Panin and Smirnov, the key aspect of an orientation of a cohomology theory is a suitable structure of Thom classes. Adopting this point of view, we give the following definition.

Definition 9.2. *Let $(X, U) \mapsto H^*_{X-U}(X)$ be a graded abelian group valued functor on $Sm^2_{\mathbb{R}}$ equipped with an external product satisfying conditions (9.1.a) - (9.1.g). Then such a multiplicative cohomology theory is said to be oriented if it is provided with the data of a Thom class*

$$t(E) \in H^*_X(E)$$

associated to any algebraic vector bundle E over a smooth variety X, such that the following properties hold:

(i.) *For any morphism $f : Y \to X$ of smooth real varieties, and any algebraic vector bundle E on X, we have*
$$f^*(t(E)) = t(f^*(E)).$$

(ii.) *If $\phi : E \to E$ is an automorphism, then*
$$\phi(t(E)) = t(E).$$

(iii.) *For any pair of algebraic vector bundles E_1, E_2 on X,*
$$p_1^*(t(E_1)) \cdot p_2^*(t(E_2)) = t(E_1 \oplus E_2) \in H^*_X(E_1 \oplus E_2).$$

(iv.) *Multiplication by the Thom class*
$$- \cdot t(E) : H^*(X) \to H^*(E)$$
is an isomorphism of $H^*(X)$-modules.

In [PS; 5.2], Panin and Smirnov assert that to give an orientation on a multiplicative cohomology theory it suffices to endow the theory with less data: namely, a first Chern class $c_1(L)$ for each smooth quasi-projective real variety X and each line bundle L over X. These Chern classes are required to satisfy the following four conditions: condition (9.2.i), condition (9.2.ii), the condition that $c_1(1)$ (i.e., the first Chern class of the trivial line bundle) vanishes, and the condition that

$$(1, c_1(\mathcal{O}(-1)) : H^*(X)^2 \to H^*(X \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}})$$

is an isomorphism.

Proposition 9.3. *The multiplicative cohomology theory $K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ is an oriented cohomology theory in the sense of Panin-Smirnov provided that one defines for a line bundle L over a smooth variety X the first Chern class $c_1(L) \in K_0^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ to be the image under the natural map $K_0^{\mathrm{alg}}(X) \to K_0^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$ of the class $[1] - [L^\vee]$, where $L^\vee$ denotes the dual of L.*

Proof. The first three conditions required of the first Chern class follow immediately from the fact that these conditions are satisfied in $K_*^{\mathrm{alg}}(-)$ with $c_1(L) = [1] - [L^\vee]$. The fourth condition follows from the projective bundle theorem (Proposition 2.7). □

The Thom class $t(E) \in K_{0,X}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} E)$ resulting from this choice of first Chern class is that constructed as discussed in the proof of Theorem 6.6.

As observed in the discussion of localization in Section 6 (which is inspired by [PS]), Thom classes determine a *Gysin map*

$$\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Z) \to \mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X)$$

for any closed immersion $i : Z \hookrightarrow X$ of smooth varieties. Recall that any projective map $f : X \to Y$ can be factored as a composition of a closed immersion $i : X \hookrightarrow Y \times \mathbb{P}^N_{\mathbb{R}}$ and the projection $p : Y \times \mathbb{P}^N_{\mathbb{R}} \twoheadrightarrow Y$ for some positive integer N. To define a push-forward map

$$f_* = p_* \circ i_* : K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y),$$

it suffices to make a choice of projection for

$$p_* : K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y \times \mathbb{P}^N) \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)$$

and choose i_* to be the Gysin map. In general for an oriented multiplicative cohomology theory $H_-^*(-)$, Panin-Smirnov define p_* using the formal group law associated to H^*. In the case of $K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$, this choice is given by the projection

$$p_* : K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y \times \mathbb{P}^N) \cong K_*^{alg}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)^{N+1} \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)$$

having the property that the composition of

$$K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) \xrightarrow{\cup \gamma^i} K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y \times \mathbb{P}^N)$$

with p_* is equal to 0 for $0 \leq i < N$ and the identity for $i = N$ (cf. Proposition 2.7).

Remark 9.4. *It seems reasonable to expect in the special case where the projective map $f : X \to Y$ is finite and étale that f_* equals the transfer map introduced in Section 5. We have not attempted to verify this compatibility.*

A key result of Panin and Smirnov is that a natural transformation of multiplicative cohomology theories $\Psi : H^*(-) \to G^*(-)$ each of which is oriented satisfies the Riemann-Roch relation. Namely, given any morphism of smooth projective varieties $f : X \to Y$, then

$$f_{G,*}(\Psi(\alpha) \cdot Td_\Psi(T_X)) = \Psi(f_{H,*}(\alpha) \cdot Td_\Psi(T_Y)),$$

where $f_{H,*}, f_{G,*}$ are the push-forward maps for H^* and G^*, T_X and T_Y are the tangent bundles of X and Y, and Td_Ψ denotes the *Todd genus*. (Panin-Smirnov require that either f is a closed immersion or that the inverse Todd genus Td_Ψ^{-1} satisfies a certain invertibility relation enabling Td_Ψ to be defined.)

One elementary application of the Panin-Smirnov Riemann-Roch Theorem, easy to verify directly without recourse to their work, is that the natural transformations

$$K_*^{\mathrm{alg}}(-) \to K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -) \to K\mathbb{R}^{-*}_{\mathrm{top}}((-)_{\mathbb{R}}(\mathbb{C}))$$

commute with push-forward maps.

Remark 9.5. *It is more interesting to apply the Panin-Smirnov Riemann-Roch Theorem to the natural transformation*

$$ch_* : K_*^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -) \to \bigoplus_{q \geq 0} L^q H\mathbb{R}^{2q-*}(-, \mathbb{Q}),$$

where the Chern character ch_* is defined in the usual way in terms of the individual Chern classes (constructed in Section 8) $c_{q,i} : K_i^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -) \to L^q H\mathbb{R}^{2q-i}(-)$ using universal polynomials (with rational coefficients). We expect to investigate such a Riemann-Roch theorem in the future.

Appendix A: Models for algebraic K-theory and multiplicative properties

In this appendix, we use the the strictly functorial model of the algebraic K-theory spectrum of a variety with supports in a subvariety introduced in [FS; App. B] to rigorously define the infinite loop space $\mathcal{K}_Z^{\mathrm{alg}}(\Delta^\bullet_{top} \times X)$ as well as the external cup product operation of such spaces. We then establish the compatibility of this cup product operation with the analogous operations in algebraic and topological K-theory.

As in [FS; App. B], for any quasi-projective real variety X and closed subscheme Z, there exits a *prespectrum* $\mathcal{K}_Z^{\mathrm{alg}}(X)$, by which we mean a sequence of pointed CW complexes

$$A_Z(X)_0, A_Z(X)_1, \ldots$$

together with maps

$$\Sigma^1 A_Z(X)_i \to A_Z(X)_{i+1}.$$

In the notation of [op. cit.], we define $A_Z(X)_i$ to be the geometric realization of the multisimplicial set $\omega(S_\bullet)^i \mathcal{C}_Z\mathcal{P}(X)$, where $\mathcal{C}_Z\mathcal{P}(X)$ is (a suitably strict model for) the Waldhausen category of bounded chain complexes of vector bundles on X which are acyclic on U, $S_\bullet$ denotes Waldhausen's S-construction, and ω denotes the class of quasi-isomorphisms of chain complexes. The prespectrum $\mathcal{K}_Z^{\mathrm{alg}}(X)$ has the property that the adjoint map

$$A_Z(X)_i \to \Omega A_Z(X)_{i+1}$$

is a homotopy equivalence of CW complexes for all $i \geq 1$, and gives the homotopy-theoretic group completion of $A_Z(X)_0$ in the case $i = 0$. The space $\mathcal{K}_Z^{\mathrm{alg}}(X)$ used throughout the body of this paper is formally defined as $\Omega_1 A_Z(X)_1$.

By a *pair of varieties*, we mean a pair (X, U) with X a quasi-projective real variety and U an open subvariety. The (reduced) complement of U in X is typically written as Z. The assignment $(X, U) \mapsto \mathcal{K}_Z^{\mathrm{alg}}(X)$ is a natural contravariant functor from such pairs to prespectra. (A map of pairs $(X, U) \to (X', U')$ is a morphism $f : X \to X'$ such that $f(U) \subset U'$ and a map of prespectra is a sequence of maps on the constituent spaces compatible with the structure maps.) In particular, this functor respects compositions exactly, not just up to natural homeomorphisms.

For each $i, j \geq 0$ and pair of pairs $(X, U), (Y, V)$ with $Z = X - U, W = Y - V$, there exists a continuous pairing of spaces

$$A_Z(X)_i \wedge A_W(Y)_j \to A_{Z \times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y)_{ij}$$

for all $i, j \geq 0$, defined by tensor product of chain complexes. These pairings commute with the structure maps of the prespectra in the evident manner. Thus, there is a pairing of prespectra

$$\mathcal{K}_Z^{\mathrm{alg}}(X) \times \mathcal{K}_W^{\mathrm{alg}}(Y) \to \mathcal{K}_{Z \times_{\mathbb{R}} W}^{\mathrm{alg}}(X \times_{\mathbb{R}} Y).$$

(For our purposes, a *pairing of prespectra* $E \times F \to G$, is a family of pairings $E_i \wedge F_j \to G_{i+j}$ such that the appropriate diagrams commute strictly – see [FS; B.3]) To define a multiplication pairing on total spaces and thus an external cup product operation on homotopy groups, one chooses an inverse to the displayed homotopy equivalence in the diagram

$$\begin{array}{ccc} \Omega^1 A_Z(X)_1 \wedge \Omega^1 A_W(Y)_1 & \longrightarrow & \Omega^2(A_Z(X)_1 \wedge A_W(Y)_1) \\ & & \big\downarrow \\ & & \Omega^2 A_{Z \times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y)_2 \\ & & \sim \big\uparrow \\ & & \Omega^1 A_{Z \times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y)_1. \end{array}$$

Since the original pairings were compatible with the prespectrum structures, we can equivalently define this pairing by choosing a homotopy inverse in the diagram

$$\begin{array}{ccc} \Omega^p A_Z(X)_p \wedge \Omega^q A_W(Y)_q & \longrightarrow & \Omega^{p+q}(A_Z(X)_p \wedge A_W(Y)_q) \\ & & \big\downarrow \\ & & \Omega^{p+q} A_{Z \times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y)_{p+q} \\ & & \sim \big\uparrow \\ & & \Omega^1 A_{Z \times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y)_1, \end{array}$$

for all $p, q \geq 1$. Thus, any choice of $p, q \geq 1$, determines a unique homotopy class of maps

$$\mathcal{K}_Z^{\mathrm{alg}}(X) \wedge \mathcal{K}_W^{\mathrm{alg}}(Y) \to \mathcal{K}_{Z \times_{\mathbb{R}} W}^{\mathrm{alg}}(X \times_{\mathbb{R}} Y).$$

Given a pair (X, U) and a compact CW complex T, define $A_{T \times_{\mathbb{R}} Z}(T \times_{\mathbb{R}} X)_i$ as the direct limit

$$\varinjlim_{T \to M(\mathbb{R})} A_{M \times_{\mathbb{R}} Z}(M \times_{\mathbb{R}} X)_i,$$

for all i and let $\mathcal{K}^{\mathrm{alg}}_{T\times_{\mathbb{R}}Z}(T\times_{\mathbb{R}}X)$ denote the associated prespectrum. One may readily verify that $T\mapsto\mathcal{K}^{\mathrm{alg}}_{T\times_{\mathbb{R}}Z}(T\times_{\mathbb{R}}X)$ defines a functor from compact CW complexes to prespectra. We write this functor more briefly as $T\mapsto\mathcal{K}^{\mathrm{alg}}_{Z}(T\times_{\mathbb{R}}X)$. Now associate to a pair (X,U) the sequence of spaces

$$|A_Z(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)_i|,\quad i\geq 0.$$

One may check this gives a functor as before and that the induced structure maps determine a prespectrum. Thus, we obtain a functor from pairs to prespectra, written

$$(X,U)\mapsto\mathcal{K}^{\mathrm{alg}}_{Z}(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X),$$

which satisfies properties analogous to $\mathcal{K}^{\mathrm{alg}}_{Z}(X)$.

It follows from [TT], that for any pair (X,U), with U an open subscheme of X, and for all $i\geq 0$, the sequence (induced by the evident maps of pairs $(U,\emptyset)\to(X,\emptyset)\to(X,U)$)

$$\Omega^iA_Z(X)_i\to\Omega^iA(X)_i\to\Omega^iA(U)_i$$

is a homotopy fibration sequence for all $i\geq 1$. This means that there is a chosen null-homotopy for the composition of the above maps and the induced map from $\Omega^iA_Z(X)_i$ to the homotopy fiber of $\Omega^iA(X)_i\to\Omega^iA(X-U)_i$ is a homotopy equivalence. As establish in Corollary 3.4, when X is smooth, the sequence

$$A_Z(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)_i\to A(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)_i\to A(\Delta^{\bullet}_{top}\times_{\mathbb{R}}U)_i$$

is a homotopy fibration sequence for all $i\geq 1$. Thus, the prespectrum $\mathcal{K}^{\mathrm{alg}}_{Z}(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)$ really can be said to represent the theory $\mathcal{K}^{\mathrm{alg}}(\Delta^{\bullet}_{top}\times_{\mathbb{R}}-)$ with supports in a closed subscheme (at least for smooth pairs).

To extend the multiplication pairing for $\mathcal{K}^{\mathrm{alg}}_{Z}(X)$ to $\mathcal{K}^{\mathrm{alg}}_{Z}(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)$, we define a pairing

$$A_Z(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X)_i\wedge A_W(\Delta^{\bullet}_{top}\times_{\mathbb{R}}Y)_j\to A_{Z\times_{\mathbb{R}}W}(\Delta^{\bullet}_{top}\times_{\mathbb{R}}X\times_{\mathbb{R}}Y)_{i+j},$$

for each $i,j\geq 0$, by first defining for each $d,e\geq 0$ a pairing of the form

$$A_Z(\Delta^{d}_{top}\times_{\mathbb{R}}X)_i\wedge A_W(\Delta^{e}_{top}\times_{\mathbb{R}}Y)_j\to A_{Z\times_{\mathbb{R}}W}(\Delta^{d}_{top}\times\Delta^{e}_{top}\times_{\mathbb{R}}X\times_{\mathbb{R}}Y)_{i+j}.$$

This latter pairing is defined by associating to a pair of indices involved the direct limits, $\Delta^d_{top}\to U(\mathbb{R})$ and $\Delta^e_{top}\to V(\mathbb{R})$, the index $\Delta^d_{top}\times\Delta^e_{top}\to(U\times_{\mathbb{R}}V)(\mathbb{R})$. Letting d and e vary, these pairings are readily verified to give pairings of bisimplicial spaces. The geometric realization of the bisimplicial set

$$d,e\mapsto A_{Z\times_{\mathbb{R}}W}(\Delta^{d}_{top}\times\Delta^{e}_{top}\times_{\mathbb{R}}X\times_{\mathbb{R}}Y)_{i+j},$$

is naturally homeomorphic to the geometric realization of

$$d\mapsto A_{Z\times_{\mathbb{R}}W}(\Delta^{d}_{top}\times\Delta^{d}_{top}\times_{\mathbb{R}}X\times_{\mathbb{R}}Y)_{i+j},$$

which in turn is naturally homotopy equivalent to to the geometric realization of

$$d\mapsto A_{Z\times_{\mathbb{R}}W}(\Delta^{d}_{top}\times_{\mathbb{R}}X\times_{\mathbb{R}}Y)_{i+j}$$

under the map induced by the diagonal $\Delta^d_{top} \hookrightarrow \Delta^d_{top} \times \Delta^d_{top}$. The desired multiplication pairing is thus defined by choosing an inverse of the displayed homotopy equivalence in the diagram

$$\begin{array}{ccc} A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_i \wedge A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_j & \longrightarrow & A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times \Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{i+j} \\ & & \sim\Big\uparrow \\ & & A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{i+j}. \end{array}$$

Compatibility of these pairings up to homotopy with the structure maps of the prespectra is readily verified. Finally, the external cup product operation for $\mathcal{K}^{\text{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ is defined by choosing a homotopy inverse to the map

$$\Omega^1 A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_1 \xrightarrow{\sim} \Omega^2 A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_2$$

just as before. We can equivalently define this pairing using the spaces $\Omega^p A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_p$, $\Omega^q A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_q$, and $\Omega^{p+q} A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{p+q}$, for any $p, q \geq 1$, instead.

We summarize our results with the following theorem. Note that since we choose a homotopy inverse to the map induced by the diagonal map $\Delta^\bullet_{top} \to \Delta^\bullet_{top} \times \Delta^\bullet_{top}$, we cannot obtain a pairing of prespectra as defined above. However, we do obtain a pairing on the level of spaces, unique up to homotopy. This suffices for our purposes, although with more work, one could presumably obtain a pairing of prespectra.

Theorem A.1. *For any $p, q \geq 1$, by choosing inverses of homotopy equivalences as needed in the diagram*

$$\begin{array}{c} \Omega^1 A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_1 \wedge \Omega^1 A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_1 \\ \sim\Big\downarrow \\ \Omega^p A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_p \wedge \Omega^q A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_q \\ \Big\downarrow \\ \Omega^{p+q}(A_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X)_p \wedge A_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y)_q) \\ \Big\downarrow \\ \Omega^{p+q} A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times \Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{p+q} \\ \sim\Big\uparrow \\ \Omega^{p+q} A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{p+q} \\ \sim\Big\uparrow \\ \Omega^1 A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_1, \end{array}$$

we obtain a pairing of spaces

$$\cup : \mathcal{K}^{\text{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\text{alg}}_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) \to \mathcal{K}^{\text{alg}}_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y),$$

which is independent up to homotopy with the choices made. In particular, we obtain well-defined maps on homotopy groups

$$\cup : K^{\mathrm{alg}}_{Z,p}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \otimes \mathcal{K}^{\mathrm{alg}}_{W,q}(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) \to \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W,p+q}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)$$

in the usual manner.

By replacing $\Delta^\bullet_{top}$ with $\Delta^\bullet$ in the preceding discussion, we obtain in a parallel manner a pairing arising from the diagram

$$\begin{array}{ccc}
\Omega^1 A_Z(\Delta^\bullet \times_{\mathbb{R}} X)_1 \wedge \Omega^1 A_W(\Delta^\bullet \times_{\mathbb{R}} Y)_1 & \xrightarrow{\sim} & \Omega^p A_Z(\Delta^\bullet \times_{\mathbb{R}} X)_p \wedge \Omega^q A_W(\Delta^\bullet \times_{\mathbb{R}} Y)_q \\
 & & \downarrow \\
 & & \Omega^{p+q}(A_Z(\Delta^\bullet \times_{\mathbb{R}} X)_p \wedge A_W(\Delta^\bullet \times_{\mathbb{R}} Y)_q) \\
 & & \downarrow \\
 & & \Omega^{p+q} A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet \times_{\mathbb{R}} \Delta^\bullet \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{p+q} \\
 & & \sim \uparrow \\
\Omega^1 A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_1 & \xrightarrow{\sim} & \Omega^{p+q} A_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y)_{p+q}
\end{array} \tag{A.2}$$

for each $p, q \geq 1$, which determines a unique homotopy class of maps

$$\cup : \mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\mathrm{alg}}_W(\Delta^\bullet \times_{\mathbb{R}} Y) \to \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W}(\Delta^\bullet \times_{\mathbb{R}} X \times_{\mathbb{R}} Y).$$

As we have already seen, there is a unique homotopy class of maps

$$\cup : \mathcal{K}^{\mathrm{alg}}_Z(X) \wedge \mathcal{K}^{\mathrm{alg}}_W(Y) \to \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y),$$

giving the standard cup product pairing for algebraic K-theory with supports. It is clear that these pairings fit into a homotopy commutative diagram

$$\begin{array}{ccc}
\mathcal{K}^{\mathrm{alg}}_Z(X) \wedge \mathcal{K}^{\mathrm{alg}}_W(Y) & \xrightarrow{\cup} & \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W}(X \times_{\mathbb{R}} Y) \\
\downarrow & & \downarrow \\
\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\mathrm{alg}}_W(\Delta^\bullet \times_{\mathbb{R}} Y) & \xrightarrow{\cup} & \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W}(\Delta^\bullet \times_{\mathbb{R}} X \times_{\mathbb{R}} Y) \\
\downarrow & & \downarrow \\
\mathcal{K}^{\mathrm{alg}}_Z(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \wedge \mathcal{K}^{\mathrm{alg}}_W(\Delta^\bullet_{top} \times_{\mathbb{R}} Y) & \xrightarrow{\cup} & \mathcal{K}^{\mathrm{alg}}_{Z\times_{\mathbb{R}} W}(\Delta^\bullet_{top} \times_{\mathbb{R}} X \times_{\mathbb{R}} Y).
\end{array}$$

Thus, the cup product operation for $\mathcal{K}^{\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} -)$ is compatible with the usual cup product operation in algebraic K-theory.

We proceed to give a sketch of a construction of Real topological K-theory with supports which is analogous to the above constructions. To avoid point-set topology complications, we restrict attention to pairs of Real spaces of the form $(X_{\mathbb{R}}(\mathbb{C}), U_{\mathbb{R}}(\mathbb{C}))$, where X, Z are quasi-projective real varieties as before and $U = X - Z$. Note that by [H2], in this case there exists an equivariant triangulation of $X_{\mathbb{R}}(\mathbb{C})$ and a closed subcomplex $U' \subset X_{\mathbb{R}}(\mathbb{C})$ which is a strong deformation retract of $U_{\mathbb{R}}(\mathbb{C})$.

We may replace $\mathcal{C}_Z\mathcal{P}(X)$ used in the definition of $\mathcal{K}_Z^{\mathrm{alg}}(X)$ with the *topological* category $\mathcal{C}_{Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}^{top}(X_{\mathbb{R}}(\mathbb{C}))$ consisting of bounded complexes of Real vector bundles on $X_{\mathbb{R}}(\mathbb{C})$ which are acyclic upon restriction to $U_{\mathbb{R}}(\mathbb{C})$ (equivalently, upon restriction to U'). Here, the topology on the objects and morphisms of $\mathcal{C}_{Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}^{top}(X_{\mathbb{R}}(\mathbb{C}))$ is given by using the compact-open topology on the collection of Real vector bundle maps (cf. [Se; end of §2]). Moreover, this topology is such that if we apply $\mathcal{M}\mathrm{aps}(\Delta^\bullet_{top}, -)$ to the objects and morphisms of $\mathcal{C}_X\mathcal{P}^{top}(X)$, we obtain a simplicial category $\mathcal{C}_{\Delta^\bullet_{top}\times Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}(\Delta^\bullet_{top}\times X_{\mathbb{R}}(\mathbb{C}))$, which in degree n is the *discrete* category of bounded complexes of Real vector bundles on $\Delta^n_{top}\times X_{\mathbb{R}}(\mathbb{C})$ acyclic on $\Delta^n_{top}\times U_{\mathbb{R}}(\mathbb{C})$. As in [Pal; 1.12 and 2.2], one may show that the prespectra defined by

$$\omega(S_\bullet)^i\mathcal{C}_{Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}^{top}(X_{\mathbb{R}}(\mathbb{C})),\quad i\geq 0,$$

and

$$\omega(S_\bullet)^i\mathcal{C}_{\Delta^\bullet_{top}\times Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}(\Delta^\bullet_{top}\times X_{\mathbb{R}}(\mathbb{C})),\quad i\geq 0, \tag{A.3}$$

are homotopy equivalent, and each gives a natural model for the homotopy fiber of

$$\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))\to\mathcal{K}\mathbb{R}_{\mathrm{top}}(U_{\mathbb{R}}(\mathbb{C}))\sim\mathcal{K}\mathbb{R}_{\mathrm{top}}(U').$$

(For the first of these two prespectra, the $S_\bullet$-construction is applied to a topological additive category in the obvious manner, resulting in a simplicial topological category.) We write the total space of the prespectrum (A.3) as $\mathcal{K}\mathbb{R}_{\mathrm{top}}^{Z_{\mathbb{R}}(\mathbb{C})}(X_{\mathbb{R}}(\mathbb{C}))$.

There is an evident natural map of simplicial categories

$$\mathcal{C}_{\Delta^\bullet_{top}\times_{\mathbb{R}}Z}\mathcal{P}(\Delta^\bullet_{top}\times_{\mathbb{R}}X)\to\mathcal{C}_{\Delta^\bullet_{top}\times Z_{\mathbb{R}}(\mathbb{C})}\mathcal{P}(\Delta^\bullet_{top}\times X_{\mathbb{R}}(\mathbb{C}))$$

defined by taking the Real topological bundle on $\Delta^n_{top}\times X_{\mathbb{R}}(\mathbb{C})$ associated to an algebraic vector bundle on $M\times_{\mathbb{R}}X$ and a continuous map $\Delta^n_{top}\to M(\mathbb{R})$. This defines an map of prespectra

$$\mathcal{K}_Z^{\mathrm{alg}}(\Delta^\bullet_{top}\times_{\mathbb{R}}X)\to\mathcal{K}\mathbb{R}_{\mathrm{top}}^{Z_{\mathbb{R}}(\mathbb{C})}(X_{\mathbb{R}}(\mathbb{C})).$$

Moreover, using an argument similar to that found in [GW], one can show that this map is equivalent to the map of Proposition 4.3 in the case $Z=X$.

Finally, we can repeat the construction of the cup product pairing for $\mathcal{K}_-^{\mathrm{alg}}(\Delta^\bullet_{top}\times_{\mathbb{R}} -)$ to obtain a pairing

$$\cup:\mathcal{K}\mathbb{R}_{\mathrm{top}}^{Z_{\mathbb{R}}(\mathbb{C})}(X_{\mathbb{R}}(\mathbb{C}))\wedge\mathcal{K}\mathbb{R}_{\mathrm{top}}^{W_{\mathbb{R}}(\mathbb{C})}(Y_{\mathbb{R}}(\mathbb{C}))\to\mathcal{K}\mathbb{R}_{\mathrm{top}}^{Z_{\mathbb{R}}(\mathbb{C})\times W_{\mathbb{R}}(\mathbb{C})}(X_{\mathbb{R}}(\mathbb{C})\times Y_{\mathbb{R}}(\mathbb{C}))$$

which is compatible up to homotopy with the cup product pairing on $\mathcal{K}_-^{\mathrm{alg}}(\Delta^\bullet_{top}\times_{\mathbb{R}} -)$. In the case $Z=X$ and $W=Y$, the diagram

$$\begin{array}{ccc}|i_\bullet\mathcal{P}^{top}(X_{\mathbb{R}}(\mathbb{C}))|\times|i_\bullet\mathcal{P}^{top}(Y_{\mathbb{R}}(\mathbb{C}))| & \longrightarrow & |i_\bullet\mathcal{P}^{top}(X_{\mathbb{R}}(\mathbb{C})\times Y_{\mathbb{R}}(\mathbb{C}))|\\ \downarrow & & \downarrow\\ \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))\times\mathcal{K}\mathbb{R}_{\mathrm{top}}(Y_{\mathbb{R}}(\mathbb{C})) & \xrightarrow{\cup} & \mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C})\times Y_{\mathbb{R}}(\mathbb{C})),\end{array} \tag{A.4}$$

commutes up to homotopy, where $i_\bullet \mathcal{P}^{top}(-)$ denotes the topological symmetric monoidal category consisting of Real vector bundles and isomorphisms of such. (The objects of this category form a discrete space and the morphisms are topologized using the compact-open topology as usual.) Here, the top horizontal map in (A.4) is given by tensor product of bundles and the vertical maps are induced by the evident inclusion of topological categories. Using [CCMT; 1.4], the homotopy commutativity of (A.4) and the fact that its vertical maps represent homotopy-theoretic group completions imply that the lower horizontal map of (A.4) is well-defined up to weak homotopy. Thus, the cup product pairing we have defined coincides (essentially by definition) with the usual cup product pairing for $\mathcal{K}\mathbb{R}_{\rm top}$ up to weak homotopy.

Theorem A.5. *For any quasi-projective real variety X, the natural map*

$$\mathcal{K}^{\rm alg}(\Delta^\bullet_{top} \times_\mathbb{R} X) \to \mathcal{K}\mathbb{R}_{\rm top}(X_\mathbb{R}(\mathbb{C}))$$

of Proposition 4.3 is compatible up to weak homotopy with the cup product pairing for $\mathcal{K}^{\rm alg}(\Delta^\bullet_{top} \times_\mathbb{R} -)$ defined above and the standard cup product pairing for $\mathcal{K}\mathbb{R}_{\rm top}(-)$. In particular, the natural map

$$\bigoplus_{q \geq 0} K_q^{\rm alg}(\Delta^\bullet_{top} \times_\mathbb{R} X) \to \bigoplus_{q \geq 0} K\mathbb{R}^{-q}_{\rm top}(X_\mathbb{R}(\mathbb{C}))$$

is a ring homomorphism.

If in addition X is smooth and Z is any closed subvariety, there is a natural homotopy commutative diagram

$$\begin{array}{ccccc}
\mathcal{K}^{\rm alg}_Z(\Delta^\bullet_{top} \times_\mathbb{R} X) & \longrightarrow & \mathcal{K}^{\rm alg}(\Delta^\bullet_{top} \times_\mathbb{R} X) & \longrightarrow & \mathcal{K}^{\rm alg}(\Delta^\bullet_{top} \times_\mathbb{R} U) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}\mathbb{R}^{Z_\mathbb{R}(\mathbb{C})}_{\rm top}(X_\mathbb{R}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\rm top}(X_\mathbb{R}(\mathbb{C})) & \longrightarrow & \mathcal{K}\mathbb{R}_{\rm top}(U_\mathbb{R}(\mathbb{C})),
\end{array}$$

all of whose arrows are compatible with cup product up to weak homotopy.

Appendix B: $\mathbb{Z}/2$-equivariant spectra and $\mathbb{Z}/2$-equivariant cohomology theories

In this appendix, we briefly recall some basic definitions and results concerning $\mathbb{Z}/2$-equivariant stable homotopy theory. The application of this machinery we have in mind is the construction, for a quasi-projective real variety X, of a "$\mathbb{Z}/2$-spectrum" $\mathcal{K}^{\mathbb{Z}/2-{\rm semi}}(X)$ (essentially, a spectrum equipped with a suitably defined notion of a group action by $\mathbb{Z}/2$) whose fixed point subspectrum gives $\mathcal{K}\mathbb{R}^{\rm semi}(X)$ and whose total spectrum gives $\mathcal{K}^{\rm semi}(X_\mathbb{C})$. However, as we explain below, there is a gap in the literature concerning the "recognition principle" for $\mathbb{Z}/2$-spectra. Modulo this gap, we indicate briefly how the results of this paper could be more naturally reworked and generalized into the $\mathbb{Z}/2$-equivariant setting. Basic references for the material in this appendix are [LMS], [M2], [CW]. This appendix benefitted significantly from correspondence with Pedro dos Santos.

In general, given a finite group G, a *complete G-universe* is an infinite dimensional real inner product space $\mathcal{U}$ upon which G acts through isometries, satisfying the properties that $\mathcal{U}$ is a direct sum of finite dimensional G-invariant sub inner product spaces and that every finite dimensional irreducible real representation of G is contained in $\mathcal{U}$ as a G-invariant sub inner product space infinitely often (up to isomorphism). Starting with this definition, one can define the notion of G-spectra, i.e., spectra equipped with G-actions in a suitable sense. In the remainder of this appendix, only the group $G = \mathbb{Z}/2$ is considered and the only example of a complete $\mathbb{Z}/2$-universe is given by the infinite dimensional complex space $\mathcal{U} = \mathbb{C}^\infty$ regarded as a real inner product space using the identification $\mathbb{R}\oplus\mathbb{R} \cong \mathbb{C}$, $(x, y) \mapsto x+iy$ and using the canonical real inner product. We let $\mathbb{Z}/2$ act on $\mathbb{C}^\infty$ via componentwise complex conjugation – i.e., by sending $(x_1, y_1, x_2, y_2, \dots)$ to $(x_1, -y_1, x_2, -y_2, \dots)$. Observe that this action is isometric. Moreover, the only finite dimensional irreducible real representations of $\mathbb{Z}/2$ are given by multiplication by ± 1 on $\mathbb{R}$, and thus $\mathbb{C}^\infty$ is clearly a complete $\mathbb{Z}/2$-universe in the sense defined above. Although this appendix focuses exclusively on $G = \mathbb{Z}/2$ and $\mathcal{U} = \mathbb{C}^\infty$, note that the results and constructions cited here are really special cases of a broader theory.

We call a real inner product space V equipped with an isometric $\mathbb{Z}/2$ action (i.e., equipped with an isometric involution) a $\mathbb{Z}/2$ *inner product space*. For any finite dimensional $\mathbb{Z}/2$ inner product space V, write S^V for the one-point compactification of V endowed with the evident $\mathbb{Z}/2$-action which fixes the point at infinity. The space S^V is viewed as a pointed $\mathbb{Z}/2$-space with the point at infinity serving as the base point. For any pointed $\mathbb{Z}/2$-space X, write $\Sigma^V X$ for $S^V \wedge X$ equipped with the evident $\mathbb{Z}/2$-action. Additionally, we write $\Omega^V X$ for the $\mathbb{Z}/2$-space $\mathcal{M}\mathrm{aps}_*(S^V, X)$ of pointed maps, where the action is given by conjugation. Finally, given an inclusion $V \subset W$ of finite dimensional $\mathbb{Z}/2$ inner product spaces, we write $W - V$ for the orthogonal complement of V in W. Note that $W - V$ is also a $\mathbb{Z}/2$ inner product space.

Definition B.1. *A* $\mathbb{Z}/2$-spectrum E *consists of a collection of pointed* $\mathbb{Z}/2$*-spaces* $E(V)$ *for each finite dimensional* $\mathbb{Z}/2$*-invariant inner product subspace* V *of* $\mathbb{C}^\infty$ *together with pointed* $\mathbb{Z}/2$*-maps*

$$\sigma_{V,W} : \Sigma^{W-V} E(V) \to E(W).$$

The adjoint maps are written

$$\tilde{\sigma}_{V,W} : E(V) \to \Omega^{W-V} E(W).$$

These maps are required to satisfy the conditions (1) $\sigma_{V,V} = id$ *for all* V*, (2)* $\sigma_{W,Z} \circ \Sigma^{Z-W} \sigma_{V,W} = \sigma_{V,Z}$ *for all* $V \subset W \subset Z$*, and (3) the map* $\tilde{\sigma}_{V,W} : E(V) \to \Omega^{W-V} E(W)$ *is a equivariant homeomorphism, for all* $V \subset W$*. A map of* $\mathbb{Z}/2$*-spectra* $E \to F$ *consists of a collection of equivariant maps* $E(V) \to F(V)$ *compatible with the above structure in the evident fashion.*

As shown in [LMS], to define a $\mathbb{Z}/2$-spectrum, one doesn't need to consider every $\mathbb{Z}/2$ inner product subspace V of $\mathbb{C}^\infty$ as above, but rather just suitably large collection. In particular, if we take V_i to be the (real) $2i$-dimensional subspace $\mathbb{C}^i$ of $\mathbb{C}^\infty$ defined as the $\mathbb{C}$-span of the first i standard complex basis elements, then this collection of subspaces forms a so-called *indexing space*. Consequently, we can equivalently describe a $\mathbb{Z}/2$-spectrum as a sequence of spaces $E_0, E_1, \dots$ (where E_i is thought of as being $E(V_i)$ in the above definition) together with pointed $\mathbb{Z}/2$ maps

$$\mathbb{P}^1(\mathbb{C}) \wedge E_i \to E_{i+1},$$

whose adjoints

$$E_1 \to \mathcal{M}\mathrm{aps}_*(\mathbb{P}^1(\mathbb{C}), E_{i+1})$$

are equivariant homeomorphisms (cf. [LMS; 2.4]). Here, $\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere equipped with $\mathbb{Z}/2$-action given by complex conjugation.

Given a $\mathbb{Z}/2$-spectrum E indexed by the $\mathbb{Z}/2$ inner product subspaces of $\mathbb{C}^\infty$, we form two spectra in the ordinary sense by taking the fixed points of E with respect to each of the two subgroups of $\mathbb{Z}/2$. Specifically, let $\mathbb{R}^\infty \subset \mathbb{C}^\infty$ be the fixed point subspace of $\mathbb{Z}/2$ (in other words, this is the subspace spanned by the real coordinates) and for $H \subset \mathbb{Z}/2$, define E^H to be the spectrum indexed by all the real subspaces of $\mathbb{R}^\infty$ given by

$$V \mapsto E(V)^H.$$

As is customary with ordinary spectra (and permitted by [LMS; 2.4]), we restrict V to range over the subspaces $\mathbb{R}^n$, $n = 0, 1, \dots$, where $\mathbb{R}^n \subset \mathbb{R}^\infty$ is the subspace spanned by the first n standard basis elements. Thus we have a spectrum in the classically defined sense:

$$E^H_0 = E(0)^H, E^H(1) = E(\mathbb{R}^1)^H, \dots.$$

The only possibilities for H are obviously $\mathbb{Z}/2$ and 0, and we call the associated spectra the *fixed point spectrum* and *total spectrum* respectively. Let us write these as $E^{\mathbb{Z}/2}$ and E^0. Thus, E^0 is the spectrum given by the collection of spaces $E(\mathbb{R}^n)$, $n = 0, 1, \dots$, with the action of $\mathbb{Z}/2$ forgotten, whereas $E^{\mathbb{Z}/2}$ is the subspectrum of E^0 given by taking fixed points of each $E(\mathbb{R}^n)$. In particular, there is a evident natural map

$$E^{\mathbb{Z}/2} \to E^0.$$

We now turn to the aforementioned gap in the literature. One would like a machine to produce $\mathbb{Z}/2$-spectra analogous to the machines used to produce ordinary spectra (operads, Γ-spaces, etc.). There does exist a notion of a $\mathbb{Z}/2$-operad (recalled below), and ideally given any $\mathbb{Z}/2$-operad $\mathcal{I}$ acting on a $\mathbb{Z}/2$-space X, there ought to be a functorial construction which produces a $\mathbb{Z}/2$-spectrum related to X. (In particular, the zeroth spaces of this spectrum ought to be the "equivariant group completion" of X.) We now develop this (conjectural) framework. All of this is strictly parallel to the non-equivariant case, and in fact J. P. May assures us that the construction we describe does have the desired properties, although proofs are apparently known only to the experts and have not appeared in the literature.

A $\mathbb{Z}/2$-E_∞ *operad* is a collection of $\mathbb{Z}/2$-spaces $\mathcal{M}(n)$, $n = 0, 1, \dots$, each of which has the $\mathbb{Z}/2$-homotopy type of a CW-complex, together with an action of the symmetric group Σ_n on $\mathcal{M}(n)$ for each n and pairings

$$\gamma : \mathcal{M}(n) \times \mathcal{M}(k_1) \times \cdots \times \mathcal{M}(k_n) \to \mathcal{M}(k_1 + \cdots + k_n)$$

for all non-negative integers n, $k_1, \dots, k_n$. The data are required to satisfy the appropriate $\mathbb{Z}/2$-equivariant analogues of the axioms for an ordinary E_∞-operad (cf. [LMS; VII.1]).

Our main example of a $\mathbb{Z}/2$-E_∞ operad, $\tilde{\mathcal{I}}$, is given by letting $\tilde{\mathcal{I}}(n)$ denote the collection of (non-equivariant) $\mathbb{C}$-linear injective maps from $(\mathbb{C}^\infty)^n$ into $\mathbb{C}^\infty$, where $\tilde{\mathcal{I}}(n)$ is topologized just as is $\mathcal{I}(n)$ of Section 1. The action of Σ_n on $\tilde{\mathcal{I}}(n)$ permutes the copies of $\mathbb{C}^\infty$ in $(\mathbb{C}^\infty)^n$ and the action of $\mathbb{Z}/2$ on $\tilde{\mathcal{I}}(n)$ is given by complex conjugation in the evident manner. The required pairings are defined by composition in the evident manner. We can also consider the suboperad $\tilde{\mathcal{L}}(n) \subset \tilde{\mathcal{I}}(n)$, $n \geq 0$, defined by taking the space of all *isometries* $(\mathbb{C}^\infty)^n \hookrightarrow \mathbb{C}^\infty$ with respect to the standard Hermitian inner product on $\mathbb{C}^\infty$. The operad $\tilde{\mathcal{L}}$ is similar to the $\mathbb{Z}/2$-operad considered in [LMS; VII.1.4]. Indeed, the proofs of [LMS; VII.1.4] and [EKMM; XI.1.7] carry over to show $\tilde{\mathcal{L}}$ satisfies all of the required axioms. It follows that $\tilde{\mathcal{I}}$ also satisfies the required axioms (e.g., each $\tilde{\mathcal{I}}(n)$ has the $\mathbb{Z}/2$-homotopy type of a $\mathbb{Z}/2$-CW complex) since $\tilde{\mathcal{L}}(n) \hookrightarrow \tilde{\mathcal{I}}(n)$, $n \geq 0$, is a $\mathbb{Z}/2$-homotopy equivalence.

Additionally, since the space $\tilde{\mathcal{I}}(n)$ is defined using inductive limits of algebraic varieties, we may define a sort of algebraic version of the operad $\tilde{\mathcal{I}}$ as follows. Namely, let $\Delta^\bullet_{\mathbb{C}}$ denote the standard cosimplicial object in the category of complex varieties and let $\tilde{\mathcal{I}}(n)(\Delta^\bullet_{\mathbb{C}})$ denote the simplicial set which in degree d is given by

$$\tilde{\mathcal{I}}(n)(\Delta^d_{\mathbb{C}}) \equiv \varprojlim_N \varinjlim_M \operatorname{Hom}_{\mathbb{C}}(\Delta^d_{\mathbb{C}}, \tilde{\mathcal{I}}(n)_{N,M})$$

where $\tilde{\mathcal{I}}(n)_{N,M}$ is the complex affine variety parameterizing injective $\mathbb{C}$-linear maps from $(\mathbb{C}^N)^n$ to $\mathbb{C}^M$. Writing $|\tilde{\mathcal{I}}(n)(\Delta^\bullet_{\mathbb{C}})|$ for the geometric realization of this simplicial set, we clearly have a natural homotopy equivalence

$$|\tilde{\mathcal{I}}(n)(\Delta^\bullet_{\mathbb{C}}))| \to \tilde{\mathcal{I}}(n).$$

Moreover, $\tilde{\mathcal{I}}(n)(\Delta^\bullet_{\mathbb{C}})$ admits an evident action of $\mathbb{Z}/2$ given by complex conjugation and this map is in fact a $\mathbb{Z}/2$-homotopy equivalence. It is easy to check that the collection $|\tilde{\mathcal{I}}(n)(\Delta^\bullet_{\mathbb{C}})|$, $n = 0, 1, \ldots$ forms a $\mathbb{Z}/2$-E_∞ operad and the above is a map of such objects.

If $\mathcal{M}$ is any $\mathbb{Z}/2$-E_∞ operad, a $\mathbb{Z}/2$-space X is said to be an $\mathcal{M}$-*space* if there are a collection of $\mathbb{Z}/2$-equivariant maps

$$\mathcal{M}(n) \times X^{\times n} \to X$$

satisfying the appropriate $\mathbb{Z}/2$-equivariant analogues of the axioms required of an action of an ordinary operad on a space (cf. [CW; 2.4]).

The primary construction needed in this paper to form $\mathbb{Z}/2$-spectra is described in the following conjecture. As mentioned, J. P. May assures us that thus result is true, but no one has yet to produce a proof in the literature.

Conjecture B.2. *A $\mathbb{Z}/2$-space X having the $\mathbb{Z}/2$-homotopy type of a CW complex endowed with an action by the $\mathbb{Z}/2$-E_∞ operad $\mathcal{M}$ determines functorially a $\mathbb{Z}/2$-spectrum EX. In particular, for such an X, there is a natural "equivariant homotopy-theoretic group completion"*

$$X \to (EX)(0),$$

by which we mean the map induces ordinary homotopy-theoretic group completions upon taking $\mathbb{Z}/2$-fixed points and upon forgetting the $\mathbb{Z}/2$ action.

Moreover, let $\mathcal{M}^{\mathbb{Z}/2}$ denote the E_∞-operad obtained by taking fixed points of $\mathcal{M}(n)$, $n \geq 0$. Then the ordinary spectrum $(EX)^0$ (respectively, $(EX)^{\mathbb{Z}/2}$) coincides *up to a natural weak equivalence of spectra with the spectrum obtained from the space X (respectively, $X^{\mathbb{Z}/2}$) endowed with the induced structure of an $\mathcal{M}^{\mathbb{Z}/2}$-space.*

We now discuss the *generalized homotopy groups* and the equivariant cohomology theory associated to a $\mathbb{Z}/2$-spectrum. The generalized homotopy groups of a pointed G-space X are defined as $[(G/H)_+ \wedge S^n, X]_G$, where $H \subset G$ is any subgroup, n is any integer, and, in general, $[-,-]_G$ denotes the collection of pointed G-maps modulo G-equivariant homotopy. In particular, for $G = \mathbb{Z}/2$, we have two families of generalized homotopy groups

$$[T^0_+ \wedge S^n, X]_{\mathbb{Z}/2} \quad \text{and} \quad [S^n, X]_{\mathbb{Z}/2},$$

where T^0 denotes the unique two-point space with non-trivial $\mathbb{Z}/2$-action. The notion of generalized homotopy groups extends to the level of spectra in that one can define a *generalized sphere spectrum* $S^n_H \equiv ((\mathbb{Z}/2)/H)_+ \wedge S^n$, where $H \subset \mathbb{Z}/2$ and S^n is the spectrum $\Sigma^n \Sigma^\infty S^0$. The generalized homotopy groups of a $\mathbb{Z}/2$-spectrum E, $\pi^H_n(E)$, are define as homotopy classes of $\mathbb{Z}/2$-equivariant maps (of spectra) from S^n_H to E. By [LMS; 4.5], the generalized homotopy groups of E can be more simply described by the formulas

$$\begin{cases} \pi^H_n(E) \cong \pi_n(E(0)^H), & \text{for } n \geq 0 \text{ and} \\ \pi^H_{-n}(R) \cong \pi_0(E(\mathbb{R}^n)^H), & \text{for } n < 0. \end{cases}$$

Thus, as for $\mathbb{Z}/2$-spaces, there are just two families of generalized homotopy groups of $\mathbb{Z}/2$-spectra:

$$\pi_n(E^0) \quad \text{and} \quad \pi_n(E^{\mathbb{Z}/2}).$$

These definitions extend to negative indices in the usual manner. A *weak equivalence* of $\mathbb{Z}/2$-spectra is a map inducing isomorphisms on π^H_n for all n, $H \subset \mathbb{Z}/2$ – in other words, a weak equivalence of $\mathbb{Z}/2$-spectra is a map inducing weak equivalences of ordinary spectra for both the fixed point and total spectra.

We record the following direct consequence of the preceding discussion.

Proposition B.3. *Naturally associated to any $\mathbb{Z}/2$-spectrum E we have the total spectrum E^0 and the fixed point spectrum $E^{\mathbb{Z}/2}$, between which there is a natural map*

$$E^{\mathbb{Z}/2} \to E^0.$$

A map of $\mathbb{Z}/2$-equivariant spectra $E \to F$ is a weak equivalence if and only if the maps $E^0 \to F^0$ and $E^{\mathbb{Z}/2} \to F^{\mathbb{Z}/2}$ are weak equivalences in the ordinary sense.

Given a $\mathbb{Z}/2$-spectrum E indexed by the universe $\mathbb{C}^\infty$, the associated cohomology theory on $\mathbb{Z}/2$-spaces is a theory indexed by the $\mathbb{Z}/2$-invariant inner product subspaces of $\mathbb{C}^\infty$. Specifically, to such a $V \subset \mathbb{C}^\infty$ and a pointed $\mathbb{Z}/2$-space X, we form the group

$$\tilde{E}^V_{\mathbb{Z}/2}(X) = [X, E(V)]_{\mathbb{Z}/2}.$$

For unbased spaces, define $E^V_{\mathbb{Z}/2}(X)$ to be $\tilde{E}^V_{\mathbb{Z}/2}(X_+)$. More generally, given a pair of such subspaces $V, W \subset \mathbb{C}^\infty$, the group $\tilde{E}^{V \ominus W}_{\mathbb{Z}/2}(X)$ is defined as $[X \wedge S^W, E(V)]_{\mathbb{Z}/2}$. This should be thought of as the cohomology group associated to the "formal difference" of V and W.

Since every action of $\mathbb{Z}/2$ on real vector space is a direct sum of the trivial action on $\mathbb{R}$ and the action on $\mathbb{R}$ given by multiplication by -1, it turns out that the $\mathbb{Z}/2$-equivariant cohomology associated to a $\mathbb{Z}/2$-spectrum reduces to a bigraded theory:

$$\tilde{E}^{p,q}_{\mathbb{Z}/2}(X) \equiv [X, E(\mathbb{R}^{p,q})]_G,$$

for $p, q \geq 0$, where $\mathbb{R}^{p,q}$ denotes the $\mathbb{Z}/2$ inner product space with $\mathbb{Z}/2$ acting trivially on the first p components and by multiplication by -1 on the last q components. Taking formal differences as above, we extend this definition to $p, q \in \mathbb{Z}$ by defining

$$\tilde{E}^{p,q}_{\mathbb{Z}/2}(X) \equiv [X \wedge S^{\mathbb{R}^{m,n}}, E(\mathbb{R}^{p+m,q+n})],$$

where m, n are any positive integers such that $p+m \geq 0, q+n \geq 0$. The axioms of a $\mathbb{Z}/2$-spectrum ensure that this definition is independent of the choice of m, n. From now on, we write $S^{p,q}$ for the space $S^{\mathbb{R}^{p,q}}$. (Note that $S^{p,q}$ here is not the same as the object referred to by the same notation in [At].)

Notation B.4. *If E is a $\mathbb{Z}/2$-spectrum and X is a (unbased) $\mathbb{Z}/2$-space, then we define*

$$E^{p,q}(X) \equiv E^{\mathbb{R}^{p,q}}_{\mathbb{Z}/2}(X_+) \equiv [X_+, E(\mathbb{R}^{p,q})]_G \cong [X_+ \wedge S^{\mathbb{R}^{m,n}}, E(\mathbb{R}^{p+m,q+n})]_G,$$

where $m, n \geq 0$ are such that $p+m \geq 0, q+n \geq 0$.

By [LMS; I.4.6], a map of $\mathbb{Z}/2$-spectra $E \to F$ is a weak equivalence if and only if $E(V) \to F(V)$ is a weak equivalence of $\mathbb{Z}/2$-spaces for all $\mathbb{Z}/2$-invariant inner product subspaces $V \subset \mathbb{C}^\infty$. It follows that the equivariant cohomology theories $E^{*,*}$, $F^{*,*}$ associated to weakly equivalent $\mathbb{Z}/2$-spectra E and F are isomorphic.

In this paper, we are primarily concerned with the generalized homotopy groups of a $\mathbb{Z}/2$-spectrum E – i.e., the (non-equivariant) homotopy groups of the spectra E^0 and $E^{\mathbb{Z}/2}$. Notice that these groups are encoded as specific equivariant cohomology groups. Namely, we have the natural isomorphisms

$$\pi_p E^0 \cong E^{-p,0}(T^0), \quad \text{and} \quad \pi_p E^{\mathbb{Z}/2} \cong E^{-p,0}(pt).$$

With this in mind, we now give a completely formal treatment of the transfer map

$$\pi_n E^0 \to \pi_n E^{\mathbb{Z}/2}$$

of the type established in Section 5 through more explicit means. We begin by defining a $\mathbb{Z}/2$-equivariant map

$$\tau : S^{0,1} \to T^0_+ \wedge S^{0,1},$$

which in the stable category gives a map from pt_+ to T^0_+. To define this map, view $S^{0,1}$ as the unit circle in the complex plane equipped with the action by complex conjugation. Collapse $\{1, -1\}$ to a single point, forming the wedge product of two circles. The $\mathbb{Z}/2$-action on this quotient space interchanges the two circles and in fact is $\mathbb{Z}/2$-homeomorphic to $T^0_+ \wedge S^{0,1}$. We take τ to be this quotient map.

Now we use the natural $\mathbb{Z}/2$-homotopy equivalences (of spectra)

$$\mathcal{M}\text{aps}_*(Y \wedge S^{0,1}, E(\mathbb{R}^{0,1})) \cong \mathcal{M}\text{aps}_*(Y, \mathcal{M}\text{aps}_*(S^{0,1}, E(\mathbb{R}^{0,1}))) \cong \mathcal{M}\text{aps}_*(Y, E),$$

where Y is any $\mathbb{Z}/2$-space. Here, the second equivalence holds since E is a $\mathbb{Z}/2$-spectrum. In particular, taking $Y = pt_+$ and $Y = T^0_+$, we see that the map τ determines a natural map of spectra

$$tr = \tau^* : E \equiv \mathcal{M}\mathrm{aps}_*^{\mathbb{Z}/2}(T^0_+, E) \to \mathcal{M}\mathrm{aps}_*^{\mathbb{Z}/2}(pt_+, E) \equiv E^{\mathbb{Z}/2},$$

which serves as our transfer map.

The unique map $T^0 \to pt$ induces a $\mathbb{Z}/2$-map $T^0_+ \wedge S^{0,1} \to S^{0,1}$. Note that the composition of

$$S^{0,1} \xrightarrow{\tau} T^0_+ \wedge S^{0,1} \to S^{0,1}$$

gives the standard 2-fold $\mathbb{Z}/2$-covering space map $\eta : S^{0,1} \to S^{0,1}$. Moreover, under the equivariant homeomorphisms

$$\mathcal{M}\mathrm{aps}_*(S^{0,1}, E(\mathbb{R}^{0,1})) \xrightarrow{\cong} \mathcal{M}\mathrm{aps}_*(S^{1,1}, E(\mathbb{R}^{1,1})) \xleftarrow{\cong} \mathcal{M}\mathrm{aps}_*(S^{1,0}, E(\mathbb{R}^{1,0})),$$

the involution η^* on $\mathcal{M}\mathrm{aps}_*(S^{0,1}, E(\mathbb{R}^{0,1}))$ coincides with an involution on $\mathcal{M}\mathrm{aps}_*(S^{1,0}, E(\mathbb{R}^{1,0}))$ induced by a degree two map $\eta' : S^{1,0} \to S^{1,0}$. Taking $\mathbb{Z}/2$ fixed points, we see that the involution on

$$\mathcal{M}\mathrm{aps}_*^{\mathbb{Z}/2}(S^{1,0}, E(\mathbb{R}^{1,0})) \cong \mathcal{M}\mathrm{aps}_*(S^1, E(\mathbb{R}^{1,0})^{\mathbb{Z}/2}) \cong E^{\mathbb{Z}/2}$$

induced by η' is homotopic to multiplication by two with respect to the H-space structure on $E^{\mathbb{Z}/2}$.

We summarize the preceding observations with the following result.

Proposition B.5. *Given a $\mathbb{Z}/2$-spectrum E, the transfer map*

$$tr : E^0 \to E^{\mathbb{Z}/2}$$

defined as above is a map of spectra. Moreover, the composition of induced maps of H-spaces

$$E(0)^{\mathbb{Z}/2} \hookrightarrow E(0) \xrightarrow{tr} E(0)^{\mathbb{Z}/2}$$

is equivalent to multiplication by 2.

In the remainder of this appendix, we indicate how to apply the above constructions and results (including the conjectural B.2) to rewrite much of the body of this paper in an equivariant setting. Recall that for a quasi-projective real variety X, the space $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ is homeomorphic to the fixed point space $\mathcal{M}or_{\mathbb{C}}(X_{\mathbb{C}}, \mathrm{Grass}_{\mathbb{C}})$ under the $\mathbb{Z}/2$-action given by complex conjugation. Moreover, the action of $\mathcal{I}$ on $\mathcal{M}or_{\mathbb{R}}(X, \mathrm{Grass}_{\mathbb{R}})$ extends in the evident manner to an equivariant action of $\tilde{\mathcal{I}}$ on the $\mathbb{Z}/2$-space $\mathcal{M}or_{\mathbb{C}}(X_{\mathbb{C}}, \mathrm{Grass}_{\mathbb{C}})$. Thus, using B.2, associated to this action there is (conjecturally) a $\mathbb{Z}/2$-spectrum

$$\mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(X).$$

Moreover (and still conjecturally), the fixed point spectrum of $\mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(X)$ is weakly equivalent to the spectrum $\mathcal{K}\mathbb{R}^{\mathrm{semi}}(X)$ defined in the body of this paper and the total spectrum of $\mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(X)$ is weakly equivalent to $\mathcal{K}^{\mathrm{semi}}(X_{\mathbb{C}})$ defined in [FW2].

Similarly, we can define (conjecturally) a $\mathbb{Z}/2$-spectrum encoding both $\mathcal{K}^{\mathrm{alg}}(X)$ and $\mathcal{K}^{\mathrm{alg}}(X_{\mathbb{C}})$ as follows. The $\mathbb{Z}/2$-E_∞ operad $|\tilde{\mathcal{I}}(\Delta^\bullet)|$ acts on the $\mathbb{Z}/2$-space $\mathrm{Hom}_{\mathbb{C}}(\Delta^\bullet_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}}, \mathrm{Grass}_{\mathbb{C}})$ and by B.2 this gives the $\mathbb{Z}/2$-spectrum which we write as $\mathcal{K}^{\mathbb{Z}/2-\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X)$.

Finally, as by now should be apparent by the pattern, we can generalize the construction of $\mathcal{K}\mathbb{R}_{\mathrm{top}}(X_{\mathbb{R}}(\mathbb{C}))$ to give a $\mathbb{Z}/2$-spectrum

$$\mathcal{K}_{\mathbb{Z}/2\text{-top}}(X_{\mathbb{R}}(\mathbb{C})).$$

(Note that in [M2; XIV] a very similar $\mathbb{Z}/2$-spectrum is defined. The conjectural recognition principle B.2 should imply that $\mathcal{K}_{\mathbb{Z}/2\text{-top}}(X_{\mathbb{R}}(\mathbb{C}))$ is equivalent to a connective version of that found in [op. cit.].) Also, the obvious variation on the construction in Section 1 gives the $\mathbb{Z}/2$-spectrum

$$\mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X).$$

We close this appendix by listing equivariant versions of results from the body of this paper, [FW2], and [FW3] which follow easily from the verification of Conjecture B.2. We list these all as conjectures so as to avoid misinterpretation by the casual reader, but we claim they all follow easily from Conjecture B.2 and previously established results.

Conjecture B.6. *For a quasi-projective real variety X, there are natural maps of $\mathbb{Z}/2$-spectra*

$$\mathcal{K}^{\mathbb{Z}/2-\mathrm{alg}}(\Delta^\bullet \times_{\mathbb{R}} X) \to \mathcal{K}^{\mathbb{Z}/2-\mathrm{alg}}(\Delta^\bullet_{top} \times_{\mathbb{R}} X) \to \mathcal{K}^{\mathbb{Z}/2-\mathrm{semi}}(X) \to \mathcal{K}_{\mathbb{Z}/2\text{-top}}(X_{\mathbb{R}}(\mathbb{C}))$$

such that the induced maps on fixed point spectrum coincides with the maps of Sections 2 and 4 and the induced maps on total spectra coincides with the maps of [FW2].

Notice that, in particular, the establishment of Conjecture B.6 would eliminate the need for Section 5 of this paper, since then Theorem 5.8 and Corollary 5.9 would become immediate consequences of this conjecture and Proposition B.5.

Conjecture B.7. *In reference to the maps of Conjecture B.6, (i) the first induces an isomorphism on generalized homotopy groups with finite coefficients for any quasi-projective real variety X (i.e., induces isomorphisms on homotopy groups with finite coefficients for both the fixed point and total spectra), (ii) the second is a weak equivalence of $\mathbb{Z}/2$-spectra for any projective real variety X, and (iii) the third is a weak equivalence of $\mathbb{Z}/2$-spectra if X is as in Propositions 6.1 or 6.2.*

Conjecture B.8. *For any quasi-projective real variety X, there is a commutative diagram of $|\tilde{\mathcal{I}}(\Delta^\bullet)|$-spaces*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{C}}(\Delta^\bullet_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}}, \mathrm{Grass}_{\mathbb{C}}) & \longrightarrow & \coprod_{r\geq 0} Z_{r-1}(\Delta^\bullet_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}}, \mathbb{P}^\infty_{\mathbb{C}})_1 \\
\downarrow & & \downarrow \\
\mathrm{Mor}_{\mathbb{C}}(\Delta^\bullet_{top} \times_{\mathbb{C}} X_{\mathbb{C}}, \mathrm{Grass}_{\mathbb{C}}) & \longrightarrow & \coprod_{r\geq 0} \mathcal{Z}_{r-1}(X_{\mathbb{C}}, \mathbb{P}^\infty_{\mathbb{C}})_1 \\
\downarrow & & \downarrow \\
\mathcal{M}\mathrm{aps}(X_{\mathbb{R}}(\mathbb{C}), \mathrm{Grass}_{\mathbb{R}}(\mathbb{C})) & \longrightarrow & \coprod_{r\geq 0} \mathcal{M}\mathrm{aps}(X_{\mathbb{R}}(\mathbb{C}), \mathcal{Z}_{r-1}(\mathbb{P}^\infty_{\mathbb{R}}(\mathbb{C})))_1,
\end{array}$$

which induces a corresponding diagram of $\mathbb{Z}/2$-spectrum. This diagram specializes to give the statement of Theorem 8.8 by taking fixed point spectra and the statement of [FW2; 6.11] *for $X_{\mathbb{C}}$ by taking total spectra.*

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